J. Math. Sci. Univ. Tokyo
3 (1996), 57–72.

A Hamiltonian Path Integral for a Degenerate Parabolic Pseudo-Differential Operator

By Naoto Kumano-go

Dedicated to Professor Hikosaburo Komatsu on his sixtieth birthday

Abstract. The symbol of the fundamental solution for a degenerate parabolic pseudo-differential operator of order $m\,(>0)$ can be described in terms of a Hamiltonian path integral. This Hamiltonian path integral converges in the topology of the symbol class $\boldsymbol{S}^{2m}_{\lambda,\rho,\delta}$ and in the weak topology of the symbol class $\boldsymbol{S}^{0}_{\lambda,\rho,\delta}$.

0. Introduction

In this paper, we construct the fundamental solution for a degenerate parabolic pseudo-differential operator of order $m\,(>0)$ in a different way from that in C.Tsutsumi [10]. In [10], she constructed the fundamental solution by Levi-Mizohata method. On the other hand, in this paper, we construct the fundamental solution by a Hamiltonian path integral. If we use a Hamiltonian path integral, we can actually give an expression of the symbol of the fundamental solution. Furthermore, this Hamiltonian path integral converges in the topology of the symbol class $\boldsymbol{S}^{2m}_{\lambda,\rho,\delta}$ and in the weak topology of the symbol class $\boldsymbol{S}^{0}_{\lambda,\rho,\delta}$.

In Section 1, we introduce some basic properties of pseudo-differential operators, which we use in Section 2. For the details, see Chapter 7 § 1 and § 2 in H.Kumano-go [6]. In Section 2, we construct the fundamental solution for a degenerate parabolic pseudo-differential operator by a Hamiltonian path integral. Theorem 2.1 is the main theorem in this paper.

1991 *Mathematics Subject Classification.* Primary 35S10; Secondary 47G05, 58D30,. . . .

1. Pseudo-Differential Operators

For $x = (x_1, \dots, x_n) \in \boldsymbol{R}^n_x$, $\xi = (\xi_1, \dots, \xi_n) \in \boldsymbol{R}^n_\xi$ and multi-indices of non-negative integers $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$, we employ the usual notation:

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad |\beta| = \beta_1 + \cdots + \beta_n,$$

$$\alpha! = \alpha_1! \cdots \alpha_n!, \quad \beta! = \beta_1! \cdots \beta_n!,$$

$$x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n, \quad \langle x \rangle = (1 + |x|^2)^{1/2}, \quad \langle \xi \rangle = (1 + |\xi|^2)^{1/2},$$

$$\partial_{\xi_j} = \frac{\partial}{\partial \xi_j}, \quad D_{x_j} = -i\frac{\partial}{\partial x_j}, \quad \partial_\xi^\alpha = \partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_n}^{\alpha_n}, \quad D_x^\beta = D_{x_1}^{\beta_1} \cdots D_{x_n}^{\beta_n}.$$

$\mathcal{S}$ denotes the Schwartz space of rapidly decreasing C^∞-functions on $\boldsymbol{R}^n$. For $u \in \mathcal{S}$, we define semi-norms $|u|_{l,\mathcal{S}} (l = 0, 1, 2, \dots)$ by

$$|u|_{l,\mathcal{S}} \equiv \max_{k+|\alpha| \le l} \sup_x |\langle x \rangle^k \partial_x^\alpha u(x)| \quad (l = 0, 1, 2, \dots).$$

Then, $\mathcal{S}$ is a Fréchet space with these semi-norms.
For simplicity, we set $đ\eta \equiv (2\pi)^{-n} d\eta$ and $đ\xi \equiv (2\pi)^{-n} d\xi$.
Oscillatory integral of a function $a(\eta, y)$, is defined by the equality

$$\mathrm{O_s}\!-\!\iint e^{-iy\cdot\eta} a(\eta, y)\, dy\, đ\eta \equiv \lim_{\epsilon \to 0} \iint e^{-iy\cdot\eta} \chi(\epsilon\eta, \epsilon y) a(\eta, y)\, dy\, đ\eta\,,$$

where $\chi(\eta, y) \in \mathcal{S}$ in $\boldsymbol{R}^{2n}_{\eta,y}$ and $\chi(0, 0) = 1$. For the details, see Chapter 1 § 6 in H.Kumano-go [6].

Definition 1.1 (A weight function $\lambda(\xi)$).
We say that a real-valued C^∞-function $\lambda(\xi)$ on $\boldsymbol{R}^n_\xi$ is a weight function, if there exist constants $A_0, A_\alpha > 0$ such that

$$1 \le \lambda(\xi) \le A_0 \langle \xi \rangle\,, \tag{1.1}$$

$$|\partial_\xi^\alpha \lambda(\xi)| \le A_\alpha \lambda(\xi)^{1-|\alpha|}\,. \tag{1.2}$$

Examples.
1° $\lambda(\xi) = \langle \xi \rangle$.
2° $\lambda(\xi) = \left\{1 + \sum_{j=1}^n |\xi_j|^{2m_j}\right\}^{1/(2m)}$, $\quad (m_j \in \mathbb{N}, \quad m \equiv \max_{1 \le j \le n} \{m_j\})$.

Definition 1.2 (Pseudo-differential operators).

We say that a C^∞-function $p(x,\xi)$ on $\boldsymbol{R}^{2n}_{x,\xi}$ is a symbol of class $\boldsymbol{S}^m_{\lambda,\rho,\delta}$ ($m \in \boldsymbol{R}$, $0 \le \delta \le \rho \le 1$, $\delta < 1$), if for any α, β, there exists a constant $C_{\alpha,\beta}$ such that

$$(1.3) \qquad |p^{(\alpha)}_{(\beta)}(x,\xi)| \le C_{\alpha,\beta}\lambda(\xi)^{m+\delta|\beta|-\rho|\alpha|}\,,$$

where $p^{(\alpha)}_{(\beta)}(x,\xi) \equiv \partial^\alpha_\xi D^\beta_x p(x,\xi)$.

The pseudo-differential operator $p(X,D_x)$ with the symbol $p(x,\xi)$ is defined by

$$(1.4) \qquad p(X,D_x)u(x) \equiv \iint e^{i(x-x')\cdot\xi} p(x,\xi)u(x')dx'đ\xi \quad (u \in \mathcal{S})\,,$$

where $đ\xi \equiv (2\pi)^{-n}d\xi$.

Remark.

1° For simplicity, we set $p^{(\alpha)}_{(\beta)}(x,\xi) \equiv \partial^\alpha_\xi D^\beta_x p(x,\xi)$, $p^{(\alpha)}(x,\xi) \equiv \partial^\alpha_\xi p(x,\xi)$ and $p_{(\beta)}(x,\xi) \equiv D^\beta_x p(x,\xi)$ for any α, β.

2° The symbol class $\boldsymbol{S}^m_{\lambda,\rho,\delta}$ is a Fréchet space with the semi-norms

$$(1.5) \quad |p|^{(m)}_l \equiv \max_{|\alpha+\beta|\le l} \sup_{(x,\xi)} \{|p^{(\alpha)}_{(\beta)}(x,\xi)|\lambda(\xi)^{-(m+\delta|\beta|-\rho|\alpha|)}\} \quad (l = 0,1,2,\dots)\,.$$

3° The continuity of $p(X,D_x) : \mathcal{S} \longrightarrow \mathcal{S}$ is clear. Furthermore, we can extend $p(X,D_x) : \mathcal{S} \longrightarrow \mathcal{S}$ to $p(X,D_x) : \mathcal{S}' \longrightarrow \mathcal{S}'$ by means of

$$(1.6) \qquad (p(X,D_x)u, v) \equiv (u, p(X,D_x)^*v) \quad \text{for} \quad u \in \mathcal{S}'\,, v \in \mathcal{S}\,.$$

Theorem 1.3 (Multi-products).

Let M be a positive constant and let $\{m_j\}_{j=1}^\infty$ be a sequence of real numbers satisfying

$$(1.7) \qquad \sum_{j=1}^\infty |m_j| \le M < \infty\,.$$

For any $\nu = 1, 2, \ldots$ *and* $p_j(x,\xi) \in \boldsymbol{S}^{m_j}_{\lambda,\rho,\delta}(j = 1, 2, \ldots, \nu+1)$*, there exists* $q_{\nu+1}(x,\xi) \in \boldsymbol{S}^{\bar{m}_{\nu+1}}_{\lambda,\rho,\delta}\,(\bar{m}_{\nu+1} \equiv m_1 + m_2 + \cdots + m_{\nu+1})$ *such that*

$$q_{\nu+1}(X, D_x) = p_1(X, D_x)p_2(X, D_x)\cdots p_{\nu+1}(X, D_x)\,. \tag{1.8}$$

Furthermore, for any l*, there exist a constant* A_l *and an integer* l' *such that*

$$|q_{\nu+1}|_l^{(\bar{m}_{\nu+1})} \leq (A_l)^\nu \prod_{j=1}^{\nu+1} |p_j|_{l'}^{(m_j)}\,, \tag{1.9}$$

where A_l *and* l' *depend only on* M *and* l*, but are independent of* ν*.*

PROOF. See Theorem 2.4 in Chapter 7 § 2 of H.Kumano-go [6]. □

THEOREM 1.4.
Let $p_j(x,\xi) \in \boldsymbol{S}^{m_j}_{\lambda,\rho,\delta}(j = 1, 2)$*. Define* $q_\theta(x,\xi)$ $(|\theta| \leq 1)$ *by*

$$q_\theta(x,\xi) \equiv \mathrm{O_s}-\iint e^{-iy\cdot\eta}p_1(x,\xi+\theta\eta)p_2(x+y,\xi)dy\,đ\eta\,. \tag{1.10}$$

Then $\{q_\theta(x,\xi)\}_{|\theta|\leq 1}$ *is a bounded set of* $\boldsymbol{S}^{m_1+m_2}_{\lambda,\rho,\delta}$*. Furthermore, for any* l*, there exist a constant* A_l *and an integer* l' *independent of* θ *such that*

$$|q_\theta|_l^{(m_1+m_2)} \leq A_l |p_1|_{l'}^{(m_1)} |p_2|_{l'}^{(m_2)}\,. \tag{1.11}$$

PROOF. See Lemma 2.4 in Chapter 2 § 2 or Lemma 2.2 in Chapter 7 §2 of H.Kumano-go [6]. □

2. The Main Theorem

THEOREM 2.1 (The main theorem).
Let $K(t,x,\xi) \in \mathcal{C}^0([0,T]; \boldsymbol{S}^m_{\lambda,\rho,\delta})$ $(m > 0\,, 0 \leq \delta < \rho \leq 1)$*. Assume that* $K(t,x,\xi)$ *satisfies the following conditions* (a1), (a2) *:*

(a1) *There exist constants* $c > 0$ *and* $m'(0 \leq m' \leq m)$ *such that*

$$Re\ K(t,x,\xi) \leq -c\lambda(\xi)^{m'} \ \ on\ [0,T] \times \boldsymbol{R}^{2n}_{x,\xi}\,. \tag{2.1}$$

(a2) *For any* α, β*, there exists a constant* $C_{\alpha,\beta}$ *such that*

$$|K^{(\alpha)}_{(\beta)}(t,x,\xi)/Re\ K(t,x,\xi)| \leq C_{\alpha,\beta}\lambda(\xi)^{\delta|\beta|-\rho|\alpha|} \ \ on\ [0,T] \times \boldsymbol{R}^{2n}_{x,\xi}\,. \tag{2.2}$$

Then we have the following (1) – (5) *:*

(1) *Let* $\mathit{\Delta}_{t,s} : (T \geq) t \equiv t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu+1} \equiv s(\geq 0)$ *be an arbitrary division of interval* $[s,t]$ *into subintervals, and let* $e^{(t_j - t_{j+1})K(t_{j+1})}(X, D_x)$ *be an operator defined by*

$$\begin{aligned} e^{(t_j-t_{j+1})K(t_{j+1})}&(X,D_x)u(x) \\ &\equiv \iint e^{i(x-x')\cdot\xi} e^{(t_j-t_{j+1})K(t_{j+1},x,\xi)} u(x')dx' đ\xi\,. \end{aligned} \tag{2.3}$$

Then there exists $p(\mathit{\Delta}_{t,s}; x, \xi) \in \boldsymbol{S}^0_{\lambda,\rho,\delta}$ *such that*

$$\begin{aligned} p(\mathit{\Delta}_{t,s}; X, D_x) = e^{(t-t_1)K(t_1)}&(X,D_x) e^{(t_1-t_2)K(t_2)}(X,D_x) \\ &\cdots e^{(t_\nu - s)K(s)}(X,D_x)\,. \end{aligned} \tag{2.4}$$

(2) *For any* l*, there exist constants* C_l, C'_l *and an integer* l' *such that*

$$|p(\mathit{\Delta}_{t,s})|^{(0)}_l \leq C_l\,, \tag{2.5}$$

and

$$\begin{aligned} &|p(\mathit{\Delta}_{t,s}) - p(\mathit{\Delta}'_{t,s})|^{(2m)}_l \\ &\leq C'_l(t-s)\Big(|\mathit{\Delta}_{t,s}| + \sup_{|t'-t''|\leq|\mathit{\Delta}_{t,s}|} |K(t') - K(t'')|^{(m)}_{l'}\Big)\,. \end{aligned} \tag{2.6}$$

Here, $\mathit{\Delta}_{t,s} : (T \geq) t \equiv t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu+1} \equiv s(\geq 0)$ *is an arbitrary division of interval* $[s,t]$ *into subintervals,* $\mathit{\Delta}'_{t,s}$ *is an arbitrary refinement of* $\mathit{\Delta}_{t,s}$*,* $|\mathit{\Delta}_{t,s}|$ *denotes the size of division defined by* $|\mathit{\Delta}_{t,s}| \equiv \max_{0\leq j\leq\nu} |t_j - t_{j+1}|$*, and the constants* C_l, C'_l *and the integer* l' *are independent of* ν*,* $\mathit{\Delta}_{t,s}$ *and* $\mathit{\Delta}'_{t,s}$*.*

(3) *There exists* $p^\star(t,s;x,\xi) \in \boldsymbol{S}^0_{\lambda,\rho,\delta}$ *such that* $p(\mathit{\Delta}_{t,s}; x, \xi)$ $(\in \boldsymbol{S}^0_{\lambda,\rho,\delta})$ *converges to* $p^\star(t,s;x,\xi)$ $(\in \boldsymbol{S}^0_{\lambda,\rho,\delta})$ *in* $\boldsymbol{S}^{2m}_{\lambda,\rho,\delta}$ *as* $|\mathit{\Delta}_{t,s}|$ *tends to* 0*. Furthermore,* $p^\star(t,s;x,\xi)$ *has the following expression:*

$$\begin{aligned} p^\star(t,s;x,\xi) = \lim_{|\mathit{\Delta}_{t,s}|\to 0} \mathrm{O_s}\!-\!\iint \cdots \iint e^{-i\sum_{j=1}^{\nu} y^j\cdot\eta^j} \\ \times \exp\Big(\sum_{j=0}^{\nu} (t_j - t_{j+1}) K(t_{j+1}, x + \bar{y}^j, \xi + \eta^{j+1})\Big) \\ \times dy^1 đ\eta^1 \cdots dy^\nu đ\eta^\nu\,, \end{aligned} \tag{2.7}$$

where $\bar{y}^0 \equiv 0$, $\bar{y}^j \equiv y^1 + y^2 + \cdots + y^j$, *and* $\eta^{\nu+1} \equiv 0$.

(4) *For* $u \in \boldsymbol{L}^2$, *the pseudo-differential operator* $U(t,s) \equiv p^\star(t,s;X,D_x)$ *satisfies the following relation:*

(2.8)
$$\begin{aligned}
&U(t,s)u(x) \\
&= \lim_{|\Delta_{t,s}|\to 0} e^{(t-t_1)K(t_1)}(X,D_x)e^{(t_1-t_2)K(t_2)}(X,D_x) \\
&\qquad \cdots e^{(t_\nu - s)K(s)}(X,D_x)u(x) \\
&= \lim_{|\Delta_{t,s}|\to 0} \iint \cdots \iint \exp\Bigg(\sum_{j=0}^{\nu} i(x^j - x^{j+1}) \cdot \xi^{j+1} \\
&\qquad\qquad + (t_j - t_{j+1})K(t_{j+1}, x^j, \xi^{j+1}) \Bigg) \\
&\qquad\qquad \times u(x^{\nu+1}) dx^{\nu+1} đ\xi^{\nu+1} \cdots dx^1 đ\xi^1 ,
\end{aligned}$$

in $\boldsymbol{L}^2$ *where* $x^0 \equiv x$.

(5) $U(t,s) \equiv p^\star(t,s;X,D_x)$ *is the fundamental solution for the operator* $L \equiv \partial_t - K(t,X,D_x)$ *such that*

(2.9)
$$\begin{cases} LU(t,s) = 0 & \text{on } (s,T] \\ U(s,s) = I & (0 \le s \le T) . \end{cases}$$

Remark.

1° It is sufficient to satisfy the conditions $(a1)$ and $(a2)$ for $|\xi| \ge M$, with a constant $M \ge 0$. In fact, in this case, there exists a sufficiently large $R > 0$ such that the symbol $K_R(t,x,\xi) \equiv K(t,x,\xi) - R$ satisfies $(a1)$ and $(a2)$ for any ξ. Let $U_R(t,s)$ be the fundamental solution of $L_R \equiv \partial_t - K_R(t,X,D_x)$. Then $U(t,s) \equiv e^{(t-s)R}U_R(t,s)$ is the fundamental solution of L.

2° We can replace $(t_j - t_{j+1})K(t_{j+1},\cdot,\cdot)$ with $\int_{t_{j+1}}^{t_j} K(\tau,\cdot,\cdot)d\tau$. Furthermore, in this case, we can replace (2.6) with

(2.6')
$$|p(\Delta_{t,s}) - p(\Delta'_{t,s})|_l^{(2m)} \le C'_l (t-s)|\Delta_{t,s}| ,$$

and the proof of Theorem 2.1 becomes a little easier.

Example.
Consider

$$L \equiv \partial_t + a(t)|x|^{2l}(-\triangle)^m + (-\triangle)^{m'} \quad (0 \le a(t) \in \mathcal{C}[0,T]\,,\ m - m' < l)\,.$$

If we set $\rho = 1$, $\delta = (m - m')/l$, $m \to 2m$ and $m' \to 2m'$, then the symbol $a(t)|x|^{2l}|\xi|^{2m} + |\xi|^{2m'}$ satisfies the conditions $(a1)$ and $(a2)$. Therefore, we see that these conditions are satisfied not only by the usual parabolic operators, but also by parabolic operators of a degenerate type.

Before we prove Theorem 2.1, we prepare some lemmas:
To begin with, for $T \ge t \ge s \ge 0$, we define $p(t,s;x,\xi)$ by

$$p(t,s;x,\xi) \equiv \exp\Big((t-s)K(s,x,\xi)\Big)\,. \tag{2.10}$$

The next lemma is a generalization of asymptotic expansion formulas, and an essential part in this paper. Especially, it is important that all constants are independent of $\Delta_{t_0,t_{\nu+1}}$ and ν.

Lemma 2.2 (A key lemma).
Let $\Delta_{t_0,t_{\nu+1}}$: $(T \ge)t_0 \ge t_1 \ge \cdots \ge t_\nu \ge t_{\nu+1}(\ge 0)$, $\nu = 1,2,\ldots,$ and let N_0 be a fixed positive integer such that $(\rho-\delta)N_0 \ge 2m$. Define $q(\Delta_{t_0,t_1};x,\xi)$, $q(\Delta_{t_0,t_{\nu+1}};x,\xi)$, and $r(\Delta_{t_0,t_{\nu+1}};x,\xi)$ respectively by

$$q(\Delta_{t_0,t_1};x,\xi) \equiv p(t_0,t_1;x,\xi)\,, \tag{2.11}$$

$$\begin{aligned} q(\Delta_{t_0,t_{\nu+1}};x,\xi) \equiv & \sum_{|\alpha^1|+|\alpha^2|+\cdots+|\alpha^\nu|<N_0} \frac{1}{\alpha^1!\alpha^2!\cdots\alpha^\nu!} \\ & \times p_{(\alpha^\nu)}(t_\nu,t_{\nu+1};x,\xi)\partial_\xi^{\alpha^\nu}\Big(p_{(\alpha_{\nu-1})}(t_{\nu-1},t_\nu;x,\xi) \\ & \times \partial_\xi^{\alpha^{\nu-1}}\Big(\cdots p_{(\alpha^2)}(t_2,t_3;x,\xi)\partial_\xi^{\alpha^2}\Big(p_{(\alpha^1)}(t_1,t_2;x,\xi) \\ & \times \partial_\xi^{\alpha^1}\Big(p(t_0,t_1;x,\xi)\Big)\Big)\cdots\Big)\Big)\,. \end{aligned} \tag{2.12}$$

and

$$
\begin{aligned}
(2.13)\qquad & r(\varDelta_{t_0,t_{\nu+1}};x,\xi) \equiv \sum_{|\alpha^1|+|\alpha^2|+\cdots+|\alpha^\nu|=N_0,\,|\alpha^\nu|\neq 0} \frac{|\alpha^\nu|}{\alpha^1!\alpha^2!\cdots\alpha^\nu!} \\
& \times \int_0^1 (1-\theta)^{|\alpha^\nu|-1} \mathrm{O_s}\!-\!\iint e^{-iy\cdot\eta} p_{(\alpha^\nu)}(t_\nu,t_{\nu+1};x+y,\xi) \\
& \times \partial_\xi^{\alpha^\nu}\Big(p_{(\alpha_{\nu-1})}(t_{\nu-1},t_\nu;x,\xi+\theta\eta) \\
& \times \partial_\xi^{\alpha^{\nu-1}}\Big(\cdots p_{(\alpha^2)}(t_2,t_3;x,\xi+\theta\eta) \\
& \times \partial_\xi^{\alpha^2}\Big(p_{(\alpha^1)}(t_1,t_2;x,\xi+\theta\eta) \\
& \times \partial_\xi^{\alpha^1}\Big(p(t_0,t_1;x,\xi+\theta\eta)\Big)\Big)\cdots\Big)\Big) dy\,đ\eta\, d\theta\,.
\end{aligned}
$$

Then it follows that

$$
\begin{aligned}
(2.14)\qquad & q(\varDelta_{t_0,t_\nu};X,D_x)p(t_\nu,t_{\nu+1};X,D_x) \\
& = q(\varDelta_{t_0,t_{\nu+1}};X,D_x) + r(\varDelta_{t_0,t_{\nu+1}};X,D_x)\,.
\end{aligned}
$$

Furthermore, there exist constants $C_{1,l}, C_{2,l}, C_{3,l}$ *such that*

$$
(2.15)\qquad |q(\varDelta_{t_0,t_\nu})|_l^{(0)} \le C_{1,l}\,,
$$

$$
\begin{aligned}
(2.16)\qquad & |q(\varDelta_{t_0,t_{\nu+1}}) - p(t_0,t_{\nu+1})|_l^{(2m)} \\
& \le C_{2,l}(t_0-t_{\nu+1}) \\
& \quad \times \Big((t_0-t_{\nu+1}) + \sup_{t_0\ge t'\ge t''\ge t_{\nu+1}} |K(t')-K(t'')|_l^{(m)}\Big)\,,
\end{aligned}
$$

and

$$
(2.17)\qquad |r(\varDelta_{t_0,t_{\nu+1}})|_l^{(0)} \le C_{3,l}(t_0-t_\nu)(t_\nu-t_{\nu+1})\,,
$$

for any $\varDelta_{t_0,t_{\nu+1}} : (T\ge)t_0 \ge t_1 \ge \cdots \ge t_\nu \ge t_{\nu+1}(\ge 0)$ *and* $\nu = 1,2,\ldots$.

Proof.

1° For $T \geq t \geq s \geq 0$, we set

$$\eta(t,s;x,\xi) \equiv -(t-s)\mathrm{Re}\, K(s,x,\xi)\,(\geq 0)\,. \tag{2.18}$$

Furthermore, for $\Delta_{t_0,t_{\nu+1}}: (T \geq) t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu+1} (\geq 0)$ and $\nu = 1,2,\ldots$, we define $d(\Delta_{t_0,t_\nu};x,\xi)$ by

$$d(\Delta_{t_0,t_\nu};x,\xi) \equiv \prod_{j=0}^{\nu-1} p(t_j,t_{j+1};x,\xi)\,, \tag{2.19}$$

and we set

$$\eta(\Delta_{t_0,t_\nu};x,\xi) \equiv \sum_{j=0}^{\nu-1} \eta(t_j,t_{j+1};x,\xi)\,. \tag{2.20}$$

Clearly, we have

$$|d(\Delta_{t_0,t_\nu};x,\xi)| = \exp\Big(-\eta(\Delta_{t_0,t_\nu};x,\xi)\Big)\,. \tag{2.21}$$

2° Define $d_{\alpha,\beta}(\Delta_{t_0,t_\nu};x,\xi)$ by

$$d^{(\alpha)}_{(\beta)}(\Delta_{t_0,t_\nu};x,\xi) \equiv d_{\alpha,\beta}(\Delta_{t_0,t_\nu};x,\xi)\, d(\Delta_{t_0,t_\nu};x,\xi)\,. \tag{2.22}$$

Then, by induction, for any α,β ($|\alpha+\beta| \geq 1$) and α',β', there exists a constant $C_{\alpha,\beta,\alpha',\beta'}$ such that

$$\begin{aligned} &|d_{\alpha,\beta}{}^{(\alpha')}_{(\beta')}(\Delta_{t_0,t_\nu};x,\xi)| \\ &\quad \leq C_{\alpha,\beta,\alpha',\beta'}\,\eta(\Delta_{t_0,t_\nu};x,\xi)\Big(\eta(\Delta_{t_0,t_\nu};x,\xi)+1\Big)^{|\alpha+\beta|-1} \\ &\qquad \times \lambda(\xi)^{\delta|\beta+\beta'|-\rho|\alpha+\alpha'|}\,, \end{aligned} \tag{2.23}$$

for any $\Delta_{t_0,t_{\nu+1}}: (T \geq) t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu+1} (\geq 0)$ and $\nu = 1,2,\ldots$.

3° Let $\widetilde{\alpha}^\nu \equiv (\alpha^1,\dots,\alpha^\nu)$ denote a multi-index of $\boldsymbol{R}^{\nu n}$. Define $f_{\widetilde{\alpha}^\nu}(\Delta_{t_0,t_{\nu+1}};x,\xi)$ by

$$
\begin{aligned}
(2.24)\quad & f_{\widetilde{\alpha}^\nu}(\Delta_{t_0,t_{\nu+1}};x,\xi)d(\Delta_{t_0,t_{\nu+1}};x,\xi)\\
&\equiv p_{(\alpha^\nu)}(t_\nu,t_{\nu+1};x,\xi)\partial_\xi^{\alpha^\nu}\Big(p_{(\alpha_{\nu-1})}(t_{\nu-1},t_\nu;x,\xi)\partial_\xi^{\alpha^{\nu-1}}\Big(\\
&\cdots p_{(\alpha^2)}(t_2,t_3;x,\xi)\partial_\xi^{\alpha^2}\Big(p_{(\alpha^1)}(t_1,t_2;x,\xi)\partial_\xi^{\alpha^1}\Big(p(t_0,t_1;x,\xi)\Big)\Big)\cdots\Big)\Big)\,.
\end{aligned}
$$

Then, by induction, for any $N=1,2,\dots$ and α,β, there exists a constant $C_{N,\alpha,\beta}$ such that

$$
\begin{aligned}
(2.25)\quad & |f_{\widetilde{\alpha}^\nu}{}^{(\alpha)}_{(\beta)}(\Delta_{t_0,t_{\nu+1}};x,\xi)|\\
&\le C_{N,\alpha,\beta}\Big(\prod_{k=1}^{J}\eta(t_{j_k},t_{j_k+1};x,\xi)\Big)\eta(\Delta_{t_0,t_{\nu+1}};x,\xi)\\
&\quad\times\Big(\eta(\Delta_{t_0,t_{\nu+1}};x,\xi)+1\Big)^{2(N-1)}\lambda(\xi)^{-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|}\,,
\end{aligned}
$$

where

$$1\le j_1<j_2<\dots<j_J\le\nu\,,\quad |\alpha^{j_k}|\ne 0\,(k=1,2,\dots,J)\,,$$

and

$$\sum_{j=1}^{\nu}|\alpha^j|=\sum_{k=1}^{J}|\alpha^{j_k}|=N\,,$$

for any $\Delta_{t_0,t_{\nu+1}}:(T\ge)t_0\ge t_1\ge\dots\ge t_\nu\ge t_{\nu+1}(\ge 0)$ and $\nu=1,2,\dots$.

4° For $N=1,2,\dots$, define $g_N(\Delta_{t_0,t_{\nu+1}};x,\xi)$ by

$$
\begin{aligned}
(2.26)\quad & g_N(\Delta_{t_0,t_{\nu+1}};x,\xi)\\
&\equiv\sum_{|\alpha^1|+|\alpha^2|+\dots+|\alpha^\nu|=N}\frac{1}{\alpha^1!\alpha^2!\cdots\alpha^\nu!}f_{\widetilde{\alpha}^\nu}(\Delta_{t_0,t_{\nu+1}};x,\xi)\,.
\end{aligned}
$$

By (2.25), we have

$$
\begin{aligned}
(2.27)\quad & |g_N{}^{(\alpha)}_{(\beta)}(\varDelta_{t_0,t_{\nu+1}};x,\xi)| \\
&\le \sum_{J=1}^{N}\ \sum_{1\le j_1<j_2<\cdots<j_J\le\nu}\ \sum_{\sum_{k=1}^{J}|\alpha^{j_k}|=N,\,|\alpha^{j_k}|\neq 0} \frac{1}{\alpha^{j_1}!\alpha^{j_2}!\cdots\alpha^{j_J}!} \\
&\quad\times C_{N,\alpha,\beta}\Big(\prod_{k=1}^{J}\eta(t_{j_k},t_{j_k+1};x,\xi)\Big)\eta(\varDelta_{t_0,t_{\nu+1}};x,\xi) \\
&\quad\times\Big(\eta(\varDelta_{t_0,t_{\nu+1}};x,\xi)+1\Big)^{2(N-1)}\lambda(\xi)^{-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|} \\
&\le (nN)^N C_{N,\alpha,\beta}\eta(\varDelta_{t_0,t_{\nu+1}};x,\xi) \\
&\quad\times\Big(\eta(\varDelta_{t_0,t_{\nu+1}};x,\xi)+1\Big)^{2(N-1)}\lambda(\xi)^{-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|} \\
&\quad\times\left(\sum_{J=1}^{N}\sum_{1\le j_1<j_2<\cdots<j_J\le\nu}\prod_{k=1}^{J}\eta(t_{j_k},t_{j_k+1};x,\xi)\right).
\end{aligned}
$$

Hence, for any $N=1,2,\ldots$ and α,β, there exists a constant $C'_{N,\alpha,\beta}$ such that

$$
\begin{aligned}
(2.28)\quad & |g_N{}^{(\alpha)}_{(\beta)}(\varDelta_{t_0,t_{\nu+1}};x,\xi)| \\
&\le C'_{N,\alpha,\beta}\Big(\eta(\varDelta_{t_0,t_{\nu+1}};x,\xi)\Big)^2 \\
&\quad\times\Big(\eta(\varDelta_{t_0,t_{\nu+1}};x,\xi)+1\Big)^{3(N-1)}\lambda(\xi)^{-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|},
\end{aligned}
$$

for any $\varDelta_{t_0,t_{\nu+1}}:(T\ge)t_0\ge t_1\ge\cdots\ge t_\nu\ge t_{\nu+1}(\ge 0)$ and $\nu=1,2,\ldots$.

5° Set

$$
(2.29)\qquad h_N(\varDelta_{t_0,t_{\nu+1}};x,\xi)\equiv g_N(\varDelta_{t_0,t_{\nu+1}};x,\xi)d(\varDelta_{t_0,t_{\nu+1}};x,\xi).
$$

Here we note that

$$
(2.30)\qquad \sup_{\eta>0}\eta^k e^{-\eta}<\infty\quad (k=0,1,2,\ldots).
$$

By (2.21), (2.23) and (2.28), there exist constants $C'_{\alpha,\beta}$, $C''_{\alpha,\beta}$, $C''_{N,\alpha,\beta}$, $C'''_{N,\alpha,\beta}$, $C''''_{N,\alpha,\beta}$ such that

$$(2.31)\quad |d^{(\alpha)}_{(\beta)}(\varDelta_{t_0,t_\nu};x,\xi)| \le \begin{cases} C'_{\alpha,\beta}\lambda(\xi)^{\delta|\beta|-\rho|\alpha|} \\ C''_{\alpha,\beta}(t_0-t_\nu)\lambda(\xi)^{m+\delta|\beta|-\rho|\alpha|} \quad (|\alpha+\beta|\ge 1)\,, \end{cases}$$

and

$$(2.32)\quad |h_N{}^{(\alpha)}_{(\beta)}(\varDelta_{t_0,t_{\nu+1}};x,\xi)| \le \begin{cases} C''_{N,\alpha,\beta}\lambda(\xi)^{-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|} \\ C'''_{N,\alpha,\beta}(t_0-t_{\nu+1})\lambda(\xi)^{m-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|} \\ C''''_{N,\alpha,\beta}(t_0-t_{\nu+1})^2\lambda(\xi)^{2m-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|}\,, \end{cases}$$

for any $\varDelta_{t_0,t_{\nu+1}}: (T\ge)t_0\ge t_1\ge\cdots\ge t_\nu\ge t_{\nu+1}(\ge 0)$ and $\nu=1,2,\dots$.

6° Now we note that

$$(2.33)\quad q(\varDelta_{t_0,t_{\nu+1}};x,\xi) = d(\varDelta_{t_0,t_{\nu+1}};x,\xi) + \sum_{N=1}^{N_0-1} h_N(\varDelta_{t_0,t_{\nu+1}};x,\xi)\,,$$

and

$$\begin{aligned}(2.34)\quad & d(\varDelta_{t_0,t_{\nu+1}};x,\xi) - p(t_0,t_{\nu+1};x,\xi) \\ &= \sum_{j=0}^{\nu}(t_j-t_{j+1})\Big(K(t_{j+1},x,\xi)-K(t_{\nu+1},x,\xi)\Big) \\ &\times \int_0^1 \exp\Big(\theta\sum_{j=0}^{\nu}(t_j-t_{j+1})K(t_{j+1},x,\xi)\Big) \\ &\times \exp\Big((1-\theta)(t_0-t_{\nu+1})K(t_{\nu+1},x,\xi)\Big)d\theta\,.\end{aligned}$$

By (2.31) and (2.32), we get (2.15) and (2.16). Furthermore, we note that

$$
\begin{aligned}
(2.35)\quad & r(\varDelta_{t_0,t_{\nu+2}};x,\xi)\\
&= \sum_{0<|\alpha^{\nu+1}|<N_0} \frac{|\alpha^{\nu+1}|}{\alpha^{\nu+1}!} \int_0^1 (1-\theta)^{|\alpha^{\nu+1}|-1}\\
&\quad \times \mathrm{Os}\text{—}\iint e^{-iy\cdot\eta} h_{N_0-|\alpha^{\nu+1}|}{}^{(\alpha^{\nu+1})}(\varDelta_{t_0,t_{\nu+1}};x,\xi+\theta\eta)\\
&\quad \times d_{(\alpha^{\nu+1})}(\varDelta_{t_{\nu+1},t_{\nu+2}};x+y,\xi)dy\bar{d}\eta d\theta\\
&+ \sum_{|\alpha^{\nu+1}|=N_0} \frac{|\alpha^{\nu+1}|}{\alpha^{\nu+1}!} \int_0^1 (1-\theta)^{|\alpha^{\nu+1}|-1}\\
&\quad \times \mathrm{Os}\text{—}\iint e^{-iy\cdot\eta} d^{(\alpha^{\nu+1})}(\varDelta_{t_0,t_{\nu+1}};x,\xi+\theta\eta)\\
&\quad \times d_{(\alpha^{\nu+1})}(\varDelta_{t_{\nu+1},t_{\nu+2}};x+y,\xi)dy\bar{d}\eta d\theta\,.
\end{aligned}
$$

By (2.31), (2.32) and Theorem 1.4 , we get (2.17).

7° By induction, we get (2.14). □

The idea of the next lemma is found in Fujiwara [3].

Lemma 2.3 (Fujiwara's skip).
Define $\varUpsilon(\varDelta_{t_0,t_{\nu+1}};x,\xi) \in \boldsymbol{S}^0_{\lambda,\rho,\delta}$ *by*

$$
\begin{aligned}
(2.36)\qquad & p(t_0,t_1;X,D_x)p(t_1,t_2;X,D_x)\cdots p(t_\nu,t_{\nu+1};X,D_x)\\
&\equiv q(\varDelta_{t_0,t_{\nu+1}};X,D_x) + \varUpsilon(\varDelta_{t_0,t_{\nu+1}};X,D_x)\,.
\end{aligned}
$$

Then it follows that

$$
\begin{aligned}
(2.37)\qquad & \varUpsilon(\varDelta_{t_0,t_{\nu+1}};X,D_x)\\
&= {\sum}' r(\varDelta_{t_0,t_{j_1+1}};X,D_x)r(\varDelta_{t_{j_1+1},t_{j_2+1}};X,D_x)\\
&\qquad \cdots r(\varDelta_{t_{j_{J-1}+1},t_{j_J+1}};X,D_x)q(\varDelta_{t_{j_J+1},t_{\nu+1}};X,D_x)\,,
\end{aligned}
$$

where ${\sum}'$ *stands for the summation with respect to the sequences of integers* $(j_1,j_2,\dots,j_J)$ *with the property*

$$
(2.38)\quad 0<j_1<j_1+1<j_2<j_2+1<\cdots<j_{J-1}<j_{J-1}+1<j_J\le\nu\,,
$$

and, in the special case of $j_J=\nu$, *we set* $q(\varDelta_{t_{j_J+1},t_{\nu+1}};X,D_x)\equiv I$. *Furthermore, there exists a constant* $C_{4,l}$ *such that*

$$|\varUpsilon(\varDelta_{t_0,t_{\nu+1}})|_l^{(0)}\le C_{4,l}(t_0-t_{\nu+1})^2\,, \tag{2.39}$$

for any $\varDelta_{t_0,t_{\nu+1}}:(T\ge)t_0\ge t_1\ge\cdots\ge t_\nu\ge t_{\nu+1}(\ge 0)$ *and* $\nu=1,2,\dots$.

Proof. Using (2.14) inductively, we get (2.37). Now let A_l, l' be the same constants in Theorem 1.3, and let $C_{1,l}, C_{3,l}$ be the same constants in Lemma 2.2. By (2.15), (2.17) and Theorem 1.3, we have

$$\begin{aligned}
&|\varUpsilon(\varDelta_{t_0,t_{\nu+1}})|_l^{(0)}\\
&\le\sum{}'(A_l)^J|r(\varDelta_{t_0,t_{j_1+1}})|_{l'}^{(0)}|r(\varDelta_{t_{j_1+1},t_{j_2+1}})|_{l'}^{(0)}\\
&\quad\cdots|r(\varDelta_{t_{j_{J-1}+1},t_{j_J+1}})|_{l'}^{(0)}|q(\varDelta_{t_{j_J+1},t_{\nu+1}})|_{l'}^{(0)}\\
&\le\sum{}'(A_l)^J\Big(\prod_{k=1}^{J}C_{3,l'}(t_0-t_{\nu+1})(t_{j_k}-t_{j_k+1})\Big)C_{1,l'}\\
&\le C_{1,l'}\bigg(\prod_{j=0}^{\nu}\Big(1+A_lC_{3,l'}(t_0-t_{\nu+1})(t_j-t_{j+1})\Big)-1\bigg)\\
&\le C_{4,l}(t_0-t_{\nu+1})^2\,.\ \square
\end{aligned}\tag{2.40}$$

Now we prove Theorem 2.1:

Proof of Theorem 2.1.
1° Define $p(\varDelta_{t,s};x,\xi)$ by

$$p(\varDelta_{t,s};x,\xi)\equiv q(\varDelta_{t,s};x,\xi)+\varUpsilon(\varDelta_{t,s};x,\xi)\,. \tag{2.41}$$

Then (1) is clear.
2° By (2.14) and (2.39), we get (2.5). Next, we note that

$$\begin{aligned}
&p(\varDelta'_{t_j,t_{j+1}};x,\xi)-p(t_j,t_{j+1};x,\xi)\\
&\quad=\Big(q(\varDelta'_{t_j,t_{j+1}};x,\xi)-p(t_j,t_{j+1};x,\xi)\Big)+\varUpsilon(\varDelta'_{t_j,t_{j+1}};x,\xi)\,,
\end{aligned}\tag{2.42}$$

Hence, by (2.16) and (2.39), there exists a constant $C_{5,l}$ such that

$$
\begin{aligned}
(2.43)\quad & |p(t_j, t_{j+1}) - p(\Delta'_{t_j,t_{j+1}})|_l^{(2m)} \\
& \leq C_{5,l}(t_j - t_{j+1})\Big((t_j - t_{j+1}) + \sup_{t_j \geq t' \geq t'' \geq t_{j+1}} |K(t') - K(t'')|_l^{(m)}\Big) .
\end{aligned}
$$

Here we can write

$$
\begin{aligned}
(2.44)\quad & p(\Delta_{t,s}; X, D_x) - p(\Delta'_{t,s}; X, D_x) = \sum_{j=0}^{\nu} p(\Delta'_{t_0,t_j}; X, D_x) \\
& \circ \Big(p(t_j, t_{j+1}; X, D_x) - p(\Delta'_{t_j,t_{j+1}}; X, D_x)\Big) \\
& \circ p(\Delta_{t_{j+1},t_{\nu+1}}; X, D_x) .
\end{aligned}
$$

By (2.5), (2.43) and Theorem 1.3, we get (2.6).

3° By (2.6) and (2.5), there exists $p^\star(t, s; x, \xi) \in \boldsymbol{S}^0_{\lambda,\rho,\delta}$ such that

$$
(2.45)\qquad |p^\star(t, s)|_l^{(0)} \leq C_l ,
$$

and

$$
\begin{aligned}
(2.46)\quad & |p(\Delta_{t,s}) - p^\star(t, s)|_l^{(2m)} \\
& \leq C'_l(t - s)\Big(|\Delta_{t,s}| + \sup_{|t'-t''| \leq |\Delta_{t,s}|} |K(t') - K(t'')|_{l'}^{(m)}\Big) .
\end{aligned}
$$

Hence we get (3).

4° By the result of (3), we get (4). See Chapter 3 § 7 in H.Kumano-go [6].

5° Using the results of (2) and (3), it is easy to check (5). □

Acknowledgements. The author would like to express his sincere gratitude to Professor K. Kataoka, Professor H. Komatsu and Professor K. Taniguchi, for helpful discussions and suggestions. He would also like to thank the referee for the important comments about this paper.

References

[1] Feynman, R. P., Space-time approach to non relativistic quantum mechanics, Rev. of Modern Phys. **20** (1948), 367–387.

[2] Fujiwara, D., A remark on Taniguchi-Kumano-go theorem for product of Fourier integral operators, Pseudo-differential operators, Proc. Oberwolfach 1986, Lecture notes in Math, Springer **1256** (1987), 135–153.

[3] Fujiwara, D., The stationary phase method with an estimate of the remainder term on a space of large dimension, Nagoya Math. J. **124** (1991), 61–97.

[4] Fujiwara, D., Some Feynman Path Integrals As Oscillatory Integrals Over A Sobolev Manifold, preprint (1993).

[5] Kitada, H. and H. Kumano-go, A family of Fourier integral operators and the fundamental solution for a Schrödinger equation, Osaka J. Math. **18** (1981), 291–360.

[6] Kumano-go, H. Pseudo-Differential Operator, MIT press, Cambridge, Massachusetts and London, England, 1983.

[7] Kumano-go, H. and K. Taniguchi, Fourier integral operators of multi-phase and the fundamental solution for a hyperbolic system, Funkcial. Ekvac. **22** (1979), 161–196.

[8] Shinkai, K., The symbol calculus for the fundamental solution of a degenerate parabolic system with applications, Osaka J. Math. **14** (1977), 55–84.

[9] Taniguchi, K., Multi-products of Fourier integral operators and the fundamental solution for a hyperbolic system with involutive characteristics, Osaka J. Math. **21** (1984), 169–224.

[10] Tsutsumi, C., The fundamental solution for a degenerate parabolic pseudo-differential operator, Proc. Japan Acad. **50** (1974), 11–15.

(Received December 20, 1994)

Graduate School of Mathematical Sciences
University of Tokyo
3-8-1, Komaba
Meguro-ku, Tokyo 153
Japan