論文

Stone–Čech boundaries of discrete groups and measure equivalence theory

(離散群のストーン–チェック境界と測度同値理論)

酒匂宏樹

STONE–ČECH BOUNDARIES OF DISCRETE GROUPS AND MEASURE EQUIVALENCE THEORY

HIROKI SAKO

ABSTRACT. We get three types of results on measure equivalence theory: direct product groups of Ozawa's class $\mathcal{S}$ groups, wreath product groups and amalgamated free products. We prove measure equivalence factorization results on direct product groups of Ozawa's class $\mathcal{S}$ groups. As consequences, Monod–Shalom type orbit equivalence rigidity theorems follow. We prove that if two wreath product groups $A \wr G$, $B \wr \Gamma$ of non-amenable exact direct product groups G, Γ with amenable bases A, B are measure equivalent, then G and Γ are measure equivalent. Rigidity results on amalgamated free products of non-amenable exact direct product groups are also shown. We also prove that being in Ozawa's class $\mathcal{S}$ of countable discrete groups is invariant under measure equivalence.

1. INTRODUCTION

Measure equivalence theory is a discipline which deals with the question how much structure of countable groups is preserved through measure equivalence. The notion of measure equivalence was introduced by Gromov [Gr, 0.5.E] as a variant of quasi-isometry. The field recently has attracted much attention since small measure equivalence classes were found. The following is the definition of measure equivalence and ME couplings given by M. Gromov.

Definition 1.1. [Gr, 0.5.E] *Let G and Γ be countable groups. The group G is said to be **measure equivalent** (**ME**) to Γ, if there exist a standard measure space (Σ, ν), a measure preserving action of $G \times \Gamma$ on Σ and measurable subsets $X, Y \subset \Sigma$ with the following properties:*

$$\Sigma = \bigsqcup_{\gamma \in \Gamma} \gamma X = \bigsqcup_{g \in G} gY, \quad \nu(X) < \infty, \quad \nu(Y) < \infty.$$

*Then we use the notation $G \sim_{\mathrm{ME}} \Gamma$. The measure space Σ equipped with the $(G \times \Gamma)$-action is called an **ME coupling of** G **with** Γ. If the $(G \times \Gamma)$-action is ergodic, then Σ is said to be **ergodic**.*

The relation $\sim_{\mathrm{ME}}$ is an equivalence relation among countable groups. The equivalence relation sometimes forgets much structure of groups. For example, arbitrary two amenable countable groups are measure equivalent. This was shown by Ornstein–Weiss [OrWe, THEOREM 6], Connes–Feldman–Weiss [ConFeWe, THEOREM 10] and the correspondence between measure equivalence and stable (weak) orbit equivalence [Fu2, Section 3]. On the other hand, for a specific group Γ, the other group G is forced to have some algebraic structure when G and Γ are ME (higher rank lattices [Fu1], mapping class groups with high complexity [Kid]). This

Key words and phrases. measurable equivalence; orbit equivalence relation; bi-exactness.

means that the measure equivalence class to which Γ belongs is small. Such phenomena are called measure equivalence rigidity.

In order to classify groups in terms of measure equivalence, it is important to find invariants and properties which are invariant under measure equivalence. Famous known examples are the following.

- The Kazhdan's property (T), Furman [Fu1].
- The Haagerup property, Popa [Po3] and Jolissaint [Jo].
- The Cowling–Haagerup constants of weak amenability, Cowling–Zimmer [CowZi] and Jolissaint [Jo].
- Ratios of the ℓ^2-Betti numbers $\mathbb{R}(\beta_n(G))_{n\geq 0}$, Gaboriau [Ga].

Measure equivalence theory is closely related to ergodic theory of measure preserving group actions. By Furman's observation [Fu2, Section 3], if two group actions on standard probability space X essentially have a common orbit, or more generally if they are stably orbit equivalent, we naturally get an associated ME coupling. We get a cross-sectional links with variegated fields at this point (see surveys [Sh] by Shalom, [Fu3] by Furman). By Murray–von Neumann's group measure space construction [MvN, Chapter 12], we can introduce operator algebraic structures on orbit equivalence relations and find a connection between measure equivalence theory and the theory of operator algebra.

The first purpose of this paper is to show measure equivalence and orbit equivalence rigidity results on three types of countable groups; direct product groups, wreath product groups and amalgamated free product groups. The second purpose is to show that being in the Ozawa's class $\mathcal{S}$ of countable discrete groups is an invariant under measure equivalence. We employ Stone–Čech boundaries of discrete groups as strategy for these results. In the paper [Ad] S. Adams used Gromov boundaries of word hyperbolic groups to show some indecomposability of orbit equivalence relations given by such groups. The class $\mathcal{S}$ includes the class of word hyperbolic groups. In stead of the geometrical boundaries, the Stone–Čech boundaries in the universal compactifications are useful in getting our purpose.

2. Main results

Our argument begins with a general principle, which can be used for the three types of rigidity results. In the following subsections, we state the principle and explain main results on individual cases.

2.1. Measurable Embedding of Subgroups.

When we consider that the ME coupling Σ gives an identification of two groups G and Γ, we can understand that the following notion decides inclusions of subgroups in the coupling Σ.

Definition 2.1. *Let Σ be an ME coupling of G with Γ (or measure embedding defined in Definition 3.1). We say that a subgroup $H \subset G$* ***measurably embeds into*** *a subgroup $\Lambda \subset \Gamma$* ***in*** *Σ, if there exists a non-null measurable subset $\Omega \subset \Sigma$ which is invariant under the $(H \times \Lambda)$-action so that the measure of a Λ fundamental domain is finite. Then we use the notation $H \preceq_\Sigma \Lambda$. The measurable subset Ω is called a* ***partial embedding of*** *H* ***into*** *Λ.*

We remark that for every Λ-invariant measurable subset Ω', there exists a Λ-fundamental domain.

We will make use of strategy which was developed for group von Neumann algebras. In the book [BrOz], Brown and Ozawa introduced the notion of bi-exactness defined on a discrete group Γ and a family $\mathcal{G}$ of its subgroups. The notion was characterized by topological amenability of a relative boundary. Brown and Ozawa showed the following criterion: If Γ is bi-exact relative to $\mathcal{G}$, then for any von Neumann subalgebra $N \subset L\Gamma$ with non-amenable (non-injective) relative commutant, we have $N \preceq_{L\Gamma} L\Lambda$ for some $\Lambda \in \mathcal{G}$. The symbol $\preceq_{L\Gamma}$ stands for the embedding of corners, which was defined by Popa ([Po1, Po3]). Bi-exactness also gives a criterion for measure embedding. It will be a key ingredient of the three kinds of rigidity results. In Section 4, we will quickly review its definition and basic properties.

Theorem 2.2 (Theorem 5.3). *Let Σ be an ergodic ME coupling between G and Γ. Suppose that Γ is bi-exact relative to $\mathcal{G}$. Let H be a subgroup of G. If the centralizer $Z_G(H) = \{g \in G \mid gh = hg, h \in H\}$ is non-amenable, then there exists $\Lambda \in \mathcal{G}$ satisfying $H \preceq_\Sigma \Lambda$.*

2.2. **Results on Direct Products.** We will show Monod–Shalom type theorems for class $\mathcal{S}$ groups (see Section 4 for the definition of $\mathcal{S}$). In the paper [MoSh], Monod and Shalom proved measure equivalence and orbit equivalence rigidity theorems on class $\mathcal{C}$ groups. Both families of groups contains non-elementary word-hyperbolic groups. But there exist class $\mathcal{S}$ groups which have normal infinite amenable subgroups (Ozawa [Oz3, Oz5]), while the class $\mathcal{C}$ does not contain such groups.

Theorem 2.3 (Theorem 6.4). *Let $\{G_i \mid 1 \le i \le m\}$ be a finite family of non-amenable groups and let $\{\Gamma_j \mid 1 \le j \le n\}$ be a finite family of $\mathcal{S}$ groups. Denote $G = \prod_i G_i$, $\Gamma = \prod_j \Gamma_j$ and $H_i = \prod_{k \ne i} G_k$. Suppose $m \ge n$. If $G \sim_{\mathrm{ME}} \Gamma$, then $m = n$ and there exists $\sigma \in \mathfrak{S}_n$ satisfying $G_{\sigma(j)} \sim_{\mathrm{ME}} \Gamma_j$ $(1 \le j \le n)$.*

Ozawa and Popa [OzPo] got factorization results on type II_1-factors. The above theorem is a factorization result in the setting of measure equivalence theory. By the correspondence between measure equivalence and stable orbit equivalence given by Furman [Fu2], we also get orbit equivalence rigidity theorems. The most typical one is the following theorem.

Theorem 2.4 (Theorem 6.15). *Let G, Γ be groups as above. Let α be a free ergodic measure preserving G-action on a standard probability measure space X and let β be a free ergodic measure preserving Γ-action on a standard probability space Y. Suppose that any G_i has no non-trivial normal finite subgroup and that any Γ_j is ICC (group with no finite conjugacy class $\ne \{1\}$).*

If the actions are stably orbit equivalent and the actions of $H_i = \prod_{k \ne i} G_k$ on X are ergodic, then $m = n$ and there exist $\sigma \in \mathfrak{S}_n$ and embeddings of groups $\phi_i \colon G_{\sigma(j)} \to \Gamma_j$ such that the Γ-action β is conjugate to the induced action $\mathrm{Ind}_G^\Gamma(\alpha, \prod \phi_i)$.

See Subsection 6.4 for the definition of induced actions. In Section 6, we will get a result on symmetric groups $\mathrm{Out}(\mathcal{R}), \mathcal{F}(\mathcal{R})$ of relations $\mathcal{R}$ and prove rigidity results on groups with amenable direct product factors. By using Furman's technique [Fu1], we have the following. A suitable description for our cases has been written in Monod and Shalom's paper [MoSh].

Theorem 2.5 (Subsection 6.5). *Let $\{\Gamma_j \mid 1 \le j \le n\}$ be a finite family of non-amenable ICC groups in the class $\mathcal{S}$. Denote $\Gamma = \prod_{j=1}^n \Gamma_j$. Let β be a free ergodic* *measure preserving Γ-action on a standard probability space Y. Suppose that the restrictions of β on $\Lambda_j = \prod_{l\neq j}\Gamma_l$ are ergodic. Let G be an arbitrary group and let α be an arbitrary free ergodic measure preserving G-action on a standard probability space X. Suppose that α does not have non-trivial recurrent subsets (mild mixing condition). If the actions α and β are stably orbit equivalent, then these actions are virtually conjugate.*

See Definition 1.8 in Monod and Shalom's paper [MoSh] for the definition of the mild mixing condition.

2.3. **Results on Wreath Products.** The **wreath product** $A \wr G$ of a group G with **base** group A is the group obtained by the semidirect product group $A \wr G = (\oplus_{g\in G} A^{(g)}) \rtimes G$, where $A^{(g)}$ are the copies of A and G acting on the direct sum $\oplus_G A^{(g)}$ by the Bernoulli shift $h((a_g)_g) = (a_{h^{-1}g})_g$.

Theorem 2.6 (Section 7). *Let G, Γ be non-amenable exact groups and let H, Λ be infinite exact groups. Denote by $\widetilde{G}$, $\widetilde{\Gamma}$ wreath products $\widetilde{G} = A\wr(G\times H)$, $\widetilde{\Gamma} = B\wr(\Gamma\times\Lambda)$ with amenable bases A, B. The following hold true:*

1. *If $\widetilde{G} \sim_{\mathrm{ME}} \widetilde{\Gamma}$, then $G \times H \sim_{\mathrm{ME}} \Gamma \times \Lambda$. More precisely, for an ergodic ME coupling Σ of $\widetilde{G}$ with $\widetilde{\Gamma}$, there exists $((G\times H)\times(\Gamma\times\Lambda))$-invariant measurable subset $\Omega \subset \Sigma$ which gives an ME coupling of $G \times H$ with $\Gamma \times \Lambda$ and satisfies $[\widetilde{\Gamma} : \widetilde{G}]_\Sigma = [\Gamma \times \Lambda : G \times H]_\Omega$,*
2. *Let α be a free ergodic measure preserving $\widetilde{G}$-action on a standard probability space X and let β be a free ergodic measure preserving $\widetilde{\Gamma}$-action on a standard probability space Y. Suppose that the restrictions $\alpha|_{G\times H}$ and $\beta|_{\Gamma\times\Lambda}$ are ergodic. If α and β are stably orbit equivalent, then $\alpha|_{G\times H}$ and $\beta|_{\Gamma\times\Lambda}$ are stably orbit equivalent.*

Popa proved very powerful rigidity theorems on Bernoulli shift actions of w-rigid groups (von Neumann rigidity [Po1, Po2], cocycle rigidity [Po4]). He also proved a cocycle super-rigidity theorem for Bernoulli shift actions of groups, which are typically given by products of infinite groups and non-amenable groups ([Po5]). In the papers, Popa developed the deformation/spectral gap argument, which has been used for several rigidity results on Bernoulli shift actions (Ioana [Io], Chifan and Ioana [ChIo]) and amalgamated free products (Chifan and Houdayer [ChHo]). We note here that our paper was influenced by the above results, although we will not use the technique.

2.4. **Results on Amalgamated Free Products.** We will also prove the following rigidity theorem in measurable group theory. Theorem 2.7 admits an amalgamation over a amenable subgroup, while restricting each factor to direct product of two non-amenable groups.

Theorem 2.7 (Theorem 8.4). *Let G_i $(i = 0, 1)$ be a countable group which is given by a direct product of two non-amenable exact groups. Let Γ_j $(j = 0, 1)$ be also such direct product groups. Denote by $G = G_0 *_A G_1, \Gamma = \Gamma_0 *_B \Gamma_1$ free products with amalgamations by amenable subgroups $A \subset G_i, B \subset \Gamma_j$. Under the convention $1 + 1 = 0$, the following hold true:*

1. *If $G \sim_{\mathrm{ME}} \Gamma$, then $G_0 \sim_{\mathrm{ME}} \Gamma_j$ and $G_1 \sim_{\mathrm{ME}} \Gamma_{j+1}$ for some $j \in \{0, 1\}$,*

(2) *Let α be a free ergodic measure preserving G-action on a standard probability space X and let β be a free ergodic measure preserving Γ-action on a standard probability space Y. Suppose that the restrictions $\alpha|_{G_i}$ and $\beta|_{\Gamma_j}$ are ergodic. If α and β are stably orbit equivalent, then there exists $j \in \{0,1\}$ so that $\alpha|_{G_i}$ and $\beta|_{\Gamma_{i+j}}$ are stably orbit equivalent for each $i \in \{0,1\}$.*

We will also prove other results in Theorem 8.4, which are analogous to the results shown by Alvarez and Gaboriau [AlvGab]. They proved measure equivalence and stably orbit equivalence results on free products of measurably freely indecomposable ($\mathcal{MFI}$) groups. The class $\mathcal{MFI}$ is a large class including groups whose first ℓ^2-Betti numbers are 0.

In [ChHo], Chifan and Houdayer proved a von Neumann algebraic rigidity theorem for group measure space constructions $L^\infty X \rtimes \Gamma$ of free ergodic measure preserving actions, where group Γ was required to be a free product of direct product groups between infinite groups and non-amenable groups. The assertion is much stronger than rigidity of orbit equivalence relations, although amalgamation on groups is not allowed in the theorem. Prior to these results, in [IoPePo], Ioana, Peterson and Popa got Bass–Serre rigidity results on von Neumann algebras and orbit equivalence relations given by free product groups of w-rigid groups.

2.5. **Ozawa's class $\mathcal{S}$.** We also prove that Ozawa's class $\mathcal{S}$ defined in [Oz3] is a measure equivalence invariant class. The class is defined by means of topological amenability of the largest boundary. C. Anantharaman-Delaroche defined the notion of amenability for discrete group actions on C^*-algebras ([AD, Section 4]). In this paper, we sometimes identify actions on commutative C^*-algebra with corresponding actions on their Gelfand spectrums. Ozawa and Popa proved classification results on group von Neumann algebras of the class $\mathcal{S}$ ([Oz2], [OzPo]).

Definition 2.8. [Oz3, Section 4] *A countable group G is said to be in $\mathcal{S}$ if the left-times-right translation action of $G \times G$ on $\beta G \cap G^c$ is amenable, where $\beta G \cap G^c$ is the Gelfand spectrum of the commutative C^*-algebra $\ell_\infty G/c_0 G$.*

This is equivalent to the condition that G is bi-exact relative to $\{\{1\}\}$ by [BrOz, Propposittion 15.2.3]. The following is the last main theorem of this paper. We recall the definition and basic properties of bi-exactness in Section 4.

Theorem 2.9. *If G is measure equivalent to Γ and $\Gamma \in \mathcal{S}$, then $G \in \mathcal{S}$.*

The class $\mathcal{S}$ is an intermediate class between those of exact groups and amenable groups. These two classes are also invariant under measure equivalence. These classes are characterized by topological amenability. A countable group is exact if and only if there exists an amenable action on a compact space [Oz1]. A countable group is amenable if and only if any continuous action on any compact space is amenable.

By Hjorth's theorem [Hj], A countable group G is treeable in the sense of Pemantle and Peres [PemPer], if and only if G is measure equivalent to a free group ($\mathbb{Z}, \mathbb{F}_2$ or $\mathbb{F}_\infty$). As a corollary of Theorem 2.9, the class of treeable groups is an intermediate measure equivalece invariant class between $\mathcal{S}$ and the set of amenable groups. We get the following fact on group von Neumann algebras:

Corollary 2.10. *If G is measure equivalent to a free group, then the group von Neumann algebra $L(G)$ is solid, namely, the relative commutant of every diffuse subalgebra is injective.*

Since the free groups are in the class $\mathcal{S}$, if G is measure equivalent to a free group, then $\Gamma \in \mathcal{S}$. By [Oz2, Theorem 6], $L(G)$ is solid.

3. The Notion of Measure Equivalence and Measurable Embedding

3.1. Measurable Embedding.

The following notion will be useful throughout this paper, even if one is only interested in measure equivalence. It is a generalization of Gromov's measure equivalence.

Definition 3.1. *Let G and Γ be countable groups. (We admit the case that they are finite). We say that the group G* ***measurably embeds*** *into Γ, if there exist a standard measure space (Σ, ν), a measure preserving action of $G \times \Gamma$ on Σ and measurable subsets $X, Y \subset \Sigma$ with the following properties:*

$$\Sigma = \bigsqcup_{\gamma \in \Gamma} \gamma X = \bigsqcup_{g \in G} gY, \quad \nu(X) < \infty.$$

Then we use the notation $G \preceq_{\mathrm{ME}} \Gamma$. The measure space Σ equipped with the $(G \times \Gamma)$-action is called a ***measure embedding*** *of G into Γ. The constant $[\Gamma : G]_\Sigma = \nu(Y)/\nu(X)$ is called the* ***coupling constant****. The measure embedding Σ is said to be* ***ergodic****, if the $(G \times \Gamma)$-action is ergodic.*

If the measure of the G fundamental domain Y is also finite, then the measure space Σ gives an ME coupling between G and Γ. As in the case of ME couplings (Lemma 2.2 in Furman [Fu1]), if we have a measure embedding of G into Γ, there is ergodic one by using ergodic decomposition.

Example 3.2. Let β be a free measure preserving action of Γ on a standard measure space (Y, μ) and let α be a free measure preserving action of G on a measurable subset $X \subset Y$ with measure 1. Suppose that $\alpha(G)(x) \subset \beta(\Gamma)(x)$, for a.e. $x \in X$. Denote by Σ the measure space $\mathcal{R}_\beta \cap (X \times Y) \cong X \times \Gamma$. Define $(G \times \Gamma)$-action on Σ by

$$(g, \gamma)(x, y) = (\alpha_g(x), \beta_\gamma(y)), \quad (x, y) \in \Sigma.$$

The measurable subset $D = \{(x, x) \in \mathcal{R}_\beta; x \in X\} \subset \Sigma$ is a Γ fundamental domain. The existence of a G-fundamental domain is shown as follows. Enumerate the elements of Γ as $\{\gamma_n\}_{n \in \mathbb{N}}$ and define $D_n = \gamma_n(D)$. By the freeness of the G-action α, the measurable subsets $\{g(D_n)\}_{g \in G}$ are disjoint. Define a measurable subset $D_G \subset \Sigma$ by

$$D_G = \bigcup_{n \in \mathbb{N}} \left(D_n \setminus \bigcup_{m=1}^{n-1} \bigcup_{g \in G} gD_m \right).$$

The measurable subsets $\{gD_G\}_{g \in G}$ are disjoint. The union $\cup_{g \in G} gD_G$ covers $\cup_n D_n = \Sigma$. Thus we get a G fundamental domain D_G.

Definition 3.3. *For a measure embedding* (Σ, ν) *of* G *into* Γ, *the following quantity is called the* ***coupling constant*** *of* Σ *and denoted by* $[\Gamma : G]_\Sigma$*:*

$$[\Gamma : G]_\Sigma = \nu(Y)/\nu(X) \in (0, \infty],$$

where X *is a* Γ *fundamental domain and* Y *is a* G *fundamental domain. This definition does not depend on the choice of* X *and* Y.

Remark 3.4.
(1) The relation $\preceq_{\rm ME}$ is transitive, that is, if $H \preceq_{\rm ME} \Lambda$ and $\Lambda \preceq_{\rm ME} \Gamma$, then $H \preceq_{\rm ME} \Gamma$. The proof is the same as that of " $\sim_{\rm ME}$" ([Fu1]).
(2) If countable groups G and Γ satisfy $G \preceq_{\rm ME} \Gamma$ and Γ is amenable (resp. exact), then G is also amenable (resp. exact). The class $\mathcal{S}$ on countable groups has the property. This is the goal of Section 9.
(3) For a subgroup $\Lambda \subset \Gamma$, we can regard Γ as a measure embedding of Λ into Γ, letting Γ act from the right and Λ act from the left. Then the coupling constant $[\Gamma : \Lambda]_\Gamma$ coincides with the index of the group inclusion.
(4) Let $G, H \subset \Gamma$ be subgroups. We regard Γ as the standard self coupling of Γ, on which $\Gamma \times \Gamma$ acts by the left-and-right translation. The groups satisfy $G \preceq_\Gamma H$ if and only if there exists $\gamma \in \Gamma$ such that $G\gamma H$ is a finite union of left H-cosets $\cup_{i=1}^n \gamma_i H$. This is equivalent to the condition $[G : G \cap \gamma H \gamma^{-1}] < \infty$.

We introduce the notion of support of partial embeddings. For a measureable subset $\Omega \subset \Sigma$, we denote by $\chi(\Omega) \in L^\infty(\Sigma)$ the characteristic function.

Definition 3.5. *Let* $H \subset G$, $\Lambda \subset \Gamma$ *be subgroups. Let* Σ *be a measure embedding of* G *into* Γ. *Choose a* Γ *fundamental domain* X *and a* G *fundamental domain* Y. *Identify naturally* $L^\infty X, L^\infty Y$ *with* $(L^\infty \Sigma)^\Gamma, (L^\infty \Sigma)^G$ *respectively. We define* $\mathrm{supp}_X^\Gamma(H \preceq_\Sigma \Lambda) \in L^\infty X$ *by the projection which corresponds to*

$$\bigvee \{\gamma\chi(\Omega) \mid \gamma \in \Gamma, \Omega \subset \Sigma \text{ gives } H \preceq_\Sigma \Lambda\} \in (L^\infty \Sigma)^\Gamma.$$

We define $\mathrm{supp}_Y^G(H \preceq_\Sigma \Lambda) \in L^\infty Y$ *by the projection which corresponds to*

$$\bigvee \{g\chi(\Omega) \mid g \in G, \Omega \subset \Sigma \text{ gives } H \preceq_\Sigma \Lambda\} \in (L^\infty \Sigma)^G.$$

We call them Γ***-support*** *and* G***-support*** *of* $H \preceq_\Sigma \Lambda$ *respectively.*

We note that $H \preceq_\Sigma \Lambda$ in Σ if and only if $p \neq 0$ (or $q \neq 0$).

3.2. **Stable Orbit Equivalence.** For a free measure preserving Γ-action β on a standard measure space Y, we write the equivalence relation of the action as

$$\mathcal{R}_\beta = \{(\gamma x, x) \mid y \in Y, \gamma \in \Gamma\} \subset Y \times Y.$$

This gives an equivalence relation on Y with countable equivalence classes. On the set $\mathcal{R}_\beta$, we introduce a structure of a measure set by the identification $\mathcal{R}_\beta \ni (\gamma y, y) \to (\gamma, y) \in \Gamma \times Y$. The measure on $\mathcal{R}_\beta$ is the left counting measure defined in Feldman–Moore [FeMoo2].

Comparison of such relations has a correspondence with ME couplings and measure embeddings. Suppose that α are a free ergodic measure preserving action of G on a finite measure space X and that β is a free measure preserving action of Γ on a standard measure space. We denote by $\mathcal{R}_\alpha^\infty \subseteq (X \times \mathbb{N})^2$ the infinite amplification of the orbit equivalence relation $\mathcal{R}_\alpha$ for α, which is given by

$$((x_1, n_1), (x_2, n_2)) \in \mathcal{R}_\alpha^\infty \text{ if and only if } (x_1, x_2) \in \mathcal{R}_\alpha.$$

This is an equivalence relation on $X \times \mathbb{N}$. A measure on $\mathcal{R}_\alpha^\infty$ is given by the identification $\mathcal{R}_\alpha^\infty \cong \mathcal{R}_\alpha \times \mathbb{N} \times \mathbb{N}$. For a measurable subset $Z \subset X \times \mathbb{N}$, we denote by $\mathcal{R}_\alpha^\infty|_Z$ the restriction of the relation $\mathcal{R}_\alpha^\infty \cap (Z \times Z)$ on Z. If the action α is ergodic and the measure of Z is finite, the equivalence relation $\mathcal{R}_\alpha^\infty|_Z$ does not depend on the choice of Z up to isomorphism of standard equivalence relations. We sometimes write $\mathcal{R}_\alpha^{\mu(Z)/\mu(X)} = \mathcal{R}_\alpha^\infty|_Z$, where μ is the measure on X or $X \times \mathbb{N}$.

Definition 3.6. *The actions α and β are said to be* ***stably orbit equivalent*** *if there exists a measure preserving bijective map $\theta : Y \to Z$ onto a measurable subset $Z \subset X \times \mathbb{N}$ such that θ gives an isomorphism between the relations $\mathcal{R}_\alpha^\infty|_Z$ and $\mathcal{R}_\beta$. Namely, the measures of $\{(x_1, x_2) \in \mathcal{R}_\alpha^\infty|_Z; (\theta^{-1}(x_1), \theta^{-1}(x_2)) \notin \mathcal{R}_\beta\}$ and $\{(y_1, y_2) \in \mathcal{R}_\beta; (\theta(y_1), \theta(y_2)) \notin \mathcal{R}_\alpha^\infty\}$ are zero. The constant $[\beta : \alpha]$ defined by $\mu(Z)/\mu(X)$ is called the* ***SOE constant***.

See Furman's paper [Fu2] with terminology *weak orbit equivalence* for the basic notions of measured equivalence relations. As in the case of ME coupling, we have the following.

Lemma 3.7. *There exists an ergodic measure embedding of G into Γ with coupling constant $s = [\Gamma : G]_\Sigma \in (0, \infty]$, if and only if there exist a free ergodic measure preserving G-action on a standard probability space X and a free ergodic measure preserving Γ-action on a standard measure space Y so that they are stably orbit equivalent with $[\beta : \alpha] = s$.*

For the case of ME coupling, we are done in Lemma 3.2 in Furman [Fu2] and Remark 2.14 in Monod–Shalom [MoSh]. We only have to concentrate on the cases $[\Gamma : G]_\Sigma = \infty$ and $[\beta : \alpha] = \infty$.

Proof. Suppose that there exist a free ergodic measure preserving G-action α on X and a free ergodic measure preserving Γ-action β on Y which are stably orbit equivalent with SOE constant ∞. In the setting of the definition of stable orbit equivalence, we identify X with $X \times \{0\} \subset X \times \mathbb{N}$ and Y with $Z \subset X \times \mathbb{N}$ by θ. Since the measure $\mu(Y)$ is infinity and the action α is ergodic, we may assume that $X \subset Y$. The product group $G \times \Gamma$ acts on $\Sigma = \mathcal{R}_\alpha^\infty \cap (X \times Y)$ by

$$(g, \gamma) \cdot (x, y) = (\alpha_g(x), \beta_\gamma(y)), \quad g \in G, \gamma \in \Gamma, (x, y) \in \Sigma.$$

The measurable subset $D_\Gamma = \{((x, 0), (x, 0)) \in \Sigma; x \in X\}$ is a Γ fundamental domain, and $D_G = \{((x, 0), (x, n)) \in \Sigma; x \in X, n \in \mathbb{N}, (x, n) \in Y\}$ is a G fundamental domain for the action. Since $\nu(D_\Gamma) < \infty$, Σ gives a measure embedding for G with Γ. The coupling constant is $\nu(D_G)/\nu(D_\Gamma) = \infty$.

Conversely, suppose that there exists a measure embedding Σ of G with Γ and that the constant $[\Gamma : G]_\Sigma = \infty$. We may assume that the coupling Σ is ergodic. We take a standard probability space (X_1, μ) equipped with a measure preserving, weakly mixing and free G-action. Let Γ act on X_1 trivially. We regard $\Sigma' = \Sigma \times X_1$ as an ergodic measure embedding, on which G and Γ act by the diagonal actions. Thus we get the ergodic measure embedding Σ', for which the natural G-action on $X \cong \Gamma \backslash \Sigma'$ is (essentially) free and ergodic. Replacing Σ with Σ', we may further assume that G-action on $X \cong \Gamma \backslash \Sigma$ is free.

Denote the actions $G \curvearrowright X \cong \Gamma \backslash \Sigma$ and $\Gamma \curvearrowright Y \cong G \backslash \Sigma$ by α and β respectively. Label the elements in G as $\{g_n\}_{n \in \mathbb{N}}$ and define $Y_n = Y \cap g_n^{-1} X$. Define the measurable map $\theta\colon Y \to X \times \mathbb{N}$ by $\theta(y) = (g_n(y), n),\ y \in Y_n$. It is straight forward to show that $\theta\colon Y \to \theta(Y)$ gives stable orbit equivalence between α and β. The SOE constant is $\mu(\theta(Y))/\nu(X) = \sum_n \nu(Y_n)/\nu(X) = \nu(Y)/\nu(X) = \infty$. □

3.3. **Function Valued Measures.** Let (Σ, ν) be a standard measure space equipped with a measure preserving free action of a countable group Γ. Assume that the Γ-action has a fundamental domain X. For a subgroup $\Lambda \subset \Gamma$, there exists a fundamental domain X_Λ for the Λ-action (for instance $X_\Lambda = \bigsqcup_{i\in I} \gamma_i X$, where $\{\gamma_i\}_{i\in I}$ are representatives of the right cosets $\Lambda\backslash\Gamma$).

We denote by Tr the integration of elements in $L^\infty\Sigma$ given by the measure ν. We naturally define the Γ-action on the function space $L^\infty\Sigma$. We define an application Tr_Λ on Λ-invariant positive functions $(L^\infty\Sigma)^\Lambda_+$ by the integration on X_Λ, that is, $\mathrm{Tr}_\Lambda(\phi) = \mathrm{Tr}(\chi(X_\Lambda)\phi) \in [0,\infty]$. This definition does not depend on the choice of X_Λ. For a Λ-invariant measurable set $\Omega \subset \Gamma$, we write $\mathrm{Tr}_\Lambda(\Omega) = \mathrm{Tr}_\Lambda(\chi(\Omega))$.

Consider the natural inclusion $L^\infty X \ni f \to \iota(f) \in (L^\infty\Sigma)^\Lambda$, defined as

$$\iota(f)(\gamma x) = f(x), \quad x \in X, \gamma \in \Gamma.$$

We denote by $\mathfrak{E}^\Lambda_X$ the pull back of the preduals:

$$\iota^* = \mathfrak{E}^\Lambda_X\colon L^1((L^\infty\Sigma)^\Lambda, \mathrm{Tr}_\Lambda) \longrightarrow L^1(X).$$

The space $L^1((L^\infty\Sigma)^\Lambda, \mathrm{Tr}_\Lambda)$ can be identified with the space of the measurable Λ-invariant functions which are integrable on X_Λ.

Let $\widehat{(L^\infty\Sigma)}^\Lambda_+$ and $\widehat{L^\infty X_+}$ be the extended positive cones. The former set consists of the $[0,\infty]$-valued Λ-invariant measurable functions on Σ, and the latter set consists of the $[0,\infty]$-valued measurable functions on X. The completely additive extension $\mathfrak{E}^\Lambda_X$ of ι^* is unique. We call the extension $\mathfrak{E}^\Lambda_X$ the **function valued measure** on $\Lambda\backslash\Sigma$. By choosing the fundamental domain as $X_\Lambda = \bigsqcup_{i\in I}\gamma_i X$, the function valued measure is written by

$$\mathfrak{E}^\Lambda_X(\phi)(x) = \sum_{i\in I} \phi(\gamma_i x), \quad \phi \in \widehat{(L^\infty\Sigma)}^\Lambda_+,$$

because every positive measurable function on X_Λ can be written as a countable sum of integrable functions and the equation holds true for all integrable functions. It turns out that for any Λ-invariant measurable subset $\Omega \subset \Sigma$, the function $\mathfrak{E}^\Lambda_X(\chi(\Omega))$ is a $(\{0,1,\cdots\} \sqcup \{\infty\})$-valued function. For a Λ-invariant measurable set $\Omega \subset \Sigma$, we also write $\mathfrak{E}^\Lambda_X(\Omega) = \mathfrak{E}^\Lambda_X(\chi(\Omega))$.

We get the following basic properties of function valued measures.

Lemma 3.8. *The function valued measure satisfies the following:*

(1) *For $\phi \in \widehat{(L^\infty\Sigma)}^\Lambda_+$, we get*

$$\mathrm{Tr}_\Lambda(\iota(f)\phi) = \int_X f\mathfrak{E}^\Lambda_X(\phi)d\nu, \quad f \in L^\infty X.$$

This condition determines $\mathfrak{E}^\Lambda_X(\phi)$.

(2) *Let θ be a measure preserving transformation on Σ commuting with the Γ-action. Denote by α a transformation on $X \cong \Sigma/\Gamma$ given by θ. We get*

$$\alpha(\mathfrak{E}^\Lambda_X(\phi)) = \mathfrak{E}^\Lambda_X(\theta(\phi)), \quad \phi \in \widehat{(L^\infty\Sigma)}^\Lambda_+.$$

(3) *For a measurable subset $W \subset X$, we get*

$$\chi(W)\mathfrak{E}_X^{\Lambda}(\phi) = \mathfrak{E}_X^{\Lambda}(\chi(\Gamma W)\phi). \quad \phi \in (\widehat{L^\infty\Sigma})_+^{\Lambda}.$$

Proof. When $\phi \in (\widehat{L^\infty\Sigma})_+^{\Lambda}$ is integrable on X_Λ, the first condition is the definition of $\mathfrak{E}_X^{\Lambda}(\phi)$. By the complete additivity of $\mathfrak{E}_X^{\Lambda}$, the first assertion holds for a general ϕ.

For the second assertion, we note that $\theta(X_\Lambda)$ is also a fundamental domain for the Λ-action on Σ. For $\phi \in (\widehat{L^\infty\Sigma})_+^{\Lambda}$ and $f \in L^\infty X$, we have

$$\mathrm{Tr}_\Lambda(\iota(f)\theta(\phi)) = \int_{\theta(X_\Lambda)} \iota(f)\theta(\phi)d\nu = \int_{X_\Lambda} \theta^{-1}(\iota(f))\phi d\nu = \mathrm{Tr}_\Lambda(\iota(\alpha^{-1}(f))\phi).$$

Since α is measure preserving, we get

$$\mathrm{Tr}_\Lambda(\iota(\alpha^{-1}(f))\phi) = \int_X \alpha^{-1}(f)\mathfrak{E}_X^{\Lambda}(\phi)d\nu = \int_X f\alpha(\mathfrak{E}_X^{\Lambda}(\phi))d\nu.$$

By the first assertion, we conclude $\alpha(\mathfrak{E}_X^{\Lambda}(\phi)) = \mathfrak{E}_X^{\Lambda}(\theta(\phi))$.

For a measurable subset $W \subset X$, we get

$$\mathrm{Tr}_\Lambda(\iota(f)\chi(\Gamma W)\phi) = \mathrm{Tr}_\Lambda(\iota(f\chi(W))\phi) = \int_X f\chi(W)\mathfrak{E}_X^{\Lambda}(\phi)d\nu$$

By the first assertion, we get the third assertion. □

Lemma 3.9. *Let $H \subset G$ and $\Lambda \subset \Gamma$ be subgroups. Let Σ be a measure embedding of G into Γ. Choose a Γ fundamental domain $X \subset \Sigma$. Then $H \preceq_\Sigma \Lambda$ if and only if there exists an $(H \times \Lambda)$-invariant measurable subset $\Omega \subset \Sigma$ so that the essential range of $\mathfrak{E}_X^{\Lambda}(\Omega)$ satisfies* $\mathrm{range}(\mathfrak{E}_X^{\Lambda}(\Omega)) \not\subset \{0, \infty\}$.

Proof. If there exists a partial embedding Ω for $H \preceq_\Sigma \Lambda$, then the function $\mathfrak{E}_X^{\Lambda}(\Omega)$ is non-zero, non-negative, integrable and integer-valued. Hence the essential range of the function intersects with positive integers.

Suppose that there exists an $(H \times \Lambda)$-invariant measurable subset Ω with the above property. Denote $F = \mathfrak{E}_X^{\Lambda}(\Omega)$. Then there exists a positive integer n such that the preimage $F^{-1}([1,n]) = W \subset X$ is non-null. Since the function $\chi(\Omega)$ is H-invariant, the function F on X is H-invariant under the action $\alpha: H \curvearrowright X \cong \Gamma\backslash\Sigma$. Therefore the measurable subset $W \subset X$ is H-invariant under the action α, and the measurable subset $\Omega' = \Omega \cap \Gamma W$ is $(H \times \Lambda)$-invariant. By Lemma 3.8, we get

$$0 < \mathrm{Tr}_\Lambda(\Omega') = \int_X \mathfrak{E}_X^{\Lambda}(\Omega \cap \Gamma W)d\nu = \int_X F\chi(W)d\nu < \infty.$$

For a Λ fundamental domain X_Λ for Σ, the measurable set $\Omega' \cap X_\Lambda$ is a Λ fundamental domain for Ω' and has finite measure. It follows that Ω' gives a partial embedding $H \preceq_\Sigma \Lambda$. □

4. Definition and Basic Properties of Bi-exactness

We recall the definition and basic properties of bi-exactness. This notion was introduced in the 15th chapter of the book [BrOz]. This section entirely relies on that book.

Definition 4.1. *A subset Γ_1 of Γ is said to be* ***small relative to*** $\mathcal{G}$ *if there exist* $s_1, t_1, \cdots, s_n, t_n \in \Gamma$ *and* $\Lambda_1, \cdots, \Lambda_n \in \mathcal{G}$ *satisfying* $\Gamma_1 \subset \bigcup_{i=1}^n s_i \Lambda_i t_i$.

Let $c_0(\Gamma; \mathcal{G})$ be a C^*-subalgebra of $\ell_\infty \Gamma$ generated by functions whose supports are small relative to $\mathcal{G}$.

Definition 4.2. *The group Γ is said to be* ***bi-exact relative to*** $\mathcal{G}$ *if there exists a map* $\mu : \Gamma \to \mathrm{Prob}(\Gamma) \subset \ell_1 \Gamma$, *with the property that for any* ϵ *and* $s, t \in \Gamma$, *there exists a small subset* Γ_1 *relative to* $\mathcal{G}$ *which satisfies*

$$\|\mu(sxt) - s\mu(x)\|_1 < \epsilon, \quad x \in \Gamma \cap \Gamma_1^c.$$

The following is a useful characterization of bi-exactness. The notion of topological amenability ([AD]) played a vital role in the proof . In the second condition, l and r stand for the left-times-right translation action of $\Gamma \times \Gamma$ on $\ell_\infty \Gamma / c_0(\Gamma; \mathcal{G})$, that is, $[l_\gamma r_{\gamma'}(\phi + c_0(\Gamma; \mathcal{G}))](\lambda) = \phi(\gamma^{-1}\lambda\gamma') + c_0(\Gamma; \mathcal{G})$, $\phi \in \ell_\infty \Gamma$.

Proposition 4.3. *The following conditions are equivalent:*

- *The group Γ is bi-exact relative to $\mathcal{G}$.*
- *Let E be a C^*-algebra of which center contains $\ell_\infty \Gamma / c_0(\Gamma; \mathcal{G})$. Suppose that $\Gamma \times \Gamma$ act on on E by $\mathfrak{A}$. If the action $\mathfrak{A}$ satisfies*

  $$\mathfrak{A}(\gamma, \gamma')(\phi) = l_\gamma r_{\gamma'}(\phi), \quad (\gamma, \gamma') \in \Gamma \times \Gamma, \phi \in \ell_\infty \Gamma / c_0(\Gamma; \mathcal{G}),$$

  then the full crossed product $E \rtimes_{\mathrm{full}, \mathfrak{A}} (\Gamma \times \Gamma)$ is naturally isomorphic to the reduced crossed product $E \rtimes_{\mathrm{red}, \mathfrak{A}} (\Gamma \times \Gamma)$.
- *The left-times-right translation action of $\Gamma \times \Gamma$ on the Gelfand spectrum of the commutative C^*-algebra $\ell_\infty \Gamma / c_0(\Gamma; \mathcal{G})$ is amenable.*

The equivalence of the first condition and the third one is proved in Proposition 15.2.3 in [BrOz]. By using the characterization of topological amenability proved in Proposition 4.8 in [AD], we get the equivalence of the second condition and the third one.

Remark 4.4. The class $\mathcal{S}$ defined in Ozawa's paper [Oz3, Section 4] is the same as the set of countable groups Γ which are bi-exact relative to $\{\{1\}\}$. Gromov's word hyperbolic groups are in $\mathcal{S}$. Discrete subgroups of connected simple Lie groups of rank one are in $\mathcal{S}$ (by using [HiGu], [Sk]). The class of amenable countable groups is a subclass of $\mathcal{S}$. A wreath product $A \wr G$ is in $\mathcal{S}$ if $G \in \mathcal{S}$ and A is amenable. The group $\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ is in $\mathcal{S}$ (by Ozawa [Oz3, Oz5]).

The notion of bi-exactness well behaves under being taken direct product, wreath product and free product with amenable amalgamation.

Lemma 4.5 (Lemma 15.3.3, Lemma 15.3.5 in [BrOz]). *Let Γ_i $(1 \le i \le n)$ be countable groups and let Γ_0 be an amenable group. We denote by Γ the direct product $\Gamma_0 \times \prod_{i=1}^n \Gamma_i$. Let $\mathcal{G}_i$ be a non-empty family of subgroups of Γ_i $(1 \le i \le n)$ and let $\mathcal{G}$ be the family of subgroups*

$$\mathcal{G} = \bigcup_{i=1}^n \left\{ \Gamma_0 \times \Lambda \times \prod_{j \ne i} \Gamma_j \;\middle|\; \Lambda \in \mathcal{G}_i \right\}.$$

If Γ_i is bi-exact relative to $\mathcal{G}_i$, then Γ is bi-exact relative to $\mathcal{G}$.

Lemma 4.6 (Lemma 15.3.6 in [BrOz]). *If A is amenable and G is exact, then the wreath product $A \wr G$ is bi-exact relative to $\{G\}$.*

Lemma 4.7 (Lemma 15.3.12 in [BrOz]). *Let Γ_1, Γ_2 be countable groups and A be a common subgroup of Γ_1, Γ_2. If Γ_1, Γ_2 are exact and A is amenable, then the amalgamated free product $\Gamma_1 *_A \Gamma_2$ is bi-exact relative to $\{\Gamma_1, \Gamma_2\}$.*

5. Location of Subgroups

The goal of this section is Theorem 5.3, which is a consequence of the following proposition.

Proposition 5.1. *Let H be a subgroup of G and Γ be bi-exact relative to $\mathcal{G}$. Let β be a free measure preserving action of Γ on a standard measure space (Y, μ) and let α be a free measure preserving action of G on a measurable subset $X \subset Y$ with measure 1. Suppose that $\alpha(G)(x) \subset \beta(\Gamma)(x)$, for a.e. $x \in X$. We regard the infinite measure space $\Sigma = \mathcal{R}_\beta \cap (X \times Y)$ as a measure embedding of G into Γ, on which G acts on the first entry and Γ acts on the second entry. If for any $\Lambda \in \mathcal{G}$, there exists no partial embedding of H into Λ in Σ, then the centralizer $Z_G(H)$ is amenable.*

Before starting the proof of Proposition 5.1, we fix some notations and prove a C^*-algebraical continuity property for Γ-action on Y. The action β (resp. α) gives a group action of Γ (resp. G) on $L^\infty(Y)$ (resp. $L^\infty(X)$). We use the same notation β (resp. α) for this action. Let $p \in L^\infty(Y)$ be the characteristic function of X. The algebra $L^\infty(Y)$ and the group Γ are represented on $L^2(\mathcal{R}_\beta, \nu)$ as

$$\begin{aligned}(f\xi)(x,y) &= f(x)\xi(x,y), \quad f \in L^\infty(Y),\\ (u_\gamma \xi)(x,y) &= \xi(\beta_{\gamma^{-1}}(x), y), \quad \gamma \in \Gamma,\ \xi \in L^2(\mathcal{R}_\beta),\ (x,y) \in \mathcal{R}_\beta.\end{aligned}$$

We denote by B the C^*-algebra generated by the images, which is the reduced crossed product algebra $B = L^\infty(Y) \rtimes_{\mathrm{red}} \Gamma$. Its weak closure is the group measure space construction $\mathcal{M} = L^\infty(Y) \rtimes \Gamma$ (Murray and von Neumann [MvN]). We denote by tr the canonical faithful normal semi-finite trace on $\mathcal{M}$. The unitary involution J of $(\mathcal{M}, \mathrm{tr})$ is written as

$$(J\xi)(x,y) = \overline{\xi(y,x)}, \quad \xi \in L^2(\mathcal{R}_\beta),\ (x,y) \in \mathcal{R}_\beta.$$

The group G is represented on $pL^2(\mathcal{R}_\beta) = L^2(\mathcal{R}_\beta \cap (X \times Y))$ by

$$(v_g \xi)(x,y) = \xi(\alpha_{g^{-1}}(x), y), \quad g \in G,\ \xi \in pL^2(\mathcal{R}_\beta).$$

We denote by $C^*_\lambda(G)$ the C^*-algebra generated by these operators. The algebra is isomorphic to the reduced group C^*-algebra of G. The Hilbert space $L^2(\mathcal{R}_\alpha, \nu)$ can be identified with a closed subspace of $pL^2(\mathcal{R}_\beta)$. The algebra $C^*_\lambda(G)$ is also represented on $L^2(\mathcal{R}_\alpha)$ faithfully. We denote by P the orthogonal projection from $L^2(\mathcal{R}_\beta)$ onto $L^2(\mathcal{R}_\alpha)$. We note that the algebra pBp does not contain $C^*_\lambda(G)$ in general, although there exists an inclusion between their weak closures.

Let e_Δ be the projection from $L^2(\mathcal{R}_\beta)$ onto the set of L^2-functions supported on the diagonal subset of $\mathcal{R}_\beta$. This is the Jones projection for $L^\infty(Y) \subset \mathcal{M}$. Consider $L^\infty(\mathcal{R}_\beta) \subset \mathcal{B}(L^2(\mathcal{R}_\beta))$ by multiplications. For $\gamma \in \Gamma$ and a subset $\Gamma_0 \subset \Gamma$, we define the projections $e(\gamma), e(\Gamma_0)$ by

$$e(\gamma) = Ju_\gamma J e_\Delta J u_\gamma^* J, \quad e(\Gamma_0) = \sum_{\gamma \in \Gamma_0} e(\gamma) \in L^\infty(\mathcal{R}_\beta).$$

For $g \in G$ and a subset $G_0 \subset G$, we define the projections $f(g), f(G_0)$ by

$$f(g) = v_g e_\Delta v_g^* = v_g P e_\Delta v_g^*, \quad f(G_0) = \sum_{g \in G_0} f(g) \in L^\infty(\mathcal{R}_\beta \cap (X \times Y)).$$

Let $K \subset \mathcal{B}(L^2(\mathcal{R}_\beta))$ be the hereditary subalgebra of $\mathcal{B}(L^2(\mathcal{R}_\beta))$ with approximate units $\{e(\Gamma_0) \mid \Gamma_0 \text{ is small relative to } \mathcal{G}\}$, that is,

$$K = \overline{\bigcup_{\Gamma_0} e(\Gamma_0)\mathcal{B}(L^2(\mathcal{R}_\beta))e(\Gamma_0)}^{\|\cdot\|}.$$

The algebras B and JBJ are in the multiplier of K, so is $D = C^*(B, JBJ)$.

The algebra B satisfies the following continuity property. The proof is conceptually identical to Proposition 4.2 of Ozawa's paper [Oz3].

Proposition 5.2. *The following map is continuous with respect to the minimal tensor norm:*

$$\Psi\colon B \otimes_{\mathbb{C}} JBJ \ni \sum_{i=1}^k b_i \otimes Jc_iJ \to \sum_{i=1}^k b_i Jc_i J + K \in (D + K)/K.$$

In the case of $\mu(Y) < \infty$, if Ψ were continuous without taken quotient by K, this condition would deduce amenability on the group Γ. The above Proposition can be regarded as a weakened amenability property for the Γ-action. We prove the above by using an assist of $\ell_\infty\Gamma/c_0(\Gamma;\mathcal{G})$. A property of topological amenability proved by C. Anantharaman-Delaroche [AD] plays a vital role. In the proof, "$\otimes$" stands for the minimal tensor of C^*-algebras.

Proof. Define a representation $m.$ of $\ell_\infty\Gamma$ on $L^2(\mathcal{R}_\beta)$ by the multiplication

$$[m_\phi(\xi)](\gamma x, x) = \phi(\gamma)\xi(\gamma x, x), \quad \xi \in L^2\mathcal{R}_\beta, \gamma \in \Gamma, \phi \in \ell_\infty\Gamma.$$

Let $\widetilde{D}$ be the C^*-algebra generated by D and the image of m. It is easy to see that $\widetilde{D}$ is in the multiplier of K. The preimage $m^{-1}(m(\ell_\infty\Gamma) \cap K)$ is $c_0(\Gamma;\mathcal{G})$. The homomorphism m also gives an injective homomorphism of $\ell_\infty\Gamma/c_0(\Gamma;\mathcal{G})$ into $(\widetilde{D} + K)/K$.

Let E be the minimal tensor product $E = L^\infty Y \otimes JL^\infty YJ \otimes \ell_\infty\Gamma/c_0(\Gamma,\mathcal{G})$. The product group $\Gamma \times \Gamma$ acts on E by

$$\begin{aligned}&\mathfrak{A}(g,h)(f_1 \otimes Jf_2J \otimes (\phi + c_0(\Gamma;\mathcal{G})))\\ =\ &\beta_g(f_1) \otimes J\beta_h(f_2)J \otimes (l_g r_h(\phi) + c_0(\Gamma;\mathcal{G})).\end{aligned}$$

Let $\widetilde{E}$ be the reduced crossed product $E \rtimes_{\mathrm{red}} (\Gamma \times \Gamma)$.

We claim that there exists a $*$-homomorphism $\Psi\colon \widetilde{E} \to (\widetilde{D} + K)/K$ satisfying

$$\Psi(f_1 \otimes Jf_2J \otimes (\phi + c_0(\Gamma;\mathcal{G}))) = f_1 Jf_2 J m_\phi + K,$$
$$\Psi(g,h) = u_g J u_h J + K, \quad f_1, f_2 \in L^\infty Y,\ \phi \in \ell_\infty\Gamma,\ (g,h) \in \Gamma \times \Gamma.$$

We consider the $*$-homomorphism from $L^\infty Y \otimes_{\mathbb{C}} JL^\infty YJ \otimes_{\mathbb{C}} \ell_\infty\Gamma/c_0(\Gamma;\mathcal{G})$ to $(\widetilde{D}+K)/K$ given by the first equation. Since $L^\infty Y, JL^\infty YJ$ are nuclear by Takesaki's theorem [Tak], this homomorphism extends to the minimal tensor product E. The homomorphism $\Gamma\times\Gamma \ni (g,h) \to u_gJu_hJ + K \in (\widetilde{D}+K)/K$ gives the covariant system of the action $\mathfrak{A}$, that is,

$$\begin{aligned} &(u_gJu_hJ+K)\Psi(f_1\otimes Jf_2J\otimes(\phi+c_0(\Gamma;\mathcal{G})))(u_gJu_hJ+K)^* \\ =\ & u_gf_1u_g^*Ju_hf_2u_h^*Jm(l_gr_h(\phi))+K \\ =\ & \Psi(\beta_g(f_1)\otimes J\beta_h(f_2)J\otimes(l_gr_h(\phi)+c_0(\Gamma;\mathcal{G}))). \end{aligned}$$

We get a $*$-homomorphism Ψ from the full crossed product $E\rtimes_{\text{full}}(\Gamma\times\Gamma)$ to $(\widetilde{D}+K)/K$.

The subalgebra $\mathbb{C}\otimes\mathbb{C}\otimes\ell_\infty\Gamma/c_0(\Gamma,\mathcal{G})$ is in the center of E and globally invariant under the action. Since Γ is bi-exact relative to $\mathcal{G}$, the $(\Gamma\times\Gamma)$-action on E satisfies the second condition of Proposition 4.3. The full crossed product algebra $E\rtimes_{\text{full}}(\Gamma\times\Gamma)$ coincides with the reduced crossed product $\widetilde{E}$, by [AD]. The restriction of Ψ on $B\otimes JBJ \cong (L^\infty Y\otimes JL^\infty YJ)\rtimes_{\text{red}}(\Gamma\times\Gamma)\subset\widetilde{E}$ gives Ψ in Proposition 5.2. □

We proceed to prove Proposition 5.1. The idea of the proof is that when the H-action on the first entry of $\mathcal{R}_\beta\cap(X\times Y)$ flees all projections $pe(\Gamma_0)$ for small sets Γ_0, Proposition 5.2 deduces a continuity property of the reduced group C^*-algebra $C^*_\lambda(Z_G(H))$.

Proof. We may assume that the family $\mathcal{G}$ is invariant under conjugation. Indeed, by the definition, Γ is bi-exact relative to $\mathcal{G}$ if and only if Γ is bi-exact relative to $\widetilde{\mathcal{G}} = \bigcup_{\gamma\in\Gamma}\gamma\mathcal{G}\gamma^{-1}$. If there exists a partial embedding $\Omega\subset\mathcal{R}_\beta\cap(X\times Y)$ of H into $\gamma\Lambda\gamma^{-1}$ for some $\Lambda\in\mathcal{G}$, then $\gamma^{-1}\Omega$ gives a partial embedding of H into Λ. Assume that $\mathcal{G}$ is conjugation invariant.

Denote $G_1 = Z_G(H)$. The unitaries $\{v_g \mid g\in G_1\}$ give a faithful representation of $C^*_\lambda(G_1)$ on $pJpJL^2(\mathcal{R}_\beta)$. We fix this representation. We denote $C^*_\rho(G_1) = JC^*_\lambda(G_1)J$. To show the amenability of G_1, it suffices to show that the natural homomorphism

$$\Phi\colon C^*_\lambda(G_1)\otimes_{\mathbb{C}} C^*_\rho(G_1) \longrightarrow \mathcal{B}(pL^2\mathcal{M}p) = \mathcal{B}(L^2(\mathcal{R}_\beta)\cap(X\times X)),$$

is continuous with respect to the minimal tensor norm. (See Section 2.6 of [BrOz], for example). We take an arbitrary positive number $\epsilon>0$, a finite subset $\mathcal{F}\subset G_1$ and $x\in C^*_\lambda(G_1)\otimes_{\mathbb{C}} C^*_\rho(G_1)$ of the following form:

$$x = \sum_{s,t\in\mathcal{F}} c(s,t)v_s\otimes Jv_tJ, \quad c(s,t)\in\mathbb{C}.$$

Then $\Phi(x)$ is given by $\Phi(x) = \sum_{s,t\in\mathcal{F}} c(s,t)v_sJv_tJ$.

Since the norm of $\Phi(x)$ is almost attained by some vector, there exists a finite subset $\Gamma_0\subseteq\Gamma$ satisfying

$$\|\Phi(x)e(\Gamma_0)\| > \|\Phi(x)\| - \epsilon. \tag{1}$$

We claim that there exists $\delta>0$ with the property: For any projection f in $L^\infty X$ with $\operatorname{tr}(p-f)\leq\delta$,

$$\|\Phi(x)e(\Gamma_0)fJfJ\| > \|\Phi(x)e(\Gamma_0)\| - \epsilon. \tag{2}$$

Otherwise, there would exist a sequence of projections $\{f_k\} \subseteq L^\infty X$ with $\mathrm{tr}(p - f_k) < 2^{-k}$ and $\|\Phi(x)e(\Gamma_0)f_k J f_k J\| \leq \|\Phi(x)e(\Gamma_0)\| - \epsilon$. Denote $p_k = f_k \wedge f_{k+1} \wedge \ldots$. Then we get $\|\Phi(x)e(\Gamma_0)p_k J p_k J\| \leq \|\Phi(x)e(\Gamma_0)\| - \epsilon$. Since $p_k J p_k J$ is an increasing sequence converging to $pJpJ$, this is a contradiction.

The unitary v_s can be written as a Fourier expansion $v_s = \sum_\gamma u_\gamma p(s,\gamma)$, by some projections $\{p(s,\gamma)\} \subset L^\infty X$ with $\sum_\gamma p(s,\gamma) = p$. There exists an increasing sequence of projections $\{q_n(s)\} \subset L^\infty X$ with $\lim_n \mathrm{tr}(q_n(s)) = \mathrm{tr}(p)$ and $v_s q_n(s) \in B = L^\infty Y \rtimes_{\mathrm{red}} \Gamma$. Since $\mathcal{F}$ is a finite set, there exists a projection $q_1 \in L^\infty X$ satisfying $\mathrm{tr}(p - q_1) \leq \delta/3$ and $v_s q_1 \in B$ for all $s \in \mathcal{F}$.

The operator $x(q_1 \otimes Jq_1J) = \sum c(s,t) v_s q_1 \otimes J v_t q_1 J$ is in the domain of Ψ in Proposition 5.2 and its image is

$$\Psi(x(q_1 \otimes Jq_1J)) = \sum_{s,t \in G_1} c(s,t) v_s q_1 J v_t q_1 J + K = \Phi(x) q_1 J q_1 J + K.$$

Since Ψ is continuous (or equivalently contractive) and $\{e(\Gamma_1) \mid \Gamma_1 \subset \Gamma \text{ small relative to } \mathcal{G}\}$ is a net of approximate units for K, we get

$$\begin{aligned} \|x\|_{\min} &\geq \|\Psi(x(q_1 \otimes Jq_1J))\| = \|\Phi(x)q_1Jq_1J + K\|_{(D+K)/K} \\ &= \inf\{\|\Phi(x)q_1Jq_1J(1 - e(\Gamma_1))\| \mid \Gamma_1 \subset \Gamma \text{ small relative to } \mathcal{G}\}. \end{aligned}$$

We get a finite subset $\Gamma_1 \subset \Gamma$ with

$$\|x\|_{\min} + \epsilon > \|\Phi(x)q_1Jq_1J(1 - e(\Gamma_1))\|. \tag{3}$$

We may assume that $\Gamma_1 \subset \bigcup_{i=1}^n \Lambda_i\gamma_i$, for some $\Lambda_i \in \mathcal{G}$, since $\mathcal{G}$ is conjugation invariant. To get the continuity of Φ, we will show an inequality between the right hand side of (3) and the left hand side of (2) for an appropriate f.

Write $\Sigma = \mathcal{R}_\beta \cap (X \times Y)$ and regard Σ as a measure embedding of G into Γ. We make use of notations in Subsection 3.3. The projection pe_Δ corresponds to a Γ fundamental domain of Σ. We identify X with the fundamental domain. Then the projections $pe(\Lambda_i\Gamma_0), pe(\Lambda_i\gamma_i)$ are written as

$$\begin{aligned} pe(\Lambda_i\Gamma_0) &= \sum_{\lambda \in \Lambda_i, \gamma \in \Gamma_0} J u_\lambda u_\gamma J p e_\Delta J u_\gamma^* u_\lambda^* J = \chi(\Lambda_i\Gamma_0 X) \in L^\infty\Sigma \\ pe(\Lambda_i\gamma_i) &= \sum_{\lambda \in \Lambda_i} J u_\lambda u_{\gamma_i} J p e_\Delta J u_{\gamma_i}^* u_\lambda^* J = \chi(\Lambda_i\gamma_i X) \in L^\infty\Sigma. \end{aligned}$$

They are elements in $(L^\infty\Sigma)^{\Lambda_i}$ and their values of $\mathrm{Tr}_i = \mathrm{Tr}_{\Lambda_i}$ are finite. Let e_0, e_1 be the projections in $\widetilde{\mathcal{A}} = (L^\infty\Sigma)^{\Lambda_1} \oplus \cdots \oplus (L^\infty\Sigma)^{\Lambda_n}$ defined by

$$\begin{aligned} e_0 &= \chi(\Lambda_1\Gamma_0X) \oplus \chi(\Lambda_2\Gamma_0X) \oplus \cdots \oplus \chi(\Lambda_n\Gamma_0X), \\ e_1 &= \chi(\Lambda_1\gamma_1X) \oplus \chi(\Lambda_2\gamma_2X) \oplus \cdots \oplus \chi(\Lambda_n\gamma_nX). \end{aligned}$$

Let Tr be the trace on $\widetilde{\mathcal{A}}$ given by the summation $\mathrm{Tr} = \mathrm{Tr}_1 + \mathrm{Tr}_2 + \cdots + \mathrm{Tr}_n$. The values $\mathrm{Tr}(e_0)$ and $\mathrm{Tr}(e_1)$ are finite.

Let $\mathcal{C} \subseteq \widetilde{\mathcal{A}} \cap L^2(\widetilde{\mathcal{A}}, \mathrm{Tr})$ be the set of convex combinations

$$\mathrm{conv}\{h(e_1) = \chi(h\Lambda_1\gamma_1X) \oplus \chi(h\Lambda_2\gamma_2X) \oplus \cdots \oplus \chi(h\Lambda_n\gamma_nX) \mid h \in H\}.$$

We take the unique element $x = x_1 \oplus x_2 \oplus \cdots \oplus x_n$ with the smallest 2-norm in 2-norm closure $\overline{\mathcal{C}}$. Since the set $\overline{\mathcal{C}}$ is globally fixed under the action of H, x is fixed under the action of H. Since x is a L^2-limit of positive functions, x is positive. For $t>0$, its preimage $\Omega_t = \bigsqcup_{i=1}^n \Omega_{i,t} \subset \Sigma \times \{1,2,\cdots,n\}$ of $[t,\infty)$ has a finite value of Tr. Since i-th entry of every element $y \in \mathcal{C}$ is Λ_i-invariant, so is x. The i-th measurable subset $\Omega_{i,t} \subset \Sigma$ is H-invariant and Λ_i-invariant, and the measure of its Λ_i fundamental domain is finite. The assumption of Proposition 5.1 tells that Ω_i is a null set. This means that $e_{[t,\infty)} = 0$. We get $x = 0 \in \overline{\mathcal{C}}$. Since the elements $\{k^{-1}\sum_{i=1}^k h_i(e_1); h_i \in H\}$ is 2-norm dense in $\overline{\mathcal{C}}$, there exist $h_1, h_2, \ldots, h_k \in G$ satisfying

$$\mathrm{Tr}\left(\frac{1}{k}\sum_{i=1}^k h_i(e_1)e_0\right) \leq \delta/3.$$

We choose $h \in \{h_1, h_2, \ldots, h_k\}$ satisfying $\mathrm{Tr}(h(e_1)e_0) \leq \delta/3$.

Let $\mathfrak{E}_X^{(i)}$ be the function valued measure from $\widehat{(L^\infty\Sigma)_+^{\Lambda_i}}$ to $\widehat{L^\infty X_+}$ defined in Subsection 3.3. Each measurable function $\mathfrak{E}_X^{(i)}(h\Lambda_i\gamma_i X \cap \Lambda_i\Gamma_0 X)$ is integer valued on X. The function $F = \sum_{i=1}^n \mathfrak{E}_X^{(i)}(h\Lambda_i\gamma_i X \cap \Lambda_i\Gamma_0 X)$ is also integer valued. Let $p - q_2 \in L^\infty X$ be the support of F. It follows that

$$\mathrm{tr}(p-q_2) \leq \int_X F d\mu = \mathrm{Tr}(h(e_1)e_0) \leq \delta/3.$$

Since $q_2\mathfrak{E}_X^{(i)}(h\Lambda_i\gamma_i X \cap \Lambda_i\Gamma_0 X) = 0$, we also get

$$\chi(h\Lambda_i\gamma_i X)\chi(\Lambda_i\Gamma_0 X)q_2 = v_h e(\Lambda_i\gamma_i)v_h^* e(\Lambda_i\Gamma_0)q_2 = 0.$$

Since $e(\Lambda_i\Gamma_0)q_2 = q_2 e(\Lambda_i\Gamma_0)$, it follows that

$$\begin{aligned} v_h e(\Lambda_i\gamma_i)v_h^* &\perp q_2 e(\Lambda_i\Gamma_0), \\ v_h e(\Gamma_1)v_h^* \leq \bigvee_{i=1}^n v_h e(\Lambda_i\gamma_i)v_h^* &\perp \bigwedge_{i=1}^n q_2 e(\Lambda_i\Gamma_0) \geq q_2 e(\Gamma_0), \\ v_h(1-e(\Gamma_1))v_h^* &\geq q_2 e(\Gamma_0). \end{aligned}$$

Since $[v_s, v_h] = 0$ for $s \in G_1$, letting $f = \alpha_h(q_1)q_1q_2$,

$$\begin{aligned} \|\Phi(x)q_1Jq_1J(1-e(\Gamma_1))\| &= \|v_h\Phi(x)q_1Jq_1J(1-e(\Gamma_1))v_h^*\| \\ &= \|\Phi(x)\alpha_h(q_1)Jq_1Jv_h(1-e(\Gamma_1))v_h^*\| \\ &\geq \|\Phi(x)\alpha_h(q_1)Jq_1Jq_2e(\Gamma_0)\| \\ &\geq \|\Phi(x)e(\Gamma_0)fJfJ\|. \end{aligned}$$

Since $\mathrm{tr}(p-f) \leq \mathrm{tr}(p-\alpha_h(q_1)) + \mathrm{tr}(p-q_1) + \mathrm{tr}(p-q_2) \leq \delta$, we can use the equation (2). Combining the above inequality, (1), (2) and (3), we get

$$\|x\|_{\min} + 3\epsilon > \|\Phi(x)\|.$$

Since the positive number ϵ is arbitrary, we get the desired continuity of Φ and Proposition 5.1. □

The following is a key result in this paper, which deduces three types of results on direct product groups, wreath product groups and amalgamated free products.

Theorem 5.3. *Let Γ be a countable group which is bi-exact relative to $\mathcal{G}$ and let $H \subset G$ be an inclusion of countable groups. Suppose that there exists an ergodic* *measure embedding Σ of G into Γ and that $\Sigma_H \subset \Sigma$ is an $(H\times\Gamma)$-invariant non-null measurable subset.*

If the centralizer $Z_G(H)$ of H is non-amenable, then there exists a partial embedding Ω of H into Λ satisfying $\Omega \subset \Sigma_H$. In particular, if $G \preceq_{\mathrm{ME}} \Gamma$ and $Z_G(H)$ is non-amenable, then $H \preceq_{\mathrm{ME}} \Lambda$ for some $\Lambda \in \mathcal{G}$.

Proof. Let Σ be an arbitrary ergodic measure embedding of G into Γ. We denote by $\widetilde{G}$ the subgroup of G generated by H and $Z_G(H)$. Let $\Sigma_H \subset \Sigma$ be a non-null measurable subset invariant under $H \times \Gamma$. To show that there exists a partial embedding of $H \preceq_\Sigma \Lambda \in \mathcal{G}$ in Σ_H, we only have to find a partial embedding Ω in $\Sigma_1 = \bigcup\{g\Sigma_H \mid g \in \widetilde{G}\}$. Suppose that $Z_G(H)$ is non-amenable.

We first consider the case of $[\Gamma : G]_\Sigma \geq 1$. We take a standard probability space (X', μ) which is equipped with a weakly mixing free measure preserving G-action. Let Γ act on X' trivially. We regard $\Sigma^{\mathrm{free}} = \Sigma \times X'$ as a measure embedding, on which G and Γ act by diagonal actions respectively. Since the G-action on the set $\Gamma\backslash\Sigma^{\mathrm{free}} \cong (\Gamma\backslash\Sigma) \times X'$ is free and ergodic, Σ^{free} is an ergodic measure embedding coming form stable orbit equivalence. The coupling constant is $[\Gamma : G]_{\Sigma^{\mathrm{free}}} = [\Gamma : G]_\Sigma \geq 1$. There exist a Γ-action β on a standard measure space Y, a measurable subset $X \subset Y$ and a G-action α on a standard probability space X satisfying $\Sigma^{\mathrm{free}} \cong \mathcal{R}_\beta \cap (X \times Y)$. The measurable subset $\Sigma_1^{\mathrm{free}} = \Sigma_1 \times X' \subset \Sigma^{\mathrm{free}}$ is a measure embedding of $\widetilde{G}$ into Γ. Since Σ_1^{free} is Γ-invariant, $\Sigma_1^{\mathrm{free}} = \mathcal{R}_\beta \cap (X_1 \times Y)$ for some $\widetilde{G}$-invariant measurable subset $X_1 \subset X$. We apply the contrapositive of Proposition 5.1 for $\alpha|_{\widetilde{G}}\colon \widetilde{G} \curvearrowright X_1$ and $\beta\colon \Gamma \curvearrowright Y$. We get some $\Lambda \in \mathcal{G}$ and an $(H \times \Lambda)$-invariant measurable subset $\Omega_1^{\mathrm{free}} \subset \Sigma_1^{\mathrm{free}}$ so that the measure of a Λ fundamental domain of Ω_1^{free} is finite.

We define the measurable function ϕ on Σ_1 by

$$\phi(s) = \mu(\{x \in X' \mid (s, x) \in \Sigma_1 \times X' = \Omega_1^{\mathrm{free}}\}),$$

which is defined almost everywhere on $s \in \Sigma_1$. The function ϕ is invariant under the H-action and Λ-action on Σ_1 outside a null set. Take a fundamental domain $D_1 \subset \Sigma_1$ for the Λ-action on Σ_1. Since $\Omega_1^{\mathrm{free}} \cap (D_1 \times X')$ is the Λ-fundamental domain of Ω_1^{free} and has finite measure, the function $\phi|_{D_1}$ is integrable, by Fubini's Theorem. Any non-trivial level set of ϕ gives a partial embedding of H into Λ in Σ_1.

We consider the case of $[\Gamma : G]_\Sigma < 1$. We take an integer n with $n[\Gamma : G]_\Sigma \geq 1$. We define $\widetilde{\Gamma} = \Gamma \times \mathbb{Z}/n\mathbb{Z}$ and $\widetilde{\Sigma} = \Sigma \times \mathbb{Z}/n\mathbb{Z}$. Let $\widetilde{\Gamma}$ act on $\widetilde{\Sigma}$ by the product action and G act on $\mathbb{Z}/n\mathbb{Z}$ trivially. We note that $\widetilde{\Gamma}$ is bi-exact relative to $\mathcal{G} \times \{1\}$. Since $[\widetilde{\Gamma} : G]_{\widetilde{\Sigma}} = n[\Gamma : G]_\Sigma \geq 1$, by the above argument there exist $\Lambda \in \mathcal{G}$ and a partial embedding $\widetilde{\Omega} \subset \widetilde{\Sigma}$ of H into $\Lambda \times \{1\}$. Then we define a non-null subset $\Omega \subset \Sigma$ by a non-null $\Omega \times \{k\} = (\Sigma \times \{k\}) \cap \widetilde{\Omega}$. This measurable subset gives an embedding of H into Λ. □

6. Factorization of Product Groups

Before stating main theorems in this section, we prove a general criterion (Proposition 6.3) on partial embeddings of normal subgroups.

6.1. **ME Coupling between Quotient Groups.** Let $(\mathcal{A}, \mathrm{Tr})$ be a pair of an abelian von Neumann algebra and its faithful normal semi-finite trace. Let Γ be a countable group acting on $\mathcal{A}$ in trace preserving way. We do not need a condition on freeness. The following notation will be useful.

Definition 6.1. *A pair (f, Λ_f) of a non-zero projection $f \in \mathcal{A}$ and a subgroup $\Lambda_f \subset \Gamma$ is said to be* ***a fundamental pair*** *if the following conditions hold:*

(1) *The projection f is an absolute invariant projection of the Λ_f-action, namely, for any projection $f' \leq f$ in $\mathcal{A}$ and $\lambda \in \Lambda_f$, we have $\lambda(f') = f'$.*
(2) *For any $\gamma \in \Gamma \cap (\Lambda_f)^c$, the projection $\gamma(f)$ is orthogonal to f.*
(3) *The projection $\bigvee_{\gamma \in \Gamma} \gamma(f)$ is 1.*

Let Γ_{nor} be the normalizing subgroup for Λ_f, namely, $\Gamma_{\mathrm{nor}} = \{\gamma \in \Gamma \mid \gamma \Lambda_f \gamma^{-1} = \Lambda_f\}$. The group $\Gamma_{\mathrm{nor}}/\Lambda_f$ naturally acts on $\mathcal{A}q$, where q is the projection $q = \bigvee_{\gamma \in \Gamma_{\mathrm{nor}}} \gamma(f)$. The group Λ_f acts on $\mathcal{A}q$ trivially. If we regard $\mathcal{A}q$ as an L^∞ function space, a measurable subset corresponding to f is a fundamental domain for the $(\Gamma_{\mathrm{nor}}/\Lambda_f)$-action on $\mathcal{A}q$.

Lemma 6.2. *Let $H \subset G$, $\Lambda \subset \Gamma$ be normal subgroups and let (Σ, ν) be a standard measure space on which an ergodic $(G \times \Gamma)$-action is given. Suppose that the Γ-action on Σ has a fundamental domain $X \subset \Sigma$.*

If there exists an $(H \times \Lambda)$-invariant projection $e \in L^\infty \Sigma$ with $\mathrm{range}(\mathfrak{E}_X^\Lambda(e)) \not\subset \{0, \infty\}$, *then there exist an $(H \times \Lambda)$-invariant projection f and an intermediate subgroup $\Lambda \subset \Lambda_f \subset \Gamma$ such that $[\Lambda_f \colon \Lambda] < \infty$ and that the pair $(f, \Lambda_f/\Lambda)$ is a fundamental pair for the (Γ/Λ)-action on $(L^\infty \Sigma)^{H \times \Lambda}$.*

Before the proof, we note that the action of Γ on $L^\infty \Sigma$ globally fixes the fixed point subalgebras $(L^\infty \Sigma)^\Lambda$, $(L^\infty \Sigma)^{H \times \Lambda}$, since Λ is a normal subgroup of Γ. Furthermore, this action preserves the trace Tr_Λ defined in Subsection 3.3. This is because the definition of Tr_Λ does not depend on the choice of a Λ fundamental domain of Σ.

Proof. Let k be the minimal element among the positive integers

$$\bigcup \{\mathrm{range}(\mathfrak{E}_X^\Lambda(e)) \mid e \in (L^\infty \Sigma)^{H \times \Lambda}\} \cap \{0, \infty\}^c.$$

We assume $k \in \mathrm{range}(\mathfrak{E}_X^\Lambda(e))$. Let $U \subset X$ be the preimage of k. We replace e with the restriction $e\chi(\Gamma U)$. Since the subset U is invariant under the H-action on $X \cong \Gamma \backslash \Sigma$, the restriction is also $(H \times \Lambda)$-invariant. The function $\mathfrak{E}_X^\Lambda(e)$ is non-zero and $\mathrm{range}(\mathfrak{E}_X^\Lambda(e)) \subset \{0, k\}$. Let Ω be a measurable subset corresponding to e. There exists a non-null measurable subset $X_1 \subset U$ such that

$$\Omega \cap \Gamma X_1 = \bigsqcup_{\gamma_i \in \Gamma_0} \Lambda \gamma_i X_1,$$

for some finite subset $\Gamma_0 = \{\gamma_1, \gamma_2, \cdots, \gamma_k\}$. By replacing X with $\gamma_1 X_1 \sqcup (X \cap (X_1)^c)$, we may assume that $1 = \gamma_1$. We claim that the union of k-cosets $\Lambda_f = \bigsqcup_{\gamma_i \in \Gamma_0} \Lambda \gamma_i$ is a subgroup of Γ_1. Indeed, for $\gamma \in \Gamma$, we get

$$\mathfrak{E}_X^\Lambda(\gamma(e)e) 1_{X_1} = |\Lambda \backslash (\gamma \Lambda_f \cap \Lambda_f)| 1_{X_1}.$$

Since the projection $\gamma(e)e$ is also $(H \times \Lambda)$-invariant, by the minimality of k, it follows that $|\Lambda \backslash (\gamma \Lambda_f \cap \Lambda_f)| = k$ or 0. In other words, we get $\gamma \Lambda_f \cap \Lambda_f = \Lambda_f$ or $\emptyset$. It follows that Λ_f is a subgroup of Γ.

We define f by $\bigwedge_{\gamma\in\Lambda_f}\gamma(e)$. Since $\chi(\cup_{i=1}^{n}\Lambda\gamma_i X_1)\leq f\leq e$, the projection f satisfies range$(\mathfrak{E}_X^{\Lambda}(f))\subset\{0,k\}$ and

$$(4)\qquad \gamma(f)=f\ (\gamma\in\Lambda_f),\quad \gamma(f)\perp f\ (\gamma\in\Gamma\cap(\Lambda_f)^c).$$

Furthermore, there exists a projection f with the property (4) and $\mathfrak{E}_X^{\Lambda}(f)$ is $k1_X$. Let α be the G-action on X defined by the natural identification $X\cong\Gamma\backslash\Sigma$. Since the $(G\times\Gamma)$-action on Σ is ergodic, the action $\alpha\colon G\curvearrowright X\cong\Gamma\backslash\Sigma$ is also ergodic. Let $V\subset X$ be the support of $\mathfrak{E}_X^{\Lambda}(f)$. This is H-invariant. If V is not X, then there exists $g\in G$ such that $W=V\cap(\alpha_{g^{-1}}(V))^c$ is not null and H-invariant. The projection $f+g(f\chi(\Gamma W))$ is also $(H\times\Lambda)$-invariant. By Lemma 3.8 (2) and (3), the value of $\mathfrak{E}_X^{\Lambda}$ is

$$\mathfrak{E}_X^{\Lambda}(f+g(f\chi(\Gamma W)))=k\chi(V)+k\chi(\alpha_g(W)).$$

We get a projection greater than the original one with the same properties. By the maximality argument, we get an $(H\times\Lambda_f)$-invariant projection f with $\mathfrak{E}_X^{\Lambda}(f)=k1_X$.

The (Λ_f/Λ)-action on $f(L^\infty\Sigma)^{H\times\Lambda}$ is trivial. Indeed, by the minimality of k, if a projection f' is smaller than f and $(H\times\Lambda)$-invariant, then range$(\mathfrak{E}_X^{\Lambda}(f'))\subset\{0,k\}$. The projection f' must be written as $f'=f\chi(\Gamma D)$ by some $D\subset X$. The projection f' is also Λ_f-invariant. Since the support of $\mathfrak{E}_X^{\Lambda}(f)$ is X, the projection $\bigvee_{\gamma\in\Gamma}\gamma(f)$ is 1. Since f satisfies the equation 4, it turns out that $(f,\Lambda_f/\Lambda)$ is a fundamental pair for the (Γ/Λ)-action on $(L^\infty\Sigma)^{H\times\Lambda}$. □

Proposition 6.3. *Let $H\subset G$, $\Lambda\subset\Gamma$ be normal subgroups of countable groups and let (Σ,ν) be an ergodic ME coupling for G and Γ (resp. an ergodic measure embedding of G into Γ). If there exists a partial embedding from H into Λ in Σ and if there exists an $(H\times\Lambda)$-invariant projection $f\in L^\infty\Sigma$ with* range$(\mathfrak{E}_Y^{H}(f))\not\subset\{0,\infty\}$, *then $G/H\sim_{\mathrm{ME}}\Gamma/\Lambda$ (resp. $G/H\preceq_{\mathrm{ME}}\Gamma/\Lambda$).*

Proof. Let $\Omega\subset\Sigma$ be a partial embedding of H into Λ. The measurable function $\mathfrak{E}_X^{\Lambda}(\Omega)$ on a Γ fundamental domain X is integrable, since $\int_X\mathfrak{E}_X^{\Lambda}(\Omega)=\mathrm{Tr}_\Lambda(\Omega)<\infty$. Hence there exists a fundamental pair $(e,\Lambda_f/\Lambda)$ for the (Γ/Λ)-action on $(L^\infty\Sigma)^{H\times\Lambda}$ by Lemma 6.2. There also exists a fundamental pair $(f,H_f/H)$ for the (G/H)-action on $(L^\infty\Sigma)^{H\times\Lambda}$ by the other assumption. Replacing $(f,H_f/H)$ with (gf,gH_fg^{-1}), we assume that $ef\neq 0$.

We have two faithful traces Tr_Λ and Tr_H on the algebra $(L^\infty\Sigma)^{H\times\Lambda}$. We can identify $(L^\infty\Sigma)^{H\times\Lambda}$ with an L^∞-function space on a standard measure space. Let F be the Radon-Nikodym derivative $d\,\mathrm{Tr}_\Lambda/d\,\mathrm{Tr}_H$. Since $0<\mathrm{Tr}_\Lambda(ef)\leq\mathrm{Tr}_\Lambda(e)<\infty$, the function F is integrable on ef. Since both of the traces are invariant under the action of G and Γ, the function F is invariant under the action of $G\times\Gamma$. Therefore $d\,\mathrm{Tr}_\Lambda/d\,\mathrm{Tr}_H$ is a constant c. It turns out that

$$(5)\qquad \mathrm{Tr}_H(e)=c^{-1}\,\mathrm{Tr}_\Lambda(e)<\infty.$$

Let $\Gamma_{\mathrm{nor}}\subset\Gamma$ be the normalizing subgroup of Λ_f. Let $q\in(L^\infty\Sigma)^{H\times\Lambda}$ be the projection given by $\bigvee_{\gamma\in\Gamma_{\mathrm{nor}}}\gamma(e)$. The group Λ_f acts trivially on the algebra $q(L^\infty\Sigma)^{H\times\Lambda}$. For $\gamma\in\Gamma\cap(\Gamma_{\mathrm{nor}})^c$, there exists $\gamma'\in\Lambda_f$ satisfying $\gamma^{-1}\gamma'\gamma\notin\Lambda_f$. The projections $\gamma(e)$ and $\gamma'\gamma(e)=\gamma\gamma^{-1}\gamma'\gamma(e)$ are perpendicular. It follows that q can be characterized as the largest projection in $(L^\infty\Sigma)^{H\times\Lambda}$ so that Λ_f acts trivially on $q(L^\infty\Sigma)^{H\times\Lambda}$.

Therefore the projection q is invariant under the $(G \times \Gamma_{\mathrm{nor}})$-action. It follows that there exists a Γ_{nor}-invariant measurable subset $Y_f \subset Y$ with $\chi(GY_f) = q$.

Choose representatives $\{\gamma_\iota\}_{\iota \in I}$ for the left cosets $\Gamma/\Gamma_{\mathrm{nor}}$. The projections $\{\gamma_\iota(q)\}_{\iota \in I}$ give a partition of 1_Y. The projection $\gamma_\iota(q)$ is the characteristic function of $\beta(\gamma_\iota)(Y_f) \subset Y$. Since $\nu(Y_f) = \nu(\beta(\gamma_\iota)(Y_f))$, we get

$$[\Gamma\colon \Gamma_{\mathrm{nor}}]\nu(Y_f) = \sum_\iota \nu(\beta(\gamma_\iota)(Y_f)) = \nu(Y). \tag{6}$$

We note that if the measure of Y is finite, the index of $\Gamma_{\mathrm{nor}} \subset \Gamma$ is finite. We regard $\Sigma_1 = GY_f$ as a measure embedding of G into Γ_{nor}. We note that e is a fundamental domain for the $(\Gamma_{\mathrm{nor}}/\Lambda_f)$-action on $q(L^\infty\Sigma)^{H\times\Lambda} = (L^\infty\Sigma)^{H\times\Lambda}$.

The pair $(qf, H_f/H)$ is a fundamental pair for the G-action on $q(L^\infty\Sigma)^{H\times\Lambda}$. Let G_{nor} be the normalizing subgroup of $H_f \subset G$. By the same technique as above, we can find a $(G_{\mathrm{nor}} \times \Gamma_{\mathrm{nor}})$-invariant projection p in $q(L^\infty\Sigma)^{H\times\Lambda}$ such that H_f/H acts on $p(L^\infty\Sigma)^{H\times\Lambda}$ trivially and that qf gives a fundamental domain for the (G_{nor}/H_f)-action on $p(L^\infty\Sigma)^{H\times\Lambda}$. Furthermore, since the measure of X is finite, the index $[G : G_{\mathrm{nor}}]$ is finite.

The projection pe is a fundamental domain for the $(\Gamma_{\mathrm{nor}}/\Lambda_f)$-action on $p(L^\infty\Sigma)^{H\times\Lambda}$ and satisfies $\mathrm{Tr}_H(pe) < \infty$ by the equation (5). The projection qf is a fundamental domain for the (G_{nor}/H_f)-action on $p(L^\infty\Sigma_1)^{H\times\Lambda}$. Therefore the measure space representing $(p(L^\infty\Sigma)^{H\times\Lambda}, \mathrm{Tr}_H)$ gives a measure embedding of G_{nor}/H_f into $\Gamma_{\mathrm{nor}}/\Lambda_f$. Together with $G/H \sim_{\mathrm{ME}} G_{\mathrm{nor}}/H_f$ and $\Gamma_{\mathrm{nor}}/\Lambda_f \sim_{\mathrm{ME}} \Gamma_{\mathrm{nor}}/\Lambda \preceq_{\mathrm{ME}} \Gamma/\Lambda$, we get $G/H \preceq_{\mathrm{ME}} \Gamma/\Lambda$.

Suppose that Σ is an ME coupling between G and Γ. Since $\mu(Y) < \infty$, the G_{nor}/H_f fundamental domain $qf \in p(L^\infty\Sigma_1)^{G\times\Lambda}$ satisfies $\mathrm{Tr}_H(qf) < \infty$. We conclude that $p(L^\infty\Sigma_1)^{G\times\Lambda}$ gives an ME coupling between G_{nor}/H_f and $\Gamma_{\mathrm{nor}}/\Lambda_f$. In addition, since the index $[\Gamma\colon \Gamma_{\mathrm{nor}}]$ is finite, we get $\Gamma_{\mathrm{nor}}/\Lambda_f \sim_{\mathrm{ME}} \Gamma/\Lambda$. We conclude $G/H \sim_{\mathrm{ME}} \Gamma/\Lambda$. □

6.2. **Factorization up to ME.** We get factorization results on measure equivalence and measure embedding.

Theorem 6.4. *Let $G = \prod_{i=1}^m G_i$ be a product group of non-amenable groups G_i and let $\Gamma = \prod_{j=1}^n \Gamma_j$ be a product group of class $\mathcal{S}$ groups Γ_j. Suppose $m \geq n$. If $G \sim_{\mathrm{ME}} \Gamma$ (resp. $G \preceq_{\mathrm{ME}} \Gamma$), then $m = n$ and the following hold:*

1. *There exists $\sigma \in \mathfrak{S}_n$ so that $G_{\sigma(j)} \sim_{\mathrm{ME}} \Gamma_j$ (resp. $G_{\sigma(j)} \preceq_{\mathrm{ME}} \Gamma_j$),*
2. *The group Γ_j is non-amenable and $G_i \in \mathcal{S}$.*

The last claim holds true by Theorem 9.1, whose proof is independent of this section.

Theorem 6.5. *Let G_0 and Γ_0 be amenable and let G_i $(1 \leq i \leq m)$, Γ_j $(1 \leq j \leq n)$ be non-amenable groups in the class $\mathcal{S}$. Denote $G = G_0 \times \prod_{i=1}^m G_i$, $\Gamma = \Gamma_0 \times \prod_{j=1}^n \Gamma_j$. If $G \sim_{\mathrm{ME}} \Gamma$, then $m = n$ and the following hold:*

1. *There exists $\sigma \in \mathfrak{S}_n$ so that $G_{\sigma(j)} \sim_{\mathrm{ME}} \Gamma_j$,*
2. *The group Γ_0 is finite, if G_0 is finite. The converse is also true.*

Until a middle point of the proof, both theorems require the same technique. We proceed the proofs in the following assumptions.

Framework 6.6. *Positive integers m, n satisfy $m \geq n$. A group G_0 is amenable and groups G_i $(1 \leq i \leq m)$ are non-amenable groups. A group Γ_0 is amenable and and groups Γ_j $(1 \leq j \leq n)$ are in the class $\mathcal{S}$. We denote by G and Γ the product groups*

$$G = G_0 \times \prod_{i=1}^{m} G_i, \quad \Gamma = \Gamma_0 \times \prod_{j=1}^{n} \Gamma_j.$$

A measure space (Σ, ν) is an ergodic measure embedding of G into Γ. We denote by H_i, Λ_j the subgroups

$$H_i = G_0 \times \prod_{k \neq i} G_k, \ 1 \leq i \leq m, \quad \Lambda_j = \Gamma_0 \times \prod_{l \neq j} \Gamma_l, \ 1 \leq j \leq n.$$

(We do not need "H_0", "Λ_0".)

A measurable subset $X \subset \Sigma$ is a Γ fundamental domain and a measurable subset $Y \subset \Sigma$ is a G fundamental domain. We denote by $\mathrm{Tr}_j = \mathrm{Tr}_{\Lambda_j}$ the trace on $(L^\infty\Sigma)^{\Lambda_j}$ defined as $\mathrm{Tr}_j(\cdot) = \mathrm{Tr}(\cdot\chi(\Gamma_j X))$. We use the notations $\mathfrak{E}_X^{(i)}$, $\mathfrak{E}_X$, $\mathfrak{E}_Y^{(j)}$ and $\mathfrak{E}_Y$ for the function valued measures defined in Subsection 3.3*:*

$$\mathfrak{E}_Y^{(i)} = \mathfrak{E}_Y^{H_i} \colon \widehat{(L^\infty\Sigma)_+^{H_i}} \longrightarrow \widehat{(L^\infty Y)_+}, \qquad \mathfrak{E}_Y \colon \widehat{(L^\infty\Sigma)_+} \longrightarrow \widehat{(L^\infty Y)_+},$$

$$\mathfrak{E}_X^{(j)} = \mathfrak{E}_X^{\Lambda_j} \colon \widehat{(L^\infty\Sigma)_+^{\Lambda_j}} \longrightarrow \widehat{(L^\infty X)_+}, \qquad \mathfrak{E}_X \colon \widehat{(L^\infty\Sigma)_+} \longrightarrow \widehat{(L^\infty X)_+}.$$

The following proposition also proves (2) in Theorem 6.5.

Proposition 6.7. *In Framework 6.6, $m = n$ holds true. If Γ_0 is finite, then G_0 is finite.*

Proof. Proposition is proved by induction. We suppose $n = 1$. The group $G = G_1 \times H_1 = G_1 \times (G_0 \times G_2 \times \cdots \times G_m)$ measurably embeds into $\Gamma = \Gamma_0 \times \Gamma_1$ by Σ. The centralizing subgroup $Z_G(H_1)$ of H_1 is non-amenable, since it includes G_1. Since Γ_1 is bi-exact relative to $\{\{1\}\}$, Γ is bi-exact relative to $\{\Gamma_0\}$ (Lemma 4.5). There exists a partial embedding for $H_1 \preceq_\Sigma \Gamma_0$ in Σ, by Theorem 5.3. By remark 3.4 (2), H_1 is amenable. It follows that $H_1 = G_0$ and $m = 1$. If Γ_0 is finite, then $H_1 = G_0$ is also finite.

We suppose that the assertion holds true for a positive integer $n - 1$ and that $G = G_m \times H_m$ measurably embeds into Γ. The group Γ is bi-exact relative to $\{\Lambda_i \mid 1 \leq i \leq n\}$, since Γ_i is bi-exact relative to $\{1\}$ (Lemma 4.5). The centralizing subgroup of H_m in G is non-amenable as G_m is not amenable. By Theorem 5.3, we have a measure embedding $H_m \preceq_{\mathrm{ME}} \Lambda_j$ for some j. By the induction hypothesis, we get $m - 1 \leq n - 1$. It also follows that if $m = n$ (equivalently $m - 1 = n - 1$) and if Γ_0 is finite, then G_0 is also finite. □

For $1 \leq j \leq n$, there exists $1 \leq \sigma(j) \leq n = m$ satisfying $H_{\sigma(j)} \preceq_\Sigma \Lambda_j$ by Theorem 5.3. We claim that σ defines a map.

Lemma 6.8. *In Framework 6.6, if $H_i \preceq_\Sigma \Lambda_j$ and $H_k \preceq_\Sigma \Lambda_j$, then $i = k$.*

Proof. By the assumptions, there exist projections e_i, e_k in $(L^\infty\Sigma)^{\Lambda_j}$ satisfying $h_i(e_i) = e_i$ $(h_i \in H_i)$, $h_k(e_k) = e_k$ $(h_k \in H_k)$ and

$$0 < \mathrm{Tr}_j(e_i) < \infty, \quad 0 < \mathrm{Tr}_j(e_k) < \infty.$$

For any $g \in G_i$ and $\gamma \in \Gamma_j$, the projection $g\gamma(e_i)$ is also invariant under the action of H_i and Λ_j and the trace $\mathrm{Tr}_j(g\gamma(e_i))$ is equal to $\mathrm{Tr}_j(e_i)$. Since the action of $G_i \times \Gamma_j$ on $(L^\infty\Sigma)^{H_i \times \Lambda_j}$ is ergodic, the projection $\bigvee\{g\gamma(e_i) \mid g \in G_i, \gamma \in \Gamma_j\}$ is 1. It follows that there exists a projection $\hat{e}_i$ obtained by a finite union of $\{g\gamma(e_i)\}$ satisfying $h_i(\hat{e}_i) = \hat{e}_i$ ($h_i \in H_i$) and

$$0 < \mathrm{Tr}_j(\hat{e}_i) < \infty, \quad \mathrm{Tr}_j(e_k)/2 < \mathrm{Tr}_j(e_k\hat{e}_i).$$

Assume $i \neq k$. Denote by $\overline{\mathcal{C}}$ the convex norm closure of $\{g(\hat{e}_i) \mid g \in G_i\}$ in $L^2((L^\infty\Sigma)^{\Lambda_j}, \mathrm{Tr}_j)$. The element $\xi \in \overline{\mathcal{C}}$ having the minimal value of 2-norm is fixed under G_i as well as H_i. Since we have $g(e_k) = e_k$ for $g \in G_i \subset H_k$, the following inequality holds true:

$$\langle e_k, g(\hat{e}_i)\rangle = \mathrm{Tr}_j(e_k g(\hat{e}_i)) = \mathrm{Tr}_j(g(e_k\hat{e}_i)) = \mathrm{Tr}_j(e_k\hat{e}_i) > \mathrm{Tr}_j(\hat{e}_i)/2.$$

The vector ξ satisfies $\langle e_k, \xi\rangle \geq \mathrm{Tr}_j(\hat{e}_i)/2$. It follows that ξ is not zero. Since ξ is fixed under $G = G_i \times H_i$, a non-trivial level set $\Omega \in \Sigma$ of ξ is also fixed under G. The measure of an H_j fundamental domain is $\mathrm{Tr}_j(\Omega) < \infty$. The measurable subset Ω gives a measure embedding of G into H_j, which contradicts Proposition 6.7. We conclude $i = k$ □

We prove that σ defines an injective map. By $m = n$, σ is also surjective.

Lemma 6.9. *In Framework 6.6, if $H_i \preceq_\Sigma \Lambda_j$ and $H_i \preceq_\Sigma \Lambda_l$, then $j = l$.*

Proof. There exist projections $f_j, f_l \in (L^\infty\Sigma)^{H_i}$ satisfying $\lambda_j(f_j) = f_j$ ($\lambda_j \in \Lambda_j$), $\lambda_l(f_l) = f_l$ ($\lambda_l \in \Lambda_l$) and

$$0 < \mathrm{Tr}_j(f_j) < \infty, \quad 0 < \mathrm{Tr}_l(f_l) < \infty.$$

Since Σ is an ergodic measure embedding, the projection $\bigvee\{g\gamma(f_j) \mid g \in G_i, \gamma \in \Gamma_j\}$ is 1. Replacing f_j with a bigger projection, we may assume that f_jf_l is not zero.

Assuming $j \neq l$, we deduce a contradiction. Denote $\Delta = \Gamma_0 \times \prod_{k\neq j,l}\Gamma_k = \Lambda_j \cap \Lambda_k$. The function valued measures $\mathfrak{E}_X^{(j)}$ and $\mathfrak{E}_X^{(l)}$ satisfy the following:

$$\begin{aligned}
\mathfrak{E}_X^{\Delta}(f_jf_l)(x) &= \sum_{\gamma_j\gamma_l \in \Gamma_j \times \Gamma_i} f_jf_l(\gamma_j\gamma_l x) \\
&= \sum_{\gamma_j\gamma_l \in \Gamma_j \times \Gamma_i} f_j(\gamma_j x) f_l(\gamma_l x) \\
&= \sum_{\gamma_j \in \Gamma_j} f_j(\gamma_j x) \sum_{\gamma_l \in \Gamma_l} f_l(\gamma_l x) \\
&= \mathfrak{E}_X^{(j)}(f_j)(x)\mathfrak{E}_X^{(l)}(f_l)(x), \quad \text{a.e. } x \in X.
\end{aligned}$$

The projection f_jf_l is H_i-invariant. The value of the measurable function $\mathfrak{E}_X^{\Delta}(f_jf_l)$ is finite almost everywhere, since the functions $\mathfrak{E}_X^{(j)}(f_j)$ and $\mathfrak{E}_X^{(l)}(f_l)$ are integrable. It follows that $H_i \preceq_\Sigma \Delta$, by Lemma 3.9. This contradicts Proposition 6.7. □

Proof for the assertion 1 in Theorem 6.4. Let G_0 and Γ_0 be trivial groups in Framework 6.6. By redefining the indices, we may assume $H_i \preceq_\Sigma \Lambda_i$.

We take a projection $e_i \in (L^\infty\Sigma)^{H_i \times \Lambda_i}$ satisfying $0 < \mathrm{Tr}_i(e_i) < \infty$. We may assume that $\mathfrak{E}_X^{(i)}(e_i)$ is bounded. By replacing e_i with a finite union of projections

$g\gamma(e_i)$ $(g \in G_i, \gamma \in \Gamma_i)$, we may also assume that the product $e = \prod_{i=1}^n e_i$ is not zero. By direct computations, we get the following equation:

$$\begin{aligned}\mathfrak{E}_Y(e)(y) &= \sum_{g\in G} e(gy) = \sum_{(g_1,g_2,\cdots,g_n)\in G} \prod_{i=1}^n e_i(g_iy) \\ &= \prod_{i=1}^n \sum_{g_i\in G_i} e_i(g_iy) = \prod_{i=1}^n \mathfrak{E}_Y^{(i)}(e_i)(y), \qquad \text{a.e. } y \in Y.\end{aligned}$$

We also get $\mathfrak{E}_X(e) = \prod_{i=1}^n \mathfrak{E}_X^{(i)}(e_i)$. It turns out that $\mathfrak{E}_Y(e)$ is integrable, since

$$\begin{aligned}\int_Y \mathfrak{E}_Y(e)d\nu &= \int_\Sigma ed\nu = \int_X \mathfrak{E}_X(e)d\nu \\ &= \int_X \prod_{i=1}^n \mathfrak{E}_X^{(i)}(e_i)d\nu \le \nu(X)\prod_{i=1}^n \sup_x \mathfrak{E}_X^{(i)}(e_i)(x) < \infty.\end{aligned}$$

On the support $W \subset Y$ of $\mathfrak{E}_Y(e)$, the function $\mathfrak{E}_Y^{(i)}(e_i)$ satisfies

$$\mathfrak{E}_Y^{(i)}(e_i)(y) \le \mathfrak{E}_Y^{(i)}(e_i)(y) \times \prod_{j\neq i} \mathfrak{E}_Y^{(j)}(e_j)(y) = \mathfrak{E}_Y(e)(y), \qquad \text{a.e. } y \in W,$$

since $\mathfrak{E}_Y^{(j)}(e_j)$ is $(\{0, 1, \cdots, \infty\})$-valued on W. It follows that the function $\mathfrak{E}_Y^{(i)}(e_i)$ is integrable on W. Since $\mathfrak{E}_Y(e)$ is not zero, $\mathfrak{E}_Y^{(i)}(e_i)$ is also not zero on W. Applying Proposition 6.3 for the quotients $G_i \cong G/H_i$ and $\Gamma_i \cong \Gamma/\Lambda_i$, we get the conclusion in the two cases $\nu(Y) < \infty$ and $\nu(Y) = \infty$. □

Proof for the assertion 1 in Theorem 6.5. Let G_0, Γ_0 be amenable groups and let G_i, Γ_i $(1 \le i \le n)$ be non-amenable groups in the class $\mathcal{S}$. By replacing the indices, we may assume that $H_i \preceq_\Sigma \Lambda_i$ for any i. By replacing the roles on G and Γ, there exists $\rho \in \mathfrak{S}_n$ with $\Lambda_i \preceq_\Sigma H_{\rho(i)}$. By Proposition 6.3, we only have to show that $\rho(i) = i$.

Assume that $k = \rho(i) \neq i$. Since $H_i \preceq_\Sigma \Lambda_i$, there exists a projection $e \in (L^\infty\Sigma)^{H_i\times\Lambda_i}$ with $0 < \mathrm{Tr}_i(e) < \infty$. Since $\Lambda_i \preceq_\Sigma H_k$, by Lemma 6.2, there exist a projection $f \in (L^\infty\Sigma)^{H_k\times\Lambda_i}$ and a finite subgroup $G_{k,f} \subset G_k$ so that the pair $(f, G_{k,f})$ is a fundamental pair for the G_k-action on $(L^\infty\Sigma)^{H_k\times\Lambda_i}$. Let $\{g_\iota\}_{\iota\in I}$ be a set of representatives for the left cosets $G_k/G_{k,f}$. The projections $\{g_\iota(f)\}_{\iota\in I}$ give a partition of 1. Since the G_k-action preserves $\mathrm{Tr}_i = \mathrm{Tr}_{\Lambda_i}$ and fixes e, we get

$$\mathrm{Tr}_i(e) = \sum_{\iota\in I} \mathrm{Tr}_i(eg_\iota(f)) = \sum_{\iota\in I} \mathrm{Tr}_i(g_\iota(ef)) = |I|\,\mathrm{Tr}_i(ef).$$

This contradicts the conditions $0 < \mathrm{Tr}_i(e) < \infty$ and $|I| = \infty$. Therefore we get $k = i$. □

6.3. **Separately Ergodic Couplings.**

Definition 6.10. *For a measure preserving group action of $G = G_0 \times \prod_{i=1}^n G_i$ on a standard probability space X, we say that the action is **separately ergodic** when the subgroups $H_i = G_0 \times \prod_{k\neq i} G_k$ $(1 \le i \le n)$ act on X ergodically. For a measure embedding Σ of the product group G and arbitrary countable group Γ, we say that the action is **separately ergodic** when the groups $H_i \times \Gamma$ act on Σ ergodically.*

For a separately ergodic couplings, we get a stronger conclusion than the previous subsection.

Theorem 6.11. *Let G and Γ be product groups which satisfy the assumptions in Theorem 6.4. Let Σ be a measure embedding of G into Γ. If Σ is separately ergodic, then $m = n$ and there exist $\sigma \in \mathfrak{S}_n$ and subgroups $G_{i,\mathrm{fin}} \subset G$, $\Gamma_{i,\mathrm{fin}} \subset \Gamma_{i,\mathrm{nor}} \subset \Gamma_i$ $(1 \leq i \leq n)$ with the following properties:*

1. *The subgroup $G_{i,\mathrm{fin}} \subset G_i$ is finite and normal. The subgroup $\Gamma_{i,\mathrm{fin}}$ is finite and $\Gamma_{i,\mathrm{nor}}$ normalizes $\Gamma_{i,\mathrm{fin}}$,*
2. *The group $G_{\sigma(i)}/G_{\sigma(i),\mathrm{fin}}$ is isomorphic to $\Gamma_{i,\mathrm{nor}}/\Gamma_{i,\mathrm{fin}}$.*
3. *The coupling constant of Σ satisfies*
$$[\Gamma\colon G]_\Sigma = \prod_{i=1}^{n} \frac{|\Gamma_{i,\mathrm{fin}}|[\Gamma_i\colon \Gamma_{i,\mathrm{nor}}]}{|G_{\sigma(i),\mathrm{fin}}|}.$$
If Σ is an ME coupling, then $[\Gamma_i\colon \Gamma_{i,\mathrm{nor}}] < \infty$ and $G_{\sigma(i)}$ and Γ_i are commensurable up to finite kernel.

Theorem 6.12. *Let G and Γ be product groups which satisfy the assumptions in Theorem 6.5. If there exists a separately ergodic ME coupling between G and Γ, then $m = n$ and there exists $\sigma \in \mathfrak{S}_n$ so that $G_{\sigma(i)}$ and Γ_i are commensurable up to finite kernel.*

We proceed the proof for the two theorems in Framework 6.6.

Proof. Suppose that the measure embedding Σ is separately ergodic. By the previous subsection, $m = n$ and there exists $\sigma \in \mathfrak{S}_n$ satisfying $H_{\sigma(i)} \preceq_\Sigma \Lambda_i$. We change the indices on G_i so that $H_i \preceq_\Sigma \Lambda_i$.

Let a pair $(e_i, \Gamma_{i,\mathrm{fin}})$ of a projection $e_i \in (L^\infty\Sigma)^{H_i\times\Lambda_i}$ and a finite subgroup $\Gamma_{i,\mathrm{fin}} \subset \Gamma_i$ be a fundamental pair for the Γ_i-action on $(L^\infty\Sigma)^{H_i\times\Lambda_i}$ (Lemma 6.2). Since $e_i \perp \gamma(e_i)$ for $\gamma \in \Gamma_i \cap (\Gamma_{i,\mathrm{fin}})^c$ and the group $\Gamma_{i,\mathrm{fin}}$ acts on $e_i(L^\infty\Sigma)^{H_i\times\Lambda_i}$ trivially, every projection e_i' in $e_i(L^\infty\Sigma)^{H_i\times\Lambda_i}$ satisfy
$$e_i' = e_i e_i' = \sum_{\gamma\Gamma_{i,\mathrm{fin}}\in\Gamma_i/\Gamma_{i,\mathrm{fin}}} e_i\gamma(e_i') = e_i \bigvee_{\gamma\in\Gamma_i} \gamma(e_i') = e_i \bigvee_{\gamma\in\Gamma} \gamma(e_i').$$
Letting $X' \subset X$ be the support of $\mathfrak{E}_X^{(i)}(e_i')$, the projection e_i' is of the form $e_i\chi(\Gamma X')$. The measurable subset $X' \subset X \cong \Gamma\backslash\Sigma$ is H_i-invariant since $\mathfrak{E}_X^{(i)}$ is G-equivariant. Since the embedding Σ is separately ergodic, it has to be null or co-null. We get $e_i' = e_i$ or $e_i' = 0$. This means that e_i is a minimal projection in $(L^\infty\Sigma)^{H_i\times\Lambda_i}$.

Let $\mathcal{P}_i$ be the set of minimal projections in $(L^\infty\Sigma)^{H_i\times\Lambda_i}$. The G_i-action and Γ_i-action on $\mathcal{P}_i$ commute with each other. Since $(e_i, \Gamma_{i,\mathrm{fin}})$ is a fundamental pair, the action of Γ_i on $\mathcal{P}_i$ is transitive. The stabilizer of e_i is $\Gamma_{i,\mathrm{fin}}$. Let $G_{i,\mathrm{fin}} \subset G_i$ be the stabilizer of e_i, and $\Gamma_{i,\mathrm{nor}} \subset \Gamma_i$ be the collection of elements $\gamma \in \Gamma_i$ for which there exists $g \in G_i$ satisfying $\gamma(e_i) = g^{-1}(e_i)$. If $g \in G_i$ and $\gamma \in \Gamma_{i,\mathrm{nor}}$ satisfy this relation, then for $g_f \in G_{i,\mathrm{fin}}$ and $\gamma_f \in \Gamma_{i,\mathrm{fin}}$ we get
$$g^{-1}g_f g(e_i) = g^{-1}g_f\gamma^{-1}(e_i) = g^{-1}\gamma^{-1}g_f(e_i) = g^{-1}\gamma^{-1}(e_i) = e_i,$$
$$\gamma^{-1}\gamma_f\gamma(e_i) = \gamma^{-1}\gamma_f g^{-1}(e_i) = \gamma^{-1}g^{-1}\gamma_f(e_i) = \gamma^{-1}g^{-1}(e_i) = e_i.$$

It turns out that G_i, $\Gamma_{i,\mathrm{nor}}$ normalize $G_{i,\mathrm{fin}}$, $\Gamma_{i,\mathrm{fin}}$ respectively. If $g_a, g_b \in G_i$ and $\gamma_a, \gamma_b \in \Gamma_{i,\mathrm{nor}}$ satisfy relations $\gamma_a(e_i) = g_a^{-1}(e_i)$, $\gamma_b(e_i) = g_b^{-1}(e_i)$, then $\gamma_a^{-1}(e_i) = g_a(e_i)$ and

$$\gamma_a\gamma_b(e_i) = \gamma_a g_b^{-1}(e_i) = g_b^{-1}\gamma_a(e_i) = g_b^{-1}g_a^{-1}(e_i) = (g_a g_b)^{-1}(e_i).$$

It follows that $\Gamma_{i,\mathrm{nor}}$ is a subgroup of Γ_i and that when we define a map

$$\phi_i : G_i/G_{i,\mathrm{fin}} \ni gG_{i,\mathrm{fin}} \to \gamma\Gamma_{i,\mathrm{fin}} \in \Gamma_{i,\mathrm{nor}}/\Gamma_{i,\mathrm{fin}}$$

by $\gamma(e_i) = g^{-1}(e_i)$. This gives a group isomorphism.

We next claim that the function valued measures satisfy

$$\mathfrak{E}_X^{(i)}(e_i) = |\Gamma_{i,\mathrm{fin}}|1_X, \quad \mathfrak{E}_Y^{(i)}(e_i) = |G_{i,\mathrm{fin}}|1_{Y_i},$$

where Y_i is the support of $\mathfrak{E}_Y^{(i)}(e_i)$. Define projections $q_i, q \in L^\infty Y$ by $q = 1_{Y_i}$ and $q = \prod_{i=1}^n q_i$. The measurable subset $Y_0 = \bigcap_{i=1}^n Y_i$ corresponds to q. Take a Γ fundamental domain $X_i \subset \Sigma$ as $\chi(X_i) \leq e_i$. The measurable set corresponding to e_i can be written as $\Gamma_{i,\mathrm{fin}}\Lambda_i X_i$, since $\gamma(e_i) = e_i$ $(\gamma \in \Gamma_{i,\mathrm{fin}})$, $\gamma(e_i) \perp e_i$ $(\gamma \in \Gamma_i \cap \Gamma_{i,\mathrm{fin}}^c)$ and e_i is Λ_i-invariant. The function valued measure satisfies $\mathfrak{E}_{X_i}^{\Lambda_i}(e_i) = |\Gamma_{i,\mathrm{fin}}|1_{X_i}$ and this confirms the first equation by the identification $X_i \cong \Gamma\backslash\Sigma \cong X$. The proof for the second equation is given by the same way. Define $e = \prod_{i=1}^n e_i \in L^\infty\Sigma$. The function valued measures of e with respect to Γ_0 and G_0 are

$$\mathfrak{E}_X^{\Gamma_0}(e) = \prod_{i=1}^n \mathfrak{E}_X^{(i)}(e_i) = \prod_{i=1}^n |\Gamma_{i,\mathrm{fin}}|1_X, \tag{7}$$

$$\mathfrak{E}_Y^{G_0}(e) = \prod_{i=1}^n \mathfrak{E}_Y^{(i)}(e_i) = \prod_{i=1}^n |G_{i,\mathrm{fin}}|q. \tag{8}$$

Define subgroups $G_{\mathrm{fin}} \subset G$ and $\Gamma_{\mathrm{fin}} \subset \Gamma_{\mathrm{nor}} \subset \Gamma$ by

$$G_{\mathrm{fin}} = G_0 \times \prod_{i=1}^n G_{i,\mathrm{fin}}, \quad \Gamma_{\mathrm{fin}} = \Gamma_0 \times \prod_{i=1}^n \Gamma_{i,\mathrm{fin}}, \quad \Gamma_{\mathrm{nor}} = \Gamma_0 \times \prod_{i=1}^n \Gamma_{i,\mathrm{nor}}.$$

We next claim

$$\nu(Y) = [\Gamma : \Gamma_{\mathrm{nor}}]\nu(Y_0). \tag{9}$$

The projection $q_i \in L^\infty Y$ corresponds to the union of G_i-orbits of e_i,

$$\bigvee_{g\in G_i} g(e_i) = \bigvee_{\gamma\in\Gamma_{i,\mathrm{nor}}} \gamma(e_i) \in (L^\infty\Sigma)^G \cong L^\infty Y.$$

The measurable subset $Y_i \subset Y \cong G\backslash\Sigma$ corresponds to q_i. We note that for $\gamma, \gamma' \in \Gamma_i$, we get either $\gamma(q_i) = \gamma'(q_i)$ $(\gamma^{-1}\gamma' \in \Gamma_{i,\mathrm{nor}})$ or $\gamma(q_i) \perp \gamma'(q_i)$ $(\gamma^{-1}\gamma' \in \Gamma_i \cap (\Gamma_{i,\mathrm{nor}})^c)$. It follows that for $\gamma, \gamma' \in \Gamma$, we get

$$\begin{aligned} \gamma(q) &= \gamma'(q), \ (\gamma^{-1}\gamma' \in \Gamma_{\mathrm{nor}}), \\ \gamma(q) &\perp \gamma'(q), \ (\gamma^{-1}\gamma' \in \Gamma \cap (\Gamma_{\mathrm{nor}})^c). \end{aligned}$$

It follows that representatives $\{\gamma_\iota\}_\iota$ for $\Gamma/\Gamma_{\mathrm{nor}}$ give a partition $\{\gamma_\iota q\}_\iota$ of 1_Σ. Since the measurable sets $\{\gamma_\iota Y_0\}_\iota$ have the same measure, we have the equation (9).

Suppose $G_0 = \Gamma_0 = \{1\}$. Since $|\Gamma_{i,\mathrm{fin}}|$ is finite, by (7) and (8), we get

$$|G_{\mathrm{fin}}|\nu(Y_0) = \int_Y \mathfrak{E}_Y(e)\nu = \int_\Sigma e d\nu = \int_X \mathfrak{E}_X(e)\nu = |\Gamma_{\mathrm{fin}}|\nu(X) < \infty.$$

It follows that the subgroups $G_{i,\mathrm{fin}}$ are finite. Furthermore, the coupling constant of Σ is given by

$$[\Gamma : G]_\Sigma = \nu(Y)/\nu(X) = [\Gamma : \Gamma_{\mathrm{nor}}]\nu(Y_0)/\nu(X) = [\Gamma : \Gamma_{\mathrm{nor}}]|\Gamma_{\mathrm{fin}}|/|G_{\mathrm{fin}}|.$$

In particular, if $[\Gamma : G]_\Sigma < \infty$, then $[\Gamma : \Gamma_{\mathrm{nor}}] < \infty$. The map ϕ_i gives an isomorphism between $G_i/G_{i,\mathrm{fin}}$ and $\Gamma_{i,\mathrm{nor}}/\Gamma_{i,\mathrm{fin}}$. Theorem 6.11 is confirmed.

In turn, we suppose that $\nu(Y) < \infty$ and G, Γ are product groups satisfying the assumptions in Theorem 6.5. The proof of Theorem 6.5 has shown that $\Lambda_i \preceq_\Sigma H_i$ and that Tr_{H_i} is a scalar multiple of Tr_{Λ_i}. It follows that the projection e_i satisfies $0 < \mathrm{Tr}_{H_i}(e_i) = \int_Y \mathfrak{E}_Y^{(i)}(e_i)d\nu < \infty$. The group $G_{i,\mathrm{fin}}$ is finite as $\mathfrak{E}_Y^{(i)}(e_i) = |G_{i,\mathrm{fin}}|1_{Y_i}$ is integrable. The index $[\Gamma\colon \Gamma_{\mathrm{nor}}] = \nu(Y)/\nu(Y_0)$ is also finite by the equation (9). It follows that $\Gamma_{i,\mathrm{nor}}$ is a finite index subgroup of Γ_i. The map ϕ_i gives an isomorphism between $G_i/G_{i,\mathrm{fin}}$ and $\Gamma_{i,\mathrm{nor}}/\Gamma_{i,\mathrm{fin}}$. This confirms Theorem 6.12. □

6.4. **OE Strong Rigidity Theorems.**

Definition 6.13. *Let G and Γ be arbitrary countable groups. Suppose that α is a free ergodic measure preserving action of G on a standard probability space X and that $\phi : G \to \Gamma$ be a group homomorphism with finite kernel. Consider the $(G \times \Gamma)$-action $\mathfrak{A}$ defined on $\Sigma = \Gamma \times X$ by*

$$\mathfrak{A}(\gamma_0, g)(\gamma, x) = (\gamma_0\gamma\phi(g)^{-1}, \alpha_g(x))$$

and choose a fundamental domain Y for the G action. ***The induced action*** $\mathrm{Ind}_G^\Gamma(\alpha, \phi)$ *is a Γ-action on $Y \cong G\backslash\Sigma$ defined by*

$$\gamma_0(\mathfrak{A}(G)(\gamma, x)) = \mathfrak{A}(G)(\gamma_0\gamma, x).$$

The induced action is free, ergodic and measure preserving. If the group homomorphism ϕ is an isomorphism, then the induced action $\beta = \mathrm{Ind}_G^\Gamma(\alpha, \phi)$ is conjugate to the action α. The measure of Y is finite if and only if the image of ϕ is a finite index subgroup of Γ.

Definition 6.14. *A group Γ is said to be in $\mathcal{S}_0$ if $\Gamma \in \mathcal{S}$ and does not have a pair of subgroups $\{1\} \neq \Gamma_{\mathrm{fin}} \subset \Gamma_{\mathrm{nor}} \subset \Gamma$ satisfying* (1) *Γ_{fin} is finite,* (2) *Γ_{nor} normalizes Γ_{fin},* (3) *$\Gamma_{\mathrm{nor}} \subset \Gamma$ is finite index.*

ICC groups in the class $\mathcal{S}$ are in $\mathcal{S}_0$.

Theorem 6.15. *Let $G = \prod_{i=1}^n G_i$ be a product group of non-amenable groups and let $\Gamma = \prod_{i=1}^n \Gamma_i$ be a product group of groups in $\mathcal{S}_0$. Suppose that α is a free ergodic measure preserving G-action on a standard probability space X and that β is a free ergodic measure preserving Γ-action on a standard measure space Y. If the two group actions α and β are stably orbit equivalent with SOE constant $s \in (0, \infty]$, and if α is separately ergodic, then there exist $\sigma \in \mathfrak{S}_n$ and a group homomorphism from $\phi_i\colon G_{\sigma(i)} \to \Gamma_i$ with the following properties:*

(1) *The kernel of ϕ_i is finite.*

(2) *The* Γ*-action* β *is conjugate to the induced action* $\mathrm{Ind}_G^\Gamma(\alpha,\phi)$*, where* ϕ *is the group homomorphism from* G *to* Γ *given by* $\phi((g_i)_{\sigma(i)}) = (\phi_i(g_i))$.

(3) *The SOE constant* s *satisfies*

$$s = \prod_{i=1}^{n} \frac{[\Gamma_i \colon \mathrm{image}(\phi_i)]}{|\ker(\phi_i)|}.$$

If $s < \infty$*, then* $[\Gamma_i \colon \mathrm{image}(\phi_i)] < \infty$ *and* $G_{\sigma(i)}, \Gamma_i$ *are commensurable up to finite kernel.*

Proof. Let $\mathcal{R}$ be a type II relation on a standard measure space (Z,ν), which gives SOE between α and β. Namely, $X, Y \subset Z$ be measurable subsets with $\mu(X) = 1, \mu(Y) = s$ and that $\mathcal{R}_\alpha = \mathcal{R} \cap (X \times X)$, $\mathcal{R}_\beta = \mathcal{R} \cap (Y \times Y)$. The measure space $\Sigma = \mathcal{R} \cap (X \times Y)$ is a measure embedding of G into Γ. Since the finite measure space X is a separately ergodic G-space, the embedding Σ is separately ergodic.

We use the notations in Framework 6.6. We may assume that $H_i \preceq_\Sigma \Lambda_i$. Let $e_i \in (L^\infty\Sigma)^{H_i \times \Lambda_i}$ be a minimal projection and let $G_{i,\mathrm{fin}} \subset G_i$ and $\Gamma_{i,\mathrm{fin}} \subset \Gamma_{i,\mathrm{nor}} \subset \Gamma_i$ be subgroups given in the previous proof. The subgroup $\Gamma_{i,\mathrm{fin}}$ is finite. If $s < \infty$, the inclusion $\Gamma_{i,\mathrm{nor}} \subset \Gamma_i$ has finite index. The condition $\Gamma \in \mathcal{S}_0$ means $\Gamma_{i,\mathrm{fin}} = \{1\}$. We get a surjective group homomorphism $\phi_i \colon G_i \to \Gamma_{i,\mathrm{nor}}$ with kernel $G_{i,\mathrm{fin}}$ by $\phi(g)e_i = g^{-1}e_i$. Defining $\phi \colon G \to \Gamma$ by $\phi((g_i)) = (\phi_i(\gamma_i))$, we get $\phi(g)e = g^{-1}e$ for $g \in G$. By using (7), the projection $e = \prod_{i=1}^n e_i$ satisfies $\mathfrak{E}_X(e) = \prod_{i=1}^n |\Gamma_{i,\mathrm{fin}}| 1_X = 1_X$. We identify the measurable set X and the support of e. We also identify the measurable set Σ and $\Gamma \times X$ by $\Gamma \times X \ni (\gamma, x) \to \gamma(x) \in \Sigma$. The G-action on $L^\infty\Sigma$ satisfies

$$g(f\gamma(e)) = \alpha_g(f)\gamma g(e) = \alpha_g(f)\gamma\phi(g)^{-1}(e), \quad f \in L^\infty X \cong (L^\infty\Sigma)^\Gamma.$$

It follows that the G-action on $\Gamma \times X$ can be written as

$$g(\gamma, x) = (\gamma\phi(g)^{-1}, \alpha_g(x)), \quad \gamma \in \Gamma, \text{ a.e. } x \in X.$$

Since Y is isomorphic to $G\backslash\Sigma$ as a Γ-space, the Γ-action β is isomorphic to the action $\mathrm{Ind}_G^\Gamma(\alpha,\phi)$. The SOE constant s is equal to the coupling constant $[\Gamma : G]_\Sigma$. By Theorem 6.11 (3), we get the description of the SOE constant. □

Corollary 6.16. *Let* $G = \prod_{i=1}^n G_i$ *and* $\Gamma = \prod_{i=1}^n \Gamma_i$ *be product groups of non-amenable groups in* $\mathcal{S}_0$*. Suppose that* α *is a free ergodic measure preserving* G*-action on a standard probability space* X *and that* β *is a free ergodic measure preserving* Γ*-action on a standard finite measure space* Y*. If the two group actions* α *and* β *are stably orbit equivalent with constant* $s \in (0,\infty)$*, that is* $\mathcal{R}_\alpha^s \cong \mathcal{R}_\beta$*, and if* α *and* β *are separately ergodic, then* $s = 1$*. In particular, the fundamental group* $\mathcal{F}(\mathcal{R}_\alpha)$ *is* $\{1\}$.

Proof. Since the group G_i is in $\mathcal{S}_0$, G_i has no normal finite subgroup other than $\{1\}$. We get $s = \prod_{i=1}^n [\Gamma_i \colon \Gamma_{i,\mathrm{nor}}] \geq 1$ by Theorem. By replacing the roles on G and Γ, we also get $s^{-1} \geq 1$. □

Corollary 6.17. *Let* G *and* Γ *be as in Corollary 6.16. Suppose that* G *and* Γ *act on a common standard probability space* Z *by* α *and* β*, respectively, in free ergodic measure preserving ways. If the two group actions* α *and* β *give the same equivalence relation* $\mathcal{R}$ *on* Z*, and if* α *and* β *are separately ergodic, then there exists a measure preserving map* θ *on* Z *so that its graph is essentially included in* $\mathcal{R}$ *and that it gives* *conjugacy between α and β. In particular, the outer automorphism group* $\mathrm{Out}(\mathcal{R}_\alpha)$ *is* $\{1\}$.

Proof. We regard $\mathcal{R}$ as an ME coupling between G and Γ with coupling constant 1, letting G act on the first entry and Γ act on the second entry. Let X be a Γ fundamental domain and Y be a G fundamental domain. Although the subset X and Y can be identical (for example, the diagonal set), we distinguish them. We may assume that $H_i \preceq_{\mathcal{R}} \Lambda_i$. The product e of minimal projections $e_i \in (L^\infty \mathcal{R})^{H_i \times \Lambda_i}$ satisfies $\mathfrak{E}_X(e) = 1_X$, since the groups $\Gamma_{i,\mathrm{fin}}$ in the proof of Theorem 6.11 are $\{1\}$. By replacing the roles on G and Γ, we also get $\mathfrak{E}_Y(e) = 1_Y$. There exists a measure preserving map θ on Z which satisfies $\chi(\{(y, \theta(y)) \mid y \in Z\}) = e$.

The group homomorphism $\phi : G \to \Gamma$ given in the proof of Theorem 6.15 is bijective, since $G_{i,\mathrm{fin}} = \{1\}$, $\prod_{i=1}^n [\Gamma_i \colon \Gamma_{i,\mathrm{nor}}] = 1$. For $g \in G$, we get

$$\begin{aligned} g^{-1}e &= \chi(\{\alpha(g^{-1})(y), \theta(y)) \mid y \in Z\}) = \chi(\{y, \theta(\alpha(g)(y)) \mid y \in Z\}),\\ \phi(g)e &= \chi(\{(y, \beta(\phi(g))\theta(y)) \mid y \in Z\}). \end{aligned}$$

Since $g^{-1}e = \phi(g)e$, there exists a co-null subset $Z' \subset Z$ satisfying

$$\theta(\alpha(g)(y)) = \beta(\phi(g))\theta(y), \quad y \in Z', g \in G,$$

□

Theorem 6.18. *Let G_0 (resp. Γ_0) be an amenable group and let G_i $(1 \le i \le n)$ (resp. Γ_i) be non-amenable groups in $\mathcal{S}$ with no finite normal subgroup. Denote $G = G_0 \times \prod_{i=1}^n G_i$ (resp. $\Gamma = \Gamma_0 \times \prod_{i=1}^n \Gamma_i$). Suppose that α (resp. β) is a free measure preserving G-action (resp. Γ-action) on a standard probability space X (resp. Y) on which G_0 acts (resp. Γ_0) ergodically. If the two group actions α and β are orbit equivalent, then there exist $\sigma \in \mathfrak{S}_n$, group isomorphisms $\phi_i : G_{\sigma(i)} \to \Gamma_i$ and measure preserving map $\theta : X \to Y$ which satisfy:*

Define ϕ by ϕ: $\prod_{i=1}^n G_i \ni (g_i)_{\sigma(i)} \to (\phi_i(g_i))_i \in \prod_{i=1}^n \Gamma_i$. For almost every $x \in X$ and every $g \in \prod_{i=1}^n G_i$, $\theta(\alpha(gG_0)x) = \beta(\phi(g)\Gamma_0)\theta(x)$.

Proof. We may assume that both of α and β are actions on a standard probability space Z and that they give the same equivalence relation Σ. We regard Σ as an ME coupling between G and Γ with coupling constant 1, letting G act on the first entry and Γ act on the second entry. We choose a G fundamental domain X and a Γ fundamental domain Y. Define a bijection σ by $H_{\sigma(i)} \preceq_\Sigma \Lambda_i$ and $\Lambda_i \preceq_\Sigma H_{\sigma(i)}$. We may assume that $\sigma = \mathrm{id}$.

By the previous subsection, $(L^\infty\Sigma)^{H_i \times \Lambda_i}$ is atomic and the Γ_i-action on the set of minimal projections is transitive. Since the assumptions are symmetric on G and Γ, the G_i-action is also transitive. It follows that for a minimal projection $e_i \in (L^\infty\Sigma)^{H_i \times \Lambda_i}$, the stabilizers $\Gamma_{i,\mathrm{fin}} \subset \Gamma_i$ and $G_{i,\mathrm{fin}} \subset G_i$ are finite normal subgroups. It follows that they are $\{1\}$. Let $\phi_i : G_i \to \Gamma_i$ be the group isomorphism given by $g^{-1}e_i = \phi_i(g)e_i$. The product of projections $e = \prod_{i=1}^n e_i$ satisfies $\mathfrak{E}_X^{G_0}(e) = \prod_{i=1}^n |G_{i,\mathrm{fin}}| 1_X = 1_X$. By replacing the roles on G and Γ, we also get $\mathfrak{E}_Y^{\Gamma_0}(e) = 1_Y$.

We claim that there exists a measure preserving map θ on X whose graph is included in the support of e. Let e_0 be maximal among projections dominated by e with the properties $\mathfrak{E}_X(e_0) \le 1_X, \mathfrak{E}_Y(e_0) \le 1_Y$. Suppose that $\int_X \mathfrak{E}_X(e_0) d\nu = \int_Y \mathfrak{E}_Y(e_0) d\nu = \nu(e_0) < 1$. By replacing X and Y, we may assume that $e_0 \le \chi(X) \le$

e and $e_0 \leq \chi(Y) \leq e$. There exists a non-null measurable subset $Y_0 \subset Y$ so that $\chi(Y_0)$ is perpendicular with e_0 and that the graph of Y_0 gives partial isomorphism on Z. Since the G_0-action on $\Gamma\backslash Z$ is ergodic, replacing Y_0 with a smaller non-null measurable subset, there exists $g \in G_0$ satisfying $\alpha_g(\mathfrak{E}_X(Y_0)) \perp \mathfrak{E}_X(e_0)$. The projection $e_0 + g_0\chi(Y_0)$ is dominated by e and satisfies

$$\begin{aligned}\mathfrak{E}_X(e_0 + g\chi(Y_0)) &= \mathfrak{E}_X(e_0) + \alpha_g(\mathfrak{E}_X(\chi(Y_0))) \leq 1_X,\\ \mathfrak{E}_Y(e_0 + g\chi(Y_0)) &= \mathfrak{E}_Y(e_0) + 1|_{Y_0} \leq 1_Y.\end{aligned}$$

This contradicts the maximality of e_0. Therefore we get $\mathfrak{E}_X(e_0) = 1_X$ and $\mathfrak{E}_Y(e_0) = 1_Y$. This means that the projection e_0 corresponds to a graph of a measure preserving map $\theta : Z \to Z$, that is, $\chi(\{(x, \theta(x)) \mid x \in Z\}) = e_0$. For $g \in \prod_{i=1}^n G_i$, we have the following equality of projections:

$$\begin{aligned}g^{-1}e &= \sum_{g_0\in G_0} g^{-1}g_0^{-1}e_0 = \chi(\{(\alpha(g_0^{-1}g^{-1})(x), \theta(x)) \mid x \in Z, g_0 \in G_0\})\\ &= \chi(\{(x, \theta\alpha(gg_0)(x)) \mid x \in Z, g_0 \in G_0\}),\\ \phi(g)e &= \sum_{\gamma_0\in G_0} \phi(g)\gamma_0 e_0 = \chi(\{(x, \beta(\phi(g)\gamma_0)\theta(x)) \mid x \in Z, \gamma_0 \in \Gamma_0\}).\end{aligned}$$

Since $g^{-1}e = \phi(g)e$, it follows that $\theta(\alpha(gG_0)x) = \beta(\phi(g)\Gamma_0)\theta(x)$, a.e. $x \in Z$. □

6.5. **OE Super Rigidity Type Theorems.**

Theorem 6.19. *Let $\Gamma = \prod_{i=1}^n \Gamma_i$ be a direct product group of non-amenable ICC groups in $\mathcal{S}$ and let G be an arbitrary countable group.*

1. *Suppose that there exists an ME coupling Σ of G with Γ. If the Γ-action on $G\backslash\Sigma$ is separately ergodic and if the G-action on $\Gamma\backslash\Sigma$ is mildly mixing, then there exists a group homomorphism ϕ: $G \to \Gamma$ with finite kernel and the coupling constant satisfies $[\Gamma : \phi(G)] = |\ker(\phi)|[\Gamma : G]_\Sigma$.*
2. *Suppose that there exist a free separately ergodic measure preserving Γ-action on a standard probability space X and a free mildly mixing measure preserving G-action on a standard finite measure space Y. If the actions α and β are stably orbit equivalent with finite constant, then there exists a homomorphism ϕ: $G \to \Gamma$ with finite kernel and finite index image such that the induced action $\mathrm{Ind}_G^\Gamma(\alpha, \phi)$ is conjugate to β.*

To deduce the above theorem from Theorem 6.11 and Theorem 6.15, we need technique by Furman [Fu1]. The technique we need here has already given by Monod–Shalom. The above theorems are obtained by verbatim translations of the sixth chapter of Monod and Shalom's paper [MoSh]. We remark that we use the ICC condition on Γ_i to construct Furman's homomorphism.

7. Measure Equivalence between Wreath Product Groups

The goal of this section is Theorem 2.6.

Lemma 7.1. *Let $H \subset G$ be an infinite subgroup of a countable group and let $\widetilde{\Gamma} = B \wr \Gamma$ be a countable wreath product group with $B \neq \{1\}$. Suppose that Σ is a measure embedding of G into $B \wr \Gamma$.*

If H measurably embeds into Γ in Σ, then there exists a partial embedding Ω of H into Γ satisfying of $H \preceq_\Sigma \Gamma$, we get $\Omega' \subset \Omega$ for any partial embedding Ω'. The Γ-support of $H \preceq_\Sigma \Gamma$ (Definition 3.5) satisfies $\mathfrak{E}_X^\Gamma(\Omega) = \mathrm{supp}_X^\Gamma(H \preceq_\Sigma \Gamma) \in L^\infty X$.

Proof. We denote by $\widetilde{B}$ the subgroup $\bigoplus_\Gamma B \in \widetilde{\Gamma}$. Let $\Omega \subset \Sigma$ be an arbitrary partial embedding of H into Γ and let X be a fundamental domain of Σ under the $\widetilde{\Gamma}$-action. We can write Ω as $\Omega = \bigsqcup_{b \in \widetilde{B}} \Gamma b X_b$, for some measurable subsets $X_b \subset X$. The measurable function $\mathfrak{E}_X^\Gamma(\Omega)$ is written as $\sum_{b \in \widetilde{B}} \chi(X_b)$ and it is integrable. We first claim that $\mathfrak{E}_X^\Gamma(\Omega)$ is a projection.

Suppose that the essential range of $\mathfrak{E}_X^\Gamma(\Omega)$ is not contained in $\{0, 1\}$. There exist a non-null measurable subset $W \subset X$ and finite subset $\{b_1, b_2, \cdots, b_k\} \subset \widetilde{B}$ satisfying $k \geq 2, b_i \neq b_j$ $(i \neq j)$ and $\Omega \cap \Gamma W = \bigsqcup_{i=1}^k \Gamma b_i W$. The measurable set $b_1^{-1}\Omega \cap b_2^{-1}\Omega$ is H-invariant and satisfies

$$b_1^{-1}\Omega \cap b_2^{-1}\Omega \cap \Gamma W = \bigcup_i b_1^{-1}\Gamma b_i W \cap \bigcup_j b_2^{-1}\Gamma b_j W = \bigcup_{i,j} (b_1^{-1}\Gamma b_i \cap b_2^{-1}\Gamma b_j) W.$$

Applying the function valued measure $\mathfrak{E}_X \colon \widehat{L^\infty(\Sigma)}_+ \to \widehat{L^\infty(X)}_+$, we get

$$\mathfrak{E}_X(b_1^{-1}\Omega \cap b_2^{-1}\Omega) 1_W = \left| \bigcup_{i,j} (b_1^{-1}\Gamma b_i \cap b_2^{-1}\Gamma b_j) \right| 1_W.$$

Since $\bigcup_{i,j} b_1^{-1}\Gamma b_i \cap b_2^{-1}\Gamma b_j$ is a finite set and non-empty, we get $H \preceq_\Sigma \{1\}$ (Lemma 3.9). This contradicts $|H| = \infty$. This means that the essential range of $\mathfrak{E}_X^\Gamma(\Omega)$ is included in $\{0, 1\}$ and $\mathfrak{E}_X^\Gamma(\Omega)$ is a projection.

When Ω, Ω' are partial embeddings of H into Γ, the union $\Omega \cup \Omega'$ is also a partial embedding of H into Γ. By the above argument, $\mathfrak{E}_X^\Gamma(\Omega \cup \Omega')$ is a projection.

There exists an increasing sequence of Ω_n of partial embeddings of H into Γ with $\bigvee_n \mathfrak{E}_X^\Gamma(\Omega_n) = \mathrm{supp}_X^\Gamma(H \preceq_\Sigma \Gamma)$. Let Ω be the union of $\{\Omega_n\}$. Applying $\mathfrak{E}_X^\Gamma$, we get

$$\mathfrak{E}_X^\Gamma(\chi(\Omega)) = \sup_n \mathfrak{E}_X^\Gamma(\Omega_n) = \mathrm{supp}_X^\Gamma(H \preceq_\Sigma \Gamma).$$

It follows that Ω is again a partial embedding of H into Γ. Let Ω' be another partial embedding. We get $\mathfrak{E}_X^\Gamma(\Omega) \leq \mathfrak{E}_X^\Gamma(\Omega \cup \Omega') \leq p = \mathfrak{E}_X^\Gamma(\Omega)$. Since $\mathfrak{E}_X^\Gamma$ is faithful, we conclude $\chi(\Omega \cup \Omega') = \chi(\Omega)$ and that Ω dominates all partial embedding after subtracting a null set. □

Proposition 7.2. *Let $G \times H \subset \widetilde{G}$ be a subgroup of an exact group $\widetilde{G}$. Let $\widetilde{\Gamma}$ be an exact wreath product group $B \wr \Gamma$ with amenable base $B \neq \{1\}$. Suppose that G is non-amenable and that H is infinite.*

If Σ is an ergodic measure embedding of $\widetilde{G}$ into $\widetilde{\Gamma}$, then there exists a maximal partial embedding Ω of $G \times H$ into Γ. Moreover, the embedding satisfies $\mathfrak{E}_X^\Gamma(\Omega) = 1_X$.

Proof. The group $\widetilde{\Gamma}$ is bi-exact relative to $\{\Gamma\}$ by Lemma 4.6. By Theorem 5.3, H measurably embeds into Γ in Σ. Furthermore, its $\widetilde{\Gamma}$-support of the embedding is 1_X. Let Ω be the largest embedding of H into Γ (Lemma 7.1).

Since $g \in G$ commutes with all elements in H, the measurable subsets $g\Omega, g^{-1}\Omega$ also give embeddings of H into Γ. The maximality of Ω means that $g\Omega \subset \Omega$ and $g^{-1}\Omega \subset \Omega$ after null sets are subtracted. It follows that the difference between $g\Omega$

and Ω is null. We may assume that Ω is $(G \times H)$-invariant. The measurable subset Ω gives a measure embedding of $G \times H$ into Γ. The embedding Ω of $G \times H$ into Γ is maximal, since it is maximal as an embedding of H. □

Proof for Theorem 2.6. Let Σ be an ergodic ME coupling between two wreath products $\widetilde{G}$ and $\widetilde{\Gamma}$. By Proposition 7.2, we take the largest embedding $\Omega_l \subset \Sigma$ of $G \times H$ into $\Gamma \times \Lambda$ and the largest embedding $\Omega_r \subset \Sigma$ of $\Gamma \times \Lambda$ into $G \times H$. It suffices to show that the difference between Ω_l, Ω_r is null. Since the assumptions are symmetric, we only prove that $\Omega_l \cap \Omega_r^c$ is null.

By the equality $\mathfrak{E}_Y^{G \times H}(\Omega_r) = 1_Y$, there exists a measurable subset $Y' \subset \Omega_r$ so that Y' is a fundamental domain for the $\widetilde{G}$-action on Σ and that $\chi((G \times H)Y') = \chi(\Omega_r)$. Denote $\widetilde{A} = \oplus_{G \times H} A$. The measurable subset Ω_r is an $\widetilde{A}$-fundamental domain for the action $\widetilde{A} \curvearrowright \Sigma$.

Suppose that $\Omega_l \cap (\Omega_r)^c$ is not null. There exists $1 \neq a \in \widetilde{A}$ such that $\Omega_l \cap a\Omega_r$ is not null. We note that this is $(\Gamma \times \Lambda)$-invariant. There exist infinitely many elements $\{g_i\}_{i \in I}$ in $G \times H$ such that $\{g_i(a)\}_{i \in I}$ are different from each other. The following equation holds true

$$\mathrm{Tr}_{\Gamma \times \Lambda}(\Omega_l \cap g_i(a)\Omega_r) = \mathrm{Tr}_{\Gamma \times \Lambda}(g_i(\Omega_l \cap a\Omega_r)) = \mathrm{Tr}_{\Gamma \times \Lambda}(\Omega_l \cap a\Omega_r).$$

Since the measurable subsets $\{g_i(a)\Omega_r\}$ are disjoint, we get

$$0 < \sum_{i \in I} \mathrm{Tr}_{\Gamma \times \Lambda}(\Omega_l \cap g_i(a)\Omega_r) \leq \mathrm{Tr}_{\Gamma \times \Lambda}(\Omega_l) < \infty.$$

This contradicts $|I| = \infty$. We conclude that $\Omega_l \subset \Omega_r$ after subtracting a null set. Since the assumptions are symmetric, we get $\Omega_l = \Omega_r$ after subtracting null sets. The measurable subset $\Omega_l = \Omega_r$ gives an ME coupling of $G \times H$ with $\Gamma \times \Lambda$.

For the second assertion, we suppose that the coupling Σ comes from stable orbit equivalence, in other words, the dot actions $\alpha\colon \widetilde{G} \curvearrowright X$ and $\beta\colon \widetilde{\Gamma} \curvearrowright Y$ are free. We further assume that the actions $\alpha|_{G \times H}$, $\beta|_{\Gamma \times \Lambda}$ are ergodic. Since $\mathfrak{E}_X^{\Gamma \times \Lambda}(\Omega_l) = 1_X$, the action $\alpha|_{G \times H}$ is conjugate to the dot action $G \times H \curvearrowright (\Gamma \times \Lambda)\backslash\Omega_l$. By symmetricity, the action $\beta|_{\Gamma \times \Lambda}$ is conjugate to the dot action $\Gamma \times \Lambda \curvearrowright (G \times H)\backslash\Omega_l$. Choose an embedding from X to a $\Gamma \times \Lambda$ fundamental domain of Ω_l and an embedding from Y to a $G \times H$ fundamental domain of Ω_l. The compositions $p\colon X \hookrightarrow \Omega_l \to (G \times H)\backslash\Omega_l \cong Y$ and $q\colon Y \hookrightarrow \Omega_l \to (\Gamma \times \Lambda)\backslash\Omega_l \cong X$ give stable orbit equvialence between $\alpha|_{G \times H}$ and $\beta|_{\Gamma \times \Lambda}$. □

8. Factorization of Amalgamated Free Products

The goal of this section is Theorem 8.4. We start with an argument on Bass–Serre trees.

Lemma 8.1. *Let Γ be an amalgamated free product $\Gamma_1 *_B \Gamma_2$ of countable groups. Let i be either 1 or 2 and u be an element of Γ. If $u\Gamma_i \neq \Gamma_1$, then there exist $\gamma \in \Gamma$ and a subgroup $B_u \subset \gamma B \gamma^{-1}$ with the following property:*
For all $s, t \in \Gamma$, $S = s\Gamma_i \cap t\Gamma_1 u \subset \Gamma$ is either empty or a left coset of B_u.

Proof. Fix u in this proof. Let s, t be arbitrary elements in Γ. Let $T = \Gamma/\Gamma_1 \sqcup \Gamma/\Gamma_2$ be the Bass–Serre tree for $\Gamma = \Gamma_1 *_B \Gamma_2$, on which the group Γ acts. The set $t\Gamma_1 u$

is identical to the collection of elements which move $u^{-1}\Gamma_1 \in T$ to $t\Gamma_1 \in T$. The set $s\Gamma_i$ is the collection of elements which move $\Gamma_i \in T$ to $u\Gamma_i \in T$.

Let B_u be the set of elements which stabilize all points

$$\{u^{-1}\Gamma_1 = p_1, p_2, \cdots, p_l = \Gamma_i\} \subset T$$

on the geodesic from $u^{-1}\Gamma_1$ to Γ_i. Suppose that S is not empty. We take an element $v \in S$. The set S is of the form vB_u. Any element $b \in B_u$ stabilizes the edge $\{p_{l-1}, p_l = \Gamma_i\}$. The stabilizer of $\{p_{l-1}, p_l = \Gamma_i\}$ is of the form $\gamma B \gamma^{-1}$ for some $\gamma \in \Gamma$. It follows that B_u is a subgroup of $\gamma B \gamma^{-1}$. □

Lemma 8.2. *Let $H \subset G$ be an inclusion of countable groups and let $\Gamma = \Gamma_0 *_B \Lambda$ be a free product with amalgamation over a common subgroup B. Suppose that Σ is a measure embedding of G into Γ.*

If $H \preceq_\Sigma \Gamma_0$ and $H \not\preceq_{\mathrm{ME}} B$, then there exists a partial embedding Ω of H into Γ_0 in Σ, which is maximal. Namely, for any partial embedding Ω' of H into Γ_0, $\Omega^c \cap \Omega'$ is a null set. Furthermore, the Γ-support of $H \preceq_\Sigma \Gamma_0$ satisfies $\mathrm{supp}_X^\Gamma(H \preceq_\Sigma \Gamma_0) = \mathfrak{E}_X^{\Gamma_0}(\Omega)$.

Proof. We choose and fix representatives $\{s_\iota\}_{\iota \in I}$ of the right cosets $\Gamma_0 \backslash \Gamma$. Let $\Omega \subset \Sigma$ be an arbitrary partial embedding of H into Γ_0 and let X be a fundamental domain of Σ under the Γ-action. We can write $\Omega = \bigsqcup_{\iota \in I} \Gamma_0 s_\iota X_\iota$, for some measurable subsets $X_\iota \subset X$. The measurable function $\mathfrak{E}_X^{\Gamma_0}(\Omega) = \sum_{\iota \in I} \chi(X_\iota)$ is integrable.

Suppose that the essential range of $\mathfrak{E}_X^{\Gamma_0}(\Omega)$ is not included in $\{0,1\}$. There exist a non-null measurable subset $W \subset X$ and a finite subset $\{s_1, s_2, \cdots, s_k\} \subset \{s_\iota\}_{\iota \in I}$ satisfying $k \geq 2, s_i \neq s_j$ $(i \neq j)$ and $\Omega \cap \Gamma W = \bigcup_{i=1}^k \Gamma_0 s_i W$. Replacing X with $s_1 W \sqcup (X \cap (W)^c)$ and $\{s_i\}$ with $\{s_i s_1^{-1}\}$, we may assume $s_1 = 1$.

The measurable set $s_2\Omega \cap \Omega$ is H-invariant and satisfies

$$s_2\Omega \cap \Omega = \bigcup_i s_2 \Gamma_0 s_i W \cap \bigcup_j \Gamma_0 s_j W = \bigcup_{i,j} (s_2 \Gamma_0 s_i \cap \Gamma_0 s_j) W.$$

The set $S = \bigcup_{i,j} (s_2 \Gamma_0 s_i \cap \Gamma_0 s_j)$ is not empty since s_2 is an element of S. By Lemma 8.1, there exists a subgroup $B_2 \subset \gamma^{-1} B \gamma$ for some $\gamma \in \Gamma$ so that S is a finite union of right cosets of B_2. The function valued measure of $b_1\Omega \cap b_2\Omega$ with respect to B_2 satisfies

$$\mathfrak{E}_X^{B_2}(s_2\Omega \cap \Omega)|_W = |B_2 \backslash S| \, 1_W.$$

We get $H \preceq_\Sigma B_2 \subset \gamma^{-1} B \gamma$ (Lemma 3.9). Since $H \not\preceq_{\mathrm{ME}} B$, this is a contradiction. We conclude that the essential range of $\mathfrak{E}_X^{\Gamma_0}(\Omega)$ is included in $\{0,1\}$. For the rest of the proof, we apply the same argument as Lemma 7.1. □

Proposition 8.3. *Let $G_1 \times H \subset G$ be a direct product type subgroup of an exact group G. Let Γ be an exact free product group $*_B \Gamma_i$ $(1 \leq i \leq n)$ with amalgamation over a common amenable subgroup B. Suppose that G_1, H are non-amenable. If Σ is an ergodic measure embedding of G into Γ, then*

1. *The Γ-supports $p_i = \mathrm{supp}_X^\Gamma(G_1 \times H \preceq_\Sigma \Gamma_i)$ of $G_1 \times H \preceq_\Sigma \Gamma_i$ are mutually orthogonal and satisfy $\sum_{i=1}^n p_i = 1_X$.*
2. *There exist maximal measure embeddings $\Omega_i \subset \Sigma$ of $G_1 \times H$ into Γ_i. Their function valued measure $\mathfrak{E}_X^{\Gamma_i} = \mathfrak{E}_X^{(i)}$ satisfies $\mathfrak{E}_X^{(i)}(\Omega_i) = p_i$.*

Proof. Since Γ is bi-exact relative to $\{\Gamma_i\}$ and G_1 is non-amenable, H measurably embeds into some Γ_i in Σ (Theorem 5.3). Define p_i as the Γ-support of the embedding $H \preceq_\Sigma \Gamma_i$ (instead of $G_1 \times H \preceq_\Sigma \Gamma_i$). By the maximality argument, Theorem 5.3 tells that the union of the Γ-supports covers X, that is, $\bigvee_i p_i = 1_X$. Since non-amenable group H does not measurably embed into amenable group B, we can take the largest partial embedding Ω_i of H into Γ_i (Lemma 8.2). The function valued measure satisfies $\mathfrak{E}_X^{(i)}(\Omega_i) = \mathrm{supp}_X^\Gamma(H \preceq_\Sigma \Gamma_i)$.

Since $g \in G_1$ commutes with all elements in H, the measurable subsets $g\Omega, g^{-1}\Omega$ also give embeddings of H into Γ. By the maximality of Ω, we have $g\Omega \subset \Omega$ and $g^{-1}\Omega \subset \Omega$ after subtracting null sets. We may assume that Ω is $(G_1 \times H)$-invariant. The measurable subset Ω_i gives a measure embedding of $G_1 \times H$ into Γ_i. The maximal embedding Ω of H into Γ_i is also maximal as an embedding of $G_1 \times H$. The support of $G_1 \times H \preceq_\Sigma \Gamma_i$ satisfies

$$p_i = \mathfrak{E}_X^{(i)}(\Omega_i) \leq \mathrm{supp}_X^\Gamma(G_1 \times H \preceq_\Sigma \Gamma_i) \leq \mathrm{supp}_X^\Gamma(H \preceq_\Sigma \Gamma_i).$$

It follows that $p_i = \mathrm{supp}_X^\Gamma(G_1 \times H \preceq_\Sigma \Gamma_i)$.

We claim that the projections p_i are mutually orthogonal. It suffices to show that the Γ-support P_i for the embedding $G_1 \times H \preceq_\Sigma *_{B,j\neq i}\Gamma_j$ is perpendicular to p_i. Denote $\Lambda = *_{B,j\neq i}\Gamma_j$. Suppose that $P_i p_i \neq 0$. There exists a partial embedding $\Omega' \subset \Sigma$ of $G_1 \times H$ into Λ which satisfies $\mathfrak{E}_X^\Lambda(\Omega')p_i \neq 0$. Since $\mathfrak{E}_X^\Lambda(\Omega')$ is a projection, there exist a non-null measurable subset $W \subset X$ and $s, t \in \Gamma$ with

$$\Omega \cap \Gamma W = \Gamma_i sW, \quad \Omega' \cap \Gamma W = \Lambda tW.$$

By Lemma 8.1, the set $ts^{-1}\Gamma_i s \cap \Lambda t$ is a right coset of a subgroup $C = C_{ts^{-1}} \subset \Gamma$ which is isomorphic to a subgroup of B. The function valued measure $\mathfrak{E}_X^C$ of $ts^{-1}\Omega \cap \Omega'$ satisfies

$$\mathfrak{E}_X^C(ts^{-1}\Omega \cap \Omega')1_W = \mathfrak{E}_X^C((ts^{-1}\Gamma_i s \cap \Lambda t)W) = 1_W.$$

This means that $G_1 \times H \preceq_\Sigma C$, since $ts^{-1}\Omega \cap \Omega'$ is $(G_1 \times H)$-invariant. This contradicts non-amenability of $G_1 \times H$. It follows that p_i is perpendicular to P_i and that $\{p_i\}$ are mutually orthogonal. □

Theorem 8.4. *Let G_i $(1 \leq i \leq m)$ and Γ_j $(1 \leq j \leq n)$ be direct products of two non-amenable exact groups. Suppose that $\{G_i\}$ have a common amenable subgroup A and that $\{\Gamma_j\}$ have a common amenable subgroup B. Denote by G, Γ the amalgamated free products $G = *_A G_i$, $\Gamma = *_B \Gamma_j$. We have the following:*

(1) *If $G \sim_{\mathrm{ME}} \Gamma$, then for any $1 \leq i \leq m$ there exists $1 \leq \sigma(i) \leq n$ satisfying $G_i \sim_{\mathrm{ME}} \Gamma_{\sigma(i)}$ and for any $1 \leq j \leq n$ there exists $1 \leq \rho(j) \leq m$ satisfying $G_{\rho(j)} \sim_{\mathrm{ME}} \Gamma_j$,*

(2) *If $m = n = 2$ and $G \sim_{\mathrm{ME}} \Gamma$, then there exists $i \in \{1, 2\}$ satisfying $G_1 \sim_{\mathrm{ME}} \Gamma_i$, $G_2 \sim_{\mathrm{ME}} \Gamma_{i+1}$, where $i + 1 \in \{1, 2\} \cap \{i\}^c$,*

(3) *Let Σ be an ME coupling between G and Γ. If the $(G_i \times \Gamma)$-action on Σ is ergodic for any i and if $(G \times \Gamma_j)$-action on Σ is ergodic for any j, then $m = n$ and there exists $\sigma \in \mathfrak{S}_n$ satisfying $G_i \sim_{\mathrm{ME}} \Gamma_{\sigma(i)}$. More precisely, there exists $(G_i \times \Gamma_{\sigma(i)})$-invariant measurable subset $\Omega(i, \sigma(i)) \subset \Sigma$ which gives an ME coupling of G_i with $\Gamma_{\sigma(i)}$ and satisfies $[\Gamma : G]_\Sigma = [\Gamma_{\sigma(i)} : G_i]_{\Omega(i,\sigma(i))}$,*

(4) *Let α be a free measure preserving G-action on standard probability space X and let β be a free measure preserving Γ-action on a standard finite measure*

space Y. Suppose that the G_i-action $\alpha|_{G_i}$ on X and the Γ_j-action $\beta|_{\Gamma_j}$ on Y are ergodic for any i, j. If the G-action and Γ-action are stably orbit equivalent, then $m = n$ and there exists $\sigma \in \mathfrak{S}_n$ so that $\alpha|_{G_i}$ and $\beta|_{\Gamma_{\sigma(i)}}$ are stably orbit equivalent.

Proof. Let Σ be an ergodic ME coupling between two amalgamated free products G and Γ and let X, Y be fundamental domains for the Γ-action and G-action, respectively. We write $G = G_i *_A H_i, \Gamma = \Gamma_j *_B \Lambda_j$.

Denote by $\Omega(i,j) \subset \Sigma$ the (possibly null) maximal partial embedding of G_i into Γ_j in Σ in Proposition 8.3. The functions $\{\mathfrak{E}_X^{\Gamma_j}(\Omega(i,j))\}$ are characteristic functions and satisfy

$$\sum_{j=1}^{n} \mathfrak{E}_X^{\Gamma_j}(\Omega(i,j)) = 1_X, \quad i = 1, 2, \cdots, m.$$

Since the assumptions are symmetric, again by Proposition 8.3, we get the maximal partial embeddings $\Xi(i,j)$ of $\Gamma_j \preceq_\Sigma G_i$. The functions $\mathfrak{E}_Y^{G_i}(\Xi(i,j))$ are characteristic functions and satisfy

$$\sum_{i=1}^{m} \mathfrak{E}_Y^{G_i}(\Xi(i,j)) = 1_X, \quad j = 1, 2, \cdots, n.$$

We first claim that $\mathrm{supp}_Y^G(\Omega(i,j)) \leq \mathrm{supp}_Y^G(\Xi(i,j))$. We only have to show that if $1 \leq i, k \leq m$ satisfy $\mathfrak{E}_Y^{G_i}(\Omega(i,j))\mathfrak{E}_Y^{G_k}(\Xi(k,j)) \neq 0$, then $i = k$. Under the assumption $\mathfrak{E}_Y^{G_i}(\Omega(i,j))\mathfrak{E}_Y^{G_k}(\Xi(k,j)) \neq 0$f, there exists $h \in G$ such that $\Omega(i,j) \cap h(\Xi(k,j))$ is non-null. Since the essential range of $\mathfrak{E}_Y^{G_k}(\Xi(k,j))$ is contained by $\{0,1\}$, there exists a measurable subset $Y_k \subset \Sigma$ satisfying

$$\Xi(k,j) = G_k Y_k, \quad h\Xi(k,j) = hG_k Y_k,$$

after subtracting null sets. Suppose $k \neq i$. For $g \in G_i \cap A^c$, the Γ_i-invariant measurable subsets $h\Xi(k,j)$ and $gh\Xi(k,j)$ are almost disjoint. Letting $\{g_\iota\}_{\iota \in I}$ be representatives for the left cosets G_i/A, we get that $\{g_\iota h\Xi(k,j)\}_{\iota \in I}$ are almost disjoint and

$$0 < \mathrm{Tr}_{\Gamma_j}\left(\Omega(i,j) \cap \bigsqcup_{\iota \in I} g_\iota h\Xi(k,j)\right) \leq \mathrm{Tr}_{\Gamma_j}(\Omega(i,j)) < \infty.$$

The measurable subsets $\Omega(i,j) \cap g_\iota h\Xi(k,j)$ are equal to $g_\iota(\Omega(i,j) \cap h\Xi(k,j))$ and have the same value of Tr_{Γ_j}. This contradicts $|I| = [G_i : A] = \infty$. The first claim is confirmed.

We next claim that $\Omega(i,j)$ is essentially included in $\Xi(i,j)$. By the last paragraph, we get $\chi(\Omega(i,j)) \leq \bigvee_{h \in G} h\chi(\Xi(i,j))$. it suffices to deduce a contradiction, assuming that $h \in G \cap G_i^c$ satisfies $\chi(\Omega(i,j))h\chi(\Xi(i,j)) \neq 0$. For $g \in G_i \cap A^c$, the measurable subsets

$$h\Xi(i,j) = hG_i Y_i, \quad gh\Xi(i,j) = ghG_i Y_i$$

are disjoint. By the same calculation as the last paragraph, we get

$$\begin{aligned} 0 &< |I| \operatorname{Tr}_{\Gamma_j}(\Omega(i,j) \cap h\Xi(i,j)) \\ &= \operatorname{Tr}_{\Gamma_j}\left(\bigcup_{gA \in G_i/A} g(\Omega(i,j) \cap h\Xi(i,j)) \right) \\ &= \operatorname{Tr}_{\Gamma_j}\left(\bigcup_{gA \in G_i/A} \Omega(i,j) \cap gh\Xi(i,j) \right) \\ &\leq \operatorname{Tr}_{\Gamma_j}(\Omega(i,j)) < \infty. \end{aligned}$$

We get a contradiction with $|I| = [G_i : A] = \infty$. We conclude that $\chi(\Omega(i,j)) \leq \chi(\Xi(i,j))$. Since the assumptions are symmetric on G and Γ, it follows that that $\Omega(i,j) = \Xi(i,j)$ after subtracting null sets.

The measurable set $\Omega(i,j) = \Xi(i,j)$ gives an ME coupling of G_i with Γ_j if it is non-null. For every $1 \leq i \leq m$ there exists $1 \leq j \leq n$ satisfying $\mathfrak{E}_X^{\Gamma_j}(\Omega(i,j)) \neq 0$. This means that $\Omega(i,j)$ is non-null and $G_i \sim_{\mathrm{ME}} \Gamma_j$. By the same way, for $1 \leq j \leq n$ there exists $1 \leq i \leq m$ satisfying $G_i \sim_{\mathrm{ME}} \Gamma_j$. We get the assertion (1).

Suppose $m = n = 2$. By the first assertion, there exist $i, j \in \{1,2\}$ with $G_1 \sim_{\mathrm{ME}} \Gamma_i$ and $G_2 \sim_{\mathrm{ME}} \Gamma_j$. If $i \neq j$, then we get the assertion (2). If $i = j = 1$, then there exists $k \in \{1,2\}$ satisfying $G_k \sim_{\mathrm{ME}} \Gamma_2$ by the assertion (1). If $i = j = 2$, then there exists $k \in \{1,2\}$ satisfying $G_k \sim_{\mathrm{ME}} \Gamma_1$. Therefore we get the assertion (2).

Suppose that the $(G_i \times \Gamma)$-action on Σ is ergodic for any $1 \leq i \leq m$ and that the $(G \times \Gamma_j)$-action on Σ is ergodic for any $1 \leq j \leq n$. Since the G_i-action on $X \cong \Gamma\backslash\Sigma$ is ergodic, the function $\mathfrak{E}_X^{\Gamma_j}(\Omega(i,j))$ is either 0 or 1_X. It follows that for $1 \leq i \leq m$ there exists a unique $1 \leq j = \sigma(i) \leq n$ such that $\Omega(i,j)$ is non-null. Since the assumptions are symmetric, for $1 \leq j \leq n$ there exists a unique $1 \leq \rho(j) \leq m$ such that $\Omega(i,j)$ is non-null. The maps σ and ρ must be the inverse maps of each other, and in particular $m = n$. Since the measure of a $\Gamma_{\sigma(i)}$ fundamental domain of $\Omega(i,\sigma(i))$ is

$$\operatorname{Tr}_{\Gamma_{\sigma(i)}}(\Omega(i,\sigma(i))) = \int_X \mathfrak{E}_X^{\Gamma_{\sigma(i)}}(\Omega(i,\sigma(i)))d\nu = \nu(X),$$

and that of a G_i fundamental domain is

$$\operatorname{Tr}_{G_i}(\Omega(i,\sigma(i))) = \int_Y \mathfrak{E}_Y^{G_i}(\Omega(i,\sigma(i)))d\nu = \nu(Y),$$

we get the following equation between two coupling indices,

$$[\Gamma : G]_\Sigma = \nu(Y)/\nu(X) = [\Gamma_{\sigma(i)} : G_i]_{\Omega(i,\sigma(i))}.$$

Suppose that the coupling Σ comes from stable orbit equvialence, in other words, the actions $G \curvearrowright X \cong \Gamma\backslash\Sigma$ and $\Gamma \curvearrowright Y \cong G\backslash\Sigma$ are essentially free and that the actions $G_i \curvearrowright X$, $\Gamma_j \curvearrowright Y$ is ergodic. The actions $G_i \curvearrowright \Gamma_\sigma(i)\backslash\Omega(i,\sigma(i))$ and $\Gamma_{\sigma(i)} \curvearrowright G_i\backslash\Omega(i,\sigma(i))$ are conjugate to the original dot actions. It follows that the coupling $\Omega(i,\sigma(i))$ gives the stable orbit equivalence between two actions $G_i \curvearrowright X$ and $\Gamma_{\sigma(i)} \curvearrowright Y$ in the sense of [Fu2, Section 3]. □

9. The class $\mathcal{S}$ is Invariant under measure equivalence

Theorem 9.1 is stronger than Theorem 2.9 and follows from Proposition 9.2.

Theorem 9.1. *If G is measurably embeds into Γ and Γ is in the class $\mathcal{S}$, then G is also in the class $\mathcal{S}$.*

Proposition 9.2. *Suppose $\Gamma \in \mathcal{S}$. Let β be a free measure preserving action of Γ on a standard measure space (Y, μ) and let α be a free measure preserving action of G on a measurable subset $X \subset Y$ with measure 1. If their orbits satisfy $\alpha(G)(x) \subset \beta(\Gamma)(x)$ for* a.e. *$x \in X$, then $G \in \mathcal{S}$.*

In this proposition, Y can be an infinite standard measure space.

Proof of Theorem 9.1 from Proposition 9.2. As in the proof of Theorem 5.3, given an ergodic measure embedding Σ of G with Γ, we can find a ergodic measure preserving free G-action α on a probability space X and an ergodic measure preserving free Γ-action β on a standard measure space Y which are stably orbit equivalent with constant $[\beta : \alpha] = [\Gamma : G]_\Sigma$.

By multiplying $\mathbb{Z}/n\mathbb{Z}$ with Γ, we may assume that $[\beta : \alpha] \geq 1$, since $\Gamma \times \mathbb{Z}/n\mathbb{Z}$ is in the class $\mathcal{S}$ if and only if Γ is in the class $\mathcal{S}$. We may assume that X is a measurable subset of Y and $\mathcal{R}_\alpha = \mathcal{R}_\beta \cap (X \times X)$. By Proposition 9.2, $\Gamma \in \mathcal{S}$ implies $G \in \mathcal{S}$. □

To prove Proposition 9.2, we fix the G-action α and Γ-action β. We use the same notations given in Section 6 such as

$$\begin{aligned} L^\infty(Y), \ell_\infty(\Gamma), \ell_\infty(G) &\subset \mathcal{B}(L^2(\mathcal{R}_\beta, \nu)), \\ p &= \text{the unit of } L^\infty(X) \subset L^\infty(Y), \\ B = L^\infty(Y) \rtimes_{\mathrm{red}} \Gamma &\subset \mathcal{B}(L^2(\mathcal{R}_\beta, \nu)), \\ D &= C^*(B, JBJ). \end{aligned}$$

In this section, we define $\mathcal{G}$ as $\{\{1_\Gamma\}\}$, so the algebra K is

$$K = \overline{\bigcup_{\Gamma_0 \text{ finite}} e(\Gamma_0)\mathcal{B}(L^2(\mathcal{R}_\beta))e(\Gamma_0)}^{\|\cdot\|} = c_0(\Gamma)\mathcal{B}(L^2(\mathcal{R}_\beta))c_0(\Gamma).$$

The group G is also represented on $L^2(\mathcal{R}_\beta \cap (X \times Y))$. This representation generates the algebra which is isomorphic to the reduced group C^*-algebra of G. We denote by $C^*_\lambda(G)$ the C^*-algebra generated by these operators. We denote by P the orthogonal projection from $L^2(\mathcal{R}_\beta)$ onto $L^2(\mathcal{R}_\alpha)$.

Denote by e_Δ the projection corresponding to the diagonal subset of $\mathcal{R}_\beta$. For $\gamma \in \Gamma$, $g \in G$ and finite subsets $\Gamma_0 \subset \Gamma$, $G_0 \subset G$, we use the notations $e(\gamma), e(\Gamma_0)$ by

$$\begin{aligned} e(\gamma) &= Ju_\gamma J e_\Delta J u_\gamma^* J, & f(g) &= v_g e_\Delta v_g^* = v_g p e_\Delta v_g^*, \\ e(\Gamma_0) &= \sum_{\gamma \in \Gamma_0} e(\gamma), & f(G_0) &= \sum_{g \in G_0} f(g), \end{aligned}$$

The algebra B satisfies the Proposition 5.2. We have shown that $\widetilde{E} = (L^\infty Y \otimes JL^\infty Y J \otimes \ell_\infty \Gamma / c_0(\Gamma)) \rtimes_{\mathrm{full}} (\Gamma \times \Gamma)$ can be identified with $(L^\infty Y \otimes JL^\infty Y J \otimes \ell_\infty \Gamma / c_0(\Gamma)) \rtimes_{\mathrm{red}} (\Gamma \times \Gamma)$ and this algebra is nuclear.

We make use of the following characterization of $\mathcal{S}$. Proposition 15.2.3 and a variant of Lemma 15.1.4 in [BrOz] imply the following.

Proposition 9.3. *A countable group G is in $\mathcal{S}$ if and only if G is exact and there exists a contractive completely positive map $\Phi\colon C^*_\lambda(G)\otimes C^*_\rho(G)\to \mathcal{B}(\ell_2 G)$, satisfying*

$$\Phi(b\otimes c)-bc\in\mathcal{K}(\ell_2 G),\quad b\in C^*_\lambda(G), c\in C^*_\rho(G),$$

where $C^*_\lambda(G)$ and $C^*_\rho(G)$ are the C^*-algebras generated by the left and right regular representations, respectively.

Remark 9.4. By using the notion of weak exactness introduced in Kirchberg [Kir], we get that the exactness of Γ implies that of G. Indeed, the algebra $\mathcal{M}$ is weakly exact by the exactness of Γ. The subalgebra $L(G)\subset p\mathcal{M}p$ is also weakly exact. It follows that G is exact by Ozawa's theorem [Oz4].

We only have to show the existence of Φ in Proposition 9.3. However, lack of inclusion "$C^*_\lambda(G)\subset B$" requires some technical elaboration.

Lemma 9.5. *There exists a sequence $\{q_n\}_{n=1,2,\cdots}$ of projections in $L^\infty(X)$ satisfying*

1. *The sequence $\{q_n\}$ is increasing and strongly converges to p,*
2. *For any finite subset $G_0\subset G$ and n, there exists a finite subset $\Gamma_0\subset\Gamma$ satisfying $q_n f(G_0)\le e(\Gamma_0)$,*
3. *For any finite subset $\Gamma_0\subset\Gamma$ and n, there exists a finite subset $G_0\subset G$ satisfying $q_n e(\Gamma_0)P\le f(G_0)$.*

We note that projections $P, p, e(\gamma), f(g), (\gamma\in\Gamma, g\in G)$ are in the commutative von Neumann algebra $L^\infty(\mathcal{R}_\beta)\subset\mathcal{B}(L^2(\mathcal{R}_\beta))$. Every projection in $L^\infty(\mathcal{R}_\beta)$ which is less than $f(g)$ (resp. $pe(\gamma)$) is of the form $qf(g)$ (*resp.* $qe(\gamma)$) for some $q\in L^\infty(X)$. We also note that $qf(g)\le e(\Gamma_0)$ if and only if there exists a partition $\{q_\gamma\in L^\infty(X)\mid \gamma\in\Gamma_0\}$ of q with $qv_g=\sum_{\gamma\in\Gamma_0}q_\gamma u_\gamma$.

Proof. We label the elements of G as $\{g_1,g_2,\ldots\}=G$. For any g_k, the projection $e(\Gamma_0)f(g_k)$ can be written as $Q(g_k,\Gamma_0)f(g_k)$ by some projection $Q(g_k,\Gamma_0)\in L^\infty(X)$. The net of projections $\{e(\Gamma_0)f(g_k)\mid \Gamma_0\subset\Gamma \text{ finite}\}$ strongly converges to $f(g_k)$. For any natural number n, there exists a finite subset $\Gamma_{k,n}\subset\Gamma$ with $\mathrm{tr}(Q(g_k,\Gamma_{k,n}))\ge 1-2^{-(n+k)}$. The projections $Q_n=\bigwedge_{k=1}^\infty Q(g_k,\Gamma_{k,n})$ satisfy $\mathrm{tr}(Q_n)\ge 1-2^{-n}$ and

$$Q_n f(g_k)\le Q(g_k,\Gamma_{k,n})f(g_k)=e(\Gamma_{k,n})f(g_k)\le e(\Gamma_{k,n}).$$

Let $\{q_n\}$ be the increasing sequence of projections $\{\bigvee_{l=1}^n Q_l\}$. We have $\mathrm{tr}(q_n)\ge 1-2^{-n}$ and

$$q_n f(g_k)\le e\left(\bigcup_{l=1}^n \Gamma_{k,l}\right),\quad g_k\in G.$$

It turns out that the sequence $\{q_n\}$ satisfies (1) and (2).

By a similar technique, we get a sequence $\{p_n\}$ with (1) and (3), since

$$\operatorname{str}\lim_{G_0} f(G_0)e(\gamma)=e(\gamma)P,\quad \gamma\in\Gamma.$$

Taking products $\{p_nq_n\}$, we get a sequence which satisfies (1), (2) and (3) at the same time. ∎

The Hilbert space $\ell_2 G$ embeds into $L^2(\mathcal{R}_\alpha)$ by the isometry

$$\ell_2 G \ni \delta_g \to v_g \xi_\Delta = J v_g^* J \xi_\Delta \in L^2(\mathcal{R}_\alpha),$$

where the L^2-function ξ_Δ is the characteristic function of the diagonal subset of $\mathcal{R}_\alpha$. We regard $\ell_2 G$ as a subspace of $L^2(\mathcal{R}_\alpha)$ by this map. The subspace $\ell_2 G$ is invariant under the action of $C^*_\lambda G$ and $JC^*_\lambda GJ$.

Lemma 9.6. *For a projection $q \in L^\infty(X)$, the following inequality on operator norm holds true: $\|(1 - qJqJ)|_{\ell_2 G}\| \leq (2 - 2\mathrm{tr}(q))^{1/2}$.*

Proof. It suffices to show $\|\eta - qJqJ\eta\|^2 \leq (2 - 2\mathrm{tr}(q))\|\eta\|^2$ for any vector $\eta \in \ell_2 G$. Since $qJqJP = PqJqJ$ and $\eta = P\eta$, we get

$$\begin{aligned} \|\eta - qJqJ\eta\|^2 &= \sum_{g\in G} \|f(g)(\eta - qJqJ\eta)\|^2 = \sum_{g\in G} \|f(g)\eta - qJqJf(g)\eta\|^2, \\ \|\eta\|^2 &= \sum_{g\in G} \|f(g)\eta\|^2, \end{aligned}$$

The claim reduces to the inequality $\|f(g)\eta - qJqJf(g)\eta\|^2 \leq (2 - 2\mathrm{tr}(q))\|f(g)\eta\|^2$.

We note that η takes a constant value $\eta(g)$ on the set $\{(\alpha(g)(x), x) \in \mathcal{R}_\alpha \mid x \in X\}$. By a direct computation, we get

$$\|f(g)\eta\|^2 = \int_{\mathcal{R}_\alpha} |(f(g)\eta)(y, x)|^2 d\nu = \int_{x\in X} |\eta(\alpha(g)(x), x)|^2 d\mu = |\eta(g)|^2.$$

Let $X_q \subset X$ be a measurable subset with $\chi(X_q) = p - q$. The measure of subset $X_0 = \{x \in X \mid x \in X_q \text{ or } \alpha(g)(x) \in X_q\}$ satisfies $\nu(X_0) \leq 2 - 2\mathrm{tr}(q)$. We get

$$\|f(g)\eta - qJqJf(g)\eta\|^2 = \int_{x\in X_0} |\eta(\alpha(g)(x), x)|^2 d\mu = \mu(X_0)|\eta(g)|^2.$$

Our claim is confirmed. □

We finish the proof of Proposition 9.2.

Proposition 9.2. We will show the existence of Φ in Proposition 9.3. We consider that $C^*_\lambda(G)$ is a subalgebra of $\mathcal{B}(pL^2(\mathcal{R}_\beta))$ and $C^*_\rho(G)$ is $JC^*_\lambda(G)J$. By P_0 we denote the orthogonal projection from $L^2(\mathcal{R}_\beta)$ onto $\ell_2 G$.

Let $\{q_n\}$ be the sequence satisfying Lemma 9.5. and let B_0 be the C^*-algebra generated by p and $\bigcup_n q_n C^*_\lambda(G) q_n$. The condition (2) in Lemma 9.5 means $B_0 \subset B$. We recall that $B_0 \otimes JB_0J$ is separable and that $\widetilde{E} = (L^\infty Y \otimes JL^\infty YJ \otimes \ell_\infty\Gamma/c_0(\Gamma)) \rtimes_{\mathrm{red}} (\Gamma \times \Gamma)$ in the proof of Proposition 5.2 is nuclear. By the Choi–Effros lifting theorem [ChEf], there exists a contractive completely positive lifting $\Psi\colon B_0 \otimes JB_0J \to D + K$ for $\widetilde{\Psi}|_{B_0\otimes JB_0J}$. We define contractive completely positive maps $\Phi_n\colon C^*_\lambda(G) \otimes C^*_\rho(G) \to \mathcal{B}(\ell_2 G)$ by

$$\Phi_n(b \otimes JcJ) = P_0 Q_n \Psi(q_n b q_n \otimes Jq_n c q_n J) Q_n P_0,$$

where $Q_n = q_n J q_n J$. By the condition (3) in Lemma 9.5, we get $P_0 Q_n K Q_n P_0 \subset \mathcal{K}(\ell_2 G)$. The element $\Phi_n(b \otimes JcJ)$ is in

$$P_0 Q_n (bJcJ + K) Q_n P_0 \subset P_0 Q_n bJcJ Q_n P_0 + \mathcal{K}(\ell_2 G).$$

The sequence $\{P_0Q_nbJcJQ_nP_0+\mathcal{K}(\ell_2G)\} \subset \mathcal{B}(\ell_2G)/\mathcal{K}(\ell_2G)$ converges to $P_0bJcJP_0+\mathcal{K}(\ell_2G)$, by the inequality

$$\begin{aligned} & \|P_0bJcJP_0 - P_0Q_nbJcJQ_nP_0\| \\ \leq\ & \|P_0(1-Q_n)bJcJP_0\| + \|P_0Q_nbJcJ(1-Q_n)P_0\| \\ \leq\ & 2\|(1-Q_n)P_0\|\|b\|\|c\| \\ \leq\ & 2(2-2\mathrm{tr}(q_n))^{1/2}\|b\|\|c\|. \end{aligned}$$

It follows that the natural $*$-homomorphism $\widetilde{\Phi}\colon C^*_\lambda(G)\otimes C^*_\rho(G) \to \mathcal{B}(\ell_2G)/\mathcal{K}(\ell_2G)$ is given and $\widetilde{\Phi}$ is a limit of liftable maps. By Theorem 6 of [Ar], there exists a contractive completely positive lifting Φ for $\widetilde{\Phi}$. □

As a consequence of Proposition 9.2, we get the following indecomposability of an equivalence relation given by a group in the class $\mathcal{S}$.

Corollary 9.7. *Let Γ be a countable group and let $H \subset G$ be an inclusion of countable groups. Suppose that $\Gamma \in \mathcal{S}$ and that the centralizer $Z_G(H)$ is non-amenable. Let β be a free measure preserving action of Γ on a standard measure space (Y,μ) and let α be a free measure preserving action of G on a measurable subset $X \subset Y$ with measure* 1. *If the orbits satisfy $\alpha(G)(x) \subset \beta(\Gamma)(x)$ for* a.e. *$x \in X$, then H is finite.*

Proof. The class $\mathcal{S}$ has the following property: If $G \in \mathcal{S}$ and $Z_G(H)$ is non-amenable, then H is finite. □

In particular, the group G is not a direct product group of an infinite group and a non-amenable group. Word hyperbolic groups are typical examples of $\mathcal{S}$ groups. Adams [Ad] showed a measurable indecomposability of non-amenable word hyperbolic groups. The above corollary covers some part of the theorem of Adams.

The class $\mathcal{C}$ defined in [MoSh] also contains non-amenable word-hyperbolic groups. A group $G \in \mathcal{C}$ satisfies an indecomposability property, that is, G has no infinite normal amenable subgroup. On the other hand, a non-amenable class $\mathcal{S}$ group can have an infinite normal amenable subgroup (for example, $\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}) \in \mathcal{S}$ [Oz5]).

Acknowledgment . *The main part of his paper was written during the author's stay in University of California, Los Angeles. He is supported by JSPS Research Fellowships for Young Scientists. The author is grateful to Professor Sorin Popa, Professor Narutaka Ozawa for their encouragement and fruitful conversations. I appreciate the feedback offered by Adrian Ioana and Cyril Houdayer. He would like to express my gratitude to Professor Yasuyuki Kawahigashi for his constant support. Special thanks to Maki and our family.*

References

[Ad] S. Adams, *Indecomposability of equivalence relations generated by word hyperbolic groups*, Topology **33** (1994), no. 4, 785–798.

[AlvGab] A. Alvarez and D. Gaboriau, *Free products, Orbit Equivalence and Measure Equivalence Rigidity*, preprint, arXiv:0806.2788.

[AD] C. Anantharaman-Delaroche, *Systèmes dynamiques non commutatifs et moyennabilité*, Math. Ann. **279** (1987), no. 2, 297–315.

[Ar] W. Arveson, *Notes on extensions of C^*-algebras*, Duke Math. J. **44** (1977), no. 2, 329–355.

[BrOz] N. P. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, **88**. American Mathematical Society, Providence, RI, 2008.

[ChHo] I. Chifan and C. Houdayer, *Prime factors and amalgamated free products*, preprint, arXiv:0805.1566.

[ChIo] I. Chifan and A. Ioana, *Ergodic Subequivalence Relations Induced by a Bernoulli Action*, preprint, arXiv:0802.2353.

[ChEf] M.D. Choi and E.D. Effros, *The completely positive lifting problem for C^*-algebras,* Ann. of Math. (2) **104** (1976), no. 3, 585–609.

[ConFeWe] A. Connes, J. Feldman and B. Weiss, *An amenable equivalence relation is generated by a single transformation,* Ergodic Theory Dynam. Systems **1** (1981), no. 4, 431–450 (1982).

[CowZi] M. Cowling and R. Zimmer, *Actions of lattices in Sp(1, n),* Ergodic Theory Dynam. Systems, **9** (1989),221–237.

[FeMoo1] J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. I,* Trans. Amer. Math. Soc. **234** (1977), no. 2, 289–324.

[FeMoo2] J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. II,* Trans. Amer. Math. Soc. **234** (1977), no. 2, 325–359.

[Fu1] A. Furman, *Gromov's measure equivalence and rigidity of higher rank lattices*, Ann. of Math. (2) **150** (1999), no. 3, 1059–1081.

[Fu2] A. Furman, *Orbit equivalence rigidity*, Ann. of Math. (2) **150** (1999), no. 3, 1059–1081.

[Fu3] A. Furman, *A survey of Measured Group Theory*, preprint, arXiv:0901.0678.

[Ga] D. Gaboriau, *Invariants l^2 de relations d'equivalence et de groupes,* Publ. Math. Inst. Hautes Etudes Sci. **95** (2002), 93–150.

[Gr] M. Gromov, *Asymptotic invariants of infinite groups. Geometric group theory*, Vol. 2, 1–295, London Math. Soc. Lecture Note Ser., **182**, Cambridge Univ. Press, Cambridge, 1993.

[HiGu] N. Higson and E. Guentner, *Group C^*-algebras and K-theory*, Noncommutative geometry, 137–251, Lecture Notes in Math., **1831**, Springer, Berlin, 2004.

[Hj] G. Hjorth, *A lemma for cost attained,* Ann. Pure Appl. Logic **143** (2006), no. 1-3, 87–102.

[Io] A. Ioana, *Rigidity results for wreath product* II_1 *factors.* J. Funct. Anal. **252** (2007), no. 2, 763–791.

[IoPePo] A. Ioana, J. Peterson and S. Popa, *Amalgamated free products of weakly rigid factors and calculation of their symmetry groups*, Acta Math. **200** (2008), no. 1, 85–153.

[Jo] P. Jolissaint, *The Haagerup property for measure-preserving standard equivalence relations*, Ergodic Theory Dynam. Systems, **25**(2005), 161–174.

[Kid] Y. Kida, *Measure equivalence rigidity of the mapping class group*, to appear in Ann. of Math.

[Kir] E. Kirchberg, *Commutants of unitaries in UHF algebras and functorial properties of exactness*, J. Reine Angew. Math. **452** (1994), 39–77.

[MoSh] N. Monod and Y. Shalom, *Orbit equivalence rigidity and bounded cohomology,* Ann. of Math. (2) **164** (2006), no. 3, 825–878.

[MvN] F.J. Murray and J. von Neumann, *On rings of operators,* Ann. of Math. (2) **37** (1936), 116-229

[OrWe] D. S. Ornstein and B. Weiss, *Ergodic theory of amenable group actions. I. The Rohlin lemma,* Bull. Amer. Math. Soc. (N.S.) **2** (1980), no. 1, 161–164.

[Oz1] N. Ozawa, *Amenable actions and exactness for discrete groups*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), no. 8, 691–695.

[Oz2] N. Ozawa, *Solid von Neumann algebras,* Acta Math. **192** (2004), no. 1, 111–117.

[Oz3] N. Ozawa, *A Kurosh-type theorem for type* II_1 *factors,* Int. Math. Res. Not. (2006), Art. ID 97560, 21 pp.

[Oz4] N. Ozawa, *Weakly exact von Neumann algebras*, J. Math. Soc. Japan **59** (2007), no. 4, 985–991.

[Oz5] N. Ozawa, *An example of a solid von Neumann algebra*, preprint, arXiv:0804.0288.

[OzPo] N. Ozawa and S. Popa, *Some prime factorization results for type* II_1 *factors,* Invent. Math. **156** (2004), no. 2, 223–234.

[PemPer] R. Pemantle and Y. Peres, *Nonamenable products are not treeable,* Israel J. Math. **118** (2000), 147–155.

[Po1] S. Popa, *Strong rigidity of* II_1 *factors arising from malleable actions of w-rigid groups. I*, Invent. Math. **165** (2006), no. 2, 369–408.
[Po2] S. Popa, *Strong rigidity of* II_1 *factors arising from malleable actions of w-rigid groups. II*, Invent. Math. **165** (2006), no. 2, 409–451.
[Po3] S. Popa, *On a class of type* II_1 *factors with Betti numbers invariants,* Ann. of Math. (2) **163** (2006), no. 3, 809–899.
[Po4] S. Popa, *Cocycle and orbit equivalence superrigidity for malleable actions of w-rigid groups*, Invent. Math. **170** (2007), no. 2, 243–295.
[Po5] S. Popa, *On the superrigidity of malleable actions with spectral gap*, J. Amer. Math. Soc. **21** (2008), no. 4, 981–1000.
[Sh] Y. Shalom, *Measurable group theory.* European Congress of Mathematics, 391–423, Eur. Math. Soc., Zürich, 2005.
[Sk] G. Skandalis, *Une notion de nucléarité en K-théorie (d'après J. Cuntz), K*-Theory **1** (1988), no. 6, 549–573.
[Tak] M. Takesaki, *On the cross-norm of the direct product of* C^**-algebras,* Tôhoku Math. J. (2) **16** (1964), 111–122.
[Vaes] S. Vaes, *Rigidity results for Bernoulli actions and their von Neumann algebras (after Sorin Popa),* Séminaire Bourbaki. Vol. 2005/2006. Astérisque No. **311** (2007), No. 961, viii, 237–294.

Graduate school of Mathematical Sciences, University of Tokyo, Komaba, Tokyo, 153-8914, Japan
E-mail address: hiroki@ms.u-tokyo.ac.jp