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The Cauchy-Kovalevsky Theorem and Noncompactness Measures

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Abstract. We give an abstract version of the Cauchy-Kovalevsky Theorem for the equation $u' = A(t, u)$ where A is a Caratheodory operator having properties based on noncompactness measures, including Lipschitz and compactness conditions. We give an application of this result to the equation $\partial_t^n u + \sum_{i=1,n} f_i(u) B^{(n-i+1)} \partial_t^{i-1} u = 0$ that generalizes the Kirchhoff equation for the vibrating string, when B is *not* a compact operator. Our technique is based on Nagumo's weights and on Tonelli delayed problems.

1. Introduction

We give a version of the abstract Cauchy-Kovalevsky Theorem for the local existence of a solution of the problem:

$$u' = A(t, u) \quad (t \in I) \tag{1.1}$$

$$u(0) = u_0 \in X_{r_0} \tag{1.2}$$

where $I = [0, a]$, $(\cdot)' = \dfrac{d}{dt}$ and, for every $t \in I$, $A(t, \cdot)$ is a continuous (but not necessarily Lipschitz) operator in a scale of Banach spaces $(X_r)_{0<r\leq r_0}$ (cf. Def 2.1).

Equation (1.1) is the abstract version of the system of partial differential equations $\partial_t u = F(t, x, \nabla u)$ that has been considered by [15], [25], [16], and [26] (see also [41]).

The problem of existence of local solutions of (1.1)–(1.2) has been considered in the autonomous linear case by [21], and in the nonautonomous linear case by [32], assuming that A is continuous and $\|A(t, u)\|_s \leq C(t)\|u\|_r (r - s)^{-1}$. The case when A is a nonlinear operator continuous in t and Lipschitz

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continuous in u, that is, for $u,\ v \in B_{r,R},\ s < r$:

$$\begin{cases} \|A(t,u) - A(t,v)\|_s \le C\,\|u-v\|_r\,(r-s)^{-1} \\ \\ \|A(t,0)\|_s \le M(r_0-s)^{-1}; \end{cases} \tag{1.3}$$

where $B_{r,R}$ is the ball of radius R in X_r, has been treated by [33], [29], [30], [23] and [10] (see also [42]). Later on, [8] proved existence of local solutions for a functional generalization of equation (1.1) under conditions of Lipschitz type and [13] treated (1.1)–(1.2) under Caratheodory hypotheses on a regularizing operator $A(t,u)$ (*i.e.* $A(t,\cdot) : X_s \to X_r,\ s < r$) such that $\|A(t,u)\|_r$ is majored by a linear operator $D : X_r \to X_s,\ s < r$.

Later, interest arisen upon the question whether the Lipschitz type condition (1.3) is removable, of course giving up the uniqueness of the solution.

K. Deimling [18] assumed that $A(t,u) = B(t)u + f(t,u)$ where $B(t)$ is a linear operator and f is a regularizing, continuous and α-Lipschitz operator (*i. e.* there exists a constant k such that $\alpha_{r_0}(A(\{t\} \times B)) \le k\alpha_s(B)$ for every bounded subset B of $B_{r,R}$, where α is the Hausdorff noncompactness measure (see Def 2.2)), whose image lies in X_{r_0}.

Later on, [11] considered the case in which $(X_r)_r$ is a scale of holomorphic functions and proved local existence of solutions by assuming that $A(t,u)$ preserves the order of singularity in the scale, *i.e.* for every $s < r$:

$$\|A(t,u)\|_s \le \frac{M}{r-s} \quad \text{if} \quad \|u\|_r \le \frac{R}{1-s}.$$

V. I. Nazarov [28] treated problem (1.1)–(1.2) when A is a continuous nonlinear operator $A : X_r \to X_s$ $(s < r)$. Moreover he supposed that the imbeddings $i : X_r \hookrightarrow X_s$ $(s < r)$ are compact and

$$\|A(t,u)\|_s \le \frac{C\|u\|_r}{r-s} + \frac{M}{r_0-s}. \tag{1.4}$$

M. Reissig [35] treated problem (1.1)–(1.2) assuming that A is a continuous operator defined only on the balls $B_{r,R}$ and that the imbeddings are compact. Moreover he supposed that A satisfies an estimate stronger that (1.4):

$$\|A(t,u)\|_s \le \frac{C\|u\|_r}{r-s} + \frac{M}{(r_0-s)^{\varepsilon}} \tag{1.5}$$

with $0 < \varepsilon < 1$ and, for technical reasoning, $u_0 = 0$.

In the case when (1.1) is an ordinary differential equation (*i.e.* the spaces of the scale coincide), many authors have considered the Cauchy problem under hypotheses that generalize both conditions: A is Lipschitz continuous and A is compact operator, by using the Hausdorff measure of noncompactness.

In this case, [2] proved local existence of solutions by supposing that A is uniformly continuous in (t, u) and α-Lipschitz in u. Later on, [38] considered the case when A is a continuous operator and there exists a constant K such that, for every bounded subset W of X:

$$\alpha(A(I \times W)) \leq K\,\alpha(W), \tag{1.6}$$

where α is the Hausdorff measure of noncompactness.

Many other authors treated this case by assuming an hypothesis of Kamke type, that generalize (1.6), but we can *not* treat this case by our technique. Some other authors considered the weak noncompactness measure α_w (see Def. 2.3), instead of noncompactness measure (for further information see [19]).

At this point it is natural to consider (1.1)–(1.2) under hypotheses of noncompactness type, like (1.6). In fact, the Cauchy problem (1.1)–(1.2) has at least a solution under two sets of hypotheses:

1. (Theorem 2.4) A is a Caratheodory operator and:
   - $\|A(t,u)\|_s \leq \dfrac{C\|u\|_r + M}{r-s} \qquad (s < r,\ u \in X_r);$
   - there exists a constant K such that for every bounded subset C of X_r:
   $$\alpha_s(A(I \times C)) \leq \frac{K\alpha_r(C)}{r-s} \qquad (s < r);$$
   where α_r is the Hausdorff measure of noncompactness in X_r.

2. (Theorem 2.5) A is a weakly Caratheodory operator and:
   - $\|A(t,u)\|_s \leq \dfrac{C\|u\|_r + M}{r-s} \qquad (s < r,\ u \in X_r);$

- there exists a constant K such that for every bounded subset C of X_r:
$$\alpha_{w,s}(A(I \times C)) \leq \frac{K\alpha_{w,r}(C)}{r-s} \qquad (s<r);$$
where $\alpha_{w,r}$ is the weak noncompactness measure in X_r.

Our technique is based on Nagumo's weights [26] (as in [10]) and on the introduction of Tonelli delayed problems (see [39]). We also use the method of [19] mixed with the method of [34].

This paper is organized as follows:
in section 2. we state the result;
in section 3. we give the proofs;
in section 4. we give an application of the result when (X_r) is the scale of B-analytic vectors obtained by a selfadjoint positively defined operator B on a Hilbert space H, and the considered equation is:

$$\partial_t^n u + \sum_{i=1,n} f_i(u) B^{(n-i+1)} \partial_t^{i-1} u = 0 \qquad (t>0). \tag{1.7}$$

This equation is a generalization of the concrete equation, already considered by [37]:

$$u_{tt} - m\left(\int_\Omega f(u, \nabla_x u, \dots, \nabla_x^\nu)\, d\xi\right) \triangle_x u = 0 \quad x \in \Omega,\ t \geq 0,\ \nu \in \mathbf{N}.$$

This equation for $n = \nu = 1$, and $f(u, \nabla_x u) = |\nabla_x u|^2$, has been introduced by [24] (see [22]) as a nonlinear model for the small vibrations of an elastic string fixed at the extremes (see [12], [14], [27], [31]).

2. Preliminaries and main theorem

Definition 2.1. A scale of Banach spaces is defined as a family of Banach spaces $(X_r)_{0<r\leq r_0}$ with norms $\|\cdot\|_r$, such that $X_r \subseteq X_s$ and $\|\cdot\|_s \leq \|\cdot\|_r$ for $s \leq r$.

Let X be a Banach space, C a bounded subset of X. We recall the following:

Definition 2.2. The Hausdorff measure of noncompactness of C is:

$$\alpha(C) := \inf\{\varepsilon > 0 : \\ C \text{ can be covered by a finite number of balls of radius } \varepsilon\}.$$

Let us denote by KW the set of weakly compact subsets of X. Then

Definition 2.3. The weak noncompactness measure of C is:

$$\alpha(C) := \inf\{\varepsilon > 0 : \text{ there exists } K \in KW \text{ such that } C \subseteq K + \varepsilon\mathbf{B}\},$$

where $\mathbf{B}$ is the ball of center 0 and radius 1 in X.

Notations. Let C be a subset of X, let us denote **cl(C)** its closure in X and **co(C)** its convex hull.

Let us recall some properties of α (for the proofs see [1], [9], [36]). Let A and B be bounded subsets of X, then:

1. $\alpha(co(B)) = \alpha(B)$;
2. $\alpha(cl(B)) = \alpha(B)$;
3. $\alpha(A \cup B) \leq \max\{\alpha(A),\ \alpha(B)\}$;
4. $\alpha(A) = 0$ if and only if A is relatively compact;
5. $\alpha(\lambda B) = |\lambda|\, \alpha(B)$ for every $\lambda \in \mathbf{R}$;
6. $\alpha(A + B) \leq \alpha(A) + \alpha(B)$;
7. $\alpha(B) \leq \alpha(A)$ if $B \subseteq A$.

The same properties hold true, with respect to weak topology, for α_w (see [17]).

If $\phi : (a, b) \to (0, +\infty[$ is a nonincreasing function, we define:

$$\mathbf{C}((a, b);\, \phi) := \bigcap_{t \in (a,b)} C^{\circ}((a, t]; X_{\phi(t)}).$$

Let $I := [0,\, a_0]$, let A be an operator such that $A : I \times X_r \to X_s$ for every $0 < s < r \leq 1$. We assume that A verifies the following properties:

– for $s < r$, for every u in X_r

$$A(\cdot, u) : I \to X_s \quad \text{is strongly measurable;} \tag{2.1}$$

– for $s < r$, for almost every t in I

$$A(t, \cdot) : X_r \to X_s \quad \text{is continuous;} \tag{2.2}$$

– there exist two constants C and M such that

$$\|A(t,u)\|_s \le \frac{C\|u\|_r + M}{r-s} \qquad \text{for} \quad s < r,\ u \in X_r; \tag{2.3}$$

– there exists a constant K such that for every bounded subset C of X_r:

$$\alpha_s(A(I \times C)) \le \frac{K\ \alpha_r(C)}{r-s} \quad (s < r). \tag{2.4}$$

Now we can state the result:

Theorem 2.4. *Let us assume that hypotheses (2.1)–(2.4) are satisfied. Then the Cauchy problem (1.1)–(1.2) has at least a solution.*

Let $I := [0, a_0]$, let A be an operator such that $A : I \times X_r \to X_s$ for every $0 < s < r \le 1$. We assume that A verifies the following properties:
– for $s < r$, for every u in X_r

$$A(\cdot, u) : I \to X_s \quad \text{is weakly measurable} \tag{2.5}$$

– for $s < r$, for almost every t in I

$$A(t, \cdot) : X_r \to X_s \quad \text{is weakly continuous} \tag{2.6}$$

– there exist two constants C and M such that

$$\|A(t,u)\|_s \le \frac{C\|u\|_r + M}{r-s} \qquad \text{for} \quad s < r,\ u \in X_r; \tag{2.7}$$

– there exists a constant K such that for every bounded subset C of X_r:

$$\alpha_{w,s}(A(I \times C)) \le \frac{K\ \alpha_{w,r}(C)}{r-s} \quad (s < r). \tag{2.8}$$

Now we can state the result:

THEOREM 2.5. *Let us assume that hypotheses (2.5)–(2.8) are satisfied. Then the Cauchy problem (1.1)–(1.2) has at least a solution.*

REMARK 2.6. We can reduce the case $r_0 \neq 1$ to the case $r_0 = 1$.

We give some examples in which hypothesis (2.4) is verified.

PROPOSITION 2.7. *For simplicity we consider that A is autonomous. The condition (2.4) is verified if:*

1. *A is Lipschitz continuous, that is:*

$$\|A(u) - A(v)\|_s \leq \frac{C\,\|u - v\|_r}{r - s} \qquad (u,\, v \in X_r),$$

   or

2. *A is compact, that is A takes bounded subsets of $I \times X_r$ into relatively compact subsets of X_s,*

   or

3. *$A = A_1 + A_2$ where A_1 is a Lipschitz operator and A_2 is a compact operator,*

   or

4. *$A(u) = F(u, u)$ where $F : X_r \times X_r \to X_s$, $(s < r)$ and $F(\cdot, v)$ is Lipschitz continuous uniformly in v (with Lipschitz constant $\leq C$) and $F(u, \cdot)$ is compact for every u.*

Now we give some examples in which hypothesis (2.8) is verified.

PROPOSITION 2.8. *For simplicity we consider that A is autonomous. The condition (2.8) is verified if:*

1. *A is Lipschitz continuous and weakly continuous,*

   or

2. *A is weakly compact, that is A takes bounded subsets of* $I \times X_r$ *into weakly relatively compact subsets of* X_s*,*

   or

3. $A = A_1 + A_2$ *where* A_1 *is a Lipschitz, weakly continuous operator and* A_2 *is a weakly compact operator,*

   or

4. $A(u) = F(u,u)$ *where* $F : X_r \times X_r \to X_s$, $(s < r)$ *and* $F(\cdot, v)$ *is Lipschitz and weakly continuous uniformly in* v *(with Lipschitz constant* $\leq L$*) and* $F(u,\cdot)$ *is weakly compact for every* u*.*

Remark 2.9. The operator A in Proposition 2.8 is weakly compact if it is weakly continuous and the spaces of the scale are reflexive Banach spaces.

3. Proofs

Proof of Theorem 2.4.

Step 0 Preliminaries

If $M = 0$ and $C = 0$, then $A = 0$ identically, therefore a solution is the constant u_0. Therefore we can assume $M + C > 0$.

a) We can assume that $u_0 = 0$. Indeed, if this is not the case, we define $v := u - u_0$, and we obtain the equivalent problem:

$$v' = A^*(t,v), \tag{3.1}$$

$$v(0) = 0, \tag{3.2}$$

where $A^*(t,v) := A(t, v+u_0)$. It is easy to verify that A^* verifies (2.1)–(2.4).

b) By extending if necessary A, we can assume that $a_0 = +\infty$.

c) Plane of the proof.

Let $R > 0$.

Let (ε_n) be a sequence such that, for every n, $0 < \varepsilon_n < \frac{1}{2}$ and $\varepsilon_n \to 0$. We fix ε_n, and for simplicity we denote ε_n only by ε. We show that problem (1.1)–(1.2) has at least a solution $v_{\varepsilon,R}$ with:

$$v_{\varepsilon,R} \in \mathbf{C}([0, a_{\varepsilon,R}];\, \phi_R - 2^{-1} - \varepsilon), \tag{3.3}$$

where

$$\begin{array}{rcl} L_R & := & \max\{4K+\varepsilon,\ 4(C+MR^{-1})\} \\ a_R & := & L_R^{-1} \\ a_{\varepsilon,R} & := & a_R(2^{-1}-\varepsilon) \\ \phi_R(t) & := & \begin{cases} 1 & \text{if} \quad t \le 0 \\ 1-tL_R & \text{if} \quad t>0 \end{cases} \end{array}$$

Let us denote

$$S_R := \max\{4K,\ 4(C+MR^{-1})\}.$$

We show, by a diagonal argument, that problem (1.1)–(1.2) has at least a solution $u \in \mathbf{C}([0,(2S_R)^{-1}[;\ \frac{1}{2}-S_Rt)$. Then we show that the problem has at least a solution $u \in \mathbf{C}([0,(2S)^{-1}[;\ \frac{1}{2}-St)$, where

$$S := \max\{4K, 4C\}.$$

Step 1 Approximating solutions of Tonelli type

Let $a_1 := S^{-1}$. We introduce for $n \in \mathbf{N}$ the following approximating problems of Tonelli type:

$$(P_n) \begin{cases} u_n(t) = 0 & \text{for} \quad t \le 0 \\ u_n(t) = \displaystyle\int_0^t A(\tau, u_n(\tau - \frac{a_1}{n}))\, d\tau & \text{for} \quad 0 \le t \le a_1. \end{cases}$$

We show that for every $n \in \mathbf{N}$ the problem (P_n) has a solution on the interval $[0,a_1]$. At this end, we define $t_k := \dfrac{ka_1}{n}$ for $k = 0,\dots,n$ and we show by finite induction on k that (P_n) has a solution on the interval $]-\infty,t_k]$. Since problem (P_n) is a delayed problem and A is defined on the whole X_r (for every r), it is enough to show that the integrals in (P_n) exist. Thanks to the boundedness of A on the bounded sets, it is enough to show (see [20], p. 212, Prop. 4 and [3]) that:

$$(3.4) \qquad \begin{array}{lll} u_n \in & \mathbf{C}(]-\infty,0],1) & k=0 \\ u_n \in & \mathbf{C}(]0,t_k],1-St_k) & k=1,\dots,n. \end{array}$$

Expression (3.4) is true on $]-\infty,0]$. If (3.4) is true until k, then u_n is a bounded function from $]-\infty,t_k]$ to X_{1-St_k}, therefore (P_n) has a solution on $]-\infty,t_{k+1}]$, with values in $X_{1-St_{k+1}}$ and this solution is continuous.

Step 2 basic estimate

We remark that actually $u_n(t)\in X_{1-St}$ for each $0\le t\le a_1$ and therefore $u_n(t)\in X_{\phi_R(t)}$ for each $0\le t\le a_R$. Let us define for $0<r\le 1$:

$$|||v|||_{r,t} := \|v\|_r\,(1-r-L_Rt).$$

We need the following basic estimate:

$$|||u_n(t)|||_{s,t}\le \frac{R}{2}\qquad \forall\, t\le a_R,\ \forall\, s<\phi_R(t). \tag{3.5}$$

Now we prove (3.5) by finite induction. Let $(t_k)_{k=1,\dots,n}$ as in step 1.

For $t\le \min\{t_1,\ a_R\}$ by (2.3) we have:

$$\|u_n(t)\|_s\le \int_0^t \frac{M}{1-s}\,d\tau=\frac{Mt}{1-s}$$

then

$$|||u_n(t)|||_{s,t}\le \frac{R}{2}.$$

Now let us suppose that (3.5) is true until $\min\{t_k,a_R\}$.

Following [26] (see [10]) we define for $0<r\le 1$:

$$h_r(t):=\begin{cases}1 & \text{if}\quad t<0,\\ \dfrac{\phi_R(t)+r}{2} & \text{otherwise.}\end{cases}$$

Let us fix $t_k\le t\le \min\{t_{k+1},a_R\}$. Let us consider $0<s<\phi_R(t)$. We have:

$$s<h_s(\tau)<\phi_R(\tau)\qquad \forall\, 0<\tau\le t.$$

By (2.3) we obtain:

$$\begin{aligned}\|u_n(t)\|_s &\le \int_0^t \|A(\sigma,u_n(\sigma-\frac{a_1}{n}))\|_s\,d\sigma\\ &\le \int_{-\frac{a_1}{n}}^{t-\frac{a_1}{n}} \frac{M}{h_s(\tau)-s}\,d\tau\ +\int_0^{t-\frac{a_1}{n}}\frac{CR}{2(h_s(\tau)-s)^2}\,d\tau\\ &\le \int_0^t \frac{M}{h_s(\tau)-s}\,d\tau\ +\int_0^t\frac{CR}{2(h_s(\tau)-s)^2}\,d\tau\\ &\le \frac{2\,(CR+M)\,a_R}{(1-s-L_Rt)}.\end{aligned}\tag{3.6}$$

Thanks to (3.6) we have:

$$\|u_n(t)\|_{s,t} \leq 2\,(CR+M)\,a_R = \frac{R}{2}.$$

Step 3 compactness of the sequence u_n

Let us fix $0 \leq t \leq a_R$. Let us define:

$$\Omega(0,t) := \bigcup_{\tau\in[0,t]} \{u_n(\tau) : \ n \in \mathbf{N}\};$$

$$\alpha(\Omega(0,a_R)) := \sup_{0\leq t\leq a_R} \{\alpha_s(\Omega(0,t))(1-s-L_R t) : \ 0 < s < \phi_R(t)\}.$$

Now let us fix $m \in \mathbf{N}$. Following [34], let us divide $[0,t]$ in m equal parts $[t_j, t_{j+1}]$. For $0 \leq \tau \leq t$, $t_{k(\tau)} \leq \tau \leq t_{k(\tau)+1}$ we have:

$$u_n(\tau) = \sum_{j=1,k(\tau)} \int_{t_j}^{t_{j+1}} A(\sigma, u_n(\sigma - \frac{a_1}{n}))\, d\sigma \ + \int_{t_{k(\tau)}}^{\tau} A(\sigma, u_n(\sigma - \frac{a_1}{n}))\, d\sigma.$$

Let us fix $0 < s < \phi_R(t)$, and let us indicate $\mathbf{cl}_s$ the closure in X_s. We have:

$$\begin{aligned}\Omega(0,t) \ \subseteq \ & \bigcup_{\tau\in[0,t]} \left(\left[\sum_{j=1,k(\tau)} (t_{j+1}-t_j) cl_s co(A([0,t_{j+1}]\times\Omega(0,t_{j+1}))) \right] \right. \\ & \left. +(\tau - t_{k(\tau)}) cl_s co(A([0,\tau]\times\Omega(0,\tau))) \right).\end{aligned}$$

We recall that if $0 \in A$ is a convex subset of a vector space, and if $0 < b \leq c$ then we have $bA \subseteq cA$. Furthermore if $A_1, \ldots, A_n$ are subsets of a vector space, $k(\tau)$ is an integer $\leq n$, then

$$\bigcup_{\tau\in[0,t]} \sum_{j=1,k(\tau)} A_j \subseteq \sum_{j=1,n} (A_j \cup \{0\}).$$

Therefore:

$$\begin{aligned}\Omega(0,t) \ \subseteq \ & \bigcup_{\tau\in[0,t]} \left(\left[\sum_{j=1,k(\tau)} (t_{j+1}-t_j) cl_s co(A([0,t_{j+1}]\times\Omega(0,t_{j+1}))) \right] \right. \\ & \left. +(t_{k(\tau)+1} - t_{k(\tau)}) cl_s co(A([0,t_{k(\tau)+1}]\times\Omega(0,t_{k(\tau)+1})) \cup \{0\}) \right) \\ \subseteq \ & \sum_{j=1,m} (t_{j+1}-t_j) cl_s co(A([0,t_{j+1}]\times\Omega(0,t_{j+1})) \cup \{0\}).\end{aligned}$$

By properties 1)–3), 5)–6) of the Hausdorff measure of noncompactness and by (2.4) we have:

$$
\begin{aligned}
\alpha_s(\Omega(0,t)) &\leq \sum_{j=1,m} (t_{j+1}-t_j)\alpha_s(A([0,t_{j+1}]\times\Omega(0,t_{j+1}))) \\
&\leq K\sum_{j=1,m} (t_{j+1}-t_j)(h_s(t_{j+1})-s)^{-1}\alpha_{h_s(t_{j+1})}(\Omega(0,t_{j+1})) \\
&\leq K\sum_{j=1,m} (t_{j+1}-t_j)\alpha(\Omega(0,a_R))(h_s(t_{j+1})-s)^{-2}.
\end{aligned}
$$

Since $\sum_{j=1,m}(t_{j+1}-t_j)(h_s(t_{j+1})-s)^{-2}$ is an integral sum of $(h_s(\cdot)-s)^{-2}$, passing to the limit for $m\to+\infty$, we obtain:

$$
\begin{aligned}
\alpha_s(\Omega(0,t)) &\leq K\alpha(\Omega(0,a_R))\int_0^t (h_s(\tau)-s)^{-2}\,d\tau \\
&\leq 4K\,a_R\,\frac{\alpha(\Omega(0,a_R))}{(1-s-L_Rt)}.
\end{aligned}
\tag{3.7}
$$

By (3.7) we have:

$$
\alpha(\Omega(0,a_R)) \leq 4K\,a_R\alpha(\Omega(0,a_R)).
$$

Thanks to our choice of a_R:

$$
\alpha(\Omega(0,a_R))(1-4K\,a_R)\leq 0 \quad \text{if and only if} \quad \alpha(\Omega(0,a_R))=0.
\tag{3.8}
$$

By (3.8), $\alpha_s(\Omega(0,t))=0$ for every $s<\phi_R(t)$, then by property 4) of the Hausdorff measure of noncompactness, $\Omega(0,t)$ is a relatively compact subset of X_s.

There exists a solution $v_{\varepsilon,R}$

Let us fix $m=m(\varepsilon)\in\mathbf{N}$ such that:

$$
|\phi_R(t)-\phi_R(\tau)|<\frac{\varepsilon}{3} \quad \text{if} \quad |t-\tau|\leq\frac{a_{\varepsilon,R}}{m},
$$

and let us define:

$$
T_1 := \frac{a_{\varepsilon,R}}{m}.
$$

Let us define $s_0:=\phi_R(T_1)-2^{-1}$. We have:

$$
s_0>0, \quad \text{and} \quad s_0\leq\phi_R(t)-\frac{1}{2} \quad \text{for} \quad 0\leq t\leq T_1.
$$

Therefore $\|u_n(t)\|_{s_0,t}$ is defined for $0 \leq t \leq T_1$, and by (3.5) we have:

$$\|u_n(t)\|_{s_0} \leq \frac{2R}{2} = R.$$

Thanks to (2.3), $A(B_{s_0,R})$ is bounded in $X_{s_0-\frac{\varepsilon}{3}}$. Therefore by the integral form of problems (P_n), the functions u_n are equicontinuous and take their values in this space. Therefore by step 3 the sequence (u_n) is compact in $X_{s_0-\frac{\varepsilon}{3}}$. By Ascoli Theorem, there exists a subsequence (u_{n_k}) of (u_n) that converges uniformly in $[0, T_1]$ to a function $v_{\varepsilon,1,R} \in C^\circ([0, T_1]; X_{s_0-\frac{\varepsilon}{3}})$. Then, by the Lebesgue Theorem for the dominate convergence in the space $X_{s_0-\frac{2\varepsilon}{3}}$, we pass the limit under the integral, and we see that $v_{\varepsilon,1,R}$ is a solution of problem (1.1) - (1.2). By our choice of m, we have:

$$v_{\varepsilon,1,R} \in \mathbf{C}([0, T_1]; \phi_R - 2^{-1} - \varepsilon).$$

Now we repeat the previous argument on the interval $[0, \dfrac{2a_{\varepsilon,R}}{m}]$ with respect to the sequence (u_{n_k}). We obtain a solution $v_{\varepsilon,2,R}$ of (1.1)–(1.2) that extends $v_{\varepsilon,1,R}$ and such that:

$$v_{\varepsilon,2,R} \in \mathbf{C}([0, \frac{2a_{\varepsilon,R}}{m}]; \phi_R - 2^{-1} - \varepsilon).$$

Since s_0 is always positive, we can repeat m times the previous argument and we obtain a subsequence of (u_n) that converges to a solution $v_{\varepsilon,R}$ of (1.1)–(1.2).

Step 5 There exists a solution v

Let (ε_k) be a sequence like in step 0. We indicate by $v_{\varepsilon_k,R}$ a solution of (1.1) - (1.2) satisfying (3.3) where $\varepsilon = \varepsilon_k$. Since the sequence (u_n) of solutions of the problems (P_n) does *not* depend on ε, we use the following argument. We denote by (u_n^1) the subsequence of (u_n) converging to $v_{\varepsilon_1,R}$. We repeat step 2, step 3, and step 4 with respect to (u_n^1), where $\varepsilon = \varepsilon_2$. We find a subsequence (u_n^2) of (u_n^1) converging to $v_{\varepsilon_2,R}$. We find $v_{\varepsilon_k,R}$ for every k in the same way. Then, by a diagonal argument, we obtain a subsequence of (u_n) that converges to a solution v_R of (1.1) - (1.2), with

$$v_R \in \mathbf{C}([0, (2S_R)^{-1}[; \frac{1}{2} - S_R t).$$

Now let $R_k \to +\infty$. By a diagonal argument, like the previous, it is easy to obtain a solution

$$v \in \mathbf{C}([0,(2S)^{-1}[;\frac{1}{2}-St)$$

of the problem (1.1)–(1.2). □

Proof of Theorem 2.5.

We can follow the outline of the proof of Theorem 2.4 by using the weak noncompactness measure instead of the Hausdorff noncompactness measure, but it is necessary to specify some technical details.

The integrals in problems (P_n) are Pettis integrals.

The step 4 of the proof is almost as in Theorem 2.4, by applying the Lebesgue Theorem for the dominate convergence to

$$\langle \psi_s, \int_0^t A(s, u_n(s-\frac{a_1}{n}))\, ds\rangle$$

for each ψ_s in X_s' (= dual space of X_s) and $s = s_0 - \dfrac{2\varepsilon}{3}$. □

Proof of Proposition 2.7.

We observe that 1), 2), 3) $\Longrightarrow$ 4). We prove 4) $\Longrightarrow$ (2.4).

Let V, W be bounded subsets of X_r. Let $s<r$, $\eta > \alpha_r(V)$, $v_1,\dots,v_n$ be the centers of a finite covering of V of radius η. Let $\varepsilon > 0$, and $w_{j,k}$ ($j=1,\dots,m_k$; $k=1\dots n$) be the centers of a finite covering of radius ε in X_s of $F(v_k,W)$. Let us fix (v,w) in $V\times W$. Let v_k, $w_{h,k}$ be such that:

$$\|v-v_k\|_r \le \eta, \qquad \|F(v_k,w)-F(v_k,w_{h,k})\|_s \le \varepsilon.$$

We have:

$$\begin{aligned}\|F(v,w)-F(v_k,w_{h,k})\|_s &\le \|F(v,w)-F(v_k,w)\|_s\\ &\quad +\|F(v_k,w)-F(v_k,w_{h,k})\|_s\\ &\le \frac{C\,\eta}{r-s}+\varepsilon.\end{aligned}$$

Since η, ε are arbitrary, we obtain:

$$\alpha_s(F(V,W)) \le \frac{C\,\alpha_r(V)}{r-s}.$$

Therefore $\alpha_s(A(V)) = \alpha_s(F(V,V)) \leq \dfrac{C\,\alpha_r(V)}{r-s}$. $\Box$

Proof of Proposition 2.8.

It is obvious that 1), 2), 3) $\Longrightarrow$ 4). Let us show that 4) $\Longrightarrow$ (2.8).

Let us indicate by $\mathbf{B}_r$ the unit ball in X_r. Let V, W be bounded subsets of X_r. Let $s < r$, $\eta > \alpha_{w,r}(V)$. Let K be a weakly compact subset of X_r such that:

$$V \subseteq K + \eta\,\mathbf{B}_r.$$

Then

$$\begin{aligned} F(V,W) &\subseteq \bigcup_{w\in W} F(K,w) + \frac{L\,\eta\,\mathbf{B}_s}{r-s} \\ &= F(K,W) + \frac{L\,\eta\,\mathbf{B}_s}{r-s}. \end{aligned}$$

Let us show that $F(K,W)$ is relatively weakly compact in X_s. Let us show equivalently that it is relatively sequentially weakly compact. Let us set $x_n := F(k_n, w_n)$, where $k_n \in K$, $w_n \in W$. There exists a subsequence $(k_{n_h})_{n_h}$ of $(k_n)_n$ that weakly converges to some k. Let $y_{n_h} := F(k, w_{n_h})$. This sequence has a subsequence $y_{n_{h_l}}$ that weakly converges to some y in X_s. Then for each $\psi \in X_s'$ (the dual space of X_s), and for $n_{h_l} \to +\infty$, we have:

$$\langle \psi_s,\, x_{n_{h_l}} - y\rangle = \langle \psi_s,\, F(k_{n_{h_l}}, w_{n_{h_l}}) - F(k, w_{n_{h_l}})\rangle + \langle \psi_s,\, F(k, w_{n_{h_l}}) - y\rangle \to 0.$$

Therefore $\alpha_{w,s}(F(W)) \leq \dfrac{L\,\alpha_{w,r}(W)}{r-s}$. $\Box$

4. Application

Let H be an Hilbert space, with inner product $(\cdot,\cdot)$ (and norm $|\cdot|$), and let $B : D(B) \subseteq H \to H$ be a selfadjoint, positively defined operator. Let us define:

$$D_\infty(B) := \bigcap_{j\in\mathbf{N}} D(B^j),$$

and for u, $v \in D_\infty(B)$ and $r > 0$ let us define:

$$(u,v)_r := \sum_{j\in\mathbf{N}} (B^j u,\, B^j v) r^{2j} (j!)^{-2}.$$

Let us consider the family of Hilbert spaces:

$$X_r := \{u \in D_\infty(B) : \|u\|_r^2 := (u,\, u)_r < +\infty\} \qquad (r > 0).$$

The family $\{X_r : r > 0\}$ is a scale of Hilbert spaces. Now let us consider for $n \geq 1$ the family of spaces $(Y_r)_{r>0}$, defined by:

$$Y_r := (X_r)^n,$$

that is Y_r is the cartesian product of n copies of X_r. Let us assume that Y_r has the following inner product:

$$((u_1, \dots, u_n),\, (v_1, \dots, v_n))_{Y_r} = \sum_{i=1,n} (u_i,\, v_i)_r.$$

The spaces Y_r are clearly a scale of Hilbert spaces.

Let us set:

$$X_{0^+} := \bigcup_{r>0} X_r.$$

let us consider for $i = 1, \dots, n$ the functions $f_i : X_{0^+} \to \mathbf{R}$ such that, for each $i = 1, \dots, n$, $f_i : X_r \to \mathbf{R}$ is continuous and there exists a constant K such that:

$$|f_i(u)| \leq K \qquad \text{for each} \quad u \in X_{0^+}.$$

Let us consider for $n \geq 1$ the Cauchy problem (1.7), that is:

$$(4.1) \qquad \begin{cases} \partial_t^n u + \sum_{i=1,n} f_i(u) B^{(n-i+1)} \partial_t^{i-1} u = 0 \quad (t > 0), \\ u(0) = u_0 \\ \partial_t u(0) = u_1 \\ \dots\dots \\ \partial_t^{n-1} u(0) = u_{n-1}. \end{cases}$$

Let us assume that $u_0, \dots, u_{n-1}$ are B-analytic, that is (see [7]) there exist two constants $r_0 > 0$ and L such that:

$$|B^k u_i| \leq L \frac{k!}{r_0^k} \qquad (k \in \mathbf{N},\ i = 0, \dots, n-1).$$

If we consider for $k = 1, \dots, n$ the change of variables:

$$w_k := B^{n-k} \partial_t^{k-1} u,$$

and we set $\mathbf{W} := (w_1, \ldots, w_n)$, $\mathbf{W_0} = (w_1(0), \ldots, w_n(0)) = (B^{n-1}u_0, \ldots, u_{n-1})$, the problem (4.1) becomes equivalent to the following problem in the scale Y_r:

$$\begin{cases} \mathbf{W}' &= A(W), \qquad (t > 0) \\ \mathbf{W}(0) &= \mathbf{W_0}, \end{cases} \tag{4.2}$$

where $A : Y_{0^+} \to (H)^n$ is defined as follows:

$$A(w_1, \ldots, w_n) := (A_k)_{k=1,n} \tag{4.3}$$

where

$$A_k := \begin{cases} Bw_{k+1} & \text{if} \quad k = 1, \ldots, n-1; \\ \sum_{i=1,n} f_i(B^{-n+1}w_1)Bw_i & \text{if} \quad k = n, \end{cases}$$

and where $Y_0^+ := \bigcup_{0<r\leq r_0} Y_r$.

We have

Lemma 4.1. *Let A be the operator defined in (4.3). Then for $0 < s < r < r_0$:*

1. *$A : Y_r \to Y_s$;*
2. *there exists a constant C such that: $\|A(\mathbf{W})\|_{Y_s} \leq \dfrac{C\|\mathbf{W}\|_{Y_r}}{r-s}$;*
3. *$A : Y_r \to Y_s$ is continuous;*
4. *A satisfies the condition 4) of Proposition 2.7.*

For a proof of Lemma 4.1 see Appendix A.

By Lemma 4.1 problem (4.2) satisfies all the hypotheses of Theorem 2.4 (see Remark 2.6 and Proposition 2.7), hence it admits at least a local solution.

Remark 4.2. A particular case of our problem is given by the equation:

$$u'' + m(|Bu|^2)B^2u = 0 \tag{4.4}$$

where $m : [0, +\infty[\to \mathbf{R}$ is a bounded continuous function or $\displaystyle\int_0^{+\infty} m(s)\, ds = +\infty$.

For this equation [6] proved for B - analytic data u_0 and u_1, assuming B^{-1} compact, local existence by the Riesz - Galerkin method. Later on [4] showed that hypothesis B^{-1} compact is removable; Theorem 2.4 allows us to obtain immediately this result if m is bounded. Moreover if $\int_0^{+\infty} m(s)\, ds = +\infty$ it is possible go back to the case m bounded (see [5]) by observing that one has the conserved energy:

$$|u_t|^2 + M(|Bu|^2) = |u_1|^2 + M(|Bu_0|^2) = E(0)$$

where $M(r) = \int_0^r m(s)\, ds$, and therefore $|Bu|^2$ must be bounded (for example $|Bu|^2 \leq c$ where $M(c) = E(0)$ and $M(r) \geq E(0)$ if $r \geq c$).

A. Appendix

Proof of Lemma 4.1.

We prove 1), 2), 3).

i) We remind that for each $\mathbf{W} \in Y_r$ we have:

$$\|A(\mathbf{W})\|_{Y_s} \leq (K+1) \left(\sum_{i=1,n} (\|Bw_i\|_s)^2 \right)^{\frac{1}{2}}.$$

ii) Let us remark that for each $w \in X_r$ and for $0 < s < r$ we have (for a proof see Proposition 2.1 of [6]):

$$\begin{aligned} \|Bw\|_s^2 &= \sum_{j \in \mathbf{N}} \left(\frac{|B^{1+j} w| s^j}{j!} \right)^2 \\ &\leq \frac{\|w\|_r^2}{(r-s)^2}. \end{aligned}$$

From i) and ii), setting $C := (K+1)$ we get:

$$\begin{aligned} \|A(\mathbf{W})\|_{Y_s} &\leq \frac{C \left(\sum_{i=1,n} \|w_i\|_r^2 \right)^{\frac{1}{2}}}{r-s} \\ &\leq C \frac{\|\mathbf{W}\|_r}{r-s}. \end{aligned}$$

Furthermore A is continuous, because from ii) we obtain that $B: X_r \to X_s$ is continuous, and B^{-n+1} is continuous.

Now let us prove 4).

We remark that $A(u) = F(u,,u) = (F_i(u,u))_{i=1,n}$ satisfies 4) of Proposition 2.7 if all A_k satisfies it. Let $u,\ v \in Y_r,\ \ s < r$.

iii) For $k = 1, \ldots, n-1,\ \ F_k(u,v) = A_k(v)$ is Lipschitz continuous (by ii) and the linearity of B) in X_s;

iv) $F_n(u,v) = \sum_{i=1,n} H_i(u)G_i(v)$, where, for $u = (u_1, \ldots, u_n),\ v = (v_1, \ldots, v_n)$:

$$H_i(u) = f_i(B^{-n+1}u_1) \qquad \text{and} \qquad G_i(v) = Bv_i.$$

We observe that, for every $i = 1, \ldots, n,\ H_i(\cdot)\, G_i(v)$ is a compact function in X_s, since H_i is a bounded real function. Moreover G_i is a Lipschitz continuous operator. Then for every $i = 1, \ldots, n,\ H_i\, G_i$, and therefore F_n, satisfies 4) of Proposition 2.7. □

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