On the index problem for C^1-generic wild homoclinic classes

(C^1 通有的に野性的な ホモクリニック類の指数問題について)

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Abstract

We study the dynamics of homoclinic classes on three dimensional manifolds under the robust absence of dominated splittings. We prove that, C^1-generically, if such a homoclinic class contains a volume-expanding periodic point, then it contains a hyperbolic periodic point whose index (dimension of the unstable manifold) is equal to two. We also furnish an example which shows that a similar result is not always true in higher dimensions.

1 Preliminaries

1.1 General notations

We consider a closed (compact and boundaryless) smooth manifold M with a Riemannian metric. The space of C^1-diffeomorphisms of M is denoted by $\mathrm{Diff}^1(M)$. We fix a distance function on $\mathrm{Diff}^1(M)$ and furnish $\mathrm{Diff}^1(M)$ with the C^1-topology. For $f \in \mathrm{Diff}^1(M)$, we denote the set of periodic points of f by $\mathrm{Per}(f)$ and the set of hyperbolic periodic poits of f by $\mathrm{Per}_h(f)$. For $P \in \mathrm{Per}(f)$, by $\mathrm{per}(P)$ we denote the period of P, i.e., the least positive integer k that satisfies $f^k(P) = P$. For $x \in M$, we denote the orbit of x by $\mathcal{O}(x, f)$ or simply by $\mathcal{O}(x)$. We put $J(P) := \det(df^{\mathrm{per}(P)}(P))$ and call this value *Jacobian* of P. A periodic point P is said to be *volume-expansive* (resp. *volume-contracting, conservative*) if $|J(P)| > 1$ (resp. $|J(P)| < 1$, $|J(P)| = 1$). For g sufficiently close to f, one can define the continuation of P. We denote the continuation of P of g by $P(g)$.

In the following, we assume that P is a hyperbolic periodic point. The *index* of a hyperbolic periodic point P (denoted by $\mathrm{ind}(P)$) is defined to be the dimension of the unstable manifold of P. By $W^s(P, f)$ (resp. $W^u(P, f)$), we denote the stable (resp. unstable) manifold of P. We also use the simplified notation $W^s(P)$ (resp. $W^u(P)$). Two hyperbolic periodic points P and Q are said to be *homoclinically related* if $W^u(P)$ and $W^s(Q)$, $W^u(Q)$ and $W^s(P)$ both have non-empty transversal intersections. We say that P has a *homoclinic tangency* if there exists a point $x \in W^s(P) \cap W^u(P)$ at which $T_xW^s(P)$ and $T_xW^s(P)$ do not span T_xM.

1.2 Linear cocycles

Let (Σ, f, E, A) be a *linear cocycle*, where Σ is a topological space, f is a homeomorphism of Σ, E is a Euclidean vector bundle over Σ and A is a bundle map compatible with f, more presicely, A is a bundle map of E such that for each $x \in \Sigma$, $A(x, \cdot)$ is a linear isomorphisms from $E(x)$ to $E(f(x))$. We often write $A(x)$ in the meaning of $A(x, \cdot)$ and denote the linear cocycle only with E or A when the meaning is clear from the context. In our application, we treat linear cocycles for which Σ is some invariant set of M, f is the restriction of a diffeomorphism of M to Σ, E is the restriction of the tangent bundle of M to Σ and A is the restriction of the differential df to E. We can naturally define the n-times iteration of A, denoted by A^n, and inverse of A, denoted by A^{-1}. We say that a linear cocycle (Σ, f, E, A) is *periodic* if each point $x \in \Sigma$ is periodic for f.

On each fiber, there is a norm which we denote by $\| \cdot \|$. A linear cocycle is said to be *bounded by* $K > 0$ if the following inequality holds:

$$\max \left\{ \sup_{x \in \Sigma} ||A(x)||, \ \sup_{x \in \Sigma} ||A^{-1}(x)|| \right\} < K.$$

In the case where A is the restriction of some differential on some compact manifold, A is always bounded by some constant. For linear cocycles, we can canonically define the invariant subcocycle, direct sum between some cocycles, and quotient of the cocycle (for the details, see section 1 of [BDP]).

1.3 Dominated splittings

Let (Σ, f, E, A) be a linear cocycle and suppose that E is a direct sum of some invariant subbundles $(\Sigma, f, F, A|_F)$ and $(\Sigma, f, G, A|_G)$ of E, where $A|_F$ is the restriction to F. Given a positive integer n, we say that the splitting $E = F \oplus G$ is an n*-dominated splitting* if the following holds:

$$\|A^n(x)|_{F(x)}\| \|A^{-n}(f^n(x))|_{G(f^n(x))}\| < 1/2, \quad \text{for all } x \in \Sigma.$$

We say that a linear cocycle (Σ, f, E, A) has a dominated splitting if there exists two invariant subbundles F, G of E and an integer n such that $E = F \oplus G$ is an n-dominated splitting. An f-invariant set $\Lambda \subset M$ is said to admit a dominated splitting if the linear cocycle $(\Lambda, f, TM|_\Lambda, df)$ admits a dominated splitting.

1.4 Robust cycles

Let $f \in \mathrm{Diff}^1(M)$ and let Γ and Σ be two transitive hyperbolic invariant sets of f. We say that f has a *heterodimensional cycle associated to* Γ *and* Σ if the following holds:

1. The indices (the dimension of the unstable manifolds) of the sets Γ and Σ are different.

2. The stable manifold of Γ meets the unstable manifold of Σ and the same holds for stable manifold of Σ and the unstable manifold of Γ.

We say that the heterodimensional cycle associated to Γ and Σ is *C^1-robust* if there exists a C^1-neighborhood $\mathcal{U}$ of f such that, for each $g \in \mathcal{U}$, g has a heterodimensional cycle associated to the continuations $\Gamma(g)$ of Γ and $\Sigma(g)$ of Σ.

2 Introduction

Given a hyperbolic periodic point P of a C^1-diffeomorphism f, we define the *homoclinic class* of P, denoted by $H(P, f)$, to be the closure of the points of the transversal intersections between the stable manifold and the unstable manifold of P. The theory of Smale's generalized horseshoe tells us that $H(P, f)$ coincides with the closure of hyperbolic periodic points that are homoclinically related to P. In the study of uniformly hyperbolic systems, homoclinic classes play an important role and it is expected that they also will be important in the research of non-hyperbolic dynamics (see chapter 10 of [BDV]).

There are several studies about the properties of non-hyperbolic homoclinic classes. For example, Abdenur, et al. [ABCDW] investigated the indices of periodic points in homoclinic classes and showed that, C^1-generically, the collection of indices in a homoclinic class forms an interval of natural numbers. Bonatti, Díaz, and Pujals [BDP] proved that the robust absence of the dominated splitting on a homoclinic class implies the *C^1-Newhouse phenomenon*, i.e., locally generic coexistence of infinitely many sinks or sources and in [BD1], Bonatti and Díaz showed, under the robust absence of dominated splittings and some conditions on the Jacobians, a homoclinic class exhibits very complicated dynamics called *universal dynamics*. Recently, Gourmelon [Gou2] proved that, under the absence of dominated splittings on a homoclinic class, one can create a homoclinic tangency inside the homoclinic class.

We can find similar interests among the works about the effects of the absence of the domination. Wen [W] proved the non-existence of dominated splittings implies the creation of a homoclinic tangency. The result of Gan [Gan] says that the existence of a dominated splitting of index i is equivalent to the non-existence i-eigenvalue gap.

Keeping these results in mind, we study of non-hyperbolic homoclinic classes by pursuing the following problem: *What are the effects that the absence of dominated splittings on a homoclinic class gives rise to?* Or, *how the existence of the dominated splittings on a homoclinic class is disturbed?*

To state our problem clearly, we prepare some notations. A homoclinic class $H(P, f)$ is said to be *wild* if there exists a C^1-neighborhood $\mathcal{U}$ of f such that, for every $g \in \mathcal{U}$, the corresponding homoclinic class $H(P, g)$ does not admit dominated splittings. For a homoclinic class $H(P, f)$, we define the *index set* of $H(P, f)$, denoted by $\mathrm{ind}(H(P, f))$, as follows:

$$\mathrm{ind}(H(P,f)) := \{\, \mathrm{ind}(Q) \in \mathbb{N} \mid Q \in \mathrm{Per}_h(f) \cap H(P,f) \,\}.$$

Then, the problem that we are to study is the following: *Does the wilderness of* $H(P,f)$ *give some information about* $\mathrm{ind}(H(P,f))$*?*

There are some examples of wild homoclinic classes in the literature (for example, see [BD2]). They are created by using the heterodimensional cycles. Hence, to assure the robustness of the wilderness, every example requires at least two hyperbolic periodic points with different indices. So, it is natural to ask whether this is the only mechanism that brings the wilderness. For instance, the following question would be interesting.

Question 1. *Let M be a closed smooth manifold. If $f \in \mathrm{Diff}^1(M)$ has a hyperbolic periodic point P of f such that $H(P,f)$ is wild, then $\#\,\mathrm{ind}(H(P,f)) \geq 2$?*

Here is a partial answer to this problem. In this article, we prove the following theorem.

Theorem 1. *For C^1-generic diffeomorphisms of a three-dimensional smooth closed manifold, if there exists a wild homoclinic class $H(P,f)$ that contains an index-one volume-expanding hyperbolic periodic point, then $2 \in \mathrm{ind}(H(P,f))$.*

Let us see an immediate corollary of this theorem.

Corollary. *For C^1-generic diffeomorphisms of a three-dimensional closed smooth manifold, if there exists a wild homoclinic class that contains two hyperbolic periodic points and one of them is volume-expanding and the other is volume-contracting, then $\mathrm{ind}(H(P,f)) = \{1,2\}$.*

Thus, under some assumptions on Jacobian, we can give a positive answer to Question 1.

This theorem can be interpreted as a qualification of homoclinic classes to be "basic pieces." To explain this, let us review the idea of [BDP], that is, the wilderness of a homoclinic class scatter its hyperbolicity to any direction. Thus, by using their technique, it is not difficult to prove that, under the wilderness, one can create an index bifurcation by an arbitrarily small perturbation. However, this argument does not tell us whether the bifurcation happens inside the homoclinic class or not. If homoclinic classes are to deserve as basic pieces, then it is desireble that a phenomenon which local (linear algebraic) information hints can be observed inside the original homoclinic classes. From this viewpoint, Theorem 1 is seen to be a kind of localization result about index bifurcations.

Let us reintroduce our theorem from a different viewpoint. Aiming at the global understanding of C^1-dynamical systems, Palis suggested the famous Palis conjecture (see [P]). Recently, Bonatti and Díaz asked a stronger version of this conjecture:

Question 2 (Question 1.2 in [BD2]). *Let M be a smooth closed manifold. Does there exist a C^1-open and dense subset $\mathcal{O} \subset \mathrm{Diff}^1(M)$ such that every $f \in \mathcal{O}$ either verifies the Axiom A and the no-cycle condition or has a C^1-robust heterodimensional cycle?*

Our study gives a partial answer to this question. In fact, we prove the following:

Theorem 2. *Let f be a C^1-diffeomorphism of a three-dimensional closed smooth manifold. If f has a wild homoclinic class that contains an index-1 volume-expanding hyperbolic periodic point, then f can be approximated by a diffeomorphism with a robust heterodimensional cycle.*

Note that Wen [W] and Gourmelon [Gou2] already give positive answers to the Palis conjecture under similar hypothesis. The novelty of our result is that we can create *a connection between two saddles.* Roughly speaking, outside Axiom A diffeomorphisms with no-cycles, by linear algebraic arguments and Franks' lemma, it is not difficult to create an index bifurcation with an arbitrarily small perturbation. On the other hand, in general, it is difficult to create a cycle between two saddles, since we need the information about the reccurence between two saddles. Our proof suggests one senario to the creation of the connection between two saddles.

In Theorem 1 and 2, we confined our attention to three-dimensional cases. Let us see what happens in the other dimensions. In dimension two, it is easy to determine $\mathrm{ind}(H(P))$. It is $\{0\}$ (sink), $\{2\}$ (source), or $\{1\}$. We would like to point out that, in dimension two, there is no example of wild homoclinic classes in C^1-topology. If one can construct such a homoclinic class, it is a counterexample of the conjecture of Smale about the density of the Axiom A and no-cycle condition diffeomorphisms (see [S]).

Let us consider what happens in dimensions larger than three. Intuitively speaking, by the idea of [BDP], it seems that the wilderness of a homoclinic class $H(P)$ makes its index set large, since the wilderness scatters the hyperbolicity. So, it is natural to expect that if $H(P)$ is a wild homoclinic class, $\mathrm{ind}(H(P)) = [1, m-1]$, where m is the dimension of the ambient manifold and $[1, m-1]$ is the interval of natural numbers. However, in general, this is not true. In appendix, we give an example that says this idea has some limitation in higher dimensional cases. Let us state the precise statement of the example.

Theorem 3. *For every four-dimensional smooth closed manifold M, there exists a diffeomorphism f that satisfies the following: There exist a hyperbolic fixed point P of f, C^1-neighborhood $\mathcal{U}$ of f and a residual subset $\mathcal{R}$ of $\mathcal{U}$ such that for every $g \in \mathcal{R}$, $H(P, g)$ does not admit any kind of dominated splittings and* $\mathrm{ind}(H(P,g)) = \{2, 3\}$.

This theorem says that a wild homoclinic class may have an *index deficiency.* More precisely, it is not always true that one can construct a saddle with any prescribed index, even from the C^1-generic viewpoint.

Finally, let us explain the organization of this article. In section 3, we introduce our strategy for the proofs of Theorem 1 and 2. We furnish some part of the proof. We discuss the contents of section 4 and section 5 at the end of the section 3. In appendix, we prove Theorem 3.

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3 Outline of the proof

In this section, we explain the strategy for the proof of Theorem 1 and 2. First, we explain how to prove Theorem 2. The proof is divided into two propositions.

We say that a hyperbolic periodic point P has a *homothetic tangency*, if P has a homoclinic tangency and the restrictions of $df^{\mathrm{per}(P)}(P)$ to $TW^s(P)$ and $TW^u(P)$ are both homotheties (a linear endomorphism of a linear space is said to be a homothety if it is equal to $r\mathrm{Id}$, where r is some real number and Id is the identity map).

Roughly speaking, the first proposition states that, under the robust absence of dominated splittings, one can create a homothethic tangency inside the homoclinic class by an arbitrarily small perturbation.

Proposition 3.1. *Let $f \in \mathrm{Diff}^1(M)$ with $\dim M = 3$ and let P be a volume-expanding index-1 hyperbolic periodic point of f. If $H(P)$ is wild then one can find a C^1-diffeomorphism g arbitrarily C^1-close to f such that the following properties hold.*

1. *There exists a volume-expanding hyperbolic periodic point Q of index 1.*
2. *The differential $dg^{\mathrm{per}(Q)}(Q)$ has only positive and real eigenvalues.*
3. *Two periodic points $P(g)$ and Q are homoclinically related.*
4. *Q has a homothetic tangency.*

The second proposition says that from a homothetic tangency one can create a heterodimensional cycle by an arbitrarily C^1-small perturbation.

Proposition 3.2. *Let $f \in \mathrm{Diff}^1(M)$ with $\dim M = 3$, and let Q be a volume-expanding hyperbolic periodic point of f with $\mathrm{ind}(Q) = 1$. If $dg^{\mathrm{per}(Q)}(Q)$ has only positive and real eigenvalues and Q has a homothetic tangency, then one can find a C^1-diffeomorphism g arbitrarily C^1-close to f such that the following properties hold.*

1. *There exists a hyperbolic periodic point R of g with $\mathrm{ind}(R) = 2$.*
2. *Let $Q(g)$ be the continuation of Q of g. Then g has a heterodimensional cycle associated with two periodic points $Q(g)$ and R.*

We need the following result in [BDK].

Lemma 3.1 (Theorem 1 in [BDK]). *Let f be a C^1-diffeomorphism with a heterodimansional cycle associated to saddles Q and R with $\mathrm{ind}(Q) - \mathrm{ind}(R) =$*

± 1. *Suppose that at least one of the homoclinic classes of these saddles is non-trivial. Then there are diffeomoropshims g arbitrarily C^1-close to f with robust heterodimensionalc cycles associated to two transitive hyperbolic sets containing the continuations $Q(g)$ and $R(g)$.*

We can summarize the results of Proposition 3.1, Proposition 3.2, and Lemma 3.1 as follows:

Proposition 3.3. *Let $f \in \mathrm{Diff}^1(M)$ with $\dim M = 3$, and let P be a volume-expanding index-1 hyperbolic periodic point of f. If $H(P)$ is wild, then one can find a C^1-diffeomorphism g arbitrarily C^1-close to f and a hyperbolic periodic point R of g with $\mathrm{ind}(R) = 2$ such that g has a robust heterodimensionalc cycle associated to two transitive hyperbolic sets containing the continuations $P(g)$ and $R(g)$.*

It is clear that Proposition 3.3 implies Theorem 2.

Let us concentrate on the proof of Theorem 1. For the proof, we need some generic properties about C^1-diffeomorphisms.

($\mathcal{R}_1$) We denote the set of Kupka-Smale diffeomorphisms by $\mathcal{R}_1$ and the set of Kupka-Smale diffeomorphisms such that none of the periodic points are conservative by $\mathcal{R}'_1$. These are residual sets. One can prove the genericity of $\mathcal{R}'_1$ by modifying the usual proof of Kupka-Smale theorem (see [R], for example) with the fact that the volume-conservativeness of a hyperbolic periodic point is a closed property in the C^1-topology.

($\mathcal{R}_2$) By $\mathcal{R}_2$ we denote the set of diffeomorphisms f such that following holds: Any chain reccurence class C containing a hyperbolic periodic point P satisfies $C = H(P)$. For the proof, see [BC].

($\mathcal{R}_3$) By $\mathcal{R}_3$ we denote the set of the diffeomorphisms f satisfying the follwoing: Let P and Q be hyperbolic periodic points of f and $\mathrm{ind}(P) = \mathrm{ind}(Q)$. If $H(P) \cap H(Q) \neq \emptyset$ then P and Q are homoclinically related. We will give the proof of the genericity of $\mathcal{R}_3$ later.

The following lemma is easy to prove, so the proof is left to the reader.

Lemma 3.2. *Let P and Q be hyperbolic periodic points and assume there exists a heterodimensional cycle associated to two transitive hyperbolic invariant sets Γ and Σ such that Γ contains P and Σ contains Q. Then P and Q belong to the same chain recurrence class.*

Let us give the proof of Theorem 1 using Proposition 3.3.

Proof of Theorem 1. For $f \in \mathrm{Diff}^1(M)$, let $\mathrm{Per}_h^N(f)$ be the set of hyperbolic periodic points whose periods are less than N. For every $f \in \mathcal{R}'_1$, we take an open neighborhood $\mathcal{U}_N(f) \subset \mathrm{Diff}^1(M)$ of f such that every $g \in \mathcal{U}_N(f)$ satisfies the following conditions:

1. For each $P_i \in \mathrm{Per}_h^N(f)$, one can define the continuation $P_i(g)$.

2. There is no creation of periodic points with period less than N, in other words, the set $\{P_i(g)\}$ exhausts the periodic points whose periods are less than N of g.

3. For each $P_i \in \mathrm{Per}_h^N(f)$, the signature of $\log|J(P_i(g))|$ is independent of the choice of g.

Since f is Kupka-Smale, $\mathrm{Per}_h^N(f)$ is a finite set. From now on, for each $\mathcal{U}_N(f)$, we are going to create an open and dense set $\mathcal{O}_N \subset \mathcal{U}_N(f)$ as follows. First, for each $g \in \mathcal{U}_N(f)$, we define an open set $\mathcal{V}_{i,N}[g]$ as follows: if P_i is a volume-contracting periodic point (note that by assumption P_i cannot be conservative) or arbitrarily C^1-close to g one can find a C^1-diffeomorphism g' such that $H(P_i, g')$ admits a dominated splitting, then let $\mathcal{V}_{i,N}[g]$ be the empty set. Otherwise let $\mathcal{V}_{i,N}[g]$ be the open set obtained by Proposition 3.3. Then, put $\mathcal{V}_{i,N}(f) := \cup_{g\in\mathcal{U}_N(f)}\mathcal{V}_{i,N}[g]$. Since each $\mathcal{V}_{i,N}[g]$ are open, $\mathcal{V}_{i,N}(f)$ is open. Let us put $\mathcal{W}_{i,N}(f) := \mathcal{U}_N(f) \setminus \overline{\mathcal{V}_{i,N}(f)}$ and $\mathcal{O}_{i,N}(f) := \mathcal{V}_{i,N}(f) \cup \mathcal{W}_{i,N}(f)$. By construction, $\mathcal{O}_{i,N}(f)$ is open and dense in $\mathcal{U}_N(f)$ and for every $g \in \mathcal{O}_{i,N}(f)$, if $P_i(g)$ is volume-expanding and $H(P_i, g)$ is wild, then g has a robust cycle associated with transitive hyperbolic sets containing $P_i(g)$ and some hyperbolic periodic point of index 2.

Now, put $\mathcal{O}_N(f) := \cap_i \mathcal{O}_{i,N}(f)$. Since each $\mathcal{O}_{i,N}(f)$ is open and dense, so is $\mathcal{O}_N(f)$ in $\mathcal{U}_N(f)$ and the set $\mathcal{O}_N := \cup_{f\in\mathcal{R}'_1}\mathcal{O}_N(f)$ is an open and dense subset of $\mathrm{Diff}^1(M)$. Then, put $\mathcal{R}_* := \cap_N \mathcal{O}_N$. By Baire's category theorem, this is a residual subset of $\mathrm{Diff}^1(M)$ and satisfies the following property: If $f \in \mathcal{R}_*$ and there exists a hyperbolic and volume-expanding periodic point P whose homoclinic class is wild, then one can find a robust cycle associated with P and some periodic point with index 2.

Finally, put $\mathcal{R}_{**} = \mathcal{R}_* \cap \mathcal{R}_2 \cap \mathcal{R}_3$. We show that every $f \in \mathcal{R}_{**}$ satisfies the conclusion of Theorem 1. Suppose there is a hyperbolic periodic point P of f such that $H(P)$ is a wild homoclinic class containing a volume-expanding hyperbolic periodic point Q with $\mathrm{ind}(Q) = 1$. If $\mathrm{ind}(P) = 2$, then we have the conclusion. So let us assume $\mathrm{ind}(P) = 1$. Since $f \in \mathcal{R}_3$, P and Q are homoclinically related. For $H(P)$ is wild, so is $H(Q)$. Then, by the definition of $\mathcal{R}_*$, f has a heterodimensional cycle associated to two transitive hyperbolic set such that one of them contains Q and the other contains a hyperbolic periodic point R of index 2. By Lemma 3.2, Q and R belong to the same chain recurrence class. Since $f \in \mathcal{R}_2$, we have $R \in H(Q) = H(P)$. □

Now, we give the proof of genericity of $\mathcal{R}_3$. For the proof, we need a generic property and a C^1-perturbation lemma.

Lemma 3.3 (Lemma 2.1 in [ABCDW]). *There is a residual subset $\mathcal{R}_4 \subset \mathrm{Diff}^1(M)$ such that, for every diffeomorphism $f \in \mathcal{R}_4$ and every pair of saddles $P(f)$ and $Q(f)$ of f, there is a neighborhood $\mathcal{U}_f$ of f in $\mathcal{R}_4$ such that either $H(P, g) = H(Q, g)$ for all $g \in \mathcal{U}_f$, or $H(P, g) \cap H(Q, g) = \emptyset$ for all $g \in \mathcal{U}_f$.*

Lemma 3.4 (Lemma 2.8 in [ABCDW]). *Let P be a hyperbolic periodic point of a C^1-diffeomorphism f. Consider a homoclinic class $H(P, f)$ and any saddle*

$Q \in H(P, f)$. *Then there is g arbitrarily C^1-close to f such that $W^u(P, g)$ and $W^s(Q, g)$ have an intersection.*

Proof of the genericity of $\mathcal{R}_3$. For every $f \in \mathcal{R}_1 \cap \mathcal{R}_4$, we take an open neighborhood $\mathcal{U}'_N(f) \subset \mathcal{R}_1 \cap \mathcal{R}_4$ of f such that for every $g \in \mathcal{U}'_N(f)$ we have the following conditions: For each $P_i \in \mathrm{Per}_h^N(f)$, one can define the continuation $P_i(g)$ and these continuations exhaust periodic points whose periods are less than N of g. From now on, for each $\mathcal{U}'_N(f)$, we are going to create an open and dense set $O_N(f) \subset \mathcal{U}'_N(f)$ as follows. First, for each $g \in \mathcal{U}'_N(f)$ and $i \neq j$, let us define an open set $\mathcal{V}_{(i,j),N}[g] \subset \mathcal{U}'_N(f)$ as follows: if $P_i(g)$ and $P_j(g)$ are not homoclinically related, then $\mathcal{V}_{(i,j),N}[g]$ is the empty set. Otherwise $\mathcal{V}_{(i,j),N}[g] \subset \mathcal{U}'_N(f)$ is a non-empty open set satisfying the following: If $h \in \mathcal{V}_{(i,j),N}[g]$ then $P_i(h)$ and $P_j(h)$ are homoclinically related. Then, put $\mathcal{V}_{(i,j),N}(f) := \cup_{g \in \mathcal{U}'_N(f)} \mathcal{V}_{(i,j),N}[g]$, and $\mathcal{W}_{(i,j),N}(f) := \mathcal{U}'_N(f) \setminus \overline{\mathcal{V}_{(i,j),N}(f)}$. We put $\mathcal{O}_{(i,j),N}(f) := \mathcal{V}_{(i,j),N}(f) \cup \mathcal{W}_{(i,j),N}(f)$. By the construction, $\mathcal{O}_{(i,j),N}(f)$ is open and dense in $\mathcal{U}'_N(f)$.

Let us see that for $h \in \mathcal{O}_{(i,j),N}$, the following holds: If $P_i \in H(P_j, h)$, then P_i and P_j are homoclinically related. If $h \in \mathcal{V}_{(i,j),N}(f)$, this is clear. We show that for $h \in \mathcal{W}_{(i,j),N}$, $P_i \notin H(P_j, h)$. Indeed, if $P_i \in H(P_j, h)$, by applying Lemma 3.3, we can take a neighborhood $\mathcal{C}(h) \subset \mathcal{U}'_N(f)$ of h such that for all $h' \in \mathcal{C}(h)$, $H(P_i, h') = H(P_j, h')$. Then, Lemma 3.4 tells us there exists h_1 arbitrarily C^1-close to h such that $W^u(P_i, h_1)$ and $W^s(P_j, h_1)$ has non-empty intersection. By giving arbitrarily small perturbation to h_1, we can find a diffeomorphism h_2 such that $W^u(P_i, h_2)$ and $W^s(P_j, h_2)$ has non-empty transversal intersection. Since having non-empty transversal intersection between invariant manifolds is a C^1-open property, we can find $h_3 \in \mathcal{C}(h)$ arbitrarily C^1-close to h_2 such that $W^u(P_i, h_2)$ and $W^s(P_j, h_2)$ has non-empty transversal intersection (note that h_1, h_2 can fail to belong to $\mathcal{C}(h)$). By a similar argument, we can find $h_4 \in C(h)$ arbitrarily C^1-close to h_3 such that $P_i(h_4)$ and $P_j(h_4)$ are homoclinically related. This is a contradiction, since we take $h \in \mathcal{W}_{(i,j),N}(f) = \mathcal{U}'_N(f) \setminus \overline{\mathcal{V}_{(i,j),N}(f)}$ and h_4 can be found arbitrarily C^1-close to h.

We put $\mathcal{O}_N(f) := \cap_{i \neq j} \mathcal{O}_{(i,j),N}(f)$. Since $\#\mathrm{Per}_h^N(f)$ is finite, $\mathcal{O}_N(f)$ is an open and dense subset of $\mathcal{U}'_N(f)$. Now we define $\mathcal{U}'_N := \cup_{f \in \mathcal{R}_1} \mathcal{O}_N(f)$. This is an open and dense subset of $\mathrm{Diff}^1(M)$. Finally, we put $\mathcal{R}_{***} := \cap_N \mathcal{U}'_N$. Then, by construction, every diffeomorphism in $\mathcal{R}_{***}$ satisfies the desired condition and by Baire's category theorem this set is residual in $\mathrm{Diff}^1(M)$. □

In the rest of this section, we discuss the proofs of Proposition 3.1 and 3.2. In Section 4 we give the proof of Proposition 3.1. It is done by three techniques. The first one is the linear algebraic arguments developed in [BDP]. We conbine this technique with the second one, the Franks' lemma that preserves the invariant manifolds developed in [Gou1]. The third one is the striking result given by Gourmelon [Gou2] for the creation of a homoclinic tangency in a homoclinic class not admitting the dominated splittings.

In Section 5, we give the proof of Proposition 3.2. The proof is divided into two steps. The first one is the reduction to affine dynamics. The second one is the investigation of the reduced dynamics and this involves some calculation.

4 Creation of a homothetic tangency

In this section, we prove Proposition 3.1.

4.1 Strategy for the proof of Proposition 3.1

An important step to Proposition 3.1 is the following proposition.

Proposition 4.1. *Let $f \in \mathrm{Diff}^1(M)$ with $\dim M = 3$, and let P be a volume-expanding hyperbolic periodic point of f such that $\mathrm{ind}(P) = 1$ and $H(P)$ is wild. Then one can find a C^1-diffeomorphism g arbitrarily C^1-close to f such that the following holds:*

1. *There exists a volume-expanding hyperbolic periodic point Q of index 1.*
2. *The differential $dg^{\mathrm{per}(Q)}(Q)$ has only positive and real eigenvalues.*
3. *Two periodic points $P(g)$ and Q are homoclinically related.*
4. *The differential $dg^{\mathrm{per}(Q)}(Q)$ restricted to the stable direction of Q is a homothety.*

By this proposition together with the following result by Gourmelon, we can prove Proposition 3.1.

Lemma 4.1 (Theorem 1.1 in section 6 of [Gou2]). *If the homoclinic class $H(P, f)$ of a saddle point P for f is not trivial and does not admit a dominated splitting of the same index as P. Then, there is an arbitrarily small perturbation g of f, that perserves the dynamics on a neighborhood of P, and such that there is a homoclinic tangency associated to P.*

Let us give the proof of Proposition 3.1 assuming above two results.

Proof of Proposition 3.1. Under the hypothesis of Proposition 3.1, Proposition 4.1 tells us that we get f_1 arbitrarily C^1-close to f such that f_1 has a hyperbolic periodic points $Q(f_1)$ satisfying all the conclusions of Proposition 4.1. By taking f_1 sufficiently close to f, we can assume $H(P, f_1) = H(Q, f_1)$ does not admit dominated splittings. Then, by applying Lemma 4.1 to $Q(f_1)$ we get f_2 arbtirarily close to f_1 such that $Q(f_1) = Q(f_2)$ exhibits a homoclinic tangency. Since the perturbation preserves the local dynamics of $Q(f_1)$, $df_2^{\mathrm{per}(Q(f_2))}(Q(f_2)) = df_1^{\mathrm{per}(Q(f_1))}(Q(f_1))$ and $Q(f_2)$ is volume-expanding. Thus we have created a homothetic tangency. Furthermore, since $P(f_1)$ and $Q(f_1)$ are homoclinically related, if we take f_2 sufficiently close to f_1, we know $P(f_2)$ and $Q(f_2)$ are homoclinically related, too. Note that we can take f_2 arbitrarily close to f because f_1 can be found arbitrarily close to f. Now the proof is completed. □

Thus let us concentrate on the proof of Proposition 4.1. We divide the proof into two lemmas.

Lemma 4.2. *Let $f \in \mathrm{Diff}^1(M)$ with $\dim M = 3$, and let P be a volume-expansive hyperbolic periodic point of f and $\mathrm{ind}(P) = 1$. If $H(P)$ is wild, then C^1-arbitrarily close to f one can find a C^1-diffeomorphism g such that following holds: There exists a volume-expansive hyperbolic periodic point $Q(g)$ whose index is 1 such that $P(g)$ and $Q(g)$ are homoclinically related, and the restriction of $dg^{\mathrm{per}(Q(g))}(Q(g))$ to the stable direction has two complex eigenvalues.*

Lemma 4.3. *Let $f \in \mathrm{Diff}^1(M)$ with $\dim M = 3$, and let P be an index-one volume-expansive hyperbolic periodic point of f and the restriction of $df^{\mathrm{per}(P)}$ have two contracting complex eigenvalues. If $H(P)$ is non-trivial, then C^1-arbitrarily close to f one can find a C^1-diffeomorphism g such that following holds: There exists a volume-expansive hyperbolic periodic point $Q(g)$ whose index is 1 such that $P(g)$ and $Q(g)$ are homoclinically related, $dg^{\mathrm{per}(Q(g))}(Q(g))$ has only positive and real eigenvalues, and the restriction of $dg^{\mathrm{per}(Q(g))}(Q(g))$ to the stable direction is a homothety.*

Let us prove Proposition 4.1 assuming Lemma 4.2 and 4.3.

Proof of Proposition 4.1. Suppose f, P are given as is in the hypothesis of Proposition 4.1. First, by Lemma 4.2 arbitrarily close to f we can find f_1 such that there exists a volume expansive hyperbolic periodic point $Q(f_1)$ whose index is 1 such that $P(f_1)$ and $Q(f_1)$ are homoclinically related, and the restriction of $df_1^{\mathrm{per}(Q(f_1))}(Q(f_1))$ has two complex eigenvalues. Second, by applying Lemma 4.3 to f_1 and $Q(f_1)$, we can find f_2 in any neighborhood of f_1 such that there exists a volume expansive hyperbolic periodic point $R(f_2)$ whose index is 1, $Q(f_2)$ and $R(f_2)$ are homoclinically related and the restriction of $df_2^{\mathrm{per}(R(f_2))}(R(f_2))$ to the stable direction is a homothety. Note that if we take f_2 sufficiently close to f_1, then $P(f_2)$ and $Q(f_2)$ remain homoclinically related and thus $P(f_2)$ and $R(f_2)$ are homoclinically related, too. Since f_1 can be constructed arbitrarily close to f and f_2 can be constructed arbitrarily close to f_1, f_2 can be constructed arbitrarily close to f. This ends the proof of our proposition. $\square$

In subsection 4.2, we prepare some techniques for the proof of Lemma 4.2 and give the proof of Lemma 4.2. In subsection 4.3, we prove Lemma 4.3 and in subsection 4.4 we give the proof of Lemma 4.7 that is needed to prove Lemma 4.2.

4.2 Proof of Lemma 4.2

In this subsection, we prove Lemma 4.2 assuming some results.

First, we collect some results for the the proof of Lemma 4.2. We start from the Franks' lemma (see appendix A of [BDV]).

Lemma 4.4 (Franks' lemma). *Let f be a C^1-diffeomorphism defined on a closed manifold M and consider any $\delta > 0$. Then there is $\varepsilon > 0$ such that, given* *any finite set $\Sigma \subset M$, any neighborhood U of Σ, and any linear maps $A_x : T_xM \to T_{f(x)}M$ $(x \in \Sigma)$, such that A_x is ε-close to $df(x)$, there exists a C^1-diffeomorphism g that is δ-close to f in the C^1-topology, coinciding with f on $M \setminus U$ and on Σ, and $dg(x) = A_x$ for all $x \in \Sigma$.*

We introduce the Franks' lemma that preserves invariant manifolds [Gou1]. To state it clearly, we prepare some notations. For a hyperbolic periodic point X of a diffeomorphism f, we consider the space of linear cocycles over $\mathcal{O}(X)$ (remember that $\mathcal{O}(X)$ is the orbit of X). Let us denote this space by $C(X)$, i.e., $C(X)$ is the set of maps $\sigma\colon TM|_{\mathcal{O}(X)} \to TM|_{\mathcal{O}(X)}$ such that for all $i \in \mathbb{Z}$, $\sigma(f^i(X)) = \sigma(f^i(X), \cdot)$ is a linear isomorphism from $T_{f^i(X)}M$ to $T_{f^{i+1}(X)}M$. We denote the cocycle given by the restriction of df to $TM|_{\mathcal{O}(X)}$ by the same symbol df.

We define a metric on this space as follows. For $\sigma_1, \sigma_2 \in C(X)$, the distance betweeen σ_1 and σ_2 (denoted as $\mathrm{dist}(\sigma_1, \sigma_2)$) is defined to be the following:

$$\max\left\{\max_{x\in\mathcal{O}(X)} ||\sigma_1(x) - \sigma_2(x)||,\ \max_{x\in\mathcal{O}(X)} ||(\sigma_1(x))^{-1} - (\sigma_2(x))^{-1}||\right\}.$$

For $\sigma \in C(X)$ we denote by $\tilde{\sigma}$ the first return map of σ, i.e., the linear endomorphism of T_XM given by $\sigma(f^{\mathrm{per}(X)-1}(X)) \circ \cdots \circ \sigma(f(X)) \circ \sigma(X)$. We define the eigenvalues of σ as the eigenvalues of $\tilde{\sigma}$. We say that σ is hyperbolic if none of the eigenvalues of $\tilde{\sigma}$ has its absolute value equal to one. Note that the set of the hyperbolic cocycles forms an open set in $C(X)$. We sometimes deal with a continuous path $\gamma(t)$ in $C(X)$, where $\gamma(t)$ is a continuous map from $[0,1]$ to $C(X)$. For a path $\gamma(t)$, we define its diameter (denoted $\mathrm{diam}(\gamma(t))$) to be the number $\max_{0\le s,t\le 1} \mathrm{dist}(\gamma(s), \gamma(t))$.

Let U be a neighborhood of $\mathcal{O}(X)$. For $x \in W^s(X) \cap (M \setminus U)$, we define $\alpha(x)$ by the least number such that $f^{\alpha(x)}(x) \in U$ holds. We define the *stable manifold of X outside U* (denoted $W^s_{\mathrm{loc}\setminus U}(X)$) by the set of points that never leave U once they enter U, more precisely,

$$W^s_{\mathrm{loc}\setminus U}(X) := \{x \in W^s(X) \cap (M \setminus U) \mid \forall n \ge \alpha(x),\ f^n(x) \in U\}.$$

Let g be a diffeomorphism so close to f that we can define the continuation $X(g)$ of X for g. We say that *g preserves the stable manifold of f outside U* if $W^s_{\mathrm{loc}\setminus U}(X, g) \supset W^s_{\mathrm{loc}\setminus U}(X, f)$. Similarly, we can define the unstable manifold outside U.

Now let us state the precise statement of the lemma.

Lemma 4.5 (Gourmelon's Franks' lemma). *Let f be a C^1-diffeomorphism of M and X be a hyperbolic periodic point of f. Suppose that there exists a continuous path $\{\gamma(t) \mid 0 \le t \le 1\}$ in $C(X)$ satisfying the following:*

1. *For all $i \in \mathbb{Z}$, $\gamma(0)(f^i(X)) = df(f^i(X))$.*

2. *The diameter $\mathrm{diam}(\gamma(t))$ is less than $\varepsilon > 0$.*

3. *For all* $0 \leq t \leq 1$, $\widetilde{\gamma(t)}$ *is hyperbolic (hence the dimensions of stable and unstable spaces are constant).*

Then, given neighborhood U *of* $\mathcal{O}(X)$ *there exists a* C^1*-diffeomorphism* g *that is* ε*-*C^1*-close to* f *satisfying the following properties:*

1. *For all* i, $f^i(X) = g^i(X)$. *Especially,* X *is a periodic point for* g.

2. *As a linear cocycle in* $C(X)$, $dg = \gamma(1)$. *Especially,* X *is a hyperbolic periodic point of* g *and* $\mathrm{ind}(X, f) = \mathrm{ind}(X, g)$.

3. *The support of* g *is contained in* U, *i.e.,* $\{x \in M \mid f(x) \neq g(x)\} \subset U$.

4. g *preserves locally the stable and unstable manifolds of* X *outside* U.

Proof. Apply Theorem 2.1 in [Gou1] putting $I = \{\dim M - \mathrm{ind}(X)\}$ and $J = \{\mathrm{ind}(X)\}$. $\square$

As a consequence of Lemma 4.5, we get the following lemma.

Lemma 4.6. *In addition to the hypotheses of Lemma 4.5, suppose that the following property holds:*

4. *There is a hyperbolic periodic point* Y *that is homoclinically related to* X.

Then, there is a (small) neighborhood V *of* $\mathcal{O}(X)$ *and a* C^1*-diffeomorphism* h ε*-*C^1*-close to* f *satisfying all the conclusions of Lemma 4.5 and the following property:*

5. $X(h)$ *is homoclinically related to* $Y(h)$.

Let us prove Lemma 4.6 by Lemma 4.5.

Proof. We fix an open neighborhood W of $\mathcal{O}(Y)$ that has no intersection with $\mathcal{O}(X)$. Take a point $a \in W^s(X) \pitchfork W^u(Y)$. Since $a \in W^s(X)$, by replacing a with $f^n(a)$ for some $n > 0$ if necessary, we can assume $a \notin W$. Furthermore, by shrinking W if necessary and using the fact $a \in W^u(Y)$, we can assume the following condition holds: Let $l_0 > 0$ be the least number that satisfies $f^{-l_0}(x) \in W$. Then there exists a small neighborhood $D^u_Y(a)$ of a in $W^u(Y)$ such that $f^{-l}(D^u_Y(a)) \subset W$ for all $l \geq l_0$ and $f^{-l}(D^u_Y(a)) \cap W = \emptyset$ for all $0 \leq l < l_0$.

Similarly, we take a point $b \in W^s(Y) \pitchfork W^u(X)$ such that $b \notin W$ and the following holds: Let $l_1 > 0$ be the least number that satisfies $f^{l_1}(b) \in W$. There exists a small neighborhood $D^s_Y(b)$ of b in $W^s(Y)$ such that $f^l(D^s_Y(b)) \subset W$ for all $l \geq l_1$ and $f^l(D^s_Y(b)) \cap W = \emptyset$ for all $0 \leq l < l_1$.

Similar argument gives us an open neighborhood V of $\mathcal{O}(X)$ satisfying the following conditions:

1. For all $n \geq 0$, $f^{-n}(D^u_Y(a)) \cap V = \emptyset$, especially $a \notin V$.

2. For all $n \geq 0$, $f^n(D^s_Y(b)) \cap V = \emptyset$, especially $b \notin V$.

3. Let $k_0 > 0$ be the least number that satisfies $f^{k_0}(a) \in V$. Then there exists a small neighborhood $D^s_X(a)$ of a in $W^s(X)$ such that $f^k(D^s_X(a)) \subset V$ for all $k \geq k_0$ and $f^k(D^s_Y(b)) \cap V = \emptyset$ for all $0 \leq k < k_0$.

4. Let $k_1 > 0$ be the least number that satisfies $f^{-k_1}(b) \in V$. Then there exists a small neighborhood $D^u_X(b)$ of b in $W^u(X)$ such that $f^{-k}(D^s_X(a)) \subset V$ for all $k \geq k_1$ and $f^{-k}(D^s_X(a)) \cap V = \emptyset$ for all $\leq k < k_1$.

By applying Lemma 4.5 to X, f, and V, we can construct an ε-C^1-close perturbation h of f whose support is contained in V such that satisfies all the conclusions of Lemma 4.5. We show $X(h)$ is homoclinically related to $Y(h)$. To see this, first we check $a \in W^s(X,h) \pitchfork W^u(Y,h)$. Since for all $n \geq 0$, $f^{-n}(D^u_Y(a)) \cap V = \emptyset$ and the support of h is contained in V, $D^u_Y(a)$ is contained in $W^u(Y,h)$. Second, by the condition of $D^s_X(a)$, it is contained in the stable manifold outside V. Thus it is contained in $W^s(X,h)$ and now we know $a \in W^s(X,h) \pitchfork W^u(Y,h)$. Similarly, we get $b \in W^s(Y,h) \pitchfork W^u(X,h)$ and finished the proof. $\square$

We collect some results about linear algebra on linear cocycles from [BDP]. The first one says, in dimension two, the absence of the domination implies the creation of complex eigenvalues. The origin of this kind of arguments goes back to [M].

Lemma 4.7. *For any $K > 0$ and $\varepsilon > 0$, the following holds: Let (Σ, f, E, A) be a two-dimensional diagonalizable periodic linear cocycle with positive eigenvalues. If (Σ, f, E, A) is bounded by K and does not admit dominated splittings, then there exists a hyperbolic periodic point X and a path $\{\gamma(t) \mid 0 \leq t \leq 1\}$ in $C(X)$ such that the following holds:*

1. $\gamma(0) = A|_{\mathcal{O}(X)}$.

2. $\mathrm{diam}(\gamma(t)) < \varepsilon$.

3. $\det \widetilde{\gamma(t)}$ *is independent of t.*

4. *Let $\lambda_m(t) \leq \lambda_b(t)$ be the absolute value of the eigenvalues of $\widetilde{\gamma(t)}$. Then for $s < t$ we have $\lambda_m(s) \leq \lambda_m(t)$ and $\lambda_b(s) \geq \lambda_b(t)$.*

5. $\widetilde{\gamma(1)}$ *has two complex eigenvalues.*

We give the proof of this lemma in the next subsection.

The next lemma is used to create complex eigenvalues from a linear map which has eigenvalues with multiplicity two.

Lemma 4.8. *Let A be a linear endmorphism on a two-dimensional normed linear space such that the eigenvalues of A is positive real number λ with multiplicity 2. Then there exists B arbitrarily close to A such that B has two complex eigenvalues.*

Proof. By taking the Jordan cannonical form of A, we can assume that A has the following matrix form:

$$\begin{pmatrix} \lambda & t \\ 0 & \lambda \end{pmatrix}.$$

When $t = 0$, the matrix

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \lambda & t \\ 0 & \lambda \end{pmatrix}.$$

has two complex eigenvalues for sufficiently small $\alpha > 0$ and α can be taken arbitrarily small. If $t \neq 0$, the matrix

$$\begin{pmatrix} \lambda & t \\ -\alpha/t & \lambda \end{pmatrix}.$$

has complex eigenvalue for all $\alpha > 0$. Since α can be taken arbitrarily small, we finished the proof. □

The following lemmas enables us to"lift up" the perturbation on some subcocycle or quotient cocycle to the perturbation on the original cocycle.

Lemma 4.9. *Given $\varepsilon > 0$, a linear cocycle (Σ, f, E, A) and its invariant subcocycle $(\Sigma, f, F, A|_F)$, where $A|_F$ denotes the restriction of A to F, the following hold:*

1. *Let (Σ, f, F, B) be a cocycle with $\mathrm{dist}(A|_F, B) < \varepsilon$. Then there exists a linear cocycle $(\Sigma, f, E, \tilde{B})$ such that $\mathrm{dist}(A, \tilde{B}) < \varepsilon$ and $A/F = \tilde{B}/F$, where A/F denotes the bundle map derives from A on E/F.*

2. *If $(\Sigma, f, E/F, B)$ is a cocycle with $\mathrm{dist}(A/F, B) < \varepsilon$. Then there exists a linear cocycle $(\Sigma, f, E, \hat{B})$ such that $\mathrm{dist}(A, \hat{B}) < \varepsilon$, $\hat{B}$ leaves F invariant and $A|_F = \hat{B}|_F$.*

Proof. See Lemma 4.1 in [BDP]. □

Lemma 4.10. *Let $E_1 \oplus E_2 \oplus E_3$ be an invariant splitting of a linear cocycle. If E_1 is not dominated by $E_2 \oplus E_3$, then one of the following holds:*

1. *E_1 is not dominated by E_2.*

2. *E_1/E_2 is not dominated by E_3/E_2.*

Proof. See Lemma 4.4 in [BDP]. □

We prepare some lemmas which enables us to reduce our problem to specific sets. For a homoclinic class $H(P)$, we denote by $\mathrm{per}_{+,\mathbb{R}}(H(P))$ the set of the volume-expansive hyperbolic periodic points that have the differentials with distinct positive real eigenvalues and are homolcinically related to P.

Lemma 4.11. *For given $f \in \mathrm{Diff}^1(M)$ and a volume-expanding hyperbolic periodic point P, if $H(P)$ is non-trivial, one can find $g \in \mathrm{Diff}^1(M)$ arbitrarily C^1-close to f such that* $\mathrm{per}_{+,\mathbb{R}}(H(P,g))$ *is dense in* $H(P,g)$.

Proof. See Proposition 2.3 in [ABCDW] and Remark 4.17 in [BDP]. □

Lemma 4.12. *Let $f \in \mathrm{Diff}^1(M)$ and P be a hyperbolic periodic point of f. Suppose $H(P)$ does not admit dominated splittings. Then any dense f-invariant subset $\Sigma \subset H(P)$ does not admit dominated splittings.*

Proof. See Lemma 1.4 in [BDP]. □

Let us start from the proof of Lemma 4.2.

Proof of Lemma 4.2. By the quotations we prepared above, large part of our proof is already finished. Let us see how to combine them to give the proof.

Suppose that f and P are given as in the hypothethis of Lemma 4.2. Lemma 4.11 implies there exists f_1 arbitrarily C^1-close to f such that $\mathrm{per}_{+,\mathbb{R}}(H(P, f_1))$ is dense in $H(P, f_1)$. Let us consider the periodic linear cocycle derives from df_1 on $\mathrm{per}_{+,\mathbb{R}}(H(P, f_1))$. Since each periodic point has only real and positive distinct eigenvalues, we can define a splitting $E_1 \oplus E_2 \oplus E_3$ on this cocycle so that corresponidng eigenvalues are in the increasing order. Note that since $H(P, f_1)$ do not admit dominated splitting, by Lemma 4.12, E_1 is not dominated by $(E_2 \oplus E_3)$. Then Lemma 4.10 says that either E_1 is not dominated by E_2 or E_1/E_2 is not dominated by E_3/E_2. We show that we can create the periodic point Q in both cases.

Let us consider the first case, where E_1 is not dominated by E_2. We fix $\varepsilon > 0$. Since M is compact, the linear cocycle df_1 restricted to $E_1 \oplus E_2$ is bounded. By applying Lemma 4.7 to this cocycle, we get a periodic point $Q \in \mathrm{per}_{+,\mathbb{R}}(H(P, f_1))$ and a path $\gamma(t)$ satisfying the conclusions in Lemma 4.7. We can assume $Q \neq P$ by letting ε sufficiently small. Since $\lambda_m(t) \leq \lambda_b(t) \leq \lambda_b(0) < 1$, there is no index bifurcaion during the perturbation.

Then Lemma 4.9 tells us there exists a path $\Gamma(t) \subset C(Q)$ such that $X = Q$, $Y = P$ and $\Gamma(t)$ satisfies all the hypotheses of Lemma 4.6. Hence by applying Lemma 4.6 we get a C^1-diffeomorphism f_2 that is ε-C^1-close to f_1 such that $P(f_2)$ and $Q(f_2)$ satisfy all the conditions we need. Since f_1 can be arbitrarily close to f and ε can be taken arbitrarily small the proof is completed in this case.

Let us see the case where E_1/E_2 is not dominated by E_3/E_2. Take $\varepsilon > 0$. Since df is bounded, the cocycle induced on the quotient bundle $E_1/E_2 \oplus E_3/E_2$ is also bounded. By using Lemma 4.7 to this cocycle, we get a periodic point $Q \in \mathrm{per}_{+,\mathbb{R}}(H(P, f_1))$, a path $\gamma(t)$ with $\mathrm{diam}(\gamma(t)) < \varepsilon$ such that they satifsy all conclusions of the Lemma 4.7. We claim that there is $t_0 \in (0, 1)$ such that $\lambda_m(t_0)$ (not bigger eigenvalue of $\widetilde{\gamma(t_0)}$) is equal to the eigenvalue of $df_1^{\mathrm{per}(Q)}|_{E_2}(Q)$. Let us put the eigenvalues of $df_1^{\mathrm{per}(Q)}(Q)$ for the E_i-direction as μ_i ($i = 1, 2, 3$). Since Q is volume-expanding, $\mu_1\mu_2\mu_3 > 1$ and Q has index 1, $\mu_2 < 1$. Then, we have $\mu_1\mu_3 > \mu_1\mu_2\mu_3 > 1$. Note that $\lambda_m(0) = \mu_1 < \mu_2$. When $t = 1$, by Lemma

4.7, we have $\lambda_m(1) = \lambda_b(1)$. Since $\det \widetilde{\gamma(t)}$ is independent of t and $\mu_1\mu_3 > 1$, we get $1 < \mu_1\mu_3 = \lambda_m(1)\lambda_b(1) = (\lambda_m(1))^2$. Here $\lambda_m(1)$ is positive and therefore $\lambda_m(1) > 1$. Finally, by $\lambda_m(0) = \mu_1 < \mu_2 < 1 < \lambda_m(1)$ and the continuity of $\gamma(t)$, there is $t_0 \in (0,1)$ such that $\lambda_m(t_0) = \mu_2$.

Then, we redefine $\gamma(t)$ as follows: $\gamma(t)$ is equal to the original $\gamma(t)$ when $t \in [0, t_0]$, otherwise $\gamma(t) = \gamma(t_0)$. Note that for this modified $\gamma(t)$, $\lambda_b(t) \geq \lambda_b(t_0) = \mu_1\mu_3/\lambda_m(t_0) = \mu_1\mu_3/\mu_2 > \mu_1\mu_3 > 1$ for all t.

By Lemma 4.9 we can create a path $\Gamma(t) \subset C(X)$ such that $X = Q$, $Y = P$ and $\Gamma(t)$ satisfy all the hypotheses of Lemma 4.6. Applying Lemma 4.6, we take a C^1-diffeomorphism f_2 that is ε-C^1-close to f_1 such that $P(f_2)$ and Q homoclinically related to $P(f_2)$ with differential $df_2^{\mathrm{per}(Q)}(Q)$ restricted to the stable direction of T_QM has the eigenvalue μ_2 with multiplicity 2. Now Lemma 4.8 gives f_3 which is arbitrarily C^1-close to f_2 such that $P(f_3)$ and $Q(f_3)$ satisfy all the conditions we need. $\square$

Remark 4.1. We point out two mistakable arguments in the proof of second case.

1. The argument of Lemma 4.8 is necessary. In general, $df_2^{\mathrm{per}(Q)}(Q)$ restricted to the stable direction of T_QM is not diagonalizable. We only know there are two eigenvectors one in E_2 and the other in E_1/E_2. These facts do not guarantee that we have two linearly independent eigenvectors in $E_1 \oplus E_2$.

2. One may wonder why we can get homoclinic relation between $P(f_3)$ and $Q(f_3)$ just by assuring the closeness of f_2 and f_3. It is because we fix the periodic points and never change till the end. On the contrary, for example, in the proof of Lemma 4.7, we need to change the periodic point we choose to decrease the size of the perturbation.

4.3 Proof of Lemma 4.3

Here we give the proof of Lemma 4.3. Before going into the detail, we give the idea of our proof. We start from a non-trivial homoclinic class $H(P)$ such that $\mathrm{ind}(P) = 1$ and $df^{\mathrm{per}(P)}(P)$ has two complex eigenvalues. First, we create a hyperbolic periodic point Q which is homoclinically related to P and $df^{\mathrm{per}(P)}(P)$ has positive and distinct eigenvalues by small perturbations (this step is carried out inside the proof of Lemma 4.13). Then we pick up a periodic point R_n whose differential restricted to the stable direction is "mixed up" under the influence of $df^{\mathrm{per}(P)}(P)$. Now we perturb the diffeomorphism along the orbit of R_n with the Franks' lemma so that the restriction of $df^{\mathrm{per}(R_n)}(R_n)$ to the stable direction is the homothety. We pick up R_n sufficiently close to Q so that the resulted periodic point has the homoclinic relation with Q.

To demonstrate this naive idea rigorously and clearly, we need the techniques developed in [BDP] about linear cocycles admitting transitions. We do not give the detailed review of [BDP] here. Instead, we give some explanation about how the techniques are used in our argument so that the reader who is not well acquainted with this techniques can understand what we do.

In the statement of Lemma 4.13, we refer to section 1 of [BDP] for the definition and fundamental properties of linear cocycle with transition. Roughly speaking, the transition matrix from a hyperbolic periodic point Q to itself is a matrix of the differential of the "return map" from a neighborhood of Q to itself. See Remark 4.2.

Lemma 4.13 (Lemma 5.4 in [BDP]). *Let (Σ, f, E, A) be a continuous three-dimensional periodic linear cocycle with transition and ε_0 be some positive real number. Assume that there exists $X \in \Sigma$ such that $\mathrm{ind}(X) = 1$ and $A^{\mathrm{per}(X)}(X)$ has two contracting complex eigenvalues. Then for every $0 < \varepsilon_1 < \varepsilon_0$ there is $Q \in \Sigma$ and an ε_1-transition $[t]$ from Q to itself with the following properties:*

1. *There is an ε_1-perturbation $\tilde{A}^{\mathrm{per}(Q)}(Q)$ of $A^{\mathrm{per}(Q)}(Q)$ such that the corresponding matrix has only real positive eigenvalues with multiplicity one. We put the eigenspaces of the matrix as E_1, E_2 and E_3 so that the corresponding eigenvalues are in the increasing order.*
2. *There is an $(\varepsilon_0 + \varepsilon_1)$-perturbation $[\tilde{t}]$ of the transition $[t]$ from Q to itself such that the corresponding matrix $\tilde{T}$ satisfies the following:*
   - $\tilde{T}(E_3) = E_3$.
   - $\tilde{T}(E_1) = E_2$ *and* $\tilde{T}(E_2) = E_1$.

Furthermore, if there exists $Y \in \Sigma$ such that $\det(A^{\mathrm{per}(Y)}(Y))$ is bigger than one, then we can choose the point Q and the perturbation $\tilde{A}$ in the lemma such that $\det(A^{\mathrm{per}(Q)}(Q)) > 1$.

Remark 4.2. The existence of the transition from Q to itself tells us that, given any matrix that is obtained as the product of some $\tilde{A}^{\mathrm{per}(Q)}(Q)$ and some $\tilde{T}$, we can find a periodic orbit whose differential is close to the matrix. In the proof, we use the existence of transition to find the periodic point R_n whose differential is close to a matrix which we want to create.

Let us begin the proof of Lemma 4.3.

Proof of Lemma 4.3. Given f and P as is in the hypothesis of the Lemma 4.3, we fix $\varepsilon > 0$ and a point $P' \in W^s(P) \pitchfork W^u(P) \setminus \mathcal{O}(P)$ (we can take such P' because $H(P)$ is non-trivial) and fix a neighborhood V of $\mathcal{O}(P) \cup \mathcal{O}(P')$ satisfying the following: For every g ε-C^1-close to f, if Z is a periodic point such that $\mathcal{O}(Z, g)$ is contained in V, then $\mathrm{ind}(Z) = \mathrm{ind}(P)$ and P and Z are homoclinically related. We can take such V because the set $\mathcal{O}(P) \cup \mathcal{O}(P')$ is uniformly hyperbolic.

Let us take a uniformly hyperbolic set Σ_1 contained in V and contains P and P' such that Σ_1 admits transition (in practice, we take a generalized horseshoe containing P and P' as Σ_1). We fix a basis of tangent space of each point in Σ_1 and we identify the differential maps between tangent spaces with some matrices (see section 1 of [BDP]).

We denote the set of hyperbolic periodic points in Σ_1 by Σ_2. Let us apply Lemma 4.13 to the periodic linear cocycle $(\Sigma_2, f, TM|_{\Sigma_2}, df)$ with $\varepsilon_0 = \varepsilon/2$ and

$\varepsilon_1 = \varepsilon/4$. Then we get a periodic point $Q \in \Sigma_2$ such that the following properties holds for $df \in C(Q)$ (remember that $C(Q)$ denotes the set of cocycles on Q): df is ε_1-close to a cocycle $\sigma \in C(Q)$ whose first return map $\widetilde{\sigma}$ has only real, positive and distinct eigenvalues. Note that we can choose Q so that $\det \widetilde{\sigma} > 1$ since P is volume-expanding. We denote the eigenspaces of $\widetilde{\sigma}$ by E_1, E_2 and E_3 so that the corresponding eigenvalues are in the increasing order. We also know the transition with matrix T from Q to itself has the property as is described in Lemma 4.13, more precisely, there exists a matrix $\tilde{T}$ that is $(\varepsilon_0 + \varepsilon_1)$-close to T such that the following holds:

- $\tilde{T}(E_3) = E_3$.
- $\tilde{T}(E_1) = E_2$ and $\tilde{T}(E_2) = E_1$.

Let us consider a matrix $D_n$given as follows:

$$D_n := \widetilde{\sigma}^{2n} \circ \tilde{T} \circ \widetilde{\sigma}^{n} \circ \tilde{T} \circ \widetilde{\sigma}^{n} \circ \tilde{T} \circ \widetilde{\sigma}^{2n} \circ \tilde{T}.$$

Since Q admits the transition with matrix T, there exists a periodic point R_n such that the cocycle $df \in C(R_n)$ is ε-close to the cocycle τ such that $\widetilde{\tau}$ is given by D_n.

Now, by applying the Franks' lemma (see Lemma 4.4), we get a diffeomorphism g that is C^1-close to f such that R_n is a hyperbolic periodic point of g and the differential $dg^{\mathrm{per}(R_n)}(R_n)$ is equal to D_n.

We show that, for each n, the linear map $dg^{\mathrm{per}(R_n)}(R_n)$ leaves $E_1 \oplus E_2$ invariant and acts as a homothetic transformation to itself with positive multiplier, and leaves E_3 invariant and the eigenvalues to the E_3-direction is positive. Let us denote the eigenvalue of $\widetilde{\tau}$ for the E_i-direction as λ_i $(i = 1, 2, 3)$, where λ_i is some positive real number and put $\tilde{T}(e_1) = \mu_1 e_2$, $\tilde{T}(e_2) = \mu_2 e_1$ and $\tilde{T}(e_3) = \mu_3 e_3$, where μ_1, μ_2 and μ_3 are some non-zero real numbers. Then direct calculations show that $dg^{\mathrm{per}(R_n)}(R_n)(e_i) = (\mu_1\mu_2)^2(\lambda_1\lambda_2)^{3n} e_i$ for $i = 1, 2$ and $dg^{\mathrm{per}(R_n)}(R_n)(e_3) = \mu_3^4 \lambda_3^6 e_3$. If n is sufficiently large, then $\det(dg^{\mathrm{per}(R_n)}(R_n))$ is greater than one and since $\det \widetilde{\sigma}$ is greater than one.

Hence we finished the proof. □

Remark 4.3. The form of the matrix D_n may look bizarre. Let us see why we need to pick up this matrix. We want to choose the matrix whose restriction is a homothety to the stable direction. For instance, the matrix $\widetilde{\sigma}^n \circ \tilde{T} \circ \widetilde{\sigma}^n \circ \tilde{T}$ satisfies this condition. However, this matrix is a power of the matrix $\widetilde{\sigma}^n \circ \tilde{T}$. Hence, in general, this matrix is not approximated by a first return map of the differential of the some periodic point (see the definition of the transition). We choose D_n so that it does not be a power of some matrix.

4.4 Proof of Lemma 4.7

Finally, we give the proof of Lemma 4.7. The idea of this proof already appears in the proof of Proposition 3.1 in [BDP]. In addition to the original proof, we need to check two things. The first one is that during the perturbation there is no index bifurcation. The second one is the perturbation is uniformly small.

The proof of Lemma 4.7 is divided into two steps. The first step (Lemma 4.14) tells us if there is a point whose eigenspaces forms a small angle, then one can create complex eigenvalues by a small perturbation. In the second step (Lemma 4.15), we prove that if a periodic linear system does not admit dominated splittings, then one can construct a periodic point with a small angle by arbitrarily small perturbations.

We prepare some notations. Given two one-dimensional subspaces V_1, V_2 in a two-dimensional Euclidean space, the *angle* between V_1, V_2 is the unique real number $0 \leq \alpha \leq \pi/2$ that satisfies $\cos\alpha = |(v_1, v_2)|/(|v_1||v_2|)$, where v_i is any non-zero vector in V_i $(i = 1, 2)$, $(\,\cdot\,,\,\cdot\,)$ is the inner product and $|\cdot|$ is the norm defined by the inner product.

Let us state the first and second steps.

Lemma 4.14. *For any $K > 0$ and $\varepsilon > 0$, there exists $\alpha = \alpha(K, \varepsilon)$ with $0 < \alpha < \pi/4$ such that the following holds: Let (Σ, f, E, A) a two-dimensional diagonalizable periodic linear cocycle with real, positive distinct eigenvalues, bounded by K and has a point X at which the angles between two eigenspaces are less than α. Then, there exists a path $\{\gamma(t) \mid 0 \leq t \leq 1\}$ in $C(X)$ such that the following holds:*

1. $\gamma(0) = A|_{\mathcal{O}(X)}$.
2. $\mathrm{diam}(\gamma(t)) < \varepsilon$.
3. $\det \widetilde{\gamma(t)}$ *is independent of* t.
4. *Let $\lambda_m(t) \leq \lambda_b(t)$ be the absolute value of the eigenvalues of $\widetilde{\gamma(t)}$. Then, for $s < t$, we have $\lambda_m(s) \leq \lambda_m(t)$ and $\lambda_b(s) \geq \lambda_b(t)$.*
5. $\det \widetilde{\gamma(1)}$ *has two complex eigenvalues.*

Furthermore, when $\varepsilon \to 0$ with K fixed, $\alpha \to 0$.

Lemma 4.15. *For any $K > 0$, $\varepsilon > 0$ and $0 < \alpha < \pi/4$, the following holds: Let (Σ, f, E, A) be a two-dimensional diagonalizable periodic linear cocycle with real positive distinct eigenvalues, bounded by K and does not admit dominated splittings. Then, there exists a hyperbolic periodic point $X \in \Sigma$ and a path $\{\gamma(t) \mid 0 \leq t \leq 1\}$ in $C(X)$ such that the following holds:*

1. $\gamma(0) = A|_{\mathcal{O}(X)}$.
2. $\mathrm{diam}(\gamma(t)) < \varepsilon$.
3. $\det \widetilde{\gamma(t)}$ *is independent of* t.
4. *Let $\lambda_m(t) \leq \lambda_b(t)$ be the absolute value of the eigenvalues of $\widetilde{\gamma(t)}$ Then, for $s < t$, we have $\lambda_m(s) \leq \lambda_m(t)$ and $\lambda_b(s) \geq \lambda_b(t)$.*

5. $\widetilde{\gamma(1)}$ *has two eigenspaces with angle less than* α.

We need an auxiliary lemma about the effect of a perturbation on the constant of the boundedness of a cocycle.

Lemma 4.16. *Let* σ *be a cocycle in* $C(X)$ *where* X *is a periodic point and let* $\gamma(t)$ *be a path with* $\mathrm{diam}(\gamma(t)) < \varepsilon$. *Then,* $\gamma(1)$ *is bounded by* $\varepsilon + \|\sigma\|$.

Proof. For every unit vector v in some $T_{f^i(X)}M$, we have

$$\begin{aligned}&\|(\gamma(1)(f^i(X)))(v)\|\\ \leq&\|(\gamma(1)(f^i(X)))(v) - (\gamma(0)(f^i(X)))(v)\| + \|(\gamma(0)(f^i(X)))(v)\| \leq \varepsilon + \|\sigma\|.\end{aligned}$$

We can prove a similar inequality for the inverse of $\gamma(1)$. $\square$

We give the proof of Lemma 4.7 using these lemmas.

Proof of Lemma 4.7. Let (Σ, f, E, A) be a two-dimensional diagonalizable periodic linear cocycle with positive distinct eigenvalues, bounded by K and suppose that it does not admit dominated splitting.

First, let us assume the angle between two eigenspaces are not bounded below, more precisely, there exists a sequence of periodic points (X_n) such that the sequence of the angles (α_n), where α_n is the angle of the eigenspaces of $A^{\mathrm{per}(X_n)}(X_n)$, converges to zero. Fix $\varepsilon > 0$ and the constant $\alpha_0 = \alpha(K, \varepsilon)$ in Lemma 4.14. Since $\alpha_n \to 0$ as $n \to \infty$, we can take n_0 such that $\alpha_{n_0} < \alpha_0$. Then Lemma 4.14 enable us to find a path $\gamma(t)$ in $C(X)$ we claimed.

Hence, let us assume that the angles between two eigenspaces are uniformly bounded below. By taking appropriate bases at every periodic point, we can assume that, the corresponding eigenspaces are orthogonal. Fix $\varepsilon > 0$ and the constant $\alpha_1 = \alpha(K + \varepsilon/2, \varepsilon/2)$ in Lemma 4.14. Applying Lemma 4.15 to (Σ, f, E, A) with constants K, $\varepsilon/2$, and α_1, we get a periodic point $X \in \Sigma$ and a path $\gamma_1(t)$ in $C(X)$ satisfying all the properties in the conclusion of Lemma 4.15 for $\gamma_1(t)$, $\varepsilon/2$ and α_1. Then, Lemma 4.16 implies that the cocycle $\widetilde{\gamma_1(1)}$ is bounded by $K + \varepsilon/2$. Since the angle between eigenspaces of $\widetilde{\gamma_1(1)}$ is less than α_1, we can apply Lemma 4.14 to K, α and ε as $K + \varepsilon/2$, α_1 and $\varepsilon/2$ respectively in order to get a path $\gamma_2(t)$.

Now we construct a continuous path $\gamma(t)$ as follows: If $0 \leq t \leq 1/2$ then $\gamma(t) = \gamma_1(2t)$. If $1/2 \leq t \leq 1$ then $\gamma(t) = \gamma_2(2t - 1)$. We have $\mathrm{diam}(\gamma(t))$ is less than ε because $\mathrm{diam}(\gamma_1(t))$, $\mathrm{diam}(\gamma_2(t))$ are less than $\varepsilon/2$. Thus we got the path we needed and the proof is completed. $\square$

Let us give the proof of Lemma 4.14. For the proof, we need an elementary lemma.

Lemma 4.17. *Let* $R(x)$ *denote the rotation map of angle* x *on a two-dimentional Euclidean space. Then there exists a positive constant* C *such that following inequality holds:*

$$\|R(s) - R(t)\| \leq C|s - t| \text{ for all } -\pi/4 \leq s, t \leq \pi/4.$$

Proof. We introduce a norm $\|\cdot\|_2$ on the space of linear maps as follows. Fix an orthonormal basis of the Euclidean space and for a linear map A, define

$$\|A\|_2 = \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_2 = \sqrt{a^2+b^2+c^2+d^2},$$

where the matrix denotes the matrix representation of A with respect to the basis. We take a constant N such that for every A the inequality $\|A\| \leq N\|A\|_2$ holds. Then,

$$\begin{aligned}
&\|R(s)-R(t)\|^2 \\
&\leq N^2\|R(s)-R(t)\|_2^2 = 2N^2\big((\cos(s)-\cos(t))^2 + (\sin(s)-\sin(t))^2\big).
\end{aligned}$$

Since $-\pi/4 \leq s, t \leq \pi/4$, we get

$$|\cos(s)-\cos(t)| \leq |s-t|/\sqrt{2},\ |\sin(s)-\sin(t)| \leq |s-t|.$$

So we have

$$2N^2\big((\cos(s)-\cos(t))^2 + (\sin(s)-\sin(t))^2\big) \leq 3N^2|s-t|^2.$$

Hence for $C = \sqrt{3}N$ we get the inequality. □

Proof of Lemma 4.14. Let X be a point at which the angle of corresponding eigenspaces is less than α. By fixing an appropriate orthonormal basis, we can take a matrix representation of $A^{\mathrm{per}(X)}(X)$ as follows:

$$A^{\mathrm{per}(X)}(X) = \begin{pmatrix} \lambda_1 & \mu \\ 0 & \lambda_2 \end{pmatrix},$$

where $\lambda_1 < \lambda_2$ is positive real numbers and μ is some non-zero real number. By a direct calculation, we get $(\lambda_2 - \lambda_1)/|\mu| < \tan\alpha$ (LHS is the tangent of the angle between two eigenspaces at X).

Given $\varepsilon > 0$, we put $\alpha = \varepsilon/(CK)$, where C is the constant given in Lemma 4.17. Let us define a path $\gamma(t)$ as follows:

$$[\gamma(t)](f^i(X)) = \begin{cases} A(f^i(X)), & \text{if } f^i(X) \neq f^{\mathrm{per}(X)-1}(X), \\ R(-\mathrm{sign}(\mu)\alpha t)A(f^i(X)), & \text{if } f^i(X) = f^{\mathrm{per}(X)-1}(X), \end{cases}$$

where $\mathrm{sign}(\mu)$ is the signature of μ.

Let us see this path enjoys all the conditions we claimed in Lemma 4.14. In the following, we treat the case when $\mu > 0$. Since the proof for the case $\mu < 0$ can be done in a similar way, it is left to the reader. We first examine the diameter of this path. To see this, we only need to check the distance of $[\gamma(t)](f^{\mathrm{per}(X)-1})$, which can be estimated using Lemma 4.17 as follows:

$$\begin{aligned}
&\|[\gamma(s)](f^{\mathrm{per}(X)-1}(X)) - [\gamma(t)](f^{\mathrm{per}(X)-1}(X))\| \\
&= \|R(-\alpha s)A(f^{\mathrm{per}(X)-1}(X)) - R(-\alpha t)A(f^{\mathrm{per}(X)-1}(X))\| \\
&\leq \|R(-\alpha s) - R(-\alpha t)\|\|A(f^{\mathrm{per}(X)-1}(X))\| \\
&\leq \alpha CK|s-t| \leq \varepsilon|s-t| \leq \varepsilon.
\end{aligned}$$

We can get similar estimates for the inverse.

The value $\det \widetilde{\gamma(t)}$ is independent of t, since $\gamma(t)$ is obtained by multiplying an orthogonal matrix to $\gamma(0)$. Let us investigate the behavior of the eigenvalues. We denote by $\lambda_m(t) \leq \lambda_b(t)$ the eigenvalues of $\widetilde{\gamma}(t)$. The characteristic equation of $\widetilde{\gamma(t)}$ is given by $x^2 - \theta x + d = 0$, where $d := \lambda_1\lambda_2$ and $\theta = \theta(t) := (\lambda_1 + \lambda_2)\cos(-\alpha t) + \mu \sin(-\alpha t)$. So two eigenvalues are given as $(\theta \pm \sqrt{\theta^2 - 4d})/2$. To complete the proof, it is enough to check the following properties hold:

- θ is monotone decreasing when t increases.
- $\theta^2 - 4d < 0$ when $t = 1$.

Indeed, $\lambda_b(t)$ is equal to $(\theta + \sqrt{\theta^2 - 4d})/2$. Therefore, if θ is monotone decreasing, so is λ_b. Moreover, it implies that $\lambda_m(t)$ is monotone increasing, since $\lambda_m(t)\lambda_b(t)$ is a positive constant.

Let us chek the items above. First, observe that

$$\theta(t) = \sqrt{(\lambda_1 + \lambda_2)^2 + \mu^2}\sin(\beta - \alpha t),$$

where $0 < \beta < \pi/2$ is a real number satisfying $\tan\beta = (\lambda_1 + \lambda_2)/\mu$. This shows $\beta > \alpha$. Thus $\theta(t)$ is monotone decreasing.

Let us show that $\theta^2 - 4d < 0$ when $t = 1$. We have

$$\theta(1) = \frac{\mu(\lambda_1 + \lambda_2) - \mu(\lambda_2 - \lambda_1)}{\sqrt{(\lambda_2 - \lambda_1)^2 + \mu^2}} = \frac{2\lambda_1}{\sqrt{1 + \left(\frac{\lambda_1 - \lambda_2}{\mu}\right)^2}} \leq 2\sqrt{\lambda_1^2} < 2\sqrt{\lambda_1\lambda_2} = 2\sqrt{d}.$$

So, we finished the proof. □

Finally, let us give the proof of Lemma 4.15.

Proof of Lemma 4.15. The proof of Lemma 4.15 is just the repetition of Lemma 3.4 in [BDP]. We can easily check the behavior of the eigenvalues during the perturbation. So the proof is left to the reader. □

5 Bifurcation of a homothetic tangency

5.1 Strategy of the proof of Proposition 2

In this section, we prove Proposition 2. The proof is divided into two steps. The first step is to reduce the problem to affine dynamics. The second step is to investigate the bifurcation of the dynamics.

To state our proof clearly, let us give the following definition.

Definition. Let $f \in \mathrm{Diff}^1(M)$ with $\dim M = 3$, and X be an index-1 hyperbolic fixed point with a homoclinic tangency. A one-parameter family of the C^1-diffeomorphism $(f_t)_{|t|<\delta} \subset \mathrm{Diff}^1(M)$ is said to be an *affine unfolding of the degenerate tangency* of f with respect to X if the following properties hold (see Figure 1):

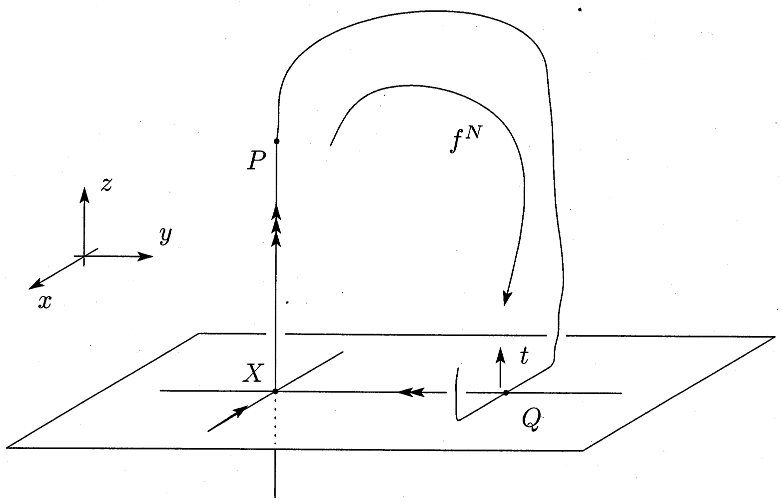

Figure 1: A degenerate tangency.

$(DT1)$ $f_0 = f$.

$(DT2)$ There exists a coordinate chart $\phi : U \to \mathbb{R}^3$ of X such that $V := \phi(U) = (-1,1)^3$ and $\phi(X)$ is the origin of $\mathbb{R}^3$.

$(DT3)$ Let $F_t := \phi \circ f_t \circ \phi^{-1}$. For $(x,y,z) \in (-1,1) \times (-1,1) \times (-\mu^{-1}, \mu^{-1})$, $F_t(x,y,z) = (\lambda x, \tilde{\lambda} y, \mu z)$, where $0 < \tilde{\lambda} < \lambda < 1 < \mu$. Thus, the z-axis in V is contained in $\phi(W^u(X))$ and the xy-plane in V is contained in $\phi(W^s(X))$.

$(DT4)$ There exist two points $P, Q \in U$ with $P' = \phi(P) = (0,0,p)$ and $Q' = \phi(Q) = (0,q,0)$, where $0 < p, q < 1$, such that the following holds: For some positive integer $N \geq 2$, $f_0^N(Q) = P$ and $f_t^i(P) \notin U$ for $0 < i < N$. This implies that $P, Q \in \phi(W^s(X) \cap W^u(X))$.

$(DT5)$ By the abuse of notation, we denote $\phi \circ f_t^N \circ \phi^{-1}$ by F_t^N. Then, there exists a small neighborhood W_P of P with $\phi(W_P) = [-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon] \times [p-\varepsilon, p+\varepsilon]$ and $f_t^N(W_P) \subset U$ such that for every $(x, y, z+p) \in \phi(W_P)$, $F_t^N(x,y,z+p) = (cz, by+q, ax+t)$, where a, b and c are some non-zero real numbers. We call the map $F_t^N|_{W_P}$ as the *return map* of f_t, and put $W_Q := F_0^N(W_P)$.

Our first step is stated as follows:

Proposition 5.1. *If X has a homothetic tangency, then one can find a diffeomorphism g C^1-arbitrarily close to f such that there exists an one-parameter family of C^1-diffeomorphisms $(g_t)_{|t|<\delta}$ that is the affine unfolding of the degenerate tangency of g with respect to $X(g)$.*

The second step is the following:

Proposition 5.2. *Let X be a volume-expanding hyperbolic periodic point with an affine unfolding of the degenerate tangency $(f_t)_{|t|<\delta}$ with respect to X. Then, for arbitrarily small $\varepsilon > 0$, there exists $0 < \tau \leq \varepsilon$ such that f_τ has an index-two periodic point Y and there exists a heterodimensional cycle associated to $X(f_\tau)$ and Y.*

It is clear that these two propositions imply Propotision 2. In the subsequent two subsections, we give the proofs of these propositions.

5.2 Proof of Proposition 5.1

We give the proof of Proposition 5.1. Before the proof, we prepare two lemmas.

The first one is a version of Franks' lemma. We omit the proof, since it is easily obtaiend from the proof of the Franks' lemma.

Lemma 5.1. *Let $f \in \mathrm{Diff}^1(M)$ with $\dim M = m$. Consider $x \in M$ and the coordinate neighborhoods $\phi : U \to \mathbb{R}^m$ and $\psi : V \to \mathbb{R}^m$ of x and $f(x)$ respectively, satisfying $x = f(x) = 0$. Then, for any $\varepsilon > 0$ and any neighborhood $U' \subset U$ of x, there exist small neighborhoods $\tilde{U}$ of x contained in U' and $\tilde{f} \in \mathrm{Diff}^1(M)$ ε-C^1-close to f, coinciding with f on $M \setminus U'$, and $\psi \circ \tilde{f} \circ \phi^{-1}$ coincides with a linear map $d(\psi \circ f \circ \phi^{-1})(0)$ on $\tilde{U}$.*

The following lemma is a version of Gourmelon's Franks' lemma.

Lemma 5.2 (Lemma 4.1 in [Gou1])**.** *Let $f \in \mathrm{Diff}^1(M)$ with $\dim M = m$. Consider a fixed point x of f and a coordinate neighborhood $\phi : U \to \mathbb{R}^m$ of x with $\phi(x) = 0$. Then, for any $\varepsilon > 0$ and any neighborhood $V \subset U$ of x, one can find a C^1-diffeomorphism $\tilde{f}$ ε-C^1-close to f and a small neighborhood $\tilde{V} \subset V$ such that $\tilde{f}(x) = f(x)$ for any $x \in M \setminus V$, $\phi \circ \tilde{f} \circ \phi^{-1}$ coincides with the linear map $d(\phi \circ f \circ \phi^{-1})(0)$ on $\phi(\tilde{V})$, and $\tilde{f}$ preserves the invariant manifolds of X outside $\tilde{V}$.*

Let us begin the proof of Proposition 5.1.

Proof of Proposition 5.1. We assume that X is a fixed point, since the general case can be reduced to this case by considering some power of f.

First, by using Lemma 5.2 to x and an appropriate coordinate neighborhood of x, we can find $f_1 \in \mathrm{Diff}^1(M)$ arbitrarily close to f that has the following properties:

- There exists a coordinate chart $\phi : U \to \mathbb{R}^3$ around X such that $V := \phi(U) = (-1, 1)^3$ and $\phi(X)$ is the origin of $\mathbb{R}^3$.
- Let $F_1 := \phi \circ f_1 \circ \phi^{-1}$. Then, for $(x, y, z) \in (-1, 1) \times (-1, 1) \times (-\mu^{-1}, \mu^{-1})$, $F_1(x, y, z) = (\lambda x, \lambda y, \mu z)$, where $0 < \lambda < 1 < \mu$.
- There exist two points $P, Q \in U$ with $P' := \phi(P) = (0, 0, p)$ and $Q' := \phi(Q) = (0, q, 0)$, where $0 < p, q < 1$ such that for some positive integer $N \geq 2$, $f_1^N(Q) = P$ and $f^i(P) \notin U$ for $0 < i < N$.

- $W^s(X)$ and $W^u(X)$ are tangent at Q, in particular, $T_Q W^u(X)$ is contained in $T_Q W^s(X)$.

We give a perturbation to f_1 so that the point of tangency is on the strong stable manifold of X. For this purpose, we take an interval $J := [j_1, j_2]$ that is contained in $(\mu^{-2}p, \mu^{-1}p)$ satisfying $J \cap \{\lambda^n q \mid n \geq 0\} = \emptyset$. Let $\rho(t)$ be a C^∞-function on $\mathbb{R}$ which satisfies the following properties: $\rho(t) = 1$ for $|t| < j_1$, and $\rho(t) = 0$ for $|t| > j_2$. We modify F_1 to F_2 as follows:

$$F_2(x, y, z) = (1 - R(X))F_1(X) + R(X)(\lambda x, \tilde{\lambda} y, \mu z),$$

where $R(x, y, z) := \rho(x)\rho(y)\rho(z)$ and $\tilde{\lambda}$ is a real number satisfying $0 < \tilde{\lambda} < \lambda < 1$. Let us define the map f_2 as follows: $f_2(x) = (\phi^{-1} \circ F_2 \circ \phi)(x)$ for $x \in \phi^{-1}((-1,1) \times (-1,1) \times (-\mu^{-1}, \mu^{-1}))$. Otherwise $f_2(x) = f_1(x)$. Then, for $\tilde{\lambda}$ sufficiently close to λ, f_2 is a diffeomorphism of M. Note that P and Q are the points of tangency of the stable and unstable manifolds of X, Q is on the strong stable manifold of X, and f_2 converges to f_1 when $\tilde{\lambda} \to \lambda$ in the C^1-topology. We fix a small $\tilde{\lambda}$ and make more perturbations.

Throughout the proof, we often change the coordinate in the following way. Given a real number $r > 1$, we define $r\mathrm{Id} : \mathbb{R}^3 \to \mathbb{R}^3$ by $r\mathrm{Id}(x, y, z) := (rx, ry, rz)$. Then, $r\mathrm{Id} \circ \phi$ gives another coordinate neighborhood of X. We replace ϕ with $r\mathrm{Id} \circ \phi$ and call this chart as the *renormalization* of ϕ with the expansion factor r. When we take a renormalization, we change U, P and Q so that $(DT2)$, $(DT3)$, $(DT4)$ in the definition of the degenerate tangency hold. More precisely, we replace U with $r^{-1}U$, P with $f^{-n_P}(P)$ where n_P is the smallest non-negative integer that satisfies $f^{-n_P}(P) \in r^{-1}U$, Q with $f^{-n_Q}(Q)$ where n_Q is the non-negative integer that satisfies $f^{n_Q}(Q) \in r^{-1}U$.

Let us see the effect of the renormalization on the differential of the return map. Given a diffeomorphism f and a coordinate chart ϕ, we have the differential of the return map $dF^N(P')$. If we take the renormalization, the differential of the return map is given by $L^{n_Q} dF^N(P') L^{n_P}$, where

$$L := \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \tilde{\lambda} & 0 \\ 0 & 0 & \mu \end{pmatrix}.$$

Direct calculations show that a component of the matrix $dF^N(P')$ is equal to zero if and only if the corresponding component of the the differential of the return map of the renormalized diffeomorphism is equal to zero.

Let us resume the proof. By taking an appropriate renormalization, we can assume $(DT2)$, $(DT3)$ and $(DT4)$ hold for f_2. Let us consider the differential of the return map F_2^N. Since $W^s(X)$ and $W^u(X)$ are tangent at Q, we can put

$$dF_2^N(P') = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & 0 \end{pmatrix}.$$

By applying Lemma 4.4 at P, we take a diffeomorphism f_3 arbitrarily close to f_2 such that $(1,3)$, $(3,1)$ and $(3,2)$-component of $dF_3^N(P')$ are not equal to zero and $(3,3)$-component remains to be zero. Note that, by making the support of the perturbation sufficiently small, we can assume that the use of the Franks' lemma does not disturb the condition that P, Q are contained in $W^s(X) \cap W^u(X)$. Let us take a positive integer l and consider the differential $dF_3^l \circ dF_3^N(P')$, which is written as

$$\begin{pmatrix} \lambda^l & 0 & 0 \\ 0 & \tilde{\lambda}^l & 0 \\ 0 & 0 & \mu^l \end{pmatrix} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & 0 \end{pmatrix} = \begin{pmatrix} \lambda^l a & \lambda^l d & \lambda^l g \\ \tilde{\lambda}^l b & \tilde{\lambda}^l e & \tilde{\lambda}^l h \\ \mu^l c & \mu^l f & 0 \end{pmatrix}.$$

By using Lemma 4.4 at $f^{l-1}(Q)$, we perturb f_3 to f_4 to make the differential $dF_4(\phi(f^{l-1}(Q)))$ into

$$\begin{pmatrix} 1 & 0 & z_l \\ x_l & 1 & y_l \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \tilde{\lambda} & 0 \\ 0 & 0 & \mu \end{pmatrix},$$

where

$$x_l := -(\tilde{\lambda}/\lambda)^l(h/g),\ y_l := -(\tilde{\lambda}/\mu)^l(ah/cg) - (\tilde{\lambda}/\mu)^l(b/c) \text{ and } z_l := -(\lambda/\mu)^l(a/c).$$

Then, by a direct calculation, we have $(1,1)$, $(2,1)$, $(2,3)$, and $(3,3)$-component of $dF_4^{N+l}(P')$ is equal to zero. We can make the distance between f_3 and f_4 being arbitrarily small by letting l large, since $x_l, y_l, z_l \to 0$ when $l \to \infty$. We fix sufficiently large l, take f_4 and continue making perturbations.

By taking appropriate renormalization, we can put

$$dF_4^N(P') := \begin{pmatrix} 0 & b & e \\ 0 & c & 0 \\ a & d & 0 \end{pmatrix},$$

where a, b, c, etc. are not necessarily equal to the corresponding numbers appeared in $dF_3^N(P')$. Since f_4 is a diffeomorphism, $a, c, e \neq 0$.

Let us take a positive integer l and consider the differential $dF_4^N \circ dF_4^l(\phi(f_4^{-l}(P)))$. This differential is given as

$$\begin{pmatrix} 0 & b & e \\ 0 & c & 0 \\ a & d & 0 \end{pmatrix} \begin{pmatrix} \lambda^l & 0 & 0 \\ 0 & \tilde{\lambda}^l & 0 \\ 0 & 0 & \mu^l \end{pmatrix} = \begin{pmatrix} 0 & b\tilde{\lambda}^l & e\mu^l \\ 0 & c\tilde{\lambda}^l & 0 \\ a\lambda^l & d\tilde{\lambda}^l & 0 \end{pmatrix}.$$

By using Lemma 4.4 at $f^{-l}(P)$, we perturb f_3 to f_4 to make the differential $dF_4(\phi(f^{-l}(P)))$ into

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \tilde{\lambda} & 0 \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} 1 & x_l & 0 \\ 0 & 1 & 0 \\ 0 & y_l & 1 \end{pmatrix},$$

where $x_l := -(\tilde{\lambda}/\lambda)^l(d/a)$ and $y_l := -(\tilde{\lambda}/\mu)^l(b/e)$, then a direct calculation shows that the non-zero components of $df_5^{N+l}(\phi(f_5^{-l}(P)))$ are $(1,3)$, $(2,2)$, and $(3,1)$.

By taking appropriate renormalization, we can put

$$dF_5^N(P) := \begin{pmatrix} 0 & 0 & c \\ 0 & b & 0 \\ a & 0 & 0 \end{pmatrix}.$$

Now, by using Lemma 5.1 at P, we can construct F_6 such that F_6^N is locally an affine map around P with $dF_6^N(P') = dF_5^N(P')$.

Let us review the properties of f_6.

- There exists a coordinate chart $\phi : U \to \mathbb{R}^3$ of X such that $V := \phi(U) = (-1,1)^3$ and $\phi(X)$ is the origin of $\mathbb{R}^3$.
- Let $F_6 := \phi \circ f_6 \circ \phi^{-1}$. For $(x,y,z) \in (-1,1) \times (-1,1) \times (-\mu^{-1}, \mu^{-1})$, $F_6(x,y,z) = (\lambda x, \tilde{\lambda} y, \mu z)$, where $0 < \tilde{\lambda} < \lambda < 1 < \mu$.
- There exist two points $P, Q \in U$ with $P' = \phi(P) = (0,0,p)$ and $Q' = \phi(Q) = (0,q,0)$, where $0 < p, q < 1$ such that for some positive integer $N \geq 2$, $f_6^N(Q) = P$ and $f_6^i(P) \notin U$ for $0 < i < N$.
- There exists a small neighborhood W_P of P with $\phi(W_P) = [-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon] \times [p-\varepsilon, p+\varepsilon]$ and $F_6^N(W_P) \subset V$ such that for every $(x, y, z+p) \in W_P$, $F_t^N(x,y,z+p) = (cz, by+q, ax)$, where a, b and c are some non-zero real numbers.

Let $\rho_2(s)$ be a C^∞-function on $\mathbb{R}$ satisfying the following properties: $\rho_2(s) = \delta$ for $|s| < \varepsilon/3$, and $\rho_2(s) = 0$ for $|s| > 2\varepsilon/3$, where δ is a positive real number, and define $R : V \to \mathbb{R}$ by $R(x,y,z) := \rho_2(x)\rho_2(y)\rho_2(z-p)$.

We define a one-parameter family of maps $\Psi_t : V \to \mathbb{R}^3$ by $\Psi_t(x,y,z) = (x + tR(x,y,z), y, z)$, for $(x,y,z) \in V$. Note that Ψ_t is a diffeomorphism of V for t with a sufficiently small absolute value. Then, we construct a one-parameter family of diffeomorphisms $\psi_t : M \to M$ as follows: For $x \in V$, $\psi_t(x) = (\phi^{-1} \circ \Psi_t \circ \phi)(x)$, otherwise $\psi_t(x) = x$. Finally, we define a one-parameter family of diffeomorphisms $(f_{7,t})$ by $f_{7,t} = f_6 \circ \psi_t$. By making W_P sufficiently small if necessary, we see that $(f_{7,t})_{|t|<\delta}$, P, Q, ϕ, W_P satisfy $(DT1)$–$(DT5)$ for sufficiently small δ.

So, we have finished the proof. □

Remark 5.1. The geometric idea behid the perturbatioins from f_2 to f_5 is simple. Let us see that.

Let us consider the differential $dF_2^{N+m}(P')$. Take the two-dimentional plane passing P' and parallel to the xy-plane, and let us consider its image by $dF_2^{N+m}(P')$. If m is sufficiently large, under some generic assumption (f_2 to f_3), this image is almost parallel to the yz-plane. Thus, by a small perturbation, we can assume this image is parallel to the yz-plane (f_3 to f_4). In a similar argument (on the inverse of f_4), we perturb f_4 so that $dF_5^N(P')$ preserves the y-direction, exchanges x- and z-direction (f_4 to f_5).

5.3 Proof of Proposition 5.2

Finally, let us prove Proposition 5.2.

Proof of Proposition 5.2. Put $I_n := [-\varepsilon, \varepsilon] \times [q-\varepsilon, q+\varepsilon] \times [(p-\varepsilon)/\mu^n, (p+\varepsilon)/\mu^n]$ and $t_n := p/\mu^n$. For n sufficiently large, we show that f_{t_n} has a hyperbolic periodic point $R_n \in I_n$ with period $n+N$ such that f_{t_n} has a heterodimensional cycle associated to X and R_n.

Let us take a point $A \in I_n$ and put $A = (x, y, z)$. Then, by definition, $F_t^n(A) = (\lambda^n x, \tilde{\lambda}^n y, \mu^n z)$. Note that $F_t^n(A)$ belongs to W_P if n is sufficiently large, and in the following we assume this property. By the definition of the return map, we can see that $F^{n+N}(A) = (c(\mu^n z - p), b\tilde{\lambda}^n y + q, a\lambda^n x + t)$. When $t = t_n$, if A is a periodic point of period $n + N$. the following equalities hold:

$$c(\mu^n z - p) = x, \ b\tilde{\lambda}^n y + q = y, \ \text{and} \ a\lambda^n x + p/\mu^n = z.$$

By a direct calculation, we have

$$x = 0, \ y = q/(1 - b\tilde{\lambda}^n), \ \text{and} \ z = p/\mu^n.$$

We put $y_n := q/(1 - b\tilde{\lambda}^n)$ and $z_n := p/\mu^n$. This result says that the point $R_n := (0, y_n, z_n)$ is a periodic point of period $n + N$ of f_{t_n}.

Second, let us check that R_n is a hyperbolic periodic point of index 2. The derivative of $F_{t_n}^n$ at R_n is given by the matrix

$$\begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \tilde{\lambda}^n & 0 \\ 0 & 0 & \mu^n \end{pmatrix}.$$

The derivative of $dF_{t_n}^N$ at $F_{t_n}^n(R_n)$ is given by the matrix

$$\begin{pmatrix} 0 & 0 & c \\ 0 & b & 0 \\ a & 0 & 0 \end{pmatrix}.$$

Hence the differential $dF_{t_n}^{n+N}(R_n)$ is equal to

$$\begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \tilde{\lambda}^n & 0 \\ 0 & 0 & \mu^n \end{pmatrix} \begin{pmatrix} 0 & 0 & c \\ 0 & b & 0 \\ a & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \lambda^n c \\ 0 & \tilde{\lambda}^n b & 0 \\ a\mu^n & 0 & 0 \end{pmatrix}.$$

Now, a direct calculation shows that the eigenvalues of this matrix is given by $\tilde{\lambda}^n b$ and $\pm\sqrt{ac\lambda^n\mu^n}$. The absolute value of $\tilde{\lambda}^n b$ is less than one when n is sufficiently large, and the absolute value of $\pm\sqrt{ac\lambda^n\mu^n}$ are greater than one when n is sufficiently large, since we have $|\tilde{\lambda}| < 1$ and $|\mu\lambda| > 1$ (the second inequality is the consequence of the volume-expansiveness of X).

Let us check that f_{t_n} has a heterodimensional cylce associated to X and R_n (see Figure 2). First, we show that $W^u(X)$ and $W^s(R_n)$ have non-empty

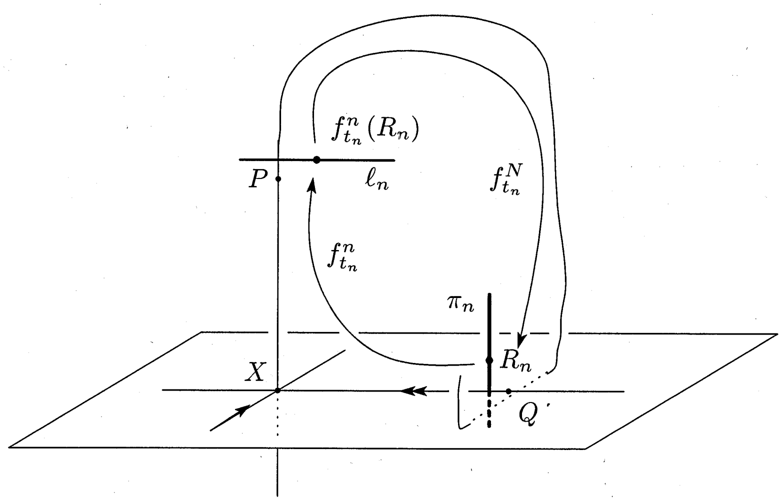

Figure 2: Creation of a heterodimensional cycle.

intersection. It is easy to see that $W^u(X)$ contains the z-axis in V. Hence, we only need to check that $W^s(R_n)$ has non-empty intersection with the z-axis. To see this, we focus on the segment

$$\ell_n := \{(0, \tilde{\lambda}^n y_n + s, \mu^n z_n) \mid |s| \leq 2\tilde{\lambda}^n y_n\}.$$

The segment ℓ_n passes $F_{t_n}^n(R_n)$ and the z-axis.

Since $F_{t_n}^n(R_n) \to Q$ and the length of ℓ_n converges to zero when $n \to \infty$, ℓ_n is contained in W_Q for n sufficiently large. In the following, we show that $\ell_n \subset W^s(R_n)$. The image of this segment by $F_{t_n}^N$ is given as

$$F_{t_n}^N(\ell_n) = \{(0, y_n + s, z_n) \mid |s| \leq 2|b|\tilde{\lambda}^n y_n\}.$$

So the image of ℓ_n by $F_{t_n}^{n+N}$ is given as follows:

$$F_{t_n}^{n+N}(\ell_n) = \{(0, \tilde{\lambda}^n y_n + s, \mu^n z_n) \mid |s| \leq (2|b|\tilde{\lambda}^n)\tilde{\lambda}^n y_n\}.$$

Thus, the restriction of F^{n+N} to ℓ_n is well defined and it is uniformly contracts ℓ_n with the factor $|b|\tilde{\lambda}^n$. This shows that ℓ_n is contained in the stable manifold of R_n.

Second, let us check that $W^s(X)$ and $W^u(R_n)$ have non-empty intersection. As is in the previous case, we only need to check that $W^u(R_n)$ has non-empty intersection with the xy-plane in V. For this purpose, we focus on the segment

$$\pi_n := \{(0, y_n, z_n + s) \mid |s| \leq 2z_n\}.$$

This segment passes R_n and have non-empty intersection with $W^s(X)$. We show that, for sufficiently large n, π_n is contained in $W^u(R_n)$. To see this, let us calculate the inverse image of π_n by $F^{-2(n+N)}$.

The image of π_n by $F_{t_n}^{-N}$ is given as

$$F_{t_n}^{-N}(\pi_n) = \{(s, \tilde{\lambda}^n y_n, \mu^n z_n) \mid |s| \leq 2z_n/|c|\}.$$

Since $2z_n/(|c|) = 2p/(|c|\mu^n)$, this set is contained in W_P if n is sufficiently large. The image of $F_{t_n}^{-N}(\pi_n)$ by $F_{t_n}^{-n}$ is given as

$$F_{t_n}^{-n-N}(\pi_n) = \{(s, y_n, z_n) \mid |s| \leq 2z_n/(|c|\lambda^n)\}.$$

Since $2z_n/(|c|\lambda^n) = 2p/(|c|\mu^n\lambda^n)$ and $\mu\lambda > 1$, this segment is also contained W_Q when n is sufficiently large.

The image of $F_{t_n}^{-n-N}(\pi_n)$ by $F_{t_n}^{-N}$ is given as follows:

$$F_{t_n}^{-n-2N}(\pi_n) = \{(0, \tilde{\lambda}^n y_n, \mu^n z_n + s) \mid |s| \leq 2z_n/(|ac|\lambda^n)\}.$$

By the same reason, if n is sufficiently large, this segment is contained in W_P.

Finally, the image of $F_{t_n}^{-n-2N}(\pi_n)$ by $F_{t_n}^{-n}$ is given as follows.

$$F_{t_n}^{-2n-2N}(\pi_n) = \{(0, y_n, z_n + s) \mid |s| \leq 2z_n/(|ac|\mu^n\lambda^n)\}.$$

These calculations show that $F_{t_n}^{-2n-2N}$ uniformly contracts π_n by the factor $(|ac|\mu^n\lambda^n)^{-1}$, whose absolute value is less than one if n is sufficiently large. Hence, we conclude that π_n is contained in $W^u(R_n)$, and our proof is completed. □

Appendix

In this appendix, we give the proof of Theorem 3.

A A necessary condition for Theorem 3

In this section, we give an abstract condition that assures the conclusion of Theorem 3 and prove it assuming the existence of the diffeomorphism that satisfies such a condition.

We denote by TM the tangent bundle of M and by $\Lambda^k(TM)$ the exterior product of TM of degree k. We furnish this bundle with the metric canonically induced from the Riemannian metric on M. For $f \in \mathrm{Diff}^1(M)$, we denote by $\Lambda^k(df)$ the bundle map of $\Lambda^k(TM)$ canonically induced from df.

For $f : V \to W$, where V and W are finite dimensional Euclidean spaces and f is a linear map, we define the value $m(f)$ to be the minimum of the length of $f(v)$, where v ranges all the unit vectors in V.

The following proposition provides the sufficient condition for Theorem 3.

Proposition A.1. *Let M be a four-dimensional smooth Riemannian manifold and $f \in \mathrm{Diff}^1(M)$. Suppose that f satisfies all the conditions below:*

(W1) There are two compact sets A and B in M such that $B \subset A$, $f(A) \subset \mathrm{int}(A)$ and $f(B) \subset \mathrm{int}(B)$ (for $U \subset M$, we denote its topological interior by $\mathrm{int}(U)$).

(W2) There exist two hyperbolic fixed points P and Q of f in $C := A \setminus B$.

(W3) f has a heterodimensional cycle associated to P and Q.

(W4) $\mathrm{ind}(P) = 3$, *and let $\sigma(P), \mu_1(P), \mu_2(P)$ and $\mu_3(P)$ be the eigenvalues of $df(P)$ in the non-decreasing order of their absolue values. Then $\mu_1(P)$ and $\mu_2(P)$ are in $\mathbb{C} \setminus \mathbb{R}$.*

(W5) $\mathrm{ind}(Q) = 2$, *and every eigenvalue of $df(Q)$ is in $\mathbb{C} \setminus \mathbb{R}$.*

(W6) There exists a constant $K > 1$ such that $m\left(\Lambda^3(df)\right) > K$ on C. In other words, df expands every three-dimensional subspace of T_xM for all $x \in C$ in volume with the expanding rate greater than K.

Then, there exist a non-empty open neighborhood $\mathcal{U} \subset$ $\mathrm{Diff}^1(M)$ *of f and a residual subset $\mathcal{R}$ of $\mathcal{U}$ satisfying the following: For every $g \in \mathcal{R}$, the homoclinic class $H(P, g)$ does not admit dominated splittings and* $\mathrm{ind}(H(P, g)) = \{2, 3\}$.

For the proof, we prepare a lemma.

Lemma A.1 (Theorem 2 in [BDK]). *Consider a diffeomorphism f of a four-dimensional manifold exhibiting a heterodimensional cycle associated to two hyperbolic fixed points P and Q with* $\mathrm{ind}(P) - \mathrm{ind}(Q) = 1$ *and* $\mathrm{ind}(Q) = 2$. *Suppose that $df(Q)$ has a complex eigenvalue with absolute value less than one. Then, there is a diffeomorphism g arbitrarily C^1-close to f exhibiting a robust heterodimensional cycle associated to transitive hyperbolic sets $\Gamma(g)$ and $\Sigma(g)$ containing $P(g)$ and $Q(g)$ respectively.*

Let us give the proof of Proposition A.1.

Proof of Proposition A.1. Let f be a diffeomorphism of a four-dimensional manifold such that f satisfies $(W1)$–$(W6)$. By applying Lemma A.1 to P and Q, we can take a diffeomorphism g, an open neighborhood $\mathcal{U}$ of g and two hyperbolic transitive invariant sets Γ, Σ such that for every $h \in \mathcal{U}$, the continuation $\Gamma(h)$ contains $P(h)$, $\Sigma(h)$ contains $Q(h)$ and f has a heterodimensional cycle assicated to $\Gamma(h)$ and $\Sigma(h)$. Note that by taking $\mathcal{U}$ sufficiently close to f, we can assume that for every $h \in \mathcal{U}$ all the properties $(W1)$–$(W6)$ hold except $(W3)$.

Let us put $\mathcal{R} = \mathcal{U} \cap \mathcal{R}_3$, where $\mathcal{R}_3$ is the residual set in section 3. We prove that the conclusion of Proposition A.1 holds for every $h \in \mathcal{R}$. First, we show that $H(P, h)$ does not admit dominated splittings. Since $h \in \mathcal{U}$, there exists a heterodimensional cycle associated to $\Gamma(h)$ and $\Sigma(h)$ with $P(h) \in \Gamma(h)$ and $Q(h) \in \Sigma(h)$. It implies that $P(h)$ and $Q(h)$ belong to the same chain recurrence class. Since $h \in \mathcal{R}_3$, this chain recurrence class coincides with $H(P, h)$ and simultaneously $H(Q, h)$, in particular $H(P, h) = H(Q, h)$. By $(W4)$, we can see that $H(P, h)$ does not admit dominated splitting of the form $E \oplus F$ with $\dim E = 2$. We can also see that $H(P, h)$ does not admit dominated splitting $E \oplus F$ with $\dim E = 1$ or 3, for $H(P, h)$ contains $Q(h)$ and h satisfies the condition $(W5)$. Thus, $H(P, h)$ does not admit any kind of dominated splittings.

Let us show that $\operatorname{ind}(H(P,h)) = \{2,3\}$. Since $H(P,h)$ contains $P(h)$ and $Q(h)$, $\operatorname{ind}(H(P,h))$ contains 2 and 3. We need to prove that $\operatorname{ind}(H(P,h))$ does not contain 1. To check this, it is enough to prove that $H(P,h) \subset C$. Indeed, $(W6)$ says that every iteration of dh expands every three-dimensional subspace of the tangent space at every point in $H(P,h)$ in volume. So the index of every periodic point in $H(P,h)$ cannot be 1.

To see $H(P,h) \subset C$, we first show $W^s(P,h) \pitchfork W^u(P,h) \subset C$. Let us take $x \in W^s(P,h) \pitchfork W^u(P,h)$. Since $x \in W^u(P,h)$, there exists $n > 0$ such that $h^{-n}(x) \in A$. So we have $x \in h^n(A)$. Since A satisfies $h(A) \subset \operatorname{int}(A)$, we have $h^n(A) \subset A$ for $n > 0$ and this implies $x \in A$. We show $x \notin B$. Since $h(B) \subset \operatorname{int}(B)$, we get $h^n(B) \subset B$. If $x \in B$, then for all $n > 0$ we have $h^n(x) \in B$. By definition, $x \in W^s(P,h)$ and therefore $f^n(x)$ converges to $P(h)$ as $n \to \infty$. This contradicts the fact $P(h) \notin B$. Thus we have proved that $W^s(P,h) \pitchfork W^u(P,h) \subset C$.

In the following we show $H(P,h) \subset C$. Take $y \in H(P,h)$. By definition there exists a sequence $(x_n) \subset W^s(P,h) \pitchfork W^u(P,h)$ converging to y as $n \to \infty$. Since $W^s(P,h) \pitchfork W^u(P,h) \subset A$ and A is compact, we have $H(P,h) \subset A$, in particular $y \in A$. To prove $y \notin B$, let us assume that $y \in B$. Then $h(y)$ belongs to $\operatorname{int}(B)$ and hence there exists a neighborhood U of $h(y)$ contained in $\operatorname{int}(B)$. Since h is continuous, the sequence $(h(x_n))$ converges to $h(y)$. This implies there exists N such that $h(x_N)$ belongs to U, in particular to B. Since $x_N \in W^s(P,h) \pitchfork W^u(P,h)$, we have $h(x_N) \in W^s(P,h) \pitchfork W^u(P,h)$. This is a contradiction, because we have already proved that $W^s(P,h) \pitchfork W^u(P,h)$ is disjoint from B. Therefore, we have proved $H(P,h) \subset C$ and finished the proof of Proposition A.1. □

The following proposition will be proved in section B.

Proposition A.2. *There exists $f \in \operatorname{Diff}(\mathbb{R}^4)$ satisfying the following properties:*

(w1) The support of f is compact, where the support of f is defined to be the closure of the set $\{x \in M \mid f(x) \neq x\}$.

(w2) There are two compact sets A, $B \subset \mathbb{R}^4$ with $B \subset A$, $f(A) \subset \operatorname{int}(A)$ and $f(B) \subset \operatorname{int}(B)$.

(w3) There exist two hyperbolic fixed points P and Q of f in $C := A \setminus B$.

(w4) f has a heterodimensional cycle associated to P and Q.

(w5) $\operatorname{ind}(P) = 3$, and let $\sigma(P), \mu_1(P), \mu_2(P)$ and $\mu_3(P)$ be the eigenvalues of $df(P)$ in non-decreasing order of their absolue values. Then $\mu_1(P)$ and $\mu_2(P)$ are in $\mathbb{C} \setminus \mathbb{R}$.

(w6) $\operatorname{ind}(Q) = 2$, and every eigenvalue of $df(Q)$ is in $\mathbb{C} \setminus \mathbb{R}$.

(w7) There exists a constant $K > 1$ such that $m\left(\Lambda^3(df)\right) > K$ on C (we furnish $\mathbb{R}^4$ with the standard Riemannian structure). In other words, df expands every three-dimensional subspace of $T_x\mathbb{R}^4$ in the volume for all $x \in C$ with the expanding rate greater than K.

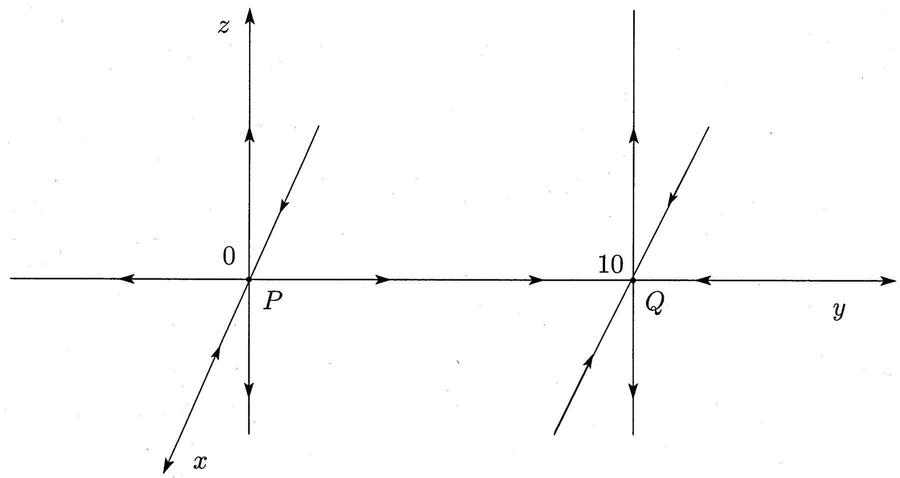

Figure 3: Dynamics of Φ.

Let us prove our 3 by using Proposition A.1 and A.2.

Proof of Theorem 3. Let us take a diffeomorphism f of $\mathbb{R}^4$ which satisfies $(w1)$–$(w7)$ by using Proposition A.2. Given a four-dimensional compact manifold M, we take a point $x \in M$ and a coordinate chart $\phi : U \to \mathbb{R}^4$ around x. By changing the coordinate if necessary, we can assume $\phi(U)$ contains the support of f. Then, we define a diffeomorphism $F \in \mathrm{Diff}^1(M)$ as follows: $F(x) = (\phi^{-1} \circ f \circ \phi)(x)$ for $x \in U$, otherwise $F(x) = x$. Let us denote the standard Riemannian metric on $\mathbb{R}^4$ by g. By the partition of unity, we construct a Riemannian metric $\tilde{g}$ on M that coincides with the pullback of g by ϕ at every point in $\phi^{-1}(A)$. Now, by applying Proposition A.1 to the triplet $(M, F, \tilde{g})$, we can find a diffeomorphism and its open neighborhood in $\mathrm{Diff}^1(M)$ that satisfy the conclusion of Theorem 3. $\square$

Now, for the proof of Theorem 3, we only need to prove Proposition A.2, whose proof will be given in the next section.

B Construction of the diffeomorphism

In this section, we give the proof of Proposition A.2.

B.1 Notations and a sketch of the proof

In this subsection, we prepare some notations and provide the idea of our proof of Proposition A.2.

We identify every point $X \in \mathbb{R}^4$ with the vector that starts from the origin and ends at X. Under this identification, we define the addition between any two points of $\mathbb{R}^4$ and multiplication by real numbers. We denote the standard Euclidean distance of $\mathbb{R}^4$ by $d(\cdot\,,\cdot)$. We also use the ℓ^∞-distance of $\mathbb{R}^4$ and denote it by $d_1(\cdot\,,\cdot)$. More presicely, given $X = (x_1, x_2, x_3, x_4)$ and $Y = (y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$, we define

$$d(X,Y) := \left(\sum_{i=1}^{4}(x_i - y_i)^2\right)^{1/2}, \quad d_1(X,Y) := \max_{i=1,\dots,4}|x_i - y_i|.$$

Let us define some points in $\mathbb{R}^4$ as follows:

$$\begin{aligned} P &:= (0,0,0,0), & Q &:= (0,10,0,0), & C_1 &:= (0,10,5,0), \\ C_2 &:= (10,10,5,0), & C_3 &:= (10,0,5,0), & C_4 &:= (10,0,0,0). \end{aligned}$$

We also define three subsets in $\mathbb{R}^4$ as follows:

$$\begin{aligned} \ell_1 &:= \{(10+x,0,0,0) \mid |x| < 0.2\}, & \ell_2 &:= \{(0,5+y,0,0) \mid |y| < 1\}, \\ \ell_3 &:= \{(0,10,z,w) \mid |z|,|w| < 0.2\}. \end{aligned}$$

For $X \in \mathbb{R}^4$ and $l > 0$, we denote by $\mathcal{B}(X,l)$ the four-dimensional cube with edges of length $2l$ centered at X. More presicely, we put

$$\mathcal{B}(X,l) := \{Z \in \mathbb{R}^4 \mid d_1(X,Z) \leq l\}.$$

With this notation, we define

$$\begin{aligned} B_l(P) &:= \mathcal{B}(P,1/100), & B_l(Q) &:= \mathcal{B}(Q,1/100), \\ B_s(P) &:= \mathcal{B}(P,1/300), & B_s(Q) &:= \mathcal{B}(Q,1/300). \end{aligned}$$

Given $X, Y \in \mathbb{R}^4$ and $l < 2d(X,Y)$, we denote by $\mathcal{C}(X,Y,l)$ the four-dimensional box defined as the collection of points whose ℓ^∞-distance from the segment joining X and Y is less than l. More presicely, we put

$$\mathcal{C}(X,Y,l) := \bigcup_{0 \leq t \leq 1} \mathcal{B}(tX + (1-t)Y, l).$$

With this notation, we define

$$D := \mathcal{C}(C_1, C_2, 1) \cup \mathcal{C}(C_2, C_3, 0.7) \cup \mathcal{C}(C_3, C_4, 0.4).$$

Throughout this section, λ denotes a real number greater than 20. We define as follows:

$$\begin{aligned} A &:= [-\lambda^2, \lambda^2]^4, \\ C &:= [-\lambda^2, \lambda^2]^2 \times (-1/2, 1/2)^2, & C' &:= [-\lambda^2, \lambda^2]^2 \times (-7,7)^2, \\ B &:= A \setminus C, & B' &:= A \setminus C'. \end{aligned}$$

Note that these sets depend on the value of λ, while the points and sets defined above are independent of λ.

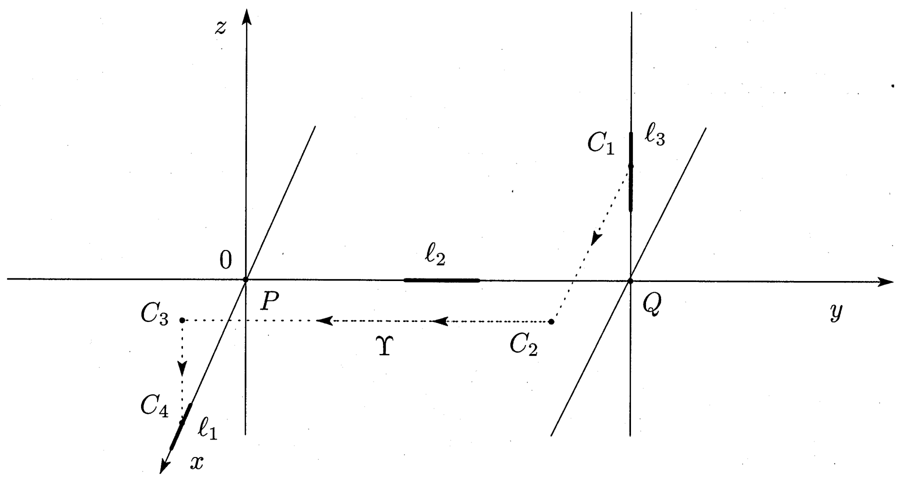

Figure 4: The role of Υ.

Let us give the idea of the proof of Proposition A.2. For the proof, we construct two diffeomorphisms Φ and Υ of $\mathbb{R}^4$. The composition $\Upsilon \circ \Phi$ gives a diffeomorphism that satisfies $(w1)$–$(w7)$ of Proposition A.2 except $(w4)$. Let us explain the role of each diffeomorphism. Φ has two hyperbolic fixed points P and Q whose eigenvalues satisfies $(w5)$ and $(w6)$, and there is a non-empty intersection between $W^u(P)$ and $W^s(Q)$ (see Figure 3). Furthermore, Φ is constructed so that once a point escapes, then it never returns close to P and Q (this corresponds to $(w2)$).

To obtain the property $(w4)$, we need to connect $W^u(Q)$ and $W^s(P)$. The diffeomorphism Υ is utilized for this purpose. Υ pushes $W^u(Q, \Phi)$ so that it has an intersection with $W^s(P)$. We need to guaranatee that this perturbation has little effect on the connection between $W^u(P)$ and $W^s(Q)$ and on the whole structure of the dynamics. So the push makes a detour (see Figure 4).

The diffeomorphism Υ is constructed independent of the value of λ. After the construction of Υ, we choose λ sufficiently large so that the effect of $d\Upsilon$ is ignorable when we check $(w7)$ for $\Upsilon \circ \Phi$.

Finally, we prepare a lemma that we frequently use throughout this section. We omit its proof.

Lemma B.1. *Given two closed intervals $[a, b] \subset [c, d]$ (a, b, c, d may be $\pm\infty$), there exists a C^∞-function $\rho[a, b, c, d](t) : \mathbb{R} \to \mathbb{R}$ that satisfies the following properties:*

- *For all $t \in \mathbb{R}, 0 \leq \rho[a, b, c, d](t) \leq 1$.*
- *For $t \in [a, b]$, $\rho[a, b, c, d](t) = 1$.*
- *For $t \notin [c, d]$, $\rho[a, b, c, d](t) = 0$.*

B.2 Construction of Φ

In this section, we construct the diffeomorphism Φ.

Proposition B.1. *There exists a diffeomorphism $\Phi : \mathbb{R}^4 \to \mathbb{R}^4$ that has the following properties:*

(Φ1) The support of Φ is contained in $[-\lambda^3 - 1, \lambda^3 + 1]^4$.

(Φ2) $\Phi(A) \subset \mathrm{int}(A)$ and $\Phi(B) \subset \mathrm{int}(B')$.

(Φ3) P and Q are hyperbolic fixed points of Φ.

(Φ4) $\mathrm{ind}(P) = 3$, and let $\sigma(P), \mu_1(P), \mu_2(P)$ and $\mu_3(P)$ be the eigenvalues of $df(P)$ in the non-decreasing order of their absolue values. Then, $\mu_1(P)$ and $\mu_2(P)$ are in $\mathbb{C} \setminus \mathbb{R}$.

(Φ5) $\ell_1 \subset W^s(P, \Phi)$ and $\cup_{n\geq 1}\Phi^n(\ell_1) \cap D = \emptyset$.

(Φ6) $\ell_2 \subset W^u(P, \Phi)$ and $\cup_{n\geq 0}\Phi^{-n}(\ell_2) \cap D = \emptyset$.

(Φ7) $\mathrm{ind}(Q) = 2$, and every eigenvalue of $d\Phi(Q)$ is in $\mathbb{C} \setminus \mathbb{R}$.

(Φ8) $\ell_2 \subset W^s(Q, \Phi)$ and $\cup_{n\geq 0}\Phi^n(\ell_2) \cap D = \emptyset$.

(Φ9) $\ell_3 \subset W^u(Q, \Phi)$ and $\cup_{n\geq 1}\Phi^{-n}(\ell_3) \cap D = \emptyset$.

(Φ10) There exists a constant $c_\Phi > 0$ (independent of λ) such that the inequality $m\left(\Lambda^3(d\Phi)(X)\right) > c_\Phi\lambda$ holds for every $X \in C$.

To prove Proposition B.1, we create auxiliary diffeomorphisms F, G and H of $\mathbb{R}$ and Θ of $\mathbb{R}^4$. After that, we give the proof of Proposition B.1.

Lemma B.2. *There exists a C^∞-diffeomorphism F of $\mathbb{R}$ that satisfies the following properties:*

(F1) The support of F is contained in $[-\lambda^3, \lambda^3]$.

(F2) For any $x \in [-\lambda^2, \lambda^2] \setminus \{0\}$, the inequality $0 < F(x)/x < 1/9$ holds.

(F3) There is a constant $c_1 > 0$ (independent of λ) such that the inequality $\min_{x\in[-\lambda^2,\lambda^2]} F'(x) > c_1$ holds.

Proof. Let us consider the vector field on $\mathbb{R}$ given by $\dot{x}(t) = f(x)$, where $f(x)$ is a C^∞-function on $\mathbb{R}$ that has the following properties:

- If $|x| < \lambda^3 - 2, f(x) = -(\log 10)x$.
- If $|x| > \lambda^3 - 1, f(x) = 0$.

We can construct such f as follows:

$$f(x) = -(\log 10)x\rho[-(\lambda^3-2), \lambda^3-2, -(\lambda^3-1), \lambda^3-1](x).$$

Let us take the time-1 map of this vector field and denote it by $F(x)$ (we can consider the time-1 map for all $x \in \mathbb{R}$ since f has compact support and Lipschitz continuous). Then, it is not difficult to check that $F(x)$ satisfies all the properties $(F1)$–$(F3)$ and the detail is left to the reader. $\square$

Remark B.1. *The conditions (F1)–(F3) imply that F satisfies the followoing conditions:*

(F4) 0 *is an attracting fixed point of* F.

(F5) $F([-\lambda^2, \lambda^2]) \subset (-\lambda^2, \lambda^2)$.

Lemma B.3. *There exists a C^∞-diffeomorphism G of $\mathbb{R}$ satisfies the following properties:*

(G1) The support of G is contained in $[-\lambda^3, \lambda^3]$.

(G2) $G([-\lambda^2, \lambda^2]) \subset (-\lambda^2, \lambda^2)$.

(G3) For any $y \in [-1/100, 1/100] \setminus \{0\}$, the inequality $9 < G(y)/y < 10$ holds.

(G4) For any $y \in [-1/100, 1/100] \setminus \{0\}$, the inequality $0 < (G(y+10)-10)/y < 1/10$ holds.

(G5) For any $y \in (0, 10)$, $\lim_{n\to\infty} G^{-n}(y) = 0$ and $\lim_{n\to\infty} G^n(y) = 10$.

(G6) There is a constant $c_2 > 0$ (independent of λ) such that the inequality $\min_{y\in[-\lambda^2,\lambda^2]} G'(y) > c_2$ holds.

Proof. Let us consider the vector field $\dot{y}(t) = g(y)$, where $g(y)$ is a C^∞-function on $\mathbb{R}$ satisfying the following properties:

- If $|y| < \lambda^3 - 2, g(y) = -\{(\log 10)/100\}(y^3 - 100y)$.
- If $|y| > \lambda^3 - 1, g(y) = 0$.

We can construct such g as follows:

$$g(y) = -\{(\log 10)/100\}(y^3 - 100y)\rho[-(\lambda^3-2), \lambda^3-2, -(\lambda^3-1), \lambda^3-1](y).$$

Let us take the time-1 map of this vector field and denote it by $G(y)$ (we can consider the time-1 map for all $y \in \mathbb{R}$ by the same reason for $F(x)$). Then, it is not difficult to check $G(y)$ satisfies all the property $(G1)$–$(G6)$. The detail is left to the reader. $\square$

Remark B.2. *The conditions (G1)–(G6) imply that G satisfies the followoing conditions:*

(G7) 0 *is a repelling fixed point of* G.

(G8) 10 *is an attracting fixed point of* G.

Lemma B.4. *There exists a* C^∞*-diffeomorphism* H *of* $\mathbb{R}$ *satisfying the following properties:*

(H1) *For every* z, $H(-z) = -H(z)$.

(H2) *The support of* $H(z)$ *is contained in* $[-\lambda^3, \lambda^3]$.

(H3) $H([-\lambda^2, \lambda^2]) \subset (-\lambda^2, \lambda^2)$.

(H4) $H([1/2, \lambda^2]) \subset (7, \lambda^2)$.

(H5) *For* $0 \le z \le 1$, $H(z) = \lambda z$. *In particular,* 0 *is a repelling fixed point of* H.

Proof. First, we construct a C^∞-function $h(t)$ on $\mathbb{R}_{\ge 0}$ satisfying the following properties:

($h1$) For all $t \ge 0, h(t) > 0$.

($h2$) For $0 \le t \le 1$, $h(t) = \lambda$.

($h3$) $\int_1^2 h(t)\,dt = 1$.

($h4$) For $2 \le t \le \lambda^2, h(t) = \lambda^{-2}$.

($h5$) $\int_0^{\lambda^3 - 1} h(t)\,dt = \lambda^3 - 1$

($h6$) For $t \ge \lambda^3 - 1, h(t) = 1$.

Let us discuss how to construct such $h(t)$. We put

$$\begin{aligned}
\rho_1(t) &:= \left(\lambda - \lambda^{-2}\right) \rho[-\infty, 1, -\infty, 1 + \lambda^{-2}](t), \\
\rho_2(t; \alpha) &:= \alpha \rho[7/5, 8/5, 6/5, 9/5](t), \\
\rho_3(t) &:= \left(1 - \lambda^{-2}\right) \rho[\lambda^3 - 1, +\infty, \lambda^3 - 2, +\infty](t), \\
\rho_4(t; \beta) &:= \beta \rho[\lambda^3 - 8/5, \lambda^3 - 7/5, \lambda^3 - 9/5, \lambda^3 - 6/5](t),
\end{aligned}$$

and take a function $\eta(t; \alpha, \beta) := \rho_1(t) + \rho_2(t; \alpha) + \rho_3(t) + \rho_4(t; \beta) + \lambda^{-2}$. Note that, for all $\alpha, \beta \ge 0$, the function $\eta(t; \alpha, \beta)$ satisfies the conditions ($h1$)–($h6$) except ($h3$) and ($h5$). We show there exist positive real numbers α_0 and β_0 for which $\eta(t; \alpha_0, \beta_0)$ satisfies ($h3$) and ($h5$).

First, let us see how to take α_0 for which $\eta(t; \alpha_0, \beta)$ satisfies ($h3$). Let us put $\Xi(\alpha) = \int_1^2 \eta(t; \alpha, \beta) dt$. Then, $\Xi(\alpha)$ is independent of β, continuous, monotone increasing and $\Xi(\alpha) \to +\infty$ as $\alpha \to +\infty$. Moreover, $\Xi(0) \le \lambda \cdot \lambda^{-2} + 1 \cdot \lambda^{-2} < 1$ (remember that we assume $\lambda > 20$). Hence, by the intermediate value theorem, we get α_0 for which $\eta(t; \alpha_0, \beta)$ satisfies ($h3$). In a similar way, we can find β_0 so that ($h5$) holds for $\eta(t; \alpha_0, \beta_0)$.

Then, by putting $H(z) := \int_0^z h(t)dt$ for $z \geq 0$, and $H(z) = -H(-z)$ for $z \leq 0$, we construct a map $H(z)$. It is easy to see that $H(z)$ is a C^∞-diffeomorphism and enjoys $(H1)$, $(H2)$ and $(H5)$. Let us check $(H3)$ and $(H4)$. We have $H(1/2) = \lambda/2 > 7$ and $H(\lambda^2)$ can be estimated as follows:

$$H(\lambda^2) = H(2) + (\lambda^2 - 2)\lambda^{-2} = \lambda + 1 + 1 - 2/\lambda^2 < \lambda + 2 < \lambda^2.$$

Hence we have proved $(H3)$ and $(H4)$ for $H(z)$. □

Lemma B.5. *There exists a C^∞-diffeomorphism Θ of $\mathbb{R}^4$ that has the following properties:*

(Θ*1) The support of Θ is contained in $B_l(P) \cup B_l(Q)$.*

(Θ*2) For $X = (x, y, z, w) \in B_s(P), \Theta(x, y, z, w) = (x, -z, y, w)$, in particular, P is a fixed point of Θ.*

(Θ*3) Θ fixes every point in the x-axis, more presicely, for $X = (x, 0, 0, 0), \Theta(X) = X$.*

(Θ*4) For any $X \in B_l(P)$, Θ preserves the d-distance between P and X, more precisely, $d(P, \Theta(X)) = d(P, X)$.*

(Θ*5) In $B_l(P)$, Θ preserves the yz-plane, more presicely, for $X = (0, y, z, 0) \in B_l(P)$, the x-coordinate and the w-coordinate of $\Theta(X)$ are 0.*

(Θ*6) For $(x, y + 10, z, w) \in B_s(Q), \Theta(x, y + 10, z, w) = (-y, x + 10, -w, z)$, in particular, Q is a fixed point of Θ.*

(Θ*7) For any $X \in B_l(Q)$, Θ preserves the d-distance between Q and X.*

(Θ*8) In $B_l(Q)$, Θ preserves the xy-plane. More presicely, for $X = (x, y, 0, 0) \in B_l(Q)$, the z-coordinate and w-coordinate of $\Theta(X)$ are 0.*

(Θ*9) In $B_l(Q)$, Θ preserves the plane that passes Q and parallel to the zw-plane, more presicely, for $X = (0, 10, z, w) \in B_l(Q)$, the x-coordinate of $\Theta(X)$ is 0 and the y-coordinate of $\Theta(X)$ is 10.*

Proof. First, we define three functions $\rho_1(t), \omega_1(X), \omega_2(X)$ as follows:

$$\begin{aligned}
\rho_1(t) &:= \rho[-1/300, 1/300, -1/200, 1/200](t), \\
\omega_1(x, y, z, w) &:= (\pi/2)\rho_1(x)\rho_1(y)\rho_1(z)\rho_1(w), \\
\omega_2(x, y, z, w) &:= (\pi/2)\rho_1(x)\rho_1(y - 10)\rho_1(z)\rho_1(w).
\end{aligned}$$

We also define a map $R[\alpha] : \mathbb{R}^2 \to \mathbb{R}^2$ to be the rotation of angle α, more precisely, for $(x, y) \in \mathbb{R}^2$, we put

$$R[\alpha](x, y) := ((\cos\alpha)x - (\sin\alpha)y, \ (\sin\alpha)x + (\cos\alpha)y).$$

Then, we define Θ as follows:

- For $X = (x, y, z, w) \in B_l(P)$, $\Theta(X) := (x, R[\omega_1(X)](y, z), w)$.
- For $X = (x, y + 10, z, w) \in B_l(Q)$,
$$\Theta(X) := (R[\omega_2(X)](x, y), R[\omega_2(X)](z, w)) + Q.$$
- Otherwise, $\Theta(X) := X$.

Now, it is not difficult to see that Θ is a C^∞-diffeomorphism satisfying all the properties (Θ1)–(Θ9). $\square$

Lemma B.6. *There is a C^∞-diffeomorphism Ψ of $\mathbb{R}^4$ satisfying the following properties:*

(Ψ1) The support of Ψ is contained in $[-\lambda^3 - 1, \lambda^3 + 1]^4$.

(Ψ2) For $(x, y, z, w) \in [-\lambda^3, \lambda^3]^4$, $\Psi(x, y, z, w) = (F(x), G(y), H(z), H(w))$.

Proof. We define the functions ρ_1 on $\mathbb{R}$ and R_i ($i = 1, \ldots, 4$) on $\mathbb{R}^4$ as follows:

$$\rho_1(t) := \rho[-\lambda^3, \lambda^3, -\lambda^3 - 1, \lambda^3 + 1](t),$$
$$R_1(X) := \rho_1(x_2)\rho_1(x_3)\rho_1(x_4), \qquad R_2(X) := \rho_1(x_1)\rho_1(x_3)\rho_1(x_4),$$
$$R_3(X) := \rho_1(x_1)\rho_1(x_2)\rho_1(x_4), \qquad R_4(X) := \rho_1(x_1)\rho_1(x_2)\rho_1(x_3),$$

where $X = (x_1, x_2, x_3, x_4)$. Then, we define a map $\Psi : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$\Psi_i(X) := R_i(X)F_i(x_i) + (1 - R_i(X))x_i, \quad \text{for } i = 1, 2, 3, 4,$$

where $\Psi_i(X)$ denotes the i-th coordinate of $\Psi(X)$ and we put $F_1 := F$, $F_2 := G$ and $F_3 = F_4 := H$ as the matter of convenience.

It is easy to see that Ψ is a C^∞-map and satisfies (Ψ1) and (Ψ2). The perplexing part is to confirm that Ψ is a diffeomorphism. To see this, we put $I_0 := [-\lambda^3, \lambda^3]$, $I_1 := \mathbb{R} \setminus I_0$, $S := \{(\sigma_i)_{i=1}^4 \mid \sigma_i = 0, 1\}$ and for every $(\sigma_i) \in S$ we put $I[(\sigma_i)] := \prod_{i=1}^4 I_{\sigma_i}$. Then, divide $\mathbb{R}^4$ into sixteen subsets as follows:

$$\mathbb{R}^4 = \coprod_{(\sigma_i) \in S} I[(\sigma_i)].$$

We claim that, for every $(\sigma_i) \in S$, the restriction of Ψ to $I[(\sigma_i)]$ is a diffeomorphism. Let us fix $(\sigma_i) \in S$ and put $j := \sum \sigma_i$. We define

$$P[(\sigma_i)] := \{(\sigma_i x_i) \in \mathbb{R}^4 \mid x_i \in I_1\},$$

and for $Y = (\sigma_i y_i) \in P[(\sigma_i)]$,

$$S([(\sigma_i)], Y) := \{Y + ((1 - \sigma_i)z_i) \mid z_i \in I_0\}.$$

Then, we have the following decomposition:

$$I[(\sigma_i)] = \coprod_{Y \in P[(\sigma_i)]} S([(\sigma_i)], Y).$$

Intuitively speaking, we divided $I[(\sigma_i)]$ into $(4-j)$-dimensional cubes that are parametrized by j-dimensional parameters.

First, we prove that $\Psi_k|_{I[(\sigma_i)]}(X) = x_k$ for k with $\sigma_k = 1$. One important observation to see this is that each F_k is the identity map on I_1 by $(F1)$, $(G1)$ and $(H2)$. So, for k with $\sigma_k = 1$, we have $\Psi_k|_{I[(\sigma_i)]}(X) = R_k(X)x_k + (1 - R_k(X))x_k = x_k$ independent of the value of $R_k(X)$.

Second, we investigate the behavior of the restriction of Ψ to $S([(\sigma_i)], Y)$. We show that $\Psi_k|_{S([(\sigma_i)],Y)}(X) = \tilde{F}_k(x_k)$ for k with $\sigma_k = 0$, where $\tilde{F}_k(x_i)$ is some diffeomorphism of I_0. The key point is that the change of X in $S([(\sigma_i)], Y)$ never varies $R_k(X)$. Indeed, for l with $\sigma_l = 1$, the l-th coordinate of X is fixed, and for l that satisfies $\sigma_l = 0$, l-th coordinate of X is in I_0. Hence the change of X in $S([(\sigma_i)], Y)$ give no effect on $R_k(X)$. Now the formula $\Psi_k|_{S([(\sigma_i)],Y)}(X) = R_k(X)F_k(x_k) + (1 - R_k(X))x_k$ tells us that the RHS gives a diffeomorphism of I_0, since it is a convex combination of diffeomorphisms of I_0. Therefore, $\Psi|_{S([(\sigma_i)],Y)}$ is a diffeomorphism of $S([(\sigma_i)], Y)$.

We have proved that $\Psi|_{I[(\sigma_i)]}$ is a diffeomorphism when it is restricted to $S([(\sigma_i)], Y)$ and behaves as the identity map in the $P[(\sigma_i)]$ direction. These two facts say that $\Psi|_{I[(\sigma_i)]}$ is a diffeomorphism of $I[(\sigma_i)]$. $\square$

Now, let us give the proof of Proposition B.1.

Proof of Proposition B.1. We put $\Phi := \Theta \circ \Psi$. From the properties of F, G, H, Θ and Ψ, we can see that Φ satisfies $(\Phi1)$–$(\Phi10)$. Indeed,

- $(\Phi1)$ follows from $(\Psi1)$ and $(\Theta1)$.
- $(\Phi2)$ follows from $(F5)$, $(G2)$, $(H3)$, $(H4)$ and $(\Theta1)$.
- $(\Phi3)$ follows from $(F4)$, $(G7)$, $(G8)$, $(H5)$, $(\Theta2)$ and $(\Theta6)$.
- $(\Phi4)$ follows from $(F2)$, $(G3)$, $(H5)$ and $(\Theta2)$.
- $(\Phi5)$ follows from $(F2)$, $(\Theta1)$ and $(\Theta3)$.
- $(\Phi6)$ follows from $(F2)$, $(G3)$, $(G5)$, $(\Theta1)$, $(\Theta4)$ and $(\Theta5)$.
- $(\Phi7)$ follows from $(F4)$, $(G8)$, $(H5)$ and $(\Theta6)$.
- $(\Phi8)$ follows from $(F2)$, $(G4)$, $(G5)$, $(\Theta1)$, $(\Theta7)$ and $(\Theta8)$.
- $(\Phi9)$ follows from $(H5)$, $(\Theta1)$, $(\Theta7)$, and $(\Theta9)$.

Let us check $(\Phi10)$. We investigate the action of $\Lambda^3(d\Psi)(X)$ (we put $X = (x, y, z, w)$) to the orthonormal basis $\langle e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4 \rangle$, where $\langle e_i \rangle$ denotes the standard orthonormal basis of $T_X\mathbb{R}^4$.

$$
\begin{aligned}
\Lambda^3(d\Psi)(X)(e_1 \wedge e_2 \wedge e_3) &= F'(x)G'(y)H'(z)e_1 \wedge e_2 \wedge e_3 \\
\Lambda^3(d\Psi)(X)(e_1 \wedge e_2 \wedge e_4) &= F'(x)G'(y)H'(w)e_1 \wedge e_2 \wedge e_4 \\
\Lambda^3(d\Psi)(X)(e_1 \wedge e_3 \wedge e_4) &= F'(x)H'(z)H'(w)e_1 \wedge e_3 \wedge e_4 \\
\Lambda^3(d\Psi)(X)(e_2 \wedge e_3 \wedge e_4) &= G'(y)H'(z)H'(w)e_2 \wedge e_3 \wedge e_4
\end{aligned}
$$

This calculation tells us that $m\left(\Lambda^3(d\Psi)(X)\right)$ is the minimum of the lengths of these four vectors. Let us fix a real number $c > 0$ such that

$$\min_{(x,y)\in[-\lambda^2,\lambda^2]^2} \{|F'(x)G'(y)|, |F'(x)|, |G'(y)|\} > c.$$

We can take such c by $(F3)$ and $(G6)$. Note that c can be choosen independent of λ. Since $H'(z) = H'(w) = \lambda$ for any $(x, y, z, w) \in C$, we have proved the lengths of these four vectors are bounded below by $c\lambda$ and $m\left(\Lambda^3(d\Psi)(X)\right) > c\lambda$ for all $X \in C$.

We put $c_\Theta := \min_{x\in\mathbb{R}^4} m(\Lambda^3(d\Theta)(x))$. Since Θ is a diffeomorphism and have compact support, c_Θ is a positive number. Now, for any $X \in C$, the inequality

$$m(\Lambda^3(d\Phi)(X)) \geq \min_{x\in\mathbb{R}^4} m(\Lambda^3(d\Theta)(x)) \cdot m\left(\Lambda^3(d\Psi)(X)\right) = c_\Theta c\lambda$$

holds and this implies $(\Phi 10)$ for $c_\Phi = c_\Theta c$.

Therefore, the proof is completed. □

B.3 Construction of Υ.

Proposition B.2. *There exists a C^∞-diffeomorphism Υ of $\mathbb{R}^4$ satisfying the following properties:*

(Υ1) The support of Υ is contained in D.

(Υ2) For $X \in \mathcal{B}(C_1, 0.2)$, $\Upsilon(X) = X + (10, -10, -5, 0)$.

To construct Υ, we need the following auxiliary diffeomorphim $\chi(X)$.

Lemma B.7. *Given two points $Y, Z \in \mathbb{R}^4$ and two real numbers $a > b > 0$ with $d(Y, Z) > 2a$, there exists a diffeomorphism $\chi[Y, Z, a, b](X) = \chi(X)$ of $\mathbb{R}^4$ satisfying the following properties:*

(χ1) The support of $\chi(X)$ is contained in $\mathcal{C}(Y, Z, a)$.

(χ2) For $X \in \mathcal{B}(Y, b)$, $\Upsilon(X) = X + Z - Y$.

Before the proof of Lemma B.7, let us see how one can construct Υ from χ. We put $l_n := 1.15 - 0.15n$ and

$$\chi_i(X) := \chi(C_i, C_{i+1}, l_{2i-1}, l_{2i}),$$

for $i = 1, 2, 3$. Then, we put $\Upsilon := \chi_3 \circ \chi_2 \circ \chi_1$. It is clear that Υ satisfies $(\Upsilon 1)$ and $(\Upsilon 2)$.

Let us give the proof of Lemma B.7.

Proof of Lemma B.7. By changing the coordinate, we can assume that Y is the origin of $\mathbb{R}^4$ and $Z = (\zeta, 0, 0, 0)$ where $\zeta = d(Y, Z)$. We put $c := (a + b)/2$.

First, we construct a diffeomorphism $\kappa(x)$ of $\mathbb{R}$ satisfying the following conditions:

($\kappa 1$) The support of κ is contained in $[-a, \zeta + a]$.

($\kappa 2$) For $x \in [-b, b]$, $\kappa(x) = x + \zeta$.

We will explain how to construct such a diffeomorphism later. Then, put $\rho_1(t) := \rho[-b, b, -c, c](t)$ and $R(x, y, z, w) := \rho_1(y)\rho_1(z)\rho_1(w)$. Finally, for $X = (x, y, z, w) \in \mathbb{R}^4$, we define

$$\chi[Y, Z, a, b](X) := (R(X)\kappa(x) + (1 - R(X))x, y, z, w)\,.$$

It is not difficult to see that χ satisfies the required conditions.

Let us see how to construct $\kappa(x)$. We prepare a C^∞-function $\eta(t)$ on $\mathbb{R}$ that satisfies the following properties:

($\eta 1$) $\eta(t) > 0$ for all $t \in \mathbb{R}$.

($\eta 2$) $\eta(t) = 1$ for $t < -c$, $-b < t < b$, or $t > \zeta + c$.

($\eta 3$) $\int_{-c}^{-b} \eta(t)\, dt = \zeta + c - b$.

($\eta 4$) $\int_{b}^{\zeta+c} \eta(t)\, dt = c - b$.

Then, $\kappa(x) := \zeta + \int_0^x \eta(t)\, dt$ is a C^∞-diffeomorphism satisfying ($\kappa 1$) and ($\kappa 2$). So, let us construct the function $\eta(t)$. We fix a positive real number $e < (c-b)/2$ and define

$$\rho_2(t) = \rho[-c+e, -b-e, -c, -b](t), \quad \rho_3(t) = \rho[b+e, \zeta+c-e, b, \zeta+c](t),$$

and

$$\eta(t; \alpha, \beta) = \exp(\alpha\rho_2(t) + \beta\rho_3(t)),$$

where α, β are some real numbers.

We show there exists α_1, β_1 such that $\eta(t; \alpha_1, \beta_1)$ satisfies ($\eta 1$)–($\eta 4$). The function $\eta(t; \alpha, \beta)$ satisfies the properties ($\eta 1$) and ($\eta 2$) for all α and β. Let us consider ($\eta 3$) and ($\eta 4$). For $t \in [-c, -b]$, $\eta(t; \alpha, \beta)$ is equal to $\exp(\alpha\rho_2(t))$. We put $J(\alpha) := \int_{-c}^{-b} \eta(t; \alpha, \beta)\, dt$. Then, one can check that $J(\alpha) \to 0$ as $\alpha \to -\infty$, $J(\alpha) \to +\infty$ as $\alpha \to +\infty$ and $J(\alpha)$ is continuous and monotone increasing. So, the intermediate value theorem says there exists α_1 satisfying $J(\alpha_1) = \zeta + c - b$. In a similar way, one can find β_1 such that $\int_b^{\zeta+c} \eta(t; \alpha_1, \beta_1)\, dt = c - b$. Hence, $\eta(t; \alpha_1, \beta_1)$ satisfies ($\eta 1$)–($\eta 4$).

□

B.4 Proof of Proposition A.2

Finally, let us give the proof of Proposition A.2.

Proof of Proposition A.2. Let us put $c_\Upsilon := \min_{x \in \mathbb{R}^4} m(\Lambda^3(d\Upsilon)(x))$. Since Υ is a diffeomorphism and has compact support, $c_\Upsilon > 0$. We fix $\lambda_0 > 0$ so that $K := c_\Phi c_\Upsilon \lambda_0 > 1$ holds. Then, we put $\Omega := \Upsilon \circ \Phi$ and show that Ω, A, B, P, Q and K satisfy the properties ($w1$)–($w7$) when $\lambda = \lambda_0$. Indeed,

- $(w1)$ follows from $(\Phi 1)$ and $(\Upsilon 1)$.
- $(w2)$ follows from $(\Phi 2)$ and $(\Upsilon 1)$.
- $(w3)$ follows from $(\Phi 3)$, and $(\Upsilon 1)$.
- $(w4)$ follows from $(\Phi 5)$, $(\Phi 6)$, $(\Phi 8)$, $(\Phi 9)$, $(\Upsilon 1)$ and $(\Upsilon 2)$.
- $(w5)$ follows from $(\Phi 4)$ and $(\Upsilon 1)$.
- $(w6)$ follows from $(\Phi 7)$ and $(\Upsilon 1)$.
- $(w7)$ follows from $(\Phi 10)$ and the definition of λ_0.

So, the proof is completed. □

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