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A Brooks Type Integral with Respect to a Set-Valued Measure

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Abstract. A generalization of the set–valued Brooks integral [3] with respect to a set–valued measure whose values are subsets of a Hausdorff locally convex topological vector space is presented.

The construction of this new kind of integral is based on Weber's result [19] concerning the existence of a family of semi–invariant pseudo–metrics which generates the uniformity of a uniform semigroup (in our case, the semigroup of convex, bounded, closed subsets of a Hausdorff locally convex topological vector space).

Several properties of the new integral are given and also a theorem of Vitali type is established.

1. Introduction

In recent years, the study of set–valued measures has been developed extensively because of their applications in the mathematical economics, optimization and optimal control [11],[16],[17].

Significant contributions in this area were made by Artstein [2], Castaing–Valadier [4], Costé [5], Alò, de Korvin and Roberts [1], Brooks [3], Drewnowski [7], Godet–Thobie [9], Papageorgiou [13],[14], Hiai [10].

We recall that, recently, Papageorgiou [14] introduced a set–valued integral with respect to a h–set–valued measure in the sense of Alò, de Korvin and Roberts [1], of bounded variation using the set of Bochner integrals of a Banach valued function with respect to measure selectors of the given multimeasure.

Our purpose is to define a new kind of integral using Brooks' procedure [3] adapted in the setting of a set–valued measure whose values are subsets of a Hausdorff locally convex topological vector space X and with respect to the Γ_μ–convergence in submeasure defined with the aid of a family of

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pseudometrics which generates the uniformity of the semigroup $\mathcal{K}(X)$ of the convex, closed, bounded subsets of X.

Very briefly the organization of the paper is as follows. In section 2 we precise the terminology and notations, recall some properties of multimeasures and some basic results concerning the Γ_μ–convergence in submeasure. In section 3 we define and study some basic properties of the integral of simple functions and in section 4 we present our set–valued integral with some natural properties and we prove a theorem of Vitaly–type for this kind of integral.

2. Terminology and notations

Let X be a Hausdorff locally convex topological vector space (briefly *H.l.c.t.v.s.*), τ its topology and $\mathcal{V}$ a base of absolutely convex closed neighborhoods of the origin 0 in X. $\mathcal{A}(X)$ is the family of all nonvoid subsets of X, $\mathcal{C}(X)$ is the subfamily of $\mathcal{A}(X)$ of all closed subsets of X and $\mathcal{K}(X)$ is the subfamily of $\mathcal{A}(X)$ of convex, closed, bounded subsets of X.

On $\mathcal{A}(X)$ we define the equivalence relation "ρ" by $A\rho B$ iff $\overline{A} = \overline{B}$, where $\overline{A}$ denotes the closure of $A \subset X$ with respect to τ.

The quotient $\mathcal{A}(X)/\rho$ may be identified with $\mathcal{C}(X)$. It is easy to see that the addition $(A, B) \longrightarrow A + B$ in $\mathcal{A}(X)$ is compatible with ρ. Hence $(\mathcal{A}(X), +)$ admits a factorization by ρ and the resulting quotient semigroup may be identified with $(\mathcal{C}(X), \dot{+})$, where $\dot{+}$ is the Minkowski addition that is

$$A \dot{+} B = \overline{A + B}, \qquad \text{for every } A, B \in \mathcal{A}(X)$$

(see [7]).

Now let $\mathcal{U}$ be the invariant uniformity on X compatible with τ and $\tilde{\mathcal{U}}$ the exponential extension of $\mathcal{U}$ to $\mathcal{A}(X)$, that is the uniformity $\tilde{\mathcal{U}}$ defined by the following base of vicinities:

$$\mathcal{W}(U) = \{(A, B) \in \mathcal{A}(X) \times \mathcal{A}(X);\ A \subset B \dot{+} U,\ B \subset A \dot{+} U\},\ (\forall) U \in \mathcal{U}.$$

The topology on $\mathcal{A}(X)$ induced by $\tilde{\mathcal{U}}$ will be denoted by $\tilde{\tau}$.
Using the equivalence relation ρ we may identify $(\mathcal{A}(X), \tilde{\tau})/\rho$ with $(\mathcal{C}(X), \tilde{\tau})$ and the separated uniform space associated with $(\mathcal{A}(X), \tilde{\mathcal{U}})/\rho$ with $(\mathcal{C}(X), \tilde{\mathcal{U}})$.

In particular, if X is metrizable, then $(\mathcal{A}(X),\tilde{\mathcal{U}})$ becomes a semimetrizable space for the Hausdorff semimetric between sets; if, besides, X is complete, then so is $\mathcal{C}(X)$.

When τ is determined by a norm $\|\cdot\|$, the Hausdorff semimetric is given by

$$d(A,B) = \inf\{t > 0;\ A \subset B + tU_1,\ B \subset A + tU_1\},$$

where $U_1 = \{x \in X;\ \|x\| \leq 1\}$.

If $(X,\mathcal{U})$ is a complete uniform space, then so is $(\mathcal{C}(X),\tilde{\mathcal{U}})$. It is easy to see that $(\mathcal{C}(X),\dot{+},\tilde{\mathcal{U}})$ is a Hausdorff uniform commutative semigroup with the unit $\{0\}$ and $\mathcal{K}(X)$ is a closed subsemigroup in which the cancellation law

$$A\dot{+}C = B\dot{+}C \implies A = B$$

holds in it.

Instead of $\{0\}$ we will usually write simply 0. According to a result of Weber [19] there exists a family $\mathcal{P} = \{p\}$ of semiinvariant pseudometrics on $\mathcal{C}(X)$ taking values in $[0,1]$, which generates the uniformity $\tilde{\mathcal{U}}$. (A pseudometric p is semiinvariant if $p(A\dot{+}C, B\dot{+}C) \leq p(A,B)$ for every $A, B, C \in \mathcal{C}(X)$.)

If $p \in \mathcal{P}$ we denote by

$$|A|_p = p(A,0), \tag{2}$$

where 0 represents the set $\{0\}$.

From the semiinvariance of $p \in \mathcal{P}$ we easily obtain the following properties:

$$|A\dot{+}B|_p \leq |A|_p + |B|_p, \qquad (\forall)A,B \in \mathcal{C}(X), \tag{3}$$

$$|A\dot{+}B|_p \geq |A|_p - |B|_p, \qquad (\forall)A,B \in \mathcal{C}(X). \tag{4}$$

Beside X we consider S a nonvoid set, $\mathcal{P}(S)$ the family of all subsets of S and $\mathcal{R}$ a ring of subsets of S.

In the following we shall consider set–valued maps μ from S to X, that is set–valued functions μ defined on $\mathcal{R}$ taking values in the semigroup $\mathcal{K}(X)$ with the supplementary property $\mu(\emptyset) = 0$.

Definition 2.1. A set–valued map μ from $\mathcal{R}$ to X is said to be:

I. an *additive set–valued measure* (*multimeasure*) if

$$\mu(A \cup B) = \mu(A) \dot{+} \mu(B), \quad (\forall) A, B \in \mathcal{R} \text{ with } A \cap B = \emptyset; \tag{5}$$

II. a *σ–additive multimeasure* if for every sequence $(A_n)_{n \geq 1} \subset \mathcal{R}$ with $A_i \cap A_j = \emptyset$ $(i \neq j)$ and $\cup_{n=1}^{\infty} A_n \in \mathcal{R}$ holds

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = {}^{\bullet}\sum_{n=1}^{\infty} \mu(A_n) \tag{6}$$

(the sum from the right side of the equality is considered with respect to the topology $\tilde{\tau}$ and the Minkowski addition $\dot{+}$).

In what follows by a set–valued measure or a multimeasure we shall mean an additive set–valued measure.

Definition 2.2. Let μ be a set–valued measure from $\mathcal{R}$ to X and $p \in \mathcal{P}$ an arbitrary pseudometric from the family $\mathcal{P}$.

The function $\tilde{\mu}_p : \mathcal{R} \longrightarrow \overline{I\!R}_+$ defined by

$$\tilde{\mu}_p(A) = \sup\left\{\sum_{i=1}^{n} |\mu(A_i)|_p\right\}, \tag{7}$$

where the supremum is taken on all finite partitions $(A_i)_{i=1}^n$ of $A \in \mathcal{R}$ with $A_i \in \mathcal{R}$ is said to be the *p–variation of* μ.

It is easy to see that if μ is additive (respectively σ–additive) so is $\tilde{\mu}_p$.

$\tilde{\mu}_p$ may be extended to $\mathcal{P}(S)$ by

$$\tilde{\mu}_p(E) = \inf\{\tilde{\mu}_p(A);\ E \subseteq A \in \mathcal{R}\}, \quad (\forall) E \in \mathcal{P}(S) \tag{8}$$

which is a submeasure in Drewnowski sense [6].

Now $\mathcal{P}(S)$ may be organized as a ring with respect to symmetrical difference Δ as addition and the intersection $\cap$ as product.

Furthermore the family $\Gamma_\mu = \{\tilde{\mu}_p^*;\ p \in \mathcal{P}\}$ of submeasures on $\mathcal{P}(S)$ generates a topology τ_{Γ_μ} on $\mathcal{P}(S)$ such that $(\mathcal{P}(S), \Delta, \cap; \tau_{\Gamma_\mu})$ becomes a topological ring for which the family $\mathcal{B}_{\Gamma_\mu}$ of subsets $V_{k,\varepsilon} = \{A \subset S;\ \tilde{\mu}_p^*(A) < \varepsilon,$ $p \in K\}$, where K is a finite subset in $\mathcal{P}$ and $\varepsilon > 0$, is a base of neighborhoods of $\emptyset$ for this topology [6].

In what follows we shall write $\mathcal{P}(S)(\Gamma_\mu)$ for $(\mathcal{P}(S), \Delta, \cap; \tau_{\Gamma_\mu})$ and $\mathcal{R}(\Gamma_\mu)$ for a topological subring $\mathcal{R}$ of $\mathcal{P}(S)(\Gamma_\mu)$.

We shall also denote by $\mathcal{R}_\mu$ the hereditary subring of $\mathcal{R}$ of all Γ_μ–integrable members of $\mathcal{R}$ that is the family of all sets $A \in \mathcal{R}$ such that $\tilde{\mu}_p(A) < \infty$ for every $p \in \mathcal{P}$.

If μ is a set–valued measure from $\mathcal{R}$ to X, then μ satisfies the following property:

(α) For every $B \in \mathcal{R}_\mu$ and for every vicinity W from $\tilde{\mathcal{U}}$ in $\mathcal{K}(X)$ there exists $\varepsilon > 0$ such that for every finite family $\{(\alpha_i, \beta_i)\}_{i=1}^n$ of real numbers such that $|\alpha_i - \beta_i| < \varepsilon$, $(\forall) i = 1, 2, \dots, n$ and for every finite disjoint sequence $(A_i)_{i=1}^n$ of subsets of $\mathcal{R}$, the following relation holds

$$\left(\sum_{i=1}^n \alpha_i \mu(A_i \cap B), \sum_{i=1}^n \beta_i \mu(A_i \cap B) \right) \in W.$$

Next we denote by $I\!R^S$ the set of all functions f from S to $I\!R$. For every finite $K \subset \mathcal{P}$ and $\varepsilon > 0$ we consider

$$W_K(\varepsilon) = \{(f, g) \in I\!R^S \times R^S;\ \tilde{\mu}_p^* (\{s \in S; |f(s) - g(s)| \geq \varepsilon\}) < \varepsilon, p \in K\}.$$

The family of all subsets $W_k(\varepsilon)$ constitutes a base of vicinities for a uniformity $\mathcal{U}_{\Gamma_\mu}$ on $I\!R^S$.

We shall also use the notation $I\!R^S(\Gamma_\mu)$ for $(I\!R^S, \mathcal{U}_{\Gamma_\mu})$.

Definition 2.3. A net (f_α) from $I\!R^S$ is said to be *convergent in* Γ_μ–*submeasure* to $f \in I\!R^S$, denoted $f_\alpha \xrightarrow{\Gamma_\mu} f$, if f_α converges to f in $I\!R^S(\Gamma_\mu)$.

Remarking that the function φ which associates to every $E \in \mathcal{P}(S)$ its indicatrice function is an isomorphism between the uniform space $\mathcal{P}(S)(\Gamma_\mu)$ and the uniform subspace Y of $I\!R^S(\Gamma_\mu)$ of all indicatrice functions of subsets of S, where $I\!R$ is endowed with its natural uniform structure, it is legitimate to use the notation $E_\alpha \xrightarrow{\Gamma_\mu} E$ for the convergence in $\mathcal{P}(S)(\Gamma_\mu)$.

3. Integration of simple functions

Definition 3.1. A function $f \in I\!R^S$ is said to be a simple Γ_μ–integrable function if: a) it assumes only a finite number of distinct values $a_i \in I\!R$; b) $f^{-1}(\{a_i\}) = A_i \in \mathcal{R}$; c) if $a_i \neq 0$ then $A_i \in \mathcal{R}_\mu$.

In this case if $T \in \mathcal{R}$ the integral of f over T is defined by

$$\int_T f d\mu = \sum_{i=1}^{n} a_i \mu(T \cap A_i). \tag{9}$$

We remark that if χ_T denotes the characteristic function of T, then $f = \sum_{i=1}^{n} a_i \chi_{A_i}$.

It is easy to see that the integral of f is independent of the representation of f as a finite combination of this type.

In what follows we shall denote by $\mathcal{E}(\Gamma_\mu, X)$, or briefly $\mathcal{E}(\Gamma_\mu)$, the set of all Γ_μ–integrable simple functions.

From the definition 3.1 we immediately obtain

Theorem 3.2. *If $f, g \in \mathcal{E}(\Gamma_\mu)$, then:*

I. $f + g \in \mathcal{E}(\Gamma_\mu)$ *and* $\int_T (f+g) d\mu = \int_T f d\mu + \int_T g d\mu$, $(\forall) T \in \mathcal{R}$;

II. *for every* $p \in \mathcal{P}$, $p(\int_T f d\mu, \int_T g d\mu) \le \int_T |f - g| d\mu_p$, $(\forall) T \in \mathcal{R}$.

III. *the set–valued map* $\nu(T) = \int_T f d\mu$, $(\forall) T \in \mathcal{R}$,
is a multimeasure and if moreover μ *is* σ*–additive then so is* ν.

IV. $\lim\limits_{T \xrightarrow{\Gamma_\mu} \emptyset} \int_T f d\mu = 0$.

Proof. I), III) and IV) may be immediately obtained from the definition 3.1.

To prove II) let $f = \sum_{i=1}^{n} a_i \chi_{A_i}$ and $g = \sum_{j=1}^{m} b_j \chi_{B_j}$. We observe that it is possible to find a finite family $(E_k)_{k=1}^{s} \subset \mathcal{R}$ such that $f = \sum_{k=1}^{s} a_k \chi_{E_k}$ and $g = \sum_{k=1}^{s} b_k \chi_{E_k}$. Next, for every $p \in \mathcal{P}$ we have

$$\begin{aligned} p\left(\int_T f d\mu, \int_T g d\mu\right) &= p\left({}^{\bullet}\sum_{k=1}^{s} a_k \mu(T \cap E_k), {}^{\bullet}\sum_{k=1}^{s} b_k \mu(T \cap E_k)\right) \le \\ &\le \sum_{k=1}^{s} p(a_k \mu(T \cap E_k), b_k \mu(T \cap E_k)) \le \\ &\le \sum_{k=1}^{s} |a_k - b_k| \| \mu(T \cap E_k) \|_p \le \end{aligned}$$

$$\leq \sum_{k=1}^{s} |a_k - b_k| \tilde{\mu}_p(T \cap E_k). \; \square$$

THEOREM 3.3. *Let $(f_\alpha)_{\alpha \in I}$ be a Cauchy net in $I\!R^S(\Gamma_\mu)$ of simple Γ_μ–integrable functions.*

The net $(\int_T f_\alpha d\mu)_{\alpha \in I}$ is uniformly Cauchy with respect to $T \in \mathcal{R}$ if and only if the following two conditions hold:

I. for every neighborhood V of the origin in $\mathcal{K}(X)$ there exists $\alpha_0 \in I$ and a neighborhood $\mathcal{V}_\mu$ of $\emptyset$ in $\mathcal{R}(\Gamma_\mu)$ such that $\int_T f_\alpha d\mu \in V$ for every $\alpha \geq \alpha_0$ and every $T \in \mathcal{V}_\mu$;

II. for every neighborhood V of the origin in $\mathcal{K}(X)$ there exist $\alpha_0 \in I$ and $M \in \mathcal{R}_\mu$ such that $\int_T f_\alpha d\mu \in V$ for every $\alpha \geq \alpha_0$ and every $T \in \mathcal{R}$ with $T \subset S \setminus M$.

PROOF. First let us assume that the net $(\int_T f_\alpha d\mu)_{\alpha \in I}$ is uniform Cauchy with respect to $T \in \mathcal{R}$ and let V be an arbitrary neighborhood of the origin in $\mathcal{K}(X)$. Then there exists a symmetrical vicinity W from $\tilde{\mathcal{U}}$ such that $W^2(0) \subset V$. By virtue of hypothesis there exists $\alpha_0 \in I$ such that $(\int_T f_\alpha d\mu, \int_T f_{\alpha_0} d\mu) \in W$ for every $T \in \mathcal{R}$ and $\alpha \geq \alpha_0$. Now, from the theorem 3.2, IV), there exists a neighborhood $\mathcal{V}_\mu$ of $\emptyset$ in $\mathcal{R}(\Gamma_\mu)$ such that $\int_T f_{\alpha_0} d\mu \in W(0)$ for $T \in \mathcal{V}_\mu$. Hence $\int_T f_\alpha d\mu \in V$ for $\alpha \geq \alpha_0$ and $T \in \mathcal{V}_\mu$ that is I) is satisfied. To obtain II) it is sufficient to take $M = \{s \in S; f_{\alpha_0} \neq 0\}$. We see that $M \in \mathcal{R}_\mu$ and $\int_T f_{\alpha_0} d\mu = 0$ for every $T \in \mathcal{R}$ with $T \subset S \setminus M$, that is II) is proved.

Conversely, let W_1 be a vicinity of $\tilde{\mathcal{U}}$ and W a symmetrical vicinity of $\tilde{\mathcal{U}}$ such that that $W^2 + W^2 + W^2 \subset W_1$ (W exists because of the uniform continuity of the addition in $\mathcal{K}(X)$.)

Let $j_0 \in J$, $\mathcal{V}_\mu$ and $M \in \mathcal{R}_\mu$ corresponding to $W(0)$ by virtue of hypotheses I) and II). (α_0 can be chosen to satisfy simultaneously I) and II).

According to (α) from section 1, to M and W it corresponds $\delta > 0$ such that for every $n \in N$, every $\{(\alpha_i, \beta_i)\}_{i=0}^n$ with $\alpha_i, \beta_i \in I\!R$ such that $|\alpha_i - \beta_i| < \delta$, $i = 0, 1, 2, \ldots, n$ and $(E_i)_{i=0}^n$, a finite disjoint sequence of members of $\mathcal{R}$, the following relation holds

$$\left(\sum_{i=0}^{n} \alpha_i \mu(E_i \cap M), \sum_{i=0}^{n} \beta_i \mu(E_i \cap M) \right) \in W.$$

Let $M_{\alpha,\alpha'} = \{s \in S; |f_\alpha(s) - f_{\alpha'}(s)| \geq \delta\}$.

We see that $M_{\alpha,\alpha'} \in \mathcal{R}$ for every $\alpha, \alpha' \in J$ and since (f_α) is a Cauchy net in $I\!R^S(\Gamma_\mu)$ there exists $\alpha_1 \geq \alpha_0$ such that $M_{\alpha,\alpha'} \in \mathcal{V}_\mu$ for $\alpha \geq \alpha_1$ and $\alpha' \geq \alpha_1$.

Now, taking into account that $W^2 + W^2 + W^2 \subset W_1$ we obtain for every $T \in \mathcal{R}$,

$$\begin{aligned}\left(\int_T f_\alpha d\mu, \int_T f_{\alpha'} d\mu\right) &= \left(\int_{T\cap M_{\alpha,\alpha'}} f_\alpha d\mu, \int_{T\cap M_{\alpha,\alpha'}} f_{\alpha'} d\mu\right) + \\ &+ \left(\int_{T\setminus(M_{\alpha,\alpha'}\cup M)} f_\alpha d\mu, \int_{T\setminus(M_{\alpha,\alpha'}\cup M)} f_{\alpha'} d\mu\right) + \\ &+ \left(\int_{(T\setminus M_{\alpha,\alpha'})\cap M} f_\alpha d\mu, \int_{(T\setminus M_{\alpha,\alpha'})\cap M} f_{\alpha'} d\mu\right) \in \\ &\in W(0) \times W(0) + W(0) \times W(0) + W^2 \subseteq \\ &\subseteq W^2 + W^2 + W^2 \subset W_1\end{aligned}$$

for every $\alpha \geq \alpha_1$, $\alpha' \geq \alpha_1$; hence the net $\{\int_T f_\alpha d\mu\}$ is uniform Cauchy with respect to $T \in \mathcal{R}$. □

4. Integration with respect to a multimeasure

LEMMA 4.1. *Let $\{f_\alpha\}_{\alpha\in I}$ and $\{g_\beta\}_{\beta\in J}$ be two nets in $\mathcal{E}(\Gamma_\mu, X)$ both convergent in $I\!R^S(\Gamma_\mu)$ to the same function. If $(\int_T f_\alpha d\mu)$ and $(\int_T g_\beta d\mu)$ are Cauchy nets uniformly with respect to $T \in \mathcal{R}$, then for every vicinity W from $\tilde{\mathcal{U}}$, there are α_0 and β_0 such that for every $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$, $(\int_T f_\alpha d\mu, \int_T g_\beta d\mu) \in W$ uniformly in $T \in \mathcal{R}$.*

PROOF. Let W be a vicinity from $\tilde{\mathcal{U}}$ and let W_1 be a symmetrical vicinity of $\tilde{\mathcal{U}}$ such that $W^2 + W^2 + W^2 \subset W_1$.

Let $\delta > 0$ corresponding to W_1 by virtue of the condition (α).

We denote by $M_{\alpha,\beta} = \{s \in S; |f_\alpha(s) - g_\beta(s)| > \delta\}$.

Using the theorem 3.3 we obtain α_0, β_0, a Γ_μ–integrable set M in S and a neighborhood $\mathcal{V}_\mu$ of $\emptyset$ in $\mathcal{R}(\Gamma_\mu)$ such that $\int_T f_\alpha d\mu \in W_1(0)$ and $\int_T g_\beta d\mu \in W_1(0)$ for $\alpha \geq \alpha_0$, $\beta \geq \beta_0$, $T \in \mathcal{V}_\mu$ with $T \subset S \setminus M$, $T \in \mathcal{R}$.

By virtue of hypotheses there exist $\alpha_1 \geq \alpha_0$ and $\beta_1 \geq \beta_0$ such that $M_{\alpha,\beta} \in \mathcal{V}_\mu$ for $\alpha \geq \alpha_1$ and $\beta \geq \beta_1$ and every $T \in \mathcal{R}$.

In the same way as in theorem 3.3 we find that $(\int_T f_\alpha d\mu, \int_T g_\beta d\mu) \in W$ for $\alpha \geq \alpha_1$, $\beta \geq \beta_1$ and every $T \in \mathcal{R}$. □

Definition 4.2. A function $f \in I\!R^S$ is said to be Γ_μ*–integrable* if there exists a net $\{f_\alpha\}$ in $\mathcal{E}(\Gamma_\mu)$ such that $f_\alpha \xrightarrow{\Gamma_\mu} f$ and $\{\int_T f_\alpha d\mu\}$ is a Cauchy net in $\mathcal{K}(X)$ uniform with respect to $T \in \mathcal{R}$.

In this case the element of the completion $\tilde{\mathcal{K}}(X)$ of $\mathcal{K}(X)$ defined by $\int_T f d\mu = \lim_\alpha \displaystyle\int_T f_\alpha d\mu$ is said to be the Γ_μ*–integral* of f on $T \in \mathcal{R}$.

By virtue of Lemma 4.1 the notion of Γ_μ–integral is well–defined.

Remark. If X is a Banach space and $\mathcal{C}(X)$ is endowed with Hausdorff distance, then $\mathcal{C}(X)$ becomes a complete metric space in which $\mathcal{K}(X)$ is a closed subspace, that is also complete.

Now, let μ be a multimeasure from S to X, ν its variation and ν^* the corresponding submeasure defined as in (8). Then the Γ_μ–integral of a function $f \in I\!R^S$, defined as in definition 4.2 is just the integral studied by Brooks in [3].

In what follows we shall denote by $\mathcal{L}(\Gamma_\mu, X)$ the set of all Γ_μ–integrable functions.

Evidently $\mathcal{E}(\Gamma_\mu, X) \subset \mathcal{L}(\Gamma_\mu, X)$.

Now we can obtain some remarkable properties of Γ_μ–integrable functions.

Theorem 4.3. *If $f, g \in \mathcal{L}(\Gamma_\mu, X)$ we have:*

I. $f + g \in \mathcal{L}(\Gamma_\mu, X)$ *and* $\int_T (f+g) d\mu = \int_T f d\mu + \int_T g d\mu$, $(\forall) T \in \mathcal{R}$;

II. *the set–valued map defined by* $\nu(T) = \int_T f d\mu$, $\quad(\forall) T \in \mathcal{R}$, *is a multimeasure and if moreover μ is σ–additive then so is ν;*

III. $\lim\limits_{\substack{E \xrightarrow{\Gamma_\mu} \emptyset \\ E \in \mathcal{R}}} \nu(E) = 0$;

IV. *if $S \notin \mathcal{R}_\mu$, then the family $\{B_M\}_{M\in\mathcal{R}_\mu}$, where the nonvoid set B_M is associated to $M \in \mathcal{R}_\mu$ by $B_M = \{T \in \mathcal{R}; T \cap M = \emptyset\}$ constitutes a filterbase $\mathcal{F}$ on $\mathcal{R}$ and for every $f \in \mathcal{L}(\Gamma_\mu, X)$, $\lim_{\mathcal{F}} \int_T f d\mu = 0$.*

Proof. We immediately obtain I) and II) from the definition 4.2 and the analogous properties for simple integrable functions (see theorem 3.2).

To prove III) let W be a symmetrical vicinity of $\tilde{\mathcal{K}}(X)$, the completion of $\mathcal{K}(X)$, and let $g \in \mathcal{E}(\Gamma_\mu, X)$ such that $(\int_T f d\mu, \int_T g d\mu) \in W$ for every $T \in \mathcal{R}$.

Let $\mathcal{V}_\mu$ be a neighborhood of $\emptyset$ in $\mathcal{R}(\Gamma_\mu)$ such that $\int_T g d\mu \in W(0)$ for $T \in \mathcal{V}_\mu$. But $\int_T f d\mu \in W^2(0)$ for $T \in \mathcal{V}_\mu$ that is III) is satisfied.

To prove IV) it is sufficient to remark that $M = \{s \in S; g(s) \neq 0\}$ is Γ_μ–integrable and then for $T \in B_M$ we have $\int_T f d\mu \in W(0)$ that is IV). $\square$

Theorem 4.4. *Let $\{f_\alpha\}_{\alpha\in D}$ be a net of Γ_μ–integrable functions, Cauchy in $I\!R^S(\Gamma_\mu)$. The net $\{\int_T f_\alpha d\mu\}_{\alpha\in D}$ is uniform Cauchy with respect to $T \in \mathcal{R}$ if and only if the following two conditions are satisfied:*

I. *for every neighborhood V of the origin in $\tilde{K}(X)$ there exist an index α_0 and a neighborhood $\mathcal{V}_\mu$ of $\emptyset$ in $\mathcal{R}(\Gamma_\mu)$ such that $\int_T f_\alpha d\mu \in V$ for $\alpha \geq \alpha_0$ and $T \in \mathcal{V}_\mu$;*

II. *for every neighborhood V of the origin in $\tilde{K}(X)$ there exist an index $\alpha_0 \in D$ and a Γ_μ–integrable set M such that $\int_T f_\alpha d\mu \in V$ for $\alpha \geq \alpha_0$, and $T \in \mathcal{R}$, $T \subset S \setminus M$.*

Proof. The necessity can be obtained in the same way as in the theorem 3.3 using the theorem 4.3, III) and IV).

Conversely, let $\{f_\alpha\}_{\alpha\in D}$ be a Cauchy net in $I\!R^S(\Gamma_\mu)$ of Γ_μ–integrable functions which satisfies the conditions I) and II) from theorem.

By virtue of the Γ_μ–integrability of f_α, for every $\alpha \in D$, there exists a net $\{f_{\alpha,\beta}\}_{\beta\in D_\alpha}$ in $\mathcal{E}(\Gamma_\mu, X)$ such that $f_{\alpha,\beta} \xrightarrow{\Gamma_\mu} f_\alpha$ and $\lim_\beta \int_T f_{\alpha,\beta} d\mu = \int_T f_\alpha d\mu$ uniformly in $T \in \mathcal{R}$.

Now let us consider $\{f_{\alpha,\varphi(\alpha)}; (\alpha, \varphi) \in D \times \prod_\alpha D_\alpha\}$ the diagonal approximation associated to $\{f_{\alpha,\beta}; \alpha \in D, \beta \in D_\alpha\}$ (see Kelley [12], Chap.II).

If U is a symmetrical vicinity in $I\!\!R^S(\Gamma_\mu)$ there exists $\varphi_U \in \prod_\alpha D_\alpha$ such that for every $\varphi \in \prod_\alpha D_\alpha$ with$\varphi \geq \varphi_U$ we obtain $(f_\alpha, f_{\alpha,\varphi(\alpha)}) \in U$ for every $\alpha \in D$. But $\{f_\alpha\}_{\alpha\in D}$ is a Cauchy net in $I\!\!R^S(\Gamma_\mu)$ and then for U there exists $\alpha_1 \in D$ such that $(f_\alpha, f_{\alpha'}) \in U$ for $\alpha \geq \alpha_1$ and $\alpha' \geq \alpha_1$. From here we obtain $(f_{\alpha,\varphi(\alpha)}, f_{\alpha',\varphi(\alpha')}) \in U^3$ for $(\alpha, \varphi) \geq (\alpha_1, \varphi_U)$ and $(\alpha', \varphi') \geq (\alpha_1, \varphi_U)$, that is $\{f_{\alpha,\varphi(\alpha)}\}_{\alpha\in D}$ is a net in $\mathcal{E}(\Gamma_\mu, X)$ which is Cauchy in $I\!\!R^S(\Gamma_\mu)$.

Now let V_0 be a neighborhood of the origin in $\tilde{\mathcal{K}}(X)$ and let W be a symmetrical vicinity of $\tilde{\mathcal{K}}(X)$ such that $W^2(0) \in V_0$.

For this $W(0)$ there exist $\alpha_0, \mathcal{V}_\mu$ and M such that the conditions I) and II) of theorem are simultaneously satisfied.

Then there exists $\varphi_1 \in \prod_{\alpha\in D} D_\alpha$ such that for every $\varphi \in \prod_\alpha D_\alpha$ with $\varphi \geq \varphi_1$ we have $(\int_T f_{\alpha,\varphi(\alpha)} d\mu, \int_T f_\alpha d\mu) \in W$ for every $\alpha \in D$ and every $T \in \mathcal{R}$.

Now if $(\alpha, \varphi) \geq (\alpha_0, \varphi_1)$ we obtain $\int_T f_{\alpha,\varphi(\alpha)} d\mu \in V_0$ for $T \in \mathcal{V}_\mu$ or $T \in \mathcal{R}$ with $T \subset S \setminus M$, that is the net $\{\int_T f_{\alpha,\varphi(\alpha)} d\mu\}$ is a Cauchy net uniform with respect to $T \in \mathcal{R}$. Consequently, there exists α_1 such that for $\alpha \geq \alpha_1$ and $\alpha' \geq \alpha_1$ we have $(\int_T f_\alpha d\mu, \int_T f_{\alpha'} d\mu) \in W^2$ uniformly in $T \in \mathcal{R}$ whence the theorem follows. $\square$

Remark. On $\mathcal{L}(\Gamma_\mu, X)$ we can introduce a uniform structure which we shall call *the uniform structure of the* Γ_μ*–mean.*

To see this let $\mathcal{W}$ be the uniform structure on $\tilde{\mathcal{K}}(X)$. For every $W \in \mathcal{W}$ we consider

$$E_W = \{(f, g) \in \mathcal{L}(\Gamma_\mu, X) \times \mathcal{L}(\Gamma_\mu, X); (\int_T f d\mu, \int_T g d\mu) \in W,\ (\forall) T \in \mathcal{R}\}.$$

It is easy to see that the family $\{E_W\}_{W\in\mathcal{W}}$ is a base of vicinities for a uniform structure $\mathcal{T}_W$ on $\mathcal{L}(\Gamma_\mu, X)$.

Let $\mathcal{M}(\mathcal{L}(\Gamma_\mu, X))$ be the uniform structure induced on $\mathcal{L}(\Gamma_\mu, X)$ by $I\!\!R^S(\Gamma_\mu)$.

The uniform structure on $\mathcal{L}(\Gamma_\mu, X)$ defined by $\sup\{\mathcal{M}(\mathcal{L}(\Gamma_\mu, X)), \mathcal{T}_W\}$ is said to be *the uniform structure of the* Γ_μ*–mean convergence.*

The convergence of a net $\{f_\alpha\}_{\alpha\in D}$ to f with respect to this uniformity considered on $\mathcal{L}(\Gamma_\mu, X)$ is called then *convergence in* Γ_μ*–mean* and we denote it by $f_\alpha \longrightarrow f$ (Γ_μ–mean).

Definition 4.5. A net $\{f_\alpha\}_{\alpha\in D}$ of Γ_μ–integrable functions is said to be Γ_μ–*equiintegrable* if $\{f_\alpha\}$ satisfies the conditions I) and II) of theorem 4.4.

Now, we can prove a Vitali type theorem.

Theorem 4.6. *Let $\{f_\alpha\}_{\alpha\in D}$ be a net of $\mathcal{L}(\Gamma_\mu, X)$. Then a function $f \in I\!R^S$ is Γ_μ–integrable and $f_\alpha \longrightarrow f$ (Γ_μ–mean) if and only if the following two conditions hold:*

I. $f_\alpha \xrightarrow{\Gamma_\mu} f$;

II. $\{f_\alpha\}_{\alpha\in D}$ *is a Γ_μ-equiintegrable net.*

Proof. The necessity may be easily obtained from the theorem 4.4.

Conversely, let $\{f_\alpha\}_{\alpha\in D}$ be a net of $\mathcal{L}(\Gamma_\mu, X)$ which satisfies the conditions I) and II) of the theorem. For every $\alpha \in D$ there exists a net $\{f_{\alpha,\beta}\}_{\beta\in D_\alpha}$ of $\mathcal{E}(\Gamma_\mu, X)$ such that $f_{\alpha,\beta} \longrightarrow f$ (Γ_μ–mean).

Let $\{f_{\alpha,\varphi(\alpha)}; \alpha \in D, \varphi \in \prod\limits_\alpha D_\alpha\}$ be the corresponding diagonale approximation. As in the proof of the theorem 4.4 we obtain that $f_{\alpha,\varphi(\alpha)} \longrightarrow f$ (Γ_μ–mean) and then $f \in \mathcal{L}(\Gamma_\mu, X)$.

Moreover $\lim\limits_{\alpha,\varphi} \displaystyle\int_T f_{\alpha,\varphi(\alpha)} d\mu = \int_T f d\mu$ uniformly with respect to $T \in \mathcal{R}$.

Now let W be a symmetrical vicinity of $\tilde{\mathcal{K}}(X)$.

There exists $(\alpha_1, \varphi_1) \in D \times \prod\limits_\alpha D_\alpha$ such that for every $(\alpha, \varphi) \geq (\alpha_1, \varphi_1)$, $(\int_T f d\mu, \int_T f_{\alpha,\varphi(\alpha)} d\mu) \in W$ and $\int_T f_{\alpha,\varphi(\alpha)} d\mu, \int_T f_\alpha d\mu) \in W$, uniformly in $T \in \mathcal{R}$.

If $\alpha \geq \alpha_1$, then $(\int_T f_\alpha d\mu, \int_T f d_\mu) \in W^3$ uniformly in T whence the theorem follows. $\square$

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