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The Penrose Transform for Certain Non-Compact Homogeneous Manifolds of $U(n,n)$

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Abstract. We construct "the Penrose transform" as an intertwining operator between two different geometric realization of infinite dimensional representations of $U(n,n)$, namely, from the space of the Dolbeault cohomology group on a non-compact complex homogeneous manifold to the space of holomorphic functions over the bounded domain of type $AIII$. We show that the image of the Penrose transform satisfies the system $(\mathcal{M}_k)$ of partial differential equations of order $k+1$ which we find in explicit forms. Conversely, we also prove that any solution of the system $(\mathcal{M}_k)$ is uniquely obtained as the image of the Penrose transform, by using the theory of prehomogeneous vector spaces.

§0. Introduction

Although our result in this paper is a small example, it is motivated by several different fields of mathematics:

1) Penrose correspondence massless field equations.

In 1966, R. Penrose found twistor construction of solutions of the so-called massless field equations (e. g. Maxwell's equations). In 1981, M. Eastwood - R. Penrose - R. Wells proved that this Penrose transform is an isomorphism between cohomology and massless fields. A very special case of our main results ($n = 2$, $k = 1$ with the notation later) gives an alternative proof of their results.

2) Construction of global solutions of partial differential equations.

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It is a basic problem in global analysis to construct all solutions of partial differential equations by means of integral transforms. This problem has been also one of the main problems in integral geometry. In 1938 ([J]), F. John constructed all solutions with a mild growth condition of the ultrahyperbolic equation

$$\frac{\partial^2 F}{\partial a \partial d} - \frac{\partial^2 F}{\partial b \partial c} = 0$$

as the image of an integral transform

$$f(x, y, z) \mapsto F_f(a, b, c, d) = \int_{-\infty}^{\infty} f(t, at + b, ct + d) dt.$$

The Poisson transform for a Riemannian symmetric space G/K is another example which is an integral transform from the space of hyperfunction valued spherical principal series (functions on the Martin boundary of G/K) onto the space of all solutions of certain partial differential equations on G/K. Each of these examples presents a bijective integral transform:

$$\begin{aligned}&\{\textit{all} \text{ functions on a manifold } N\}\\ &\quad \to \{\textit{all} \text{ solutions of partial differential equations on a manifold } M\}.\end{aligned}$$

A natural requirement in this formulation is an inequality of dimensions, namely, $\dim N < \dim M$.

In this paper, we shall give another explicit example of integral transforms which constructs all solutions of the system of partial differential equations $(\mathcal{M}_k)$ on a classical bounded domain of type $AIII$

$$M := \{Z = (z_{ij})_{1 \le i,j \le n} \in M(n, \mathbb{C}) : I_n - Z^* Z \gg 0\}.$$

Here for each fixed integer k $(1 \le k \le n)$, the system of partial differential equations $(\mathcal{M}_k)$ is defined by

$$(\mathcal{M}_k) \quad \det \left(\frac{\partial}{\partial z_{ij}} \right)_{i \in I, j \in J} F = 0 \qquad (F(Z) \in \mathcal{O}(M))$$
$$\text{for all } I, J \subset \{1, \cdots, n\} \text{ such that } |I| = |J| = k + 1.$$

If $k = 1$, then $(\mathcal{M}_k)$ is a system of differential equations of the form

$$(\frac{\partial^2}{\partial z_{il}\partial z_{jm}} - \frac{\partial^2}{\partial z_{im}\partial z_{jl}})F(Z) = 0 \quad (1 \leq i, j, l, m \leq n).$$

Such a system has been recently intensively studied in the context of hypergeometric functions with "multivariables" due to Gelfand. To construct all solutions of the system $(\mathcal{M}_k)$, we define a non-compact complex manifold $N = U(n,n)/U(k) \times U(n-k,n)$, which is a generalization of the upper half plane (see §1.3). Then our main theorem asserts:

THEOREM (see Theorem in §1). *The Penrose transform $\mathcal{R}$ gives a bijection from*

$\{$*the space of Dolbeault cohomology of holomorphic line bundles on* $N\}$

to

$\{$*the space of all solutions of* $(\mathcal{M}_k)$ *on* $M\}$.

We note that $\dim_{\mathbb{C}} N = k(2n-k) \leq \dim_{\mathbb{C}} M = n^2$. A precise formulation of the main theorem will be given in §1. We shall prove in §7 that the dimension of the space of solutions of generalized Aomoto-Gelfand equations is finite. There are two remarkable features in our setting, that is,

i) The manifold $N = U(n,n)/U(k) \times U(n-k,n)$ depends on $k+1$, which is the order of the system of partial differential equations $(\mathcal{M}_k)$.
ii) We use arbitrary elements in the Dolbeault cohomology group on N instead of arbitrary "functions on N".

Another motivation of this paper is to understand:

3) Irreducible unitary representations with singular parameter.

In the late twenty years, a number of powerful methods have been developed in representation theory of semisimple Lie groups, based on geometric quantization, asymptotics of matrix coefficients, algebraic theory of Harish-Chandra modules such as Vogan-Zuckerman theory, $\mathcal{D}$-modules on flag varieties and so on. Now, irreducible representations with regular parameter are fairly well-understood, while the status of those with singular parameter is still mysterious. Irreducible infinite dimensional representations that we treat in this paper have two features:

i) they have singular parameter (difficult aspect),

ii) they have highest weight vectors (easy aspect).

Among representations that enjoy (i) and (ii), N. Wallach captured some part of unitary highest weight modules with singular parameter as a coherent continuation of holomorphic discrete series out of the canonical Weyl chamber, which is now known as *the Wallach set* ([Wal]). The Penrose transform in our theorem gives a bridge between a holomorphic realization over a Hermitian symmetric space M and another realization in the Dolbeault cohomology space on a non-compact complex manifold N. The parameter lies in the Wallach set for the first realization (namely, out of the canonical Weyl chamber), while the parameter lies on the "wall" of the weakly fair range in the sense of Vogan for the second realization, after a careful computation of "ρ-shift" with respect to two different parabolic subgroups. In general, the representations in the Wallach set are difficult to understand. So we hope that our approach by the Penrose transform gives a better understanding of these singular representations.

Fourth motivation of this paper is to find:

4) Explicit differential operators that characterize irreducible representations.

Discrete series representations with very regular parameter can be realized as the space of all solutions of an elliptic differential equation of first order acting on sections of an equivariant vector bundles on a Riemannian symmetric space by a result of W. Schmid ([Sch1]). This result is recently generalized by H. Wong for $A_{\mathfrak{q}}(\lambda)$ in the sense of Vogan-Zuckerman ([Wo]). However, their results deal with only representations having very regular parameter. As the parameter of representations tends to be singular, one might expect that the corresponding representation spaces become "smaller" and satisfy more differential equations. Our results can be interpreted as an explicit model of this phenomenon. Namely, we consider a coherent family of holomorphic discrete series of $U(n,n)$ which are realized as the space of all solutions of the Schmid operator (which reduces to the Cauchy-Riemann equation in our setting) for very regular parameters. Our main results assert that the coherent family of holomorphic discrete series in the Wallach set (out of the canonical Weyl chamber) satisfies not only the

Schmid differential equation but also another system of differential equations $(\mathcal{M}_k)$ of order $k+1$. The latter equations make the representation space (i. e. the space of all solutions) smaller as the parameter tends to be more singular.

Generalized hypergeometric equations and Representations

This paper treats the interaction of the following four objects.

A) The characterization of singular unitary representations (in particular, in the Wallach set) by differential equations (see Motivations 3 and 4).

B) A generalization of the Aomoto-Gelfand hypergeometric differential equations to higher order.

C) Construction of all global solutions of the above equations by the intertwining operators from degenerate standard representations; — the Penrose transform from Dolbeault cohomology representations (imaginary polarization), the Poisson transform from degenerate principal series representations (real polarization) (see Motivation 2).

D) Prehomogeneous vector spaces and the b-function of Bernstein-Sato.

Some connection of (A) with (D) was previously observed for certain special unitary highest weight modules. The connection of (B) with integral geometry was due to the Gelfand school. (C) was studied in special cases (either maximal parabolic or regular parameters) such as differential equations of second or third order by Hua, Johnson, Berline and Vergne in the real polarization; by Penrose, Eastwood, Wells, Schmid, Mantini and Gindikin in the imaginary polarization; but the viewpoint of (B) was not emphasized at that time. The explicit idea and formulation of interacting (A), (B), (C) and (D) in the above generality were proposed and clarified by T. Kobayashi in 1994.

Historical Notes

Several historical remarks on the Penrose transform are in order.

The "Penrose transform" was named after the work of R. Penrose in mathematical physics, which corresponds to the case $n = 2$ and $k = 1$ in our main theorem ([P], see also a survey paper of R. Wells [We1]).

The injectivity of the Penrose transform has been studied extensively in a more general setting by C. Patton - H. Rossi based on results due to Grauert and Kodaira-Spencer ([PR]). In §3, we give an alternative and elementary proof of the injectivity of the Penrose transform.

The image of the Penrose transform satisfies the system of differential equations $(\mathcal{M}_k)$. Such a system was studied by a series of papers of Mantini

([Ma1], [Ma2], [Ma3]). We give a simple and different proof of her results by using a theory of prehomogeneous vector space in §5.

The main novelty of this paper is to prove the surjectivity of the Penrose transform $\mathcal{R}$ onto the space of all solutions of $(\mathcal{M}_k)$ for arbitrary n and k in §6. The surjectivity of $\mathcal{R}$ has been known only in the lower dimensional case $n = 2,\ k = 1$ ([P], [We1]). In the lectures at University of Tokyo in 1994, Gindikin explained an alternative proof (still assuming $n = 2, k = 1$) of the surjectivity of the Penrose transform using a method of integral geometry. The latter half of this paper is devoted to the proof of the surjectivity of $\mathcal{R}$ for arbitrary n and k. Our approach here is new even in the special case $n = 2$ and $k = 1$. The key step is to find an explicit formula of the radial part of the generators of the system $(\mathcal{M}_k)$ with respect to the Bruhat decomposition, which is proved in §4. Our method relies on the theory of prehomogeneous vector spaces in the sense of M. Sato ([Sa]).

Another novelty of this paper is to formulate a generalization of Aomoto-Gelfand hypergeometric equations (of higher order) and give the finite dimensionality theorem (see §7). Our proof uses a special case of Kobayashi's theory of the "admissible restriction" of irreducible unitary representations ([Ko3],[Ko4]).

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Contents.

§1. Statement of main results

1.1. Definition of $V(n,k)$

We define an indefinite unitary group by

$$G := U(n,n) \equiv \{g \in GL(2n,\mathbb{C}) : g^* J_{n,n} g = J_{n,n}\},$$
$$J_{n,n} := \operatorname{diag}(\underbrace{1,\cdots,1}_{n},\underbrace{-1,\cdots,-1}_{n}) \in GL(2n,\mathbb{C}).$$

Let $K = U(n) \times U(n)$, a maximal compact subgroup of G. We define a family of maximal parabolic subgroups $Q(k)$ $(k = 0, 1, \cdots, n)$ of $G_{\mathbb{C}} = GL(2n,\mathbb{C})$ by

$$\begin{aligned} Q(k) &\equiv Q_{2n}(k) \\ &:= \{g = (g_{ij}) \in GL(2n,\mathbb{C}) : g_{ij} = 0 \ \ k+1 \le i \le 2n,\ 1 \le j \le k\}. \end{aligned}$$

Then the complex homogeneous space $G_{\mathbb{C}}/Q(k)$ is naturally biholomorphic to the complex Grassmannian manifold $Gr_k(\mathbb{C}^{2n})$. We define

$$L \equiv L(k) := G \cap Q(k) \simeq U(k) \times U(n-k,n).$$

Then we have an open embedding (the Borel embedding)

$$G/L \subset G_{\mathbb{C}}/Q(k). \tag{1.1.1}$$

Note that if $n = k = 1$, then (1.1.1) corresponds to a classical embedding:

$$\text{the Poincaré plane } \subset \mathbb{P}^1\mathbb{C}.$$

As an open set of a complex manifold $G_{\mathbb{C}}/Q(k)$, we equip G/L with a G-invariant complex structure. Let

$$\begin{aligned} \mathfrak{g} &:= \operatorname{Lie}(G) \underset{\mathbb{R}}{\otimes} \mathbb{C} \simeq \mathfrak{gl}(2n,\mathbb{C}), \\ \mathfrak{l} \equiv \mathfrak{l}(k) &:= \operatorname{Lie}(L(k)) \underset{\mathbb{R}}{\otimes} \mathbb{C} \simeq \mathfrak{gl}(k,\mathbb{C}) \oplus \mathfrak{gl}(2n-k,\mathbb{C}), \\ \mathfrak{q} \equiv \mathfrak{q}(k) &:= \operatorname{Lie}(Q(k)) = \mathfrak{l} + \mathfrak{u} \quad \text{(Levi decomposition)}, \\ \mathfrak{k} &:= \operatorname{Lie}(K) \underset{\mathbb{R}}{\otimes} \mathbb{C} \simeq \mathfrak{gl}(n,\mathbb{C}) \oplus \mathfrak{gl}(n,\mathbb{C}), \\ \mathfrak{h} &:= \sum_{i=1}^{2n} \mathbb{C} E_{ii} \ (\subset \mathfrak{g}). \end{aligned}$$

Here E_{ij} is the matrix unit. We remark that $\mathfrak{h}$ is a Cartan subalgebra of any of the reductive subalgebras $\mathfrak{k}$, $\mathfrak{l}$ or $\mathfrak{g}$, because $\mathfrak{h} \subset \mathfrak{k}$, $\mathfrak{h} \subset \mathfrak{l}$ and because $\operatorname{rank} G = \operatorname{rank} K = \operatorname{rank} L = 2n$. Let $\{e_i : 1 \le i \le 2n\}$ be the dual basis of the basis $\{E_{ii} : \ 1 \le i \le 2n\}$ of $\mathfrak{h}$. We identify $\mathfrak{h}^*$ with $\mathbb{C}^{2n}$ via the basis $\{e_j : 1 \le j \le 2n\}$. Then the weight of the nilpotent radical $\mathfrak{u} \equiv \mathfrak{u}(k)$ of a parabolic subalgebra $\mathfrak{q}(k)$ with respect to $\mathfrak{h}$ is given by

$$\Delta(\mathfrak{u}, \mathfrak{h}) = \{e_i - e_j : 1 \le i \le k,\ k+1 \le j \le 2n\}.$$

We put

$$\begin{aligned}\rho(\mathfrak{u}) \equiv \rho(\mathfrak{u}(k)) &:= \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{u},\mathfrak{h})} \alpha \ \in \mathfrak{h}^* \\ &= \frac{1}{2}(2n-k, \cdots, 2n-k, -k, \cdots, -k) \equiv \frac{2n-k}{2}\mathbf{1}_k \oplus (-\frac{k}{2})\mathbf{1}_{2n-k}.\end{aligned}$$

Next, we define a character parametrized by $m \in \mathbb{Z}$:

$$\nu_m \equiv \nu_m^{(k)} : L = U(k) \times U(n-k, n) \to \mathbb{C}^\times,\ (a, d) \mapsto (\det a)^m.$$

This one dimensional representation $(\nu_m^{(k)}, \mathbb{C})$ of L will be simply denoted by $\mathbb{C}_m$.

We are particularly interested in the case $m = n$, here. Let

$$G \underset{L}{\times} \mathbb{C}_n = U(n,n) \underset{U(k)\times U(n-k,n)}{\times} (\nu_n^{(k)}, \mathbb{C})$$

be an associated holomorphic line bundle over a complex manifold $G/L \equiv G/L(k) \equiv U(n,n)/U(k) \times U(n-k, n)$. Let $V(n,k)$ be the $k(n-k)$-th Dolbeault cohomology group of the holomorphic line bundle $G \underset{L}{\times} \mathbb{C}_n \to G/L$, which is equipped with a Fréchet topology by a recent result of Wong ([Wo]). Thus we have defined a finite family of Fréchet representations $(\pi_{n,k}, V(n,k))$ of $U(n,n)$ with parameter $k = 0, 1, \cdots, n$. These representations are our main subject. Similarly, we consider a holomorphic line bundle over a complex manifold $G/K \equiv G/L(n)$:

$$G \underset{K}{\times} \mathbb{C}_k \equiv G \underset{K}{\times} (\nu_k^{(n)}, \mathbb{C}) \to G/K$$

associated with a character

$$\nu_k^{(n)} : K = U(n) \times U(n) \to \mathbb{C}^\times,\ (a,d) \mapsto (\det a)^k.$$

Let $\mathcal{E}(G \underset{K}{\times} \mathbb{C}_k)$, $\mathcal{O}(G \underset{K}{\times} \mathbb{C}_k)$ be the space of smooth sections, the space of global holomorphic sections, respectively. Then $\mathcal{O}(G \underset{K}{\times} \mathbb{C}_k)$ is a Fréchet representation of $G = U(n,n)$ which has a Jordan-Hölder series of finite length (Fact (2.2)).

1.2. Penrose transform

In this section, we define the "Penrose transform", which is an intertwining map between representations of G:

$$\mathcal{R} : H_{\bar\partial}^{k(n-k)}(G \underset{L(k)}{\times} \mathbb{C}_n) \to \mathcal{E}(G \underset{K}{\times} \mathbb{C}_k) \qquad (k = 0, 1, \cdots, n).$$

First, let

$$i : K/L \cap K \subset G/L$$

be the natural embedding, which is holomorphic if we define a complex structure on $K/L \cap K$ through an isomorphism $K/L \cap K \simeq K_{\mathbb{C}}/Q(k) \cap K_{\mathbb{C}}$. We note that i is a K-equivariant map. The left action of G on G/L is given by

$$l_g : G/L \to G/L, \quad xL \mapsto gxL \quad (\text{ for each } g \in G).$$

Then the action of G on $V(n,k) = H_{\bar\partial}^{k(n-k)}(G \underset{L}{\times} \mathbb{C}_n)$ is by definition:

$$\pi_{n,k}(g)[\omega] = [{l_{g^{-1}}}^* \omega] \ \text{ for } [\omega] \in V(n,k).$$

Let

$$Z^{0,k(n-k)}(G \underset{L}{\times} \mathbb{C}_n) := \operatorname{Ker}(\bar\partial \colon \mathcal{E}^{0,k(n-k)}(G \underset{L}{\times} \mathbb{C}_n) \to \mathcal{E}^{0,k(n-k)+1}(G \underset{L}{\times} \mathbb{C}_n)),$$

the subspace of $\bar\partial$-closed form on G/L. If $\omega \in Z^{0,k(n-k)}(G \underset{L}{\times} \mathbb{C}_n)$, then

$$i^* {l_g}^* \omega \in \mathcal{E}^{0,k(n-k)}(K \underset{L \cap K}{\times} \mathbb{C}_n)$$

is also a $\bar\partial$-closed form on $K/L\cap K$, giving rise to a cohomology class

$$[i^* {l_g}^* \omega] \in H^{k(n-k)}_{\bar\partial}(K \underset{L\cap K}{\times} \mathbb{C}_n) =: W.$$

Thus we have defined a map

$$\begin{aligned}(1.2.1)\qquad \widetilde{\mathcal{R}}\colon Z^{0,k(n-k)}(G\underset{L}{\times}\mathbb{C}_n) \times G \to W = H^{k(n-k)}_{\bar\partial}(K \underset{L\cap K}{\times} \mathbb{C}_n),\\ (\omega, g) \mapsto [i^* {l_g}^* \omega].\end{aligned}$$

If $\omega \in \mathcal{E}^{0,k(n-k)}(G\underset{L}{\times}\mathbb{C}_n)$ is a $\bar\partial$-exact form $\omega = \bar\partial\eta$ on G/L, then $i^*{l_g}^*\omega = i^*{l_g}^*\bar\partial\eta = \bar\partial i^*{l_g}^*\eta \in \mathcal{E}^{0,k(n-k)}(K \underset{L\cap K}{\times} \mathbb{C}_n)$ is also a $\bar\partial$-exact form on $K/L\cap K$. Therefore $\widetilde{\mathcal{R}}(\omega,\cdot)$ depends only on the cohomology class $[\omega] \in H^{k(n-k)}_{\bar\partial}(G\underset{L}{\times}\mathbb{C}_n)$. Hence (1.2.1) is well-defined on the level of cohomology:

$$\widetilde{\mathcal{R}}\colon H^{k(n-k)}_{\bar\partial}(G\underset{L}{\times}\mathbb{C}_n) \times G \to W,\ ([\omega], g) \mapsto [i^*{l_g}^*\omega].$$

Then $\widetilde{\mathcal{R}}$ respects the action of G and K in the following way:

$$\begin{aligned}\widetilde{\mathcal{R}}(\pi_{n,k}(g_0)[\omega], g) = \widetilde{\mathcal{R}}([{l_{g_0^{-1}}}^*\omega], g) = [i^*{l_g}^*{l_{g_0^{-1}}}^*\omega] = \widetilde{\mathcal{R}}([\omega], g_0^{-1}g),\\ \widetilde{\mathcal{R}}([\omega], gh) = [i^*{l_h}^*{l_g}^*\omega] = [{l_h}^*i^*{l_g}^*\omega] = h^{-1}\widetilde{\mathcal{R}}([\omega], g),\end{aligned}$$

where $g, g_0 \in G,\ h \in K$. These two relations imply that $\widetilde{\mathcal{R}}$ induces a G-intertwining operator between representations of G:

$$(1.2.2)\qquad \mathcal{R}\colon H^{k(n-k)}_{\bar\partial}(G \underset{L(k)}{\times} \mathbb{C}_n) \to \mathcal{E}(G \underset{K}{\times} W),\ [\omega] \mapsto \widetilde{\mathcal{R}}([\omega], \cdot).$$

This is our definition of the Penrose transform.

By the Borel-Weil-Bott theorem, the highest weight of

$$W = H^{k(n-k)}_{\bar\partial}(K \underset{L\cap K}{\times} \mathbb{C}_n) = H^{k(n-k)}_{\bar\partial}(K \underset{L\cap K}{\times} \mathbb{C}_{n\mathbf{1}_k \oplus 0\mathbf{1}_{2n-k}})$$

with respect to $\Delta^+(\mathfrak{k})$ is given by

$$\begin{aligned}&(n\mathbf{1}_k \oplus 0\mathbf{1}_{2n-k}) - 2\rho(\mathfrak{u}\cap\mathfrak{k})\\ =&(n\mathbf{1}_k \oplus 0\mathbf{1}_{2n-k}) - ((n-k)\mathbf{1}_k \oplus (-k)\mathbf{1}_{n-k} \oplus 0\mathbf{1}_n)\\ =&k\mathbf{1}_n \oplus 0\mathbf{1}_n.\end{aligned}$$

This means that W is isomorphic to one dimensional representation of $U(n)\times U(n)$:

$$(\det)^k \boxtimes \mathbf{1} \in \widehat{U(n)} \boxtimes \widehat{U(n)},$$

which is equal to $\nu_k^{(n)}$ in our notation as before. Thus (1.2.2) is rewritten as

$$\mathcal{R}\colon H_{\bar\partial}^{k(n-k)}(G \underset{L(k)}{\times} \mathbb{C}_n) \to \mathcal{E}(G \underset{K}{\times} \mathbb{C}_k). \tag{1.2.3}$$

1.3. System of differential equations on a bounded symmetric domain

First, we recall a realization of a Hermitian symmetric space $G/K = U(n,n)/U(n)\times U(n)$ as a classical bounded domain in $\mathbb{C}^{n^2} \simeq M(n,\mathbb{C})$ ([Hel]). Let

$$\bar U := \{\begin{pmatrix} I_n & 0 \\ Z & I_n \end{pmatrix} : Z \in M(n,\mathbb{C})\}.$$

Retain notation in §1.1. Then we have

$$G/K \overset{j_1}{\subset} G_{\mathbb{C}}/Q(n) \overset{j_2}{\hookleftarrow} \bar U \simeq M(n,\mathbb{C}). \tag{1.3.1}$$

The point here is that the image of the Borel embedding G/K is contained in the Bruhat cell $\bar U$. This gives a global coordinate of G/K by a biholomorphic map:

$$\bar U \simeq M(n,\mathbb{C}), \quad \begin{pmatrix} I_n & 0 \\ Z & I_n \end{pmatrix} \mapsto Z.$$

The image $D := j_2^{-1} j_1(G/K) \subset M(n,\mathbb{C})$ is precisely given by

$$D = \{Z \in M(n,\mathbb{C}) : I_n - Z^* Z \gg 0\}.$$

The case $n = 1$ corresponds to a well-known realization:

$$U(1,1)/U(1) \times U(1) \simeq \{z \in \mathbb{C} : |z| < 1\} \quad \text{(the Poincaré disk)}.$$

Hereafter, we shall identify G/K with D and trivialize vector bundles over G/K by a global coordinate on D. With the identification $G/K \simeq D$, the action of G on D is defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Z = (c + dZ)(a + bZ)^{-1}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(n,n),\ Z \in D \subset M(n,\mathbb{C})$, because

$$\begin{aligned} &\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} I_n & 0 \\ Z & I_n \end{pmatrix} \\ =&\begin{pmatrix} I_n & 0 \\ (c+dZ)(a+bZ)^{-1} & I_n \end{pmatrix} \\ &\times \begin{pmatrix} a+bZ & b \\ 0 & d-(c+dZ)(a+bZ)^{-1}b \end{pmatrix} \\ \in&\begin{pmatrix} I_n & 0 \\ (c+dZ)(a+bZ)^{-1} & I_n \end{pmatrix} Q(n). \end{aligned} \tag{1.3.2}$$

Suppose

$$\chi = (\chi_1, \chi_2) \colon K = U(n) \times U(n) \to \mathbb{C}^\times$$

is a character, which we also extend to a holomorphic character

$$\chi = (\chi_1, \chi_2) \colon Q(n) \to \mathbb{C}^\times$$

letting the restriction of χ to the unipotent radical trivial. According to (1.3.1), we have a commutative diagram of associated line bundles:

$$\begin{array}{ccccc} G \underset{K}{\times} \mathbb{C}_\chi & \subset & G_{\mathbb{C}} \underset{Q(n)}{\times} \mathbb{C}_\chi & \supset & \bar{U} \times \mathbb{C} \\ \downarrow & & \downarrow & & \downarrow \\ G/K & \underset{j_1}{\subset} & G_{\mathbb{C}}/Q(n) & \underset{j_2}{\supset} & \bar{U}. \end{array}$$

So we have a homeomorphism of Fréchet spaces

$$\mathcal{E}(G\underset{K}{\times}\mathbb{C}_\chi) \simeq C^\infty(D). \tag{1.3.3}$$

We can identify by the correspondence, $f \mapsto \tilde{F} \mapsto F$ in the following:

$$\begin{aligned}\mathcal{E}(G\underset{K}{\times}\mathbb{C}_\chi) &= \{f\colon G \to \mathbb{C} : f(gk) = \chi(k)^{-1} f(g),\ g \in G, k \in K\}\\ &= \{\tilde{F}\colon GQ(n) \to \mathbb{C}\colon \tilde{F}(gk) = \chi(k)^{-1}\tilde{F}(g),\\ &\qquad\qquad g \in GQ(n), k \in Q(n)\},\end{aligned}$$

$$\tilde{F}(\begin{pmatrix} I_n & 0 \\ Z & I_n \end{pmatrix}) = F(Z) \quad (Z \in M(n,\mathbb{C})).$$

Then the action π of G on $C^\infty(D)$ is given by

$$\pi(g)F(Z) = \chi_1^{-1}(a+bZ)\chi_2^{-1}(d-(c+dZ)(a+bZ)^{-1}b)F((c+dZ)(a+bZ)^{-1})$$

if $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(n,n)$, because of (1.3.2).

In particular, if $\chi = \nu_k^{(n)}$, then the corresponding representation of G on $C^\infty(D)$ is given by

$$\tilde{\pi}_{n,k}(g)F(Z) = (\det(a+bZ))^{-k} F((c+dZ)(a+bZ)^{-1}). \tag{1.3.4}$$

Similarly, we have an isomorphism:

$$\mathcal{O}(G\underset{K}{\times}\mathbb{C}_k) \simeq \mathcal{O}(D), \tag{1.3.5}$$

the subrepresentation $\mathcal{O}(G\underset{K}{\times}\mathbb{C}_k) \subset \mathcal{E}(G\underset{K}{\times}\mathbb{C}_k)$ is realized in

$$\mathcal{O}(D) \subset C^\infty(D).$$

(Note that the representation space is the same for all k. And the multiplier depends on the parameter k.)

Now we are ready to introduce a system of partial differential equations on D as follows. For $Z \in M(n,\mathbb{C})$, we write z_{ij} for its (i,j) element. In particular, $\{z_{ij} : 1 \leq i, j \leq n\}$ is global coordinate of a bounded domain D. For $I, J \subset \{1, 2, \cdots, n\}$ such that $|I| = |J| = l$, we define a differential operator of order l:

$$P(I, J) = \det(\frac{\partial}{\partial z_{ij}})_{i \in I, j \in J}.$$

Here, $\dfrac{\partial}{\partial z_{ij}}$ is a holomorphic derivative of the variable z_{ij}. We define a system of differential equations on D by

$$(\mathcal{M}_k) \quad P(I, J)F(Z) = 0$$
$$\text{for any } I, J \subset \{1, 2, \cdots, n\} \text{ such that } |I| = |J| = k + 1.$$

We regard as $(\mathcal{M}_n) = \emptyset$. Denote by $\mathcal{S}ol(\mathcal{M}_k)$ the subspace of holomorphic functions on D that satisfy the system $(\mathcal{M}_k)$. From the Laplace expansion formula of the determinant of matrices, we have

$$\mathbb{C} = \mathcal{S}ol(\mathcal{M}_0) \subset \mathcal{S}ol(\mathcal{M}_1) \subset \mathcal{S}ol(\mathcal{M}_2) \subset \cdots \subset \mathcal{S}ol(\mathcal{M}_n) = \mathcal{O}(D).$$

In other words,

$$\begin{aligned} \mathcal{S}ol(\mathcal{M}_k) &= \{F \in \mathcal{O}(D) : P(I, J)F = 0 \ \text{ for any } I, J \ |I| = |J| = k + 1\} \\ &= \{F \in \mathcal{O}(D) : P(I, J)F = 0 \ \text{ for any } I, J \ |I| = |J| \geq k + 1\}. \end{aligned}$$

Now we are ready to state the main result of this paper.

THEOREM. (1) *The Penrose transform*

$$\mathcal{R} \colon H_{\bar{\partial}}^{k(n-k)}(G \underset{L}{\times} \mathbb{C}_n) \to \mathcal{E}(G \underset{K}{\times} \mathbb{C}_k) \quad (k = 0, 1, \cdots, n)$$

is a well-defined continuous and G-intertwining operator.
(2) $\mathcal{R}$ *is injective.*
(3) *The image of* $\mathcal{R}$ *is contained in* $\mathcal{S}ol(\mathcal{M}_k)$*, the solutions of a system of partial differential equations* $(\mathcal{M}_k)$.
(4) $\mathcal{R}$ *is a bijection between the* K*-finite vectors of* $H_{\bar{\partial}}^{k(n-k)}(G \underset{L}{\times} \mathbb{C}_n)$ *and* $\mathcal{S}ol(\mathcal{M}_k)$.

REMARK. Wong proved that $H_{\bar{\partial}}^{k(n-k)}(G \underset{L}{\times} \mathbb{C}_n)$ is a maximal globalization of its Harish-Chandra module ([Wo], 1991). Then, in view of [Sch1], our result would also imply the bijection between $H_{\bar{\partial}}^{k(n-k)}(G \underset{L}{\times} \mathbb{C}_n)$ and $\mathcal{S}ol(\mathcal{M}_k)$.

§2. Preliminary results

We review some algebraic results on the representations $V(n,k)$ and $\mathcal{O}(G\underset{K}{\times}\mathbb{C}_m)$.

Let W be a Weyl group of the root system $\Delta(\mathfrak{g},\mathfrak{h})$. Then W is identified with the $2n$-th symmetric group $\mathfrak{S}_{2n}$ if we identify $\mathfrak{h}$ with $\mathbb{C}^{2n}$ by the basis $\{E_{ii} : 1 \leq i \leq 2n\}$. Fix a positive system of $\Delta(\mathfrak{k},\mathfrak{h})$ by

$$\Delta^+(\mathfrak{k},\mathfrak{h}) := \{e_i - e_j : 1 \leq i < j \leq n \text{ or } n+1 \leq i < j \leq 2n\}.$$

Then $\lambda = (\lambda_1, \cdots, \lambda_{2n}) \in \mathfrak{h}^* \simeq \mathbb{C}^{2n}$ is a dominant integral weight with respect to $\Delta^+(\mathfrak{k},\mathfrak{h})$, if and only if $\lambda_j \in \mathbb{Z}$ $(1 \leq j \leq 2n)$ and $\lambda_1 \geq \cdots \geq \lambda_{2n}$.

We denote by $F(K,\lambda)$ the irreducible finite dimensional representation of K with highest weight $\lambda \in \mathfrak{h}^*$. According to the direct product $K \simeq U(n) \times U(n)$, any irreducible representation of K is isomorphic to the outer tensor product of two irreducible representations of $U(n)$:

$$F(U(n) \times U(n), \lambda) \simeq F(U(n), (\lambda_1, \cdots, \lambda_n)) \boxtimes F(U(n), (\lambda_{n+1}, \cdots, \lambda_{2n})).$$

It is a theorem due to Schmid, Wong that the Harish-Chandra module of $V(n,k)$ is isomorphic to a derived functor module $\mathcal{R}_{\mathfrak{q}(k)}^{k(n-k)}(\mathbb{C}_{\lambda_k})$ in the sense of Zuckerman-Vogan ([Vo]), where $\mathbb{C}_{\lambda_k}$ is a character of $\mathfrak{l}(k)$ determined by its restriction to $\mathfrak{h}$:

$$\lambda_k := d\nu_n^{(k)} - \rho(\mathfrak{u}) = \frac{k}{2}\mathbf{1}_{2n} = (\frac{k}{2}, \cdots, \frac{k}{2}) \in \mathfrak{h}^*.$$

Then we see that the representation $V(n,k)_K$ is a special case of those representations (not necessarily highest weight modules), whose algebraic properties are studied in ([Ko1]). The correspondence is given by

$$p \Rightarrow n, q \Rightarrow n,\ r \Rightarrow k,\ s \Rightarrow 0,$$
$$\lambda_1, \cdots, \lambda_r \Rightarrow k,\ \lambda_{r+1}, \cdots, \lambda_{r+s} \Rightarrow \emptyset,$$

in the notation loc. cit. In particular, we have

FACT 2.1. (0) *$V(n,k)$ has a $Z(\mathfrak{g})$-infinitesimal character*

$$(\underbrace{k-\frac{1}{2}, k-\frac{3}{2}, \cdots, \frac{3}{2}, \frac{1}{2}}_{k}, \underbrace{n-\frac{1}{2}, n-\frac{3}{2}, \cdots, -n+k+\frac{1}{2}}_{2n-k}) \in \mathfrak{h}^*/W$$

in the Harish-Chandra parametrization, namely, $\mathrm{Hom}_{\mathbb{C}\text{-alg}}(Z(\mathfrak{g}), \mathbb{C}) \simeq \mathfrak{h}^*/W \simeq \mathbb{C}^{2n}/\mathfrak{S}_{2n}$.
(1) *$V(n,k)$ $(k = 0, 1, 2, \cdots, n)$ is a non-zero irreducible representation of $U(n,n)$.*
(2) *(Blattner formula) The K-module structure of $V(n,k)$ is given by*

$$V(n,k)|_K \simeq \bigoplus_{\substack{\mu_1 \geq \cdots \geq \mu_k \\ \mu_j \in \mathbb{N}}} F(U(n), (\mu_1 + k, \cdots, \mu_k + k, \underbrace{k, \cdots, k}_{n-k})) \boxtimes F(U(n), (\underbrace{0, \cdots, 0}_{n-k}, -\mu_k, \cdots, -\mu_1)).$$

We note that $V(n,0)$ is a one dimensional representation, and that $V(n,k)$ $(n \geq k \geq 1)$ is an infinite dimensional representation.

Similarly, the Harish-Chandra module of $\mathcal{O}(G \underset{K}{\times} \mathbb{C}_k)$ is isomorphic to $\mathcal{R}^0_{\mathfrak{q}(n)}(\mathbb{C}_{(k-\frac{n}{2})\mathbf{1}_n \oplus \frac{n}{2}\mathbf{1}_n})$. This follows from the computation:

$$k\mathbf{1}_n \oplus 0\mathbf{1}_n - \rho(\mathfrak{u}(n)) = (k\mathbf{1}_n \oplus 0\mathbf{1}_n) - (\frac{n}{2}\mathbf{1}_n \oplus (-\frac{n}{2})\mathbf{1}_n) = (k - \frac{n}{2})\mathbf{1}_n \oplus \frac{n}{2}\mathbf{1}_n.$$

Thus, we have

FACT 2.2. *Suppose $k \in \mathbb{Z}$.*
(0) *$\mathcal{O}(G \underset{K}{\times} \mathbb{C}_k)$ is a representation of $G = U(n,n)$ with a $Z(\mathfrak{g})$-infinitesimal character*

$$(\underbrace{k-\frac{1}{2}, k-\frac{3}{2}, \cdots, k-n+\frac{1}{2}}_{n}, \underbrace{n-\frac{1}{2}, n-\frac{3}{2}, \cdots, \frac{1}{2}}_{n}).$$

(1) *$\mathcal{O}(G \underset{K}{\times} \mathbb{C}_k)$ is irreducible if $k \geq n$.*

(2)

$$\mathcal{O}(G\underset{K}{\times}\mathbb{C}_k)|_K \simeq \bigoplus_{\substack{\mu_1\geq\cdots\geq\mu_n \\ \mu_j\in\mathbb{N}}} F(U(n),(\mu_1+k,\cdots,\mu_n+k)) \boxtimes F(U(n),(-\mu_n,\cdots,-\mu_1)),$$

if $k \geq 0$.

§3. Injectivity

In this section, we prove the injectivity of $\mathcal{R}$ in Corollary (3.12). Because $V(n,k)$ is irreducible, it suffices to show $\mathcal{R}$ is not identically zero. To see this, we first study some geometry of $G/L(k)$. The setting in this section is slightly more general than what we need, that is, we suppose $G = U(p,q)$. The injectivity of $\mathcal{R}$ in our main Theorem is proved in Corollary (3.12), where we assume $p = q = n$.

In this section, we suppose

$$\begin{aligned} G_{\mathbb{C}} &:= GL(p+q,\mathbb{C}), \\ Q(k) &:= \{g=(g_{ij}) \in GL(p+q,\mathbb{C}) : g_{ij}=0, \\ &\qquad\qquad k+1 \leq i \leq p+q,\ 1 \leq j \leq k\}. \end{aligned}$$

This notation coincides with that of other sections, if we put $p = q = n$.

We recall the notation in §1. In particular, the quadratic form associated to $J_{p,q} = \mathrm{diag}(\underbrace{1,\cdots,1}_{p},\underbrace{-1,\cdots,-1}_{q})$ is given by

$$(\overrightarrow{u},\overrightarrow{v})_{p,q} = \overrightarrow{u}^{*} J_{p,q} \overrightarrow{v} \quad (\overrightarrow{u},\overrightarrow{v} \in \mathbb{C}^{p+q}).$$

Lemma 3.1. *Let* $(\ ,\ )_{p,q}$ *be the quadratic form of signature* (p,q). *If* $\overrightarrow{u_1},\cdots,\overrightarrow{u_k} \in \mathbb{C}^{p+q}$ $(k \leq p)$ *satisfy*

$$(\overrightarrow{u_i},\overrightarrow{u_j})_{p,q} = \delta_{ij} \quad (1 \leq i,j \leq k),$$

then there exists $g \in U(p,q)$ *such that* $g\overrightarrow{e_j} = \overrightarrow{u_j}$ $(1 \leq j \leq k)$, *where* $\overrightarrow{e_j}$ *is the* j*-th unit vector* ${}^t(\underbrace{0,\cdots,0}_{j-1},1,\underbrace{0,\cdots,0}_{p+q-j})$.

PROOF. Although this lemma is standard and well-known, we give a sketch of proof for the sake of completeness.
1) Assume $p = q = k = 1$. Let $\overrightarrow{u_1} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2$ such that $(\overrightarrow{u_1}, \overrightarrow{u_1})_{1,1} = |\alpha|^2 - |\beta|^2 = 1$. Then we can find $g \in U(1,1)$ such that $g \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ because of an explicit description:

$$SU(1,1) = \{ \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} : |\alpha|^2 - |\beta|^2 = 1 \}.$$

2) For general p, q, k, we can find $g_1 \in U(p) \times U(q)$ such that

$$g_1 \cdot \overrightarrow{u_1} = {}^t(\underbrace{\alpha, 0, \cdots, 0}_{p}, \underbrace{\beta, 0, \cdots, 0}_{q}).$$

Since $|\alpha|^2 - |\beta|^2 = 1$, we find $g_2 \in U(1,1) \subset U(p,q)$ such that $g_2 \cdot {}^t(\underbrace{\alpha, 0, \cdots, 0}_{p}, \underbrace{\beta, 0, \cdots, 0}_{q}) = {}^t(1, \underbrace{0, \cdots, 0}_{p+q-1})$. Thus, we can replace $\overrightarrow{u_1}$ by ${}^t(1, \underbrace{0, \cdots, 0}_{p+q-1})$. Then the first coordinate of $\overrightarrow{u_2}, \cdots, \overrightarrow{u_k}$ must vanish because $(\overrightarrow{u_1}, \overrightarrow{u_j})_{p,q} = 0$ $(2 \le j \le k)$. So we can regard $\overrightarrow{u_2}, \cdots, \overrightarrow{u_k} \in \mathbb{C}^{p+q-1}$. Repeating this argument, we find $g \in U(p,q)$ such that $g \cdot \overrightarrow{e_j} = \overrightarrow{u_j}$ $(1 \le j \le k)$. $\square$

DEFINITION 3.2. For $m \ge k$, we define a set of regular $m \times k$ matrices by

$$M'(m,k;\mathbb{C}) := \{A \in M(m,k;\mathbb{C}) : \operatorname{rank} A = k\}.$$

Then $M'(m,k;\mathbb{C})$ is an open dense subset of $M(m,k;\mathbb{C})$. We note that $M'(m,m;\mathbb{C}) = GL(m,\mathbb{C})$.

The general linear group $GL(m,\mathbb{C})$ acts on $M'(m,k;\mathbb{C})$ from the left. This action is obviously transitive. The isotropy subgroup at $I_{m,k} = \sum_{j=1}^{k} E_{jj} \in M'(m,k;\mathbb{C})$ is given by

$$\begin{aligned} Q'(k) = \{g = (g_{ij}) \in GL(m,\mathbb{C}) : \ & g_{ij} = \delta_{ij} \ (1 \le i,j \le k), \\ & g_{ij} = 0 \ (k+1 \le i \le m,\ 1 \le j \le k)\}. \end{aligned}$$

Thus we have a $GL(m,\mathbb{C})$-equivariant biholomorphic map

$$GL(m,\mathbb{C})/Q'(k) \simeq M'(m,k;\mathbb{C}).$$

DEFINITION 3.3. Let $\mathrm{pr}\colon M(p+q,k;\mathbb{C}) \to M(p,k;\mathbb{C})$ be the projection given by

$$\mathrm{pr}\begin{pmatrix} a \\ b \end{pmatrix} = a \quad \text{for } a \in M(p,k;\mathbb{C}),\ b \in M(q,k;\mathbb{C}).$$

Then pr respects the right action of $GL(k,\mathbb{C})$. We put

$$M''(p+q,k;\mathbb{C}) := \mathrm{pr}^{-1}(M'(p,k;\mathbb{C})).$$

Clearly $M''(p+q,k;\mathbb{C})$ is an open dense set of $M'(p+q,k;\mathbb{C})$.

LEMMA 3.4. *Let $p \geq k$ and*

$$\iota : M'(p,k;\mathbb{C}) \subset M''(p+q,k;\mathbb{C})$$

be a natural embedding given by $a \mapsto \begin{pmatrix} a \\ 0 \end{pmatrix}$, which respects the right $GL(k,\mathbb{C})$ action and the left $GL(p,\mathbb{C}) \simeq GL(p,\mathbb{C}) \times \mathbf{1}_q$ action. Then

$$\iota(M'(p,k;\mathbb{C})) \subset U(p,q) I_{p+q,k} GL(k,\mathbb{C}).$$

Here, $I_{p+q,k} := \sum_{j=1}^{k} E_{jj} \in M'(p+q,k;\mathbb{C})$.

PROOF. By the Gram-Schmidt orthogonalization, we have a decomposition:

$$a = ub$$

where $u = (\overrightarrow{u_1}, \cdots, \overrightarrow{u_k}) \in M'(p,k;\mathbb{C})$ satisfies $(\overrightarrow{u_i}, \overrightarrow{u_j})_{p,0} = \delta_{ij}$ $(1 \leq i,j \leq k)$ and $b \in GL(k,\mathbb{C})$ is a upper triangular matrix. Because

$$(\iota(\overrightarrow{u_i}), \iota(\overrightarrow{u_j}))_{p,q} = (\overrightarrow{u_i}, \overrightarrow{u_j})_{p,0} = \delta_{ij}\ (1 \leq i,j \leq k),$$

we find $g \in U(p,q)$ such that $g\overrightarrow{e_j} = \iota(\overrightarrow{u_j})$ from Lemma (3.1). This means that $gI_{p+q,k} = \iota(u)$.

Hence $\iota(a) = \iota(ub) = \iota(u)b = gI_{p+q,k}b \in U(p,q)I_{p+q,k}GL(k,\mathbb{C})$. $\square$

LEMMA 3.5.

$$U(p,q)I_{p+q,k}GL(k,\mathbb{C}) \subset M''(p+q,k;\mathbb{C}).$$

PROOF. Because $M''(p+q,k;\mathbb{C})$ is preserved by the action of $GL(k,\mathbb{C})$ from the right, it suffices to show $U(p,q)I_{p+q,k} \subset M''(p+q,k;\mathbb{C})$. Let $g \in U(p,q)$ and we put

$$\begin{pmatrix} a \\ b \end{pmatrix} := gI_{p+q,k} \quad (a \in M(p,k;\mathbb{C}), b \in M(q,k;\mathbb{C})).$$

We want to see $\operatorname{rank} a = k$. Then

$$\begin{aligned} a^*a - b^*b &= \begin{pmatrix} a^* & b^* \end{pmatrix} J_{p,q} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= {I_{p+q,k}}^* g^* J_{p,q} g I_{p+q,k} \\ &= {I_{p+q,k}}^* J_{p,q} I_{p+q,k} \\ &= I_{k,k} \equiv \operatorname{diag}(1,\cdots,1). \end{aligned}$$

Hence $a^*a = b^*b + I_{k,k}$ is a positive definite matrix. Therefore $\operatorname{rank} a = k$. $\square$

We write the natural projection as

$$\pi_k^{p+q}\colon M'(p+q,k;\mathbb{C}) \to M'(p+q,k;\mathbb{C})/GL(k,\mathbb{C}),$$

which respects the left action of $GL(p+q,\mathbb{C})$. The same notation will be used for

$$G_{\mathbb{C}}/Q'(k) \to G_{\mathbb{C}}/Q(k)$$

by

$$\begin{array}{ccc} G_{\mathbb{C}}/Q'(k) & \xrightarrow{\pi_k^{p+q}} & G_{\mathbb{C}}/Q(k) \\ \simeq\big\downarrow & & \big\downarrow\simeq \\ M'(p+q,k;\mathbb{C}) & \xrightarrow{\pi_k^{p+q}} & M'(p+q,k;\mathbb{C})/GL(k,\mathbb{C}), \end{array}$$

where the vertical isomorphism is given by

$$G_{\mathbb{C}}/Q'(k) \to M'(p+q,k;\mathbb{C}),\ g \mapsto gI_{p+q,k}.$$

Similarly, we have

$$\begin{array}{ccc} K_{\mathbb{C}}/Q'(k)\cap K_{\mathbb{C}} & \xrightarrow{\pi_k^p} & K_{\mathbb{C}}/Q(k)\cap K_{\mathbb{C}} \\ \simeq\Big\downarrow & & \Big\downarrow\simeq \\ M'(p,k;\mathbb{C}) & \xrightarrow{\pi_k^p} & M'(p,k;\mathbb{C})/GL(k,\mathbb{C}). \end{array}$$

Then the following lemma is immediate from the definition.

Lemma 3.6. *Let $G = U(p,q) \supset L = U(k) \times U(p-k,q)$. The full inverse of an open set $G/L \subset G_{\mathbb{C}}/Q(k)$ by π_k^{p+q} is given by*

$$(\pi_k^{p+q})^{-1}(G/L) = U(p,q)I_{p+q,k}GL(k,\mathbb{C}) \subset M''(p+q,k;\mathbb{C}).$$

Proposition 3.7.

$$\iota(M'(p,k;\mathbb{C})) \subset (\pi_k^{p+q})^{-1}(G/L) \subset M''(p+q,k;\mathbb{C}).$$

Proof. The first inclusion follows from

$$\begin{aligned} \iota(M'(p,k;\mathbb{C})) &= \iota(U(p)I_{p,k}GL(k,\mathbb{C})) \\ &= (U(p)\times \mathbf{1}_q)\cdot I_{p+q,k}GL(k,\mathbb{C}) \subset (\pi_k^{p+q})^{-1}(G/L) \end{aligned}$$

and the second inclusion follows from Lemma (3.5) and Lemma (3.6). □

Definition 3.8.

$$\varpi\colon G/L \to K/K\cap L,\ gL \mapsto \pi_k^p \circ \mathrm{pr}(gI_{p+q,k}).$$

To see ϖ is a well-defined map, first we note that we have the following diagram by Proposition (3.7).

$$\begin{tikzcd}
K/L\cap K \arrow[r, shift left, "i"] & G/L \arrow[l, shift left, "\varpi"] \arrow[r, phantom, "\overset{\text{open}}{\subset}"] & G_{\mathbb{C}}/Q(k) \\
 & (\pi_k^{p+q})^{-1}(G/L) \arrow[u, "\pi_k^{p+q}"'] \arrow[d, phantom, "\cap"] & \\
M'(p,k;\mathbb{C}) \arrow[uu, "\pi_k^p"] \arrow[ur, hook, "\iota"'] & M''(p+q,k;\mathbb{C}) \arrow[l, "\mathrm{pr}"] \arrow[r, phantom, "\subset"] & M'(p+q,k;\mathbb{C}) \arrow[uu, "\pi_k^{p+q}"']
\end{tikzcd}$$

In particular,

$$gI_{p+q,k} \in M''(p+q,k;\mathbb{C}) \quad \text{if } g \in U(p,q).$$

Thus $\mathrm{pr}(gI_{p+q,k})$ is well-defined. Therefore

$$G \to K/L\cap K, \quad g \mapsto \pi_k^p \circ \mathrm{pr}(gI_{p+q,k}) \tag{3.8.1}$$

is well-defined.

Next, let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G = U(p,q),$$
$$l = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in L = U(k)\times U(p-k,q),$$

where $a, x \in M(k,\mathbb{C})$, $d, y \in M(p+q-k,\mathbb{C})$, $b \in M(k,p+q-k;\mathbb{C})$, $c \in M(p+q-k,k;\mathbb{C})$. If $c \in M(p+q-k,k;\mathbb{C})$, we write $c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ where $c_1 \in M(p-k,k;\mathbb{C})$ and $c_2 \in M(q,k;\mathbb{C})$. Analogous notation is used for each $(p+q-k)\times k$ matrix. Then $\mathrm{pr}(gI_{p+q,k}) = \begin{pmatrix} a \\ c_1 \end{pmatrix}$, $\mathrm{pr}(glI_{p+q,k}) = \begin{pmatrix} ax \\ (cx)_1 \end{pmatrix}$. Therefore it follows from the definition of π_k^p that $\pi_k^p(a) = \pi_k^p(ax)$, because $x \in U(k)$. This implies that (3.8.1) induces a map

$$G/L \to K/L\cap K, \quad g \mapsto \pi_k^p \circ \mathrm{pr}(gI_{p+q,k}).$$

Hence ϖ is well-defined.

LEMMA 3.9. *We have the following relation among the maps defined above.*

$$\begin{aligned}
&(1) && \mathrm{pr}\circ\iota = \mathrm{id} && : M'(p,k;\mathbb{C}) \to M'(p,k;\mathbb{C}),\\
&(2) && i\circ\pi_k^p = \pi_k^{p+q}\circ\iota && : M'(p,k;\mathbb{C}) \to G/L,\\
&(3) && \varpi\circ\pi_k^{p+q} = \pi_k^p\circ\mathrm{pr} && : (\pi_k^{p+q})^{-1}(G/L) \to K/K\cap L,\\
&(4) && \varpi\circ i = \mathrm{id} && : K/K\cap L \to K/K\cap L.
\end{aligned}$$

PROOF. The relation (1),(2) are obvious from definition. For the equation (3), we take $gI_{p+q,k}a \in (\pi_k^{p+q})^{-1}(G/L)$ where $g\in U(p,q)$, $a\in GL(k,\mathbb{C})$ (see Lemma (3.6)). Then we have

$$\begin{aligned}
\varpi\circ\pi_k^{p+q}(gI_{p+q,k}a) &= \varpi(gL) = \pi_k^p(\mathrm{pr}(gI_{p+q,k})),\\
\pi_k^p\circ\mathrm{pr}(gI_{p+q,k}a) &= \pi_k^p(\mathrm{pr}(gI_{p+q,k})).
\end{aligned}$$

Hence $\varpi\circ\pi_k^{p+q} = \pi_k^p\circ\mathrm{pr}$.
Now the last relation (4) follows from

$$\varpi\circ i\circ\pi_k^p = \varpi\circ\pi_k^{p+q}\circ\iota = \pi_k^p\circ\mathrm{pr}\circ\iota = \pi_k^p$$

and from that π_k^p is surjective. $\square$

PROPOSITION 3.10.

$$\varpi\colon G/L \to K/K\cap L,\ gL \mapsto \pi_k^p\circ\mathrm{pr}(gI_{p+q,k})$$

is holomorphic.

PROOF. First we note that all of the following maps

$$\begin{array}{ccc}
\pi_k^{p+q}\colon (\pi_k^{p+q})^{-1}(G/L) & \to & G/L\\
\text{open}\ \cap & & \cap\ \text{open}\\
M''(p+q,k;\mathbb{C}) & \to & GL(p+q,\mathbb{C})/Q(k)
\end{array}$$

$$\pi_k^p \colon M'(p,k;\mathbb{C}) \to K/K \cap L$$
$$\mathrm{pr} \colon (\pi_k^{p+q})^{-1}(G/L) \underset{\text{open}}{\subset} M''(p+q,k;\mathbb{C}) \to M'(p,k;\mathbb{C})$$

are holomorphic. Because $\pi_k^{p+q} \colon (\pi_k^{p+q})^{-1}(G/L) \to G/L$ is a holomorphic fiber bundle with typical fiber $GL(k,\mathbb{C})$, there is a local holomorphic section, say s. That is $\pi_k^{p+q} \circ s = \mathrm{id}_W$ for an open neighborhood W of G/L. Because $\varpi \circ \pi_k^{p+q} = \pi_k^p \circ \mathrm{pr}$ (see Lemma (3.9)(3)), we have $\varpi|_W = \varpi \circ \pi_k^{p+q} \circ s = \pi_k^p \circ \mathrm{pr} \circ s$. Therefore ϖ is holomorphic as a composition of holomorphic maps. Thus ϖ is holomorphic. □

Theorem 3.11. *Let $G = U(p,q) \supset L = U(k) \times U(p-k,q)$. Retain notation as above. In particular, we recall the complex structure on G/L is induced from $G_{\mathbb{C}}/Q(k)$. Then the Penrose transform*

$$\mathcal{R} \colon H_{\bar\partial}^{k(p-k)}(G \underset{L}{\times} \mathbb{C}_m) \to \mathcal{E}(G \underset{K}{\times} H_{\bar\partial}^{k(p-k)}(K \underset{L\cap K}{\times} \mathbb{C}_m))$$

is non-zero if $H_{\bar\partial}^{k(p-k)}(K \underset{L\cap K}{\times} \mathbb{C}_m) \neq 0$.

Proof. Take $\eta \in Z^{0,k(p-k)}(K \underset{L\cap K}{\times} \mathbb{C}_m)$ such that $[\eta] \neq 0$ in a cohomology group $H_{\bar\partial}^{k(p-k)}(K \underset{L\cap K}{\times} \mathbb{C}_m)$. We put

$$\omega := \varpi^* \eta \in \mathcal{E}^{0,k(p-k)}(G \underset{L}{\times} \mathbb{C}_m).$$

Then we have

$$\bar\partial_G \omega = \bar\partial_G(\varpi^* \eta) = \varpi^* \bar\partial_K \eta = 0,$$

since ϖ is holomorphic (Proposition (3.10)). Hence ω gives a cohomology class $[\omega] \in H_{\bar\partial}^{k(p-k)}(G \underset{L}{\times} \mathbb{C}_m)$. We recall the definition of $\mathcal{R}$ (see (1.2.3)):

$$\mathcal{R}([\omega])(g) = [i^* {l_g}^* \omega].$$

Consequently,

$$\mathcal{R}([\omega])(e) = [i^* \omega] = [i^* \varpi^* \eta] = [(\varpi \circ i)^* \eta] = [\eta].$$

Because $[\eta] \neq 0$, $\mathcal{R}([\omega])(e) \neq 0$. This means that $\mathcal{R}([\omega]) \neq 0$. Hence we have proved that $\mathcal{R}$ is a non-zero map. $\square$

Corollary 3.12. *Let $G = U(n,n) \supset L = U(k) \times U(n-k,n)$. Let $\lambda = \lambda_k = \frac{k}{2}\mathbf{1}_{2n}$ so that*

$$\lambda + \rho(\mathfrak{u}) = d\nu_n^{(k)} = n\mathbf{1}_k \oplus 0\mathbf{1}_{2n-k}.$$

Then

$$\mathcal{R}\colon H_{\bar\partial}^{k(n-k)}(G \underset{L}{\times} \mathbb{C}_{n\mathbf{1}_k \oplus 0\mathbf{1}_{2n-k}}) \to \mathcal{E}(G \underset{K}{\times} H_{\bar\partial}^{k(n-k)}(K \underset{L\cap K}{\times} \mathbb{C}_{n\mathbf{1}_k \oplus 0\mathbf{1}_{2n-k}}))$$

is injective.

Proof. $H_{\bar\partial}^{k(n-k)}(K \underset{L\cap K}{\times} \mathbb{C}_{n\mathbf{1}_k \oplus 0\mathbf{1}_{2n-k}})$ is a one dimensional representation of K by a generalization of the Borel-Weil theorem by Bott. So the assumption of Theorem (3.11) is satisfied, whence $\mathcal{R}$ is a non-zero map.

Because $V(n,k) = H_{\bar\partial}^{k(n-k)}(G \underset{L}{\times} \mathbb{C}_{n\mathbf{1}_k \oplus 0\mathbf{1}_{2n-k}})$ is irreducible by Fact (2.1), $\mathcal{R}$ is injective. $\square$

§4. Bruhat decomposition and the radial part of $\det\left(\frac{\partial}{\partial Z}\right)$

In this section, we compute the radial part of $\det\left(\frac{\partial}{\partial Z}\right)$, a differential operator on $D\,(\subset M(n,\mathbb{C}))$ with respect to the Bruhat decomposition.

We define subgroups of $GL(n,\mathbb{C})$ by

$$\begin{aligned}
N'_+ &:= \{g = (g_{ij}) \in GL(n,\mathbb{C}) : \ g_{ij} = 0 \ (1 \le j < i \le n), \\
&\qquad\qquad\qquad\qquad\qquad\quad g_{ll} = 1 \ (1 \le l \le n)\}, \\
N'_- &:= \{g = (g_{ij}) \in GL(n,\mathbb{C}) : \ g_{ij} = 0 \ (1 \le i < j \le n), \\
&\qquad\qquad\qquad\qquad\qquad\quad g_{ll} = 1 \ (1 \le l \le n)\}, \\
H' &:= \{\mathrm{diag}(y_1, \cdots, y_n) \in GL(n,\mathbb{C}) : \ y_l \in \mathbb{C}^\times \ (1 \le l \le n)\} \simeq (\mathbb{C}^\times)^n.
\end{aligned}$$

We also define a subgroup of $GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$ by $N(K_{\mathbb{C}}) := N'_- \times N'_+$. The l-th principal minors of $n \times n$ matrices $(1 \le l \le n)$ are defined by

$$p_l\colon M(n,\mathbb{C}) \to \mathbb{C}, \ Z \mapsto \det(z_{ij})_{1 \le i,j \le l}.$$

Note that $p_1(Z) = z_{11}$ and $p_n(Z) = \det(Z)$. Let

$$\begin{aligned} p &:= (p_1, \cdots, p_n) : M(n, \mathbb{C}) \to \mathbb{C}^n, \\ M'''(n, \mathbb{C}) &:= p^{-1}((\mathbb{C}^\times)^n) \\ &= \{Z \in M(n, \mathbb{C}) :\ p_l(Z) \neq 0\ (1 \leq l \leq n)\} \subset GL(n, \mathbb{C}) \end{aligned}$$

Then we have a biholomorphic map (the open cell in the Bruhat decomposition):

$$N'_- \times H' \times N'_+ \simeq M'''(n, \mathbb{C}),\ (n_-, h, n_+) \mapsto n_- h n_+ \tag{4.1}$$

The action $N'_- H' \times H' N'_+$ on $M(n, \mathbb{C})$ given by

$$g \mapsto (n_- h_1) g (h_2 n_+)^{-1} \quad (h_1, h_2 \in H', n_\pm \in N'_\pm)$$

forms a prehomogeneous vector space in the sense of M. Sato. The b-function of relative invariants is known as follows.

FACT 4.2 ([Sa]). *Let*

$$b(\lambda) := (\lambda_1 + \lambda_2 + \cdots + \lambda_n + n - 1)(\lambda_2 + \cdots + \lambda_n + n - 2) \cdots (\lambda_{n-1} + \lambda_n + 1)\lambda_n.$$

$$\det\left(\frac{\partial}{\partial Z}\right) \prod_{l=1}^{n} p_l(Z)^{\lambda_l} = b(\lambda)\,(\prod_{l=1}^{n} p_l(Z)^{\lambda_l})\, p_n(Z)^{-1}, \tag{4.2.1}$$

$$\begin{aligned} \det(\frac{\partial}{\partial z_{ij}})_{1 \leq i,j \leq l}\, p_n(Z)^\lambda &= \lambda(\lambda + 1) \cdots (\lambda + l - 1) p_n(Z)^{\lambda - 1} \\ &\quad \times \det(z_{ij})_{l+1 \leq i,j \leq n}. \end{aligned} \tag{4.2.2}$$

LEMMA 4.3. *We define a partial differential operator with polynomial coefficient by*

$$\begin{aligned} \tilde{R} := (x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n} + n)(x_2 \frac{\partial}{\partial x_2} + \cdots + x_n \frac{\partial}{\partial x_n} + n - 1) \\ \cdots (x_{n-1} \frac{\partial}{\partial x_{n-1}} + x_n \frac{\partial}{\partial x_n} + 2) \frac{\partial}{\partial x_n} \in \mathcal{D}(\mathbb{C}^n). \end{aligned} \tag{4.3.1}$$

Then the following diagram commutes.

$$\begin{tikzcd} \mathcal{O}(M'''(n,\mathbb{C})) \arrow[r, "\det\left(\frac{\partial}{\partial Z}\right)"] & \mathcal{O}(M'''(n,\mathbb{C})) \\ \mathcal{O}((\mathbb{C}^\times)^n) \arrow[u, "p^*"] \arrow[r, "\tilde{R}"'] & \mathcal{O}((\mathbb{C}^\times)^n) \arrow[u, "p^*"'] \end{tikzcd}$$

Proof. It is sufficient to see

$$p^* \circ \tilde{R} h_\lambda(Z) = \det\left(\frac{\partial}{\partial Z}\right) \circ p^* h_\lambda(Z) \tag{4.3.2}$$

for any $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$, because $\mathbb{C}$-span$\langle h_\lambda(x) : \lambda \in \mathbb{C}^n\rangle$ is a dense subspace of $\mathcal{O}(W)$ for each small neighborhood $W \subset M'''(n,\mathbb{C})$. Here, $h_\lambda(x) := x_1^{\lambda_1} \cdots x_n^{\lambda_n} \in \mathcal{O}_{(\mathbb{C}^\times)^n}$ is a locally defined holomorphic function on $(\mathbb{C}^\times)^n$.

Put $\lambda' = (\lambda_1, \cdots, \lambda_{n-1}, \lambda_n - 1) \in \mathbb{C}^n$.

$$\begin{aligned}\text{The left side of (4.3.2)} &= p^*(b(\lambda)h_{\lambda'}(Z)) \\ &= b(\lambda)p_1(Z)^{\lambda_1} \cdots p_{n-1}(Z)^{\lambda_{n-1}} p_n(Z)^{\lambda_n - 1}.\end{aligned}$$

On the other hand, it follows from Fact (4.2) that

$$\begin{aligned}\text{The right side of (4.3.2)} &= \det\left(\frac{\partial}{\partial Z}\right) p_1(Z)^{\lambda_1} \cdots p_n(Z)^{\lambda_n} \\ &= b(\lambda)p_1(Z)^{\lambda_1} \cdots p_{n-1}(Z)^{\lambda_{n-1}} p_n(Z)^{\lambda_n - 1}.\end{aligned}$$

Hence, we have (4.3.2). □

Lemma 4.4. *We define a partial differential operator R on $(\mathbb{C}^\times)^n$ by*

$$R = (y_1\frac{\partial}{\partial y_1} + n)(y_2\frac{\partial}{\partial y_2} + n - 1) \cdots (y_{n-1}\frac{\partial}{\partial y_{n-1}} + 2)\frac{1}{y_1 \cdots y_{n-1}}\frac{\partial}{\partial y_n}. \tag{4.4.1}$$

We define a change of variables by

$$\varphi \colon (\mathbb{C}^\times)^n \to (\mathbb{C}^\times)^n,\ (x_1, \cdots, x_n) \mapsto (y_1, \cdots, y_n) := (x_1, \frac{x_2}{x_1}, \cdots, \frac{x_n}{x_{n-1}}).$$

Then the following diagram commutes (see (4.3.1) for the definition of $\tilde{R}$):

$$\begin{array}{ccc} \mathcal{O}((\mathbb{C}^\times)^n) & \xrightarrow{\tilde{R}} & \mathcal{O}((\mathbb{C}^\times)^n) \\ {\scriptstyle\varphi^*}\Big\uparrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\Big\uparrow{\scriptstyle\varphi^*} \\ \mathcal{O}((\mathbb{C}^\times)^n) & \xrightarrow[R]{} & \mathcal{O}((\mathbb{C}^\times)^n). \end{array}$$

PROOF. Using $y_i = \dfrac{x_i}{x_{i-1}}$, we have

$$\begin{aligned} x_j \frac{\partial}{\partial x_j} &= x_j \sum_{i=1}^n \frac{\partial y_i}{\partial x_j} \frac{\partial}{\partial y_i} \\ &= y_j \frac{\partial}{\partial y_j} - y_{j+1} \frac{\partial}{\partial y_{j+1}} \quad (1 \le j \le n-1), \\ x_n \frac{\partial}{\partial x_n} &= y_n \frac{\partial}{\partial y_n}. \end{aligned}$$

Consequently,

$$\sum_{j=k}^n x_j \frac{\partial}{\partial x_j} = y_k \frac{\partial}{\partial y_k} \quad (k = 1, \cdots, n).$$

By using $y_1 \cdots y_{n-1} = x_{n-1}$, we have $R = (\varphi^*)^{-1} \circ \tilde{R} \circ (\varphi^*)$. $\square$

Then the following lemma is a direct consequence of the Bruhat decomposition and the definition of $p = (p_1, \cdots, p_n)$.

LEMMA 4.5. (1) *The inclusion map with respect to*

$$H' \simeq (\mathbb{C}^\times)^n \hookrightarrow M'''(n, \mathbb{C}),\ (y_1, \cdots, y_n) \mapsto \mathrm{diag}(y_1, \cdots, y_n)$$

induces a bijection

$$\mathrm{Rest} : \mathcal{O}(M'''(n, \mathbb{C}))^{N(K_{\mathbb{C}})} \simeq \mathcal{O}((\mathbb{C}^\times)^n).$$

(2) $$p^*(\mathcal{O}((\mathbb{C}^\times)^n)) \subset \mathcal{O}(M'''(n, \mathbb{C}))^{N(K_{\mathbb{C}})}.$$

$$(3) \qquad (\varphi \circ p)^* \colon \mathcal{O}((\mathbb{C}^\times)^n) \to \mathcal{O}(M'''(n,\mathbb{C}))^{N(K_{\mathbb{C}})}$$

is the inverse of the bijection Rest *in* (1).

Now we are ready to state the main result in this section.

THEOREM 4.6. *There is a transversal section $H' \simeq (\mathbb{C}^\times)^n$ with respect to the action of $N(K_{\mathbb{C}}) = N'_- \times N'_+$ on $M'''(n,\mathbb{C})$ by the Bruhat decomposition* (4.1)*. Then the radial part of* $\det\left(\frac{\partial}{\partial Z}\right) \in \mathcal{D}(M'''(n,\mathbb{C}))$ *with respect to $N(K_{\mathbb{C}})$ is equal to $R \in \mathcal{D}((\mathbb{C}^\times)^n)$. (See* (4.4.1) *for definition.)*

PROOF. We want to show that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{O}(M'''(n,\mathbb{C})) \arrow[r, "\det\left(\frac{\partial}{\partial Z}\right)"] & \mathcal{O}(M'''(n,\mathbb{C})) \\
\mathcal{O}(M'''(n,\mathbb{C}))^{N(K_{\mathbb{C}})} \arrow[u, phantom, "\cup"] \arrow[r, "\det\left(\frac{\partial}{\partial Z}\right)"] \arrow[d, "\mathrm{Rest}"', "\simeq"] & \mathcal{O}(M'''(n,\mathbb{C}))^{N(K_{\mathbb{C}})} \arrow[u, phantom, "\cup"] \arrow[d, "\simeq"', "\mathrm{Rest}"] \\
\mathcal{O}((\mathbb{C}^\times)^n) \arrow[r, "R"'] & \mathcal{O}((\mathbb{C}^\times)^n)
\end{tikzcd}$$

The point of the proof is that $(\varphi \circ p)^*$ is the inverse of the bijective map Rest (see Lemma (4.5)(3)). Thus it suffices to show that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{O}(M'''(n,\mathbb{C}))^{N(K_{\mathbb{C}})} \arrow[r, "\det\left(\frac{\partial}{\partial Z}\right)"] & \mathcal{O}(M'''(n,\mathbb{C}))^{N(K_{\mathbb{C}})} \\
\mathcal{O}((\mathbb{C}^\times)^n) \arrow[u, "(\varphi\circ p)^*"] \arrow[r, "R"'] & \mathcal{O}((\mathbb{C}^\times)^n) \arrow[u, "(\varphi\circ p)^*"']
\end{tikzcd}$$

Now, this commutativity is derived from Lemma (4.3) and Lemma (4.4). □

§5. $\mathcal{R}H_{\bar\partial}^{k(n-k)}(G \underset{L(k)}{\times} \mathbb{C}_n) \subset \mathcal{S}ol(\mathcal{M}_k)$

In this section, we prove the image of the Penrose transform satisfies the system of the differential equations $(\mathcal{M}_k)$.

LEMMA 5.1. *$\mathcal{R}V(n,k) \ni \mathbf{1}$. Here, $\mathbf{1}$ is a constant function on D and we identify*

$$\mathcal{O}(D) \simeq \mathcal{O}(G \underset{K}{\times} \mathbb{C}_k)$$

as in (1.3.5).

PROOF. It follows from Fact (2.1) that $V(n,k)$ contains a K-type

$$F(U(n),(k,\cdots,k)) \boxtimes F(U(n),(0,\cdots,0)) \simeq (\det)^k \boxtimes \mathbf{1}.$$

Because $\mathcal{R}$ is injective (Corollary (3.12)), the image $\mathcal{R}V(n,k)$ also contains the same K-type. We take such a function

$$F \in \mathcal{R}V(n,k) \subset \mathcal{E}(D).$$

This means that

$$\tilde\pi_{n,k}(g)F(Z) = (\det a)^k F(Z)$$

if $g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in K \simeq U(n) \times U(n)$. Because $g^{-1} = \begin{pmatrix} a^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}$, we have from (1.3.4)

$$\tilde\pi_{n,k}(g)F(Z) = (\det a)^k F(d^{-1}Za).$$

Therefore

$$F(d^{-1}Za) = F(Z) \quad \text{for any } a, d \in U(n).$$

Let $\mathbb{D}(G \underset{K}{\times} \mathbb{C}_k)$ be the ring of the left G-invariant differential operators acting on the sections of $\mathcal{E}(G \underset{K}{\times} \mathbb{C}_k)$. The spherical function of type

$$\xi \in \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}(G \underset{K}{\times} \mathbb{C}_k), \mathbb{C})$$

is defined to be a function $f \in \mathcal{E}(G \underset{K}{\times} \mathbb{C}_k)$ satisfying

$$f(hgh^{-1}) = f(g) \quad (h \in K,\ g \in G), \tag{5.1.1}$$

$$Df = \xi(D)f. \tag{5.1.2}$$

By a result of Godement, such a function is unique up to constant multiples. On the other hand, it is known that the composition $Z(\mathfrak{g}) \subset U(\mathfrak{g})^K \to \mathbb{D}(G \underset{K}{\times} \mathbb{C}_k)$ is surjective, if G is a classical reductive Lie group and if $k = 0$. This in turn implies that

$$Z(\mathfrak{g}) \to \mathbb{D}(G \underset{K}{\times} \mathbb{C}_k)$$

is surjective for any k. Because the $Z(\mathfrak{g})$-infinitesimal character of both of the representations $V(n,k)$ and $\mathcal{O}(G \underset{K}{\times} \mathbb{C}_k)$ coincides by Fact (2.1) and Fact (2.2), both of $F \in \mathcal{R}V(n,k)$ and $\mathbf{1} \in \mathcal{O}(G \underset{K}{\times} \mathbb{C}_k)$ satisfy (5.1.2) with the same $\xi \in \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}(G \underset{K}{\times} \mathbb{C}_k), \mathbb{C})$. Because both of the vectors $F \in \mathcal{R}V(n,k)$ and $\mathbf{1} \in \mathcal{O}(G \underset{K}{\times} \mathbb{C}_k)$ satisfy also (5.1.1), we conclude that F is a constant scalar multiple of $\mathbf{1}$, namely a constant function on D. $\square$

Lemma 5.2. *$\mathcal{R}V(n,k)$ contains a subspace*

$$\mathbb{C}\text{-}span\langle \det(a+bZ)^{-k} : a, b \in M(n,\mathbb{C}) \quad \textit{such that } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(n,n) \textit{ for some } c, d \in M(n,\mathbb{C})\rangle$$

as a dense set in the Fréchet topology on $\mathcal{O}(D)$.

Proof. Because $V(n,k)$ is irreducible and because $\mathcal{R}$ is a non-zero map, the image $\mathcal{R}V(n,k)$ contains

$$\mathbb{C}\text{-span}\langle \tilde{\pi}_{n,k}(g)v : g \in U(n,n)\rangle$$

as a dense subspace for any non-zero $v \in \mathcal{R}V(n,k) \subset \mathcal{O}(D)$. In particular, we can choose $v = \mathbf{1}$ by Lemma (5.1). Then from (1.3.4), we have

$$\tilde{\pi}_{n,k}(g)\mathbf{1} = \det(a+bZ)^{-k}.$$

Therefore we have proved Lemma (5.2). □

We recall

$$P(I,J) = \det(\frac{\partial}{\partial z_{ij}})_{i\in I, j\in J} \in \mathcal{D}(M(n,\mathbb{C}))$$

for $I, J \subset \{1, \cdots, n\}$, $|I| = |J|$.

Lemma 5.3. *Let $k \in \{0, 1, \cdots, n-1\}$. Suppose $a, b \in M(n,\mathbb{C})$. Let*

$$V \equiv V_{a,b} = \{Z \in M(n,\mathbb{C}) : \det(a + bZ) \neq 0\}.$$

Then

$$P(I,J)\det(a+bZ)^{-k} = 0 \qquad \text{for } Z \in V_{a,b},$$

for any $I, J \subset \{1, \cdots, n\}$ such that $|I| = |J| = k+1$.

We remark that $V_{a,b} \supset D$ if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(n,n)$ for some $c, d \in M(n,\mathbb{C})$.

Proof. Case I) $\det b \neq 0$.
By a change of variables, it is sufficient to show

$$P(I,J)(\det Z)^{-k} = 0.$$

Without loss of generality, we may and do assume $I = J = \{1, \cdots, k+1\}$. Then the substitution of $\lambda = -k$, $l = k+1$ into (4.2.2) yields

$$\det(\frac{\partial}{\partial z_{ij}})_{1\leq i,j\leq k+1} (\det Z)^{-k} = 0,$$

which is what we wanted to prove.
Case II) $\det b = 0$.
Take a sequence $b_j \in GL(n,\mathbb{C})$ such that $\lim_{j\to\infty} b_j = b$. If $Z \in V_{a,b}$, then we find an open neighborhood $Z \in W \subset V_{a,b}$ and $j_0 \in \mathbb{N}$ such that

$$\det(a + b_j Z) \neq 0 \quad (j \geq j_0,\ Z \in W).$$

From Case (I), we have

$$P(I,J)\det(a+b_jZ)^{-k}=0.$$

Taking the limit $j\to\infty$, we have

$$P(I,J)\det(a+bZ)^{-k}=0 \quad (Z\in W).$$

Hence we have proved Lemma. $\square$

THEOREM 5.4. *Let $G=U(n,n)$. Retain the notation as before. Then the image of the Penrose transform $\mathcal{R}$ satisfies the system of differential equations $(\mathcal{M}_k)$. That is, we have*

$$\mathcal{R}V(n,k)\subset \mathcal{S}ol(\mathcal{M}_k).$$

PROOF. If there is $c,d\in M(n,\mathbb{C})$ such that $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in U(n,n)$, then we have

$$V_{a,b}\supset D.$$

As $P(I,J)\det(a+bZ)^{-k}=0 \quad (Z\in V_{a,b})$ from Lemma (5.3), we have

$$P(I,J)\det(a+bZ)^{-k}=0 \quad (Z\in D).$$

Because $\mathcal{S}ol(\mathcal{M}_k)$ is a closed subspace of $\mathcal{O}(D)$ in the Fréchet topology, and because $\mathcal{S}ol(\mathcal{M}_k)$ contains a dense subspace of $\mathcal{R}V(n,k)$ by Lemma (5.2) and Lemma (5.3), we have $\mathcal{R}V(n,k)\subset\mathcal{S}ol(\mathcal{M}_k)$. $\square$

§6. Surjectivity

We recall that the action of $G=U(n,n)$ on the function space $\mathcal{O}(D)$ is given by

$$\tilde{\pi}_{n,k}(g)F(Z)=(\det(a+bZ))^{-k}F((c+dZ)(a+bZ)^{-1})$$
$$\text{for } g^{-1}=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in U(n,n).$$

The differential action induces a Lie algebra homomorphism

$$d\tilde{\pi}_{n,k}\colon \mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{gl}(2n,\mathbb{C}) \to \mathcal{D}(D),$$

the ring of differential operators on D. For later use, we compute $d\tilde{\pi}_{n,k}$ for a base of $\mathfrak{k}_{\mathbb{C}} \simeq \mathfrak{gl}(n,\mathbb{C}) \oplus \mathfrak{gl}(n,\mathbb{C})$ $(\subset \mathfrak{g}_{\mathbb{C}})$.

LEMMA 6.1. *Let E_{ij} $(1 \le i,j \le 2n)$ be the matrix unit of $\mathfrak{gl}(2n,\mathbb{C})$. For $1 \le a,b \le n$, we have*

$$\begin{aligned} Q_{ab} &:= d\tilde{\pi}_{n,k}(E_{ab}) = k\delta_{ab} + \sum_{l=1}^{n} z_{la}\frac{\partial}{\partial z_{lb}}, \\ P_{ab} &:= d\tilde{\pi}_{n,k}(E_{n+a,n+b}) = -\sum_{l=1}^{n} z_{bl}\frac{\partial}{\partial z_{al}}. \end{aligned}$$

PROOF. It follows from

$$\tilde{\pi}_{n,k}(\exp(sE_{ab}))F(Z) = e^{sk\delta_{ab}}F(Z(I+sE_{ab})) + o(s)$$

that

$$\begin{aligned} d\tilde{\pi}_{n,k}(E_{ab})F(Z) &= k\delta_{ab}F(Z) + \sum_{l=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial s}_{|s=0}(Z(I+sE_{ab}))_{lj}\frac{\partial F}{\partial z_{lj}}(Z) \\ &= k\delta_{ab}F(Z) + \sum_{l=1}^{n}\sum_{j=1}^{n} z_{la}\delta_{bj}\frac{\partial F}{\partial z_{lj}}(Z) \\ &= k\delta_{ab}F(Z) + \sum_{l=1}^{n} z_{la}\frac{\partial F}{\partial z_{lb}}(Z). \end{aligned}$$

Similarly, it follows from

$$\tilde{\pi}_{n,k}(\exp(sE_{n+a,n+b}))F(Z) = F((I-sE_{ab})Z) + o(s)$$

that

$$\begin{aligned} d\tilde{\pi}_{n,k}(E_{n+a,n+b})F(Z) &= \frac{\partial}{\partial s}_{|s=0} F((I - sE_{ab})Z) \\ &= \sum_{i=1}^{n}\sum_{l=1}^{n} \frac{\partial}{\partial s}_{|s=0} ((I - sE_{ab})Z)_{il} \frac{\partial F}{\partial z_{il}}(Z) \\ &= -\sum_{i=1}^{n}\sum_{l=1}^{n} \delta_{ai} z_{bl} \frac{\partial F}{\partial z_{il}}(Z) = -\sum_{l=1}^{n} z_{bl} \frac{\partial F}{\partial z_{al}}(Z). \ \square \end{aligned}$$

We recall a definition (4.4.1)

$$R = (y_1 \frac{\partial}{\partial y_1} + n)(y_2 \frac{\partial}{\partial y_2} + n - 1) \cdots (y_{n-1} \frac{\partial}{\partial y_{n-1}} + 2) \frac{1}{y_1 \cdots y_{n-1}} \frac{\partial}{\partial y_n} \\ \in \mathcal{D}((\mathbb{C}^\times)^n).$$

Then we need a simple Lemma.

LEMMA 6.2. *Let $\mathcal{O}_0$ denote the germ of $\mathcal{O}(\mathbb{C}^n)$ at $0 \in \mathbb{C}^n$. Then*

$$\{f \in \mathcal{O}_0 : Rf = 0 \textit{ in } W \cap (\mathbb{C}^\times)^n \textit{ for some open set } 0 \in W \subset \mathbb{C}^n\} \\ = \{f \in \mathcal{O}_0 : \frac{\partial f}{\partial y_n} = 0\}.$$

PROOF. Let $f(y) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha y^\alpha$ be the Taylor expansion of $f \in \mathcal{O}_0$, where $\alpha = (\alpha_1, \cdots, \alpha_n)$ denotes a multi-index. Then

$$Rf(y) = \sum_{\alpha \in \mathbb{N}^n} (n + \alpha_1 - 1) \cdots (2 + \alpha_{n-1} - 1) \alpha_n a_\alpha y^{\alpha - (1, \cdots, 1)}.$$

Therefore $Rf = 0$ if and only if $a_{\alpha_1 \cdots \alpha_n} = 0$ for all $\alpha_n \geq 1$. This condition is equivalent to $\frac{\partial f}{\partial y_n}(y) = 0$. $\square$

PROPOSITION 6.3. *Assume $F \in \mathcal{O}(D)$ satisfies*

$$\det\left(\frac{\partial}{\partial Z}\right) F = 0, \tag{6.3.1}$$

$$P_{ab}F \equiv -\sum_{l=1}^{n} z_{bl}\frac{\partial F}{\partial z_{al}} = 0 \quad (1 \leq b < a \leq n), \tag{6.3.2}$$

$$Q_{ab}F \equiv \sum_{l=1}^{n} z_{la}\frac{\partial F}{\partial z_{lb}} = 0 \quad (1 \leq a < b \leq n). \tag{6.3.3}$$

Then we have
(1) $\dfrac{\partial F}{\partial z_{nn}} = 0.$
(2) $Q_{nn}F(Z) = kF(Z)$, $P_{nn}F(Z) = 0$.

PROOF.
(1) The differential equations (6.3.2) and (6.3.3) imply that F is invariant under the action of $N(K_{\mathbb{C}}) \simeq N'_- \times N'_+$ (see the beginning of §4). Therefore if $F \in \mathcal{O}(D)$ satisfies (6.3.2) and (6.3.3), then F defines a germ of $\mathcal{O}(D)^{N(K_{\mathbb{C}})}$ at 0. We put

$$D_1 := \{y \in \mathbb{C} : |y| < 1\},$$
$$f(y_1, \cdots, y_n) := F(\mathrm{diag}(y_1, \cdots, y_n)) \in \mathcal{O}(D_1^n).$$

If $F \in \mathcal{O}(D)$ satisfies (6.3.1) in addition to (6.3.2) and (6.3.3), then $Rf = 0$ by Theorem (4.6). Now, we have $\dfrac{\partial f}{\partial y_n} = 0$ by Lemma (6.2). In turn, this implies

$$\frac{\partial F}{\partial z_{nn}} = 0, \tag{6.3.4}$$

proving (1).
(2) It follows from

$$\frac{\partial F}{\partial z_{nn}} = P_{nb}F(Z) = 0 \quad (1 \leq b \leq n-1)$$

that

$$0 = P_{nb}F = -\sum_{l=1}^{n} z_{bl}\frac{\partial F}{\partial z_{nl}} = -\sum_{l=1}^{n-1} z_{bl}\frac{\partial F}{\partial z_{nl}} \quad (1 \le b \le n-1).$$

Here, we used $\dfrac{\partial F}{\partial z_{nn}} = 0$ in the last equality. Therefore we have

$$\begin{pmatrix} z_{11} & \cdots & z_{1,n-1} \\ \vdots & & \vdots \\ z_{n-1,1} & \cdots & z_{n-1,n-1} \end{pmatrix} \begin{pmatrix} \dfrac{\partial F}{\partial z_{n1}} \\ \vdots \\ \dfrac{\partial F}{\partial z_{n,n-1}} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Because $F \in \mathcal{O}(D) \subset C^1(D)$ and $\det \begin{pmatrix} z_{11} & \cdots & z_{1,n-1} \\ \vdots & & \vdots \\ z_{n-1,1} & \cdots & z_{n-1,n-1} \end{pmatrix}$ is not identically zero on D, we have

$$\text{(6.3.5)} \qquad \frac{\partial F}{\partial z_{n1}}(Z) = \cdots = \frac{\partial F}{\partial z_{n,n-1}}(Z) = 0 \qquad (Z \in D).$$

Using again $\dfrac{\partial F}{\partial z_{nn}} = 0$ and (6.3.5), we have $P_{nn}F = 0$.

Similarly, we have $\dfrac{\partial F}{\partial z_{1n}} = \cdots = \dfrac{\partial F}{\partial z_{n-1,n}} = 0$, yielding $Q_{nn}F = kF$. $\square$

We recall

$$\begin{aligned} \mathcal{S}ol(\mathcal{M}_k) &= \{F \in \mathcal{O}(D) : P(I,J)F = 0 \text{ for any } I, J\ |I| = |J| \ge k+1\} \\ &\subset \mathcal{O}(D) \end{aligned}$$

for $0 \le k \le n$. Now we are ready to state the $K_{\mathbb{C}}$-module structure of the representation space of solutions $\mathcal{S}ol(\mathcal{M}_k)$:

Theorem 6.4. *Let $\mathcal{S}ol(\mathcal{M}_k)_K$ be the set of K-finite vectors of $\mathcal{S}ol(\mathcal{M}_k)$. Then $\mathcal{S}ol(\mathcal{M}_k)_K$ is decomposed into irreducible representations* *of K with multiplicity free as follows.*

$$\mathcal{S}ol(\mathcal{M}_k)_K \simeq \bigoplus_{\substack{\mu_1 \geq \cdots \geq \mu_k \\ \mu_j \in \mathbb{N}}} F(U(n), (\mu_1 + k, \cdots, \mu_k + k, \underbrace{k, \cdots, k}_{n-k})) \boxtimes F(U(n), (\underbrace{0, \cdots, 0}_{n-k}, -\mu_k, \cdots, -\mu_1)). \tag{6.4.1}$$

PROOF. Because $V(n,k)$ has the same K-type formulas as (6.4.1), because $\mathcal{R}V(n,k) \subset \mathcal{S}ol(\mathcal{M}_k)$ (see Theorem (5.4)), and because $\mathcal{R}$ is injective (see §3), it suffices to see that any K-type occurring in $\mathcal{S}ol(\mathcal{M}_k)$ is contained in the right side of (6.4.1).

Let $\sigma = \sigma_1 \boxtimes \sigma_2$ be an arbitrary K-type which occurs in the representation space $\mathcal{S}ol(\mathcal{M}_k)$. Because $\mathcal{S}ol(\mathcal{M}_k) \subset \mathcal{O}(G \underset{K}{\times} \mathbb{C}_k)$, it follows from Fact (2.2) that $\sigma = \sigma_1 \boxtimes \sigma_2$ is of the form:

$$\sigma = \sigma_1 \boxtimes \sigma_2 = F(U(n), (\mu_1 + k, \cdots, \mu_n + k)) \boxtimes F(U(n), (-\mu_n, \cdots, -\mu_1)).$$

Let F be a non-zero vector which is a highest weight vector with respect to the first factor $U(n)$, and a lowest weight vector with respect to the second factor $U(n)$. This means that F satisfies the differential equations (6.3.2) and (6.3.3). As $F \in \mathcal{S}ol(\mathcal{M}_k)$, F also satisfies $P(I,J)F = 0$ for $|I| = |J| \geq k+1$, in particular $\det\left(\frac{\partial}{\partial Z}\right) F = 0$. Hence we have from Proposition (6.3):

$$\begin{cases} \dfrac{\partial F}{\partial z_{ij}}(Z) = 0 \quad \text{if } i = n \text{ or } j = n, 1 \leq i,j \leq n, \\ P_{nn}F = 0, \quad Q_{nn}F = kF. \end{cases}$$

Using a downward induction on the size of matrix from n to $k+1$, we have

$$\frac{\partial F}{\partial z_{ij}}(Z) = 0 \tag{6.4.2}$$
$$\text{for } 1 \leq i,j \leq n \text{ such that } k+1 \leq i \text{ or } k+1 \leq j,$$

$$Q_{jj}F = kF \quad (k+1 \leq j \leq n), \tag{6.4.3}$$

$$P_{jj}F = 0 \quad (k+1 \leq j \leq n). \tag{6.4.4}$$

In view of the definition:

$$Q_{jj} = d\tilde{\pi}_{n,k}(E_{jj}), \quad P_{jj} = d\tilde{\pi}_{n,k}(E_{n+j,n+j}),$$

the equations (6.4.3),(6.4.4) determine the action of the Cartan subalgebra $\mathfrak{h} = \sum_{j=1}^{2n} \mathbb{C}E_{jj}$ on the $N(K_{\mathbb{C}})$-fixed vector $F \in \mathcal{O}(D)$, so that we have

$$\mu_{k+1} = \cdots = \mu_n = 0.$$

Therefore $\sigma = \sigma_1 \boxtimes \sigma_2$ occurs in the right side of (6.4.1). Thus we have completed the proof of Theorem. □

Corollary 6.5. *$\mathcal{R}\colon V(n,k)_K \to \mathcal{S}ol(\mathcal{M}_k)_K$ is a bijective $(\mathfrak{g}, K)$-map.*

Proof. First we recall that the Penrose transform $\mathcal{R}\colon V(n,k) \to \mathcal{S}ol(\mathcal{M}_k)$ is injective from Corollary (3.12). Because $\mathcal{R}V(n,k) \subset \mathcal{S}ol(\mathcal{M}_k)$ and because the K-type formula of $V(n,k)_K$ and $\mathcal{S}ol(\mathcal{M}_k)_K$ coincides (Fact (2.1), Theorem (6.4)), we have $\mathcal{R}V(n,k)_K = \mathcal{S}ol(\mathcal{M}_k)_K$. □

§7. Finiteness of the dimension of generalized Aomoto-Gelfand hypergeometric functions

We write H for the Cartan subgroup of $U(n,n)$ with Lie algebra $\mathfrak{h}$ (see notation §1.1). Any character of H is of the form:

$$\chi_\nu\colon H \to \mathbb{C}^\times, \quad \exp(\sum_{i=1}^{2n} \sqrt{-1} t_i E_{ii}) \mapsto \exp(\sum_{i=1}^{2n} \sqrt{-1}\nu_i t_i),$$

where $\nu = (\nu_1, \ldots, \nu_{2n}) \in \mathbb{Z}^{2n}$. We set:

$$H_{\bar\partial}^{k(n-k)}(G \underset{L}{\times} \mathbb{C}_n)(\nu) := \{v \in H_{\bar\partial}^{k(n-k)}(G \underset{L}{\times} \mathbb{C}_n) : \pi_{n,k}(h)v = \chi_\nu(h)v \, (h \in H)\},$$
$$\mathcal{S}ol(\widetilde{\mathcal{M}_k(\nu)}) := \{F \in \mathcal{S}ol(\mathcal{M}_k) : \tilde{\pi}_{n,k}(h)F = \chi_\nu(h)F \, (h \in H)\}.$$

Because the Penrose transform $\mathcal{R}$ is a bijective intertwining operator between $H_{\bar\partial}^{k(n-k)}(G \underset{L}{\times} \mathbb{C}_n)$ and $\mathcal{S}ol(\mathcal{M}_k)$ (see Theorem and Remark in §1), we have a bijection:

$$\mathcal{R}\colon H_{\bar\partial}^{k(n-k)}(G \underset{L}{\times} \mathbb{C}_n)(\nu) \simeq \mathcal{S}ol(\widetilde{\mathcal{M}_k(\nu)})$$

for each $\nu \in \mathbb{Z}^{2n}$.

It follows from Lemma (6.1) that we have

$$\mathcal{S}ol(\widetilde{\mathcal{M}_k}(\nu)) = \{F \in \mathcal{O}(D) : F \text{ satisfies } (\widetilde{\mathcal{M}_k}(\nu))\}.$$

Here we define:

$(\widetilde{\mathcal{M}_k}(\nu))$

$$\begin{cases} P(I,J)F(Z) = 0 \text{ for any } I, J \subset \{1, 2, \cdots, n\} \text{ with } |I| = |J| = k+1, \\ \displaystyle\sum_{i=1}^{n} z_{ij} \frac{\partial}{\partial z_{ij}} F(Z) = (\nu_j - k)F(Z) \qquad (1 \le j \le n), \\ \displaystyle\sum_{j=1}^{n} z_{ij} \frac{\partial}{\partial z_{ij}} F(Z) = -\nu_{n+i}F(Z) \qquad (1 \le i \le n). \end{cases}$$

If $k = 1$, this system $(\widetilde{\mathcal{M}_k}(\nu))$ is the hypergeometric differential equations introduced by Aomoto and I. M. Gelfand.

We may regard $(\widetilde{\mathcal{M}_k}(\nu))$ as a generalization of Aomoto-Gelfand hypergeometric equations. Let us show the finiteness of the dimension of holomorphic solutions of $(\widetilde{\mathcal{M}_k}(\nu))$ defined on a bounded domain $D \subset M(n, \mathbb{C}) \simeq \mathbb{C}^{n^2}$.

We put

$$\begin{aligned} H_1 &:= \sum_{i=1}^{n} E_{ii} - \sum_{i=n+1}^{2n} E_{ii}, \\ H_2 &:= \sum_{i=k+1}^{n} E_{ii}, \\ \mathfrak{t}'_0 &:= \sqrt{-1}\mathbb{R}H_1 + \sqrt{-1}\mathbb{R}H_2 \ (\subset \mathfrak{h}), \\ \mathfrak{t}' &:= \mathfrak{t}'_0 \otimes \mathbb{C}, \\ T' &:= \exp(\mathfrak{t}'_0)(\subset K \subset G). \end{aligned}$$

Then the tensor product of two symmetric tensor algebra $S(\mathfrak{u}(k) \cap \mathfrak{p})$ and $S(\overline{\mathfrak{u}(k)} \cap \mathfrak{k})$ is decomposed into irreducible representations of T' as follows:

$$S(\mathfrak{u}(k) \cap \mathfrak{p}) \otimes S(\overline{\mathfrak{u}(k)} \cap \mathfrak{k}) \simeq \bigoplus_{m,l \in \mathbb{N}} S^m(\mathfrak{u}(k) \cap \mathfrak{p}) \otimes S^l(\overline{\mathfrak{u}(k)} \cap \mathfrak{k}).$$

With notation in §1.1, we have

$$\Delta(\mathfrak{u}(k)\cap\mathfrak{p},\mathfrak{h}) = \{e_i - e_j : 1 \le i \le k, n+1 \le j \le 2n\},$$
$$\Delta(\overline{\mathfrak{u}(k)}\cap\mathfrak{k},\mathfrak{h}) = \{e_i - e_j : k+1 \le i \le n, 1 \le j \le k\}.$$

Therefore, $\mathrm{ad}(H_1)$ acts on $\mathfrak{u}(k)\cap\mathfrak{p}$ and $\overline{\mathfrak{u}(k)}\cap\mathfrak{k}$ by scalars 2 and 0, respectively. Similarly, $\mathrm{ad}(H_2)$ acts on $\mathfrak{u}(k)\cap\mathfrak{p}$ and $\overline{\mathfrak{u}(k)}\cap\mathfrak{k}$ by scalars 0 and 1, respectively. Hence $\mathrm{ad}(H_1)$ and $\mathrm{ad}(H_2)$ act on $S^m(\mathfrak{u}(k)\cap\mathfrak{p})\otimes S^l(\overline{\mathfrak{u}(k)}\cap\mathfrak{k})$ by scalars $2m$ and l, respectively.

Now let us apply Theorem 4.1 in [Ko4] with $G' := K' = T'$ (here G' and K' are the notation loc. cit.). In this case, $\mathfrak{t}'\subset\mathfrak{h}\subset\mathfrak{l}(k)$ so that

$$(\overline{\mathfrak{u}(k)}\cap\mathfrak{t}')\oplus(\mathfrak{l}(k)\cap\mathfrak{t}')\oplus(\mathfrak{u}(k)\cap\mathfrak{t}') = \{0\}\oplus\mathfrak{t}'\oplus\{0\} = \mathfrak{t}'.$$

Also, in view of the actions of $\mathrm{ad}(H_1)$ and $\mathrm{ad}(H_2)$ as above,

$$S(\mathfrak{u}(k)\cap\mathfrak{p})\otimes S(\overline{\mathfrak{u}(k)}\cap\mathfrak{k}/\overline{\mathfrak{u}(k)}\cap\mathfrak{k}')) \simeq S(\mathfrak{u}(k)\cap\mathfrak{p})\otimes S(\overline{\mathfrak{u}(k)}\cap\mathfrak{k})$$

is decomposed into irreducible modules of $L(k)\cap K'\simeq T'$ with finite multiplicities. Hence, the assumptions (4.1)(a) and (4.1)(b) in Theorem 4.1 in [Ko4] are satisfied and we conclude that $V(n,k)$ is T'-admissible. Because $T'\subset H$, $V(n,k)$ is also H-admissible (see Theorem 1.2 in [Ko4]). This is nothing but

$$\dim_{\mathbb{C}} H^{k(n-k)}_{\bar\partial}(G\underset{L}{\times}\mathbb{C}_n)(\nu) < \infty$$

for any $\nu\in\mathbb{Z}^{2n}$. Now we have proved

Theorem 7.1. *$\dim_{\mathbb{C}} \mathcal{S}ol(\widetilde{\mathcal{M}_k}(\nu)) < \infty$ for any $\nu\in\mathbb{Z}^{2n}$.*

Remark 7.2. As the proof indicates, analogous theorem of finite dimensionality of the space of the solutions is also true if we replace H by an arbitrary subgroup of G provided the Lie algebra of H contains the elements H_1 and H_2.

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