J. Math. Sci. Univ. Tokyo
4 (1997), 595–619.

Asymptotic Self-Similarity and Short Time Asymptotics of Stochastic Flows

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Dedicated to Professor Shinzo Watanabe on the occasion of his 60th birthday

Abstract. We study asymptotic properties of Lévy flows, changing scales of the space and the time. Let $\xi_t(x), t \geq 0$ be a Lévy flow on a Euclidean space $\mathbf{R}^d$ determined by a SDE driven by an operator stable Lévy process. Consider the Lévy flows $\xi_t^{(r)}(x) = \gamma_{1/r}^{(x)}(\xi_{rt}(x)), t \geq 0$, where $\{\gamma_r^{(x)}\}_{r>0}$ is a dilation, i.e., a one parameter group of diffeomorphisms of $\mathbf{R}^d$ with invariant point x such that $\gamma_{1/r}^{(x)}(y) \to \infty$ as $r \to 0$ whenever $y \neq x$. We show that as $r \to 0$ $\{\xi_t^{(r)}(x), t \geq 0\}$ converge weakly to a stochastic flow $\{\xi_t^{(0)}(x), t \geq 0\}$, if we choose a suitable dilation. Further, the limit flow is self-similar with respect to the dilation, i.e., its law is invariant by the above changes of the space and the time. This fact enables us to prove that the short time asymptotics of the density function of the distribution of $\xi_t(x)$ coincides with that of the density function of the distribution of $\xi_t^{(0)}(x)$.

1. Introduction

This paper is concerned with the asymptotic self-similarity of Lévy flows driven by operator-stable Lévy processes. Our goal is to obtain the short time asymptotics of density functions of probability distributions of these Lévy flows.

Let us consider the Lévy flow $\{\xi_t, t \geq 0\}$ on a Euclidean space $\mathbf{R}^d$ generated by the following canonical SDE:

$$d\xi_t = \sum_{j=1}^{k} X_j(\xi_t) \diamond dZ^j(t), \tag{1.1}$$

1991 *Mathematics Subject Classification.* 60B15, 60H10, 60J30.

where $X_1, \dots, X_k$ are complete C^∞ vector fields on $\mathbf{R}^d$ and $\{Z(t) = (Z^1(t), \dots, Z^k(t)), t \geq 0\}$ is an operator-stable Lévy process. Suppose that we are given a family of one parameter group of diffeomorphisms $\{\gamma_r^{(x)}\}_{r>0}, x \in \mathbf{R}^d$ such that $\lim_{r\to 0} \gamma_r^{(x)}(y) = x$ uniformly on compact sets of $\mathbf{R}^d$ and $\gamma_r^{(x)}(x) = x, \forall r > 0$ for any $x \in \mathbf{R}^d$. It is called a family of dilations. We change scales of the space and the time of the stochastic processes $\{\xi_t(x), t \geq 0\}$ by the family of dilations and consider the stochastic processes

$$\xi_t^{(r)}(x) = \gamma_{1/r}^{(x)}(\xi_{rt}(x)), \quad t \geq 0. \tag{1.2}$$

The Lévy flow $\{\xi_t, t \geq 0\}$ is called *self-similar with respect to the family of dilations* $\{\gamma_r^{(x)}\}_{r>0}, x \in \mathbf{R}^d$ if the law of the process $\{\xi_t^{(r)}(x), t \geq 0\}$ coincides with the law of the process $\{\xi_t(x), t \geq 0\}$ for any x and $r > 0$.

In the previous paper [11], the author studied the strict self-similarity of the Lévy flow (1.1) (the definiton of the strict self-similarity is slightly stronger than the present one). It turned out that the class of Lévy flows with the strict self-similarity is not a big one. Indeed, if a Lévy flow is strictly self-similar, the Lie algebra generated by vector fields $X_1, \dots, X_k$ is nilpotent and satisfies a specific property (Thorem 3.1).

In the first part of this paper (Sections 2, 3), we show that under Conditions (A.1) and (A.2), there exists a family of dilations $\{\gamma_r^{(x)}\}, x \in \mathbf{R}^d$ such that the family of the stochastic processes $\{\xi_t^{(r)}(x), t \geq 0\}, r > 0$ converges weakly to a stochastic process $\{\xi_t^{(0)}(x), t \geq 0\}$ as $r \to 0$ for all x (Theorem 3.2). The limiting flow $\{\xi_t^{(0)}, t \geq 0\}$ is no longer a solution of a certain SDE and does not define a Markov process any more. However, it has the self-similarity with respect to the dilation $\{\gamma_r^{(x)}\}$ for all x. The Lévy flow $\{\xi_t, t \geq 0\}$ with the above property is called *asymptotically self-similar* and the limiting flow $\{\xi_t^{(0)}, t \geq 0\}$ is called a *self-similar approximation* of the flow $\{\xi_t(x), t \geq 0\}$.

Our argument is based on the asymptotic expansion of the Lévy flow by Champbell Hausdorff formula (Theorem 2.1). In the case where the flow $\{\xi_t, t \geq 0\}$ is driven by a standard Brownian motion (instead of operator-stable Lévy process), similar expansion formulas have been studied by many authors. See Yamato [21], Kunita [8], Ben Arous [2] and Castell [6]. We obtain the Champbell-Hausdorff representation: $\xi_t^{(0)} = \exp \eta_t^{(0)}$, where $\eta_t^{(0)}$ is a linear sum of multiple Wiener-Stratonovich integrals of $Z^1(t), \dots, Z^k(t)$ and it is operator-self-similar (Theorem 3.2).

In the second half of this paper (Sections 4, 5), we study properties of the density functions of probability distributions of self-similar and asymptotically self-similar Lévy flows. Probability distributions of self-similar Lévy flows and those of the approximating self-similar flows have some nice properties. Specially, their density functions can be represented similarly as the heat kernel (Gaussian kernel) of the Laplacian provided that they exist. Indeed, if $\{Z(t), t \geq 0\}$ is a standard Brownian motion with demension $k = d$ and the vector fields $X_1(x), \ldots, X_d(x)$ are linearly independent at any x (nondegenerate), then the density function $p_t^{(0)}(x, y)$ of the distribution of the random variable $\xi_t^{(0)}(x)$ is represented by

$$(1.3)\ p_t^{(0)}(x,y) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|\psi(x,y)|^2}{2t}\right) \left|\det\left(\frac{\partial \varphi_x}{\partial z}(\psi(x,y))\right)\right|^{-1},$$

where $\varphi_x(z) = \exp \sum_j z_j X_j(x)$, $(\partial\varphi_x/\partial z)$ is the Jacobian matrix of $\varphi_x(z)$ and $\psi(x,y) = \varphi_x^{-1}(y)$ (Theorem 4.3 and the remark after the theorem).

We shall then study the short time asymptotics of the density function $p_t(x, y)$ of the distribution of the random variable $\xi_t(x)$ determined by SDE (1.1) in the case where the driving process $Z(t)$ is a standard Brownian motion. We show that its short time asymptotics coincide with the short time asymptotics of $p_t^{(0)}(x, y)$ mentioned above. See Theorem 5.2. This fact enables us to obtain a new result on the short time asymptotics for degenerate heat kernels. See Theorem 5.5.

A lot of works has been done for the short time asymptotics and upper-lower estimates of the heat kernels or fundamental solutions of diffusion equations. See Ben Arous [2], Kusuoka-Stroock [13] [14], Takanobu [17] [18] and others (found in the references of these papers). Our approach to the problem is quite different from those works. We conjecture that the similar short time asymptotics would be valid for the case where the driving process $Z(t)$ is an operator stable Lévy process, but we have not yet succeeded in proving it.

2. Asymptotic expansions of Lévy flows

Let $\{Z(t) = (Z^1(t), \ldots, Z^k(t)), t \geq 0\}$ be a Lévy process with values in $\mathbf{R}^k$ cadlag with respect to t defined on a probability space $(\Omega, \mathcal{F}, P)$. In this paper, we assume that the Lévy process $\{Z(t), t \geq 0\}$ is operator-stable as is stated below. Let Q be a $k\times k$-matrix such that the real parts of its eigen values are positive. A Lévy process $\{Z(t), t\geq 0\}$ is called *operator-stable with exponent* Q (or simply Q*-stable*) if the law of the process $\{r^Q Z(t), t\geq 0\}$ is equal to that of the process $\{Z(rt), t\geq 0\}$ for any $r>0$. The one parameter group of linear transformations $\{r^Q\}_{r>0}$ is called a *dilation*. We assume that the exponent Q is a diagonal matrix with diagonal elements $\alpha_1,\dots,\alpha_k$ such that $\alpha_j\geq 1/2$ for any j.

Let $X_1,\dots,X_k$ be linearly independent complete C^∞ vector fields on a Euclidean space $\mathbf{R}^d$ of dimension d. We will often identify these vector fields with first order partial differential operators. Throughout this paper, we assume:

(A.1) The Lie algebra $\mathcal{L}\equiv\mathcal{L}(X_1,\dots,X_k)$ generated by vector fields X_1, $\dots,X_k$ is a finite dimensional space. Let $\mathcal{L}(x)$ be the projection of $\mathcal{L}$ to the point $x\in\mathbf{R}^d$. Then it holds $\dim\mathcal{L}(x)=d$ for any $x\in\mathbf{R}^d$.

We consider a canonical stochastic differential equation (SDE) on the Euclidean space $\mathbf{R}^d$:

$$\tag{2.1} \begin{cases} d\xi_t &= \sum_{j=1}^k X_j(\xi_t)\diamond dZ^j(t), \\ \xi_0 &= x. \end{cases}$$

By the solution of the above *canonical equation*, we mean a cadlag process $\{\xi_t, t\geq 0\}$ with values in $\mathbf{R}^d$ adapted to $\mathcal{F}_t=\sigma(Z(s); s\leq t)$ satisfying

$$\tag{2.2} \begin{aligned} f(\xi_t) \;=\; & f(x)+\sum_{j=1}^k\int_0^t X_j f(\xi_s)dZ^j(s)+\frac{1}{2}\sum_{i,j}a_{ij}\int_0^t X_iX_jf(\xi_s)ds \\ & +\sum_{0\leq s\leq t}\{f(\varphi_{\xi_{s-}}(\Delta Z(s)))-f(\xi_{s-})-\sum_j X_jf(\xi_{s-})\Delta Z^j(s)\}, \end{aligned}$$

for any $f\in C^\infty(M)$. Here $(a_{ij})t$ is the covariance of the continuous part $Z_c(t)$ of the Lévy process $\{Z(t), t\geq 0\}$. $\varphi_x(z)=\exp\{\sum_{j=1}^k z_jX_j\}(x)$ and $\Delta Z(s)=Z(s)-Z(s-)$, where for a given complete vector field Y on $\mathbf{R}^d$, $\exp tY(x)$ denotes the solution φ_t of the ordinary differential equation $d\varphi_t/dt=Y(\varphi_t)$ with the initial condition $\varphi_0=x$.

In Applebaum-Kunita [1], it is shown that equation (2.1) has a unique global solution and that it has a modification of a *stochastic flow of diffeomorphisms*, i.e., there exists a Diffeo($\mathbf{R}^d$)-valued stochastic process $\{\xi_t, t\geq 0\}$, cadlag with respect to t a.s. such that its projection to $x\in\mathbf{R}^d$ denoted by $\xi_t(x)$ satisfies (2.2). The stochastic flow $\{\xi_t, t \geq 0\}$ has independent increments, i.e., $\xi_{t_i}^{-1}\xi_{t_{i+1}}, i = 0, \ldots, n-1$ are independent for any $0 = t_0 < t_1 < \cdots < t_n$, and is temporally homogeneous, i.e., the laws of $\xi_h^{-1}\xi_{t+h}$ do not depend on $h > 0$. It is called a *Lévy flow* determined by equation (2.1).

For the convenience of the later discussion, we will recall that under Condition (A.1) there exists a Lie group G of dimension n with properties (i)–(iv) below.

(i) G is a Lie transformation group of $\mathbf{R}^d$, i.e., there exists a C^∞-map $\psi : G \times \mathbf{R}^d \to \mathbf{R}^d$ such that $\psi(e, \cdot) = identity$ and $\psi(\tau\sigma, \cdot) = \psi(\sigma, \psi(\tau))$.

(ii) The map $\tau \to \psi(\tau)$ is an isomorphism from G into Diffeo$(\mathbf{R}^d)$.

(iii) Let $\hat{\mathcal{G}}$ be the Lie algebra of G. For any X of $\mathcal{L}$ there exists $\hat{X}$ of $\hat{\mathcal{G}}$ such that

$$\hat{X}(f \circ \psi_x)(\tau) = Xf(\psi_x(\tau))$$

holds for all f of $C^\infty(M)$, where $\psi_x : G \to \mathbf{R}^d$ is defined by $\psi_x(\tau) = \psi(\tau, x)$.

(iv) The Lie transformation group G acts on $\mathbf{R}^d$ transitively so that $\mathbf{R}^d$ is a homogeneous space. (See [1])

We shall make an asymptotic expansion of the flow ξ_t determined by equation (2.1) through Champbell-Hausdorff formula. Our expansion formula is similar to Castell [6]. Let $J = (j_1, \ldots, j_l)$ be a multi-index where $j_1 \ldots, j_l \in \{1, \ldots, k\}$. We set

$$|J| = \text{length of } J, \quad \|J\|_Q = \sum_{j \in J} \alpha_j. \tag{2.3}$$

We define

$$X_J = [X_{j_1}, [X_{j_2}, [\cdots [X_{j_{k-1}}, X_{j_l}] \cdots]. \tag{2.4}$$

The Lie algebra $\mathcal{L}$ is called *nilpotent of step p* if $X_J = 0$ holds for any J such that $|J| \geq p$.

Associated with the multi-index $J = (j_1, \ldots, j_l)$ of length l, we shall define a multiple Wiener-Stratonovich integral

$$Z^J(t) = \int \cdots \int_{A_t(J)} \circ dZ^{j_1}(t_1) \cdots \circ dZ^{j_l}(t_l), \tag{2.5}$$

where $A_t(J) = \{(t_1, \ldots, t_l); 0 < t_1 \leq \cdots \leq t_l \leq t\}$. Then the stochastic process $\{Z^J(t), t \geq 0\}$ is self-similar with exponent $\|J\|_Q$, i.e., the law of the stochastic process $\{r^{\|J\|_Q} Z^J(t), t \geq 0\}$ coincides with that of the stochastic process $\{Z^J(rt), t \geq 0\}$, since the $\{Z(t), t \geq 0\}$ is operator-stable with exponent Q.

THEOREM 2.1. *Let $\{\xi_t, t \geq 0\}$ be the Lévy flow determined by equation (2.1) and let $\{\mathcal{L}\}$ be the Lie algebra generated by vector fields $\{X_1, \ldots, X_k\}$ defining equation (2.1).*

(1) If $\mathcal{L}$ is nilpotent of step p, the Lévy flow is represented by

$$\xi_t(x) = \exp \eta_t(x), \tag{2.6}$$

where

$$\eta_t = \sum_{J;1 \leq |J| \leq p-1} c_t^J X_J, \tag{2.7}$$

and

$$c_t^J = \sum_{\tau;|\tau|=|J|} \frac{(-1)^{e(\tau)}}{|J|^2 \begin{pmatrix} |J|-1 \\ e(\tau) \end{pmatrix}} Z_t^{J \circ \tau^{-1}}. \tag{2.8}$$

Here τ is a permutation of order $|\tau| = l$ and $e(\tau)$ is the cardinality of the set $\{j; j \in \{1, \ldots, l\}; \tau(j) > \tau(j+1)\}$.

(2) Suppose that $\mathcal{L}$ is not necessarily nilpotent. Let $R_J(t,x)$ be the process defined by

$$\xi_t(x) = \exp \eta_t(x) + \sum_{J;|J|=p} t^{\|J\|_Q} R_J(t,x). \tag{2.9}$$

Then

$$\lim_{\delta \to \infty} \sup_{0 \leq t \leq 1} P(|R_J(t,x)| > \delta) = 0. \tag{2.10}$$

For the proof of the theorem, we consider an SDE with multi-parameter $\epsilon = (\epsilon_1, \ldots, \epsilon_k)$:

$$d\varphi_t^\epsilon = \sum_{j=1}^k \epsilon_j X_j(\varphi_t^\epsilon) \diamond dZ^j(t). \tag{2.11}$$

Let $\varphi_t^\epsilon(x)$ be the Lévy flow generated by the above SDE. If $\epsilon_j = r^{\alpha_j}, j = 1, \ldots, k$ where $r > 0$, we denote φ_t^ϵ by $\varphi_t^{(r)}$. Then the law of the stochastic process $\{\varphi_t^{(r)}, t \geq 0\}$ is equal to that of the stochastic process $\{\xi_{rt}, t \geq 0\}$, because of the operator-stable property of $\{Z(t), t \geq 0\}$.

LEMMA 2.2. *(1) If $\mathcal{L}$ is nilpotent of step p, the Lévy flow $\{\varphi_t^\epsilon, t \geq 0\}$ determined by equation (2.11) is represented by*

$$\varphi_t^\epsilon(x) = \exp \zeta_t^\epsilon(x), \tag{2.12}$$

where

$$\zeta_t^\epsilon = \sum_{J;1\leq |J| \leq p-1} \epsilon^J c_t^J X_J, \quad \epsilon^J = \epsilon_{j_1} \cdots \epsilon_{j_l} \quad \text{if} \quad J = (j_1, \ldots, j_l). \tag{2.13}$$

(2) Suppose that $\mathcal{L}$ is not necessarily nilpotent. For an arbitrary fixed positive integer p, define ζ_t^ϵ by (2.13). Let $\{R_J^\epsilon(t,x), t \geq 0\}$ be the stochastic process such that

$$\varphi_t^\epsilon(x) = \exp \zeta_t^\epsilon(x) + \sum_{J;|J|=p} \epsilon^J R_J^\epsilon(t,x). \tag{2.14}$$

Then

$$\lim_{\delta\to\infty} \sup_{0\leq|\epsilon|\leq 1} P(|R^\epsilon(t,x)| > \delta) = 0, \tag{2.15}$$

for any $t > 0$ and $x \in \mathbf{R}^d$.

PROOF. Consider a sequence of stochastic ordinary differential equations:

$$\frac{d\varphi_t^{\epsilon,n}}{dt} = A^{\epsilon,n}(\varphi_t^{\epsilon,n}), \tag{2.16}$$

where

$$A^{\epsilon,n}(s,x) = 2^n \sum_{j=1}^k \epsilon_j \delta_i^n Z^j X_j(x), \quad \text{if} \quad s \in [t_i, t_{i+1}), \tag{2.17}$$

and $\delta_i^n Z^j = Z^j(t_{i+1}) - Z^j(t_i)$ and $t_i = i/2^n, i = 0, 1, 2, \ldots$. Let $\{\varphi_t^{\epsilon,n}(x), t \geq 0\}$ be the solution of (2.16) such that $\varphi_0^{\epsilon,n}(x) = x$. It is known that finite dimensional distributions of $\{\varphi_t^{\epsilon,n}(x), t \geq 0\}$ converge weakly to those of $\{\varphi_t^\epsilon(x), t \geq 0\}$. See Kunita [10].

On the other hand, we can show similarly as in Castell [6] that for any fixed positive integer p $\varphi_t^{\epsilon,n}(x)$ is represented by

$$\varphi_t^{\epsilon,n}(x) = \exp \zeta_t^{\epsilon,n}(x) + \sum_{J;|J|=p} \epsilon^J R_J^{\epsilon,n}(t,x). \tag{2.18}$$

Here,

$$\zeta_t^{\epsilon,n} = \sum_{J;1\le|J|\le p-1} \epsilon^J c_t^{J,n} X_J, \tag{2.19}$$

$$c_t^{J,n} = \sum_{\tau;|\tau|=|J|} \frac{(-1)^{e(\tau)}}{|J|^2 \begin{pmatrix} |J|-1 \\ e(\tau) \end{pmatrix}} Z_t^{J\circ\tau^{-1},n}, \tag{2.20}$$

and

$$Z_t^{J,n} = \int\cdots\int_{A_t(J)} \dot{Z}^{j_1,n}(t_1)\cdots\dot{Z}^{j_l,n}(t_l)dt_1\cdots dt_l,$$

where $\dot{Z}^{j,n}(t) = 2^n\delta_i^n Z^j$ if $t\in[t_i,t_{i+1})$.

Suppose first that $\mathcal{L}$ is nilpotent of step p. Then it holds $R^{\epsilon,J}(x,t) = 0$. Further $\{Z_t^{J,n}, t\ge 0\}$ converges to $\{Z_t^J, t\ge 0\}$ strongly as $n\to\infty$. Thererfore, $\{\zeta_t^{\epsilon,n}, t\ge 0\}$ converges to

$$\zeta_t^{\epsilon} = \sum_{J;1\le|J|\le p-1} \epsilon^J c_t^J X_J, t\ge 0 \tag{2.21}$$

strongly as $n\to\infty$. Consequently, we have $\varphi_t^\epsilon(x) = \exp\zeta_t^\epsilon(x)$, proving the first assertion (1).

Suppose next that $\mathcal{L}$ is not nilpotent. Since $\varphi_t^{\epsilon,n}(x) \to \varphi_t^\epsilon(x)$ and $\zeta_t^{\epsilon,n}(x) \to \zeta_t^\epsilon(x)$ hold as $n\to\infty$ in equation (2.18), $\{R_J^{\epsilon,n}(t,x), t\ge 0\}$ converges to $\{R_J^\epsilon(t,x)\}$. Therefore we obtain (2.14). For the proof of the last assertion (2.15), we consider the follwing two ϵ expansions:

$$\varphi_t^\epsilon(x) = x + \sum_{J;1\le|J|\le p-1} \epsilon^J q_J(t,x) + \sum_{J;|J|=p} \epsilon^J Q_J^\epsilon(t,x), \tag{2.22}$$

and

$$\exp\zeta_t^\epsilon(x) = x + \sum_{J;1\le|J|\le p-1} \epsilon^J h_J(t,x) + \sum_{J;|J|=p} \epsilon^J P_J^\epsilon(t,x). \tag{2.23}$$

Since (2.14) holds, we have $q_J = h_J$ for J such that $|J|\le p-1$. Therefore we have $R_J^\epsilon(t,x) = Q_J^\epsilon(t,x) - P_J^\epsilon(t,x)$. Since both $\varphi_t^\epsilon(x)$ and $\exp\zeta_t^\epsilon(x)$ are infinitely differentiable with respect to ϵ, $Q_J^\epsilon(t,x)$ and $P_J^\epsilon(t,x)$ have the same properties. Then $R_J^\epsilon(t,x)$ has also the same property. This implies (2.15). The proof is complete. □

PROOF OF THEOREM 2.1. The first assertion (1) is immediate from Lemma 2.2 (1) by setting $\epsilon = 1$. Next, set $\epsilon_j = r^{\alpha_j}, j = 1,\dots,k$ and $t = 1$ in equation (2.14). Note that the law of the random variable $\varphi_1^{(r)}(x)$ coincides with that of $\xi_r(x)$ and the law of the random variable $\zeta_1^{(r)}$ coincides with that of η_r for any $r > 0$ and $x \in \mathbf{R}^d$. Then the law of the random variable $R_J^{(r)}(1, x)$ coincides with that of $R_J(r, x)$. Then we get the equalities (2.9) and (2.10) for $t = r$. The proof is complete. □

3. Self-similar and asymptotically self-similar Lévy flows

The strict self-similarity of a Lévy flow is studied in Kunita [11]. Let us recall it quickly. Let $\{\gamma_r\}_{r>0}$ be a one parameter group of diffeomorphisms of the Euclidean space $\mathbf{R}^d$ such that $\lim_{r\to 0} \gamma_r(x) = x_0$ holds, where x_0 is a certain fixed point of $\mathbf{R}^d$. It is called a dilation. Let $d\gamma_r$ be the differential of the map γ_r. Then $d\gamma_r$ is an automorphism of the Lie algebra $\mathcal{V}(\mathbf{R}^d)$ of C^∞-vector fields of $\mathbf{R}^d$, i.e., it is a linear invertible map of $\mathcal{V}(\mathbf{R}^d)$ and satisfies $d\gamma_r[X, Y] = [d\gamma_r X, d\gamma_r Y]$. Thus $\{d\gamma_r\}_{r>0}$ is a one parameter group of automorphisms of $\mathcal{V}(\mathbf{R}^d)$.

The law of the Lévy flow $\{\xi_t, t \geq 0\}$ can be defined on the Skorohod space $D[[0,\infty), \mathrm{Diffeo}(\mathbf{R}^d))$ (the space of cadlag maps from $[0,\infty)$ to $\mathrm{Diffeo}(\mathbf{R}^d)$). The Lévy flow is called *strictly self-similar with respect to the dilation* $\{\gamma_r\}$ if the law of the Lévy flow $\{\gamma_r \circ \xi_t \circ \gamma_r^{-1}, t \geq 0\}$ coincides with that of $\{\xi_{rt}, t \geq 0\}$ for any $r > 0$. Denote the distribution of $\xi_t(x)$ by $P_t(x, E)$. Then if the Lévy flow is strictly self-similar, we have

$$P_t(x, E) = P_{rt}(\gamma_r(x), \gamma_r(E)), \quad \forall r > 0, t > 0. \tag{3.1}$$

A Lévy flow driven by an operator-stable Lévy process $\{Z(t), t \geq 0\}$ is not always strictly self-similar with respect to a certain dilation. A characterization of the strictly self-similar Lévy flow was given in [11]. Before we state the result, let us remark that the exponent Q of the driving opeator-stable Lévy process $\{Z(t), t \geq 0\}$ can be regarded as a linear transformation on $l.s.\{X_1, \ldots, X_k\}$.

Throughout the rest of this paper, we assume:

(A.2) For any point $x \in \mathbf{R}^d$, $\exp X(x) = x, X \in \mathcal{L}$ implies $X = 0$.

The above condition means that the isotropic subgroup

$$H_x = \{\sigma \in G; \psi(\sigma, x) = x\}$$

of the Lie transformation group G is trivial, i.e., $H_x = \{e\}$ for any x. Therefore, G is diffeomorphic to $\mathbf{R}^d$. Then the group G should be solvable by the Iwasawa decomposition.

THEOREM 3.1. (Kunita [11]) *Assume that there exists an extension* $\bar{Q}$ *of the linear transformation* Q *on the space* $\mathcal{L}$ *such that*

$$\bar{Q}[X,Y] = [\bar{Q}X,Y] + [X,\bar{Q}Y], \quad \forall X,Y \in \mathcal{L}. \tag{3.2}$$

Then there exists a dilation $\{\gamma_r\}_{r>0}$ *with respect to which the Lévy flow* $\{\xi_t, t \geq 0\}$ *determined by equation (2.1) is strictly self-similar. Furthermore, their differentials* $d\gamma_r$ *map* $\mathcal{L}$ *into itself linearly and are represented by*

$$d\gamma_r = r^{\bar{Q}}. \tag{3.3}$$

Moreover, the Lie algebra $\mathcal{L}$ *is nilpotent.*

Conversely assume that the Lévy flow $\{\xi_t, t \geq 0\}$ *determined by equation (2.1) is strictly self-similar with respect to a certain dilation* $\{\gamma_r\}_{r>0}$*. Then the Lie algebra* $\mathcal{L}$ *is nilpotent and the differentials* $\{d\gamma_r\}$ *define a one parameter group of automorphsims of* $\mathcal{L}$ *and these are represented by (3.3), where* $\bar{Q}$ *is an extension of* Q *satisfying (3.2).*

We call $\bar{Q}$ the *exponent of the dilation* $\{\gamma_r\}_{r>0}$.

In the sequel, we shall define a weaker self-similarity of the stochastic flow. We want to relax the notion of the dilation. Let S be a simply connected domain of the Euclidean space $\mathbf{R}^d$. Let $\{\gamma_r\}_{r>0}$ be a one parameter group of diffeomorphims of S such that $\gamma_r(y) \to x$ holds unifromly on compact sets of S as $r \to 0$, where x is a point of S. It is called a dilation on S.

Suppose we are given an arbitrary point $x \in \mathbf{R}^d$ and an arbitrary linear transformation R on $\mathcal{L}$ such that the real parts of its eigen values are all positive. We shall construct a dilation with the invariant point x and the exponent R. Set

$$S_x = \{y = \exp X(x); X \in \mathcal{L}\}. \tag{3.4}$$

Then it is a domain of $\mathbf{R}^d$ including the point x so that it can be regarded as a neighborhood of x. Define $\gamma_r^{(x)} : S_x \to S_x$ by

$$\gamma_r^{(x)}(\exp X(x)) = \exp r^R X(x). \tag{3.5}$$

Then $\gamma_r^{(x)}$ is an onto C^∞-map of S_x. Moreover, it is one to one by Condition (A.2). Then S_x is a domain of $\mathbf{R}^d$ and $\gamma_r^{(x)}$ is a diffeomorphism of S_x. Further, the family of diffeomorphisms $\{\gamma_r^{(x)}\}_{r>0}$ satisfies $\gamma_t^{(x)}\gamma_s^{(x)} = \gamma_{st}^{(x)}$ for all $s, t > 0$ and $\lim_{r\to 0}\gamma_r^{(x)}(y) = x$ for all $y \in S_x$. Therefore $\{\gamma_r^{(x)}\}_{r>0}$ is a dilation on the simply connected manifold S_x with the invariant point x and exponent R. The family of dilations $\{\gamma_r^{(x)}\}_{r>0}, x \in \mathbf{R}^d$ satisfying (3.5) for all $x \in \mathbf{R}^d$ is said to have the *common exponent* R.

Let $\{\xi_t, t \geq 0\}$ be a stochastic flow on $\mathbf{R}^d$. It may or may not be a Lévy flow determined by SDE. It is called *self-similar with respect to the family of dilations* $\{\gamma_r^{(x)}\}$ on S_x, $x \in \mathbf{R}^d$, if the stochastic process $\{\xi_t(x), t \geq 0\}$ takes values in S_x and the law of the stochastic process $\{\gamma_r^{(x)} \circ \xi_t(x), t \geq 0\}$ coincides with the law of the stochastic process $\{\xi_{rt}(x), t \geq 0\}$ for any $x \in \mathbf{R}^d$.

Remark. If a Lévy flow $\{\xi_t, t \geq 0\}$ is strictly self-similar with respect to a certain dilation with exponent $\bar{Q}$, then it is self-similar with respect to a family of dilations $\{\gamma_r^{(x)}\}$ on $\mathbf{R}^d$, $x \in \mathbf{R}^d$ with the common exponent $\bar{Q}$. Indeed we define dilations $\{\gamma_r^{(x)}\}$ on $S_x = \mathbf{R}^d$ through (3.5). Then, the stochastic process $\{\xi_t(x), t \geq 0\}$ is self-similar with respect to the dilation $\{\gamma_r^{(x)}\}_{r>0}$ for any $x \in \mathbf{R}^d$.

Conversely, a self-similar Lévy flow is not necessarily strictly self-similar. Indeed, we do not require that the law of the Lévy flow $\{\gamma_r^{(x)} \circ \xi_t \circ \gamma_{1/r}^{(x)}, t \geq 0\}$ coincides with the law of the Lévy flow $\{\xi_{rt}, t \geq 0\}$. Then the differential $d\gamma^{(x)}$ of the map $\gamma_r^{(x)}$ maps $\mathcal{V}(S_x)$ into itself but does not map $\mathcal{L}$ into itself. Further $\mathcal{L}$ is not necessarily nilpotent. Here is an example. Consider a Brownian flow $\xi_t(x)$ on $\mathbf{R}^1$ determined by the Ito SDE $d\xi_t = \cos(\xi_t)dB^1(t) + \sin(\xi_t)dB^2(t)$, where $(B^1(t), B^2(t))$ is a standard Brownian motion. The solution $\xi_t(x)$ is a Brownian motion on $\mathbf{R}^1$ starting from x, since its infinitesimal generator is $2^{-1}\cos^2 x(d^2/dx^2) + 2^{-1}\sin^2 x(d^2/dx^2) = 2^{-1}d^2/dx^2$. Therefore it is self-similar (stable) with exponent $1/2$. However it is not strictly self-similar, since the Lie algebra generated by $X_1 = (\cos x)d/dx$ and $X_2 = (\sin x)d/dx$ is not nilpotent.

We shall next define the asymptotic self-similarity of the stochastic flow $\{\xi_t, t \geq 0\}$. Suppose we are given a family of dilations $\{\gamma_r^{(x)}\}, x \in \mathbf{R}^d$ on domains $S_x, x \in \mathbf{R}^d$. Let $\tau(x)$ be the first leaving time of the trajectry

$\{\xi_t(x), t \geq 0\}$ from the domain S_x. Then the stopped process $\{\xi'_t(x) = \xi_{t\wedge\tau(x)}(x), t \geq 0\}$ defines a stochastic flow on S_x. We define a family of stochastic processes $\{\xi_t^{(r)}(x), t \geq 0\}$ with parameter $\{r > 0\}$ by

$$\xi_t^{(r)}(x) = \gamma_{1/r}^{(x)} \circ \xi'_{rt}(x). \tag{3.6}$$

If $S_x = \mathbf{R}^d$, the law of the stochastic process $\{\xi_t^{(r)}(x), t \geq 0\}$ coincides with that of $\{\xi_t(x), t \geq 0\}$ for all r if and only if the Lévy flow $\{\xi_t, t \geq 0\}$ is self-similar with respect to the family of dilations $\{\gamma_r^{(x)}\}, x \in \mathbf{R}^d$. The Lévy flow $\{\xi_t, t \geq 0\}$ is called *asymptotically self-similar (with respect to the family of dilations* $\{\gamma_r^{(x)}\}, x \in \mathbf{R}^d$*)* if there exists a stochastic flow $\{\xi_t^{(0)}, t \geq 0\}$ such that for any x, the family of stochastic processes $\{\xi_t^{(r)}(x), t \geq 0\}, r > 0$ converges weakly to the stochastic process $\{\xi_t^{(0)}(x), t \geq 0\}$ as $r \to 0$. The limiting flow $\{\xi_t^{(0)}, t \geq 0\}$ is always self-similar with respect the dilation $\{\gamma_r^{(x)}\}$ if it exists. Indeed, we have

$$\begin{aligned}\{\gamma_s^{(x)} \circ \xi_t^{(0)}(x), t \geq 0\} &= \lim_{r\to 0}\{\gamma_s^{(x)}\gamma_{1/r}^{(x)} \circ \xi_{rt}(x), t \geq 0\} \\ &= \lim_{r/s\to 0}\{\gamma_{s/r}^{(x)} \circ \xi_{(r/s)st}(x) : t \geq 0\} = \{\xi_{st}^{(0)}(x), t \geq 0\}\end{aligned}$$

in the sense of distributions. We call $\{\xi_t^{(0)}, t \geq 0\}$ a *self-similar approximation* of $\{\xi_t, t \geq 0\}$ based on $\{\gamma_r^{(x)}\}, x \in \mathbf{R}^d$.

In the sequel we show that any Lévy flow determined by SDE (2.1) is asymptotically self-similar. We shall introduce exponents and dilations adapted to SDE (2.1). Let $\{X_1, \dots, X_k, X_{k+1}, \dots, X_n\}$ be a basis of $\mathcal{L}$ such that $X_1, \dots, X_k$ are vector fields defining SDE (2.1). We denote the projection of X_J to $\{X_j\}$ by $\mathrm{P}_{X_j}X_J$. We set

$$\alpha_j^{(0)} = \min\{\|J\|_Q; P_{X_j}X_J \neq 0\}, \qquad j = 1, \dots, n. \tag{3.7}$$

Let $\beta_j, j = 1, \dots, n$ be arbitrary numbers satisfying $1/2 \leq \beta_j \leq \alpha_j^{(0)}, j = 1, \dots, n$. Then it holds $\beta_j \leq \alpha_j$ for any $j = 1, \dots, k$. The $n \times n$ diagonal matrix R with diagonal elements $\beta_1, \dots, \beta_n$ is called an exponent *adapted to* SDE (2.1) and the family of dilations $\{\gamma_r^{(x)}\}, x \in \mathbf{R}^d$ with the common exponent R is said to be *adapted to* SDE (2.1).

THEOREM 3.2. *The Lévy flow $\{\xi_t, t \geq 0\}$ determined by equation (2.1) is asymptotically self-similar with respect to any family of adapted dilations. Let (2.9) be the representation of the Lévy flow. Then the self-similar approximation $\{\xi_t^{(0)}, t \geq 0\}$ is represented by*

$$\xi_t^{(0)} = \exp \eta_t^{(0)}, \tag{3.8}$$

where

$$\eta_t^{(0)} = \sum_{j \in K} \left(\sum_{J; \|J\|_Q = \beta_j} a_J^j c_t^J \right) X_j. \tag{3.9}$$

Here $\beta_1, \ldots, b_n$ are diagonal elements of the adapted exponent matrix R, $K = \{j; \beta_j = \alpha_j^{(0)}\}$ and $a_J^j, j \in K$ are constants such that $a_J^j X_j = P_{X_j} X_J$.

PROOF. Let $\{\varphi_t^{(r)}, t \geq 0\}$ be the Lévy flow determined by equation (2.11) where $\epsilon_j = r^{\alpha_j}, j = 1, \ldots, k$. Then we have by Lemma 2.2,

$$\varphi_t^{(r)} = \exp \zeta_t^{(r)} + \sum_{|J|=p} r^{\|J\|_Q} R_J^{(r)}(t, \cdot), \tag{3.10}$$

where $p > 2 \max_j \beta_j + 1$ and

$$\zeta_t^{(r)} = \sum_{J; 1 \leq |J| \leq p-1} r^{\|J\|_Q} c_t^J X_J. \tag{3.11}$$

Let $\{\gamma_r^{(x)}\}, x \in \mathbf{R}^d$ be a family of dilations with the common exponent R. We have

$$\begin{aligned} \gamma_{1/r}^{(x)} \circ \varphi_t^{(r)}(x) &= \gamma_{1/r}^{(x)} \circ \exp \zeta_t^{(r)}(x) \\ &\quad + D\gamma_{1/r}^{(x)}(x + \theta) \sum_{J; |J| = p} r^{\|J\|_Q} R_J^{(r)}(t, x), \end{aligned} \tag{3.12}$$

where θ is a vector such that $|\theta| \leq \sum_{J;|J|=p} r^{\|J\|_Q} |R_J^{(r)}(t,x)|$ and $D\gamma_{1/r}^{(x)}$ is the Jacobian matrix of the map $\gamma_{1/r}^{(x)}$. We have

$$\begin{aligned} \gamma_{1/r}^{(x)} \circ \exp \zeta_t^{(r)}(x) &= \exp \left\{ r^{-R} \sum_{J; 1 \leq |J| \leq p-1} r^{\|J\|_Q} c_t^J X_J \right\} (x) \\ &= \exp \left\{ \sum_{j \in K} \left(\sum_{J; \|J\|_Q = \beta_j} a_J^j c_t^J \right) X_j + O(r) \right\} (x), \end{aligned} \tag{3.13}$$

where

$$O(r) = \sum_{j\in K}\left(\sum_{J;1\le|J|\le p-1,\|J\|_Q>\beta_j} r^{\|J\|_Q-\beta_j} a_J^j c_t^J\right) X_j.$$

Clearly $O(r)$ converges to 0 a.s. as $r\to 0$. Therefore (3.13) converges to $\xi_t^{(0)}(x)$ of (3.8) a.s. as $r\to 0$. Since $\|D\gamma_{1/r}^{(x)}\| = O(r^{-\max_j\beta_j})$, and $\|J\|_Q \ge |J|/2 > \max_j\beta_j$ holds if $|J| = p$, the last member of (3.12) converges to 0 a.s. as $r\to 0$. Consequently, $\{\gamma_{1/r}^{(x)}\circ\varphi_t^{(r)}(x), t\ge 0\}$ converges to $\{\xi_t^{(0)}(x), t\ge 0\}$ a.s. Now, since the law of $\{\xi_{rt}, t\ge 0\}$ coincides with the law of $\{\varphi_t^{(r)}, t\ge 0\}$, the law of $\{\xi_t^{(r)}(x); t\ge 0\}$ coincides with that of $\{\gamma_{1/r}^{(x)}\circ\varphi_t^{(r)}(x); t\ge 0\}$. Therefore the former converges weakly to $\{\xi_t^{(0)}(x), t\ge 0\}$. The proof is complete. □

Remark. The stochastic process $\{\eta_t^{(0)}, t\ge 0\}$ of (3.9) can be identified with $\mathbf{R}^{|K|}$-valued process $\{(\sum_{J:\|J\|_Q=\beta_j} a_J^j c_t^J)_{j\in K}, t\ge 0\}$. It is self-similar with respect to the linear transformations $\{r^{R_0}\}_{r>0}$, where R_0 is the restriction of the exponent R to $\mathbf{R}^{|K|}$.

The stochastic flow $\{\xi_t^{(0)}, t\ge 0\}$ is not a Lévy flow in general. Thus the stochastic process $\{\xi_t^{(0)}(x), t\ge 0\}$ can not be obtained by solving a certain SDE and it is not always Markovian.

Corollary 3.3. *Let $P_t(x,\cdot)$ and $P_t^{(0)}(x,\cdot)$ be the probability distributions of the random variables $\xi_t(x)$ and $\xi_t^{(0)}(x)$, respectively. Then, for any Borel subset E of S_x such that $P_t^{(0)}(x,\partial E) = 0$,*

$$\exists \lim_{r\to 0} P_{rt}(x,\gamma_r^{(x)}(E)) = P_t^{(0)}(x,E). \tag{3.14}$$

Further, $P_t^{(0)}(x,E)$ is self-similar with respect to the dilation $\{\gamma_r^{(x)}\}$, i.e., for any Borel subset E of S_x,

$$P_t^{(0)}(x,E) = P_{rt}^{(0)}(x,\gamma_r^{(x)}(E)),\quad \forall r>0. \tag{3.15}$$

Remark. Self-similar approximations of a given Lévy flow $\{\xi_t, t\ge 0\}$ are not unique. In the next section, we will discuss a canonical dilation, which plays an important role for the study of the short time asymptotics of the transition density function.

4. Representations of probability densities of self-similar stochastic flows

Let $Y_1, \ldots, Y_d$ be linear independent vectors in $\mathcal{L}$ such that $l.s.\{Y_1, \ldots, Y_d\}$ is invariant with respect to the linear transformation $\bar{Q}$. Let $dy_1, \ldots, dy_d$ be 1-forms such that $(dy_i)_x((Y_j)_x) = \delta_{ij}$ holds for all $x \in \mathbf{R}^d$. Since $Y_i, i = 1, \ldots, d$ are invariant with respect to the Lie transformation group G, the 1-forms $dy_1, ..., dy_d$ are also G-invariant. Consider the d-form $m = dy_1 \wedge \cdots \wedge dy_d$. It is a G-invariant d-form. The form m is a volume form if $\{(Y_1)_x, \ldots, (Y_d)_x\}$ are linear independent for any x.

Let $\{\xi_t, t \geq 0\}$ be the Lévy flow determined by SDE (2.1) and let $P_t(x, E)$ be the distribution of the random variable $\xi_t(x)$. The existence of the density function of $P_t(x, E)$ with respect to a volume form m has been studied in details by Malliavin [16], Ikeda-Watanabe [7], Kusuoka-Stroock [12] in the case where the driving process $Z(t)$ is a standard Brownian motion. Indeed, it has beeen shown by these authors that under Condition (A.1) $P_t(x, E)$ has a C^∞ density function. A similar problem has beeen studied by Bismut [5], Bichiteler-Gravereaux-Jacod [4] and Leandre [15] in the case where $Z(t)$ is Lévy process with jumps. However, it seems to us that those results are not sufficiently sharp for our purpose. Thus in this section, we assume the existence of the continuous density function $p_t(x, y)$ with respect to the d-form m.

We shall obtain the representation of the density function $p_t(x, y)$ by means of the dilation. We first consider the case where the Lévy flow is strictly self-similar.

Theorem 4.1. *Let $\{\xi_t, t \geq 0\}$ ba a Lévy flow determined by SDE (2.1), which is strictly self-similar with respect to the dilation $\{\gamma_r\}$ and let $P_t(x, E)$ be the distribution of the random variable $\xi_t(x)$. Suppose that $P_t(x, E)$ has a density funtion $p_t(x, y)$ with respect to the G-invariant d-form m. If it is continuous in (t, x, y), it satisfies*

$$p_t(x, y) = \frac{1}{t^{\mathrm{tr} Q_0}} p_1(\gamma_{1/t}(x), \gamma_{1/t}(y)), \quad \forall x, y \in \mathbf{R}^d, \tag{4.1}$$

where Q_0 is the restriction of the exponent $\bar{Q}$ of the dilation $\{\gamma_r\}$ to the space $\{Y_1, \ldots, Y_d\}$.

PROOF. The density function $p_t(x,y)$ satisfies

$$\begin{aligned}\int_A p_t(x,y)m(dy) &= P(\xi_t(x)\in A) = P(\gamma_t(\varphi_1\circ\gamma_{1/t}(x))\in A)\\ &= P_1(\gamma_{1/t}(x),\gamma_t^{-1}(A)),\end{aligned}$$

and

$$P_1(x,\gamma_t^{-1}(A)) = \int_{\gamma_t^{-1}(A)} p_1(x,y)m(dy) = \int_A p_1(x,\gamma_{1/t}(y))\gamma_t m(dy),$$

where $\gamma_t m$ is the measure such that $\gamma_t m(A) = m(\gamma_t^{-1}(A))$. We can regard $\gamma_t m$ as the pull back of the d-form m by the map γ_t. Then, $\gamma_t m$ satisfies

$$\begin{aligned}(\gamma_t m)_y((Y_1)_y,\dots,(Y_d)_y) &= m_{\gamma_t(y)}((d\gamma_t Y_1)_{\gamma_t(y)},\dots,(d\gamma_t Y_d)_{\gamma_t(y)})\\ &= |\det d\gamma_t|^{-1} m_{\gamma_t(y)}((Y_1)_{\gamma_t(y)},\dots,(Y_d)_{\gamma_t(y)})\\ &= |\det d\gamma_t|^{-1} m_y((Y_1)_y,\cdots,(Y_d)_y),\end{aligned}$$

because the d-form m and vector fields $Y_1,\dots,Y_d$ are G-invariant. Since $d\gamma_t = t^{Q_0}$ holds by (3.3), we have $|\det d\gamma_t| = t^{\mathrm{tr}Q_0}$. Therefore we have the equality $\gamma_t m = t^{-\mathrm{tr}Q_0}m$. Consequently we obtain

$$\int_A p_t(x,y)m(dy) = \int_A p_1(\gamma_{1/t}(x),\gamma_{1/t}(y))\frac{1}{t^{\mathrm{tr}Q_0}}m(dy)$$

for any Borel set A. This implies the formula (4.1). The proof is complete. □

We shall next consider the density function of the distribution of a self-similar approximation $\xi_t^{(0)}(x)$ of the Lévy flow determined by SDE (2.1), in the case where the associated dilation is canonical. Let $K=\{j_1,\dots,j_d\}$ be a subset of $\{1,\dots,n\}$ such that $(X_{j_1})_x,\dots,(X_{j_d})_x$ are linearly independent. Set $\mathcal{L}_K = l.s.\{X_j; j\in K\}$. It is a subspace of $\mathcal{L}$. An adapted exponent R is called *canonical* if their diagonal elements $\beta_j, j=1,...,n$ satisfy $\beta_j=\alpha_j^{(0)}$ if $j\in K$ and $\beta_j<\alpha_j^{(0)}$ if $j\notin K$. Let $\{\gamma_r^{(x)}\}$ be a family of dilations with the common canonical exponent R. It is called a family of *canonical dilations* and the corresponding self-similar approximation is called the *canonical self-similar approximation*. We need a lemma.

Lemma 4.2. *Let $\varphi_x : \mathcal{L}_K \to \mathbf{R}^d$ be a map such that $\varphi_x(z) = \exp(\sum_{j\in K} z_j X_j)(x)$. Let ν be the Lebesgue measure on $\mathcal{L}_K$. For each x, define a measure $\hat{m}_x$ on S by $\hat{m}_x = \varphi_x \nu$:*

$$\hat{m}_x(A) = \nu(\varphi_x^{-1}(A)). \tag{4.2}$$

Then it is a volume form at a neighborhood of x. Let $\{\gamma_r^{(x)}\}$ be a family of dilations with common canonical exponent R. Then it holds $\gamma_r^{(x)}\hat{m}_x = r^{\mathrm{tr}R_0}\hat{m}_x$, where R_0 is the restriction of R to $\mathcal{L}_K$.

Proof. Since the map $\varphi_x : \mathcal{L}_K \to \mathbf{R}^d$ is a local diffeomorphism at a neighborhood of $0 \in \mathcal{L}_K$, the measure $\hat{m}_x$ is a volume form at a neighborhood of x. For a Borel set A in $\mathbf{R}^d$, set $\tilde{A} = \{z : \varphi_x(z) \in A\}$. Then we have $\{z : \gamma_r^{(x)}\varphi_x(z) \in A\} = \{r^{R_0} z \in \tilde{A}\}$. Therefore we have

$$\begin{aligned} \gamma_r^{(x)}\hat{m}_x(A) &= \nu(\{z : \gamma_r^{(x)}\varphi_x(z) \in A\}) = \nu(r^{-R_0}\tilde{A}) \\ &= r^{\mathrm{tr}R_0}\nu(\tilde{A}) = r^{\mathrm{tr}R_0}\hat{m}_x(A). \ \square \end{aligned}$$

Theorem 4.3. *Let $\{\xi_t^{(0)}, t \geq 0\}$ be the stochastic flow determined by (3.8) and (3.9). Suppose that the distribution of $\eta_1^{(0)} = \log \xi_1^{(0)}$ has a continuous density function $f(x)$ with respect to the Lebesgue measure. Then the distribution $P_t^{(0)}(x, E)$ of the random variable $\xi_t^{(0)}(x)$ has a continuous density function $\hat{p}_t^{(0)}(x, y)$ with respect to the measure $\hat{m}_x$ of (4.2). Further, it is represented by*

$$\hat{p}_t^{(0)}(x, y) = \frac{1}{t^{\mathrm{tr}R_0}} f(t^{-R_0}\psi(x, y)), \quad \forall y \in S_x, \tag{4.3}$$

where $f(z)$ is the C^∞ density function of the probability distribution of $\eta_1^{(0)} = \log \xi_1^{(0)}$ with respect to the Lebesgue measure, $\psi(x, y) = \varphi_x^{-1}(y)$, R_0 is the restriction of the exponent R to $\mathcal{L}_K$ and $\mathrm{tr}R_0 = \sum_{j\in K} \alpha_j^{(0)}$.

Proof. By Corollary 3.3 we have

$$\begin{aligned} P_t^{(0)}(x, (\gamma_t^{(x)})^{-1}(E)) &= P_1^{(0)}(x, E) = P(\exp \eta_1^{(0)}(x) \in E) \\ &= P(\eta_1^{(0)} \in \varphi_x^{-1}(E)) \\ &= \int_{\varphi_x^{-1}(E)} f(z)\nu(dz) = \int f(\varphi_x^{-1}(y))\hat{m}_x(dy). \end{aligned}$$

On the other hand, by Lemma 4.2 we have

$$\begin{aligned} P_t^{(0)}(x,(\gamma_t^{(x)})^{-1}(E)) &= \int_{(\gamma_t^{(x)})^{-1}(E)} \hat{p}_t^{(0)}(x,y)\hat{m}_x(dy) \\ &= \int_E t^{\mathrm{tr}R_0}\hat{p}_t^{(0)}(x,\gamma_t^{(x)}(y))\hat{m}_x(dy). \end{aligned}$$

Therefore we have

$$t^{\mathrm{tr}R_0}\hat{p}_t^{(0)}(x,\gamma_t^{(x)}(y)) = f(\varphi_x^{-1}(y)). \tag{4.4}$$

Substitute $(\gamma_t^{(x)})^{-1}(y)$ in place of y in the above formula. It is immediate to see

$$\varphi_x^{-1}((\gamma_t^{(x)})^{-1}(y)) = t^{-R_0}\varphi_x^{-1}(y) = t^{-R_0}\psi(x,y).$$

Then we obtain the formula (4.3). □

REMARK. Let μ be the Lebesgue measure and let $\rho(x,y)$ be the density function of the measure $\hat{m}_x$ with respect to μ. Then $P_t^{(0)}(x,E)$ has a density function $p_t^{(0)}(x,y)$ with respect to μ and it is represented by

$$p_t^{(0)}(x,y) = \frac{1}{t^{trR_0}}f(t^{-R_0}\psi(x,y))\rho(x,y), \quad \forall y \in S_x \tag{4.5}$$

Now $\rho(x,y)$ is given by

$$\rho(x,y) = |\det\frac{\partial\varphi_x}{\partial z}\circ\psi(x,y)|^{-1}, \quad y \in S_x, \tag{4.6}$$

where $(\partial\varphi_x/\partial z)$ is the Jacobian matrix of the map φ_x. In particular, $\rho(x,x) = |\det(X_j^i(x))|^{-1}$ holds, where $(X_j^i(x))$ is the matrix of coefficients of vector fields $X_{j_1},\dots,X_{j_d} \in \mathcal{L}_K$.

5. Short time asymptotics of probability densities of Brownian flows

In this section we shall assume that the driving process $\{Z(t), t \geq 0\}$ of equation (2.1) is a standard Brownian motion. Then the stochastic flow $\xi_t(x)$ determined by equatin (2.1) is called a *Brownian flow* instead of a Lévy flow. Let $P_t(x,E)$ be the distribution of the random variable $\xi_t(x)$.

Then under Condition (A.1), $P_t(x, E)$ has a C^∞ density funtion $p_t(x, y)$ with respect to a volume form m as we have remarked in the previous section.

The next lemma shows that the distribution $P_t^{(0)}(x, E)$ of the self-similar approximation $\xi_t^{(0)}(x)$ has also a C^∞ density function.

LEMMA 5.1. *Let $\{\eta_t^{(0)}, t \geq 0\}$ be the stochastic process of (3.9). Then the distribution of $\eta_t^{(0)}$ has a C^∞ density function with respect to the Lebesgue measure ν.*

PROOF. We shall first show that $\eta_t^{(0)}$ can be obtained by taking a projection of a certain stochastic process determined by an SDE. We introduce some notions following Yamato [21]. Let $E = \{1, \ldots, k\}$ and $E(p) = \{I = (i_1, \ldots, i_l); i_1, \ldots, i_l \in E, 1 \leq l \leq p\}$, where p is a positive integer. The set $\{y = (y^I)_{I \in E(p)}; y^I \in \mathbf{R}\}$ is identified with $\mathbf{R}^m$, where $m = \sharp E(p)$. We define vector fields V_i on $\mathbf{R}^m$ by

$$V_i = \frac{\partial}{\partial y^i} + \sum_{l+1 \leq p, j_1, \ldots, j_l \in E} y^{j_1, \ldots, j_l} \frac{\partial}{\partial y^{j_1, \ldots, j_l, i}}.$$

Consider the SDE on $\mathbf{R}^m$:

$$dY_t = \sum_{j \in E} V_j(Y_t) \diamond dZ^j(t).$$

The solution sarting from 0 at time 0 is $Y_t = (Z^I(t))_{I \in E(p)}$, where $Z^I(t)$ is the multiple Wiener-Stratonovich integral defined by (2.5).

Let $\mathcal{L}(V_1, \ldots, V_k)$ be the Lie algebra generated by $V_1, \ldots, V_k$. Then $\dim\mathcal{L}(V_1, \ldots, V_k)(y) = m$ holds for any y. Therefore the distribution of Y_t has a C^∞ density function.

Define $\pi : \mathbf{R}^m \to \mathcal{L}_K$ by

$$\pi(y) = \sum_{j \in K} \sum_{J : \|J\|_Q = \beta_j} \sum_{\tau : |\tau| = |J|} \frac{(-1)^{e(\tau)}}{|J|^2 \begin{pmatrix} |J| - 1 \\ e(\tau) \end{pmatrix}} y^J P_{X_j} X_J.$$

Then we have $\pi(Y_t) = \eta_t^{(0)}, t \geq 0$. Further $\{\pi^* V_J; J \in E(p)\} = \mathcal{L}_K$ holds. Therefore, the distribution of $\eta_t^{(0)}$ has a C^∞ function. See Taniguchi [19] and Kusuoka-Stroock [13]. The proof is complete. □

We shall study the short time asymptotics of transition density functions of the Brownian flow. The next theorem indicates that it coincides with the short time asymptotics of the probability density of a canonical self-similar approximation.

THEOREM 5.2. *Let $\{\xi_t, t \geq 0\}$ be a Brownian flow determined by equation (2.1) and let $\{\xi_t^{(0)}, t \geq 0\}$ be its canonical self-similar approximation. Let $P_t(x, E)$ and $P_t^{(0)}(x, E)$ be the probability distributions of random variables $\xi_t(x)$ and $\xi_t^{(0)}(x)$ respectively. Let $p_t(x, y)$ and $p_t^{(0)}(x, y)$ be their density functions with respect to the Lebesgue measure μ. Then we have*

$$p_t(x, y) \sim p_t^{(0)}(x, y), \quad \text{as} \quad t \to 0 \quad \forall y \in S_x. \tag{5.1}$$

For the proof of the theorem, we shall introduce an intermediate stochastic flow. For a positive integer p, define $\eta_t^p \equiv \eta_t$ by (2.7) and $\tilde{\xi}_t^p \equiv \tilde{\xi}_t$ by

$$\tilde{\xi}_t \equiv \exp \eta_t. \tag{5.2}$$

Then the law of the stochastic process $\{\tilde{\xi}_t(x), t \geq 0\}$ coincides with the law of the stochastic process $\{\gamma_t^{(x)}(\exp \eta_1^{(t)}(x)), t \geq 0\}$ for any x. Here $\{\gamma_r^{(x)}\}$ is a canonical dilation with exponent R and $\eta_s^{p,(t)} \equiv \eta_s^{(t)}$ is given by

$$\eta_s^{(t)} = \sum_j \sum_{J:|J| \leq p-1} t^{\|J\|_Q - \beta_j} c_s^J P_{X_j} X_J, \tag{5.3}$$

where β_j are diagonal elements of R. The following lemma can be verified similarly as in Lemma 5.1 and Theorem 4.3.

LEMMA 5.3. *The distribution of $\eta_1^{(t)}$ has a C^∞ density function $f_t(z)$ with respect to the Lebesgue measure. Further, the distribution of $\tilde{\xi}_t(x)$ has a C^∞ density function $\tilde{p}_t(x, y)$ with respect to the Lebesgue measure and it is represented by*

$$\tilde{p}_t(x, y) = \frac{1}{t^{\operatorname{tr} R_0}} f_t(t^{-R_0} \psi(x, y)) \rho(x, y), \quad \forall y \in S_x, \tag{5.4}$$

where R_0 is the restriction of R to $\mathcal{L}_K$.

We shall apply Malliavin calculus of Ikeda-Watanabe [7]. Let $q > 1$ and $s \in \mathbf{R}$. Let $\mathbf{D}_{q,s}$ be the Sobolev space defined on the Wiener space $(Z(t), P)$, where $Z(t), t \geq 0$ is a k-dimensional Brownian motion defining SDE (2.1). The associated Sobolev norm is defined by $\| \quad \|_{q,s}$. It is known that the solution $\xi_t(x)$ of equation (2.1) belongs to $\mathbf{D}_{q,s}$ for any q and s.

LEMMA 5.4. *For a sufficiently large p, there exists $s \in \mathbf{R}$ such that*

$$\lim_{t\to 0} \|\delta_y(\xi_t(x)) - \delta_y(\tilde{\xi}_t^p(x))\|_{q,s} = 0, \quad \forall x, y \in \mathbf{R}^d \tag{5.5}$$

$$\lim_{t\to 0} \sup_{z\in\mathbf{R}^n} \|\delta_z(\eta_1^{p,(t)}) - \delta_z(\eta_1^{(0)})\|_{q,s} = 0 \tag{5.6}$$

holds for any $q > 1$ and $M > 0$, where δ_y and δ_z are Dirac delta functions at y and z respectively.

PROOF. We shall follow Watanabe's argument ([7] and [20]). For a given positive integer p, set $\tilde{\varphi}_t^\epsilon = \exp \zeta_t^\epsilon$, where ζ_t^ϵ is defined by (2.13). When $\epsilon_j = r^{1/2}, j = 1, \ldots, k$, we denote $\tilde{\varphi}_t^\epsilon$ by $\tilde{\varphi}_t^{(r)}$. Then the laws of $\varphi_1^{(t)}(x)$ and $\tilde{\varphi}_1^{(t)}(x)$ coincide with those of $\xi_t(x)$ and $\tilde{\xi}_t^p(x)$ respectively. Let $\sigma^{(t)}(x)$ and $\tilde{\sigma}^{(t)}(x)$ be the Malliavin covariances of $\varphi_1^{(t)}(x)$ and $\tilde{\varphi}_1^{(t)}(x)$, respectively. Then for any $q > 1$, there exists a positive constant c and N such that

$$\| |\det \sigma^{(t)}(x)|^{-1}\|_q \leq ct^{-N}, \quad \| |\det \tilde{\sigma}^{(t)}(x)|^{-1}\|_q \leq ct^{-N} \tag{5.7}$$

for $0 \leq t \leq 1$.

Let T be a tempered distribution. Then there exists $s \in \mathbf{R}$ such that both $T \circ \varphi_1^{(t)}(x)$ and $T \circ \tilde{\varphi}_1^{(t)}(x)$ are in the Sobolev space $\mathbf{D}_{q,s}$ for all $q > 1$. Further, these have the $t^{1/2}$-expansions:

$$T \circ \varphi_1^{(t)}(x) = \Phi_0 + t^{1/2}\Phi_1 + \cdots + t^{(p-1)/2}\Phi_{p-1} + O(\epsilon^{p/2-N}),$$

$$T \circ \tilde{\varphi}_1^{(t)}(x) = \tilde{\Phi}_0 + t^{1/2}\tilde{\Phi}_1 + \cdots + t^{(p-1)/2}\tilde{\Phi}_{p-1} + O(t^{p/2-N}),$$

in $\mathbf{D}_{q,s}$, where p is the positive integer in Lemma 2.2 (2). (Theorem 9.4 and its proof in Ikeda-Watanabe [7]). Since (2.14) holds, we have $\Phi_i = \tilde{\Phi}_i$ for $i \leq p - 1$. Therefore we have $T \circ \varphi_1^{(t)}(x) - T \circ \tilde{\varphi}_1^{(t)}(x) = O(t^{p/2-N})$ in $\mathbf{D}_{q,s}$. Consequently we obtain

$$\|T \circ \xi_t(x) - T \circ \tilde{\xi}_t^p(x)\|_{q,s} = O(t^{p/2-N}). \tag{5.8}$$

Now set $T=\delta_y$ and take $p>2N$. Then we get (5.5).

We shall next prove (5.6). Let $\tau^{(t)}$ be the Malliavin covariance of $\eta_1^{(t)}$. Then it satisfies

$$\sup_{0\le t\le 1}\||\det\tau^{(t)}|^{-1}\|_q<\infty \tag{5.9}$$

for any $q>0$. Then we can show similarly as the above that there exists $s\in\mathbf{R}$ such that

$$\sup_z\|\delta_z(\eta_1^{(t)})-\delta_z(\eta_1^{(0)})\|_{q,s}=O(t^\gamma) \tag{5.10}$$

for some $\gamma>0$. This implies (5.6). The proof is complete. □

PROOF OF THEOREM 5.2. It holds $p_t(x,y) = E[\delta_y(\xi_t(x))]$ and $\tilde p_t(x,y)=E[\delta_y(\tilde\xi_t(x))]$. Therefore we have by Lemma 5.3,

$$|p_t(x,y)-\tilde p_t(x,y)|\le\|\delta_y(\xi_t(x))-\delta_y(\tilde\xi_t(x))\|_{q,s}\to 0,$$

as $t\to 0$.

We shall next compare $\tilde p_t(x,y)$ with $p_t^{(0)}(x,y)$. Note that $p_t^{(0)}(x,y)$ and $\tilde p_t(x,y)$ are represented by (4.5) and (5.4) with the density functions f and f_t, respectively. We shall prove that $f_t(z)$ converges to $f(z)$ uniformly in z as $t\to 0$. Since $f_t(z)=E[\delta_z(\eta_1^{(t)})]$ and $f(z)=E[\delta_z(\eta_1^{(0)})]$, we have

$$|f_t(z)-f(z)|=|E[\delta_z(\eta_1^{(t)})-\delta_z(\eta_1^{(0)})]|\le\|\delta_z(\eta_1^{(t)})-\delta_z(\eta_1^{(0)})\|_{q,s}$$

which converges to 0 uniformly in z by Lemma 5.4. Therefore we have

$$\tilde p_t(x,y)\sim\frac{f(t^{-R_0}\psi(x,y))}{t^{\operatorname{tr}R_0}}\rho(x,y)=p_t^{(0)}(x,y),$$

in view of Theorem 4.3. The proof is complete. □

We shall rewrite the kernel $p_t^{(0)}(x,y)$ of Theorem 5.2 more explicitly. Assume $k\le d$ and $\{X_1(x),\dots,X_k(x)\}$ is linearly independent at every x. Set $V_1=\{X_1,\dots,X_k\},V_2=[V_1,V_1],\cdots,V_n=[V_{n-1},V_1]$. Then $\mathcal{L}=\cup_nV_n$. We can choose $X_1,\dots,X_d$ of $\mathcal{L}$ and positive integers $k<k_2<\cdots<k_l=d$ such that a) $\{X_1(x),\dots,X_d(x)\}$ is linearly independent for any x and b) $\cup_{i=1}^jV_i=\{X_1,\dots,X_{k_j}\}$ for $j=2,\dots,l$. Then the diagonal elements $\beta_j,j=$

$1, \ldots, d$ of the canonical exponent R associated with the above $\{X_1, \ldots, X_d\}$ is given by

$$\beta_1 = \cdots = \beta_{k_1} = \frac{1}{2},\ \beta_{k_1+1} = \cdots = \beta_{k_2} = 1, \cdots, \beta_{k_{l-1}+1} = \cdots = \beta_{k_l} = \frac{l}{2},$$

where $0 = k_0 < k = k_1 < k_2 < \cdots < k_l = d$. Then we have $\mathrm{tr} R_0 = (1/2)\sum_{i=1}^{l} i(k_i - k_{i-1})$ and we have the expression

$$t^{-R_0}\psi(x,y) = \Big(\frac{\psi_{k_1}(x,y)}{t^{1/2}}, \frac{\psi_{k_2}(x,y)}{t}, \cdots, \frac{\psi_{k_l}(x,y)}{t^{l/2}}\Big),$$

where $\psi(x,y) = (\psi_{k_1}(x,y), \ldots, \psi_{k_l}(x,y))$. Further, $\eta_t^{(0)} = \sum_{j=1}^{d} \hat{c}_t^j X_j$, where $\hat{c}_t^j$ is a linear sum of multiple Wiener-Stratonovich integrals of order i if $k_{i-1} < j \le k_i$, which is self-similar with exponent $i/2$. Then Theorem 5.2 can be reformulated as:

THEOREM 5.5. *We have*

$$p_t(x,y) \sim \frac{1}{t^{N/2}} f\Big(\frac{\psi_{k_1}(x,y)}{t^{1/2}}, \frac{\psi_{k_2}(x,y)}{t}, \ldots, \frac{\psi_{k_l}(x,y)}{t^{k_l/2}}\Big)\rho(x,y), \tag{5.11}$$

as $t \to 0$ for all $y \in S_x$, where $f(z)$ is the density funtion of the distribution of the random variable $\eta_1^{(0)}$ with respect to the Lebesgue measure and

$$N = \sum_{i=1}^{l} i(k_i - k_{i-1}). \tag{5.12}$$

In the case where $l = 1$ in the above theorem, $f(z)$ is a Gaussian density with mean 0 and covariance I. Therefore, we have

COROLLARY 5.6. *Assume $k = d$ and the dimension of l.s.$\{X_1, \ldots, X_k\}(x) = d$ for any $x \in \mathbf{R}^d$. Then we have:*

$$p_t(x,y) \sim \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|\psi(x,y)|^2}{2t}\right)\rho(x,y) \quad \text{as } t \to 0, \quad \forall y \in S_x. \tag{5.13}$$

REMARK. (Kusuoka-Stroock [13]) It holds

$$|\psi(x,y)|^2 = \inf\{\int_0^1 \sum_{i=1}^k |u_i(s)|^2 ds; \quad \frac{d\varphi_t}{dt} = \sum_{i=1}^k X_i(\varphi_t)u_i(t), \\ \varphi(0)=x,\ \varphi(1)=y\}$$

Thus $|\psi(x,y)|$ defines a metric on S_x.

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(Received June 11, 1996)
(Revised May 16, 1997)

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