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The logarithmic forms of k-generic arrangements

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Abstract. A (central) arrangement $\mathcal{A}$ is a finite family of one codimensional subspaces of a vector space V. Relations between the module of logarithmic forms of $\mathcal{A}$ and the module of logarithmic forms of $\mathcal{A} \setminus \{H\}$ are studied. It is found that the logarithmic q-forms of $\mathcal{A}$ are generated by the logarithmic forms of the type $\frac{d\alpha}{\alpha}$ if $\mathcal{A}$ is k-generic and $q \leq k-2$.

1. Introduction

1.1. The Setup of This Paper

Let V be an l-dimensional vector space over a field $\mathbf{K}$. Let $\mathcal{A}$ be a central arrangement in V: $\mathcal{A}$ is a finite family of one-codimensional subspaces of V. Let $L(\mathcal{A})$ be the set of intersections of elements of $\mathcal{A}$. We agree that $V \in L(\mathcal{A})$ as the empty intersection. For $X \in L(\mathcal{A})$, let $r(X)$ be the codimension of X in V. We say that a hyperplane H is $k-generic$ to $\mathcal{A}$ if $X \not\subseteq H$ for every $X \in L(\mathcal{A})$ with $r(X) < k$. An arrangement $\mathcal{A}$ is said to be $k-generic$ if every hyperplane H in $\mathcal{A}$ is k-generic to $\mathcal{A} \setminus \{H\}$. Let S denote the symmetric algebra $S(V^*)$ of the dual space V^* of V. Then S can be considered as the $\mathbf{K}$-algebra of all polynomial functions on V. Let $0 \leq q \leq l$. Let $\Omega^q[V]$ denote the S-module of all regular(=polynomial) q-forms on V. Then each $\Omega^q[V]$ is a free S-module of rank $\binom{l}{q}$. For each $H \in \mathcal{A}$, choose $\alpha_H \in V^*$ such that $\ker(\alpha_H) = H$. Let

$$Q = Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H \ \in S.$$

Define

$$\begin{aligned} \Omega^q(\mathcal{A}) \ &= \{\omega \mid \omega \text{ is a rational } q\text{-form on } V \text{ such that} \\ &\qquad Q\omega \in \Omega^q[V] \text{ and } Q(d\omega) \in \Omega^{q+1}[V]\}, \end{aligned}$$

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which is called the *module of logarithmic q-forms with pole along $\mathcal{A}$.* Define

$$\Omega^q \langle \mathcal{A} \rangle = \bigwedge^q < d\alpha_1/\alpha_1, \cdots, d\alpha_n/\alpha_n, dx_1, \cdots, dx_l >$$

where $\mathcal{A} = \{\ker(\alpha_1), \cdots, \ker(\alpha_n)\}$. Obviously $\Omega^q \langle \mathcal{A} \rangle \subseteq \Omega^q(\mathcal{A})$ for each q.

1.2. The Aim

Kita and Noumi [1], Rose and Terao [3] proved that $\Omega^q(\mathcal{A}) = \Omega^q \langle \mathcal{A} \rangle$ for an l-generic arrangement $\mathcal{A}$, where $q = 0, \cdots, l-2$. The aim of this paper is to generalize the result to k-generic arrangements. We will prove:

Theorem 1. *We have*

$$\Omega^q(\mathcal{A}) = \Omega^q \langle \mathcal{A} \rangle$$

for a k-generic arrangement $\mathcal{A}$, where $q = 0, \cdots, k-2$.

Ziegler announced exactly the same generalization in [4, Theorem 7.4] with incomplete proof. However he gave a detailed proof for k=3, in other words, he proved that "$\Omega^1(\mathcal{A}) = \Omega^1 \langle \mathcal{A} \rangle$, if $\mathcal{A}$ is 3-generic".

2. The Residue map

2.1. The setup

Fix $H \in \mathcal{A}$ in this section. Let $\mathcal{A}' = \mathcal{A} \backslash \{H\}$ and $\mathcal{A}'' = \{K \cap H | K \in \mathcal{A}'\}$. Then $\mathcal{A}'$ is an arrangement in V called the *deletion* of $\mathcal{A}$. The arrangement $\mathcal{A}''$ is called the *restriction* of $\mathcal{A}$ to H. We call $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$ a *triple* of arrangements and H the *distinguished* hyperplane. Choose $\alpha_H \in V^*$ with $\ker(\alpha_H) = H$. Let $Q' = Q/\alpha_H$. Then Q' defines $\mathcal{A}'$. Let $x_1, \cdots, x_l$ be a basis for V^*.

2.2. The Residue

Lemma 2. [4, Lemma 3.6]
For any $\omega \in \Omega^q(\mathcal{A})$, there exist a rational $(q-1)$-form ω' and a rational q-form ω'' such that

1. $\omega = \omega' \wedge (d\alpha_H/\alpha_H) + \omega''$,
2. *$Q'\omega'$ and $Q'\omega''$ are both regular (no pole).*

Lemma 3. [3, Lemma 2.2.2]
Let $\omega \in \Omega^q(\mathcal{A})$. Choose a rational $(q-1)$-form ω' and a rational q-form ω'' such that

1. $\omega = \omega' \wedge (d\alpha_H/\alpha_H) + \omega''$,
2. *neither ω' nor ω'' has a pole along H.*

Then the restriction $\omega'|_H$ of the ω' to H depends only upon ω (and H).

Definition. For $\omega \in \Omega^q(\mathcal{A})$, the restriction of ω' in Lemma 3 to H is called the *residue* of ω and is denoted by $\mathrm{res}(\omega)$.

2.3. A Short Exact Sequence

Let H in $\mathcal{A}$ be 3-generic to $\mathcal{A} \setminus \{H\}$. Let $\mathcal{A}, \mathcal{A}', \mathcal{A}''$ be the *triple* of arrangements with H the *distinguished* hyperplane.

Lemma 4. *For any $\omega \in \Omega^q(\mathcal{A})$, $\mathrm{res}(\omega) \in \Omega^{q-1}(\mathcal{A}'')$. In other words, one can define a* $\mathbf{K}$*-linear map*

$$\mathrm{res} : \Omega^q(\mathcal{A}) \to \Omega^{q-1}(\mathcal{A}'').$$

Proof. Write

$$\omega = \omega' \wedge (d\alpha_H/\alpha_H) + \omega''$$

as in Lemma 2. Since H is 3-generic, $\mathrm{res}(\omega) = \omega'|_H$ has at most a simple pole along $\mathcal{A}''$. Since $d\omega \in \Omega^{q+1}(\mathcal{A})$, write

$$d\omega = \eta' \wedge (d\alpha_H/\alpha_H) + \eta'',$$

where both $Q'\eta'$ and $Q'\eta''$ are regular. Then one has

$$d\omega' \wedge (d\alpha_H/\alpha_H) + d\omega'' = d\omega = \eta' \wedge (d\alpha_H/\alpha_H) + \eta''.$$

Since neither $d\omega'$ nor $d\omega''$ has a pole along H, from Lemma 3,

$$\eta'|_H = d\omega'|_H = d(res(\omega)).$$

Since H is 3-generic, $d(\mathrm{res}(\omega))$ has at most a simple pole along $\mathcal{A}''$. Therefore $\mathrm{res}(\omega) \in \Omega^{q-1}(\mathcal{A}'')$. $\square$

Lemma 5. *The sequence* $0 \to \Omega^q(\mathcal{A}') \to \Omega^q(\mathcal{A}) \to \Omega^{q-1}(\mathcal{A}'')$ *is exact for* $q \geq 1$.

Proof. For $\omega \in \Omega^q(\mathcal{A}')$, we can choose $\omega' = 0$ and $\omega'' = \omega$ in lemma 3. Thus $res(\omega) = 0$. We can assume that $\alpha_H = x_1$. We may (uniquely) choose ω' and ω'' such that $\omega = \omega' \wedge (dx_1/x_1) + \omega''$ and neither ω' nor ω'' contains dx_1. Suppose

$$\omega'|_H = res(\omega) = 0.$$

Then every coefficient of ω' is divisible by x_1. So $Q'\omega$ is regular. Also, $d\omega$ has no pole along H either. Therefore $\omega \in \Omega^q(\mathcal{A}')$. □

3. Proof of Theorem 1

In section 3.1, we give a proof for 3-generic arrangement. In section 3.2, we prove Theorem 1.

3.1. 3-Generic Case

Let $\mathcal{A}$ be a 3-generic arrangement. The aim is to show that $\Omega^1(\mathcal{A}) = \Omega^1\langle\mathcal{A}\rangle$. We want to show by induction on $|\mathcal{A}|$. If $\mathcal{A}$ is an empty arrangement, $\Omega^1(\mathcal{A}) = \Omega^1\langle\mathcal{A}\rangle =< dx_1, \cdots, dx_l >$. Suppose $|\mathcal{A}| \geq 1$ and $H \in \mathcal{A}$. Since $|\mathcal{A}'| < |\mathcal{A}|$, by induction assumption it is enough to show that

$$\Omega^1(\mathcal{A}) = Sd\alpha_H/\alpha_H + \Omega^1(\mathcal{A}')$$

where $H = \ker(\alpha_H), \mathcal{A}' = \mathcal{A} \setminus \{H\}$.

Lemma 6. *If* H *in* $\mathcal{A}$ *is 3-generic to* $\mathcal{A} \setminus \{H\}$,

$$\Omega^1(\mathcal{A}) = Sd\alpha_H/\alpha_H + \Omega^1(\mathcal{A}')$$

where $H = \ker(\alpha_H), \mathcal{A}' = \mathcal{A} \setminus \{H\}$.

Remark. The above lemma was first proved by Ziegler in [4, Lemma 6.1]. It was derived from his "Strong Preparation Lemma for 1-forms" [4, theorem 5.1]. Here, we give a new proof of it.

Proof. Let Q and Q' be the defining equations for $\mathcal{A}$ and $\mathcal{A}'$ respectively. Then $Q = Q'\alpha_H$. Let $\eta := \omega Q$. Then $\eta \in \Omega^1[V]$. We can choose η_1 and η_2 such that

$$\eta = \eta_1 d\alpha_H + \alpha_H \eta_2,$$

where $\eta_1 \in S, \eta_2 \in \Omega^1[V]$. Similarly, for any $K \in \mathcal{A}'$,

$$\eta = \eta_1' d\alpha_K + \alpha_K \eta_2'.$$

We have

$$\eta_1 d\alpha_H - \eta_1' d\alpha_K = \alpha_K \eta_2' - \alpha_H \eta_2.$$

Now, wedge $d\alpha_K$ both sides. Then

$$\eta_1 d\alpha_H \wedge d\alpha_K = (\eta_2' \alpha_K - \eta_2 \alpha_H) \wedge d\alpha_K.$$

We get

$$\eta_1 \in (\alpha_H, \alpha_K).$$

But K is an arbitrary element in $\mathcal{A}$. Thus, $\eta_1 \in \bigcap_{K \in \mathcal{A}'} (\alpha_H, \alpha_K)$. Since H is 3-generic, $\eta_1 \in (\alpha_H, \prod_{K \in \mathcal{A}'} \alpha_K) = (\alpha_H, Q')$. For some $f_1, f_2 \in S$, $\eta_1 = f_1 \alpha_H + f_2 Q'$.

$$\begin{aligned} \omega &= \eta / Q' \alpha_H \\ &= (\eta_1 d\alpha_H + \alpha_H \eta_2) / Q' \alpha_H \\ &= (f_1 \alpha_H d\alpha_H + f_2 Q' d\alpha_H + \alpha_H \eta_2) / Q' \alpha_H \\ &= f_2 d\alpha_H / \alpha_H + (f_1 d\alpha_H + \eta_2) / Q'. \\ &= f_2 d\alpha_H / \alpha_H + \omega'', \end{aligned}$$

where $\omega'' = (f_1 d\alpha_H + \eta_2)/Q'$. Since $\omega \in \Omega^1(\mathcal{A})$ and $f_2 d\alpha_H/\alpha_H \in \Omega^1(\mathcal{A})$, we have $\omega'' \in \Omega^1(\mathcal{A})$. This implies $\omega'' \in \Omega^1(\mathcal{A}')$ because ω'' has no pole along H. $\square$

REMARK. It may be natural to try to generalize the above lemma as follows:

Let k be an integer such that $3 \leq k \leq l$. If H in $\mathcal{A}$ is k-generic to $\mathcal{A}'$, then $\Omega^q(\mathcal{A}) = \Omega^{q-1}(\mathcal{A}') \wedge d\alpha_H/\alpha_H + \Omega^q(\mathcal{A}')$ for $q = 1, \cdots, k-2$.

We may call the above generalization a "Strong Preparation Lemma for k-forms $(k \geq 1)$". If it were true, we could prove Theorem 1 easily applying the " Strong Preparation Lemma for k-forms $(k \geq 1)$" repeatedly. But, unfortunately, we have some counter examples to this generalization: Let $\mathcal{A}'$ be an arrangement defined by $Q' = xyzw(x+z)(x+w)(y+z)(y+w)(x+$

$y+z+w)$. Let H be a hyperplane defined by $\alpha_H = x - y + 2z + 3w$. Let $\mathcal{A} = \mathcal{A}' \cup \{H\}$. Then H is 4-generic to $\mathcal{A}'$. But in this case, $\Omega^2(\mathcal{A}) \neq \Omega^1(\mathcal{A}') \wedge d\alpha_H/\alpha_H + \Omega^2(\mathcal{A}')$. This fact can be shown by use of the Gröbner basis program "Macaulay" by D. Bayer and M. Stillman, and "Matroid" by P. Edelman to compute the intersection lattice.

3.2. General Case

The proof of Theorem 1 will be done by double induction on $|\mathcal{A}|$ and k. $|\mathcal{A}|=0$ or $k=3$ cases are the starting points of the induction. Let us consider the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^q \langle \mathcal{A}' \rangle \arrow[r, "j_1"] \arrow[d, "i'"] & \Omega^q \langle \mathcal{A} \rangle \arrow[r, "res_1"] \arrow[d, "i"] & \Omega^{q-1} \langle \mathcal{A}'' \rangle \arrow[r] \arrow[d, "i''"] & 0 \\
0 \arrow[r] & \Omega^q(\mathcal{A}') \arrow[r, "j_2"] & \Omega^q(\mathcal{A}) \arrow[r, "res_2"] & \Omega^{q-1}(\mathcal{A}'') &
\end{tikzcd}$$

where i', i, i'', j_1 and j_2 are inclusions, res_1 and res_2 are the residue maps. We know that $|\mathcal{A}'| < |\mathcal{A}|$, and $\mathcal{A}''$ is $(k-1)$-generic if $\mathcal{A}$ is k-generic. So by induction assumption, i', i'' are the identity maps. Let $\omega \in \Omega^q(\mathcal{A})$. Since res_1 is surjective, there is $\eta \in \Omega^q \langle \mathcal{A} \rangle$ such that $res_1(\eta) = res_2(\omega)$. Note $res_1(\eta) = res_2(\eta)$. Then $\xi := \omega - \eta$ is in $\ker(res_2)$. From Lemma 5., we know that $\ker(res_2)=\mathrm{im}(j_2)$. Now $\xi \in \Omega^q(\mathcal{A}') = \Omega^q \langle \mathcal{A}' \rangle$. This says that $\omega = \eta + \xi$ is in $\Omega^q \langle \mathcal{A} \rangle$. So $\Omega^q(\mathcal{A}) = \Omega^q \langle \mathcal{A} \rangle$.

Since the converse of Theorem 1 can be easily proved as in [4, Theorem 7.4], we have

Theorem 7. *An arrangement $\mathcal{A}$ is k-generic if and only if $\Omega^q(\mathcal{A}) = \Omega^q \langle \mathcal{A} \rangle$ for $0 \le q \le k-2$.*

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