The abelianization of the level d mapping class group
（レベルd写像類群のアーベル化）

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Abstract

In this paper, we determine the abelianization of the level d mapping class group for $d = 2$ and odd d when $g \geq 3$. For an even d greater than 2, we determine the abelianization of the group up to a cyclic group of order 2. To compute them, we extend the homomorphism of the Torelli group defined by Heap to a homomorphism of the level 2 mapping class group.

Contents

1 Introduction

Let g be a positive integer, and r either 0 or 1. We denote by $\Sigma_{g,r}$ a closed oriented connected surface of genus g with r boundary components. We denote by $\mathrm{Diff}_+(\Sigma_{g,r}, \partial\Sigma_{g,r})$ the topological group of orientation-preserving diffeomorphisms of $\Sigma_{g,r}$ which fix the boundary pointwise with C^∞ topology.

The mapping class group $\mathcal{M}_{g,r}$ of $\Sigma_{g,r}$ is defined by $\mathcal{M}_{g,r} = \pi_0 \operatorname{Diff}_+(\Sigma_{g,r}, \partial\Sigma_{g,r})$. Fix the symplectic basis $\{A_i, B_i\}_{i=1}^g$ of the first homology group $H_1(\Sigma_{g,r}; \mathbf{Z})$ such that $A_i \cdot A_j = B_i \cdot B_j = 0$ and $A_i \cdot B_j = \delta_{ij}$ for $1 \leq i \leq j \leq g$, where δ_{ij} is the Kronecker delta. Then the natural action of $\mathcal{M}_{g,r}$ on this group gives rise to the classical representation $\rho : \mathcal{M}_{g,r} \to \operatorname{Sp}(2g; \mathbf{Z})$ onto the integral symplectic group. The kernel $\mathcal{I}_{g,r}$ of this representation is called the Torelli group.

For an integer $d \geq 2$, the level d mapping class group $\mathcal{M}_{g,r}[d] \subset \mathcal{M}_{g,r}$ is defined by the kernel of the mod d reduction $\mathcal{M}_{g,r} \to \operatorname{Sp}(2g; \mathbf{Z}_d)$ of ρ. The level d congruence subgroup $\Gamma_g[d]$ of the symplectic group is defined by the kernel of mod d reduction map $\operatorname{Ker}(\operatorname{Sp}(2g; \mathbf{Z}) \to \operatorname{Sp}(2g; \mathbf{Z}_d))$. This is equal to the image of $\mathcal{M}_{g,r}[d]$ under ρ. The group $\mathcal{M}_g[d]$ arises as the orbifold fundamental group of the moduli space of nonsingular curves of genus g with level d structure. In particular, for $d \geq 3$, the level d mapping class groups are torsion-free, and the abelianizations of the level d mapping class groups are equal to the first homology groups of the corresponding moduli spaces.

In this paper, we compute the abelianizations, that is the first integral homology groups, of $\mathcal{M}_{g,r}[d]$ and $\Gamma_g[d]$. We determine especially $H_1(\mathcal{M}_{g,r}[d]; \mathbf{Z})$ for $d = 2$ and odd $d \geq 3$, and $H_1(\Gamma_g[d]; \mathbf{Z})$ for all $d \geq 2$. This is an analogous result in Satoh [34] and Lee-Szczarba [26] for the abelianizations of the level d congruence subgroups of $\operatorname{Aut} F_n$ and $\operatorname{GL}(n; \mathbf{Z})$. To determine the abelianization $H_1(\mathcal{M}_{g,r}[2]; \mathbf{Z})$, we construct an injective homomorphism $\beta_\sigma : \mathcal{M}_{g,1}[2] \to \operatorname{Map}(H_1(\Sigma_g; \mathbf{Z}_2), \mathbf{Z}_8)$. This function is defined using Rochlin functions of mapping tori. We will show that this is an extension of a homomorphism of the Torelli group defined by Heap [11]. To determine the abelianization $H_1(\mathcal{M}_{g,r}[d]; \mathbf{Z})$ for odd d, we construct the Johnson homomorphism of modulo d on $\mathcal{M}_{g,r}[d]$.

Historically, McCarthy [27] proved that the first rational homology group of a finite index subgroup of $\mathcal{M}_{g,r}$ which includes the Torelli group vanishes for $r = 0$. More generally, Hain [10] proved that this group vanishes for any $r \geq 0$.

Theorem 1.1 ((McCarthy [27], Hain [10])). *Let $g \geq 3$ and $r \geq 0$. If $\mathcal{M}$ be a finite index subgroup of $\mathcal{M}_{g,r}$ that includes the Torelli group, then we have*

$$H_1(\mathcal{M}; \mathbf{Q}) = 0.$$

Farb raised the problem to compute the abelianization of the group $\mathcal{M}_{g,r}[d]$ in Farb [9] Problem 5.23 p.43. Recently, Putman [32] also determined the abelianization of the level d congruence subgroup of the symplectic group and the level d mapping class group for odd d when $g \geq 3$. See also [33].

This paper is organized as follows. In Section 2, we construct the injective homomorphism $\beta_\sigma : \mathcal{M}_{g,1}[2] \to \operatorname{Map}(H_1(\Sigma_g; \mathbf{Z}_2), \mathbf{Z}_8)$, for spin structures σ of Σ_g, to determine the abelianization of the level 2 mapping class group. Let n be a positive integer. For a $(4n-1)$-manifold M and a spin structure σ of M, the Rochlin function $R(M, \sigma)$ is defined as the signature of a compact $4n$-manifold which spin bounds (M, σ). See, for example, Turaev [35]. The homomorphism $\beta_\sigma(\varphi)$ is defined as the difference $R(M_\varphi, \sigma) - R(M_\varphi, \sigma')$ for a mapping torus M_φ of $\varphi \in \mathcal{M}_{g,1}[2]$ (Definition 2.1). Turaev [35] proved that it is written as the Brown invariant of a pin^- bordism class represented by a surface embedded in the spin manifold M_φ (Lemma 2.9). In section 4, we see that this homomorphism is an extension of Heap's homomorphism (Lemma 2.12). In section 5, We compute the value of it by examining the pin^- bordism class of the surface F in M_φ. The main theorem in this paper is illustrated as follows. For homology classes $\{x_i\}_{i=1}^n$ in $H_1(\Sigma_{g,1}; \mathbf{Z}_2)$, define a map $I : H_1(\Sigma_{g,1}; \mathbf{Z}_2)^n \to \mathbf{Z}_2$ by

$$I(x_1, x_2, \cdots, x_n) := \sum_{1 \leq i < j \leq n} (x_i \cdot x_j) \bmod 2,$$

where $x_i \cdot x_j$ is the intersection number of x_i with x_j. We denote by $\mathbf{Z}_8[H_1(\Sigma_{g,1}; \mathbf{Z}_2)]$ the free $\mathbf{Z}_8$-module generated by all formal symbol $[X]$ for $X \in H_1(\Sigma_{g,1})$. Define a map $\Delta_0^n : H_1(\Sigma_{g,1}; \mathbf{Z})^n \to$

$\mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]$ by

$$\begin{aligned}\Delta_0^n(x_1,x_2,\cdots,x_n) = [0] + \sum_{i=1}^{n}[x_i] + \sum_{1\le i<j\le n}(-1)^{I(x_i,x_j)}[x_i+x_j] \\ + \sum_{1\le i<j<k\le n}(-1)^{I(x_i,x_j,x_k)}[x_i+x_j+x_k] \\ +\cdots+(-1)^{I(x_1,x_2,\cdots,x_n)}[x_1+x_2+x_3+\cdots+x_n] \in \mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)].\end{aligned}$$

Theorem 1.2. *Let $g \geq 3$ be an integer. Denote by $L_{g,1} \subset \mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]$ the submodule generated by*

$$[0],\ 4\Delta_0^2(x_1,x_2),\ 2\Delta_0^3(x_1,x_2,x_3),\ \Delta_0^n(x_1,x_2,\cdots,x_n) \in \mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)],$$

for $n \geq 4$ and $\{x_i\}_{i=1}^n \subset H_1(\Sigma_{g,1};\mathbf{Z}_2)$. Then, we have

$$H_1(\mathcal{M}_{g,1}[2];\mathbf{Z}) \cong \mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]/L_{g,1},$$

as $\mathcal{M}_{g,1}$-module.

For a group G and a G-module M, denote by M_G the coinvariant of the action of G on M. We will show that the kernel of the homomorphism $\iota : H_1(\mathcal{I}_{g,r};\mathbf{Z})_{\Gamma_g[2]} \to H_1(\mathcal{M}_{g,r}[2];\mathbf{Z})$ induced by the inclusion is isomorphic to $\mathbf{Z}_2$ (See Remark 3.16). By the 5-term exact sequence coming from the exact sequence $1 \longrightarrow \mathcal{I}_{g,r} \longrightarrow \mathcal{M}_{g,r}[2] \longrightarrow \Gamma_g[2] \longrightarrow 1$, we have:

Corollary 1.3. *Let $g \geq 3$ be an integer, and r either 0 or 1. The sequence*

$$0 \longrightarrow \mathbf{Z}_2 \longrightarrow H_1(\mathcal{I}_{g,r};\mathbf{Z})_{\Gamma_g[2]} \longrightarrow H_1(\mathcal{M}_{g,r}[2];\mathbf{Z}) \longrightarrow H_1(\Gamma_g[2];\mathbf{Z}) \longrightarrow 0$$

is exact.

In Section 3, we will prove Theorem 1.2 using the result of Section 4 on the abelianization of the level d congruence subgroup of the symplectic group. We also determine the abelianization of the level 2 mapping class group of a closed surface in Section 3.4. In Section 4, we determine the abelianization of $\Gamma_g[d]$ for every integer $d \geq 2$ (Corollary 4.2). To do it, we investigate the commutator subgroup of the level d congruence subgroup of the symplectic group. This mainly relies on the work of Mennicke [29] and Bass-Milnor-Serre [3] on congruence subgroups of the symplectic group. In Section 5, we compute the abelianization of the level d mapping class group for $d \geq 3$. The main tool is the Johnson homomorphism of modulo d on the level d mapping class group. This derives from the extension of the Johnson homomorphism defined by Kawazumi [23]. Let H denote the first homology group $H_1(\Sigma_{g,r};\mathbf{Z})$. Denote by Λ^3H/H the cokernel of the homomophism

$$\begin{array}{ccc} H & \to & \Lambda^3 H \\ x & \mapsto & \sum_{i=1}^{g}(A_i \wedge B_i)\wedge x. \end{array}$$

Then, the abelianization of the level d mapping class group is written as:

Theorem 1.4. *Let $g \geq 3$ be an integer. For an odd integer $d \geq 3$, we have*

$$\begin{aligned} H_1(\mathcal{M}_g[d];\mathbf{Z}) &= (\Lambda^3H/H \otimes \mathbf{Z}_d) \oplus H_1(\Gamma_g[d];\mathbf{Z}) \\ &= \mathbf{Z}_d^{(4g^3-g)/3}, \\ H_1(\mathcal{M}_{g,1}[d];\mathbf{Z}) &= (\Lambda^3H \otimes \mathbf{Z}_d) \oplus H_1(\Gamma_g[d];\mathbf{Z}) \\ &= \mathbf{Z}_d^{(4g^3+5g)/3}. \end{aligned}$$

For even $d \geq 4$, we do not know the abelianization of $\mathcal{M}_{g,r}[d]$. But we have the following exact sequence.

Proposition 1.5. *Let $g \geq 3$ be an integer , and r either 0 or 1. For an even integer $d \geq 4$,*

$$\mathbf{Z}_2 \longrightarrow H_1(\mathcal{I}_{g,r};\mathbf{Z})_{\Gamma_g[d]} \longrightarrow H_1(\mathcal{M}_{g,r}[d];\mathbf{Z}) \longrightarrow H_1(\Gamma_g[d];\mathbf{Z}) \longrightarrow 0$$

is exact.

2 The abelianization of the level 2 mapping class group

In this section, we will define a family of homomorphisms

$$\beta_{\sigma,x} : \mathcal{M}_{g,1}[2] \to \mathbf{Z}_8,$$

for a spin structure σ of Σ_g and an element x in $H_1(\Sigma_g; \mathbf{Z}_2)$ (Definition 2.1). This family determines the abelianization of the level 2 mapping class group. In Section 2.4, the homomorphism $\beta_{\sigma,x}$ is proved to be an extension of the homomorphism $\omega_{\sigma,y} : \mathcal{I}_{g,1} \to \mathbf{Z}_2$ defined in Heap [11] p.26 to the level 2 mapping class group. We calculate the values of this homomorphism on generators of the level 2 mapping class group using the Brown invariant in Section 2.5.

2.1 Spin structures

In this section, we define spin structures of an oriented vector bundle and introduce some properties. We also define spin structures of mapping tori which come from those of a fiber Σ_g to construct the homomorphism $\beta_{\sigma,x} : \mathcal{M}_{g,1}[2] \to \mathbf{Z}_8$.

2.1.1 Definition of Spin structures

Let $f : E \to V$ be a smooth oriented real vector bundle of rank $n \geq 2$ with a metric on a smooth manifold V. We denote by $P(E)$ the oriented frame bundle associated to this bundle. When the Stiefel-Whitney class w_2 of E vanishes, we define a spin structure of E by a right inverse homomorphism of the homomorphism $f_* : H_1(P(E); \mathbf{Z}_2) \to H_1(V; \mathbf{Z}_2)$. Denote by $\mathrm{spin}(E)$ the set of spin structures of E.

Since $P(E)$ is a principal $\mathrm{GL}_+(n)$ bundle and w_2 vanishes, we have the exact sequence

$$0 \longrightarrow \mathbf{Z}_2 \xrightarrow{\ j\ } H_1(P(E); \mathbf{Z}_2) \xrightarrow{\ f_*\ } H_1(V; \mathbf{Z}_2) \longrightarrow 0$$

arising from the fibration $\mathrm{GL}_+(n) \to P(E) \to V$. For $\sigma \in \mathrm{spin}(E)$, consider a splitting $H_1(P(E); \mathbf{Z}_2) \cong H_1(V; \mathbf{Z}_2) \oplus \mathbf{Z}_2$ induced by $\sigma : H_1(V; \mathbf{Z}_2) \to H_1(P(E); \mathbf{Z}_2)$. Under this splitting, we denote by $k_\sigma : H_1(P(E); \mathbf{Z}_2) \to \mathbf{Z}_2$ the projection to the second factor. Spin structures σ of E are equivalent to left inverse homomorphisms k_σ of $j : \mathbf{Z}_2 \to H_1(P(E); \mathbf{Z}_2)$ as above. In this way, we can consider a spin structure as an element of $H^1(P(E); \mathbf{Z}_2) = \mathrm{Hom}(H_1(P(E); \mathbf{Z}_2), \mathbf{Z}_2)$ which evaluates to $1 \bmod 2 \in \mathbf{Z}_2$ on a homotopically nontrivial loop in a fiber $f^{-1}(x) \cong \mathrm{GL}_+(n)$. We will define the $\mathrm{Spin}(n)$ group in Section 2.3 which is the double cover of the special orthogonal group $\mathrm{SO}(n)$. Thus, if we endow a fiber metric on E, spin structures are equivalent to principal $\mathrm{Spin}(n)$ bundles which are double covers of oriented orthonormal frame bundles associated to E. In detail, for example, see Lee-Miller-Weintraub [25] Section 1.1.

For an oriented smooth n-manifold V, we define the spin structure of V by the spin structure of the tangent bundle TV. We denote $\mathrm{spin}(TV)$, the set of spin structures of TV, simply by $\mathrm{spin}(V)$. Note that a spin structure of V is equivalent to a spin structure of $V \times (-\epsilon, \epsilon)^k$, for $\epsilon > 0$ and $k > 0$.

2.1.2 Spin structures of mapping tori

Fix a closed disk neighborhood $N(c_0)$ of a point c_0 in Σ_g. The mapping class group $\pi_0 \operatorname{Diff}_+(\Sigma_g, N(c_0))$ is the group consisting of isotopy classes of orientation-preserving diffeomorphisms of Σ_g which fix the neighborhood $N(c_0)$ pointwise. By restricting each diffeomorphism to $\Sigma_g - \operatorname{Int} N(c_0)$, the group $\pi_0 \operatorname{Diff}_+(\Sigma_g, N(c_0))$ is isomorphic to $\mathcal{M}_{g,1}$. Hence, we identify these two groups. We also identify the kernel $\mathrm{Ker}(\pi_0 \operatorname{Diff}_+(\Sigma_g, N(c_0)) \to \mathrm{Sp}(2g; \mathbf{Z}_2))$ of the mod 2 reduction of the homomorphism $\rho : \pi_0 \operatorname{Diff}_+(\Sigma_g, N(c_0)) \to \mathrm{Sp}(2g; \mathbf{Z}))$ with $\mathcal{M}_{g,1}[2]$.

For $\varphi = [f] \in \mathcal{M}_{g,1}$, denote the mapping torus of φ by $M_\varphi := \Sigma_g \times [0,1]/\sim$, where the equivalence relation is given by $(f(x), 0) \sim (x, 1)$. We denote the mapping torus M_φ simply by M. To construct the homomorphism $\beta_{\sigma,x} : \mathcal{M}_{g,1}[2] \to \mathbf{Z}_8$ on the level 2 mapping class group, we must consider spin structures of the mapping tori M induced from those of Σ_g. In the following, we define a map $\theta : \mathrm{spin}(\Sigma_g) \to \mathrm{spin}(M_\varphi)$.

Fix a spin structure on Σ_g. Since $\varphi \in \mathcal{M}_{g,1}[2]$ acts on $H_1(\Sigma_g; \mathbf{Z}_2)$ trivially, the Wang exact sequence is written as

$$0 \longrightarrow H_1(\Sigma_g; \mathbf{Z}_2) \longrightarrow H_1(M; \mathbf{Z}_2) \longrightarrow H_1(S^1; \mathbf{Z}_2) \longrightarrow 0.$$

Since $\varphi \in \mathcal{M}_{g,1}[2]$ preserves $N(c_0)$ pointwise, we have a natural embedding $l : N(c_0) \times S^1 \to M$. This embedding gives the splitting

$$H_1(M; \mathbf{Z}_2) = H_1(\Sigma_g; \mathbf{Z}_2) \oplus H_1(S^1; \mathbf{Z}_2).$$

In order to define the spin structure on M, we will construct homomorphisms from each direct summand to $H_1(P(M); \mathbf{Z}_2)$.

Let $\{v_0, v_1\}$ be a frame of $T_{c_0}N(c_0)$, and $v_{S^1}(t) \in T_tS^1$ a nonzero tangent vector. For $N(c_0) \times S^1 \subset M$, define the framing $\hat{l} : S^1 \to P(N(c_0) \times S^1)$ by $\hat{l}(t) = (v_0 \cos 2\pi t + v_1 \sin 2\pi t, v_1 \cos 2\pi t - v_0 \sin 2\pi t, v_{S^1}(t))$. This framing induces the homomorphism

$$H_1(S^1; \mathbf{Z}_2) \xrightarrow{\hat{l}_*} H_1(P(N(c_0) \times S^1); \mathbf{Z}_2) \xrightarrow{\mathrm{inc}_*} H_1(P(M); \mathbf{Z}_2), \tag{1}$$

where inc_* is the homomorphism induced by the inclusion map.

Next, consider the natural smooth map $P(\Sigma_g \times (-\epsilon, \epsilon)) \to P(M)$ induced by the inclusion of a tubular neighborhood $\Sigma_g \times (-\epsilon, \epsilon) \subset M$ for small $\epsilon > 0$. Using the spin structure σ of Σ_g, we have the homomorphism

$$H_1(\Sigma_g; \mathbf{Z}_2) \xrightarrow{\sigma} H_1(P(\Sigma_g \times (-\epsilon, \epsilon)); \mathbf{Z}_2) \xrightarrow{\mathrm{inc}_*} H_1(P(M); \mathbf{Z}_2). \tag{2}$$

Thus, we have constructed the homomorphism $H_1(M; \mathbf{Z}_2) \to H_1(P(M); \mathbf{Z}_2)$. In this way, we obtain the map $\theta : \mathrm{spin}(\Sigma_g) \to \mathrm{spin}(M)$.

2.2 A homomorphism $\beta_{\sigma,x} : \mathcal{M}_{g,1}[2] \to \mathbf{Z}_8$

In this section, we will construct a homomorphism $\beta_{\sigma,x}$ which determines the abelianization of the group $\mathcal{M}_{g,1}[2]$, using Rochlin functions of mapping tori. In Section 2.2.1, we define the map $\beta_{\sigma,x} : \mathcal{M}_{g,1}[2] \to \mathbf{Z}_8$. In Section 2.2.2, we show that it is a homomorphism.

2.2.1 Rochlin functions of mapping tori

First we review the simply transitive action of $H_1(\Sigma_g; \mathbf{Z}_2)$ on $\mathrm{spin}(\Sigma_g)$. Let M be an oriented n-manifold with second Stiefel-Whitney class $w_2 = 0$. By the Serre spectral sequence of the fibration $GL_+(n) \to P(M) \to M$, we have the exact sequence

$$1 \longrightarrow \mathbf{Z}_2 \xrightarrow{j} H_1(P(M); \mathbf{Z}_2) \longrightarrow H_1(M; \mathbf{Z}_2) \longrightarrow 1.$$

For $x \in H^1(M; \mathbf{Z}_2)$, we denote again by $x : H_1(M; \mathbf{Z}_2) \to H_1(P(M); \mathbf{Z}_2)$ the composite of $x : H_1(M; \mathbf{Z}_2) \to \mathbf{Z}_2$ and the inclusion $j : \mathbf{Z}_2 \to H_1(P(M); \mathbf{Z}_2)$. Let σ be a spin structure of M. Since $x : H_1(M; \mathbf{Z}_2) \to H_1(P(M); \mathbf{Z}_2)$ factors through $j : \mathbf{Z}_2 \to H_1(P(M); \mathbf{Z}_2)$, the homomorphism $\sigma + x : H_1(M; \mathbf{Z}_2) \to H_1(P(M); \mathbf{Z}_2)$ is also a spin structure. In this way, $H^1(M; \mathbf{Z}_2)$ acts on $\mathrm{spin}(M)$ simply transitively.

Next, we review the definition of Rochlin functions. It is known as Rochlin's theorem that every spin 3-manifold bounds a spin 4-manifold. For $\varphi \in \mathcal{M}_{g,1}[2]$, choose a compact oriented spin manifold V which is spin bounded by the mapping torus $M = M_\varphi$. Denote by $\mathrm{Sign}\, V$ the signature of the 4-manifold V. Then a Rochlin function of (M, σ) is defined by

$$R(M, \sigma) := \mathrm{Sign}\, V \bmod 16 \in \mathbf{Z}_{16}.$$

This is well-defined by Rochlin's theorem, and is called the Rochlin function. Let $\theta : \mathrm{spin}(\Sigma_g) \to \mathrm{spin}(M_\varphi)$ be the map defined in Section 2.1.2.

Definition 2.1. *For $x \in H_1(\Sigma_g;\mathbf{Z}_2)$ and $\sigma \in \mathrm{spin}(\Sigma_g)$, define the map*

$$\beta_{\sigma,x} : \mathcal{M}_{g,1}[2] \to \left(\frac{1}{2}\mathbf{Z}\right)/8\mathbf{Z}$$

by $\beta_{\sigma,x}(\varphi) := (R(M_\varphi, \theta(\sigma)) - R(M_\varphi, \theta(\sigma+x)))/2 \mod 8$.

As we will show in Section 2.3, the image $\operatorname{Im}\beta_{\sigma,x}$ is in $\mathbf{Z}_8$. Denote by $\mathrm{Map}(H_1(\Sigma_g;\mathbf{Z}_2);\mathbf{Z}_8)$ the free $\mathbf{Z}_8$-module consisting of all maps $H_1(\Sigma_g;\mathbf{Z}_2) \to \mathbf{Z}_8$. We define the map $\beta_\sigma : \mathcal{M}_{g,1}[2] \to \mathrm{Map}(H_1(\Sigma_g;\mathbf{Z}_2);\mathbf{Z}_8)$ by $\beta_\sigma(\varphi)(x) = \beta_{\sigma,x}(\varphi)$.

2.2.2 A spin manifold bounded by Mapping tori

In the following, we will prove that the map $\beta_\sigma : \mathcal{M}_{g,1}[2] \to \mathrm{Map}(H_1(\Sigma_g;\mathbf{Z}_2);\mathbf{Z}_8)$ is a homomorphism. To prove this, we construct a compact spin 4-manifold W which is spin bounded by mapping tori with spin structure $\theta(\sigma)$.

Lemma 2.2. *Let g be a positive integer. For a spin structure σ of Σ_g, the map $\beta_\sigma : \mathcal{M}_{g,1}[2] \to \mathrm{Map}(H_1(\Sigma_g;\mathbf{Z}_2);\mathbf{Z}_8)$ is a homomorphism.*

Proof. Let $P_0 := S^2 - \amalg_{i=1}^3 \operatorname{Int} D_i$ denote a pair of pants, where $\{D_i\}$ are mutually disjoint disks and $\operatorname{Int} D_i$ is the interior of D_i in S^2. Pick paths $\alpha, \beta, \gamma \in \pi_1(P_0, x_0)$ going once round boundary components as in Figure 1. Denote by $\mathrm{Diff}_+(\Sigma_g, N(c_0))[2]$ the kernel of the representation of $\mathrm{Diff}_+(\Sigma_g, N(c_0))$ on $H_1(\Sigma_g;\mathbf{Z}_2)$. Consider Σ_g bundles with its structure group $\mathrm{Diff}_+(\Sigma_g, N(c_0))[2]$. For $\varphi, \psi \in \mathcal{M}_{g,1}[2]$, there exists a Σ_g bundle $p : W = W_{\varphi,\psi} \to P_0$ such that the topological monodromy $\pi_1(P_0, x_0) \to \mathcal{M}_{g,1}[2]$ sends α, β, and $\gamma \in \pi_1(P_0, x_0)$ to φ, ψ, and $(\varphi\psi)^{-1} \in \mathcal{M}_{g,1}[2]$, respectively. This bundle is unique up to diffeomorphism. For example, see Husemöller [14] Thm 13.1 p.59. Note that the boundary ∂W is diffeomorphic to the disjoint sum $M_\varphi \amalg M_\psi \amalg M_{(\varphi\psi)^{-1}}$. First, assume that for $\varphi, \psi \in \mathcal{M}_{g,1}[2]$

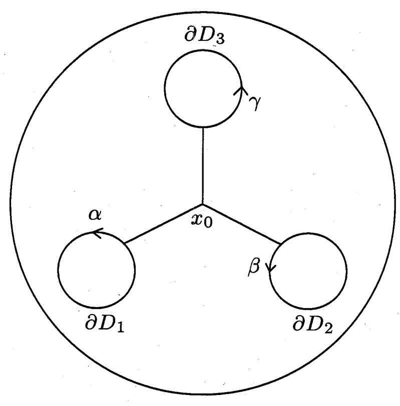

Figure 1: loops in a pair of pants

and $\sigma \in \mathrm{spin}(\Sigma_g)$, there exists a spin structure on the 4-manifold W which is spin bounded by the mapping tori $M_\varphi \amalg M_\psi \amalg M_{(\varphi\psi)^{-1}}$ with spin structure $\theta(\sigma)$. By Rochlin's theorem, we have

$$R(M_\varphi, \theta(\sigma)) + R(M_\psi, \theta(\sigma)) - R(M_{\varphi\psi}, \theta(\sigma)) \equiv \operatorname{Sign} W_{\varphi,\psi},$$
$$R(M_\varphi, \theta(\sigma+x)) + R(M_\psi, \theta(\sigma+x)) - R(M_{\varphi\psi}, \theta(\sigma+x)) \equiv \operatorname{Sign} W_{\varphi,\psi} \mod 16.$$

Hence, we obtain $\beta_{\sigma,x}(\varphi\psi) = \beta_{\sigma,x}(\varphi) + \beta_{\sigma,x}(\psi)$. This shows that the map β_σ is a homomorphism.

In the following, we define a spin structure of W which induces the spin structure $\theta(\sigma)$ on the mapping tori. Since $\varphi, \psi \in \mathcal{M}_{g,1}[2]$ act on $H_1(\Sigma_g;\mathbf{Z}_2)$ trivially, we have the splitting

$$H_1(W;\mathbf{Z}_2) = H_1(\Sigma_g;\mathbf{Z}_2) \oplus H_1(P_0;\mathbf{Z}_2)$$

by the inclusion map $N(c_0) \times P_0 \to W$. In order to define the spin structure on W, we will construct homomorphisms from each direct summand to $H_1(P(W); \mathbf{Z}_2)$. By the local triviality of the bundle $W \to P_0$, we have a neighborhood $\Sigma_g \times (-\epsilon, \epsilon)^2 \subset W$ of the fiber on $x_0 \in P_0$. For a spin structure $\sigma \in \mathrm{spin}(\Sigma_g)$, define the homomorphism

$$H_1(\Sigma_g; \mathbf{Z}_2) \xrightarrow{\ \sigma\ } H_1(P(\Sigma_g \times (-\epsilon, \epsilon)^2); \mathbf{Z}_2) \xrightarrow{\ \mathrm{inc}_*\ } H_1(P(W); \mathbf{Z}_2). \tag{3}$$

We will construct the homomorphism $H_1(P_0; \mathbf{Z}_2) \to H_1(P(W); \mathbf{Z}_2)$. In the disk $D^2 = \{(x, y) \in \mathbf{R}^2 \,|\, x^2 + y^2 \leq 1\}$, choose two mutually disjoint disks $D_1, D_2 \subset \mathrm{Int}\, D^2$. Choose an orthonormal frame $\{v'_0, v'_1\}$ of $\mathbf{R}^2$. Let $s : D^2 - D_1 - D_2 \to P(D^2 - D_1 - D_2) = (D^2 - D_1 - D_2) \times \mathbf{R}^2$ be the trivial framing defined by $s(x) = (x, v'_0, v'_1)$. By identifying P_0 with $D^2 - D_1 - D_2$, we have the map $\hat{l}' : P_0 \to P(N(c_0) \times P_0)$ by $\hat{l}'(x) = (v_0, v_1, s(x))$, where $\{v_0, v_1\}$ is a frame of $T_{c_0}N(c_0)$ in Section 2.1.2. This map and the inclusion $N(c_0) \times P_0 \to W$ induce the homomorphism

$$H_1(P_0; \mathbf{Z}_2) \xrightarrow{\ \hat{l}'_*\ } H_1(P(N(c_0) \times P_0); \mathbf{Z}_2) \xrightarrow{\ \mathrm{inc}_*\ } H_1(P(W); \mathbf{Z}_2). \tag{4}$$

Define the spin structure of W by the homomorphisms (3) and (4).

To prove that the spin structure of W induces the spin structure $\theta(\sigma)$ on each boundary component M, we must check that some diagrams commute. The homomorphism (3) is equal to the composite of (2) and the homomorphism $H_1(P(M); \mathbf{Z}_2) \cong H_1(P(M \times [0, \epsilon)); \mathbf{Z}_2) \to H_1(P(W); \mathbf{Z}_2)$ induced by inclusion, where $M \times [0, \epsilon)$ is a collar neighborhood of M in W. Hence we only have to show that the diagram

$$\begin{tikzcd}
H_1(\partial D_i; \mathbf{Z}_2) \arrow[r, "\mathrm{inc}_*"] \arrow[d, "\hat{l}_*"'] & H_1(P_0; \mathbf{Z}_2) \arrow[d, "\hat{l}'_*"'] \\
H_1(P(N(c_0) \times S^1); \mathbf{Z}_2) \arrow[r, "\mathrm{inc}_*"] & H_1(P(N(c_0) \times P_0); \mathbf{Z}_2)
\end{tikzcd}$$

commutes. Let $\hat{l}_0 : \partial D_i \to P(N(c_0) \times \partial D_i)$ be the map defined by $x \mapsto (v_0, v_1, s_{S^1}(x))$. Then, the homomorphism $(\mathrm{inc}\, \hat{l}_0)_* : H_1(\partial D_i; \mathbf{Z}_2) \to H_1(P(N(c_0) \times P); \mathbf{Z}_2)$ is different from each of $(\hat{l}'\, \mathrm{inc})_*$ and $(\mathrm{inc}\, \hat{l})_*$. Since there are only two kinds of right inverse homomorphisms of the homomorphism $H_1(P(N(c_0) \times P)|_{N(c_0) \times \partial D_i}; \mathbf{Z}_2) \to H_1(\partial D_i; \mathbf{Z}_2)$ induced by the projection, the homomorphisms $(\hat{l}'\, \mathrm{inc})_*$ and $(\mathrm{inc}\, \hat{l})_*$ coincide.

As above, the manifold W induces the spin structure $\theta(\sigma)$ on boundary components M_φ, M_ψ, and $M_{(\varphi\psi)^{-1}}$. $\square$

2.3 Pin$^-$ structures and Brown invariants

Brown defined an invariant of a closed surface F with a pin$^-$ structure, called the Brown invariant. In this section, we will define pin$^-$ structures of the surface, and review the Brown invariants and its relation to Rochlin functions stated by Turaev [35].

2.3.1 Definition of Pin$^-$ structures

Let $n \geq 2$ be an integer. The spin group $\mathrm{Spin}(n)$ is defined as the central $\mathbf{Z}_2$ extension of $\mathrm{SO}(n)$. Similarly, the pin$^-$ group is a central $\mathbf{Z}_2$ extension of the group $\mathrm{O}(n)$. But there are two kinds of central extensions of this group. Hence, we first give a precise definition of the group $\mathrm{Pin}^-(n)$.

The Spin group and the pin$^-$ group are constructed through a Clifford algebra. Let V be a real vector space of dimension n with a positive definite inner product. Denote the inner product by (x, y) for $x, y \in V$, and the tensor algebra by $T(V) := \oplus_{j=0}^{\infty} V^{\otimes j}$ with $V^{\otimes 0} := \mathbf{R}$. Let I_n be the $n \times n$ identity matrix. We define the clifford algebra as $\mathrm{Cl}^-(V) := T(V)/K$, where K is the ideal generated by $\{vw + wv + 2(v, w) \,|\, v, w \in V\}$. The spin group and the pin$^-$ group are defined as

$$\mathrm{Spin}(V) := \{a \in \mathrm{Cl}^-(V) \,|\, a = v_1 v_2 \cdots v_k, |v_k| = 1, k \text{ is even}\},$$
$$\mathrm{Pin}^-(V) := \{a \in \mathrm{Cl}^-(V) \,|\, a = v_1 v_2 \cdots v_k, |v_k| = 1\}.$$

By the definition, we can consider $\mathrm{Spin}(V)$ as a subgroup of $\mathrm{Pin}^-(V)$. Denote by $\mathrm{Spin}(n) := \mathrm{Spin}(\mathbf{R}^n)$ and $\mathrm{Pin}^-(n) := \mathrm{Pin}^-(\mathbf{R}^n)$.

For $v \in V$, denote the reflection $r_v : V \to V$ across the $(n-1)$-plane orthogonal to v through the origin as $r_v(x) = x - 2(x, v)v$. Define the surjective homomorphism $\mathrm{Pin}^-(V) \to \mathrm{O}(V)$ by

$$\begin{aligned} \mathrm{Pin}^-(V) &\to \mathrm{O}(V) \\ v_1 v_2 \cdots v_k &\mapsto r_{v1} r_{v2} \cdots r_{vk}. \end{aligned}$$

Then, the group $\mathrm{Pin}^-(V)$ is proved to be a central $\mathbf{Z}_2$ extension of $\mathrm{O}(V)$, and $\mathrm{Spin}(V)$ a central $\mathbf{Z}_2$ extension of $\mathrm{SO}(V)$. For more detail, see Atiyah-Bott-Shapiro [2] Section 3.

Next, we introduce a correspondence between pin^- structure and spin structure shown by Kirby and Taylor. Let ξ be a (not necessarily orientable) bundle with a fiber metric. A pin^- structure of bundle ξ is defined by the principal $\mathrm{Pin}^-(n)$ bundle associated to the bundle ξ, where $\mathrm{Pin}^-(n)$ acts on $\mathbf{R}^n$ via its covering projection to $\mathrm{O}(n)$. Denote by $\det \xi$ the determinant line bundle of the bundle ξ. Then $\xi \oplus \det \xi$ has a canonical orientation. Kirby-Taylor [24] showed many correspondences between the set of pin^- structures, spin structures, and pin^+ structures of the vector bundles $\xi \oplus (k \det \xi)$ for integer $k \geq 0$. We need a special case of their lemma.

Lemma 2.3 ((Kirby-Taylor [24] Lemma 1.7)). *Let $n \geq 2$ be an integer, and ξ be a n-plane bundle over CW-complex X. Then, there exists a natural bijection*

$$\mathrm{Pin}^-(\xi) \to \mathrm{Spin}(\xi \oplus \det \xi).$$

Remark 2.4. *Let M be a closed non-orientable n-manifold embedded in a spin $(n+1)$-manifold W. Then, we have a spin structure $TM \oplus N(M)$ induced by W, where $N(M)$ is a normal bundle of M. Since $N(M) \cong \det TM$, M has a natural pin^- structure induced by the spin structure on $TM \oplus N(M)$.*

2.3.2 Quadratic functions and quadratic enhancements

We restrict ourselves to spin structures and pin^- structures of compact surfaces. For a closed surface F, these structures are considered as functions on $H_1(F; \mathbf{Z}_2)$.

Definition 2.5. *Let F be a compact oriented surface. If a function $q : H_1(F; \mathbf{Z}_2) \to \mathbf{Z}_2$ satisfies*

$$q(x+y) = q(x) + q(y) + x \cdot y,$$

we call q a quadratic function.

The set of spin structures on a compact oriented surface F is known to correspond bijectively to the set of quadratic functions. We review this correspondence. Let σ be a spin structure of F. For $v \in H_1(F; \mathbf{Z}_2)$, choose a simple closed curve $K \subset F$ such that $v = [K] \in H_1(F; \mathbf{Z}_2)$. Let NK denote a normal bundle of K in F. Choose a unit tangent vector field $s_K : K \to TK$ and a nonzero section $s : K \to NK$. As in Section 2.1, the spin structure σ induces a homomorphism $k_\sigma : H_1(P(F); \mathbf{Z}_2) \to \mathbf{Z}_2$. Define a quadratic function by

$$q_\sigma(v) := k_\sigma(s \oplus s_K) + 1 \in \mathbf{Z}_2. \tag{5}$$

This number does not depend on the choice of the representative of a homology class v, the orientation of K, and the choice of vector field s_K. This is equivalent to the correspondence in Johnson [21] Theorem 3A p.371. In this way, we consider a spin structure as a quadratic function.

Since the group $\mathrm{Spin}(n)$ is a subgroup of $\mathrm{Pin}^-(n)$, a pin^- structure on a vector bundle is a natural generalization of a spin structure to a nonorientable bundle. There is also a generalization of a quadratic function as follows.

Definition 2.6. *Let F be a (not necessarily orientable) compact surface. If a function $\hat{q} : H_1(F; \mathbf{Z}_2) \to \mathbf{Z}_4$ satisfies $\hat{q}(x+y) = \hat{q}(x) + \hat{q}(y) + 2x \cdot y$, we call $\hat{q}$ the quadratic enhancement.*

Similar to the correspondence between spin structures and quadratic functions, the set of quadratic enhancements also corresponds bijectively to the set of pin$^-$ structures. A pin$^-$ structure α of a (not necessarily orientable) compact surface F induces a quadratic enhancement $\hat{q}_\alpha$ as follows. For an element $v \in H_1(F;\mathbf{Z}_2)$, choose a simple closed curve $K \subset F$ which represents v. Then the bundle $E := TF \oplus \det TF$ has the canonical orientation and the spin structure corresponds to a pin$^-$ structure α. Then the restriction $E|_K$ can be written as $E|_K = TK \oplus N(K) \oplus \det TF|_K$. If we fix the orientation of K, the bundle $E' = N(K) \oplus \det TF|_K$ gets also oriented.

Let $s_K : K \to TK$ be a unit tangent vector field on TK. Choose a framing $s : K \to P(E')$ so that the induced homomorphism $s \oplus s_K : H_1(K;\mathbf{Z}_2) \to H_1(P(E);\mathbf{Z}_2)$ is equal to the homomorphism $H_1(K;\mathbf{Z}_2) \to H_1(P(E);\mathbf{Z}_2)$ induced by the spin structure of E.

Definition 2.7. *Count the number $w(s)$ of right half twists that the normal bundle $N(K)$ in TF makes in a complete traverse of K with respect to the framing $s : K \to P(E')$ along K. Define the number*

$$\hat{q}_\alpha(v) = w(s) + 2 \in \mathbf{Z}_4.$$

This induces the map $\hat{q}_\alpha : H_1(F;\mathbf{Z}_2) \to \mathbf{Z}_4$. We call it the quadratic enhancement of a pin$^-$ structure α.

This number does not depend on the choice of the representative of a homology class, the orientation of K, and the choice of framings s_K and s. In detail, see Kirby-Taylor [24] section 3.

A quadratic enhancement is a natural generalization of a quadratic function as follows. Assume that F is an oriented surface with spin structure σ. The spin structure σ can be considered as a pin$^-$ structure. By the orientability of F, the normal bundle NK has even number of half twists with respect to the trivialization $s : K \to P(E')$ in Definition 2.7. Hence, the image of the quadratic enhancement $\hat{q}_\sigma$ is in $2\mathbf{Z}_4 \subset \mathbf{Z}_4$. By the definition of the quadratic enhancement (Definition 2.6), the function $\hat{q}_\sigma/2$ satisfies the condition of quadratic functions in Definition 2.5. Moreover, it is equal to the quadratic function q_σ of the spin structure σ. This is shown by comparing Definition 2.7 and equation 5. For example, let $s : K \to NK$ be the frame such that the homology class $[s \oplus s_K] \in H_1(P(F);\mathbf{Z}_2)$ in equation 5 is in the image of $\sigma : H_1(F;\mathbf{Z}_2) \to H_1(P(F);\mathbf{Z}_2)$. Then the value $q(v)$ of the quadratic function is equal to 1, and NK makes even number of full twists with respect to the frame $s : K \to P(E')$. By the definition of the quadratic enhancement, the value $\hat{q}_\sigma(v)/2$ is also 1.

2.3.3 Brown invariants

Brown defined an invariant of second pin$^-$ bordism classes, called the Brown invariant. This invariant is a generalization of an Arf invariant. We review this invariant and its relation to Rochlin functions stated by Turaev [35].

Definition 2.8. *Let F be a closed surface with its pin$^-$ strucure α. Then, the Brown invarant $B_\alpha \in \mathbf{Z}_8$ of α is defined by the equation*

$$\sqrt{|H_1(F;\mathbf{Z}_2)|}\exp(2\pi\sqrt{-1}B_\alpha/8) = \sum_{x \in H_1(F;\mathbf{Z}_2)} \exp(2\pi\sqrt{-1}\hat{q}_\alpha(x)/4).$$

If F is an oriented surface F with pin$^-$ structure comes from a spin structure, its Brown invariant is in $4\mathbf{Z}_8$, and it is equal to 4 times the Arf invariant (See Brown [7] p.374 Theorem 1.20.).

Let M be a closed spin 3-manifold. Consider a closed surface F which represents $s \in H_2(M;\mathbf{Z}_2)$. Then, the surface F has canonical pin$^-$ structure α induced by the spin structure of the bundle $TM|_F$ as in Remark 2.4. Furthermore, the pin$^-$ bordism class $[F,\alpha] \in \Omega_2^{\mathrm{pin}^-}$ does not depend on the choice of a representative of $s \in H_2(M;\mathbf{Z}_2)$ (Kirby-Taylor [24] (4.8)). For $\sigma, \sigma' \in \mathrm{spin}(M)$, Turaev [35] showed that the difference $R(M,\sigma) - R(M,\sigma')$ is written by the Brown invariant of these pin$^-$ structures.

Lemma 2.9 ((Turaev [35] Lemma 2.3)). *Let M be a closed orientable 3-manifold with spin structure σ. For a cohomology class $x \in H^1(M;\mathbf{Z}_2)$, let $F \subset M$ denote the closed surface which represents the Poincaré dual of $x \in H^1(M;\mathbf{Z}_2)$. Denote the pin$^-$ structure α of F induced by a spin structure σ of M. Then we have*

$$R(M,\sigma) - R(M,\sigma+x) = 2B_\alpha.$$

We apply the lemma to the case when M is a mapping torus M_φ of $\varphi \in \mathcal{M}_{g,1}[2]$. By the Serre spectral sequence of the fibration $\Sigma_g \to M \to S^1$, we have

$$0 \longrightarrow H^1(\Sigma_g;\mathbf{Z}_2) \longrightarrow H^1(M;\mathbf{Z}_2) \longrightarrow \mathbf{Z}_2 \longrightarrow 0. \tag{6}$$

The natural embedding $l : N(c_0)\times S^1 \to M$ induces the splitting $H^1(M;\mathbf{Z}_2) = H^1(\Sigma_g;\mathbf{Z}_2)\oplus\mathbf{Z}_2$. Hence, by identifying $H_1(\Sigma_g;\mathbf{Z}_2)$ with $H^1(\Sigma_g;\mathbf{Z}_2) \subset H^1(M;\mathbf{Z}_2)$ by the Poincaré duality, the homology group $H_1(\Sigma_g;\mathbf{Z}_2)$ acts on spin(M). Let x be an element in $H_1(\Sigma_g;\mathbf{Z}_2)$, and $c \in H^1(\Sigma_g;\mathbf{Z}_2) \subset H^1(M_\varphi;\mathbf{Z}_2)$ its Poincaré dual. Choose a closed surface F_c in M which represents the Poincaré dual of $c \in H^1(M;\mathbf{Z}_2)$. By Lemma 2.9, we obtain $\beta_{\sigma,x}(\varphi) = B_{\alpha_c} \in \mathbf{Z}_8$, where B_{α_c} is the Brown invariant of F_c with a pin$^-$ structure induce by a spin structure $\theta(\sigma)$ of M_φ.

2.4 Heap's homomorphism

In this section, we review the homomorphism $\omega_{\sigma,y} : \mathcal{I}_{g,1} \to \mathbf{Z}_2$ defined by Heap [11], and show that the homomorphism $\beta_{\sigma,x} : \mathcal{M}_{g,1}[2] \to \mathbf{Z}_8$ defined in Section 2.2 is the extension of his homomorphism $\omega_{\sigma,y}$ to the level 2 mapping class group (Lemma 2.12). In fact, he constructs many homomorphisms on Johnson subgroups, and calculate the specific homomorphisms $\omega_{\sigma,y}$ in his paper. He claims in Theorem 6.2 that the homomorphisms $\omega_{\sigma,y}$ has all the information of the Birman-Craggs homomorphisms on the Torelli group. But unfortunately, his proof has an error, and this is not true. We will show that the kernel of the homomorphism $H_1(\mathcal{I}_{g,r};\mathbf{Z}) \to H_1(\mathcal{M}_{g,r}[2];\mathbf{Z})$ induced by the inclusion is nontrivial (Lemma 3.8), and the homomorphism $\omega_{\sigma,y} : \mathcal{I}_{g,1} \to \mathbf{Z}_2$ factors through $H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$ (Lemma 2.12). These lemmas show that $1 \in B^3_{g,1}$ is in the kernel of every $\omega_{\sigma,y}$, where $B^3_{g,1}$ is a $\mathbf{Z}_2$-module defined in Section 3.2.

First we define a spin 3-manifold M'_φ for $\varphi \in \mathcal{M}_{g,1}[2]$. Let σ be a spin structure of Σ_g, and endow the spin structure $\theta(\sigma)$ on the mapping torus M_φ. Denote by $M'_\varphi = (M_\varphi - N(c_0)\times S^1)\cup(\partial N(c_0)\times D^2)$ the manifold obtained by the elementary surgery on $N(c_0)\times S^1 \subset M_\varphi$. We can choose the spin structure of $\partial N(c_0)\times D^2$ so that it induces in the boundary $\partial N(c_0)\times S^1$ the same spin structure induced by $\theta(\sigma)$. Hence, the elementary surgery is compatible with the spin structure, and M'_φ has the induced spin structure.

Next, we define Heap's homomorphism $\omega_{\sigma,y}$. For a group G, denote by $\Omega^{\mathrm{spin}}_*(G)$ the spin bordism group of $K(G,1)$-space $\Omega^{\mathrm{spin}}_*(K(G,1))$. Let G be an abelian group. For a closed spin n-manifold (M,σ), an homotopy class of a continuous map $f : M \to K(G,1)$ depends only on the induced homomorphism $f^* : H_1(M;\mathbf{Z}) \to G$. Hence the bordism class $[M,\sigma,f] \in \Omega^{\mathrm{spin}}_n(G)$ depends only on the cohomology class $c \in H^1(M;G)$ determined by f and a spin structure σ. We denote this bordism class in $\Omega^{\mathrm{spin}}_n(G)$ by $[M,\sigma,c]$ instead of $[M,\sigma,f]$.

If φ is contained in $\mathcal{I}_{g,1}$, we have a natural isomorphism $H^1(\Sigma_g;\mathbf{Z}) \cong H_1(M'_\varphi;\mathbf{Z})$, denote by $c_\varphi \in H^1(M'_\varphi;H_1(\Sigma_g;\mathbf{Z}))$ the cohomology class determined by this isomorphism. Heap [11]defined the map

$$\eta_{\sigma,2} : \mathcal{I}_{g,1} \to \Omega^{\mathrm{spin}}_3(H_1(\Sigma_g;\mathbf{Z})),$$

which maps $\varphi \in \mathcal{I}_{g,1}$ to $[M'_\varphi,\theta(\sigma),c_\varphi]$, and showed that this is a homomorphism.

We can define a homomorphism $\eta_{\sigma,2}[2] : \mathcal{M}_{g,1}[2] \to \Omega^{\mathrm{spin}}_3(H_1(\Sigma_g;\mathbf{Z}_2))$ in the same way. If φ is contained in $\mathcal{M}_{g,1}[2]$, we have an isomorphism $H_1(\Sigma_g;\mathbf{Z}_2) \cong H_1(M'_\varphi;\mathbf{Z}_2)$. Denote by $\bar{c}_\varphi \in H^1(M'_\varphi;H_1(\Sigma_g;\mathbf{Z}_2))$ the cohomology class determined by this isomorphism.

Proposition 2.10. *Let g be a positive integer. For a spin structure σ of Σ_g, the map*

$$\eta_{\sigma,2}[2] : \mathcal{M}_{g,1}[2] \to \Omega^{\mathrm{spin}}_3(H_1(\Sigma_g;\mathbf{Z}_2))$$

defined by $\eta_{\sigma,2}[2](\varphi) = [M'_\varphi, \theta(\sigma), \bar{c}_\varphi]$ *is a homomorphism.*

Proof. The proof is almost the same as that of Theorem 6.1 in Heap [11]. In Section 2.2.2, we saw that the mapping tori M_φ, M_ψ, and $M_{(\varphi\psi)^{-1}}$ are spin bounded by $W_{\varphi,\psi}$. If we attach 2-handles $\amalg_{i=1}^3 D^2 \times D^2$ to $W_{\varphi,\psi}$ along $N(c_0) \times S^1$ in each boundary component, we obtain a compact 4-manifold $W'_{\varphi,\psi}$ with boundary $\partial W'_{\varphi,\psi} = M'_\varphi \amalg M'_\psi \amalg M'_{(\varphi\psi)^{-1}}$. We also have a natural isomorphism $H_1(W'_{\varphi,\psi}; \mathbf{Z}_2) \cong H_1(\Sigma_g; \mathbf{Z}_2)$. Let $\bar{F} : W'_{\varphi,\psi} \to K(H_1(\Sigma_g; \mathbf{Z}_2), 1)$ be a map which induces the isomorphism $H_1(W'_{\varphi,\psi}; \mathbf{Z}_2) \cong H_1(\Sigma_g; \mathbf{Z}_2)$. This map bords $(M'_\varphi, \theta(\sigma), \bar{c}_\varphi)$, $(M'_\psi, \theta(\sigma), \bar{c}_\psi)$, and $(M'_{(\varphi\psi)^{-1}}, \theta(\sigma), \bar{c}_{(\varphi\psi)^{-1}})$. Thus we have $[M'_{\varphi\psi}, \theta(\sigma), \bar{c}_{\varphi\psi}] = [M'_\varphi, \theta(\sigma), \bar{c}_\varphi] + [M'_\psi, \theta(\sigma), \bar{c}_\psi] \in \Omega_3^{\mathrm{spin}}(H_1(\Sigma_g; \mathbf{Z}_2))$. □

For a homology class $y \in H^1(\Sigma_g; \mathbf{Z}) = \mathrm{Hom}(H_1(\Sigma_g; \mathbf{Z}), \mathbf{Z})$, we have the commutative diagram

$$\begin{tikzcd}
H_1(\Sigma_g; \mathbf{Z}) \arrow[r, "y"] \arrow[d, "\bmod 2"'] & \mathbf{Z} \arrow[d, "\bmod 2"] \\
H_1(\Sigma_g; \mathbf{Z}_2) \arrow[r, "y \bmod 2"] & \mathbf{Z}_2.
\end{tikzcd}$$

Since a homomorphism between groups induces a unique continuous map between $K(G, 1)$ spaces up to homotopy, the above diagram induces the commutative diagram

$$\begin{tikzcd}
\mathcal{I}_{g,1} \arrow[r, "\eta_{\sigma,2}"] \arrow[d] & \Omega_3^{\mathrm{spin}}(H_1(\Sigma_g; \mathbf{Z})) \arrow[r, "y_*"] \arrow[d, "(\bmod 2)_*"'] & \Omega_3^{\mathrm{spin}}(\mathbf{Z}) \cong \mathbf{Z}_2 \arrow[d, "(\bmod 2)_*"'] \\
\mathcal{M}_{g,1}[2] \arrow[r, "\eta_{\sigma,2}[2]"] & \Omega_3^{\mathrm{spin}}(H_1(\Sigma_g; \mathbf{Z}_2)) \arrow[r, "(y \bmod 2)_*"] & \Omega_3^{\mathrm{spin}}(\mathbf{Z}_2) \cong \mathbf{Z}_8,
\end{tikzcd}$$

where the left vertical map is the inclusion. Heap denote by $\omega_{\sigma,y} : \mathcal{I}_{g,1} \to \mathbf{Z}_2$ the composite of the homomorphisms $\eta_{\sigma,2} : \mathcal{I}_{g,1} \to \Omega_3^{\mathrm{spin}}(H_1(\Sigma_g; \mathbf{Z}))$ and $y_* : \Omega_3^{\mathrm{spin}}(H_1(\Sigma_g; \mathbf{Z})) \to \Omega_3^{\mathrm{spin}}(\mathbf{Z})$ for $y \in H_1(\Sigma_g; \mathbf{Z})$.

Lemma 2.11. *Let* σ *be a spin structure of* Σ_g*, and* y *be an element in* $H_1(\Sigma_g; \mathbf{Z})$*. For a mapping class* $\psi \in \mathcal{I}_{g,1}$*, we have*

$$(y \bmod 2)_* \eta_{\sigma,2}[2](\psi) = 4\omega_{\sigma,y}(\psi) \in \mathbf{Z}_8$$

Proof. Since the above diagram commutes, it suffices to show that the right vertical map is the inclusion. We explain the isomorphisms $\Omega_3^{\mathrm{spin}}(\mathbf{Z}) \cong \mathbf{Z}_2$ and $\Omega_3^{\mathrm{spin}}(\mathbf{Z}_2) \cong \mathbf{Z}_8$ in more detail.

For an $[M', \sigma, c] \in \Omega_3^{\mathrm{spin}}(\mathbf{Z})$, choose a closed oriented surface $F_c \subset M'$ which represents the Poincaré dual of $c \in H^1(M; \mathbf{Z})$. Then F_c has the spin structure σ_c induced by $\sigma \in \mathrm{spin}(M)$. By the Atiyah-Hirzebruch spectral sequence, the homomorphism

$$\Omega_3^{\mathrm{spin}}(\mathbf{Z}) \cong \Omega_2^{\mathrm{spin}}$$

defined by $[M', \sigma, c] \mapsto [F_c, \sigma_c]$ is isomorphic. For example, see Conner [8] Section 7. Similarly, for $[M', \sigma, c] \in \Omega_3^{\mathrm{spin}}(\mathbf{Z}_2)$, choose an embedded closed surface $F_c \subset M'$ which represents the Poincaré dual of $c \in H^1(M'; \mathbf{Z}_2)$. The surface F_c has the pin$^-$ structure α_c induced from the spin structure of M'. Then, there is an isomorphism

$$\Omega_3^{\mathrm{spin}}(\mathbf{Z}_2) \to \Omega_2^{\mathrm{pin}^-}$$

given by $[M', \sigma, c] \mapsto [F_c, \alpha_c]$ (See, for example, Anderson-Brown-Peterson [1] Introduction).

A spin structure of a closed manifold can be considered as a pin$^-$ structure. Hence we have a natural homomorphism $\Omega_2^{\mathrm{spin}} \to \Omega_2^{\mathrm{pin}^-}$. As above, the diagram

$$\begin{tikzcd}
\Omega_3^{\mathrm{spin}}(\mathbf{Z}) \arrow[r, "\cong"] \arrow[d, "(\bmod 2)_*"'] & \Omega_2^{\mathrm{spin}} \arrow[d] \\
\Omega_3^{\mathrm{spin}}(\mathbf{Z}_2) \arrow[r, "\cong"] & \Omega_2^{\mathrm{pin}^-}
\end{tikzcd}$$

commutes. As Arf invariants gives an isomorphism $\Omega_2^{\mathrm{spin}} \cong \mathbf{Z}_2$ (For example, see Hopkins-Singer [12] pp.336-337), Brown invariants also gives the isomorphism $\Omega_2^{\mathrm{pin}^-} \cong \mathbf{Z}_8$ (Kirby-Taylor [24] Lemma 3.6). As stated in Section 2.3.3, the diagram

$$\begin{array}{ccc} \Omega_2^{\mathrm{spin}} & \xrightarrow{\cong} & \mathbf{Z}_2 \\ \downarrow & & \downarrow \\ \Omega_2^{\mathrm{pin}^-} & \xrightarrow{\cong} & \mathbf{Z}_8 \end{array}$$

also commutes, where $\mathbf{Z}_2 \to \mathbf{Z}_8$ is defined by $a \mapsto 4a$.

By the commutative diagrams as above, we have $(y \bmod 2)_* \eta_{\sigma,2}(\psi) = 4\omega_{\sigma,y}(\psi) \in \mathbf{Z}_8$ for $\psi \in \mathcal{I}_{g,1}$ and $y \in H_1(\Sigma_g; \mathbf{Z})$. □

Lemma 2.12. *Let σ be a spin structure of Σ_g, and y be an element in $H_1(\Sigma_g; \mathbf{Z}_2)$. For a mapping class $\psi \in \mathcal{M}_{g,1}[2]$, we have*

$$\beta_{\sigma,x}(\psi) = x_* \eta_{\sigma,2}[2](\varphi) \in \mathbf{Z}_8.$$

Proof. Denote by $c \in H^1(\Sigma_g; \mathbf{Z}_2) \subset H^1(M_\varphi; \mathbf{Z}_2)$ the Poincare dual of $x \in H_1(\Sigma_g; \mathbf{Z}_2)$. We can choose a surface F_c in $M - (N(c_0) \times S^1)$ which represents the Poincaré dual of $c \in H^1(M_\varphi; \mathbf{Z}_2)$ with a pin$^-$ structure α_c. Then, by Lemma 2.9 and the isomorphism $\Omega_3^{\mathrm{spin}}(\mathbf{Z}_2) \cong \mathbf{Z}_8$ above, we have

$$\beta_{\sigma,x}(\varphi) = B_{\alpha_c} = x_* \eta_{\sigma,2}[2](\varphi).$$

□

2.5 The value of $\beta_{\sigma,x}$

Let C be a nonseparating simple closed curve. In Introduction, we denote by $\rho : \mathcal{M}_{g,r} \to \mathrm{Sp}(2g; \mathbf{Z})$ the classical representation on the first homology group $H_1(\Sigma_{g,r}; \mathbf{Z})$. Mennicke found a generating set for the level d congruence subgroup of the symplectic group as follows.

Theorem 2.13 ((Mennicke [29] p.128)). *Let $g \geq 2$ be an integer. For an integer $d \geq 2$, the level d congruence subgroup $\Gamma_g[d] \subset \mathrm{Sp}(2g; \mathbf{Z})$ is generated by $\rho(t_C)^d$ for all nonseparating simple closed curves C in $\Sigma_{g,r}$.*

Using this result, Humphries ([13] p.314 Proposition 2.1) showed that the level 2 mapping class group $\mathcal{M}_{g,r}[2]$ is generated by squares of the Dehn twists along all nonseparating simple closed curve when $g \geq 3$. In this section, we will compute the value of the homomorphism β_σ defined in Section 2.2 on the generators of $\mathcal{M}_{g,1}[2]$, using Brown invariants.

For a homology class $x \in H_1(\Sigma_{g,r}; \mathbf{Z}_2)$, define the map $i_x : H_1(\Sigma_{g,r}; \mathbf{Z}_2) \to \mathbf{Z}_8$ by

$$i_x(y) = \begin{cases} 1 & \text{if } x \cdot y \equiv 1 \bmod 2, \\ 0 & \text{if } x \cdot y \equiv 0 \bmod 2. \end{cases}$$

Note that this is not a homomorphism. For example, if $x \cdot y = 1$, we have $2i_x(y) = 2$. This is not equal to $i_x(2y) = 0$. For a spin structure σ of Σ_g, denote by $q_\sigma : H_1(\Sigma_{g,r}; \mathbf{Z}_2) \to \mathbf{Z}_2$ the quadratic function as in Section 2.3.2.

Proposition 2.14. *Let g be a positive integer, and σ a spin structure of Σ_g. For a nonseparating simple closed curve C in $\Sigma_g - N(c_0)$, we have*

$$\beta_\sigma(t_C^2) = (-1)^{q_\sigma(C)} i_{[C]} \in \mathrm{Map}(H_1(\Sigma_g; \mathbf{Z}_2), \mathbf{Z}_8).$$

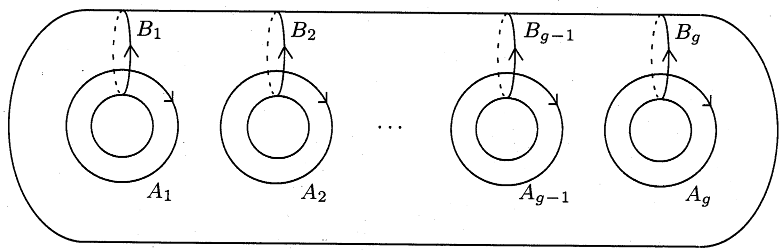

Figure 2: the symplectic basis

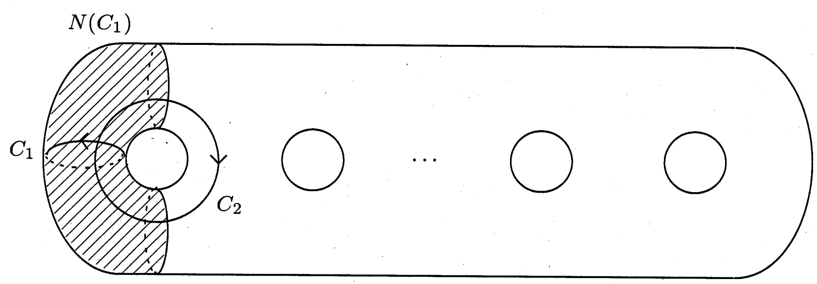

Figure 3: the neighborhood $N(C_1)$

Proof. We denote by $\{A_i, B_i\}_{i=1}^g$ the symplectic basis in $H_1(\Sigma_{g,r}; \mathbf{Z})$ represented by the simple closed curves in Figure 2. Let C_1 be the oriented simple closed curve as described in Figure 3. For a nonseparating simple closed curve C, choose a mapping class $\varphi = [f] \in \mathcal{M}_{g,1}$ such that $\varphi(C_1) = C$. By the relation (See, for example, Wajnryb [36] Lemma 20 p.430)

$$\varphi t_{C_1} \varphi^{-1} = t_{C_2}, \tag{7}$$

we have

$$\begin{aligned} \beta_{\sigma,x}(t_C^2) &= \beta_{\sigma,x}(\varphi t_{C_1}^2 \varphi^{-1}) \\ &= (R(M_{\varphi t_{C_1}^2 \varphi^{-1}}, \theta(\sigma)) - R(M_{\varphi t_{C_1}^2 \varphi^{-1}}, \theta(\sigma + x)))/2. \end{aligned}$$

The map $f^{-1} \times id : \Sigma_g \times [0,1] \to \Sigma_g \times [0,1]$ induces an diffeomorphism between the mapping tori $M_{\varphi t_{C_1}^2 \varphi^{-1}}$ and $M_{t_{C_1}^2}$. Under the diffeomorphism, the spin structures $\theta(\sigma)$ and $\theta(\sigma + x)$ of $M_{\varphi t_{C_1}^2 \varphi^{-1}}$ corresponds to $\theta(\varphi^* \sigma)$ and $\theta(\varphi^* \sigma + \varphi_*^{-1}(x))$ of $M_{t_{C_1}^2}$, respectively. Thus, we have

$$\begin{aligned} \beta_{\sigma,x}(t_C^2) &= (R(M_{t_{C_1}^2}, \theta(\varphi^* \sigma)) - R(M_{t_{C_1}^2}, \theta(\varphi^* \sigma + \varphi_*^{-1}(x))))/2 \\ &= \beta_{\varphi^* \sigma, \varphi_*^{-1}(x)}(t_{C_1}^2). \end{aligned}$$

Since we have

$$(-1)^{q_{\varphi^* \sigma}(C_1)} i_{[C_1]}(\varphi_*^{-1}(x)) = (-1)^{q_\sigma(\varphi(C_1))} i_{[\varphi(C_1)]}(x) = (-1)^{q_\sigma(C)} i_{[C]}(x),$$

it suffices to show that $\beta_\sigma(t_{C_1}^2) = (-1)^{q_\sigma(C_1)} i_{[C_1]}$.

First, we calculate the value $\beta_{\sigma, A_1+B_1}(t_{C_1}^2)$. Consider the compact submanifold $M_1 := N(C_1) \times [0,1]/\sim \subset M$. Choose the compact surface $F_1 \subset M_1$ as shown in Figure 4. For the arc $r = C_2 \cap (\Sigma_g - \operatorname{Int} N(C_1))$ as in Figure 3, denote another subsurface $F_2 := r \times S^1 \subset M$. Let $c \in H^1(\Sigma_g; \mathbf{Z}_2)$ be the Poincaré dual of the homology class $A_1 + B_1 \in H_1(\Sigma_g; \mathbf{Z}_2)$. Under the splitting $H^1(M; \mathbf{Z}_2) =$

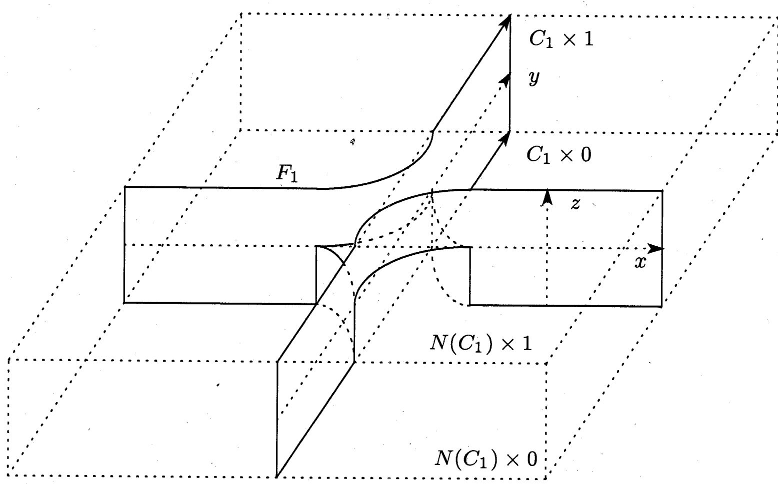

Figure 4: the surface $F_1 \subset M_1$

$H^1(\Sigma_g; \mathbf{Z}_2) \oplus H^1(S^1; \mathbf{Z}_2)$ of (6), the surface $F := F_1 \cup F_2$ represents the Poincaré dual of $c \in H^1(M; \mathbf{Z}_2)$. Let α denote the pin$^-$ structure of F induced by the spin structure of M. By Lemma 2.9 proved by Turaev, the value $\beta_{\sigma, A_1+B_1}(t_{C_1}^2)$ equals to the Brown invariant B_α. Hence we investigate the quadratic enhancement $\hat{q}_\alpha : H_1(F; \mathbf{Z}_2) \to \mathbf{Z}_4$ of the pin$^-$ structure of F. Pick the oriented simple closed curves x, y, and z which represent a generating set of $H_1(F; \mathbf{Z}_2) = \mathbf{Z}_2^3$ as in Figure 4. In the splitting $H_1(M; \mathbf{Z}) = H_1(\Sigma_g; \mathbf{Z}) \oplus H_1(S^1; \mathbf{Z})$, these curves represent the homology classes A_1, $B_1 \in H_1(\Sigma_g; \mathbf{Z})$, and $[S^1] \in H_1(S^1; \mathbf{Z})$, respectively. Consider the subsurface $\Sigma_g \times \{1/2\} \subset M$ with the orientation of Σ_g. Then, this surface inherits the spin structure σ. Denote the quadratic function of this surface by q_σ. In the last paragraph of Section 2.3.2, we state that a quadratic enhancements is a generalization of a quadratic function. Hence, the quadratic enhancement $\hat{q}_\alpha(x)$ and $\hat{q}_\alpha(y)$ of (F, α) and the quadratic function $q_\sigma(x)$ and $q_\sigma(y)$ of $(\Sigma_g \times \{1/2\}, \sigma)$ are both calculated by the number of half twists made by the surfaces along simple closed curves x and y. Since the surface F has a half twist along each of x and y with respect to $\Sigma_g \times \{1/2\}$, we have

$$\hat{q}_\alpha(x) = -1 + 2q_\sigma(A_1), \ \hat{q}_\alpha(y) = 1 + 2q_\sigma(B_1).$$

Since the framing $\hat{l} : S^1 \to P(N_{c_0} \times S^1)$ in (1) twists once with respect to TN_{c_0} and is compatible with the spin structure $\theta(\sigma)$ of M_φ, we have

$$\hat{q}_\alpha(z) = 0.$$

By the condition of the quadratic enhancements (Definition 2.6), we also have

$$\hat{q}_\alpha(x+y) = 2 + 2q_\sigma(A_1) + 2q_\sigma(B_1), \ \hat{q}_\alpha(y+z) = 1 + 2q_\sigma(B_1), \ \hat{q}_\alpha(z+x) = 1 + 2q_\sigma(A_1),$$
$$\hat{q}_\alpha(x+y+z) = 2q_\sigma(A_1) + 2q_\sigma(B_1).$$

Hence the Brown invariant B_α satisfies

$$\begin{aligned}\sqrt{|H_1(F; \mathbf{Z}_2)|} \exp(2\pi\sqrt{-1}B_\alpha/8) &= \sum_{x \in H_1(F; \mathbf{Z}_2)} \exp(2\pi\sqrt{-1}\hat{q}_\alpha(x)/4) \\ &= 2\exp(2\pi\sqrt{-1}(2q_\sigma(B_1)+1)/4) + 2.\end{aligned}$$

This shows $\beta_{\sigma, A_1+B_1}(t_{C_1}^2) = B_\alpha = (-1)^{q_\sigma(B_1)}$.

Let V be a submodule of $H_1(\Sigma_g; \mathbf{Z}_2)$ generated by $\{B_1, A_2, B_2 \cdots, A_g, B_g\}$. Next, we show that $\beta_{\sigma, x}(t_{C_1}^2) = 0$ for $x \in V$. For a homology class $x \in V$, we denote by C_x a simple closed curve in $\Sigma_g - N(C_1)$ which represents x. Let F' denote the subsurface $C_x \times S^1$ in M, and denote by $c' \in H^1(\Sigma_g; \mathbf{Z}_2)$ the Poincaré dual of $x \in H_1(\Sigma_g; \mathbf{Z}_2)$. Then, the subsurface F' represents the Poincaré dual of $c' \in H^1(M; \mathbf{Z}_2)$. Choose a generators $x' := [C_x \times t_0]$, $y' := [t_0 \times S^1]$ of $H_1(F'; \mathbf{Z}_2) = \mathbf{Z}_2^2$, where $t_0 \in C_x$ is a point. Note that F' is orientable, and a spin structure σ' of F' is induced by that of M. Since spin group naturally injects into pin$^-$ group, we can consider σ' as the pin$^-$ structure of F'. As in the previous paragraph, Since the surface F has no twist along C_x with respect to $\Sigma_g \times \{1/2\}$, we have

$$\hat{q}_{\sigma'}(x') = 2q_\sigma(x).$$

Since the framing $\hat{l} : S^1 \to P(N_{c_0} \times S^1)$ in (1) twists once with respect to TN_{c_0}, we have

$$\hat{q}_\alpha(y) = 0.$$

This shows that $\beta_{\sigma,x}(t_{C_1}^2) = B_{\sigma'} = 0$.

Finally, we prove $\beta_{\sigma,A_1+x}(t_{C_1}^2) = \beta_{\sigma,A_1}(t_{C_1}^2)$ for $x \in V$. We have

$$\begin{aligned}\beta_{\sigma,A_1+x}(t_{C_1}^2) &= (R(M,\sigma) - R(M,\sigma + A_1 + x))/2 \\ &= (R(M,\sigma) - R(M,\sigma + A_1))/2 + (R(M,\sigma + A_1) - R(M,\sigma + A_1 + x))/2 \\ &= \beta_{\sigma,A_1}(t_{C_1}^2) + \beta_{\sigma+A_1,x}(t_{C_1}^2).\end{aligned}$$

Since we have $\beta_{\sigma,A_1}(t_{C_1}^2) = 0$, it follows that $\beta_{\sigma,A_1+x}(t_{C_1}^2) = \beta_{\sigma,A_1}(t_{C_1}^2)$.

Thus, for all $x \in H_1(\Sigma_g; \mathbf{Z}_2)$, we have

$$\beta_{\sigma,x}(t_{C_1}^2) = (-1)^{q_\sigma(C_1)} i_{[C_1]}(x).$$

This completes the proof. □

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2 (which determines the $\mathrm{Sp}(2g; \mathbf{Z})$-module structure of the abelianization $H_1(\mathcal{M}_{g,r}[2]; \mathbf{Z})$), assuming Corollary 4.2 (which determines $H_1(\Gamma_g[d]; \mathbf{Z})$) in Section 4. We also determine the abelianization of the level 2 mapping class group for closed surfaces.

In Section 3.1, we construct a surjective homomorphism $\Phi : \mathbf{Z}[H_1(\Sigma_{g,r}; \mathbf{Z}_2)] \to H_1(\mathcal{M}_{g,r}[2]; \mathbf{Z})$. With this homomorphism, we calculate the order of the homology group $H_1(\mathcal{M}_{g,1}[2]; \mathbf{Z})$ in Section 3.2 and 3.3.

3.1 A homomorphism $\Phi_d : \mathbf{Z}[S_d] \to H_1(\mathcal{M}_{g,r}[d]; \mathbf{Z})$

Let R be either $\mathbf{Z}$ or $\mathbf{Z}_d$ and v an element in $H_1(\Sigma_{g,r}; R)$. If there does not exist $v' \in H_1(\Sigma_{g,r}; K)$ and a nonunit n in R such that $v = nv'$, we call the element $v \in H_1(\Sigma_{g,r}; R)$ primitive. Since the intersection form is nondegenerate, the element v in $H_1(\Sigma_{g,r}; R)$ is primitive if and only if there exists $w \in H_1(\Sigma_{g,r}; R)$ such that $v \cdot w \in K$ is a unit. We denote by $H_1(\Sigma_{g,r}; R)^{\mathrm{pri}}$ the set of primitive elements in $H_1(\Sigma_{g,r}; R)$. Let S_d denote the set $H_1(\Sigma_{g,r}; \mathbf{Z}_d)^{\mathrm{pri}}/\{\pm 1\}$.

In this section, we define the homomorphism $\Phi_d : \mathbf{Z}[S_d] \to H_1(\mathcal{M}_{g,r}[d]; \mathbf{Z})$. In particular, we will show that this homomorphism is surjective when $d = 2$ in Lemma 3.4.

3.1.1 A homomorphism $\Phi_d : \mathbf{Z}[S_d] \to H_1(\mathcal{M}_{g,r}[d]; \mathbf{Z})$ for general $d \geq 2$

The level d mapping class group acts on the set of isotopy classes of nonseparating unoriented simple closed curves. We will prove that S_d corresponds bijectively to the orbit space of this action in Lemma 3.2. Note that any element of $H_1(\Sigma_{g,r}; \mathbf{Z})^{\mathrm{pri}}$ is known to be represented by an oriented simple closed curve. For example, see Meeks-Patrusky [28].

Lemma 3.1. *Let g be a positive integer, and r either 0 or 1. For an integer $d \geq 2$, the* $\bmod d$ *reduction map $H_1(\Sigma_{g,r}; \mathbf{Z})^{\mathrm{pri}} \to H_1(\Sigma_{g,r}; \mathbf{Z}_d)^{\mathrm{pri}}$ is surjective.*

Proof. Let $v_d \in H_1(\Sigma_{g,r}; \mathbf{Z}_d)^{\mathrm{pri}}$, and choose $v \in H_1(\Sigma_{g,r}; \mathbf{Z})$ so that $v \bmod d = v_d \in H_1(\Sigma_{g,r}; \mathbf{Z}_d)^{\mathrm{pri}}$. If v is not primitive, there exists an integer $k \geq 2$ and a primitive element $w \in H_1(\Sigma_{g,r}; \mathbf{Z})^{\mathrm{pri}}$ such that $v = kw$. Since v_d is primitive, k and d are coprime. Thus there exist integers $k', d' \in \mathbf{Z}$ such that $kk' + dd' = 1$. Choose $w' \in H_1(\Sigma_{g,r}; \mathbf{Z})^{\mathrm{pri}}$ such that $w \cdot w' = 1$. We have

$$(v + dw') \cdot (-d'w + k'w') = kk' + dd' = 1.$$

Hence, $v + dw' \in H_1(\Sigma_{g,r}; \mathbf{Z})$ is primitive and $v + dw' \bmod d = v_d \in H_1(\Sigma_{g,r}; \mathbf{Z}_d)$. $\square$

Lemma 3.2. *Let g be a positive integer, r either 0 or 1, and $d \geq 2$. Let C_1 and C_1' be nonseparating simple closed curves in $\Sigma_{g,r}$ that represent the same class in $H_1(\Sigma_{g,r}; \mathbf{Z}_d)/\{\pm 1\}$. Then, there exists a mapping class $[f] \in \mathcal{M}_{g,r}[d]$ such that $f(C_1) = C_1'$.*

Note that the correspondence result of Lemma 3.2 was proved for integer coefficients and separating curves (Theorem 1A) by Johnson [19], and for nonseparating curves by Putman [31] (p.853 Lemma 6.2).

Proof. Fix orientations of C_1 and C_1' so that $[C_1] = [C_1'] \in H_1(\Sigma_{g,r}; \mathbf{Z}_d)$. Choose a simple closed curve C_2 with intersection number $[C_1] \cdot [C_2] = 1$, and let u denote the homology class $([C_1'] - [C_1])/d \in H_1(\Sigma_{g,r}; \mathbf{Z})$. When we have a primitive vector, we can find another primitive vector so that the intersection number is any integer we want, in particular $-u \cdot [C_2]$. Since $[C_1'] \in H_1(\Sigma_{g,r}; \mathbf{Z})$ is primitive, there exists $v \in H_1(\Sigma_{g,r}; \mathbf{Z})$ which satisfies $[C_1'] \cdot v = -u \cdot [C_2]$. If we set $\alpha_2' := [C_2] + dv$, we have

$$\begin{aligned}
[C_1'] \cdot \alpha_2' &= [C_1'] \cdot ([C_2] + dv) \\
&= (du + [C_1]) \cdot [C_2] + d[C_1'] \cdot v \\
&= du \cdot [C_2] + 1 + d[C_1'] \cdot v \\
&= 1.
\end{aligned}$$

In particular, the vector α_2' is primitive. By Lemma A.3 in Putman [31], there exists C_2' such that $[C_2'] = \alpha_2'$, and intersect C_1' transversely at a single point.

Choose a diffeomorphism $f : \Sigma_{g,r} \to \Sigma_{g,r}$ that satsifies $f(C_1) = C_1'$, $f(C_2) = C_2'$, $f|_{\partial \Sigma_{g,r}} = id_{\partial \Sigma_{g,r}}$. Denote by $\{Y_i\}_{i=1}^{2g-2}$ a set of elements of $H_1(\Sigma_{g,r}; \mathbf{Z})$ such that $\{[C_1], [C_2]\} \cup \{Y_i\}_{i=1}^{2g-2}$ is a symplectic basis. Since we have $f_*([C_i]) \equiv [C_i] \bmod d$ for $i = 1, 2$, the symplectic action of f on $H_1(\Sigma_{g,r}; \mathbf{Z}_d)$ induces an action on the submodule of $H_1(\Sigma_{g,r}; \mathbf{Z}_d)$ spanned by $\{[Y_i]\}_{i=2}^{2g-2}$. For $i = 1, 2$, let $N(C_i)$ be a closed tubular neighborhood of C_i, and let F be the surface $F = \Sigma_{g,r} - \cup \operatorname{Int} N(C_i)$. The action of the mapping class group $\mathcal{M}_{g-1,r+1}$ of F on $H_1(F; \mathbf{Z}_d)/\operatorname{Im}(H_1(\partial F; \mathbf{Z}_d) \to H_1(F; \mathbf{Z}_d))$ induces a surjective homomorphism $\mathcal{M}_{g-1,r+1} \to \mathrm{Sp}(2g-2; \mathbf{Z}) \to \mathrm{Sp}(2g-2; \mathbf{Z}_d)$. Hence there exists $g \in \mathrm{Diff}(F, \partial F)$ such that

$$g_*(Y_i) = f_*^{-1}(Y_i) \in H_1(F; \mathbf{Z}_d).$$

Since the diffeomorphism f of $\Sigma_{g,r}$ maps the simple closed curve C_1 to C_1', $[f(g \cup id_{\amalg N(C_i)})] \in \mathcal{M}_{g,r}[d]$ is the desired mapping class. $\square$

Denote by $t_C \in \mathcal{M}_{g,r}$ the Dehn twist along a simple closed curve $C \subset \Sigma_{g,r}$. By Lemma 3.2, for simple closed curves C_1 and C_2 in $\Sigma_{g,r}$, we have a mapping class $\varphi \in \mathcal{M}_{g,r}[d]$ represented by a diffeomorphism f such that $f(C_1) = C_2$. By the relation 7, the mapping classes $t_{C_1}^d$ and $t_{C_2}^d$ are conjugate in $\mathcal{M}_{g,r}[d]$. Hence we have the following.

Corollary 3.3. *Let g be a positive integer, r either 0 or 1, and $d \geq 2$. Denote by C_1 and C_2 the simple closed curves in $\Sigma_{g,r}$ such that their homology classes satisfy $[C_1] = [C_2] \in H_1(\Sigma_{g,r}; \mathbf{Z}_d)$. Then, the dehn twists $t_{C_1}^d$ and $t_{C_2}^d$ represent the same element of $H_1(\mathcal{M}_{g,r}[d]; \mathbf{Z})$.*

Now, we define the homomorphism $\mathbf{Z}[S_d] \to H_1(\mathcal{M}_{g,r}[d];\mathbf{Z})$. By Lemma 3.1 and Corollary 3.3, we can define the map $\Phi_d : S_d \to H_1(\mathcal{M}_{g,r}[d];\mathbf{Z})$ by $\Phi_d([C]) := [t_C^d]$. We denote it simply by $\langle [C] \rangle$. Extend this map to a homomorphism of $\mathbf{Z}$-modules

$$\Phi_d : \mathbf{Z}[S_d] \to H_1(\mathcal{M}_{g,r}[d];\mathbf{Z}).$$

3.1.2 A surjective homomorphism $\Phi : \mathbf{Z}_8[H_1(\Sigma_{g,r};\mathbf{Z}_2)] \to H_1(\mathcal{M}_{g,r}[2];\mathbf{Z})$

We consider the case when $d = 2$. Then, we have $S_2 = H_1(\Sigma_{g,r};\mathbf{Z}_2) - 0$. Define $\Phi_2([0]) := 0$ and extend $\Phi_2 : \mathbf{Z}[S_2] \to H_1(\mathcal{M}_{g,r}[2];\mathbf{Z})$ to a homomorphism $\mathbf{Z}[H_1(\Sigma_{g,r};\mathbf{Z}_2)] \to H_1(\mathcal{M}_{g,r}[2];\mathbf{Z})$.

In the following, we denote the extended homomorphism $\mathbf{Z}[H_1(\Sigma_{g,r};\mathbf{Z}_2)] \to H_1(\mathcal{M}_{g,r}[2];\mathbf{Z})$ of Φ_2 simply by Φ.

Lemma 3.4. *Let $g \geq 3$ be an integer, and r either 0 or 1. The homomorphism $\Phi : \mathbf{Z}[H_1(\Sigma_{g,r};\mathbf{Z}_2)] \to H_1(\mathcal{M}_{g,r}[2];\mathbf{Z})$ is surjective, and it induces a homomorphism $\mathbf{Z}_8[H_1(\Sigma_{g,r};\mathbf{Z}_2)] \to H_1(\mathcal{M}_{g,r}[2];\mathbf{Z})$ on the quotient module $\mathbf{Z}_8[H_1(\Sigma_{g,r};\mathbf{Z}_2)]$.*

Proof. As stated in Section 2.5, Humphries [13] proved that the level 2 mapping class group is generated by squares of Dehn twists along nonseparating curves using Mennicke's result (Theorem 2.13). Hence, Φ is surjective. Consider the 5-term exact sequence

$$H_1(\mathcal{I}_{g,r};\mathbf{Z})_{\mathcal{M}_{g,r}[2]} \xrightarrow{\ \iota\ } H_1(\mathcal{M}_{g,r}[2];\mathbf{Z}) \longrightarrow H_1(\Gamma_g[2];\mathbf{Z}) \longrightarrow 0 \tag{8}$$

of the exact sequence $1 \longrightarrow \mathcal{I}_{g,r} \longrightarrow \mathcal{M}_{g,r}[2] \longrightarrow \Gamma_g[2] \longrightarrow 1$. Johnson determined the coinvariant $H_1(\mathcal{I}_{g,r};\mathbf{Z})_{\mathcal{M}_{g,r}[2]}$. In datail, see Theorem 3.6 and 3.7. In particular, $H_1(\mathcal{I}_{g,r};\mathbf{Z})_{\mathcal{M}_{g,r}[2]}$ is proved to be a $\mathbf{Z}_2$-module. We will prove that $H_1(\Gamma_g[2];\mathbf{Z})$ is a $\mathbf{Z}_4$-module in Corollary 4.2. Hence $H_1(\mathcal{M}_{g,r}[2];\mathbf{Z})$ is a $\mathbf{Z}_8$-module. This shows that Φ factors through the quotient module $\mathbf{Z}_8[H_1(\Sigma_{g,r};\mathbf{Z}_2)]$. □

3.2 An upper bound for the order $|H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})|$

In this section, we examine the kernel of the homomorphism

$$\iota : H_1(\mathcal{I}_{g,r};\mathbf{Z})_{\mathcal{M}_{g,r}[2]} \to H_1(\mathcal{M}_{g,r}[2];\mathbf{Z})$$

induced by inclusion, and give an upper bound for the order of $H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$ as follows.

Proposition 3.5. *Let $g \geq 3$ be an integer. Then, we have*

$$|H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})| \leq |B^3_{g,1}/\langle 1 \rangle|\,|H_1(\Gamma_g[2];\mathbf{Z})|.$$

First, we review the $\mathbf{Z}_2$-module $B^3_{g,r}$ defined by Johnson [20]. To each $x \in H_1(\Sigma_{g,r};\mathbf{Z})$, we attach a formal symbol $\bar{x}$. Let R be the commutative polynomial ring in the variables $\{\bar{x}\}$, with coefficient in $\mathbf{Z}_2$. Denote by J the ideal of this polynomial ring generated by

$$\overline{x+y} - (\bar{x} + \bar{y} + x \cdot y), \quad \bar{x}^2 - \bar{x},$$

for $x, y \in H_1(\Sigma_{g,r};\mathbf{Z})$,

Let R_n denote the submodule of R consisting of polynomials with degree at most n. Define the module B^n by

$$B^n = \frac{R_n}{J \cap R_n},$$

and denote

$$B^3_{g,1} := B^3, \text{ and } B^2_{g,1} := B^2.$$

Let $\{A_i, B_i\}_{i=1}^g$ denote the symplectic basis defined in Proposition 2.14. Let α be the element $\Sigma_{i=1}^g \bar{A}_i \bar{B}_i \in B^2$. There is a homomorphism $B^1 \to B^3_{g,1}$ defined by $x \mapsto x\alpha$. Let $B^3_{g,0}$ be the cokernel of this homomorphism and let $B^2_{g,0}$ denotes the quotient module $B^2/\langle \alpha \rangle$. Then, $B^2_{g,r}$ is a submodule of $B^3_{g,r}$. Johnson showed that the coinvariant $H_1(\mathcal{I}_{g,r};\mathbf{Z})_{\mathcal{M}_{g,r}[2]}$ and the module $B^3_{g,r}$ are isomorphic as $\mathrm{Sp}(2g;\mathbf{Z})$-module as follows.

Theorem 3.6 ((Johnson [22] Theorem 1 p.139, Theorem 4 p.141)). *Let $g \geq 3$ be an integer, and r either 0 or 1. We have*

$$\frac{\mathcal{I}_{g,r}}{\mathcal{I}_{g,r}^2} \cong B_{g,r}^3.$$

Theorem 3.7 ((Johnson [22] Corollary p.134, Theorem 5 p.141)). *Let $g \geq 3$ be an integer, and r either 0 or 1. We have*

$$[\mathcal{M}_{g,r}[2], \mathcal{I}_{g,r}] = \mathcal{I}_{g,r}^2.$$

Next, we examine the kernel of $\iota : B_{g,r}^3 \cong H_1(\mathcal{I}_{g,r}; \mathbf{Z})_{\mathcal{M}_{g,r}[2]} \to H_1(\mathcal{M}_{g,r}[2]; \mathbf{Z})$ to prove Proposition 3.5.

Lemma 3.8. *Let $g \geq 3$ be an integer, and r be either 0 or 1. Then, we have*

$$1 \in \operatorname{Ker} \iota.$$

Since we have the isomorphism $H_1(\mathcal{I}_{g,r}; \mathbf{Z})_{\mathcal{M}_{g,r}[2]} \cong B_{g,r}^3$ and the exact sequence (8), Proposition 3.5 follows immediately from this lemma.

of Lemma 3.8. As in Figure 5, choose the simple closed curves C_1, C_2, D_1 so that $[C_1] = B_1, [C_2] = A_1 \in H_1(\Sigma_{g,r}; \mathbf{Z}_2)$. We denote $\Phi(X)$, the image of X in $H_1(\mathcal{M}_{g,r}[2]; \mathbf{Z})$ under the homomorphism in 3.1.2 by $\langle X \rangle$. Then, by Lemma 12a in Johnson [20], we have $\iota(\overline{A_1}\,\overline{B_1}) = [t_{D_1}]$. By the chain relation

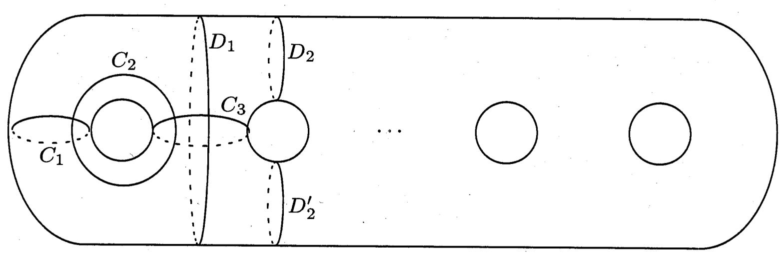

Figure 5: the curves

(See for example Wajnryb [36] Lemma 21(iii) pp.430–431), we have $t_{D_1} = (t_{C_1}t_{C_2})^6$. The commutative relation and the braid relation (Wajnryb [36] Lemma 21(ii)(iii) p.430) shows

$$(t_{C_1}t_{C_2})^6 = (t_{C_1}(t_{C_1}t_{C_2}t_{C_1})t_{C_1}t_{C_2})^2 = (t_{C_1}^2 t_{C_2}^2 (t_{C_2}^{-1} t_{C_1}^2 t_{C_2}))^2.$$

Applying the relation (7), these three equations imply

$$\iota(\overline{A_1}\,\overline{B_1}) = 2\langle A_1 \rangle + 2\langle B_1 \rangle + 2\langle A_1 + B_1 \rangle. \tag{9}$$

Note that the homomorphism ι has a naturality property

$$\iota(\varphi(x)) = \varphi_* \iota(x) \tag{10}$$

for a mapping class $\varphi \in \mathcal{M}_{g,r}$ and $x \in B_{g,r}^3$. If we choose $\varphi \in \mathcal{M}_{g,r}$ such that $\varphi(A_1) = A_1$, $\varphi(B_1) = B_1 + B_2$, by the naturality (10) and equation (9), we have

$$\iota(\overline{A_1}(\overline{B_1 + B_2})) = \varphi_*(\iota(\overline{A_1}\,\overline{B_1})) = 2\langle A_1 \rangle + 2\langle B_1 + B_2 \rangle + 2\langle A_1 + B_1 + B_2 \rangle. \tag{11}$$

In the same fashion, by Lemma 12b in Johnson [20], we have the equation $\iota(\overline{A_1}\,\overline{B_1}(\overline{B_2}+1)) = [t_{D_2}t_{D_2'}^{-1}] \in H_1(\mathcal{M}_{g,r}[2]; \mathbf{Z})$. The chain relation (Wajnryb [36] Lemma 21(iii) p.431) shows $t_{D_2}t_{D_2'} = (t_{C_1}t_{C_2}t_{C_3})^4$.

By the commutative relation and the braid relation, we also have

$$\begin{aligned}
(t_{C_1}t_{C_2}t_{C_3})^4 &= t_{C_1}t_{C_2}(t_{C_1}t_{C_3})t_{C_2}t_{C_3}t_{C_1}t_{C_2}(t_{C_1}t_{C_3})t_{C_2}t_{C_3},\\
&= t_{C_1}t_{C_2}t_{C_1}(t_{C_2}t_{C_3}t_{C_2})t_{C_1}t_{C_2}t_{C_1}(t_{C_2}t_{C_3}t_{C_2}),\\
&= t_{C_1}(t_{C_1}t_{C_2}t_{C_1})t_{C_3}(t_{C_1}t_{C_2}t_{C_1})t_{C_1}t_{C_2}t_{C_3}t_{C_2},\\
&= t_{C_1}^2t_{C_2}t_{C_1}(t_{C_1}t_{C_3})t_{C_2}t_{C_1}^2t_{C_2}t_{C_3}t_{C_2},\\
&= t_{C_1}^2(t_{C_2}t_{C_1}^2t_{C_2}^{-1})(t_{C_2}t_{C_3}t_{C_2}t_{C_1}^2t_{C_2}^{-1}t_{C_3}^{-1}t_{C_2}^{-1})(t_{C_2}t_{C_3}t_{C_2}^2t_{C_3}^{-1}t_{C_2}^{-1})(t_{C_2}t_{C_3}^2t_{C_2}^{-1})t_{C_2}^2.
\end{aligned}$$

By applying the relation (7), we have the equation

$$[t_{C_1}t_{C_2}t_{C_3}^4] = \langle B_1\rangle + \langle A_1\rangle + \langle B_1+B_2\rangle + \langle A_1+B_1\rangle + \langle A_1+B_1+B_2\rangle + \langle A_1+B_2\rangle - \langle B_2\rangle$$

in $H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$. Hence we obtain

$$\begin{aligned}
\iota(\overline{A}_1\overline{B}_1(\overline{B_2}+1)) &= [t_{D_2}t_{D_2'}^{-1}]\\
&= [(t_{C_1}t_{C_2}t_{C_3})^4] - [t_{D_2'}^2]\\
&= \langle B_1\rangle + \langle A_1\rangle + \langle B_1+B_2\rangle + \langle A_1+B_1\rangle\\
&\qquad + \langle A_1+B_1+B_2\rangle + \langle A_1+B_2\rangle - \langle B_2\rangle\,.
\end{aligned}$$

Since $2\overline{A}_1\overline{B}_1(\overline{B_2}+1) = 0$ in $B^3_{g,r}$, we have $\iota(2\overline{A}_1\overline{B}_1(\overline{B_2}+1)) = \iota(0) = 0$. Applying the previous equality, we see that

$$2\langle A_1+B_1+B_2\rangle = -2(\langle B_1\rangle + \langle A_1\rangle + \langle B_1+B_2\rangle + \langle A_1+B_1\rangle + \langle A_1+B_2\rangle - \langle B_2\rangle).$$

Put this into the equation (11), then we have

$$\begin{aligned}
\iota(\overline{A}_1(\overline{B_1+B_2})) &= 2\langle A_1\rangle + 2\langle B_1+B_2\rangle\\
&\quad - 2(\langle A_1\rangle + \langle B_1\rangle - \langle B_2\rangle + \langle A_1+B_1\rangle + \langle A_1+B_2\rangle + \langle B_1+B_2\rangle)\\
&= -2\langle B_1\rangle + 2\langle B_2\rangle - 2\langle A_1+B_1\rangle - 2\langle A_1+B_2\rangle\,. \qquad (12)
\end{aligned}$$

Since $\overline{A}_1\overline{B}_2 = \overline{A}_1(\overline{B_1+B_2}) + \overline{A}_1\overline{B}_1$, by equations (9) and (12), we have

$$\begin{aligned}
\iota(\overline{A}_1\overline{B}_2) &= \iota(\overline{A}_1(\overline{B_1+B_2}) + \overline{A}_1\overline{B}_1)\\
&= 2\langle A_1\rangle + 2\langle B_2\rangle - 2\langle A_1+B_2\rangle\,.
\end{aligned}$$

If we choose $\varphi \in \mathcal{M}_{g,r}$ so that $\varphi_*(A_1) = A_1$, $\varphi_*(B_2) = A_1+B_2$, by the naturality of ι (10), we have

$$\iota(\overline{A}_1(\overline{A}_1+\overline{B}_2)) = \iota\varphi_*(\overline{A}_1\overline{B}_2) = \varphi\iota(\overline{A_1B_2}) = 2\langle A_1\rangle + 2\langle A_1+B_2\rangle - 2\langle B_2\rangle\,.$$

Combining the last two equalities, we have

$$\begin{aligned}
\iota(\overline{A}_1) &= \iota(\overline{A}_1(\overline{A}_1+\overline{B}_2)) - \iota(\overline{A}_1\overline{B}_2)\\
&= 4\langle A_1\rangle\,.
\end{aligned}$$

Since $\overline{A_1+B_1} = \overline{A}_1 + \overline{B}_1 + 1$, by the previous equality, we have

$$\begin{aligned}
\iota(1) &= \iota(\overline{A_1+B_1} - \overline{A}_1 - \overline{B}_1)\\
&= 4(\langle A_1+B_1\rangle - \langle A_1\rangle - \langle B_1\rangle).
\end{aligned}$$

As we stated in Lemma 3.4, $H_1(\mathcal{M}_{g,r}[2];\mathbf{Z})$ is a $\mathbf{Z}_8$-module. Hence we have $8\langle A_1\rangle = 8\langle B_1\rangle = 0$. Therefore, applying equation (9), we have

$$4(\langle A_1+B_1\rangle - \langle A_1\rangle - \langle B_1\rangle) = 4(\langle A_1+B_1\rangle + \langle A_1\rangle + \langle B_1\rangle) = \iota(2\overline{A}_1\overline{B}_1) = 0.$$

Since $2\overline{A}_1\overline{B}_1 = 0 \in B^3_{g,r}$, combining last two equality, we have $\iota(1) = \iota(2\overline{A}_1\overline{B}_1) = \iota(0) = 0 \in H_1(\mathcal{M}_{g,r}[2];\mathbf{Z})$. □

3.3 A lower bound for the order $|H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})|$

In this section, we give a lower bound for the order of $H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$,

$$|H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})| \geq |\mathbf{Z}_8^{2g} \oplus \mathbf{Z}_4^{\binom{2g}{2}} \oplus \mathbf{Z}_2^{\binom{2g}{3}}|.$$

Using this result, we determine the abelianization $H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$ as $\mathbf{Z}$-module in Proposition 3.15, and complete the proof of Theorem 1.2.

Let σ be a spin structure of Σ_g, and $\{x_j\}_{j=1}^n$ elements in $H_1(\Sigma_{g,r};\mathbf{Z}_2)$. For $n \geq 1$, define an element $\Delta_\sigma^n(x_1, x_2, \cdots, x_n)$ in $\mathbf{Z}_8[H_1(\Sigma_{g,r};\mathbf{Z}_2)]$ by

$$\begin{aligned}\Delta_\sigma^n(x_1, x_2, \cdots, x_n) = [0] + \sum_{j=1}^n (-1)^{q_\sigma(x_j)}[x_j] + \sum_{1\leq j<k\leq n}^{n} (-1)^{q_\sigma(x_j+x_k)}[x_j + x_k] \\ + \cdots + (-1)^{q_\sigma(x_1+x_2+\cdots+x_n)}[x_1 + x_2 + \cdots + x_n].\end{aligned}$$

We also define $\Delta_\sigma^0 = [0]$. Note that Δ_σ^n is commutative, that is,

$$\Delta_\sigma^n(x_1, x_2, \cdots, x_i, x_{i+1}, \cdots, x_n) = \Delta_\sigma^n(x_1, x_2, \cdots, x_{i+1}, x_i, \cdots, x_n).$$

Lemma 3.9. *Let $n \geq 1$ be an integer and σ a spin structure of Σ_g. For $\{x_j\}_{j=1}^n \subset H_1(\Sigma_{g,r};\mathbf{Z}_2)$, we have*

1. $\Delta_\sigma^{n+1}(x_1, x_2, \cdots, x_{n-1}, x_n, x_{n+1}) = \Delta_\sigma^n(x_1, x_2, \cdots, x_{n-1}, x_n + x_{n+1})$ $+\Delta_\sigma^n(x_1, x_2, \cdots, x_{n-1}, x_n) + \Delta_\sigma^n(x_1, x_2, \cdots, x_{n-1}, x_{n+1}) - 2\Delta_\sigma^{n-1}(x_1, x_2, \cdots, x_{n-1})$,
2. $\Delta_\sigma^{n+1}(x_1, x_2, \cdots, x_{n-1}, x_n, x_n) = 2\Delta_\sigma^n(x_1, x_2, \cdots, x_{n-1}, x_n)$.

Proof. First we prove (i). For $X \in H_1(\Sigma_{g,r};\mathbf{Z}_2)$, let $\Delta_{\sigma,x}^n$ denote the element

$$\begin{aligned}\Delta_{\sigma,X}^n(x_1, x_2, \cdots, x_n) := (-1)^{q_\sigma(X)}[X] + \sum_{j=1}^n (-1)^{q_\sigma(x_j+X)}[x_j + X] \\ + \sum_{1\leq j<k\leq n}^{n} (-1)^{q_\sigma(x_j+x_k+X)}[x_j + x_k + X] \\ + \cdots + (-1)^{q_\sigma(x_1+x_2+\cdots+x_n+X)}[x_1 + x_2 + \cdots + x_n + X]\end{aligned}$$

in $\mathbf{Z}_8[H_1(\Sigma_{g,r};\mathbf{Z}_2)]$. The element $\Delta_\sigma^{n+1}(x_1, x_2, \cdots, x_{n-1}, x_n, x_{n+1})$ in $\mathbf{Z}_8[H_1(\Sigma_{g,r};\mathbf{Z}_2)]$ consists of two kinds of terms, the terms with valuables $[Y]$ and with valuables $[Y + x_{n+1}]$, for Y a linear combination of $\{x_i\}_{i=1}^n \in H_1(\Sigma_{g,r};\mathbf{Z}_2)$. Hence $\Delta_\sigma^{n+1}(x_1, x_2, \cdots, x_{n-1}, x_n, x_{n+1})$ is equal to the sum of $\Delta_\sigma^n(x_1, x_2, \cdots, x_{n-1}, x_n)$ and $\Delta_{\sigma,x_{n+1}}^n(x_1, x_2, \cdots, x_{n-1}, x_n)$. Thus we have

$$\Delta_{\sigma,x_{n+1}}^n(x_1, x_2, \cdots, x_{n-1}, x_n) = \Delta_\sigma^{n+1}(x_1, x_2, \cdots, x_{n-1}, x_n, x_{n+1}) - \Delta_\sigma^n(x_1, x_2, \cdots, x_{n-1}, x_n). \quad (13)$$

In the same way, we have

$$\Delta_{\sigma,x_{n+1}}^{n-1}(x_1, x_2, \cdots, x_{n-1}) = \Delta_\sigma^n(x_1, x_2, \cdots, x_{n-1}, x_{n+1}) - \Delta_\sigma^{n-1}(x_1, x_2, \cdots, x_{n-1}), \quad (14)$$
$$\Delta_{\sigma,x_n+x_{n+1}}^{n-1}(x_1, x_2, \cdots, x_{n-1}) = \Delta_\sigma^n(x_1, x_2, \cdots, x_{n-1}, x_n + x_{n+1}) - \Delta_\sigma^{n-1}(x_1, x_2, \cdots, x_{n-1}). \quad (15)$$

Similarly, the element $\Delta_{\sigma,x_{n+1}}^n(x_1, x_2, \cdots, x_{n-1}, x_n)$ consists of the terms with valuables $[Z + x_{n+1}]$ and with valuables $[Z + x_n + x_{n+1}]$, for Z a linear combination of $\{x_i\}_{i=1}^{n-1} \in H_1(\Sigma_{g,r};\mathbf{Z}_2)$. Hence, we have

$$\Delta_{\sigma,x_{n+1}}^n(x_1, x_2, \cdots, x_{n-1}, x_n) = \Delta_{\sigma,x_{n+1}}^{n-1}(x_1, x_2, \cdots, x_{n-1}) + \Delta_{\sigma,x_n+x_{n+1}}^{n-1}(x_1, x_2, \cdots, x_{n-1}).$$

Put (13), (14), and (15) into this equality, then we obtain what we intended to prove.

Next we prove (ii). By (i), we have

$$\Delta^n_\sigma(x_1, x_2, \cdots, x_n, x_n) = \\ \Delta^n_\sigma(x_1, x_2, \cdots, x_{n-1}, 0) + 2\Delta^n_\sigma(x_1, x_2, \cdots, x_{n-1}, x_n) - 2\Delta^{n-1}_\sigma(x_1, x_2, \cdots, x_{n-1}). \quad (16)$$

By the definition of Δ^n_σ, we have $\Delta^n_\sigma(x_1, x_2, \cdots, x_{n-1}, 0) = 2\Delta^n_\sigma(x_1, x_2, \cdots, x_{n-1})$. Putting it to equation 16, we obtain what we intended to prove. $\square$

We defined $\beta_{\sigma,x}$ in Definition 2.1. In the following, we also denote by $\beta_{\sigma,x}$ the homomorphism $H_1(\mathcal{M}_{g,1}[2]; \mathbf{Z}) \to \mathbf{Z}_8$ induced by $\beta_{\sigma,x}$.

Lemma 3.10. *Let $n \geq 1$ be an integer and σ a spin structure of Σ_g. For $\{x_j\}_{j=1}^n \subset H_1(\Sigma_{g,1}; \mathbf{Z}_2)$ and $x \in H_1(\Sigma_g; \mathbf{Z}_2)$, the homomorphism $\beta_{\sigma,x} : H_1(\mathcal{M}_{g,1}[2]; \mathbf{Z}) \to \mathbf{Z}_8$ satisfies*

$$\beta_{\sigma,x}\Phi(\Delta^n_\sigma(x_1, x_2, \cdots, x_n)) = \begin{cases} 0, & \text{if } x \cdot x_j \equiv 0 \bmod 2 \text{ for any } j = 1, 2, \cdots, n, \\ 2^{n-1}, & \text{otherwise.} \end{cases}$$

Proof. The statement is equivalent to

$$\beta_{\sigma,x}\Phi(\Delta^n_\sigma(x_1, x_2, \cdots, x_n)) = 2^{n-1} - 2^{n-1}\prod_{j=1}^{n}(1 - i_{x_j}(x)).$$

We prove it by induction on n. In Proposition 2.14, we proved that $\beta_{\sigma,x}\Phi((-1)^{q_\sigma(x)}x_1) = i_{x_1}(x)$. Assume that the equation holds for $1, 2, \cdots, n$. By Lemma 3.9, we have

$$\begin{aligned} &\beta_{\sigma,x}\Phi(\Delta^{n+1}_\sigma(x_1, x_2, \cdots, x_n, x_{n+1})) \\ =&\beta_{\sigma,x}\Phi(\Delta^n_\sigma(x_1, x_2, \cdots, x_{n-1}, x_n + x_{n+1})) + \beta_{\sigma,x}\Phi(\Delta^n_\sigma(x_1, x_2, \cdots, x_{n-1}, x_n)) \\ &+ \beta_{\sigma,x}\Phi(\Delta^n_\sigma(x_1, x_2, \cdots, x_{n-1}, x_{n+1})) - 2\beta_{\sigma,x}\Phi(\Delta^{n-1}_\sigma(x_1, x_2, \cdots, x_{n-1})). \end{aligned}$$

By induction, this is equal to

$$2^n - 2^{n-1}\prod_{j=1}^{n-1}(1 - i_{x_j}(x))(2 - i_{x_n+x_{n+1}}(x) - i_{x_n}(x) - i_{x_{n+1}}(x)).$$

The number $2 - i_{x_n+x_{n+1}}(x) - i_{x_n}(x) - i_{x_{n+1}}(x)$ is equal to 2, if the intersection numbers $x \cdot x_n \equiv x \cdot x_{n+1} \equiv 0 \bmod 2$, and is equal to 0, otherwise. Hence, it is equal to $2(1 - i_{x_n}(x))(1 - i_{x_{n+1}}(x))$. This proves the lemma. $\square$

Denote the homology classes X_n by $X_{2j-1} := A_j$, and $X_{2j} = B_j$ for $j = 1, 2, \cdots, g$. For convenience, we denote $X_{n+2g} = X_n$ for $n = 1, 2, \cdots 2g$.

Lemma 3.11. *Let $n \geq 1$ be an integer and σ a spin structure of Σ_g. Assume $\{i_k\}_{k=1}^p$ are mutually dintinct integers such that $1 \leq i_k \leq 2g$, and $\{j_l\}_{l=1}^q$ are also mutualy distinct integers such that $1 \leq j_l \leq 2g$. Then, the homomorphism $\beta_{\sigma, X_{i_1}+X_{i_2}+\cdots+X_{i_p}} : H_1(\mathcal{M}_{g,1}[2]; \mathbf{Z}) \to \mathbf{Z}_8$ satisfies*

$$\begin{aligned} &\beta_{\sigma, X_{i_1}+X_{i_2}+\cdots+X_{i_p}}\Phi(\Delta^n_\sigma(X_{j_1}, X_{j_2}, \cdots, X_{j_q})) \\ &\qquad = \begin{cases} 0 & \text{if } \{X_{i_1}, X_{i_2}, \cdots X_{i_p}\} \cap \{X_{j_1+g}, X_{j_2+g}, \cdots, X_{j_q+g}\} = \emptyset, \\ 2^{n-1} & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Since $\{X_k\}_{k=1}^{2g}$ is a symplectic basis, the intersection number $(X_{i_1} + X_{i_2} + \cdots + X_{i_p}) \cdot X_{j_l} \equiv 0 \bmod 2$ for any $l = 1, 2, \cdots, q$ if and only if $\{X_{i_1}, X_{i_2}, \cdots X_{i_p}\} \cap \{X_{j_1+g}, X_{j_2+g}, \cdots, X_{j_q+g}\} = \emptyset$. Under this equvalence, the statement of this lemma is a special case of Lemma 3.10. $\square$

Let f_s, f_{st}, and f_{stu} denote the homomorphisms β_{σ,X_s}, $\beta_{\sigma,X_s}+\beta_{\sigma,X_t}-\beta_{\sigma,X_s+X_t}$, and $\beta_{\sigma,X_s}+\beta_{\sigma,X_t}+\beta_{\sigma,X_u}-\beta_{\sigma,X_s+X_t}-\beta_{\sigma,X_t+X_u}-\beta_{\sigma,X_u+X_s}+\beta_{\sigma,X_s+X_t+X_u}$ from $H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$ to $\mathbf{Z}_8$, respectively. By calculating values of f_s, f_{st}, and f_{stu} on $\Phi(\Delta^n_\sigma(X_{j_1},X_{j_2},\cdots,X_{j_n}))$ using this lemma, we obtain the following.

Corollary 3.12. *Let n be a positive integer, and σ be a spin structure of Σ_g. Assume s,t,u are integers such that $1\le s<t<u\le 2g$, and $\{j_k\}$ are mutually distinct integers $1\le j_1<j_2<\cdots j_n\le 2g$.*

$$f_s(\Phi(\Delta^n_\sigma(X_{j_1},X_{j_2},\cdots,X_{j_n})))=\begin{cases}2^{n-1} & \textit{if } \{X_s\}\subset\{X_{j_1+g},X_{j_2+g},\cdots,X_{j_n+g}\},\\ 0 & \textit{otherwise},\end{cases}$$

$$f_{st}(\Phi(\Delta^n_\sigma(X_{j_1},X_{j_2},\cdots,X_{j_n})))=\begin{cases}2^{n-1} & \textit{if } \{X_s,X_t\}\subset\{X_{j_1+g},X_{j_2+g},\cdots,X_{j_n+g}\},\\ 0 & \textit{otherwise},\end{cases}$$

$$f_{stu}(\Phi(\Delta^n_\sigma(X_{j_1},X_{j_2},\cdots,X_{j_n})))=\begin{cases}2^{n-1} & \textit{if } \{X_s,X_t,X_u\}\subset\{X_{j_1+g},X_{j_2+g},\cdots,X_{j_n+g}\},\\ 0 & \textit{otherwise}.\end{cases}$$

Define a homomorphism $\Psi: H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})\to\mathbf{Z}_8^{2g}\oplus\mathbf{Z}_8^{\binom{2g}{2}}\oplus\mathbf{Z}_8^{\binom{2g}{3}}$ by

$$\Psi(\varphi):=(\{f_s(\varphi)\}_{s=1}^{2g},\{f_{st}(\varphi)\}_{1\le s<t\le 2g},\{f_{stu}(\varphi)\}_{1\le s<t<u\le 2g}).$$

By Corollary 3.12, we can calculate the image of Ψ.

Lemma 3.13. *For $g\ge 3$,*

$$\operatorname{Im}\Psi=\mathbf{Z}_8^{2g}\oplus 2\mathbf{Z}_8^{\binom{2g}{2}}\oplus 4\mathbf{Z}_8^{\binom{2g}{3}}.$$

Proof. The set $\{\Delta^n_\sigma(X_{j_1},X_{j_2},\cdots,X_{j_n})\}^{n=0,1,2,\cdots,2g}_{1\le j_1<j_2<\cdots<j_n\le 2g}$ is a basis of $\mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]$. Since $\Phi:\mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]\to H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$ is surjective as in Lemma 3.4 when $g\ge 3$, we only need to examine the value of Ψ on $\Phi(\Delta^n_\sigma(X_{j_1},X_{j_2},\cdots,X_{j_n}))\in H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$, using Corollary 3.12.

For $1\le i\le 2g$, we have

$$f_s\Phi(\Delta^1_\sigma(X_i))=\begin{cases}1, & \text{if } X_s=X_{i+g},\\ 0, & \text{otherwise},\end{cases}\qquad f_{st}\Phi(\Delta^1_\sigma(X_i))=0,$$
$$f_{stu}\Phi(\Delta^1_\sigma(X_i))=0.$$

For $1\le i<j\le 2g$,

$$f_s\Phi(\Delta^2_\sigma(X_i,X_j))=\begin{cases}2, & \text{if } \{X_s\}\subset\{X_{i+g},X_{j+g}\},\\ 0, & \text{otherwise},\end{cases},$$

$$f_{st}\Phi(\Delta^2_\sigma(X_i,X_j))=\begin{cases}2, & \text{if } \{X_s,X_t\}=\{X_{i+g},X_{j+g}\},\\ 0, & \text{otherwise},\end{cases}\qquad f_{stu}\Phi(\Delta^2_\sigma(X_i,X_j))=0.$$

For $1\le i<j<k\le 2g$,

$$f_s\Phi(\Delta^3_\sigma(X_i,X_j,X_k))=\begin{cases}4, & \text{if } \{X_s\}\subset\{X_{i+g},X_{j+g},X_{k+g}\},\\ 0, & \text{otherwise},\end{cases}$$

$$f_{st}\Phi(\Delta^3_\sigma(X_i,X_j,X_k))=\begin{cases}4, & \text{if } \{X_s,X_t\}\subset\{X_{i+g},X_{j+g},X_{k+g}\},\\ 0, & \text{otherwise},\end{cases}$$

$$f_{stu}\Phi(\Delta^3_\sigma(X_i,X_j,X_k))=\begin{cases}4, & \text{if } \{X_s,X_t,X_u\}=\{X_{i+g},X_{j+g},X_{k+g}\},\\ 0, & \text{otherwise}.\end{cases}$$

We also have $\Psi\Phi(\Delta^n_\sigma(X_{j_1},X_{j_2},\cdots,X_{j_n}))=0$ for $n\ge 4$ and $\Psi\Phi([0])=\Psi(0)=0$. Hence we have obtained the image of Ψ as stated. $\square$

By this lemma, we obtain the lower bound as follows.

Corollary 3.14. *For $g \geq 3$, we have*

$$|H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})| \geq |\mathbf{Z}_8^{2g} \oplus \mathbf{Z}_4^{\binom{2g}{2}} \oplus \mathbf{Z}_2^{\binom{2g}{3}}|.$$

Now, we determine the abelianization $H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$ as $\mathbf{Z}$-module.

Proposition 3.15. *For $g \geq 3$, we have*

$$H_1(\mathcal{M}_{g,1}[2];\mathbf{Z}) \cong \mathbf{Z}_8^{2g} \oplus \mathbf{Z}_4^{\binom{2g}{2}} \oplus \mathbf{Z}_2^{\binom{2g}{3}}.$$

Proof. We obtained the lower bound for the order of $H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$ in Corollary 3.14. In Proposition 3.5, we also obtained the upper bound. Hence we have

$$|\mathbf{Z}_8^{2g} \oplus \mathbf{Z}_4^{\binom{2g}{2}} \oplus \mathbf{Z}_2^{\binom{2g}{3}}| \leq |H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})| \leq |B_{g,1}^3/\langle 1\rangle\,||H_1(\Gamma_g[2];\mathbf{Z})|.$$

As we defined below Proposition 3.5, the $\mathbf{Z}_2$-module $B_{g,1}^3$ is isomorphic to $\mathbf{Z}_2^{1+\binom{2g}{1}+\binom{2g}{2}+\binom{2g}{3}}$, and we will prove that $H_1(\Gamma_g[2];\mathbf{Z}) \cong \mathbf{Z}_4^{2g} \oplus \mathbf{Z}_2^{2g^2-g}$ in Corollary 4.2. Hence we have

$$|B_{g,1}^3/\langle 1\rangle\,||H_1(\Gamma_g[2];\mathbf{Z})| = |\mathbf{Z}_8^{2g} \oplus \mathbf{Z}_4^{\binom{2g}{2}} \oplus \mathbf{Z}_2^{\binom{2g}{3}}|.$$

By Lemma 3.13, the homomorphism

$$\Psi : H_1(\mathcal{M}_{g,1}[2];\mathbf{Z}) \to \mathbf{Z}_8^{2g} \oplus 2\mathbf{Z}_8^{\binom{2g}{2}} \oplus 4\mathbf{Z}_8^{\binom{2g}{3}}$$

is surjective. Since the orders of these two groups coincide, the surjective homomorphism Ψ is isomorphic. □

Remark 3.16. *For $g \geq 3$, the kernel of the homomorphism $\iota : B_{g,1}^3 \to H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$ is equal to $\langle 1\rangle$.*

Let $(\beta_\sigma)_* : H_1(\mathcal{M}_{g,1}[2];\mathbf{Z}) \to \mathrm{Map}(H_1(\Sigma_g;\mathbf{Z}_2),\mathbf{Z}_8)$ be the induced homomorphism by β_σ. Since the homomorphism $\Psi : H_1(\mathcal{M}_{g,1}[2];\mathbf{Z}) \to \mathbf{Z}_8^{2g} \oplus 2\mathbf{Z}_8^{\binom{2g}{2}} \oplus 4\mathbf{Z}_8^{\binom{2g}{3}}$ factors through $(\beta_\sigma)_*$, we have the following.

Corollary 3.17. *Let σ be a spin structure of Σ_g. For $g \geq 3$,*

$$(\beta_\sigma)_* : H_1(\mathcal{M}_{g,1}[2];\mathbf{Z}) \to \mathrm{Map}(H_1(\Sigma_g;\mathbf{Z}_2),\mathbf{Z}_8)$$

is injective.

Now, we prove Theorem 1.2.

of Theorem 1.2. Let $L_{g,1}^\sigma$ denote the submodule in $\mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]$ generated by

$$[0], 4\Delta_\sigma^2(x_1,x_2), 2\Delta_\sigma^3(x_1,x_2,x_3), \Delta_\sigma^n(x_1,x_2,\cdots,x_n).$$

First, we prove that $L_{g,1}^\sigma$ is contained in the kernel of$\Phi : \mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)] \to H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$. By Lemma 3.10, the elements $[0]$, $4\Delta_\sigma^2(x_1,x_2)$, $2\Delta_\sigma^3(x_1,x_2,x_3)$, and $\{\Delta_\sigma^n(x_1,x_2,\cdots,x_n)\}_{4\leq n\leq 2g}$ in $\mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]$ are contained in the kernel of $(\beta_\sigma)_*\Phi : \mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)] \to \mathrm{Map}(H_1(\Sigma_g;\mathbf{Z}_2),\mathbf{Z}_8)$. Since the homomorphism $(\beta_\sigma)_*$ is injective as in Corollary 3.17, we have $\mathrm{Ker}\,\Phi = \mathrm{Ker}((\beta_\sigma)_*\Phi)$. Hence we have proved that $L_{g,1}^\sigma \subset \mathrm{Ker}\,\Phi$.

Next, we prove that the homomorphism $\Phi_* : \mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]/L_{g,1}^\sigma \to H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$ induced by Φ is isomorphic. By Lemma 3.9 (i)(ii), $\Delta_\sigma^n(x_1,x_2,\cdots,x_n)$ is written as a sum of the elements

$\{\Delta^n_\sigma(X_{i_1}, X_{i_2}, \cdots, X_{i_m})\}_{m\leq n}$ for distinct integers $\{i_k\}_{k=1}^m$. By the commutativity of Δ^n_σ, we may assume $i_1 < i_2 < \cdots < i_n$. In particular, $L^\sigma_{g,1}$ is generated by

$$[0], 4\Delta^2_\sigma(X_{i_1}, X_{i_2}), 2\Delta^3_\sigma(X_{i_1}, X_{i_2}, X_{i_3}), \Delta^n_\sigma(X_{i_1}, X_{i_2}, \cdots, X_{i_n})$$

for $1 \leq i_1 < i_2 < \cdots < i_n \leq 2g$. Since $\{\Delta^n_\sigma(X_{i_1}, X_{i_2}, \cdots, X_{i_n})\}^{n=0,1,\cdots,2g}_{1\leq i_1<i_2<\cdots<i_n\leq 2g}$ is a basis of the free $\mathbf{Z}_8$-module $\mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]$, we have

$$|\mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]/L^\sigma_{g,1}| = |\mathbf{Z}_8^{2g} \oplus \mathbf{Z}_4^{\binom{2g}{2}} \oplus \mathbf{Z}_2^{\binom{2g}{3}}|.$$

This is equal to the order of $H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$. Since the homomorphism Φ is surjective as in Lemma 3.4, the induced homomorphism $\Phi_* : \mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]/L^\sigma_{g,1} \to H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$ is isomorphic. In particular, the submodule $L^\sigma_{g,1}$ does not depend on the choice of a spin structure σ of Σ_g.

Finally, we will show that the submodules $L^\sigma_{g,1}$ and $L_{g,1}$ in $\mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]$ coincide. If we choose the spin structure σ_0 of Σ_g such that its quadratic function q_{σ_0} satisfies $q_{\sigma_0}(X_i) = 0$ for $1 \leq i \leq 2g$, we have

$$q_{\sigma_0}(X_{i_1} + X_{i_2} + \cdots + X_{i_n}) = \sum_{1\leq j<k\leq n} (X_{i_j} \cdot X_{i_k}) \bmod 2 = I(X_{i_1}, X_{i_2}, \cdots, X_{i_n}).$$

Hence we have $\Delta^n_0(X_{i_1}, X_{i_2}, \cdots, X_{i_n}) = \Delta^n_{\sigma_0}(X_{i_1}, X_{i_2}, \cdots, X_{i_n})$. Since $L^{\sigma_0}_{g,1}$ is generated by these elements, it is contained in the submodule $L_{g,1}$. As $\Delta^n_\sigma(x_1, x_2, \cdots, x_n)$ is written as the sum of the elements $\{\Delta^n_\sigma(X_{i_1}, X_{i_2}, \cdots, X_{i_n})\}_{m\leq n}$ for distinct integers $\{i_k\}_{k=1}^n$, we can show that $\Delta^n_0(x_1, x_2, \cdots, x_n)$ is written as the sum of $\{\Delta^n_0(X_{i_1}, X_{i_2}, \cdots, X_{i_m})\}_{m\leq n}$ in the same way. Thus, the submodules $L^{\sigma_0}_{g,1}$ and $L_{g,1}$ in $\mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]$ coincide. Hence Theorem 1.2 is proved. □

3.4 The abelianization of the level 2 mapping class group of a closed surface

In this section, we determine the abelianization of the level 2 mapping class group of a closed surface Σ_g.

As stated in Section 2.1.2, the mapping class group $\mathcal{M}_{g,1}$ is isomorphic to $\pi_0 \operatorname{Diff}_+(\Sigma_g, N(c_0))$. Thus, the inclusion $\operatorname{Diff}_+(\Sigma_g, N(c_0)) \to \operatorname{Diff}_+ \Sigma_g$ induces the homomorphism $r : \mathcal{M}_{g,1} \to \mathcal{M}_g$. It is well-known that this is surjective, for example, see Ivanov [16] p.582. Since the mod d reduction $\mathcal{M}_{g,1} \to \operatorname{Sp}(2g;\mathbf{Z}_d)$ of $\rho : \mathcal{M}_{g,1} \to \operatorname{Sp}(2g;\mathbf{Z})$ factors through the homomorphism r, the restriction $\mathcal{M}_{g,1}[d] \to \mathcal{M}_g[d]$ of the homomorphism r is also surjective.

Consider the case $d = 2$. The homomorphism $\mathcal{M}_{g,1}[2] \to \mathcal{M}_g[2]$ induces the homomorphism $H_1(\mathcal{I}_{g,1};\mathbf{Z})_{\mathcal{M}_{g,1}[2]} \to H_1(\mathcal{I}_g;\mathbf{Z})_{\mathcal{M}_g[2]}$ between the coinvariants. By Theorem 3.6 and 3.7 proved by Johnson, the coinvariant $H_1(\mathcal{I}_{g,r};\mathbf{Z})_{\mathcal{M}_{g,r}[2]}$ is isomorphic to $B^3_{g,r}$. Moreover, the diagram

$$\begin{array}{ccc} H_1(\mathcal{I}_{g,1};\mathbf{Z})_{\mathcal{M}_{g,1}[2]} & \longrightarrow & B^3_{g,1} \\ \downarrow & & \downarrow \\ H_1(\mathcal{I}_{g,0};\mathbf{Z})_{\mathcal{M}_{g,1}[2]} & \longrightarrow & B^3_{g,0} \end{array}$$

commutes (See, Johnson [22] Proof of Lemma 9 p.134), where the left vertical map is the quotient map. Hence, under the identification $B_{g,1} \cong H_1(\mathcal{I}_{g,1};\mathbf{Z})_{\mathcal{M}_{g,1}[2]}$, the kernel of the homomorphism $H_1(\mathcal{I}_{g,1};\mathbf{Z})_{\mathcal{M}_{g,1}[2]} \to H_1(\mathcal{I}_g;\mathbf{Z})_{\mathcal{M}_g[2]}$ is generated by

$$\sum_{i=1}^g \overline{A}_i\overline{B}_i, \ \sum_{i=1}^g \overline{A}_i\overline{B}_i\overline{X} \in B^3_{g,1} \quad \text{for } X = A_1, B_1, \cdots, A_g, B_g.$$

By the 5-term exact sequence, the diagram

$$\begin{array}{ccccccc} H_2(\Gamma_g[2];\mathbf{Z}) & \longrightarrow & H_1(\mathcal{I}_{g,1};\mathbf{Z})_{\mathcal{M}_{g,1}[2]} & \longrightarrow & H_1(\mathcal{M}_{g,1}[2];\mathbf{Z}) & \longrightarrow & H_1(\Gamma_g[2];\mathbf{Z}) \\ \| & & \downarrow & & \downarrow & & \| \\ H_2(\Gamma_g[2];\mathbf{Z}) & \longrightarrow & H_1(\mathcal{I}_g;\mathbf{Z})_{\mathcal{M}_g[2]} & \longrightarrow & H_1(\mathcal{M}_g[2];\mathbf{Z}) & \longrightarrow & H_1(\Gamma_g[2];\mathbf{Z}) \end{array}$$

commutes. Hence, $\mathrm{Ker}(H_1(\mathcal{M}_{g,1}[2];\mathbf{Z}) \to H_1(\mathcal{M}_g[2];\mathbf{Z}))$ is generated by the images of these elements under the homomorphism $H_1(\mathcal{I}_{g,1};\mathbf{Z})_{\mathcal{M}_{g,1}[2]} \to H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$. Therefore, $H_1(\mathcal{M}_g[2];\mathbf{Z})$ is isomorphic to the quotient of $\mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]/L_{g,1}$ by the images of these elements under ι.

We write $\iota(\overline{A_i}\overline{B_i})$, $\iota(\overline{A_i}\overline{B_i}\overline{X})$ as elements of $\mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]/L_{g,1}$. As we saw in Lemma 3.8, we have

$$\iota(\overline{A_1}\overline{B_1}) = 2\Phi\Delta_0^2(A_1,B_1) + 4\langle A_1\rangle + 4\langle B_1\rangle,$$

and

$$\begin{aligned}\iota(\overline{A_1}\overline{B_1}(\overline{B_2+1})) &= -\Phi\Delta_0^3(A_1,B_1,B_2) + 2\Phi\Delta_0^2(A_1,B_2) + 2\Phi\Delta_0^2(B_1,B_2) - 4\langle B_2\rangle \\ &= \Phi\Delta_0^3(A_1,B_1,B_2) + 2\Phi\Delta_0^2(A_1,B_2) + 2\Phi\Delta_0^2(B_1,B_2) + 4\langle B_2\rangle.\end{aligned}$$

Hence for $X = A_1, B_1, \cdots, A_g, B_g$, we have

$$\begin{aligned}\iota(\overline{A_i}\overline{B_i}) &= \Phi\{2\Delta_0^2(A_i,B_i) + 4[A_i] + 4[B_i]\}, \\ \iota(\overline{A_i}\overline{B_i}\overline{X}) &= \Phi\{\Delta_0^3(A_i,B_i,X) + 2\Delta_0^2(A_i,X) + 2\Delta_0^2(B_i,X) + 4[X]\} + \iota(\overline{A_i}\overline{B_i}).\end{aligned}$$

Proposition 3.18. *Let $g \geq 3$ be an integer. Denote by L_g the submodule of $\mathbf{Z}_8[H_1(\Sigma_g;\mathbf{Z}_2)]$ generated by*

$$[0],\ 4\Delta_0^2(x_1,x_2),\ 2\Delta_0^3(x_1,x_2,x_3),\ \Delta_0^n(x_1,x_2,\cdots,x_n),$$
$$\sum_{i=1}^{g}\{2\Delta_0^2(A_i,B_i) + 4[A_i] + 4[B_i]\},$$
$$\sum_{i=1}^{g}\{\Delta_0^3(A_i,B_i,X) + 2\Delta_0^2(A_i,X) + 2\Delta_0^2(B_i,X) + 4[X]\},$$

for $\{x_i\}_{i=1}^n \subset H_1(\Sigma_g;\mathbf{Z}_2)$ and $X = A_1, B_1, \cdots, A_g, B_g$. Then, we have

$$\mathbf{Z}_8[H_1(\Sigma_g;\mathbf{Z}_2)]/L_g \cong H_1(\mathcal{M}_g[2];\mathbf{Z}).$$

4 The abelianization of the level d congruence subgroup of the symplectic group

In this section, we determine the abelianization of the level d congruence subgroup $\Gamma_g[d]$ of the symplectic group $\mathrm{Sp}(2g;\mathbf{Z})$.

4.1 The abelianization of the level d congruence subgroup

In this section, we calculate the abelianization of the level d congruence subgroup (Corollary 4.2) assuming Proposition 4.1.

Denoting the $n \times n$ identity matrix by I_n, let J be the matrix $\begin{pmatrix} O & I_g \\ -I_g & O \end{pmatrix}$. The group $\mathrm{Sp}(2g;\mathbf{Z})$ consists of $2g \times 2g$ integral matrices A which satisfy ${}^tAJA = J$. A matrix $A \in \Gamma_g[d]$ can be written as $A = I_{2g} + dA'$ with an integral $2g \times 2g$ matrix A'. Denote the (i,j)-element of a matrix u by $u(i,j)$. For an even integer d, define the subgroup $\Gamma_g[d,2d]$ of the symplectic group by

$$\Gamma_g[d,2d] := \{A \in \Gamma_g[d] \mid A'(g+i,i) \equiv A'(i,g+i) \equiv 0 \bmod 2 \text{ for } i = 1,2,\cdots,g\}.$$

This subgroup was proved to be a normal subgroup of $\mathrm{Sp}(2g;\mathbf{Z})$ in Igusa [15] Lemma 1.(i).

We will prove in this section:

Proposition 4.1. *Let $g \geq 2$ be an integer. For an odd integer $d \geq 2$, we have*

$$\Gamma_g[d^2] = [\Gamma_g[d], \Gamma_g[d]].$$

For an even integer $d \geq 2$, we have

$$\Gamma_g[d^2, 2d^2] = [\Gamma_g[d], \Gamma_g[d]].$$

Before proving Proposition 4.1, we calculate the abelianization of the congruence subgroup $\Gamma_g[d]$ using this proposition. First, we compute the module $\Gamma_g[d]/\Gamma_g[d^2] \cong H_1(\Gamma_g[d]; \mathbf{Z})$ when d is an odd integer. Denote by $\mathfrak{sp}_{2g}(\mathbf{Z}_d)$ the additive group of all $2g \times 2g$ matrices X with entries in $\mathbf{Z}_d$ such that ${}^tXJ + JX = 0$. For $A := I_{2g} + dA'$, the equation ${}^tAJA = J$ shows $A' \bmod d$ is contained in $\mathfrak{sp}_{2g}(\mathbf{Z}_d)$. For $A, B \in \Gamma_g[d]$, we have

$$AB = I_{2g} + d(A' + B') \mod d^2. \tag{17}$$

Hence, we can define a homomorphism $m : \Gamma_g[d] \to \mathfrak{sp}_{2g}(\mathbf{Z}_d)$ by $m(A) = A'$. For $1 \leq i, j \leq 2g$, denote the $2g \times 2g$ matrix $e_{i,j}$ which has 1 in the (i,j)-element, and 0 in the other elements. The homomorphism m is surjective because the images of $I_{2g} + d(e_{i,j} - e_{g+i,g+j}), I_{2g} + de_{i,i+g}, I_{2g} + de_{i+g,i} \in \Gamma_g[d]$ generate $\mathfrak{sp}_{2g}(\mathbf{Z}_d)$. This is the restriction of the homomorphism of the level d congruence subgroup of $\mathrm{GL}(2g; \mathbf{Z})$ defined in Lee and Szczarba [26] p.16. See also Putman [32]. Then, we have the exact sequence

$$1 \longrightarrow \Gamma_g[d^2] \longrightarrow \Gamma_g[d] \xrightarrow{m} \mathfrak{sp}_{2g}(\mathbf{Z}_d) \longrightarrow 1. \tag{18}$$

This shows that $H_1(\Gamma_g[d]; \mathbf{Z}) \cong \mathfrak{sp}_{2g}(\mathbf{Z}_d) \cong \mathbf{Z}_d^{2g^2+g}$.

Next, we consider the case when d is even. We compute the group $\Gamma_g[d]/\Gamma_g[d^2, 2d^2] \cong H_1(\Gamma_g[d]; \mathbf{Z})$. Since $\Gamma_g[d^2, 2d^2]$ lies in $\mathrm{Ker}\, m$, sequence (18) gives another exact sequence

$$0 \longrightarrow \frac{\Gamma_g[d^2]}{\Gamma_g[d^2, 2d^2]} \longrightarrow \frac{\Gamma_g[d]}{\Gamma_g[d^2, 2d^2]} \xrightarrow{m} \mathfrak{sp}_{2g}(\mathbf{Z}_d) \longrightarrow 0. \tag{19}$$

For a matrix $A = I_{2g} + d^2A' \in \Gamma_g[d^2]$, define the surjective homomorphism $m_1' : \Gamma_g[d^2] \to \mathbf{Z}_2^{2g}$ by

$$m_1'(A) := (\{A'(g+i, i)\}_{i=1}^g, \{A'(g+i, i)\}_{i=1}^g) \bmod 2.$$

By the definition, the kernel of m_1' is equal to $\Gamma_g[d^2, 2d^2]$. Since the images of $I_{2g} + d^2e_{i+g,i}$, $I_{2g} + d^2e_{i,i+g}$ under m_1' generate $\mathbf{Z}_2^{2g}$, this induces the isomorphism $\Gamma_g[d^2]/\Gamma_g[d^2, 2d^2] \cong \mathbf{Z}_2^{2g}$. The exact sequence (19) is consequently wrriten as

$$0 \longrightarrow \mathbf{Z}_2^{2g} \longrightarrow \frac{\Gamma_g[d]}{\Gamma_g[d^2, 2d^2]} \xrightarrow{m} \mathfrak{sp}_{2g}(\mathbf{Z}_d) \longrightarrow 0. \tag{20}$$

The orders of $I_{2g} + de_{i+g,i}$ and $I_{2g} + de_{i,i+g}$ are equal to $2d$, because the images of d times of these elements under m_1' generates $\mathbf{Z}_2^{2g}$. When we showed that m is surjective to obtain the exact sequence (18), we saw that the images of elements $I_{2g} + d(e_{i,j} - e_{j+g,i+g})$, $I_{2g} + de_{i+g,i}$, $I_{2g} + de_{i,i+g}$ generate $\mathfrak{sp}_{2g}(\mathbf{Z}_d)$. Hence these elements generate $\Gamma_g[d]/\Gamma_g[d^2, 2d^2]$. This shows $H_1(\Gamma_g[d]; \mathbf{Z}) = \mathbf{Z}_d^{2g^2-g} \oplus \mathbf{Z}_{2d}^{2g}$.

Corollary 4.2. *Let $g \geq 2$ be an integer. For $d \geq 2$,*

$$H_1(\Gamma_g[d]; \mathbf{Z}) = \begin{cases} \mathbf{Z}_d^{2g^2+g} & \text{if } d \text{ is odd,} \\ \mathbf{Z}_d^{2g^2-g} \oplus \mathbf{Z}_{2d}^{2g} & \text{if } d \text{ is even.} \end{cases}$$

4.2 Proof of Proposition 4.1

In this section, we prove Proposition 4.1 to complete the calculation of the abelianization of the level d congruence subgroup. For a homology class $y \in H_1(\Sigma_{g,r}; \mathbf{Z})$, define the transvection $T_y \in \mathrm{Sp}(2g; \mathbf{Z})$ so that which acts on $H_1(\Sigma_g; \mathbf{Z})$ as $T_y(x) := x + (y \cdot x)y$. If y is represented by an oriented simple closed curve C, this is the image of the Dehn twist t_C under the surjective homomorphism $\rho : \mathcal{M}_{g,r} \to \mathrm{Sp}(2g; \mathbf{Z})$.

By fixing the symplectic basis $\{A_i, B_i\}_{i=1}^{2g}$ as in Introduction, we consider that the symplectic group acts on $H_1(\Sigma_{g,r}; \mathbf{Z}) \cong \mathbf{Z}^{2g}$. To prove the proposition, we need two lemmas.

Lemma 4.3. *Let $g \geq 2$ be an integer. We have*

1. $[T_{A_1}^{2d^2}] = 0 \in H_1(\Gamma_g[d]; \mathbf{Z})$ *for any integer $d \geq 2$.*
2. $[T_{A_1}^{d^2}] = 0 \in H_1(\Gamma_g[d]; \mathbf{Z})$ *for an odd integer $d \geq 2$.*

Lemma 4.4. *Let $g \geq 2$ be an integer. If d is an even integer, we have*

$$\Gamma_g[d^2, 2d^2] \subset [\Gamma_g[d], \Gamma_g[d]].$$

Before showing these lemmas, we prove the proposition assuming the lemmas. Since $\Gamma_g[d^2]$ is the kernel of the homomorphism m in (18), we have

$$[\Gamma_g[d], \Gamma_g[d]] \subset \Gamma_g[d^2]$$

for every $d \geq 2$. Further, if d is even, it is shown that

$$[\Gamma_g[d], \Gamma_g[d]] \subset \Gamma_g[d^2, 2d^2]$$

in Igusa [15] Lemma 1.(ii). By Lemma 4.4, it suffices to prove

$$\Gamma_g[d^2] \subset [\Gamma_g[d], \Gamma_g[d]], \text{ for } d \text{ odd.}$$

Theorem 2.13 proved by Mennicke is essential in this proof. By Lemma 4.3, we have $T_{A_1}^{d^2} \in [\Gamma_g[d], \Gamma_g[d]]$ when d is odd. Hence, by Theorem 2.13, we obtain

$$\Gamma_g[d^2] \subset [\Gamma_g[d], \Gamma_g[d]] \text{ if } d \text{ is odd.}$$

This shows Proposition 4.1.

In the following, we prove Lemma 4.3 and 4.4. To prove (i) of the Lemma 4.3, we prepare some lemmas. A straightforward computation shows:

Lemma 4.5. *Let $g \geq 2$ be an integer. For $d \geq 1$, we have*

$$T_{a_1A_1+b_1B_1+a_2A_2}^d = (T_{A_2}^d)^{(a_1b_1+1)a_2^2}(T_{B_1+A_2}^d T_{A_2}^{-d} T_{B_1}^{-d})^{b_1a_2}(T_{A_1+A_2}^d T_{A_1}^{-d} T_{A_2}^{-d})^{a_1a_2} T_{a_1A_1+b_1B_1}^d.$$

If we put $a_1 = 1, a_2 = -1, b_1 = 0$, we obtain

$$[T_{A_1+A_2}^d] + [T_{A_1-A_2}^d] = 2[T_{A_1}^d] + 2[T_{A_2}^d].$$

We denote by δ_{ij} the Kronecker delta. The following lemma is a special case of Corollary 2 proved by Johnson.

Lemma 4.6 ((Johnson [22] Corollary 2)). *Let $\{a_i, b_i\}_{i=1}^k$ and $\{a_i', b_i'\}_{i=1}^k$ be two sets of elements in $H_1(\Sigma_{g,r}; \mathbf{Z})$ such that $a_i \cdot b_j = a_i' \cdot b_j' = \delta_{ij}$. Then, there exists $X \in \mathrm{Sp}(2g; \mathbf{Z})$ which satisfies $X(a_i) = a_i'$, and $X(b_i) = b_i'$.*

Using the relation $XT_xX^{-1} = T_{X(x)}$ for $x \in H_1(\Sigma_{g,r}; \mathbf{Z})$ and $X \in \mathrm{Sp}(2g; \mathbf{Z})$ which comes form (7), the above lemmas show:

Lemma 4.7. *Let $g \geq 2$ be an integer. If $x, y \in H_1(\Sigma_{g,r}; \mathbf{Z})$ satisfy $x \cdot y = 0$, and $\{x, y\}$ can be extended to form a symplectic basis of $H_1(\Sigma_{g,r}; \mathbf{Z})$, then we have*

$$[T^d_{x+y}] + [T^d_{x-y}] = 2[T^d_x] + 2[T^d_y].$$

Remark 4.8. *For $i = 1, 2, 3, 4$, let $D_i \subset S^2$ be mutually disjoint disks. By the assumption of x, y in Lemma 4.7, we can choose an embedding $i : S^2 - \amalg_{i=1}^4 D_i \to \Sigma_{g,r}$ such that $[i(\partial D_1)] = -[i(\partial D_2)] = x$, $[i(\partial D_3)] = -[i(\partial D_4)] = y \in H_1(\Sigma_{g,r}; \mathbf{Z})$ as in Figure 6. If we write the Lantern relation of this embedding as*

$$t_{i(\partial D_1)} t_{i(\partial D_2)} t_{i(\partial D_3)} t_{i(\partial D_4)} t_{c_3}^{-1} = t_{c_1} t_{c_2}.$$

Since $t_{i(\partial D_i)}$ for $i = 1, 2, 3, 4$ and t_{c_3} mutually commute, the image of the d-powers of this relation

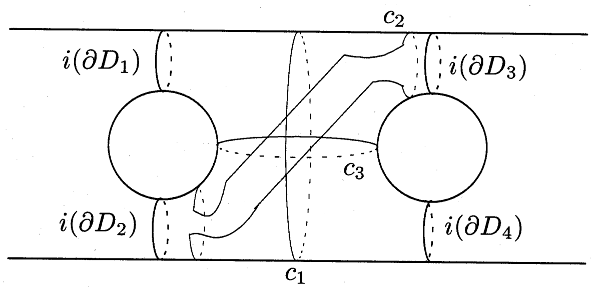

Figure 6: Lantern relation

under the homomorphism $\mathcal{M}_{g,r} \to \mathrm{Sp}(2g; \mathbf{Z})$ also shows the above relation.

Put $x = kA_1 + A_2$, $y = A_1$ in the equation of Lemma 4.7, and take the summation over $k = 1, 2, \cdots, d$. By the definition, we have $[T^{2d^2}_{A_1}] = 2d[T^d_{A_1}] = 0$, that is (i) of Lemma 4.3.

Next, we will show (ii) of Lemma 4.3. Replace x by $x + ky$ in the equation of Lemma 4.7 and take d times the equation. By (i) of Lemma 4.3, the equation is written as

$$d[T^d_{x+2ky}] = d[T^d_{x+2(k+1)y}].$$

If d is odd, we obtain

$$d[T^d_x] = d[T^d_{x+y}] = d[T^d_y]. \tag{21}$$

If we put $a_1 = b_1 = 2$, and $a_2 = 1$ in Lemma 4.5, we have

$$\begin{aligned}&[T^d_{2A_1+2B_1+A_2}] \\ =&5[T^d_{A_2}] + 2([T^d_{B_1+A_2}] - [T^d_{A_2}] - [T^d_{B_1}]) + 2([T^d_{A_1+A_2}] - [T^d_{A_1}] - [T^d_{A_2}]) + [T^d_{2A_1+2B_1}].\end{aligned} \tag{22}$$

If we apply the equation (21) to d times the equation (22), we have

$$[T^{d^2}_{A_1}] = d[T^d_{A_1}] = 0 \in H_1(\Gamma_g[d]; \mathbf{Z}). \tag{23}$$

This completes the proof of Lemma 4.3.

Next, we prove Lemma 4.4. We have already known that $[T^{2d^2}_{A_1}] = 0 \in H_1(\Gamma_g[d]; \mathbf{Z})$. By Theorem 2.13, we obtain

$$\Gamma_g[2d^2] \subset [\Gamma_g[d], \Gamma_g[d]].$$

Hence, it suffices to prove the inclusion between the quotient groups

$$\Gamma_g[d^2, 2d^2]/\Gamma_g[2d^2] \subset [\Gamma_g[d], \Gamma_g[d]]/\Gamma_g[2d^2].$$

When d is even, for matrices $A = I_{2g} + d^2 A', B = I_{2g} + d^2 B' \in \Gamma_g[d^2]$, we have

$$AB \equiv I_{2g} + d^2(A' + B') \bmod 2d^2.$$

Similar to the homomorphism m, define the homomorphism $m' : \Gamma_g[d^2] \to \mathfrak{sp}_{2g}(\mathbf{Z}_2)$ by $m'(A) = A'$. Then, we have

$$\operatorname{Ker} m' = \{A \in \Gamma_g[d^2] \mid A' \equiv 0 \bmod 2\} = \Gamma_g[2d^2].$$

The homomorphism m' induces the isomorphism $\Gamma_g[d^2]/\Gamma_g[2d^2] \cong \mathfrak{sp}_{2g}(\mathbf{Z}_2)$. We calculate the images of $\Gamma_g[d^2, 2d^2]$ and $[\Gamma_g[d], \Gamma_g[d]]$ under this isomorphism.

Define the sub $\mathbf{Z}_2$-module

$$V := \{X \in \mathfrak{sp}_{2g}(\mathbf{Z}_2) \mid X(i, g+i) = X(g+i, i) = 0\}.$$

By the definition, we have $m(\Gamma_g[d^2, 2d^2]) \subset V$. The images of

$$I_{2g} + d^2(e_{g+i,j} + e_{g+j,i}), I_{2g} + d^2(e_{i,g+j} + e_{j,g+i}), I_{2g} + d^2(e_{i,j} - e_{g+j,g+i}) \in \Gamma_g[d^2, 2d^2] \tag{24}$$

under m' generates V. This shows $m'(\Gamma_g[d^2, 2d^2]) = V$.

For $A = I_{2g} + dA', B = I_{2g} + dB' \in \Gamma_g[d]$, we have

$$ABA^{-1}B^{-1} \equiv I_{2g} + d^2(A'B' - B'A') \bmod 2d^2.$$

If we put $A' = e_{g+i,i}, B' = e_{i,j} - e_{g+j,g+i}, A' = e_{j,i} - e_{g+i,g+j}, B' = e_{i,g+i}$, and $A' = e_{i,g+j} + e_{j,g+i}, B' = e_{g+j,j}$, we get the elements (24). Hence we have

$$m'(\Gamma_g[d^2, 2d^2]) = V \subset m'([\Gamma_g[d], \Gamma_g[d]]).$$

This proves the lemma.

5 On the abelianization of the level d mapping class group for $d \geq 3$

In this section, we prove Theorem 1.4 and Proposition 1.5 which describe the abelianization of the level d mapping class group for $d \geq 3$. The exact sequence

$$1 \to \mathcal{I}_{g,r} \to \mathcal{M}_{g,r}[d] \to \Gamma_g[d] \to 1$$

plays an important role. By the 5-term exact sequence, we have the exact sequence

$$H_1(\mathcal{I}_{g,r}; \mathbf{Z}) \to H_1(\mathcal{M}_{g,r}[d]; \mathbf{Z}) \to H_1(\Gamma_g[d]; \mathbf{Z}) \to 0.$$

Lemma 5.1. *Let $g \geq 3$ be an integer, and r either 0 or 1. For $d \geq 2$, the homomorphism $H_1(\mathcal{I}_{g,r}; \mathbf{Z}) \to H_1(\mathcal{M}_{g,r}[d]; \mathbf{Z})$ factors through $H_1(\mathcal{I}_{g,r}; \mathbf{Z}) \otimes \mathbf{Z}_d$.*

Proof. For any pair of simple closed curves $\{C_1, C_1'\}$ which bounds a subsurface of genus 1 in $\Sigma_{g,1}$, the mapping class $t_{C_1} t_{C_1'}^{-1}$ is in the Torelli group $\mathcal{I}_{g,1}$. Johnson [17] showed that $\mathcal{I}_{g,1}$ is generated by all pairs of twists $t_{C_1} t_{C_1'}^{-1}$, for $g \geq 3$ and such an bounding pair $\{C_1, C_1'\}$. In particular, $\mathcal{I}_g$ is also generated by pairs of twists as above. Johnson ([19] Lemma 11) also shows that any pair of simple closed curves C_2, C_2' which bounds a subsurface in $\Sigma_{g,r}$ satisfies $(t_{C_2} t_{C_2'}^{-1})^d \in [\mathcal{M}_{g,r}[d], \mathcal{I}_{g,r}]$.

Therefore for a mapping class $\varphi \in \mathcal{I}_{g,r}$, we have $[\varphi^d] = 0 \in H_1(\mathcal{M}_{g,r}[d]; \mathbf{Z})$. This proves the lemma. □

We have already determined the abelianization of level d congruence subgroup of the symplectic group in Section 4. To prove Theorem 1.4, we will construct a splitting of the exact sequence

$$H_1(\mathcal{I}_{g,r};\mathbf{Z})\otimes \mathbf{Z}_d \to H_1(\mathcal{M}_{g,r}[d];\mathbf{Z}) \to H_1(\Gamma_g[d];\mathbf{Z}) \to 0$$

for $r=0,1$ and odd d. The splitting of the above exact sequence comes from the mod d reduction of the Johnson homomorphism defined on the level d mapping class group for $d \geq 2$. We construct this homomorphism.

For $n \geq 2$, let F_n denote the free group of rank n, and denote by $H := F_n/[F_n, F_n]$ the abelianization of F_n. Let $\mathrm{Aut}\, F_n$ be the automorphism group of the free group F_n. For a commutative ring R with unit element, denote the tensor algebra of $H \otimes R$ by

$$\hat{T} := \prod_{m=0}^{\infty} H^{\otimes m} \otimes R.$$

Let $\hat{T}_i$ be the subalgebra $\hat{T}_i := \prod_{m \geq i} H^{\otimes m} \otimes R$ in $\hat{T}$ for $i \geq 1$.

Definition 5.2. *The map $\theta : F_n \to 1+\hat{T}_1$ is called R-valued Magnus expansion of F_n if $\theta : F_n \to 1+\hat{T}_1$ is a group homomorphism, and for any $\gamma \in F_n$, the map θ satisfies*

$$\theta(\gamma) \equiv 1 + [\gamma] \ (\mathrm{mod}\, \hat{T}_2).$$

In detail, see Kawazumi [23] Section 1 and Bourbaki[5] Ch.2, Section 5, no.4, 5. In the following, we put $R := \mathbf{Z}_d$ for an integer $d \geq 2$. We denote by $\theta_m : F_n \to H^{\otimes m} \otimes \mathbf{Z}_d$ the m-th component of θ. Denote by Γ_2^d the kernel of the homomorphism $F_n \to H \otimes \mathbf{Z}_d$, then the restriction of θ_2 to $\Gamma_2^d \to H^{\otimes 2} \otimes \mathbf{Z}_d$ is an $\mathrm{Aut}\, F_n$-equivariant homomorphism (See Bourbaki[5] Ch.2, Section 5, no.5.). Define the level d IA-automorphism group as $IA_n[d] := \mathrm{Ker}(\mathrm{Aut}\, F_n \to \mathrm{GL}(n;\mathbf{Z}_d))$. Let H^* denote the dual $\mathbf{Z}$-module $\mathrm{Hom}(H,\mathbf{Z})$, and define the mod d Johnson homomorphism by

$$\begin{array}{rccll} \tau_d : & IA_n[d] & \to & \mathrm{Hom}(H, H^{\otimes 2} \otimes \mathbf{Z}_d) & \cong H^* \otimes H^{\otimes 2} \otimes \mathbf{Z}_d. \\ & \varphi & \mapsto & ([x] \to \theta_2(\varphi(x)) - \theta_2(x)) & \end{array}$$

Then, the map τ_d is an $\mathrm{Aut}\, F_n$-equivariant homomorphism, as in Johnson [18] Lemmas 2C and 2D, and Kawazumi [23] Theorem 3.1. See also Satoh [34].

Next, we define the mod d Johnson homomorphism on the level d mapping class group. Choose symplectic generators $\{a_i, b_i\}_{i=1}^g$ of $\pi_1(\Sigma_{g,1})$ which represent the symplectic basis $\{A_i, B_i\}$. Then we have the isomorphism $\pi_1(\Sigma_{g,1}) \cong F_{2g}$, and $H \cong H_1(\Sigma_{g,1};\mathbf{Z})$. The action of $\mathcal{M}_{g,1}[d]$ on the fundamental group of the surface induces the homomorphism $\mathcal{M}_{g,1}[d] \to IA_n[d]$. By composing this homomorphism, we have a homomorphism $\tau_d : \mathcal{M}_{g,1}[d] \to H^* \otimes H^{\otimes 2} \otimes \mathbf{Z}_d$. By the isomorphism

$$H^* \otimes H^{\otimes 2} \otimes \mathbf{Z}_d \cong H^{\otimes 3} \otimes \mathbf{Z}_d.$$

given by the Poincaré duality, we denote it as

$$\tau_d : \mathcal{M}_{g,1}[d] \to H^{\otimes 3} \otimes \mathbf{Z}_d.$$

By the definition, this homomorphism is independent of the choice of the generators of $\pi_1(\Sigma_{g,1})$. In Kawazumi [23] Theorem 3.1, he also showed that the restriction of τ_d to $\mathcal{I}_{g,1}$ is equal to the mod d reduction of the Johnson homomorphism. Similar mod d Johnson homomorphisms are constructed in Broaddus-Farb-Putman [6] and Perron [30], independently.

To determine the abelianization of $\mathcal{M}_{g,r}[d]$ for an odd integer d, we calculate the image of the mod d Johnson homomorphism on the level d mapping class group.

Lemma 5.3. *Let $g \geq 3$ be an integer. For an odd integer $d \geq 3$,*

$$\mathrm{Im}\, \tau_d = \Lambda^3 H \otimes \mathbf{Z}_d$$

Proof. By the Theorem 2.13, $\mathcal{M}_{g,1}[d]$ is generated by d-th powers of Dehn twists along all nonseparating curves and the Torelli group $\mathcal{I}_{g,1}$. Since $\tau_d|_{\mathcal{I}_{g,1}}$ is equal to the mod d reduction of the Johnson homomorphism τ, we have $\tau_d(\mathcal{I}_{g,1}) = \Lambda^3 H \otimes \mathbf{Z}_d$. Since τ_d is $\mathcal{M}_{g,1}$-equivalent, we only have to calculate $\tau_d(t_{C_1}^d)$ for the simple closed curve C_1 as shown in Figure 5. By the definition of τ_d, we have

$$\tau_d(t_{C_1}^d) = \frac{d(d-1)}{2} B_1 \otimes B_1 \otimes B_1 \in \Lambda^3 H \otimes \mathbf{Z}_d.$$

If d is odd, it is equal to 0. Hence we have $\operatorname{Im} \tau_d = \Lambda^3 H \otimes \mathbf{Z}_d$. □

Next, We will define the Johnson homomorphism τ_d for closed surfaces. As stated in Section 3.4, we have the surjective homomorphism $\mathcal{M}_{g,1}[d] \to \mathcal{M}_g[d]$. As in Introduction, consider H as a subspace of $\Lambda^3 H$.

Lemma 5.4. *For $g \geq 3$, the homomorphism $\tau_d : \mathcal{M}_{g,1}[d] \to \Lambda^3 H \otimes \mathbf{Z}_d$ induces the well-defined homomorphism*

$$\mathcal{M}_g[d] \to \Lambda^3 H/H \otimes \mathbf{Z}_d.$$

Proof. It is known that $\operatorname{Ker}(\mathcal{M}_{g,1}[d] \to \mathcal{M}_g[d])$ is generated by twisting pair $T_C T_{C'}^{-1}$ and separating twist $T_{\partial\Sigma_{g,1}}$, where $\{C, C'\}$ is a pair which bounds subsurface of genus $g-1$ (see Birman [4] pp156-160). Johnson [18] (Lemmas 4A p.230, Lemma 4B p.232) calculated the value of the Johnson homomorphism on the dehn twists $T_{\partial\Sigma_{g,1}}$ along the boundary curve and bounding pairs $T_C T_{C'}^{-1}$. Since $\tau_d|_{\mathcal{I}_{g,1}}$ coincide with mod d reduction of the Johnson homomorphism, we have $\tau_d(T_{\partial\Sigma_{g,1}}) = 0$, and $\tau_d(T_C T_{C'}^{-1}) \in H \otimes \mathbf{Z}_d \subset \Lambda^3 H \otimes \mathbf{Z}_d$. Since $H \otimes \mathbf{Z}_d \subset \Lambda^3 H \otimes \mathbf{Z}_d$ is a $Sp(2g; \mathbf{Z})$-invariant subspace, we see that τ_d induces homomorphism $\mathcal{M}_g[d] \to \Lambda^3 H/H \otimes \mathbf{Z}_d$. □

Now, we prove Theorem 1.4 and Proposition 1.5, using the homomorphism defined as above. We need to review Johnson's result.

Theorem 5.5 (Johnson [22] Theorems 3, 6). *For $g \geq 3$, the abelianization of the Torelli group is written as*

$$H_1(\mathcal{I}_{g,1}; \mathbf{Z}) \cong \Lambda^3 H \oplus B_{g,1}^2,$$
$$H_1(\mathcal{I}_g; \mathbf{Z}) \cong \Lambda^3 H/H \oplus B_{g,0}^2.$$

of Theorem 1.4. Let $d \geq 3$ be an odd integer. Consider the homomorphism

$$\tau_d : \mathcal{M}_{g,1}[d] \to \Lambda^3 H \otimes \mathbf{Z}_d,$$
$$\tau_d : \mathcal{M}_g[d] \to \Lambda^3 H/H \otimes \mathbf{Z}_d,$$

defined in Lemma 5.1. By Theorem 5.5 proved by Johnson, τ_d induces the isomorphism

$$H_1(\mathcal{I}_{g,1}; \mathbf{Z}) \otimes \mathbf{Z}_d \cong \Lambda^3 H \otimes \mathbf{Z}_d,$$
$$H_1(\mathcal{I}_g; \mathbf{Z}) \otimes \mathbf{Z}_d \cong \Lambda^3 H/H \otimes \mathbf{Z}_d,$$

when d is odd. Hence, we have the splitting of the exact sequence

$$H_1(\mathcal{I}_{g,r}; \mathbf{Z}) \otimes \mathbf{Z}_d \to H_1(\mathcal{M}_{g,r}[d]; \mathbf{Z}) \to H_1(\Gamma_g[d]; \mathbf{Z}) \to 0,$$

by the homomorphism τ_d. This shows that

$$H_1(\mathcal{M}_{g,r}[d]; \mathbf{Z}) = \begin{cases} (\Lambda^3 H \otimes \mathbf{Z}_d) \oplus H_1(\Gamma_g[d]; \mathbf{Z}), & \text{when } r = 1, \\ (\Lambda^3 H/H \otimes \mathbf{Z}_d) \oplus H_1(\Gamma_g[d]; \mathbf{Z}), & \text{when } r = 0. \end{cases}$$

□

of Proposition 1.5. Let d be an even integer. By Theorem 5.5 proved by Johnson, we have

$$H_1(\mathcal{I}_{g,r};\mathbf{Z})_{\Gamma_g[d]} = \begin{cases} (\Lambda^3 H \otimes \mathbf{Z}_d) \oplus B^2_{g,1}, & \text{if } r=1, \\ (\Lambda^3 H/H \otimes \mathbf{Z}_d) \oplus B^2_{g,0}, & \text{if } r=0. \end{cases}$$

Since the restriction of τ_d to $\mathcal{I}_{g,1}$ is equal to the mod d reduction of the Johnson homomorphism, we have

$$(\Lambda^3 H \otimes \mathbf{Z}_d) \cap \mathrm{Ker}(H_1(\mathcal{I}_{g,1}[d];\mathbf{Z})_{\Gamma_g[d]} \to H_1(\mathcal{M}_{g,1}[d];\mathbf{Z})) = 0,$$
$$(\Lambda^3 H/H \otimes \mathbf{Z}_d) \cap \mathrm{Ker}(H_1(\mathcal{I}_g[d];\mathbf{Z})_{\Gamma_g[d]} \to H_1(\mathcal{M}_g[d];\mathbf{Z})) = 0.$$

Since the homomorphism $H_1(\mathcal{I}_{g,r};\mathbf{Z}) \to H_1(\mathcal{M}_{g,r}[2];\mathbf{Z})$ induced by the inclusion factors through the group $H_1(\mathcal{M}_{g,r}[d];\mathbf{Z})$, we also have

$$B^2_{g,1} \cap \mathrm{Ker}(H_1(\mathcal{I}_{g,1}[d];\mathbf{Z})_{\Gamma_g[d]} \to H_1(\mathcal{M}_{g,1}[d];\mathbf{Z})) \subset \langle 1 \rangle,$$
$$B^2_{g,0} \cap \mathrm{Ker}(H_1(\mathcal{I}_g[d];\mathbf{Z})_{\Gamma_g[d]} \to H_1(\mathcal{M}_g[d];\mathbf{Z})) \subset \langle 1 \rangle.$$

Hence, the sequence

$$\mathbf{Z}_2 \longrightarrow H_1(\mathcal{I}_{g,r};\mathbf{Z})_{\Gamma_g[d]} \longrightarrow H_1(\mathcal{M}_{g,r}[d];\mathbf{Z}) \longrightarrow H_1(\Gamma_g[d];\mathbf{Z}) \longrightarrow 0$$

is exact. □

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