

## *On the Sparre Andersen Transformation for Multidimensional Brownian Bridge*

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**Abstract.** A family of law-preserving path transformations of  $d$ -dimensional Brownian bridge (pinned Brownian motion),  $d \geq 1$ , is constructed. This generalizes a result of one-dimensional cases obtained first by Embrechts, Rogers and Yor. Our approach and theirs are, however, completely different from each other.

### 1. Introduction and Statement of the Result

The study of law-preserving path transformations of one-dimensional Brownian motion is currently a popular subject. Karatzas-Shreve [7] and Bertoin [3] constructed a transformation connecting local minimum and excursions in half-lines. A generalization to a larger class of path transformations (Corollary 1.2 of the present paper) was recently obtained first by Embrechts-Rogers-Yor [5], whose approach is based on Brownian excursion theory. Also, the second author (Takaoka [9]; an English translated version is [10]) independently gives another proof of the same result; the proof is obtained by taking the continuous-time limit of the idea lying behind Richards' proof of Sparre Andersen's theorem on discrete-time processes (see Theorem 2.1 and its proof below).

In this paper, we give a further extension along the lines of [9] [10], even to multidimensional cases:

**THEOREM 1.1.** *Fix  $d \in \mathbf{N}$ ,  $A \in \mathcal{B}(\mathbf{R}^d)$  and  $b \in \mathbf{R}^d$ , where  $\mathcal{B}(\mathbf{R}^d)$  is the Borel  $\sigma$ -algebra of  $\mathbf{R}^d$ . Let  $(B_t)_{t \in [0,1]}$  be a  $d$ -dimensional Brownian bridge from 0 to  $b$  on a certain probability space  $(\Omega, \mathcal{F}, P)$ , i.e., a Brownian motion starting from the origin and conditioned to be at  $b$  at time 1. Let  $(Z_t)_{t \in [0,1]}$  denote its time-reversed process:*

$$Z_t \stackrel{\text{def}}{=} b - B_{1-t} \quad \text{for } t \in [0, 1].$$

For  $t \in [0, 1]$ , define:

$$\begin{aligned} \Gamma_+(t) &\stackrel{\text{def}}{=} \int_0^t 1_A(B_s) ds; \\ \Gamma_-(t) &\stackrel{\text{def}}{=} t - \Gamma_+(t); \\ \Gamma_{\pm}^{-1}(t) &\stackrel{\text{def}}{=} \inf\{s \in [0, 1] \mid \Gamma_{\pm}(s) \geq t \wedge \Gamma_{\pm}(1)\}; \\ Y_+(t) &\stackrel{\text{def}}{=} \int_{1-t}^1 1_A(b - Z_s) dZ_s; \\ Y_-(t) &\stackrel{\text{def}}{=} \int_{1-t}^1 1_{A^c}(b - Z_s) dZ_s; \\ B_{\pm}(t) &\stackrel{\text{def}}{=} Y_{\pm}(\Gamma_{\pm}^{-1}(t)); \end{aligned}$$

where we take the integrals  $Y_{\pm}(t)$  with respect to the backward filtration, i.e., the filtration generated by  $Z$  rather than by  $B$ . Furthermore,

$$\tilde{B}_t \stackrel{\text{def}}{=} \begin{cases} B_+(\Gamma_+(1)) - B_+(\Gamma_+(1) - t), & \text{if } t \in [0, \Gamma_+(1)]; \\ B_+(\Gamma_+(1)) + B_-(t - \Gamma_+(1)), & \text{if } t \in (\Gamma_+(1), 1]. \end{cases}$$

Then we have

$$(\tilde{B}_t)_{t \in [0,1]} \stackrel{(d)}{=} (B_t)_{t \in [0,1]}.$$

REMARKS.

(i) Following [9] [10], we propose calling this transformation the **Sparre Andersen transformation** of  $(B_t)_{t \in [0,1]}$  with respect to  $A \in \mathcal{B}(\mathbf{R}^d)$ , because, as we will see in more detail in Section 2, the starting point of our study is a combinatorial theorem of E. Sparre Andersen [1] on sums of exchangeable random variables.

(ii) Feller’s proof [6] of Sparre Andersen’s theorem has been used to derive some continuous-time properties, e.g. Bertoin [4]. It should be noted, however, that Richards’ proof of Sparre Andersen’s theorem, utilized in this paper (see Section 2), covers a wider variety of cases and thus offers a unified way of viewing the whole matter.

(iii) If  $A = \emptyset$  then this transformation is the identity; if  $A = \mathbf{R}^d$  it is the time-reversal transformation. Moreover, if we denote by  $(\tilde{B}_t^A)_{t \in [0,1]}$  the resulting process with respect to  $A$ , then  $\tilde{B}^A$  and  $\tilde{B}^{A^c}$  are time reversals of each other for any  $A \in \mathcal{B}(\mathbf{R}^d)$ .

In the special case where  $d = 1$  and  $A = (a, \infty)$ ,  $a \in \mathbf{R}$ , then Tanaka's formula recovers the above mentioned result of Embrechts-Rogers-Yor [5]:

**COROLLARY 1.2.** *Let  $(B_t)_{t \in [0,1]}$  be a one-dimensional standard Brownian motion starting from the origin on a certain probability space  $(\Omega, \mathcal{F}, P)$ . Fix  $a \in \mathbf{R}$ . For  $t \in [0, 1]$ , define:*

$$\begin{aligned} \ell_t^a &\stackrel{\text{def}}{=} \text{its local time at } a \text{ up to time } t; \\ \Gamma_+^a(t) &\stackrel{\text{def}}{=} \int_0^t 1_{(a, \infty)}(B_s) ds; \\ \Gamma_-^a(t) &\stackrel{\text{def}}{=} t - \Gamma_+^a(t); \\ \{\Gamma_{\pm}^a\}^{-1}(t) &\stackrel{\text{def}}{=} \inf\{s \in [0, 1] \mid \Gamma_{\pm}^a(s) \geq t \wedge \Gamma_{\pm}^a(1)\}; \\ Y_+^a(t) &\stackrel{\text{def}}{=} (B_t \vee a) - (a \vee 0) + \frac{\ell_t}{2}; \\ Y_-^a(t) &\stackrel{\text{def}}{=} (B_t \wedge a) - (a \wedge 0) - \frac{\ell_t}{2}; \\ B_{\pm}^a(t) &\stackrel{\text{def}}{=} Y_{\pm}^a(\{\Gamma_{\pm}^a\}^{-1}(t)). \end{aligned}$$

(Note that  $B_{\pm}^a(0) = 0$  a.s.) Furthermore

$$\tilde{B}_t^a \stackrel{\text{def}}{=} \begin{cases} B_+^a(\Gamma_+^a(1)) - B_+^a(\Gamma_+^a(1) - t), & \text{if } t \in [0, \Gamma_+^a(1)]; \\ B_+^a(\Gamma_+^a(1)) + B_-^a(t - \Gamma_+^a(1)), & \text{if } t \in (\Gamma_+^a(1), 1]. \end{cases}$$

Then the process  $(\tilde{B}_t^a)_{t \in [0,1]}$  is also a Brownian motion starting from the origin.

The rest of this paper is organized as follows. Section 2 explains the underlying discrete-time argument. In Section 3, we prove Theorem 1.1.

## 2. Path Transformation for Pinned Random Walk

As mentioned above in the Introduction, our starting point is Sparre Andersen's theorem:

**THEOREM 2.1** (Sparre Andersen [1]). *Let  $(S_k)_{k=0}^n$  be an arbitrary one-dimensional process starting from the origin and with exchangeable increments, i.e., the joint distribution of the  $n$  random variables*

$$S_1 - S_0, S_2 - S_1, \dots, S_n - S_{n-1}$$

is symmetric with respect to the  $n$  arguments. Then the two functionals

- (i)  $\min \{k \in \{0, 1, \dots, n\}; S_k = \max_{0 \leq j \leq n} S_j\}$ ,
- (ii)  $\#\{k \in \{1, \dots, n\}; S_k > 0\}$

are identically distributed.

Richards' proof (unpublished; see Baxter [2]) of Theorem 2.1 involves a path transformation of pinned random walk which we will use in the next section. The key idea is the following

LEMMA 2.2 (Richards; Baxter [2]). (i) Fix  $d \in \mathbf{N}$  and  $A \in \mathcal{B}(\mathbf{R}^d)$ . For each  $(x_1, x_2, \dots, x_n) \in (\mathbf{R}^d)^n$ , form a new arrangement of the  $x_k$ 's by placing first in decreasing order of  $k$  the terms  $x_k$  for which  $s_k \in A$  and then (afterwards) in increasing order of  $k$  the  $x_k$  for which  $s_k \notin A$ , where  $s_0 \stackrel{\text{def}}{=} 0$  and  $s_k \stackrel{\text{def}}{=} \sum_{j=1}^k x_j$  for  $k = 1, \dots, n$ . Denote this new arrangement by  $(\tilde{x}_1, \dots, \tilde{x}_n)$ . Then the transformation

$$\begin{aligned} \theta_A : \quad (\mathbf{R}^d)^n &\longrightarrow (\mathbf{R}^d)^n \\ (x_1, \dots, x_n) &\longmapsto (\tilde{x}_1, \dots, \tilde{x}_n) \end{aligned}$$

is one-to-one and onto. Furthermore, if  $\mu$  is an exchangeable measure on  $((\mathbf{R}^d)^n, \mathcal{B}((\mathbf{R}^d)^n))$ , i.e., if

$$\forall \sigma \in \mathfrak{S}_n, \forall B \in \mathcal{B}((\mathbf{R}^d)^n), \quad \mu[\sigma(B)] = \mu[B]$$

with  $\mathfrak{S}_n$  the symmetric group of order  $n$ , then

$$\forall B \in \mathcal{B}((\mathbf{R}^d)^n), \quad \mu[\theta_A(B)] = \mu[B].$$

(ii) Let  $(S_k)_{k=0}^n$  be a  $d$ -dimensional process starting from the origin and with exchangeable increments. Fix  $A \in \mathcal{B}(\mathbf{R}^d)$  and define

$$\begin{aligned} X_k &\stackrel{\text{def}}{=} S_k - S_{k-1} \quad \text{for } k = 1, \dots, n; \\ (\tilde{X}_1(\omega), \dots, \tilde{X}_n(\omega)) &\stackrel{\text{def}}{=} \theta_A(X_1(\omega), \dots, X_n(\omega)) \quad \text{for } \omega \in \Omega; \\ \tilde{S}_0 &\stackrel{\text{def}}{=} 0; \\ \tilde{S}_k &\stackrel{\text{def}}{=} \sum_{j=1}^k \tilde{X}_j \quad \text{for } k = 1, \dots, n. \end{aligned}$$

Then

$$(S_k)_{k=0}^n \stackrel{(d)}{=} (\tilde{S}_k)_{k=0}^n.$$

PROOF OF LEMMA 2.2. (i) It is straightforward to check that  $\theta_A$  is one-to-one and onto. Next, let  $\Lambda \stackrel{\text{def}}{=} \{0, 1\}^n$ . For  $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda$  define

$$C_\lambda \stackrel{\text{def}}{=} \{(x_1, \dots, x_n) \in (\mathbf{R}^d)^n \mid s_k \in A \text{ if } \lambda_k = 1, \\ s_k \notin A \text{ if } \lambda_k = 0, k = 1, 2, \dots, n\}.$$

Then it is clear that

$$C_\lambda \cap C_{\lambda'} = \emptyset \text{ if } \lambda \neq \lambda', \\ \bigcup_{\lambda \in \Lambda} C_\lambda = (\mathbf{R}^d)^n.$$

In addition, for each  $\lambda \in \Lambda$  there exists a  $\sigma_\lambda \in \mathfrak{S}_n$  such that

$$\forall B \subset (\mathbf{R}^d)^n, \quad \theta_A(B \cap C_\lambda) = \sigma_\lambda(B \cap C_\lambda).$$

Therefore, for any  $B \in \mathcal{B}((\mathbf{R}^d)^n)$ :

$$\begin{aligned} \mu[\theta_A(B)] &= \sum_{\lambda \in \Lambda} \mu[\theta_A(B \cap C_\lambda)] \\ &= \sum_{\lambda} \mu[\sigma_\lambda(B \cap C_\lambda)] \\ &= \sum_{\lambda} \mu[B \cap C_\lambda] \quad \text{by exchangeability} \\ &= \mu[B]. \end{aligned}$$

(ii) This is an immediate consequence of (i).  $\square$

REMARKS. (i) We can apply the above argument to all  $d$ -dimensional random walks and pinned random walks,  $d \geq 1$ .

(ii) We propose that  $(\tilde{S}_k)_{k=0}^n$  be called the *Sparre Andersen transformation* of  $(S_k)_{k=0}^n$  with respect to  $A \in \mathcal{B}(\mathbf{R}^d)$ .

PROOF OF THEOREM 2.1 (Richards). If  $d = 1$  and  $A = (0, \infty)$ , then  $(S_k)_{k=0}^n$  and its transformation  $(\tilde{S}_k)_{k=0}^n$  have the following relation:

$$\#\{k \in \{1, \dots, n\}; S_k > 0\} = \min\{k \in \{0, 1, \dots, n\}; \tilde{S}_k = \max_{0 \leq j \leq n} \tilde{S}_j\} \quad \text{a.s.}$$

The proof of Theorem 2.1 is therefore complete.  $\square$

Finally, a little closer look at this transformation immediately gives the following property, the proof of which we omit.

PROPOSITION 2.3. *With the notations of Lemma 2.2(ii) assumed, define for  $k = 0, \dots, n$  :*

$$\begin{aligned} \Gamma_+^S(k) &\stackrel{\text{def}}{=} \#\{j \in \{1, \dots, k\}; S_j \in A\}; & (\Gamma_+^S(0) \stackrel{\text{def}}{=} 0) \\ \Gamma_-^S(k) &\stackrel{\text{def}}{=} k - \Gamma_+^S(k); \\ \{\Gamma_{\pm}^S\}^{-1}(k) &\stackrel{\text{def}}{=} \min \{j \in \{0, 1, \dots, n\}; \Gamma_{\pm}^S(j) \geq k \wedge \Gamma_{\pm}^S(n)\}; \\ Y_+^S(k) &\stackrel{\text{def}}{=} \sum_{j=1}^k 1_A(S_j) (S_j - S_{j-1}); \\ Y_-^S(k) &\stackrel{\text{def}}{=} \sum_{j=1}^k 1_{A^c}(S_j) (S_j - S_{j-1}); \\ S_{\pm}(k) &\stackrel{\text{def}}{=} Y_{\pm}^S(\{\Gamma_{\pm}^S\}^{-1}(k)). \end{aligned}$$

Then, a.s.,

$$\tilde{S}_k = \begin{cases} S_+(\Gamma_+^S(n)) - S_+(\Gamma_+^S(n) - k), & \text{if } 0 \leq k \leq \Gamma_+^S(n); \\ S_+(\Gamma_+^S(n)) + S_-(k - \Gamma_+^S(n)), & \text{if } \Gamma_+^S(n) < k \leq n. \end{cases}$$

### 3. Proof of the Main Theorem

The idea of the proof of Theorem 1.1 is to show that our path transformation of Brownian bridge is the continuous-time limit of the Sparre Andersen transformation for pinned random walk. A quite similar method was employed in the proof of the main theorem of [9] [10] (Corollary 1.2 of the present paper). It should be noted, however, that the way we approximate the paths of  $(B_t)_{t \in [0,1]}$  with pinned random walk here is not the same as in [9] [10].

DEFINITION 3.1. Let

$$S_k^{(n)} \stackrel{\text{def}}{=} B\left(\frac{k}{2^n}\right) \quad \text{for } k = 0, 1, \dots, 2^n, \quad n \in \mathbf{N}.$$

Clearly  $(S_k^{(n)})_{k=0}^{2^n}$  is a  $d$ -dimensional random walk pinned at  $b$ . Also, let  $(\tilde{S}_k^{(n)})_{k=0}^{2^n}$  denote its Sparre Andersen transformation with respect to  $A$ .

DEFINITION 3.2. For  $t \in [0, 1]$ , define:

$$\begin{aligned}\Gamma_+^{(n)}(t) &\stackrel{\text{def}}{=} \int_0^t 1_A \left( B \left( \frac{[2^n s + 1]}{2^n} \right) \right) ds; \\ \Gamma_-^{(n)}(t) &\stackrel{\text{def}}{=} t - \Gamma_+^{(n)}(t); \\ \{\Gamma_{\pm}^{(n)}\}^{-1}(t) &\stackrel{\text{def}}{=} \inf \{ s \in [0, 1] \mid \Gamma_{\pm}^{(n)}(s) \geq t \wedge \Gamma_{\pm}^{(n)}(1) \}; \\ Y_+^{(n)}(t) &\stackrel{\text{def}}{=} \sum_{k=1}^{2^n} 1_A \left( B \left( \frac{k}{2^n} \right) \right) \left( B(t \wedge \frac{k}{2^n}) - B(t \wedge \frac{k-1}{2^n}) \right); \\ Y_-^{(n)}(t) &\stackrel{\text{def}}{=} \sum_{k=1}^{2^n} 1_{A^c} \left( B \left( \frac{k}{2^n} \right) \right) \left( B(t \wedge \frac{k}{2^n}) - B(t \wedge \frac{k-1}{2^n}) \right); \\ B_{\pm}^{(n)}(t) &\stackrel{\text{def}}{=} Y_{\pm}^{(n)}(\{\Gamma_{\pm}^{(n)}\}^{-1}(t)).\end{aligned}$$

Furthermore,

$$\tilde{B}_t^{(n)} \stackrel{\text{def}}{=} \begin{cases} B_+^{(n)}(\Gamma_+^{(n)}(1)) - B_+^{(n)}(\Gamma_+^{(n)}(1) - t), & \text{if } t \in [0, \Gamma_+^{(n)}(1)]; \\ B_+^{(n)}(\Gamma_+^{(n)}(1)) + B_-^{(n)}(t - \Gamma_+^{(n)}(1)), & \text{if } t \in (\Gamma_+^{(n)}(1), 1]. \end{cases}$$

PROPOSITION 3.3. We have

$$\tilde{S}_k^{(n)} = \tilde{B}^{(n)}\left(\frac{k}{2^n}\right) \quad k = 0, 1, \dots, 2^n, \quad a.s.,$$

PROOF. This is a straightforward consequence of Proposition 2.3 and the following fact:

$$\frac{1}{2^n} \#\{j \in \{1, \dots, k\} \mid S_j^{(n)} \in A\} = \Gamma_+^{(n)}\left(\frac{k}{2^n}\right), \quad k = 0, 1, \dots, 2^n. \quad \square$$

We shall use the next lemma to prove Proposition 3.5 below.

LEMMA 3.4. For any bounded Borel measurable function  $f : \mathbf{R}^d \rightarrow \mathbf{R}$  we have:

$$\forall t \in (0, 1), \quad \lim_{\epsilon \downarrow 0} E[|f(B_{t+\epsilon}) - f(B_t)| \mid \mathcal{F}_t] = 0 \quad a.s.,$$

where  $(\mathcal{F}_t)_{t \in [0,1]}$  is the filtration generated by  $B$ .

PROOF. Let  $p(t; x, y)$  denote the transition density function of  $d$ -dimensional Brownian motion. Then, for any fixed  $t \in (0, 1)$  and  $x \in \mathbf{R}^d$  there exists a positive constant  $C_{x,t}$  such that

$$\begin{aligned} P[B_{t+\epsilon} \in dy \mid B_t = x] / dy &= \frac{p(\epsilon; x, y) p(1-t-\epsilon; y, b)}{p(1-t; x, b)} \\ &\leq C_{x,t} p(\epsilon; x, y) \end{aligned}$$

for all  $y \in \mathbf{R}^d$  and all sufficiently small  $\epsilon > 0$ . Consequently, for each  $a > 0$  we have

$$\begin{aligned} &E[|f(B_{t+\epsilon}) - f(B_t)| \mid B_t = x] \\ &\leq C_{x,t} \left\{ (2\pi\epsilon)^{-d/2} \int_{|y-x| < a\sqrt{\epsilon}} |f(y) - f(x)| dy \right. \\ &\quad \left. + 2\|f\|_\infty \int_{|y-x| \geq a\sqrt{\epsilon}} p(\epsilon; x, y) dy \right\} \\ &= C_{x,t} \left\{ (2\pi\epsilon)^{-d/2} \int_{|y-x| < a\sqrt{\epsilon}} |f(y) - f(x)| dy \right. \\ &\quad \left. + 2\|f\|_\infty \int_{|y| \geq a} p(1; 0, y) dy \right\}. \end{aligned}$$

Furthermore, the Lebesgue differentiation theorem (see e.g. Stroock [8] §5.3) states that, for Lebesgue-almost every  $x \in \mathbf{R}^d$ :

$$\lim_{\epsilon \downarrow 0} \epsilon^{-d} \int_{|y-x| < \epsilon} |f(y) - f(x)| dy = 0,$$

and hence

$$\overline{\lim}_{\epsilon \downarrow 0} E[|f(B_{t+\epsilon}) - f(B_t)| \mid B_t = x] \leq 2C_{x,t} \|f\|_\infty \int_{|y| \geq a} p(1; 0, y) dy.$$

Since  $a$  can be made arbitrarily large, we conclude

$$\lim_{\epsilon \downarrow 0} E[|f(B_{t+\epsilon}) - f(B_t)| \mid B_t = x] = 0. \quad \square$$



PROPOSITION 3.5. *The following assertions hold:*

$$(i) \quad \lim_{n \uparrow \infty} E \left[ \sup_{t \in [0,1]} |\Gamma_{\pm}^{(n)}(t) - \Gamma_{\pm}(t)| \right] = 0,$$

$$(ii) \quad \lim_{n \uparrow \infty} E \left[ \sup_{t \in [0,1]} |Y_{\pm}^{(n)}(t) - Y_{\pm}(t)| \right] = 0.$$

Consequently, there exists a subsequence  $(n_k)_{k=1}^{\infty}$  such that

$$\lim_{k \uparrow \infty} \sup_{t \in [0,1]} |\Gamma_{\pm}^{(n_k)}(t) - \Gamma_{\pm}(t)| = 0 \quad a.s.,$$

$$\lim_{k \uparrow \infty} \sup_{t \in [0,1]} |Y_{\pm}^{(n_k)}(t) - Y_{\pm}(t)| = 0 \quad a.s.$$

PROOF. (i) It holds that

$$\begin{aligned} & \overline{\lim}_{n \uparrow \infty} E \left[ \sup_{t \in [0,1]} |\Gamma_+^{(n)}(t) - \Gamma_+(t)| \right] \\ &= \overline{\lim}_{n \uparrow \infty} E \left[ \sup_{t \in [0,1]} \left| \int_0^t \left\{ 1_A \left( B \left( \frac{[2^n s + 1]}{2^n} \right) \right) - 1_A(B_s) \right\} ds \right| \right] \\ &\leq \overline{\lim}_{n \uparrow \infty} E \left[ \int_0^1 \left| 1_A \left( B \left( \frac{[2^n s + 1]}{2^n} \right) \right) - 1_A(B_s) \right| ds \right] \\ &\leq \int_0^1 ds \overline{\lim}_{n \uparrow \infty} E \left[ \left| 1_A \left( B \left( \frac{[2^n s + 1]}{2^n} \right) \right) - 1_A(B_s) \right| \right] \\ &= 0 \quad \text{by Lemma 3.4.} \end{aligned}$$

(ii) We have

$$\begin{aligned} Y_+^{(n)}(t) &= \sum_{k=1}^{2^n} 1_A \left( b - Z \left( \frac{2^n - k}{2^n} \right) \right) \\ &\quad \cdot \left\{ Z \left( (1-t) \vee \frac{(2^n - k) + 1}{2^n} \right) - Z \left( (1-t) \vee \frac{2^n - k}{2^n} \right) \right\} \\ &= \sum_{k=0}^{2^n - 1} 1_A \left( b - Z \left( \frac{k}{2^n} \right) \right) \\ &\quad \cdot \left\{ Z \left( (1-t) \vee \frac{k+1}{2^n} \right) - Z \left( (1-t) \vee \frac{k}{2^n} \right) \right\} \quad \text{by reindexing} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{2^n-1} 1_A \left( b - Z \left( \frac{k}{2^n} \right) \right) \left\{ Z \left( \frac{k+1}{2^n} \right) - Z \left( \frac{k}{2^n} \right) \right\} \\
&\quad - \sum_{k=0}^{2^n-1} 1_A \left( b - Z \left( \frac{k}{2^n} \right) \right) \\
&\quad \cdot \left\{ Z \left( (1-t) \wedge \frac{k+1}{2^n} \right) - Z \left( (1-t) \wedge \frac{k}{2^n} \right) \right\} \\
&= \int_0^1 1_A \left( b - Z \left( \frac{[2^n s]}{2^n} \right) \right) dZ_s - \int_0^{1-t} 1_A \left( b - Z \left( \frac{[2^n s]}{2^n} \right) \right) dZ_s \\
&= \int_{1-t}^1 1_A \left( b - Z \left( \frac{[2^n s]}{2^n} \right) \right) dZ_s,
\end{aligned}$$

where the third equality holds since

$$Z \left( (1-t) \vee \frac{k}{2^n} \right) + Z \left( (1-t) \wedge \frac{k}{2^n} \right) = Z(1-t) + Z \left( \frac{k}{2^n} \right).$$

Moreover, there exists a  $\mathcal{G}$ -Brownian motion  $(W_t)_{t \in [0,1]}$  such that

$$dZ_t = dW_t + \frac{b - Z_t}{1-t} dt,$$

where  $\mathcal{G} = (\mathcal{G}_t)_{t \in [0,1]}$  is the filtration generated by  $Z$ . It follows that

$$\begin{aligned}
&E \left[ \sup_{t \in [0,1]} |Y_+^{(n)}(t) - Y_+(t)| \right] \\
&= E \left[ \sup_{t \in [0,1]} \left| \int_{1-t}^1 \left\{ 1_A \left( b - Z \left( \frac{[2^n s]}{2^n} \right) \right) - 1_A(b - Z_s) \right\} dZ_s \right| \right] \\
&\leq 2 E \left[ \sup_{t \in [0,1]} \left| \int_0^t \left\{ 1_A \left( b - Z \left( \frac{[2^n s]}{2^n} \right) \right) - 1_A(b - Z_s) \right\} dZ_s \right| \right] \\
&\leq 2 E \left[ \sup_{t \in [0,1]} \left| \int_0^t \left\{ 1_A \left( b - Z \left( \frac{[2^n s]}{2^n} \right) \right) - 1_A(b - Z_s) \right\} dW_s \right| \right] \\
&\quad + 2 E \left[ \sup_{t \in [0,1]} \left| \int_0^t \left\{ 1_A \left( b - Z \left( \frac{[2^n s]}{2^n} \right) \right) - 1_A(b - Z_s) \right\} \frac{b - Z_s}{1-s} ds \right| \right] \\
&\leq 2 C E \left[ \left\{ \int_0^1 \left| 1_A \left( b - Z \left( \frac{[2^n s]}{2^n} \right) \right) - 1_A(b - Z_s) \right|^2 ds \right\}^{\frac{1}{2}} \right] \\
&\quad + 2 E \left[ \int_0^1 \left| 1_A \left( b - Z \left( \frac{[2^n s]}{2^n} \right) \right) - 1_A(b - Z_s) \right| \frac{|b - Z_s|}{1-s} ds \right]
\end{aligned}$$

$$\begin{aligned}
 &= 2C E \left[ \left\{ \int_0^1 \left| 1_A \left( B \left( \frac{[2^n s + 1]}{2^n} \right) \right) - 1_A(B_s) \right| ds \right\}^{\frac{1}{2}} \right] \\
 &\quad + 2 E \left[ \int_0^1 \left| 1_A \left( B \left( \frac{[2^n s + 1]}{2^n} \right) \right) - 1_A(B_s) \right| \frac{|B_s|}{s} ds \right]
 \end{aligned}$$

with  $C$  the constant appearing in the Burkholder-Davis-Gundy inequality. The same reasoning as in (i) then leads to the desired property.  $\square$

The following lemma is needed to prove Proposition 3.7 below.

LEMMA 3.6. *Let*

$$\mathcal{S} \stackrel{\text{def}}{=} \left\{ (y, \gamma) \in C([0, 1]; \mathbf{R}^d) \times C([0, 1]; \mathbf{R}) \mid \begin{array}{l} \gamma \text{ non-decreasing, } \gamma(0) = 0, \gamma(1) \leq 1, \\ y(s) = y(t) \text{ if } \gamma(s) = \gamma(t), s < t \end{array} \right\}$$

equipped with the metric induced by the sup norm. Define  $\Phi : \mathcal{S} \rightarrow C([0, 1]; \mathbf{R}^d)$  by

$$\Phi(y, \gamma) \stackrel{\text{def}}{=} y(\gamma^{-1}(\cdot)),$$

where

$$\gamma^{-1}(t) \stackrel{\text{def}}{=} \inf \{ s \in [0, 1] \mid \gamma(s) \geq t \wedge \gamma(1) \}$$

for  $t \in [0, 1]$ . Then  $\Phi$  is a continuous mapping.

PROOF. We divide the proof into two steps.

Step 1. We first show that

$$\forall (y, \gamma) \in \mathcal{S}, \quad \Phi(y, \gamma) \in C([0, 1]; \mathbf{R}^d),$$

i.e., for any sequence  $(t_n)_{n=1}^\infty \subset [0, 1]$  with  $t_n \rightarrow t$

$$\lim_{n \uparrow \infty} \Phi(y, \gamma)(t_n) = \Phi(y, \gamma)(t).$$

We will prove this only for the case where  $(t_n)_n$  is non-increasing: the other cases can be proved similarly.

Since we have assumed that  $(t_n)_n$  is non-increasing,  $(\gamma^{-1}(t_n))_n$  is also non-increasing and so  $\lim_{n \uparrow \infty} \gamma^{-1}(t_n)$  exists. In addition, it is easy to verify that

$$\begin{aligned}
 \gamma(\gamma^{-1}(t_n)) &= t_n \wedge \gamma(1), \\
 \gamma(\gamma^{-1}(t)) &= t \wedge \gamma(1).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \gamma(\lim_{n \uparrow \infty} \gamma^{-1}(t_n)) &= \lim_{n \uparrow \infty} \gamma(\gamma^{-1}(t_n)) \\
 &= \lim_{n \uparrow \infty} (t_n \wedge \gamma(1)) \\
 &= t \wedge \gamma(1) \\
 &= \gamma(\gamma^{-1}(t))
 \end{aligned}$$

and thus, by definition of  $\mathcal{S}$  we have

$$\begin{aligned}
 y(\gamma^{-1}(t)) &= y(\lim_{n \uparrow \infty} \gamma^{-1}(t_n)) \\
 &= \lim_{n \uparrow \infty} y(\gamma^{-1}(t_n)).
 \end{aligned}$$

Step 2. Next we prove that  $\Phi$  is a continuous mapping. Let  $((y_n, \gamma_n))_{n=1}^\infty \subset \mathcal{S}$  and  $(y_\infty, \gamma_\infty) \in \mathcal{S}$  be such that

$$\lim_{n \uparrow \infty} (y_n, \gamma_n) = (y_\infty, \gamma_\infty) \quad \text{in } \mathcal{S}.$$

What we wish to show is:

$$\overline{\lim}_{n \uparrow \infty} \sup_{t \in [0,1]} |y_n(\gamma_n^{-1}(t)) - y_\infty(\gamma_\infty^{-1}(t))| = 0.$$

It is easy to see that

$$\sup_{t \in [0,1]} |(t \wedge \gamma_n(1)) - (t \wedge \gamma_\infty(1))| \rightarrow 0 \quad (n \uparrow \infty)$$

and also

$$\begin{aligned}
 \sup_{t \in [0,1]} |(t \wedge \gamma_n(1)) - \gamma_\infty(\gamma_n^{-1}(t))| &= \sup_{t \in [0,1]} |\gamma_n(\gamma_n^{-1}(t)) - \gamma_\infty(\gamma_n^{-1}(t))| \\
 &\leq \sup_{t \in [0,1]} |\gamma_n(t) - \gamma_\infty(t)| \\
 &\rightarrow 0 \quad (n \uparrow \infty),
 \end{aligned}$$

which combine to yield

$$(3.1) \quad \lim_{n \uparrow \infty} \sup_{t \in [0,1]} |(t \wedge \gamma_\infty(1)) - \gamma_\infty(\gamma_n^{-1}(t))| = 0.$$

Furthermore, since

$$\gamma_\infty(\gamma_\infty^{-1}(\gamma_\infty(t))) = \gamma_\infty(t), \quad t \in [0, 1],$$

we have, by definition of  $\mathcal{S}$ ,

$$(3.2) \quad y_\infty(\gamma_\infty^{-1}(\gamma_\infty(t))) = y_\infty(t), \quad t \in [0, 1].$$

It follows that

$$\begin{aligned} & \sup_{t \in [0,1]} |y_n(\gamma_n^{-1}(t)) - y_\infty(\gamma_\infty^{-1}(t))| \\ & \leq \sup_{t \in [0,1]} |y_n(\gamma_n^{-1}(t)) - y_\infty(\gamma_n^{-1}(t))| \\ & \quad + \sup_{t \in [0,1]} |y_\infty(\gamma_n^{-1}(t)) - y_\infty(\gamma_\infty^{-1}(t))| \\ & \leq \sup_{t \in [0,1]} |y_n(t) - y_\infty(t)| \\ & \quad + \sup_{t \in [0,1]} \left| y_\infty \left( \gamma_\infty^{-1} \left( \gamma_\infty(\gamma_n^{-1}(t)) \right) \right) - y_\infty(\gamma_\infty^{-1}(t)) \right| \quad \text{by (3.2)} \end{aligned}$$

and therefore

$$\begin{aligned} & \overline{\lim}_{n \uparrow \infty} \sup_{t \in [0,1]} |y_n(\gamma_n^{-1}(t)) - y_\infty(\gamma_\infty^{-1}(t))| \\ & \leq \overline{\lim}_{n \uparrow \infty} \sup_{t \in [0,1]} \left| y_\infty \left( \gamma_\infty^{-1} \left( \gamma_\infty(\gamma_n^{-1}(t)) \right) \right) - y_\infty(\gamma_\infty^{-1}(t)) \right| \\ & = 0 \quad \text{by (3.1),} \end{aligned}$$

which completes the proof of Lemma 3.6.  $\square$

PROPOSITION 3.7. (i)  $(B_\pm(t))_{t \in [0,1]}$  have continuous paths a.s.  
(ii) For the subsequence  $(n_k)_{k=1}^\infty$  in Proposition 3.5,

$$\lim_{k \uparrow \infty} \sup_{t \in [0,1]} |B_\pm^{(n_k)}(t) - B_\pm(t)| = 0 \quad \text{a.s.}$$

PROOF. We see that, for almost all  $\omega \in \Omega$  and all  $n \in \mathbf{N}$ ,

$$\begin{aligned} (Y_\pm^{(n)}(\omega, t), \Gamma_\pm^{(n)}(\omega, t))_{t \in [0,1]} & \in \mathcal{S}, \\ (Y_\pm(\omega, t), \Gamma_\pm(\omega, t))_{t \in [0,1]} & \in \mathcal{S}. \end{aligned}$$

The desired properties then follow from Proposition 3.5 and Lemma 3.6.  $\square$

PROPOSITION 3.8. (i)  $(\tilde{B}_t)_{t \in [0,1]}$  has continuous paths a.s.  
(ii) For the subsequence  $(n_k)_k$  in Propositions 3.5 and 3.7 we have:

$$\lim_{k \uparrow \infty} \sup_{t \in [0,1]} |\tilde{B}_t^{(n_k)} - \tilde{B}_t| = 0 \quad \text{a.s.}$$

PROOF. (i) The first assertion follows immediately from Proposition 3.7(i).

(ii) For the sake of brevity we introduce the following notations:

$$\begin{aligned} \alpha_n &\stackrel{\text{def}}{=} |\Gamma_+^{(n)}(1) - \Gamma_+(1)|; \\ \beta_n &\stackrel{\text{def}}{=} 1 - |\Gamma_+^{(n)}(1) - \Gamma_+(1)|. \end{aligned}$$

Then, for  $\alpha_n \leq t \leq \Gamma_+^{(n)}(1)$  we have

$$\begin{aligned} & \left| \tilde{B}_t^{(n)} - \tilde{B} \left( t + \Gamma_+(1) - \Gamma_+^{(n)}(1) \right) \right| \\ &= \left| \left\{ B_+^{(n)}(\Gamma_+^{(n)}(1)) - B_+^{(n)}(\Gamma_+^{(n)}(1) - t) \right\} \right. \\ & \quad \left. - \left\{ B_+(\Gamma_+(1)) - B_+(\Gamma_+^{(n)}(1) - t) \right\} \right| \\ &\leq \left| B_+^{(n)}(\Gamma_+^{(n)}(1)) - B_+(\Gamma_+(1)) \right| \\ & \quad + \left| B_+^{(n)}(\Gamma_+^{(n)}(1) - t) - B_+(\Gamma_+^{(n)}(1) - t) \right| \\ &\leq \left| Y_+^{(n)}(1) - Y_+(1) \right| + \sup_{t \in [0,1]} \left| B_+^{(n)}(t) - B_+(t) \right|. \end{aligned}$$

Similarly, for  $\Gamma_+^{(n)}(1) \leq t \leq \beta_n$ ,

$$\begin{aligned} \left| \tilde{B}_t^{(n)} - \tilde{B} \left( t + \Gamma_+(1) - \Gamma_+^{(n)}(1) \right) \right| &\leq \left| Y_+^{(n)}(1) - Y_+(1) \right| \\ & \quad + \sup_{t \in [0,1]} \left| B_-^{(n)}(t) - B_-(t) \right|. \end{aligned}$$

It follows that

$$\sup_{\alpha_n \leq t \leq \beta_n} \left| \tilde{B}_t^{(n)} - \tilde{B} \left( t + \Gamma_+(1) - \Gamma_+^{(n)}(1) \right) \right|$$

$$\begin{aligned} &\leq \left| Y_+^{(n)}(1) - Y_+(1) \right| + \sup_{t \in [0,1]} \left| B_+^{(n)}(t) - B_+(t) \right| \\ &\quad + \sup_{t \in [0,1]} \left| B_-^{(n)}(t) - B_-(t) \right| \quad \text{a.s.} \end{aligned}$$

and hence by Propositions 3.5 and 3.7(ii)

$$(3.3) \quad \lim_{k \uparrow \infty} \sup_{\alpha_{n_k} \leq t \leq \beta_{n_k}} \left| \tilde{B}_t^{(n_k)} - \tilde{B} \left( t + \Gamma_+(1) - \Gamma_+^{(n_k)}(1) \right) \right| = 0 \quad \text{a.s.}$$

This implies

$$\begin{aligned} &\overline{\lim}_{k \uparrow \infty} \sup_{t \in [0,1]} \left| \tilde{B}_t^{(n_k)} - \tilde{B}_t \right| \\ &\leq \overline{\lim}_{k \uparrow \infty} \sup_{0 \leq t \leq \alpha_{n_k}} \left| \tilde{B}_t^{(n_k)} - \tilde{B}_t \right| + \overline{\lim}_{k \uparrow \infty} \sup_{\beta_{n_k} \leq t \leq 1} \left| \tilde{B}_t^{(n_k)} - \tilde{B}_t \right| \\ &\quad + \overline{\lim}_{k \uparrow \infty} \sup_{\alpha_{n_k} \leq t \leq \beta_{n_k}} \left| \tilde{B}_t^{(n_k)} - \tilde{B}_t \right| \\ &= \overline{\lim}_{k \uparrow \infty} \sup_{\alpha_{n_k} \leq t \leq \beta_{n_k}} \left| \tilde{B}_t^{(n_k)} - \tilde{B}_t \right| \quad \text{by Proposition 3.5} \\ &\leq \overline{\lim}_{k \uparrow \infty} \sup_{\alpha_{n_k} \leq t \leq \beta_{n_k}} \left| \tilde{B}_t^{(n_k)} - \tilde{B} \left( t + \Gamma_+(1) - \Gamma_+^{(n_k)}(1) \right) \right| \\ &\quad + \overline{\lim}_{k \uparrow \infty} \sup_{\alpha_{n_k} \leq t \leq \beta_{n_k}} \left| \tilde{B} \left( t + \Gamma_+(1) - \Gamma_+^{(n_k)}(1) \right) - \tilde{B}_t \right| \\ &= 0 \quad \text{a.s. by (3.3) } \square \end{aligned}$$

We are now in a position to prove our main theorem.

PROOF OF THEOREM 1.1. It is clear that

$$\lim_{n \uparrow \infty} \sup_{t \in [0,1]} \left| S_{[2^n t]}^{(n)} - B_t \right| = 0 \quad \text{a.s.}$$

Also, we have

$$\begin{aligned} \sup_{t \in [0,1]} \left| \tilde{S}_{[2^n t]}^{(n)} - \tilde{B}_t \right| &= \sup_{t \in [0,1]} \left| \tilde{B}^{(n)} \left( \frac{[2^n t]}{2^n} \right) - \tilde{B}_t \right| \quad \text{by Proposition 3.3} \\ &\leq \sup_{t \in [0,1]} \left| \tilde{B}^{(n)} \left( \frac{[2^n t]}{2^n} \right) - \tilde{B} \left( \frac{[2^n t]}{2^n} \right) \right| + \sup_{t \in [0,1]} \left| \tilde{B} \left( \frac{[2^n t]}{2^n} \right) - \tilde{B}_t \right| \\ &\leq \sup_{t \in [0,1]} \left| \tilde{B}_t^{(n)} - \tilde{B}_t \right| + \sup_{t \in [0,1]} \left| \tilde{B} \left( \frac{[2^n t]}{2^n} \right) - \tilde{B}_t \right| \quad \text{a.s.,} \end{aligned}$$

which in turn implies that

$$\lim_{k \uparrow \infty} \sup_{t \in [0,1]} \left| \tilde{S}_{[2^{n_k}t]}^{(n_k)} - \tilde{B}_t \right| = 0 \quad \text{a.s.}$$

with  $(n_k)_k$  the subsequence in Proposition 3.8. The argument in Section 2 shows

$$\forall n \in \mathbf{N}, \quad \left( \tilde{S}_{[2^n t]}^{(n)} \right)_{t \in [0,1]} \stackrel{(d)}{=} \left( S_{[2^n t]}^{(n)} \right)_{t \in [0,1]},$$

and therefore, for any  $m \in \mathbf{N}$  and  $0 \leq t_1 < t_2 < \dots < t_m \leq 1$ ,

$$\left( \tilde{B}_{t_1}, \tilde{B}_{t_2}, \dots, \tilde{B}_{t_m} \right) \stackrel{(d)}{=} \left( B_{t_1}, B_{t_2}, \dots, B_{t_m} \right).$$

This and the path continuity of  $(\tilde{B}_t)_{t \in [0,1]}$  (see Proposition 3.8(i)) complete the proof.  $\square$

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