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Global Theta Liftings of General Linear Groups

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Abstract. A global theta lifting of some irreducible type 2 dual reductive pair is studied. It is proved that the image of a global theta lifting of a given irreducible automorphic cuspidal representation is nonvanishing if and only if its standard L-function is nonzero at the point $1/2$, and then the image coincides with the initial automorphic cuspidal representation. As a corollary of this result, the global Howe correspondence is obtained.

Introduction

Let k be a global field and $\mathbb{A}$ the adele ring of k. The pair $(GL_n(k), GL_n(k))$ is a type 2 dual reductive pair in the symplectic group $Sp_{n^2}(k)$ of size $2n^2$ ([3]). If ω' denotes the Weil representation of the metaplectic group $Mp_{n^2}(\mathbb{A})$ of $Sp_{n^2}(\mathbb{A})$, then the restriction of ω' to $GL_n(\mathbb{A}) \times GL_n(\mathbb{A})$ is described as follows. Let $\mathcal{S}(M_n(\mathbb{A}))$ be the space of Schwartz - Bruhat functions on the set $M_n(\mathbb{A})$ of all $n \times n$ matrices with entries in $\mathbb{A}$. Then, for $f \in \mathcal{S}(M_n(\mathbb{A}))$ and $h,\ g \in GL_n(\mathbb{A})$,

$$\omega'(h,g)f(x) = |\det h|_{\mathbb{A}}^{n/2} |\det g|_{\mathbb{A}}^{n/2} f({}^t h x g)\ .$$

Let ω be the representation of $GL_n(\mathbb{A}) \times GL_n(\mathbb{A})$ defined by $\omega(h,g) = \omega'({}^t h^{-1}, g)$. We use ω instead of ω' for convenience. The purpose of this paper is to study the theta lifting and the Howe correspondence of the irreducible automorphic cuspidal representations of $GL_n(\mathbb{A})$ with respect to ω.

In order to mention our results, we denote by $\mathcal{H}_n = \otimes'_v \mathcal{H}_{n,v}$ the global Hecke algebra of $GL_n(\mathbb{A})$ (cf. [1, Section 3]) and K_n the standard maximal

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compact subgroup of $GL_n(\mathbb{A})$. Let $\mathcal{S}_0(M_n(\mathbb{A}))$ be the subspace of $\mathcal{S}(M_n(\mathbb{A}))$ consisting of all $K_n \times K_n$-finite functions. We consider ω as a representation of $\mathcal{H}_n \otimes \mathcal{H}_n$ acting on $\mathcal{S}_0(M_n(\mathbb{A}))$. Let π be an irreducible automorphic cuspidal representation of $GL_n(\mathbb{A})$. We always assume that the representation space H_π of π is contained in the space of cusp forms on $GL_n(\mathbb{A})$. Thus H_π is an $\mathcal{H}_n$-module, but not a $GL_n(\mathbb{A})$-module. For $\varphi \in H_\pi$, $f \in \mathcal{S}_0(M_n(\mathbb{A}))$ and a complex number $s \in \mathbf{C}$, the theta lifting φ_f^s of φ is defined to be

$$\varphi_f^s(h) = \int_{GL_n(k)\backslash GL_n(\mathbb{A})} \varphi(g)|\det g|_{\mathbb{A}}^s \sum_{\substack{x \in M_n(k) \\ x \neq 0}} \omega(h,g)f(x)dg\ .$$

This integral is absolutely convergent for $\mathrm{Re}(s) >> 0$ and analytically continued to the whole s-plane as an entire function (see Lemma 3). For a fixed $s \in \mathbf{C}$, we denote by $\Theta^s(\pi)$ the space spanned by functions φ_f^s, ($\varphi \in H_\pi$, $f \in \mathcal{S}_0(M_n(\mathbb{A}))$) on $GL_n(\mathbb{A})$. Then we prove the following theorem.

THEOREM 1. *Let π be an irreducible automorphic cuspidal representation of $GL_n(\mathbb{A})$ and $L(s,\pi)$ its standard automorphic L-function. Then, the space $\Theta^0(\pi)$ is nonzero if and only if $L(1/2,\pi)$ is nonzero. In this case, $\Theta^0(\pi)$ coincides with H_π.*

We write $\pi^\vee$ for the contragredient representation of π. Next theorem is obtained as a corollary of the proof of Theorem 1 and the strong multiplicity one theorem.

THEOREM 2. *For any irreducible automorphic cuspidal representation π of $GL_n(\mathbb{A})$, one has $\mathrm{Hom}_{\mathcal{H}_n\otimes\mathcal{H}_n}(\omega, \pi\otimes\pi^\vee) \neq 0$. Furthermore, if σ is an irreducible automorphic cuspidal representation satisfying $\mathrm{Hom}_{\mathcal{H}_n\otimes\mathcal{H}_n}(\omega, \sigma \otimes \pi^\vee) \neq 0$, then σ is isomorphic to π.*

It should be noted that the explicit local theta correspondence of (GL_n, GL_n) was implicitly proved by Godement and Jacquet ([2], [12]). Thus one can formally prove the first assertion of Theorem 2 for any irreducible admissible representation of $GL_n(\mathbb{A})$ (see Remark after the proof of Theorem 2). However, it seems for the author that there is no article described these facts and the global theta liftings (cf. [10, Section 4.6.5]).

We will use the following notations. For an associative ring R with identity element, we denote by $M_n(R)$ the set of all $n \times n$ matrices with entries in R. For $A \in M_n(R)$, $\det A$ stands for its determinant. The identity matrix in $M_n(R)$ is denoted by 1_n. When a base field F is given, we set $G_n = GL_n(F)$. We denote by U_n the groups consisting of all upper triangular matrices with ones in the diagonals and by Z_n the center of G_n. If F is a local field, then $|\cdot|_F$ denotes the normalized absolute value on F and α_F denotes the character of G_n defined as $\alpha_F(g) = |\det g|_F$ for $g \in G_n$. If G is a locally compact abelian group, then $\mathcal{S}(G)$ denotes the space of Schwartz - Bruhat functions on G.

1. The local theta correspondence: Non-archimedean case

Let F be a local non-archimedean field and q the order of the residual field of F. We fix a non-trivial additive character ψ_F of F. The character $\psi_{n,F}$ of U_n is defined to be

$$\psi_{n,F}(u) = \psi_F(u_{12} + u_{23} + \cdots + u_{n-1\,n}), \qquad (u = (u_{ij}) \in U_n)\ . \tag{1.1}$$

Let $\mathbf{W}(\psi_{n,F})$ be the space of all locally constant functions W on G_n satisfying $W(ug) = \psi_{n,F}(u)W(g)$ for any $u \in U_n$ and $g \in G_n$. Then $g \in G_n$ acts on $\mathbf{W}(\psi_{n,F})$ by right translation; $\rho(g)W(g') = W(g'g)$. For an irreducible admissible generic representation π of G_n, we denote by $\mathbf{W}(\pi, \psi_{n,F})$ the Whittaker model of π in $\mathbf{W}(\psi_{n,F})$.

We define the smooth representation $(\omega_F, \mathcal{S}(M_n(F)))$ of $G_n \times G_n$ as follows: for $f \in \mathcal{S}(M_n(F))$ and $h,\ g \in G_n$,

$$\omega_F(h,g)f(x) = \alpha_F(h)^{-n/2}\alpha_F(g)^{n/2}f(h^{-1}xg)\ . \tag{1.2}$$

For $f \in \mathcal{S}(M_n(F))$, $W \in \mathbf{W}(\pi, \psi_{n,F})$ and a complex number $s \in \mathbf{C}$, we consider the integral

$$\begin{aligned} V^s_{(W,f)}(h) &= \int_{G_n} W(g)\omega_F(h,g)f(1_n)\alpha_F(g)^{s-1/2}dg\ , \qquad (h \in G_n) \\ &= \alpha_F(h)^{-n/2}\int_{G_n} W(g)f(h^{-1}g)\alpha_F(g)^{s+n/2-1/2}dg\ . \end{aligned} \tag{1.3}$$

This integral converges absolutely for $\mathrm{Re}(s)$ large and becomes a rational function of q^{-s} (cf. [7, (5.2)]). Let $I(\pi)$ be the linear span of rational functions $V^s_{(W,f)}(h)$, $(W \in \mathbf{W}(\pi,\psi_{n,F}),\ f \in \mathcal{S}(M_n(F)),\ h \in G_n)$. It is known by [2, Theorem 3.3] and [7, (5.2)] that $I(\pi)$ equals a fractional ideal $L(s,\pi)\mathbf{C}[q^{-s},q^s]$ of the polynomial ring $\mathbf{C}[q^{-s},q^s]$, where $L(s,\pi)$ stands for the local factor of π defined by Godement and Jacquet. Therefore, as a function in s, $L(s,\pi)^{-1}V^s_{(W,f)}(h)$ is holomorphic on $\mathbf{C}$ and is denoted by $\widetilde{V}^s_{(W,f)}(h)$. On the other hand, as a function in $h \in G_n$, $V^s_{(W,f)}$ is contained in $\mathbf{W}(\psi_{n,F})$. We denote by $\mathbf{V}^s(\pi,\psi_{n,F})$ the linear span of functions $\widetilde{V}^s_{(W,f)}$, $(W \in \mathbf{W}(\pi,\psi_{n,F}),\ f \in \mathcal{S}(M_n(F)))$. By the uniqueness of analytic continuation , we have

$$\rho(g_1)\widetilde{V}^s_{(\pi(g_2)W,f)} = \alpha_F(g_2)^{-s+1/2}\widetilde{V}^s_{(W,\omega(g_1,g_2^{-1})f)}\,, \qquad (g_1,\ g_2 \in G_n)$$

for all $s \in \mathbf{C}$. Therefore, $\mathbf{V}^s(\pi,\psi_{n,F})$ is a nonzero G_n-submodule of $\mathbf{W}(\psi_{n,F})$ for each s.

LEMMA 1. *The space $\mathbf{V}^s(\pi,\psi_{n,F})$ coincides with the space $\alpha_F^{s-1/2} \otimes \mathbf{W}(\pi,\psi_{n,F})$ for all $s \in \mathbf{C}$.*

PROOF. We first assume $\mathrm{Re}(s) >> 0$. By changing g to hg in the integral (1.3), we obtain

$$V^s_{(W,f)}(h) = \alpha_F(h)^{s-1/2}\int_{G_n} W(hg)f(g)\alpha_F(g)^{s+n/2-1/2}dg\,.$$

We take an open compact subgroup Ω in G_n such that $f(kg) = f(g)$ for any $k \in \Omega$. Let dk be the Haar measure on Ω normalized so that the volume of Ω equals 1. Then, we have

$$\begin{aligned}\int_{G_n} & W(hg)f(g)\alpha_F(g)^{s+n/2-1/2}dg \\ &= \int_\Omega\int_{G_n} W(hkg)f(kg)\alpha_F(kg)^{s+n/2-1/2}dgdk \\ &= \int_{G_n}\left(\int_\Omega W(hkg)dk\right)f(g)\alpha_F(g)^{s+n/2-1/2}dg\,.\end{aligned}$$

Let $\mathbf{W}(\pi,\psi_{n,F})^\Omega$ be the subspace of $\mathbf{W}(\pi,\psi_{n,F})$ consisting of all elements fixed by Ω. The admissibility of π implies that $\mathbf{W}(\pi,\psi_{n,F})^\Omega$ is of finite dimension. Thus we can take a basis $\{W_1, \cdots, W_m\}$ of $\mathbf{W}(\pi, \psi_{n,F})^{\Omega}$. Then, by the same argument as in [7, page 434], there exist matrix coefficients $\phi_1, \cdots, \phi_m$ of π such that

$$\int_{\Omega} W(hkg)dk = \sum_{j=1}^{m} W_j(h)\phi_j(g) \ .$$

Therefore, if we set

$$(1.4) \qquad Z(f, s+n/2-1/2; \phi_j) = \int_{G_n} \phi_j(g) f(g) \alpha_F(g)^{s+n/2-1/2} dg \ ,$$

then we have

$$(1.5) \qquad \widetilde{V}^s_{(W,f)}(h) = \alpha_F(h)^{s-1/2} \sum_{j=1}^{m} \frac{Z(f, s+n/2-1/2; \phi_j)}{L(s, \pi)} W_j(h) \ .$$

Since the right-hand side is holomorphic on $\mathbf{C}$ by [2, Theorem 3.3], (1.5) holds for all $s \in \mathbf{C}$, and hence $\widetilde{V}^s_{(W,f)}$ is contained in $\alpha_F^{s-1/2} \otimes \mathbf{W}(\pi, \psi_{n,F})$. The irreducibility of π concludes that $\mathbf{V}^s(\pi, \psi_{n,F}) = \alpha_F^{s-1/2} \otimes \mathbf{W}(\pi, \psi_{n,F})$. □

As we mentioned in Introduction, Godement and Jacquet ([2]) essentially proved that $\mathrm{Hom}_{G_n \times G_n}(\omega_F, \pi \otimes \pi^{\vee}) \neq 0$ for any irreducible admissible representation π of G_n. Furthermore, Weil ([12]) noted that $\dim \mathrm{Hom}_{G_n \times G_n}(\omega_F, \pi \otimes \pi^{\vee}) = 1$ if π is a supercuspidal representation.

2. The local theta correspondence: Archimedean case

In this section, we denote by F a local archimedean field. Let $\mathcal{G}_n$ be the Lie algebra of G_n as a real Lie group and K the standard maximal compact subgroup of G_n. We define a non-trivial additive character ψ_F of F as

$$\psi_F(a) = \begin{cases} \exp(2\pi\sqrt{-1}a\lambda) & \text{if } F = \mathbf{R} \\ \exp(2\pi\sqrt{-1}(a\lambda + \overline{a\lambda})) & \text{if } F = \mathbf{C} \end{cases},$$

where $\lambda \in F$ is a nonzero constant. The character $\psi_{n,F}$ of U_n is defined similarly as (1.1).

Let (π, H^∞) be an irreducible admissible representation of G_n realized as a smooth Fréchet representation of moderate growth (cf. [9, Section 2]). We denote by H the space of K-finite vectors in the Fréchet space H^∞. We assume that π is generic, i.e. there exists a nonzero continuous linear functional λ on H^∞ satisfying

$$\lambda(\pi(u)v) = \psi_{n,F}(u)\lambda(v)$$

for all $u \in U_n$ and $v \in H^\infty$. Such a λ is unique up to constant ([11, Theorem 3.1]). Then we denote by $\mathbf{W}^\infty(\pi, \psi_{n,F})$ the space of functions W_v on G_n of the form

$$W_v(g) = \lambda(\pi(g)v), \qquad (v \in H^\infty).$$

We also denote by $\mathbf{W}(\pi, \psi_{n,F})$ the subspace of $\mathbf{W}^\infty(\pi, \psi_{n,F})$ consisting of W_v with $v \in H$, so that $\mathbf{W}(\pi, \psi_{n,F})$ is an underlying irreducible $(\mathcal{G}_n, K)$-module of π. We write $L(s, \pi)$ for the local factor of π defined by Godement and Jacquet ([2, Theorem 8.7]).

We define the smooth representation $(\omega_F, \mathcal{S}(M_n(F)))$ of $G_n \times G_n$ by the same way as (1.2). Let $\mathcal{S}_0(M_n(F))$ be the subspace of $\mathcal{S}(M_n(F))$ consisting of all $K \times K$-finite functions. Then $(\omega, \mathcal{S}_0(M_n(F)))$ is a $(\mathcal{G}_n \oplus \mathcal{G}_n, K \times K)$-module. For $W \in \mathbf{W}^\infty(\pi, \psi_{n,F})$, $f \in \mathcal{S}(M_n(F))$ and $s \in \mathbf{C}$, we set

$$V^s_{(W,f)}(h) = \int_{G_n} W(g)\omega_F(h, g)f(1_n)\alpha_F(g)^{s-1/2}dg\ .$$

By [9, Section 6] (or [6, Section 9]), this integral is absolutely convergent for $\mathrm{Re}(s) >> 0$ and extends to a meromorphic function on the whole $\mathbf{C}$. Furthermore, if we set $\widetilde{V}^s_{(W,f)}(h) = L(s, \pi)^{-1}V^s_{(W,f)}(h)$, it becomes an entire function in s. By the uniqueness of analytic continuation, $\widetilde{V}^s_{(W,f)}$ satisfies the following for all $s \in \mathbf{C}$:

$$\widetilde{V}^s_{(W,f)}(uh) = \psi_{n,F}(u)\widetilde{V}^s_{(W,f)}(h), \qquad (u \in U_n),$$
$$\widetilde{V}^s_{(\pi(g_2)W,f)}(hg_1) = \alpha_F(g_2)^{-s+1/2}\widetilde{V}^s_{(W,\omega(g_1,g_2^{-1})f)}(h), \qquad (g_1,\ g_2 \in G_n).$$

Let $\mathbf{V}^s(\pi, \psi_{n,F})$ denote the linear span of $\widetilde{V}^s_{(W,f)}$, $(W \in \mathbf{W}(\pi, \psi_{n,F}),\ f \in \mathcal{S}_0(M_n(F)))$. Since the linear span of functions $\widetilde{V}^s_{(W,f)}(1_n)$, $(W \in$

$\mathbf{W}(\pi,\psi_{n,F})$, $f\in\mathcal{S}_0(M_n(F)))$ in s contains the set $\{P(s)|\lambda|_F^{-ns/2}\colon P(s)\in\mathbf{C}[s]\}$ (cf. [2, Theorem 8.7]), the space $\mathbf{V}^s(\pi,\psi_{n,F})$ is nonzero for all $s\in\mathbf{C}$.

Lemma 2. *The space $\mathbf{V}^s(\pi,\psi_{n,F})$ coincides with the space $\alpha_F^{s-1/2}\otimes\mathbf{W}(\pi,\psi_{n,F})$ for all $s\in\mathbf{C}$.*

Proof. For $f\in\mathcal{S}_0(M_n(F))$, there exists an elementary idempotent ξ in the Hecke algebra of G_n ([2, Section 8]) such that

$$\int_K f(k^{-1}x)\xi(k)dk=f(x)\ .$$

The admissibility of π implies that the image $\pi(\xi)\mathbf{W}(\pi,\psi_{n,F})$ of $\pi(\xi)$ is of finite dimension. Let $\{W_1,\cdots,W_m\}$ be a basis of $\pi(\xi)\mathbf{W}(\pi,\psi_{n,F})$. From the similar argument as in the proof of [6, Proposition 9.2], it follows that, for each $W\in\mathbf{W}(\pi,\psi_{n,F})$, there exist bi-$K$-finite matrix coefficients $\phi_1,\cdots,\phi_m$ of π such that

$$\pi(\xi)(\pi(g)W)(h)=\int_K W(hkg)\xi(k)dk=\sum_{j=1}^m W_j(h)\phi_j(g)\ .$$

Then, by the analogous calculation as in the proof of Lemma 1, we have

$$\widetilde{V}^s_{(W,f)}(h)=\alpha_F(h)^{s-1/2}\sum_{j=1}^m\frac{Z(f,s+n/2-1/2,\phi_j)}{L(s,\pi)}W_j(h)\ , \tag{2.1}$$

if $\mathrm{Re}(s)>>0$. Here $Z(f,s+n/2-1/2,\phi_j)$ is defined similarly as (1.4). It is known by [2, Theorem 8.7] or [5, Proof of Proposition 4.5] that $L(s,\pi)^{-1}Z(f,s+n/2-1/2,\phi_j)$ extends to an entire function of s. Thus the assertion follows from the same argument as in the proof of Lemma 1. □

3. The global theta correspondence

In the rest of this paper, we denote by k a global field and by $\mathbb{A}=\prod'_v k_v$ the adele ring of k. For a k-subgroup G of $G_n=GL_n(k)$, $G(\mathbb{A})=\prod'_v G(k_v)$ denotes the corresponding adele group. We fix a non-trivial additive character ψ of $k\backslash\mathbb{A}$ and define the character ψ_n of $U_n(\mathbb{A})$ similarly as (1.1). The restriction of ψ_n to $U_n(k_v)$ is denoted by $\psi_{n,v}$. We define the character α of $G_n(\mathbb{A})$ by $\alpha(g) = |\det g|_{\mathbb{A}}$. Let $\pi = \otimes_v' \pi_v$ be an irreducible automorphic cuspidal representation of $G_n(\mathbb{A})$. There exists a unique real number t so that $\alpha^{-t}\varphi$ is square integrable on $Z(\mathbb{A})G_n \backslash G_n(\mathbb{A})$ for any $\varphi \in H_\pi$. Thus $\alpha^{-t} \otimes \pi$ becomes a unitary cuspidal representation. For each $\varphi \in H_\pi$, we set

$$W_\varphi(g) = \int_{U_n \backslash U_n(\mathbb{A})} \psi_n(u)^{-1} \varphi(ug) du \ .$$

The space $\mathbf{W}(\pi, \psi_n)$ of all W_φ ($\varphi \in H$) is decomposed into the restricted tensor product of local Whittaker models $\mathbf{W}(\pi_v, \psi_{n,v})$, i.e.

$$\mathbf{W}(\pi, \psi_n) = \otimes_v' \mathbf{W}(\pi_v, \psi_{n,v}) \ .$$

Let μ_π be the central character of π. For $f \in \mathcal{S}(M_n(\mathbb{A}))$ and $s \in \mathbf{C}$, we define a modified theta series $\theta(s, \mu_\pi, f)$ as

$$\theta(s, \mu_\pi, f) = \int_{Z_n \backslash Z_n(\mathbb{A})} \mu_\pi(z) \alpha(z)^{s+n/2} \sum_{\substack{x \in M_n(k) \\ x \neq 0}} f(zx) dz \ .$$

From [2, Lemmas 11.5 and 11.6], it follows that the integral of the right-hand side is absolutely convergent for $\mathrm{Re}(s) > n/2 - t$ and the function $(h, g) \mapsto \theta(s, \mu, \omega(h, g)f)$ is slowly increasing on $G_n \backslash G_n(\mathbb{A}) \times G_n \backslash G_n(\mathbb{A})$. By using $\theta(s, \mu_\pi, f)$, the theta lifting φ_f^s of $\varphi \in H_\pi$ is written as

$$\varphi_f^s(h) = \int_{Z_n(\mathbb{A})G_n \backslash G_n(\mathbb{A})} \varphi(g) \alpha(g)^s \theta(s, \mu_\pi, \omega(h, g)f) dg \ .$$

Since $\varphi(h)$ is rapidly decreasing on $Z_n(\mathbb{A})G_n \backslash G_n(\mathbb{A})$, this integral is absolutely convergent for $\mathrm{Re}(s) > n/2 - t$.

LEMMA 3. *$\varphi_f^s(h)$ is analytically continued to an entire function of s.*

PROOF. If $\mathrm{Re}(s) > n/2 - t$, we have

$$\varphi_f^s(h) = \sum_{j=1}^{n} \int_{G_n \backslash G_n(\mathbb{A})} \varphi(g) \alpha(g)^s \sum_{\substack{x \in M_n(k) \\ \mathrm{rank}(x) = j}} \omega(h, g) f(x) dg \ .$$

It follows from [2, Lemma 12.13] that the sum over $1 \le j \le n-1$ is equal to zero. Thus, $\varphi_f^s(h)$ equals

$$\begin{aligned}\int_{G_n\backslash G_n(\mathbb{A})} &\varphi(g)\alpha(g)^s \sum_{x\in G_n} \omega(h,g)f(x)dg \\ &= \alpha(h)^{-n/2}\int_{G_n(\mathbb{A})} \varphi(g)\alpha(g)^{s+n/2} f(h^{-1}g)dg \;.\end{aligned}$$

By [2, Theorem 12.4], the last integral can be continued analytically to the whole s-plane as an entire function. □

By the above expression of φ_f^s, it is known that the space $\Theta^s(\pi)$ is contained in the space of cusp forms on $G_n(\mathbb{A})$ if $\mathrm{Re}(s) > n/2 - t$.

Proof of Theorem 1. First we assume $\mathrm{Re}(s) >> 0$. For $\varphi_f^{s-1/2} \in \Theta^{s-1/2}(\pi)$, we set

$$\begin{aligned}V^s_{(\varphi,f)}(h) &= \int_{U_n\backslash U_n(\mathbb{A})} \psi_n(u)^{-1}\varphi_f^{s-1/2}(uh)du \\ &= \alpha(h)^{s-1/2}\int_{G_n(\mathbb{A})} W_\varphi(hg)\alpha(g)^{s+n/2-1/2} f(g)dg \;.\end{aligned}$$

We may assume that W_φ and f are decomposable, i.e. they are of the forms

$$W_\varphi(g) = \prod_v W_v(g_v), \qquad f(g) = \prod_v f_v(g_v) \;.$$

Then we set

$$V^s_{(W_v,f_v)}(h_v) = |\det h_v|_v^{s-1/2}\int_{G_n(k_v)} W_v(h_v g_v)|\det g_v|_v^{s+n/2-1/2} f_v(g_v)dg_v \;.$$

Let $S(\varphi,f)$ be the finite set of places of k such that W_v is a class one Whittaker function and f_v the characteristic function of the set $M_n(\mathcal{O}_v)$ consisting of integral matrices if $v \notin S(\varphi,f)$. It follows from [2, Lemma 6.10], (1.5) and (2.1) that if $v \notin S(\varphi,f)$, then

$$V^s_{(W_v,f_v)}(h_v) = L(s,\pi_v)|\det h_v|_v^{s-1/2} W_v(h_v),$$

and if $v \in S(\varphi, f)$, then $V^s_{(W_v,f_v)}$ is of the form

$$V^s_{(W_v,f_v)}(h_v) = L(s,\pi_v)|\det h_v|_v^{s-1/2} \sum_j \Xi_{v,j}(s) W_{v,j}(h_v) ,$$

where $\Xi_{v,j}(s)$ are entire functions of s and $W_{v,j}$ are elements in $\mathbf{W}(\pi_v, \psi_{n,v})$. We have

$$V^s_{(\varphi,f)}(h) = L(s,\pi)\alpha(h)^{s-1/2} \times \prod_{v\in S(\varphi,f)} \left\{ \sum_j \Xi_{v,j}(s) W_{v,j}(h_v) \right\} \prod_{v\notin S(\varphi,f)} W_v(h_v) .$$

Consequently, we can take a finite number of entire functions $\Xi_j(s)$ and cusp forms $\varphi_j \in H_\pi$ such that

$$V^s_{(\varphi,f)}(h) = L(s,\pi)\alpha(h)^{s-1/2} \sum_j \Xi_j(s) W_{\varphi_j}(h) . \tag{3.1}$$

It is known by [11, Theorem 5.9] that

$$\varphi_f^{s-1/2}(h) = \sum_{\gamma\in U_{n-1}\backslash G_{n-1}} V^s_{(\varphi,f)}(\gamma h)$$
$$\varphi_j(h) = \sum_{\gamma\in U_{n-1}\backslash G_{n-1}} W_{\varphi_j}(\gamma h) .$$

Here we regard G_{n-1} as a subgroup of G_n by the embedding

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} .$$

Therefore, when $\mathrm{Re}(s)$ is sufficiently large, we obtain

$$\varphi_f^s(h) = L(s+1/2,\pi)\alpha(h)^s \sum_j \Xi_j(s+1/2)\varphi_j(h) .$$

Since the right-hand side is an entire function of s, this expression holds for all $s \in \mathbf{C}$. This implies the first assertion of Theorem. Furthermore, by the irreducibility of π, we have $\Theta^0(\pi) = H_\pi$ if $\Theta^0(\pi) \neq 0$. □

Proof of Theorem 2. For $\varphi \in H_\pi$, $f \in \mathcal{S}_0(M_n(\mathbb{A}))$ and $s \in \mathbf{C}$ with $\mathrm{Re}(s) >> 0$, we set

$$\widetilde{V}^s_{(\varphi,f)}(h) = L(s,\pi)^{-1} \int_{G_n(\mathbb{A})} W_\varphi(g)\alpha(g)^{s-1/2}\omega(h,g)f(1_n)dg \ .$$

By (3.1), $\widetilde{V}^s_{(\varphi,f)}(h)$ extends to an entire function of s, and as a function in h, $\widetilde{V}^s_{(\varphi,f)}$ is contained in $\alpha^{s-1/2} \otimes W(\pi, \psi_n)$. Thus we can consider

$$\theta(\varphi, f)(h) = \sum_{\gamma \in U_{n-1}\backslash G_{n-1}} \widetilde{V}^{1/2}_{(\varphi,f)}(\gamma h) \ .$$

The space spanned by $\theta(\varphi, f)$, ($\varphi \in H_\pi$, $f \in \mathcal{S}_0(M_n(\mathbb{A}))$) equals H_π. We identify the dual space $H_\pi^\vee$ of H_π with the space of functions $\alpha^{-2t}\overline{\varphi}$, ($\varphi \in H_\pi$) by the pairing

$$< \alpha^{-2t}\overline{\varphi}_1, \varphi_2 >= \int_{Z_n(\mathbb{A})G_n\backslash G_n(\mathbb{A})} \overline{\varphi}_1(g)\varphi_2(g)\alpha(g)^{-2t}dg \ .$$

Then the pairing $(\alpha^{-2t}\overline{\varphi}_1 \otimes \varphi_2) \otimes f \mapsto < \alpha^{-2t}\overline{\varphi}_1, \theta(\varphi_2, f) >$ on $(H_\pi^\vee \otimes H_\pi) \otimes \mathcal{S}_0(M_n(\mathbb{A}))$ gives rise to a nonzero $\mathcal{H}_n \otimes \mathcal{H}_n$-morphism from ω to the contragredient representation $(\pi^\vee \otimes \pi)^\vee$ of $\pi^\vee \otimes \pi$. Next, let $\sigma = \otimes'_v \sigma_v$ be an irreducible autormophic cuspidal representation satisfying $\mathrm{Hom}_{\mathcal{H}_n \otimes \mathcal{H}_n}(\omega, \sigma \otimes \pi^\vee) \neq 0$. Then we have

$$\text{(3.2)} \qquad \begin{array}{l} \mathrm{Hom}_{\mathcal{H}_{n,v} \otimes \mathcal{H}_{n,v}}(\omega_v, \sigma_v \otimes \pi_v^\vee) \neq 0 \quad \text{and} \\ \mathrm{Hom}_{\mathcal{H}_{n,v} \otimes \mathcal{H}_{n,v}}(\omega_v, \pi_v \otimes \pi_v^\vee) \neq 0 \end{array}$$

for each place v. We denote by $S(\pi)$ the finite set of finite places v where π_v is not a spherical representation. Since the local Howe duality conjecture is true for the case of real reductive dual pairs ([4, Theorem 1]) and the case of spherical representations of unramified reductive dual pairs ([3, Theorem 7.1]), we have $\sigma_v \cong \pi_v$ for all $v \notin S(\pi)$ by (3.2). Then the strong multiplicity one theorem ([8, Corollary 4.10]) implies $\sigma \cong \pi$. □

Remark. We prove $\mathrm{Hom}_{\mathcal{H}_n \otimes \mathcal{H}_n}(\omega, \pi \otimes \pi^\vee) \neq 0$ for any irreducible admissible representation π of $G_n(\mathbb{A})$. Let S be the finite set of places containing all archimedean places and all finite places v where π_v is not a spherical representation. We take nonzero spherical vectors $e_v \in H_{\pi_v}$ and $e_v^\vee \in H_{\pi_v}^\vee$ for each $v \notin S$. Then H_π and $H_\pi^\vee$ are decomposed into restricted tensor products of the H_{π_v} and the $H_{\pi_v}^\vee$ with respect to $\{e_v\}_{v \notin S}$ and $\{e_v^\vee\}_{v \notin S}$, respectively. It is known by [2, Theorems 3.3 and 8.7] that $\mathrm{Hom}_{\mathcal{H}_{n,v} \otimes \mathcal{H}_{n,v}}(\omega_v, \pi_v \otimes \pi_v^\vee) \neq 0$ for each v. If $v \notin S$, we can take a nonzero $T_v \in \mathrm{Hom}_{\mathcal{H}_{n,v} \otimes \mathcal{H}_{n,v}}(\omega_v, \pi_v \otimes \pi_v^\vee)$ normalized so that $T_v(f_v) = e_v \otimes e_v^\vee$ for the characteristic function f_v of $M_n(\mathcal{O}_v)$ (cf. [3, Theorem 10.2]). If $v \in S$, we take an arbitrary nonzero $T_v \in \mathrm{Hom}_{\mathcal{H}_{n,v} \otimes \mathcal{H}_{n,v}}(\omega_v, \pi_v \otimes \pi_v^\vee)$. Then $T = \otimes_v T_v$ gives a nonzero element in $\mathrm{Hom}_{\mathcal{H}_n \otimes \mathcal{H}_n}(\omega, \pi \otimes \pi^\vee)$.

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