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Essential Conformal Fields in Pseudo-Riemannian Geometry. II

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Abstract. We study conformal vector fields on pseudo-Riemannian manifolds. In the case of conformally flat manifolds, the main tool is the conformal development map into the projective quadric. On the other hand, we show that there exists a pseudo-Riemannian manifold carrying a complete and essential vector field which is not conformally flat. The example implies that there is no finite dimensional moduli space for such manifolds. Therefore, a pseudo-Riemannian analogue of Alekseevskii's theorem on the classification of essential conformal vector fields cannot be expected.

1. Introduction

Conformal mappings and conformal vector fields were intensively studied in both Riemannian and pseudo-Riemannian geometry. Conformally flat spaces have been characterized by Cotton, Finzi and Schouten in the early 20th century. In General Relativity conformal aspects are of importance. For global conformal geometry, the conformal development map was introduced by Kuiper in 1949, after earlier work by Brinkmann in the 1920's. Essential conformal vector fields on Riemannian spaces have been studied by Obata, Lelong-Ferrand and Alekseevskii [A1], [La2]. Conformal gradient fields are essentially solutions of the differential equation $\nabla^2\varphi = \frac{\Delta\varphi}{n} \cdot g$. This equation has been studied since the 1920's by Brinkmann, Fialkow, Yano, Obata, Kerbrat and others. In the Riemannian case the results are quite complete. In the pseudo-Riemannian case we started in part I of

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this paper [KR2] a systematic approach including a conformal classification theorem. In this part II we discuss the global structure of conformal vector fields in pseudo-Riemannian geometry using the conformal development map into the conformal compactification of pseudo-Euclidean space. Furthermore we show that there exists a manifold carrying a complete conformal vector field with zeros which is not conformally flat.

2. Conformal vector fields

From the view-point of infinitesimal transformations [Ya], a vector field V is said to preserve a certain geometric quantity if the Lie derivative $\mathcal{L}_V$ of this quantity vanishes. On a pseudo-Riemannian manifold (M, g) a vector field V is called *isometric* if it preserves the metric in the sense of $\mathcal{L}_V g = 0$. Recall that by definition $(\mathcal{L}_V g)(X, Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V)$ for arbitrary tangent vectors X, Y where ∇ denotes the Levi-Civita connection. V is called *conformal* if it preserves the conformal class of the metric in the sense that $\mathcal{L}_V g = 2\varphi \cdot g$ holds for some function φ. Necessarily this function is $\varphi = \frac{1}{n} divV$ in this case. This is equivalent to saying that for the flow (Φ_t) of local diffeomorphisms defined by V each Φ_t is conformal. A vector field is called *complete* if the flow is globally defined as a 1-parameter group of diffeomorphisms $\Phi \colon \mathbb{R} \times M \to M$. V is called *homothetic* if φ is constant. In the particular case of a gradient field $V = grad\ f$ we have $\mathcal{L}_V g = 2\nabla^2 f$, hence $grad\ f$ is conformal if and only if $\nabla^2 f = \varphi \cdot g$ where $n \cdot \varphi = \Delta f = div(grad\ f)$ is the Laplacian. If the symbol $(\quad)^\circ$ denotes the traceless part of a $(0,2)$-tensor, then $grad\ f$ is conformal if and only if $(\nabla^2 f)^\circ \equiv 0$. This equation $(\nabla^2 f)^\circ = 0$ has been extensively studied in many papers, for Riemannian as well as for pseudo-Riemannian manifolds. It arises in various contexts, in particular in connection with the behavior of the *Ricci tensor* Ric_g in a conformal class of metrics. In a similar context, *Ricci solitons* have been considered as solutions of the equation $\mathcal{L}_V g = 2\mathrm{Ric}(g)$, in particular gradient solitons as solutions of $\nabla^2 f = \mathrm{Ric}(g)$ [Iv]. A vector field V is called *concircular* if the local flow (Φ_t) consists of concircular mappings, i.e. conformal mappings preserving geodesic circles. A transformation of the metric $g \mapsto \bar{g} = \frac{1}{\psi^2} g$ is concircular if and only if $(\nabla^2 \psi)^\circ = 0$, see [T], equivalently if $\mathrm{Ric}^\circ_{\bar{g}} = \mathrm{Ric}^\circ_g$, see [KR1]. The local structure of all solutions of $(\nabla^2 \varphi)^\circ = 0$ for any function φ is well understood at least in the case where $grad\varphi$ is not a null vector on an open set. We recall the following lemma from [KR2]. (i) is originally due to Fialkow [Fi], in the Riemannian case (ii) was observed by Tashiro [T].

2.1 LEMMA. *Let (M, g) be a pseudo-Riemannian manifold admitting a non-constant solution φ of the equation $(\nabla^2\varphi)^\circ = 0$ (or admitting a non-trivial conformal gradient field). Then the following holds:*

(i) *In a neighborhood of any point with $\|grad\varphi\|^2 \neq 0$ g is a warped product $g = \eta dt^2 + \varphi'^2(t) \cdot g_*$ ($\eta = \pm 1$ is the sign of $\|grad\varphi\|^2$), φ is a function depending only on t , the trajectories of $grad\varphi/\|grad\varphi\|$ are geodesics, and φ satisfies $\varphi'' = \eta \cdot \frac{\Delta\varphi}{n}$ along these trajectories.*

(ii) *The zeros of $grad\ \varphi$ are isolated. In a neighborhood of such a zero the metric is a warped product in polar coordinates $g = \eta\ dt^2 + \frac{\varphi_\eta'^2(t)}{\varphi_\eta''^2(0)} g_\eta$ where g_η denotes the induced metric on the 'unit sphere' $S(\eta) = \{x \mid \|x\|^2 = \eta\}$ in the pseudo-euclidean space of the same signature as g. In particular, near a critical point of φ the metric g is conformally flat.*

3. The conformal compactification of $\mathbb{E}^n_k$

In the case of Euclidean space $\mathbb{E}^n$ there are the following key examples of complete conformal vector fields

1. the *radial vector field* $V_1(x) = x$,
2. the *constant vector field* $V_2(x) = x_0$.

The corresponding 1-parameter groups of conformal diffeomorphisms are

1. $\Phi_t^{(1)}(x) = exp(t) \cdot x$,
2. $\Phi_t^{(2)}(x) = x + t \cdot x_0$,

respectively. On the *conformal compactification* $S^n = \mathbb{E}^n \cup \{\infty\}$ with the standard conformal structure these two vector fields are *essential* meaning that they are not isometric with respect to any conformally equivalent metric. V_1 has two zeros at $0, \infty$. It is the gradient of a globally defined function on the sphere whereas V_2 is not a gradient and has only one zero at ∞. The standard metric on S^n is characterized by the existence of a conformal gradient field gradφ such that $\nabla_g^2\varphi + c^2\varphi \cdot g = 0$, see [T], [Ob1]. Any simply connected and conformally flat Riemannian manifold M of dimension n admits a conformal immersion $\delta\colon M \to S^n$, see 3.4. Vice versa, for getting examples of conformally flat spaces one can take the preimage under δ of any open subset $A \subset S^n$ or its universal covering. This includes the example $\mathbb{R} \times H^{n-1}$ as the covering of $S^n \setminus S^{n-2}$, called a *Mercator-manifold* in [KP]. Unfortunately, there are no nontrivial coverings which are compatible with complete and essential conformal vector fields. Compare 4.3 for the indefinite case.

3.1 Theorem (Alekseevskii [A1], [Fe1], [Fe2], [Yo]). *Assume that (M, g) is a Riemannian manifold of dimension n admitting a complete and essential conformal vector field. Then (M, g) is conformally diffeomorphic with either the sphere S^n or the Euclidean space $\mathbb{E}^n$.*

In the compact case this result was obtained also by Obata and Lelong-Ferrand [La2]. Note that under the hypotheses of 3.1 the conformal development map is injective which provides an important step in the proof. No analogous result seems to be known in the case of a pseudo-Riemannian manifold with an indefinite metric. However, an analogue of the conformal development map has been studied already quite early.

We denote by $\mathbb{E}^n_k$ the *pseudo-Euclidean space* with the metric $g = -\sum_{i\le k} dx_i^2 + \sum_{i>k} dx_i^2$. A pseudo-Riemannian metric of the same signature is called (locally) *conformally flat* if it is locally conformally equivalent to the metric of $\mathbb{E}^n_k$. For the determination of all conformal transformations of pseudo-Euclidean space see [Ha].

3.2 Lemma (Brinkmann [Br1]). *For any conformally flat pseudo-Riemannian manifold (M^n_k, g) there exists locally an isometric immersion into $\mathbb{E}^{n+2}_{k+1}$.*

Proof. Locally the metric has the form $\varphi^2(-\sum_{i\le k} dx_i^2 + \sum_{i>k} dx_i^2)$ where $x_1, \dots, x_n$ are cartesian coordinates and $\varphi \neq 0$ is a scalar function. Let $\langle x, x\rangle$ denote the pseudo-Euclidean scalar product of the point $x = (x_1, \dots, x_n)$. We define the following mapping

$$x \mapsto y = (y_0, \dots, y_{n+1}) := \Big(\tfrac{\varphi}{2}\big(\langle x, x\rangle + 1\big), \varphi x_1, \dots, \varphi x_n, \tfrac{\varphi}{2}\big(\langle x, x\rangle - 1\big)\Big).$$

Then the following conditions are easily checked:

1. $(y_0, \dots, y_{n+1}) \neq (0, \dots, 0)$,
2. y lies in the null cone $\{y \mid \langle y, y\rangle = 0\}$,
3. the induced metric of this immersion is
$-\sum_{i\le k} dy_i^2 + \sum_{i>k} dy_i^2 = \varphi^2\big(-\sum_{i\le k} dx_i^2 + \sum_{i>k} dx_i^2\big)$.

The *sphere inversion* appears essentially as the mapping $y_{n+1} \mapsto -y_{n+1}$. Note that the Riemannian case $k = 0$ is included; in this case the image does not meet the hyperplane $y_0 = 0$. However, the ambient space is $\mathbb{E}_1^{n+2}$.

With respect to the pseudo-Euclidean metric, the mapping $x \mapsto y$ is conformal in any case, independent of φ. For this aspect of conformal geometry in cosmology see [Hu]. The conformal interpretation of the pseudo-Euclidean metric on $(n+2)$-space is quite classical, see [Be], [Bô]. This motivates the following definition of a conformal development map into the real projective space $\mathbb{R}P^{n+1}$. $\square$

3.3 Definition ([Kui], see also [AD]). The *conformal development map* on a conformally flat pseudo-Riemannian manifold (M_k^n, g) is defined locally by $x \mapsto y \mapsto [y_0, \dots, y_{n+1}] \in \mathbb{R}P^{n+1}$. If M is simply connected this induces a conformal immersion $\delta\colon M \to Q_k^n \subset \mathbb{R}P^{n+1}$, the conformal development. Here Q_k^n denotes the projective quadric $\{y \mid \langle y, y\rangle = 0\}$. Q_k^n can also be regarded as the *conformal compactification* of $\mathbb{E}_k^n$, see [Bô], [Cox]. One observes that $\delta(\mathbb{E}_k^n) = \{[y_0, y, y_{n+1}] \in Q \mid y_{n+1} \neq y_0\}$. The 'points at infinity' $\{y_{n+1} = y_0\}$ can be seen as follows: If C_k^n denotes the null cone in $\mathbb{E}_k^n$ which is represented by $\delta(C_k^n) = \{+1\} \times C_k^n \times \{-1\}$ in Q_k^n, then $\{+1\} \times C_k^n \times \{+1\}$ is a second copy of the null cone after inversion, and $\{0\} \times (C_k^n \cap S^{n-1}) \times \{0\}$ is the 'null cone at infinity'. Projectively the latter is nothing but a lower dimensional quadric $Q_{k-1}^{n-2} \subset \mathbb{R}P^{n-1}$.

Since the real projective space $\mathbb{R}P^{n+1}$ can be regarded as the sphere S^{n+1} modulo identification of antipodal pairs of points, we observe that Q_k^n is diffeomorphic with $\{\langle y, y\rangle = 0\} \cap S^{n+1} \cong S^k \times S^{n-k}$ modulo the identification of antipodal pairs of points. Topologically, Q_k^n can also be regarded as a sphere bundle over $\mathbb{R}P^k$ if $k \le n - k$ [CK], an Euclidean model is the tensor product $S^k \otimes S^{n-k} \subset \mathbb{R}^{(k+1)(n-k+1)}$.

3.4 Remark (Complex structure in two complex variables). In the case of $\mathbb{E}_1^4$ there is the following complex version, using unitary groups (see [BDP]). If we multiply the classical Pauli Spin matrices by the factor i, we obtain a basis for the Lie algebra $\mathbf{su}(2)$. Together with the identity matrix (multiplied by i) we get a basis for $\mathbf{u}(2) \cong \mathbb{E}^4_1$ in a natural way such that $\mathbf{su}(2)$ appears as the spacelike 3-space. Then the Lie group $\mathbf{U}(2)$ turns out to be the conformal compactification of $\mathbf{u}(2)$ via the *Cayley map* $\delta\colon \mathbf{u}(2) \to \mathbf{U}(2)$, $\delta(x) = (1+x)(1-x)^{-1}$. Here 1 denotes the identity matrix. Note that $\mathbf{U}(2) \cong Q^4_1$ appears as a two-fold quotient of $S^1 \times \mathbf{SU}(2) \cong S^1 \times S^3$ which is again diffeomorphic to $S^1 \times S^3$.

3.5 PROPOSITION. *The conformal transformations of the projective quadric Q^n_k are in* 1-1-*correspondence with those projective transformations of $\mathbb{R}P^{n+1}$ preserving Q^n_k.*

This lemma is essentially due to Möbius for the classical case $k = 0, n = 2$ (*Möbius geometry*). For arbitrary dimensions it is stated in [Kui].

3.6 THEOREM (Kuiper [Kui]). *If M is simply connected and conformally flat then $\delta\colon M \to Q^n_k$ is globally defined. If moreover M is compact then δ is either a diffeomorphism between M and S^n (if $k = 0$) or a two-fold covering (if $2 \le k \le n-2$). For $k = 1$ or $k = n-1$ the universal covering is non-compact.*

A conformally flat manifold M is called *developable* if the conformal development map δ is globally defined. Any simply connected conformally flat manifold is developable. Compare [AD] for a Riemannian interpretation for isometric immersions.

4. Conformal vector fields on conformally flat spaces

4.1 *Examples*. The *key examples of conformal vector fields* on pseudo-Euclidean space are again the vector fields V_1 and V_2 above, extended to Q^n_k by taking limits of the flow.

The fixed points of the flow $\Phi^{(1)}_t$ are the two isolated points $[1, 0, -1]$ (the origin) and $[1, 0, 1]$ (its image under the inversion at the unit sphere) and the null cone at infinity $\{0\} \times Q^{n-2}_{k-1} \times \{0\}$.

The fixed point set of $\Phi^{(2)}_t$ depends on the type of the translation vector x_0: If $\langle x_0, x_0 \rangle \neq 0$ then the only fixed point is $[1, 0, 1] = \infty$. This is a perfect analogue of the conformal flow on the standard sphere with one fixed point. If $\langle x_0, x_0\rangle = 0$ then the fixed point set contains in addition the null cone at infinity and the inversion of the ordinary null cone.

4.2 COROLLARY. *On the conformal compactification Q^n_k there exists a conformal vector field $\overline{V}_2$ with one zero, and on $Q^n_k \setminus Q^{n-2}_{k-1}$ there exists a conformal vector field $\overline{V}_1$ with two zeros. These vector fields are essential and complete. $\overline{V}_1$ is a local gradient field, $\overline{V}_2$ is not a gradient field near the zero.*

$Q^n_k\setminus Q^{n-2}_{k-1}$ is nothing but the union of $\delta(\mathbb{E}^n)$ and its image under inversion at the 'unit sphere'. This inversion transforms $\overline{V}_1$ into $-\overline{V}_1$. This space $Q^n_k\setminus Q^{n-2}_{k-1}$ is not simply connected. In fact, its fundamental group is isomorphic to the integers $\mathbb{Z}$ if $2 \le k \le n-2$, leading to a $\mathbb{Z}$-sheeted universal covering which carries a conformal vector field with infinitely many zeros.

4.3 COROLLARY [KR2]. *For $2 \le k \le n-2$ the universal covering of $Q^n_k\setminus Q^{n-2}_{k-1}$ defines a manifold $M(\mathbb{Z})$ together with a conformal structure such that $\delta\colon M(\mathbb{Z}) \to Q^n_k \setminus Q^{n-2}_{k-1}$ becomes a conformal covering. The conformal vector field $\overline{V}_1$ can be lifted to a vector field $V_1^{(\mathbb{Z})}$ with infinitely many zeros. These zeros are in natural bijection to $(2\mathbb{Z})\cup(2\mathbb{Z}+1) \cong \mathbb{Z}$. Similarly, there are intermediate coverings with any even number of zeros of the vector field.*

Recall that $M(\mathbb{Z})$ together with a particular metric g in this conformal class is also characterized by the *pendulum equation* $\nabla^2_g\varphi + \sin\varphi\cdot g = 0$, see [KR3].

For $2 \le k \le n-2$ the universal covering of the quadric Q^n_k itself is diffeomorphic to $S^k \times S^{n-k}$. The metric can be chosen as the product of two metrics of constant curvature with opposite signs. This space carries a conformal vector field $V_2^{(2)}$ with two zeros as the lift of $\overline{V}_2$ via the conformal covering $\delta\colon S^k \times S^{n-k} \to Q^n_k$. Even if we remove one of the zeros, the vector field is still complete. The punctured $S^k\times S^{n-k}$ carries a complete conformal vector field with one zero.

In a neighborhood of a zero of a conformal gradient field the metric is conformally flat by Lemma 2.1. By using geodesic polar coordinates one can extend this to a global result, under the additional assumption that (M, g) is *C-complete* with respect to the gradient field, meaning that every point on the manifold can be joined by a geodesic with at least one critical point of the function and that every geodesic through a critical point is defined on $\mathbb{R}$.

4.4 THEOREM [KR2]. *Assume that a C-complete pseudo-Riemannian manifold is given carrying a non-isometric conformal gradient field with at least one zero. Then the manifold is (locally) conformally flat.*

As a consequence, under the same assumptions on M the conformal development map δ is well defined on the universal covering $\tilde{M}$ of M. If $\delta\colon \tilde{M} \to \delta(\tilde{M})$ itself is a covering map then we are either in the situation of $M(\mathbb{Z})$ or of $M(\mathbb{Z}) \setminus 2\mathbb{Z}$, the preimage under δ of $Q^n_k \setminus Q^{n-2}_{k-1}$ minus one of the zeros. This holds for signature $2 \le k \le n-2$ by 4.2 and 4.3.

If in addition the metric is geodesically complete and the conformal gradient field is complete, then the conformal type is determined by the number N of zeros, $N \in \mathbb{N} \cup \{\mathbb{N}\} \cup \{\mathbb{Z}\}$, see [KR2;Thm.C]. There it is claimed that the conformal type is uniquely determined by N. This is true only for $N = \mathbb{Z}$ and for finite odd numbers N. In the other cases, there are two types in general, depending on the two possibilities of ends. Therefore, the formulation of Theorem C should be slightly modified as follows.

4.5 THEOREM. *Let M^n_k be a geodesically complete pseudo-Riemannian manifold of signature (k, n) with $2 \le k \le n-2$ carrying a non-trivial conformal gradient field with at least one zero.*

1. *There are at most two diffeomorphism types of M^n_k for given number N of zeros. Here in the case of infinitely many zeros we have to distinguish between $\mathbb{N}$ and $\mathbb{Z}$.*
2. *Every manifold is conformally equivalent to a standard manifold $M(J)(\alpha, \beta)$ defined at the end of* [KR2;Sect.4].
3. *If in addition the vector field is complete then the conformal type is uniquely determined by the diffeomorphism type.*

The proof given in [KR2] is going through with only slight modifications. In any case, the manifold can be decomposed into $N + 1$ building blocks with one $(+)$-end and one $(-)$-end each. Depending on the types of the ends, the diffeomorphism type of the manifold is uniquely determined. If N is odd then there is no choice, if N is even we have the two cases $(+)(+)$ and $(-)(-)$.

Without the completeness of the metric but still under the assumption of the completeness of the vector field we obtain a huge variety of examples as follows:

4.6 Theorem. *There are uncountably many distinct pairs (M, V) of conformally flat pseudo-Riemannian manifolds M and conformal and complete gradient fields V with zeros.*

Here 'distinct' means that there is no conformal diffeomorphism preserving the trajectories of the corresponding flows of the vector fields.

Proof. We start with $M(\mathbb{Z})$ and consider an arbitrary subset A of the set $\mathbb{Z}$ of zeros. The vector field $\overline{V}_1$ is still complete on $M(\mathbb{Z}) \setminus A$ for any A, it has still at least one zero if $A \neq \mathbb{Z}$, and M is still C-complete if A does not contain two subsequent integers $x, x+1$. Two such subsets A_1, A_2 lead to distinct examples if A_1, A_2 are inequivalent under translation $x \mapsto x+1$ and reflection $x \mapsto -x$. There are uncountably many possibilities left. □

5. Non-conformally flat spaces admitting complete conformal vector fields

The same construction as in 4.6 leads to non-conformally flat examples if we drop the condition of C-completeness. Locally it is easy to get examples as warped products of a real interval with an arbitrary (M_*, g_*), just by applying Lemma 2.1 (i).

5.1 Theorem. *For $2 \leq k \leq n-2$ there is an infinite-dimensional space of non-conformally flat pseudo-Riemannian manifolds admitting a complete conformal gradient field with at least one zero.*

Proof. We start with $M(\mathbb{Z}) \setminus A$ as in 4.6 but now assume that A does contain two subsequent numbers $x, x+1$. The trajectories between these two points in the manifold are still defined over $\mathbb{R}$ but do not converge to a point of the manifold for $t \to \pm\infty$. The metric on the part covered by all those geodesics is a warped product $(-r, r) \times_{\varphi'} S(\eta)$ where $S(\eta)$ denotes the induced metric of the 'unit sphere' in $\mathbb{E}^n_k$ of dimension at least 2, see Lemma 2.1. If we perturb this induced metric in a small neighborhood of an arbitrary point lying on one of those trajectories between x and $x+1$, then the corresponding warped product with the same function will still admit a complete conformal gradient field by 2.1. But this perturbation will - in general - destroy the conformal flatness of the warped product because generically a metric in dimension > 2 is not conformally flat. Again there are uncountably many essentially distinct possibilities for such an A, and for each one there is an infinite-dimensional space of possibilities for the perturbation of the metric. $\square$

In the Riemannian case this type of perturbation is impossible unless one cuts out all the zeros of the vector field because there are at most two. In this case the warped product $(-r, r) \times_{\varphi'} M_*$ admits an inessential conformal gradient field for any Riemannian manifold M_*, i.e., one without a zero. This includes examples which are non-standard Einstein spaces if one starts with a non-standard Einstein metric on $M_* = S^{n-1}$. By adding the two 'poles' one obtains an Einstein metric on the sphere S^n with two metrical (but not topological) singularities.

6. Conformally closed vector fields

Given a metric g, a vector field V is called *closed* if is locally a gradient field or, equivalently, if the associated 1-form ω is closed: $d\omega = 0$. Closed conformal vector fields can be classified in the same way as gradient fields. Compare [Bo] for a version of Theorem 3.1 for closed conformal vector fields. Unfortunately, the closedness is not conformally invariant. Therefore it seems to be more natural to find an appropriate notion which is invariant under conformal changes of the metric. In terms of 1-forms, we observe that a conformal change of the metric induces a conformal change of the 1-form: we replace ω by $\overline{\omega} = e^f \omega$. The equation $d(\overline{\omega}) = d(e^f \omega) = e^f (d\omega + df \wedge \omega)$ leads to the following definition [Ob2]:

6.1 DEFINITION. A vector field V is called *conformally closed* if (locally) it is closed after a certain conformal change of the metric. An equivalent formulation is that the associated 1-form ω satisfies $d\omega + df \wedge \omega = 0$ for a certain function f. Locally this is equivalent to $d\omega + \eta \wedge \omega = 0$ for a certain closed 1-form η. Note that the conformal closedness is a local notion just as conformal flatness. It is much weaker than the corresponding global notion where the conformal change would have to be globally defined.

6.2 THEOREM. *Assume that V is a conformally closed conformal vector field on a manifold M with a fixed conformal structure. Assume that V has at least one zero. Then the following hold:*

1. *In a neighborhood of the zero the manifold is conformally flat.*
2. *If V is complete then M admits a globally defined conformal immersion $\mathbb{E}^n_k \to M$.*
3. *If in addition M is compact then the metric must be positive or negative definite. Consequently, M is conformally equivalent with the standard sphere.*

PROOF. 1. After a conformal change of the metric, V is a gradient field in a neighborhood of the zero p_0. By Proposition 2.1 it is conformally flat in that neighborhood. Moreover, the metric is a warped product in polar coordinates, and V is the radial vector field in these polar coordinates.

2. The warped product structure of the metric near the zero of V cannot change along the trajectories of V as long as $V \neq 0$. By the completeness of V this does not happen in finite time of the flow Φ_t. Therefore, we can extend the polar coordinates globally. This defines a conformal immersion of the (pseudo-)euclidean space into M, mapping the origin to p_0, and the radial straight lines onto the trajectories of V which are geodesics in M. It also follows that the exponential map from p_0 is globally defined. However, the metric need not be complete, as the example of the quadric Q^n_k in 3.3 shows. A limit point of the trajectories for $t \to \infty$ does not have to be in M.

3. If we assume the compactness of M then M must contain limit points of the trajectories, and V has a zero at such limit points because they are fixed points of the flow. This implies that V has at least two zeros p_0 and p_1 such that there is an open set of trajectories from one to the other. If we apply the argument of 2. to p_1 then we find that $-V$ is the radial vector field in polar coordinates around p_1. Moreover, there are conformal immersions $F_0, F_1 : \mathbb{E}^n_k \to M$ with $F_i(0) = p_i$. The composition $F_1^{-1} F_0$ is the inversion at the 'unit sphere'. So we are precisely in the position of Corollary 4.2. If the metric is indefinite then we find limit points of the trajectories also in the direction of the null cone. By the same argument as before, V has a zero at each of these limit points. On the other hand, these limit points are not a discrete set. This contradicts the conformal closedness of V if we consider a neighborhood of such a limit point. Therefore the metric cannot

be indefinite, and the conclusion follows from 3.1. □

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