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Cross Ratio Varieties for Root Systems of Type A and the Terada Model

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Abstract. The notion of cross ratio varieties for root systems is introduced in [7]. Among others, in the case of the root system of type A_{n+2}, it was conjectured (cf. Conjecture 2.2 in [7]) that the corresponding cross ratio variety is isomorphic to the n-dimensional Terada model which is a natural compactification of the complement in $\mathbf{P}^n$ of the singular locus of the holonomic system of differential equations for the Appell-Lauricella hypergeometric function F_D. The purpose of this article is to prove this conjecture.

1. Introduction

Let Δ be an irreducible root system on an Euclidean space E over $\mathbf{R}$ and let $\mathbf{P}(E_{\mathbf{C}})$ be the complex projective space associated to E. For each subroot system of type A_3 in Δ, it is possible to define an A_3-cross ratio map of $Z(\Delta)$ to $CR(\mathbf{P})$, where $Z(\Delta)$ is a Zariski open subset of $\mathbf{P}(E_{\mathbf{C}})$ and $CR(\mathbf{P}) \simeq \mathbf{P}^1$ (for the precise definition of $Z(\Delta)$ and $CR(\mathbf{P})$, see [7], §1). By taking the product of the A_3-cross ratio maps for all subroot systems of type A_3 in Δ, we obtain a map cr_{Δ,A_3} of $Z(\Delta)$ to $CR(\mathbf{P})^m$, where m is the number of subroot systems of type A_3 in Δ. We put $\mathcal{C}'(\Delta, A_3) = cr_{\Delta,A_3}(Z(\Delta))$ and denote by $\mathcal{C}(\Delta, A_3)$ its Zariski closure in $CR(\mathbf{P})^m$ following the notation in [7].

We now assume that $\Delta = \Delta(A_{n+2})$ is of type A_{n+2}. In this case, it is easy to see that $\dim \mathcal{C}(\Delta, A_3) = n$ and that $\mathcal{C}(\Delta, A_3)$ is regarded as a compactification of the complement of the hypersurface $\mathcal{S}_n$ in $\mathbf{C}^n$ defined by

$$\prod_{j=1}^{n}\{z_j(1-z_j)\}\prod_{i<j}(z_i - z_j) = 0,$$

where $z = (z_1, \cdots, z_n)$ is a standard affine coordinate system of $\mathbf{C}^n$ (cf. [7]). On the other hand, there is a natural compactification of $\mathbf{C}^n \backslash \mathcal{S}_n$ constructed

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in [9] which is called the (n-dimensional) Terada model and denotes $\mathcal{T}_n$ in this article. Moreover, both $\mathcal{C}(\Delta, A_3)$ and $\mathcal{T}_n$ admit $W(A_{n+2})$-actions. Noting these, we are led to ask a question whether $\mathcal{C}(\Delta, A_3)$ is isomorphic to $\mathcal{T}_n$ or not (cf. [7], Conjecture 2.2 (i)).

The purpose of this article is to give an answer affirmative to the question above, namely, to prove that $\mathcal{C}(\Delta, A_3)$ is isomorphic to $\mathcal{T}_n$ for each n. Conjecture 2.2 in [7] is its easy consequence. The result of this article shows that the notion of cross ratio varieties introduced in [7] is regarded as a generalization of the Terada model to the case of root systems.

We are going to explain the contents of this article briefly. In §2, we introduce a projective space with displacements which is denoted by $\mathbf{P}^n_{dsp}$ to distinguish from the usual projective space and, by using it, we define the (n-dimensional) Terada model $\mathcal{T}_n$ following [9]. In §3, we define the cross ratio variety $\mathcal{C}(\Delta(A_n), A_3)$ (cf. [7]) and its variations $\mathcal{C}(\Delta(A_n), A_2)$, $\mathcal{C}(\Delta(A_n), \{A_2, A_3\})$. Among these three varieties, there are isomorphisms

$$\mathcal{C}(\Delta(A_n), A_3) \simeq \mathcal{C}(\Delta(A_{n-1}), \{A_2, A_3\}) \simeq \mathcal{C}(\Delta(A_{n-1}), A_2),$$

which will be shown in §4. The Terada model is defined as a closed subvariety of the product of a large number of projective spaces. For our purpose, it is better to define it as a closed subvariety of the product of a smaller number of projective lines, which will be done in §5. We next show that the modification of the definition of the Terada model above is same as that of $\mathcal{C}(\Delta(A_{n-1}), A_2)$. This implies the our main result of this article.

THEOREM.
(1) $\mathcal{C}(\Delta(A_{n+1}), A_2)$ *is non-singular.*
(2) There is a $W(A_{n+2})$*-equivariant isomorphism of* $\mathcal{C}(\Delta(A_{n+1}), A_2)$ *to* $\mathcal{T}_n$.

In the last section, we give a remark on the relations among the Terada model, cross ratio varieties of type A and other compactifications of the configuration space of n points of the projective line.

The author thanks to Professor T. Oda for explaining him both the constructions of the Terada model and stable n-pointed trees of projective lines and sending him the preprint [4]. The author also thanks to the referee for explaining an idea simplifying the proof of the main result (cf. the arguments in §5.).

2. Projective spaces with displacements and the Terada model

We begin with introducing the notion of a projective space with displacement which was constructed in [9]. The argument below is based on [4].

For each $t = (t_1, t_2, \cdots, t_n) \in \mathbf{C}^n$ and $a \in \mathbf{C}\backslash\{0\}$, we put $a \cdot t = (at_1, \cdots, at_n)$. Let $[t]$ for $t \in \mathbf{C}^n$ be the set $\{a \cdot t;\ a \in \mathbf{C}\backslash\{0\}\}$. Then the $(n-1)$-dimensional projective space $\mathbf{P}^{n-1}$ is the totality of $[t]$, $t \in \mathbf{C}^n\backslash\{0\}$.

We define the diagonal map ι_n of $\mathbf{C}$ to $\mathbf{C}^n$ by $\iota_n(a) = (a, \cdots, a) \in \mathbf{C}^n$ for each $a \in \mathbf{C}\backslash\{0\}$. Moreover, we put $(t_1, \cdots, t_n) + \iota_n(a) = (t_1 + a, \cdots, t_n + a)$ for $(t_1, \cdots, t_n) \in \mathbf{C}^n$ and $a \in \mathbf{C}$. We denote by $[[t]]$ the set $[t + \iota_n(a)]$, $a \in \mathbf{C}$. The totality of $[[t]]$, $t \in \mathbf{C}^n\backslash\iota_n(\mathbf{C})$, is called the $(n-2)$-dimensional *projective space with displacement* (cf. [9], [4]) which is denoted by $\mathbf{P}^{n-2}_{dsp}$ to distinguish it from the projective space in the usual sense.

There is a natural identification between $\mathbf{P}^{n-2}_{dsp}$ and $\mathbf{P}^{n-2}$. In fact, for each $[[t]] = [[t_1, \cdots, t_n]] \in \mathbf{P}^{n-2}_{dsp}$, we put $\sigma([[t]]) = [t_1 - t_n, \cdots, t_{n-1} - t_n]$. Then it is clear that σ induces a bijection between $\mathbf{P}^{n-2}_{dsp}$ and $\mathbf{P}^{n-2}$.

For a finite set F, we put $\mathbf{C}^F = \{(t_f)_{f\in F};\ t_f \in \mathbf{C},\ f \in F\}$ stressing the affine coordinate system $t = (t_f)_{f\in F}$ parametrized by F. Using the coordinate $(t_f)_{f\in F}$ instead of $(t_1, \cdots, t_n)$, we introduce $\mathbf{P}^F$ and $\mathbf{P}^F_{dsp}$ by an argument similar to that defining $\mathbf{P}^{n-1}$ and $\mathbf{P}^{n-2}_{dsp}$. We now take $x \in \mathbf{P}^F_{dsp}$. Then there is $t = (t_f)_{f\in F}$ such that $x = [[t]]$. In this case, we write $x_F(f) = t_f\,(\forall f \in F)$ for simplicity. In spite that $x_F(f)\,(f \in F)$ depends on the choice of $t \in \mathbf{C}^F$, the ratio of $x_F(i) - x_F(j)$ and $x_F(i) - x_F(k)$ $(i, j, k \in F)$ does only on x if one of $x_F(i) - x_F(j)$, $x_F(i) - x_F(k)$ is not zero. In the argument below, it is sufficient to treat the ratio $(x_F(i) - x_F(j))/(x_F(i) - x_F(k))$.

We are going to introduce the Terada model. For this purpose, we define a product of projective spaces with displacements:

$$\tilde{\mathbf{P}}^F_{dsp} = \prod_{I\subset F, \sharp I>2} \mathbf{P}^I_{dsp}. \tag{1}$$

Let $pr_{F,I}$ be the projection of $\tilde{\mathbf{P}}^F_{dsp}$ to $\mathbf{P}^I_{dsp}$. For each $x \in \tilde{\mathbf{P}}^F_{dsp}$, we put $x_I = pr_{F,I}(x)$. Using the notation introduced above, we write $x_I = [[x_I(i)]]_{i\in I}$.

Definition 1. (cf. [9], [4].) Let $\mathcal{T}_F$ be the subvariety of $\tilde{\mathbf{P}}^F_{dsp}$ defined as follows. A point $x \in \tilde{\mathbf{P}}^F_{dsp}$ is contained in $\mathcal{T}_F$ if and only if

$$MD(I,J) \qquad \begin{aligned}&(x_I(i)-x_I(k))(x_J(j)-x_J(k))\\ &\quad = (x_I(j)-x_I(k))(x_J(i)-x_J(k)) \quad \forall i,j,k \in I,\end{aligned}$$

where (I, J) runs through all the pairs of subsets of F such that $I \subset J$, $\sharp I > 2$.

REMARK 1. Since $(x_I(i)-x_I(k))/(x_I(j)-x_I(k))$ depends only on $x_I = pr_{F,I}(x)$, the condition $MD(I, J)$ is well-defined.

If F and F' are finite sets such that $\sharp F = \sharp F'$, it is clear that $\mathcal{T}_F \simeq \mathcal{T}_{F'}$. Noting this, we frequently write $\mathcal{T}_n$ instead of $\mathcal{T}_F$ in the case where $n = \sharp F - 2$. In this note, $\mathcal{T}_n$ is called the (n-dimensional) *Terada model*. The Terada model has some nice properties. For example, $\mathcal{T}_n$ is non-singular and admits a biregular $W(A_{n+2})$-action, where $W(A_{n+2})$ is the Weyl group of type A_{n+2} which is isomorphic to the symmetric group on $n+3$ letters.

3. Cross ratio varieties for root systems of type A

In this section, we review the definition of cross ratio varieties for root systems of type A introduced in [7] and its variations.

We first recall the definition of root systems of type A (cf. [1]).

Let ε_j $(j = 1, \cdots, n, n+1)$ be a standard basis of the $(n+1)$-dimensional Euclidean space $\tilde{E}$ over $\mathbf{R}$. We identifiy $\tilde{E}$ with $\mathbf{R}^{n+1}$ by the correspondence

$$t = \sum_{i=1}^{n+1} t_i \varepsilon_i \longrightarrow (t_1, \cdots, t_n, t_{n+1}).$$

Let E be a linear subspace of $\tilde{E}$ defined by $t_1 + \cdots + t_n + t_{n+1} = 0$. The set $\Delta(A_n)$ consisting of

$$\varepsilon_j - \varepsilon_k \quad (j \neq k)$$

is a root system of type A_n on E (cf. [1]).

Let $\mathbf{P}(E_{\mathbf{C}})$ be the projective space associated to $E_{\mathbf{C}} = E \otimes_{\mathbf{R}} \mathbf{C}$. Then $\mathbf{P}(E_{\mathbf{C}})$ consists of $[t]$, $t \in E_{\mathbf{C}} \backslash \{0\}$, $t_1 + \cdots + t_n + t_{n+1} = 0$. It is clear that $\mathbf{P}(E_{\mathbf{C}})$ is identified with $\mathbf{P}^{n-1}_{dsp}$. Let $Z(\Delta(A_n))$ be the complement in $\mathbf{P}(E_{\mathbf{C}})$ of the union of hyperplanes $t_j = t_k$ $(j \neq k)$.

Subroot systems of type A_p in $\Delta(A_n)$ are parametrized by subsets of $\mathsf{N}_n = \{1, \cdots, n, n+1\}$ of cardinality $p+1$. In fact, if $I = \{i_1, i_2, \cdots, i_{p+1}\}$ is a subset of N_n such that $\sharp I = p+1$, then

$$\varepsilon_j - \varepsilon_k \quad (j, k \in I,\ j \neq k)$$

form a root system of type A_p. We denote by $\Delta(I)$ the subroot system thus defined in this note.

In the cases $p = 2, 3$, we are going to define a map of $Z(\Delta(A_n))$ to $CR(\mathbf{P})$ corresponding to I, where $CR(\mathbf{P})$ is a linear subspace of $\mathbf{P}^2$ with homogeneous coordinate $[\xi_1 : \xi_2 : \xi_3]$ defined by $\xi_1 + \xi_2 + \xi_3 = 0$ (cf. [7], §1).

We first treat the case $p = 3$. Corresponding to I, we define a map $cr_{A_3,I}$ of $Z(\Delta(A_n))$ to $CR(\mathbf{P})$ by

$$cr_{A_3,I}(t) = [(t_{i_1} - t_{i_2})(t_{i_3} - t_{i_4}) : -(t_{i_1} - t_{i_3})(t_{i_2} - t_{i_4}) : (t_{i_1} - t_{i_4})(t_{i_2} - t_{i_3})],$$

where $I = \{i_1, i_2, i_3, i_4\}$. The definition of $cr_{A_3,I}$ depends on the ordering of i_1, i_2, i_3, i_4. But for our purpose, this dependence is not important. Therefore we may take one of such orderings. Taking the product of all the maps of the form $cr_{A_3,I}$, we define

$$cr_{\Delta(A_n),A_3} = \prod_{I \subset \mathsf{N}_n, \sharp I = 4} cr_{A_3,I}.$$

We next treat the case $p = 2$. Then, corresponding to I, we define a map $cr_{A_2,I}$ of $Z(\Delta(A_n))$ to $CR(\mathbf{P})$ by

$$cr_{A_2,I}(t) = [t_{i_1} - t_{i_2} : t_{i_2} - t_{i_3} : t_{i_3} - t_{i_1}],$$

where $I = \{i_1, i_2, i_3\}$. As in the case $p = 4$, we may take one of the orderings on i_1, i_2, i_3 for the definition of $cr_{A_2,I}$. In this case, we define

$$cr_{\Delta(A_n),A_2} = \prod_{I \subset \mathsf{N}_n, \sharp I = 3} cr_{A_2,I}.$$

Definition 2. We put

$$\mathcal{C}'(\Delta(A_n), A_3) = cr_{\Delta(A_n),A_3}(Z(\Delta(A_n))),$$
$$\mathcal{C}(\Delta(A_n), A_3) = \overline{\mathcal{C}'(\Delta(A_n), A_3)},$$

$$\begin{aligned}
&\mathcal{C}'(\Delta(A_n), A_2) = cr_{\Delta(A_n),A_2}(Z(\Delta(A_n))),\\
&\mathcal{C}(\Delta(A_n), A_2) = \overline{\mathcal{C}'(\Delta(A_n), A_2)},\\
&\mathcal{C}'(\Delta(A_n), \{A_2, A_3\}) = (cr_{\Delta(A_n),A_2} \times cr_{\Delta(A_n),A_3})(Z(\Delta(A_n))),\\
&\mathcal{C}(\Delta(A_n), \{A_2, A_3\}) = \overline{\mathcal{C}'(\Delta(A_n), \{A_2, A_3\})},
\end{aligned}$$

and call $\mathcal{C}(\Delta(A_n), A_3)$ (resp. $\mathcal{C}(\Delta(A_n), A_2)$, $\mathcal{C}(\Delta(A_n), \{A_2, A_3\})$) the *cross ratio variety* for the root system $\Delta(A_n)$ of type $(\Delta(A_n), A_3)$ (resp. $(\Delta(A_n), A_2)$, $(\Delta(A_n), \{A_2, A_3\})$).

REMARK 2. (i) The relation between the map $cr_{A_3,I}$ and the cross ratio in the usual sense was explained in [7], §1.

(ii) The cross ratio variety $\mathcal{C}(\Delta(A_n), A_3)$ was introduced and studied in [7], §2 and $\mathcal{C}(\Delta(A_n), A_2)$ was referred to in [7], §6.

The set $\mathcal{C}'(\Delta(A_n), A_3)$ is identified with a Zariski open subset of $\mathbf{C}^{n-2}$, which we are going to explain. Let $z = (z_1, \cdots, z_{n-2})$ be a standard affine coordinate system of $\mathbf{C}^{n-2}$. As in the introduction, let $\mathcal{S}_{n-2}$ be the hypersurface of $\mathbf{C}^{n-2}$ defined by the equation

$$\prod_{i=1}^{n-2}\{z_i(1-z_i)\}\prod_{i<j}(z_i - z_j) = 0. \tag{2}$$

We now state a lemma which is easy to prove (cf. [7], Lemma 2.1).

LEMMA 1. *We put*

$$z_j(t) = \frac{(t_j - t_{n+1})(t_{n-1} - t_n)}{(t_j - t_n)(t_{n-1} - t_{n+1})} \quad (j = 1, 2, \cdots, n-2) \tag{3}$$

for all $t = (t_1, \cdots, t_n, t_{n+1}) \in Z(\Delta(A_n))$ *and define the map* F *of* $Z(\Delta(A_n))$ *to* $\mathbf{C}^{n-2}$ *by*

$$t \longrightarrow (z_1(t), \cdots, z_{n-2}(t)).$$

Then $F(Z(\Delta(A_n))) = \mathbf{C}^{n-2} \backslash \mathcal{S}_{n-2}$.

We now show that $\mathcal{C}(\Delta(A_n), A_3)$ is a compactification of $\mathbf{C}^{n-2} \backslash \mathcal{S}_{n-2}$. This is a consequence of the lemma below.

Lemma 2. *We have an isomorphism:*

$$\mathcal{C}'(\Delta(A_n), A_3) \simeq \mathbf{C}^{n-2}\backslash\mathcal{S}_{n-2}. \tag{4}$$

Proof. For $t = (t_1, \cdots, t_n, t_{n+1}) \in Z(\Delta(A_n))$, it is easy to compute the following identities (in the below, $j, k, l, m(= 1, 2, \cdots, n-2)$ are mutually different):

$$\begin{cases} \dfrac{z_k(t)}{z_j(t)} & = \dfrac{(t_k - t_{n+1})(t_j - t_n)}{(t_k - t_n)(t_j - t_{n+1})}, \\ \dfrac{1 - z_k(t)}{1 - z_j(t)} & = \dfrac{(t_k - t_{n-1})(t_j - t_n)}{(t_k - t_n)(t_j - t_{n-1})}, \\ \dfrac{z_j(t)(1 - z_k(t))}{z_k(t)(1 - z_j(t))} & = \dfrac{(t_j - t_{n+1})(t_k - t_{n-1})}{(t_j - t_{n-1})(t_k - t_{n+1})}, \\ \dfrac{z_j(t) - z_k(t)}{z_j(t) - z_l(t))} & = \dfrac{(t_j - t_k)(t_l - t_n)}{(t_j - t_l)(t_k - t_n)}, \\ \dfrac{z_l(t)(z_j(t) - z_k(t))}{z_j(t)(z_l(t) - z_k(t))} & = \dfrac{(t_j - t_k)(t_l - t_{n+1})}{(t_j - t_l)(t_j - t_{n+1})}, \\ \dfrac{(1 - z_l(t))(z_k(t) - z_j(t))}{(1 - z_j(t))(z_k(t) - z_l(t))} & = \dfrac{(t_k - t_j)(t_l - t_{n-1})}{(t_k - t_l)(t_j - t_{n-1})}, \\ \dfrac{(z_j(t) - z_k(t))(z_l(t) - z_m(t))}{(z_l(t) - z_k(t))(z_j(t) - z_m(t))} & = \dfrac{(t_j - t_k)(t_l - t_m)}{(t_k - t_l)(t_j - t_m)}. \end{cases} \tag{5}$$

In virtue of (3), (5), we find that for each subset I of N_n with $\sharp I = 4$, $cr_{A_3,I}(t)$ $(t \in Z(\Delta(A_n)))$ is expressed by $z_j(t), j = 1, 2, \cdots, n-2$. Therefore, noting the definition of $\mathcal{C}'(\Delta(A_n), A_3)$, we easily show the isomorphism (4). □

On the other hand, $\mathcal{T}_{n-2}$ is also regarded as a compactification of $\mathbf{C}^{n-2}\backslash\mathcal{S}_{n-2}$ as we are going to explain below briefly. By the correspondence

$$(z_1, \cdots, z_{n-2}) \longrightarrow [z_1 : \cdots : z_{n-2} : 1],$$

$\mathbf{C}^{n-2}$ is embedded in $\mathbf{P}^{n-2}$. Under the identification σ between $\mathbf{P}^{n-2}_{dsp} \simeq \mathbf{P}^{n-2}$ given in §2, $\mathbf{C}^{n-2}\backslash\mathcal{S}_{n-2}$ corresponds to the Zariski open subset $(\mathbf{P}^{n-2}_{dsp})'$ of $\mathbf{P}^{n-2}_{dsp}$ defined by

$$(\mathbf{P}^{n-2}_{dsp})' = \{[[t_1, \cdots, t_n]]\ ;\ (t_1, \cdots, t_n) \in \mathbf{C}^n,\ t_i \neq t_j (i \neq j)\}. \tag{6}$$

Moreover we put

$$\mathcal{T}'_{n-2} = \{x \in \mathcal{T}_{n-2},\ pr_{F,F}(x) \in (\mathbf{P}^{n-2}_{dsp})'\}, \tag{7}$$

where $F = \mathsf{N}_{n-1}$ and $pr_{F,F}$ is the projection of $\tilde{\mathbf{P}}^{n-2}_{dsp}$ to $\mathbf{P}^{n-2}_{dsp}$ defined in §2. Then clearly

$$\mathcal{T}'_{n-2} \simeq (\mathbf{P}^{n-2}_{dsp})' \simeq \mathbf{C}^{n-2} \backslash \mathcal{S}_{n-2}$$

and therefore $\mathcal{T}_{n-2}$ is a compactification of $\mathbf{C}^{n-2} \backslash \mathcal{S}_{n-2}$.

REMARK 3. The hypersurface $\mathcal{S}_{n-2}$ in $\mathbf{C}^{n-2}$ is nothing but the singular locus of the holonomic system of differential equations for the Appell-Lauricella hypergeometric function of $n-2$ variables F_D.

4. Isomorphisms among $\mathcal{C}(\Delta(A_n), A_3)$, $\mathcal{C}(\Delta(A_{n-1}), \{A_2, A_3\})$ and $\mathcal{C}(\Delta(A_{n-1}), A_2)$

The main purpose of this section is to prove isomorphisms in Theorem 1 below.

THEOREM 1. *We have the isomorphism:*

$$\mathcal{C}(\Delta(A_n), A_3) \simeq \mathcal{C}(\Delta(A_{n-1}), \{A_2, A_3\}) \simeq \mathcal{C}(\Delta(A_{n-1}), A_2). \tag{8}$$

This theorem is a consequence of the following two propositions.

PROPOSITION 1. $\mathcal{C}(\Delta(A_n), A_3) \simeq \mathcal{C}(\Delta(A_{n-1}), \{A_2, A_3\})$.

PROPOSITION 2. $\mathcal{C}(\Delta(A_n), A_2) \simeq \mathcal{C}(\Delta(A_n), \{A_2, A_3\})$.

This section is devoted to prove these two propositions.

PROOF OF PROPOSITION 1. For $u = (u_1, u_2, \cdots, u_n) \in \mathbf{C}^n$ such that $u_i \neq u_j$ $(i \neq j)$ and that $\sum_{j=1}^n u_j = 0$, we put

$$w_j(u) = \frac{u_{n-1} - u_n}{u_j - u_n}, \quad j = 1, 2, \cdots, n-2. \tag{9}$$

Then

$$(10)\quad \begin{cases} \dfrac{w_k(u)}{w_j(u)} & = & \dfrac{u_j-u_n}{u_k-u_n}, \\ \dfrac{w_j(u)(1-w_k(u))}{w_k(u)(1-w_j(u))} & = & \dfrac{u_k-u_{n-1}}{u_j-u_{n-1}}, \\ \dfrac{w_l(u)(w_j(u)-w_k(u))}{w_j(u)(w_l(u)-w_k(u))} & = & \dfrac{u_j-u_k}{u_j-u_l}, \end{cases}$$

$$(11)\quad \begin{cases} \dfrac{1-w_k(u)}{1-w_j(u)} & = & \dfrac{(u_k-u_{n-1})(u_j-u_n)}{(u_k-u_n)(u_j-u_{n-1})}, \\ \dfrac{w_j(u)-w_k(u)}{w_j(u)-w_l(u)} & = & \dfrac{(u_j-u_k)(u_l-u_n)}{(u_j-u_l)(u_k-u_n)}, \\ \dfrac{(1-w_l(u))(w_k(u)-w_j(u))}{(1-w_j(u))(w_k(u)-w_l(u))} & = & \dfrac{(u_k-u_j)(u_l-u_{n-1})}{(u_k-u_l)(u_j-u_{n-1})}, \\ \dfrac{(w_j(u)-w_k(u))(w_l(u)-w_m(u))}{(w_l(u)-w_k(u))(w_j(u)-w_m(u))} & = & \dfrac{(u_j-u_k)(u_l-u_m)}{(u_k-u_l)(u_j-u_m)}. \end{cases}$$

(In the above, $j,k,l,m(=1,2,\cdots,n-2)$ are mutually different.) We note that the left-hand sides of equations in (5) coincide with those in (10) and (11) by replacing $z_j(t)$ with $w_j(u)$ $(j=1,2,\cdots,n-2)$. In virtue of (9), (10) (resp. (11)), we find that for each subset I of N_{n-1} such that $\sharp I=3$ (resp. $\sharp I=4$), $cr_{A_2,I}(u)$ (resp. $cr_{A_3,I}(u)$) is expressed in terms of $w_j(u), j=1,2,\cdots,n-2$. Recalling the definition of $\mathcal{C}(\Delta(A_{n-1}),\{A_2,A_3\})$, we conclude from the arguments above that $\mathcal{C}(\Delta(A_n),A_3)\simeq\mathcal{C}(\Delta(A_{n-1}),\{A_2,A_3\})$ and the proposition follows. □

We are going to prove Proposition 2. For this purpose, we prepare the following lemma.

Lemma 3. $\mathcal{C}(\Delta(A_3),A_2)\simeq\mathcal{C}(\Delta(A_3),\{A_2,A_3\})$.

Proof. For $t=(t_1,t_2,t_3,t_4)\in Z(\Delta(A_3))$, we put

$$(12)\quad \begin{aligned} (\sigma_{1,4}:\sigma_{1,2}:\sigma_{1,3}) &= (t_2-t_3:t_3-t_4:t_4-t_2), \\ (\sigma_{2,4}:\sigma_{2,1}:\sigma_{2,3}) &= (t_1-t_3:t_3-t_4:t_4-t_1), \\ (\sigma_{3,4}:\sigma_{3,1}:\sigma_{3,2}) &= (t_1-t_2:t_2-t_4:t_4-t_1), \\ (\sigma_{4,3}:\sigma_{4,1}:\sigma_{4,2}) &= (t_1-t_2:t_2-t_3:t_3-t_1), \end{aligned}$$

$$(13)\quad \begin{aligned} (\tau_1:\tau_2:\tau_3) \;=\; & ((t_1-t_2)(t_3-t_4): \\ & -(t_1-t_3)(t_2-t_4):(t_1-t_4)(t_2-t_3)). \end{aligned}$$

The right-hand sides of these equations are images of cross ratio maps. It is easy to show the identity equations among $\sigma_{i,j}$ and τ_i given below:

$$
\begin{aligned}
&\sigma_{1,2}\sigma_{2,3}\sigma_{3,1}+\sigma_{1,3}\sigma_{2,1}\sigma_{3,2}=0, \quad \sigma_{1,2}\sigma_{2,4}\sigma_{4,1}+\sigma_{1,4}\sigma_{2,1}\sigma_{4,2}=0,\\
&\sigma_{1,3}\sigma_{3,4}\sigma_{4,1}+\sigma_{1,4}\sigma_{3,1}\sigma_{4,3}=0, \quad \sigma_{2,3}\sigma_{3,4}\sigma_{4,2}+\sigma_{2,4}\sigma_{3,2}\sigma_{4,3}=0,
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
&\tau_1\sigma_{2,4}\sigma_{3,1}+\tau_2\sigma_{2,1}\sigma_{3,4}=0, \quad \tau_1\sigma_{4,2}\sigma_{1,3}+\tau_2\sigma_{4,3}\sigma_{1,2}=0,\\
&\tau_2\sigma_{3,2}\sigma_{4,1}+\tau_3\sigma_{3,1}\sigma_{4,2}=0, \quad \tau_2\sigma_{2,3}\sigma_{1,4}+\tau_3\sigma_{2,4}\sigma_{1,3}=0,\\
&\tau_3\sigma_{4,3}\sigma_{2,1}+\tau_1\sigma_{4,1}\sigma_{2,3}=0, \quad \tau_3\sigma_{3,4}\sigma_{1,2}+\tau_1\sigma_{3,2}\sigma_{1,4}=0.
\end{aligned} \tag{15}
$$

We denote by $\sigma_1,\sigma_2,\sigma_3,\sigma_4$ the left-hand sides of equations in (12) in order and by τ the left-hand side of (13). Let X be the subvariety of $CR(\mathbf{P})^4$ with coordinate $\sigma=(\sigma_1,\sigma_2,\sigma_3,\sigma_4)$ satisfying the relation (14) and $\tilde{Y}$ be the subvariety of $CR(\mathbf{P})^5$ with coordinate $(\sigma,\tau)=(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\tau)$ satisfying the relations (14), (15). Then it follows from the definition that $\mathcal{C}(\Delta(A_3),A_2)$ (resp. $\mathcal{C}(\Delta(A_3),\{A_2,A_3\})$) is a closed subvariety of X (resp. $\tilde{Y}$). In particular, $\mathcal{C}(\Delta(A_3),A_2)$ is identified with the subvariety Y of $\tilde{Y}$ defined by

$$
Y=\{(\sigma,\tau)\in CR(\mathbf{P})^5;\ \sigma\in\mathcal{C}(\Delta(A_3),A_2),\ (\sigma,\tau)\in\tilde{Y}\}.
$$

The correspondence $(\sigma,\tau)\to\sigma$ defines a natural projection π_Y of $Y\simeq\mathcal{C}(\Delta(A_3),\{A_2,A_3\})$ to $\mathcal{C}(\Delta(A_3),A_2)$.

We are going to prove that π_Y is an isomorphism. To prove this, it suffices to show that for any $(\sigma,\tau)\in Y$ satisfying $\sigma\in\mathcal{C}(\Delta(A_3),A_2)$, $\tau\in CP(\mathbf{P})$ is uniquely determined by equation (15). If $\sigma\in\mathcal{C}'(\Delta(A_3),A_2)$, all of $\sigma_{i,j}$ do not vanish, so the claim is clear. Therefore we assume that $\sigma\notin\mathcal{C}'(\Delta(A_3),A_2)$. Then, since at least one of $\sigma_{i,j}$ is zero, we may assume that $\sigma_{4,3}=0$ without loss of generality. In this case, $\sigma_{4,1}=-\sigma_{4,2}\neq 0$ and it follows from (14) that

$$
\begin{aligned}
&\sigma_{1,2}\sigma_{2,3}\sigma_{3,1}+\sigma_{1,3}\sigma_{2,1}\sigma_{3,2}=0, \quad \sigma_{1,2}\sigma_{2,4}-\sigma_{1,4}\sigma_{2,1}=0,\\
&\sigma_{1,3}\sigma_{3,4}=0, \quad \sigma_{2,3}\sigma_{3,4}=0.
\end{aligned} \tag{16}
$$

In virtue of (16), there are two possibilities:

Case (I): $\sigma_{3,4}=0$.

Case (II): $\sigma_{3,4}\neq 0$ and $\sigma_{1,3}=\sigma_{2,3}=0$.

In Case (I), it is easy to show that

$$
\sigma_1=\sigma_2=0, \quad \tau=(0:1:-1).
$$

On the other hand, in Case (II), we find that $\tau = \sigma_3$.

By the argument above, we find that $(\sigma, \tau) \in Y$ depends only on $\sigma \in \mathcal{C}(\Delta(A_3), A_2)$ and the lemma follows. $\square$

Remark 4. Under the notation in Lemma 3, X actually coincides with $\mathcal{C}(\Delta(A_3), A_2)$. In fact, it is provable that (14) is a defining equation of $\mathcal{C}(\Delta(A_3), A_2)$.

Still we treat the cross ratio variety $\mathcal{C}(\Delta(A_3), A_2)$ for the root system $\Delta(A_3)$. We are going to define a map ϕ of $\mathcal{C}(\Delta(A_3), A_2)$ to $CR(\mathbf{P})$. For each $x \in \mathcal{C}(\Delta(A_3), A_2)$, it follows from Lemma 3 that there is a unique $\tau \in CR(\mathbf{R})$ such that $(x, \tau) \in \mathcal{C}(\Delta(A_3), \{A_2, A_3\})$. Then $\phi(x) = \tau$.

We return to the general case, that is, the case where n is arbitrary and start to prove Propisition 2.

Proof of Propisition 2. We are going to define a map of $\mathcal{C}(\Delta(A_n), A_2)$ to $\mathcal{C}(\Delta(A_n), A_3)$. If $I = \{i_1, i_2, i_3, i_4\}$ is a subset of N_n such that $\sharp I = 4$, we put

$$\varpi_{A_n, I} = \prod_{k=1}^{4} \pi_{(A_n, A_2), I \setminus \{i_k\}}.$$

Then

$$\varpi_{A_n, I}(\mathcal{C}(\Delta(A_n), A_2)) = \mathcal{C}(\Delta(I), A_2).$$

Since $\Delta(I)$ is a root system of type A_3, we can define a surjective map of $\mathcal{C}(\Delta(I), A_2)$ to $CR(\mathbf{P})$ by an argument similar to that constructing the map ϕ. We denote this map by $\phi_{\Delta(I)}$. Then $\phi_{\Delta(I)} \circ \varpi_{A_n, I}$ is a surjective map of $\mathcal{C}(\Delta(A_n), A_2)$ to $CR(\mathbf{P})$. Moreover we define

$$\tilde{\phi}_{\Delta(A_n)} = \prod_{I \subset \mathsf{N}_n, \sharp I = 4} \phi_{\Delta(I)} \circ \varpi_{A_n, I}.$$

For each $x \in \mathcal{C}(\Delta(A_n), A_2)$, we put

$$\eta_{\Delta(A_n)}(x) = (x, \tilde{\phi}_{\Delta(A_n)}(x)).$$

Then $\eta_{\Delta(A_n)}$ defines a map of $\mathcal{C}(\Delta(A_n), A_2)$ to $\mathcal{C}(\Delta(A_n), \{A_2, A_3\})$.

To prove Proposition 2, it suffices to show that the map $\eta_{\Delta(A_n)}$ is an isomorphism of $\mathcal{C}(\Delta(A_n), A_2)$ to $\mathcal{C}(\Delta(A_n), \{A_2, A_3\})$. But this is clear from the definition of $\eta_{\Delta(A_n)}$. In fact, for each $x \in \mathcal{C}(\Delta(A_n), A_2)$, we find that both $\pi_{A_2,A_3} \circ \eta_{\Delta(A_n)}$ and $\eta_{\Delta(A_n)} \circ \pi_{A_2,A_3}$ are the identity maps, where π_{A_2,A_3} denotes the natural projection of $\mathcal{C}(\Delta(A_n), \{A_2, A_3\})$ to $\mathcal{C}(\Delta(A_n), A_2)$.

We have thus proved Proposition 2. $\square$

5. A simplification of the Terada model

For our purpose, it is better to simplify the definition of $\mathcal{T}_n$. This section is devoted to this subject. We begin with introducing a product space of projective spaces with displacements modifying the definition of $\tilde{\mathbf{P}}^F_{dsp}$ (cf. (1)):

$$\left(\tilde{\mathbf{P}}^F_{dsp}\right)_{rest} = \prod_{I \subset F, \sharp I = 3} \mathbf{P}^I_{dsp}$$

and a natural projection

$$TR : \tilde{\mathbf{P}}^F_{dsp} \longrightarrow \left(\tilde{\mathbf{P}}^F_{dsp}\right)_{rest}$$

defined by $TR((x_I)_{I \subset F, \sharp I > 2}) = (x_I)_{I \subset F, \sharp I = 3}$. Then

$$\mathcal{T}_{F,rest} = TR(\mathcal{T}_F)$$

is a closed subvariety of $\left(\tilde{\mathbf{P}}^F_{dsp}\right)_{rest}$.

LEMMA 4. *The restriction of TR to $\mathcal{T}_F$ gives an isomorphism between $\mathcal{T}_F$ and $\mathcal{T}_{F,rest}$.*

PROOF. For any $x \in \mathcal{T}_F$, we write $x = (x_I)_{I \subset F, \sharp I > 2}$. We are going to prove that for each subset J of F with $\sharp I > 3$, x_J is uniquely determined by $TR(x)$. From the definition, there are $i, j \in J$ such that $x_J(i) - x_J(j) \neq 0$. For any $k \in J$ with $k \neq i, j$, we put $I = \{i, j, k\}$. Then it follows from the condition $MD(I, J)$ of Definition 1 that

$$(x_I(i) - x_I(j))(x_J(k) - x_J(i)) = (x_I(k) - x_I(i))(x_J(i) - x_J(j)). \tag{17}$$

If $x_I(i) - x_I(j) = 0$, the assumption combined with (17) implies that $x_I(k) - x_I(j) = 0$, which contradicts the definition of $\mathbf{P}^I_{dsp}$. Therefore $x_I(i) - x_I(j) \neq 0$. Then (17) turns out to be

$$x_J(k) - x_J(i) = \frac{x_I(k) - x_I(i)}{x_I(i) - x_I(j)} \cdot (x_J(i) - x_J(j)). \tag{18}$$

Since, for any $k' \in J$ with $k' \neq i, j$, an equation for $x_J(k') - x_J(i)$ similar to (18) holds, we conclude that $x_J = [[x_J(j)]]_{j \in J}$ is uniquely determined by $TR(x)$.

As a consequence, $TR|\mathcal{T}_F$ is bijective. Moreover, in virtue of the equations of the form (18) and that $\mathcal{T}_F$ is non-singular, we find that $\mathcal{T}_{F,rest}$ is also non-singular and

$$TR : \mathcal{T}_F \longrightarrow \mathcal{T}_{F,rest}$$

is an isomorphism between algebraic varieties. $\square$

At the first appearance, the conditions $MD(I, J)$ of Definition 1 for all pair (I, J) induce no relation for $\mathcal{T}_{F,rest}$ as a subvariety of $\left(\tilde{\mathbf{P}}^F_{dsp}\right)_{rest}$. We are going to mention equations defining $\mathcal{T}_{F,rest}$. Let $J = \{i, j, k, l\}$ be a subset of F with $\sharp J = 4$. We put $I_m = J \backslash \{m\}$ for each $m \in J$. Then we have the following.

Lemma 5. *Let $x = (x_I)_{I \subset F, \sharp I = 3} \in \left(\tilde{\mathbf{P}}^F_{dsp}\right)_{rest}$. If x is contained in $\mathcal{T}_{F,rest}$, then*

$$\begin{aligned}(19) \qquad & (x_{I_i}(k) - x_{I_i}(l))(x_{I_j}(i) - x_{I_j}(l))(x_{I_k}(j) - x_{I_k}(l)) \\ & \quad = (x_{I_i}(j) - x_{I_i}(l))(x_{I_j}(k) - x_{I_j}(l))(x_{I_k}(i) - x_{I_k}(l)).\end{aligned}$$

This lemma is proved by an easy but a little lengthy argument, using $MD(I_m, J)$ $(m \in J)$.

Remark 5. The author does not know whether the equations of the form (19) for all $i, j, k, l \in F$ actually define the variety $\mathcal{T}_{F,rest}$ or not.

Last in this section, we give an identification between $\mathbf{P}^I_{dsp}$ and $CR(\mathbf{P})$ for each $I \subset F$, $\sharp I = 3$. This is eatablished by the correspondence

$$x_I \longrightarrow [x_I(j) - x_I(k) : x_I(k) - x_I(i) : x_I(i) - x_I(j)]$$

in the case where $I = \{i, j, k\}$.

6. The main theorem

In this section, we consider the case $\sharp F = \mathsf{N}_{n+2}$. Therefore $\mathcal{T}_{F,rest} \simeq \mathcal{T}_F \simeq \mathcal{T}_n$. In this case, it follows from the argument at the last part of the previous section that $\mathcal{T}_{F,rest}$ is regarded as a closed subvariety of $CR(\mathbf{P})^m$, where $m = \sharp\{I; I \subset \mathsf{N}_{n+2}, \sharp I = 3\}$. On the other hand, $\mathcal{C}(\Delta(A_{n+1}), A_2)$ is also a closed subvariety of $CR(\mathbf{P})^m$.

We are going to show that the construction of $\mathcal{T}_{F,rest}$ is same as that of $\mathcal{C}(\Delta(A_{n+1}), A_2)$. In fact, we recall the definition of $\mathcal{T}_F$. We denote by $(\mathbf{P}^F_{dsp})'$ the totality of $x_F \in \mathbf{P}^F_{dsp}$ such that $x_F(i) - x_F(j) \neq 0$ for $i \neq j$ (cf. (6)). Moreover, we put (cf. (7))

$$\begin{aligned} \mathcal{T}'_F &= \{x = (x_I)_{I \subset F, \sharp I > 2} \in \mathcal{T}_F; x_F \in (\mathbf{P}^F_{dsp})'\}, \\ \mathcal{T}'_{F,rest} &= TR(\mathcal{T}'_R). \end{aligned}$$

Cleraly, $\mathcal{T}'_{F,rest}$ is Zariski open in $\mathcal{T}_{F,rest}$. Taking a subset $I = \{i, j, k\}$ of F, we write down the relation $MD(I, F)$. Then

$$(x_I(i) - x_I(k))(x_F(j) - x_F(k)) = (x_I(j) - x_I(k))(x_F(i) - x_F(k)).$$

This implies that if $x = (x_I)_{I \subset F, \sharp I = 3}$ is contained in $(\mathcal{T}_{F,rest})'$, there is $x_F \in (\mathbf{P}^F_{dsp})'$ such that $x_I = [x_F(j) - x_F(k) : x_F(k) - x_F(i) : x_F(i) - x_F(j)]$ for all subset I of F with $\sharp I = 3$. Comparing the argument above with the definition of $\mathcal{C}(\Delta(A_{n+1}), A_2)$, we have proved the following.

Theorem 2. *The varieties $\mathcal{T}_{F,rest}(\simeq \mathcal{T}_F)$ and $\mathcal{C}(\Delta(A_{n+1}), A_2)$ are isomorphic.*

As an easy consequence of Theorem 2, we obtain the theorem below which is nothing but Conjecture 2.2 in [7].

Theorem 3. *(i) The variety $\mathcal{C}(\Delta(A_n), A_3)$ is non-singular.*

(ii) The complement of $\mathcal{C}'(\Delta(A_n), A_3)$ in $\mathcal{C}(\Delta(A_n), A_3)$ is the union of hypersurfaces $Y_{\Delta(A_n),A_3}(\Delta(I))$, where I runs through all the subsets of N_n such that $1 < \sharp I < n$.

Proof. The claim (i) is a consequence of Theorem 2 and (8).

The claim (ii) follows from Theorem 2 and the arguments in [9], [4] and [7], §2. □

REMARK 6. It is known (cf. [9]) that the Weyl group $W(A_{n+2})$ acts on $\mathcal{T}_n$ biregularly. This coincides with the $W(A_{n+2})$-action on the cross ratio variety $\mathcal{C}(\Delta(A_{n+2}), A_3)(\simeq \mathcal{T}_n)$.

We are going to explain a relation between this action and the $W(A_n)$-action on a Cartan subgroup H of $SL(n+1, \mathbf{C})$. We take $\alpha_1, \cdots, \alpha_n \in \Delta(A_n)$ as a system of simple roots with Dynkin diagram:

$$\alpha_1 \text{———} \alpha_2 \text{———} \cdots\cdots \text{———} \alpha_n$$

We regard each $\alpha \in \Delta(A_n)$ as a character on H which we denote by χ_α. Putting $\beta_j = \sum_{i=1}^{j} \alpha_i \quad (j = 1, 2, \cdots, n)$, we define a map χ of H to $\mathbf{P}^n \simeq \mathbf{P}^n_{dsp}$ by $\chi(g) = [\chi_{\beta_1}(g) : \cdots : \chi_{\beta_n}(g) : 1]$ for each $g \in H$. If $H' = \{g \in H;\ \chi_\alpha(g) \neq 1\ (\forall \alpha \in \Delta(A_n))\}$, then H' coincides with $\mathbf{C}^n \backslash \mathcal{S}_n$ under the identification of $\mathbf{C}^n$ with a subset of $\mathbf{P}^n$ explained in §3. Therefore $\mathcal{T}_n \simeq \mathcal{C}(\Delta(A_{n+2}), A_3)$ is regarded as a compactification of H'. The $W(A_n)$-action on H' induces the biregular $W(A_n)$-action on $\mathbf{P}^n$ which is identified with the group of permutations among homogeneous coordinates. This action on $\mathbf{P}^n$ is extended to a birational $W(A_{n+2})$-action which coincides with that induced from the $W(A_{n+2})$-action on $\mathcal{C}(\Delta(A_{n+2}), A_3)$.

A Cartan subalgebra of the Lie algebra $\mathbf{sl}(n+3, \mathbf{R})$ is regarded as a standard representation space of $W(A_{n+2})$. Then, roughly speaking, there is a $W(A_{n+2})$-equivariant map of a Cartan subalgebra of $\mathbf{sl}(n+3, \mathbf{C})$ with a natural linear $W(A_{n+2})$-action to a Cartan subgroup of $SL(n+1, \mathbf{C})$ with the birational $W(A_{n+2})$-action explained before.

A similar situation occurs when we consider the $W(D_4)$-action on a Cartan subgroup H of the simple group $SO(8, \mathbf{C})/\{\pm 1\}$. In this case, the $W(D_4)$-action on H is biregular and is extended to a birational $W(E_6)$-action (cf. [3], [6]).

7. Concluding remarks

After the work was done, the author was informed by N. Takayama (Kobe Univ.) of the work of M. M. Kapranov(cf. [12]). It is stated in [11], [12] that there are several works on compactifications of the configuration space of n-points of the projective line and the subject goes back to A. Grothendieck (cf. [10]). We here list up such compactifications.

(C1) The Grothendieck-Knudsen moduli space $\overline{\mathcal{M}_{0,n}}$ (cf. [10], 1972, [13], 1983).

(C2) The n-dimensional Terada model $\mathcal{T}_n$ (cf. [9], 1983).

(C3) The Gerritzen-Herrlich-van der Put compactification of stable n-pointed trees of projective lines (cf. [2], 1988).

(C4) Kapranov's Chow quotient $G(2,n)//H$ (cf. [12]).

(C5) The cross ratio variety $\mathcal{C}(\Delta(A_{n+2}), A_3)$.

It seems to be true the equivalence of (C1) and (C3), because the definitions of them are quite similar. T. Oda ([4]) proved the isomorphism between $\mathcal{T}_n$ and the compactification (C3). On the other hand, M. M. Kapranov ([12] proved the isomorphism $\overline{\mathcal{M}_{0,n}} \simeq G(2,n)//H$. For these reasons, it is plausible that the compactifications (C1)-(C5) are mutually isomorphic.

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