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*Asymptotic behaviour of the sequence of norms of derivatives**

By Ha Huy Bang

Abstract. Two exact asymptotic inequalities for derivatives, which show a relation between behaviour of the sequence of norms of derivatives of a function and the support of its Fourier transform, are given in this paper.

Let I be an unbounded set of multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n, 1 \leq p_\alpha \leq \infty$ and let $f(x)$ be a measurable function such that its generalized derivative $D^\alpha f(x)$ belongs to $L_{p_\alpha}(\mathbb{R}^n)$ for any $\alpha \in I$. In this paper we will describe behaviour of the sequence $||D^\alpha f||_{p_\alpha}, \alpha \in I$, in the connection with suppFf, where $Ff(\xi) = \tilde{f}(\xi)$ is the Fourier transform of the function $f(x)$. The necessity of the consideration is clear from the definition of the Sobolev spaces of infinite order [5 - 6]. Note that Sobolev spaces of infinite order, which arise in the study of nonlinear (or linear) differential equations of infinite order, were introduced by Ju.A. Dubinskii in 1975 and studied by him, T.D. Van, G.S. Balashova, L.I. Klenina, Ju.A. Konjaev, A.Ja. Kobilov, S.R. Umarov, A.N. Agadzhanov and the author (see, for example, [5 - 6] and their references). The obtained results improve the corresponding results in [1 - 2], which are helpful for establishing nontriviality criteria and imbedding theorems for Sobolev spaces of infinite order (see, for example, [2 - 4]).

We will use the following standard notations: $D = (D_1, ..., D_n), D_j = \frac{\partial}{\partial x_j}, j = 1, ..., n, D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, sp$(f)$ =supp$\tilde{f}$ and $W_{m,2}(G)$, $W^0_{m,2}(G)$ -

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the classical Sobolev spaces (see, for example, [7 - 8]). And we presuppose that $0^0 = \frac{0}{0} = 1, \frac{\lambda}{0} = \infty$ for $\lambda > 0, f(x) \in \mathcal{S}'$ and $f(x) \not\equiv 0$.

We will show the following

THEOREM 1. *Let I be an unbounded set of integral multi-indices $\alpha = (\alpha_1, ..., \alpha_n), \alpha_j \geq 0,\ j = 1, ..., n,\ 1 \leq p_\alpha \leq \infty$ and let $f(x)$ be a measurable function such that its generalized derivative $D^\alpha f(x)$ belongs to $L_{p_\alpha}(\mathbb{R}^n), \alpha \in I$. Then*

$$(I) \varliminf_{|\alpha|\to\infty} (||D^\alpha f||_{p_\alpha}/|\xi^\alpha|)^{1/|\alpha|} \geq 1 \tag{1}$$

for any point $\xi \in \mathrm{sp}(f)$, where the notation (I) means that we take the limit only for $\alpha \in I$.

PROOF. Let $\xi^0 \in \mathrm{sp}(f), \xi_j^0 \neq 0, j = 1, ..., n$. For the sake of convenience, we assume that $\xi_j^0 > 0, j = 1, ..., n$. We fix a number $0 < \epsilon < \frac{1}{2} \min_{1\leq j\leq n} \xi_j^0$ and choose a domain G with a smooth boundary such that $\xi^0 \in G$ and $G \subset \{\xi : \xi_j^0 - \epsilon \leq \xi_j \leq \xi_j^0 + \epsilon, j = 1, ..., n\}$. Further we fix a function $\tilde{v}(\xi) \in C_0^\infty(G)$ such that $\xi^0 \in \mathrm{supp}(\tilde{v}\tilde{f})$. Then

$$< \tilde{v}(\xi)\tilde{f}(\xi), \tilde{w}(\xi) > = < f(x), \varphi(x) >, \tag{2}$$

where $\tilde{w}(\xi) \in C_0^\infty(G)$ is an arbitrary function, $\varphi(x) = \overset{\vee}{v} * \overset{\vee}{w}(x)$ and $\overset{\vee}{u}(x) = u(-x)$. Since the distribution $\tilde{v}(\xi)\tilde{f}(\xi)$ has a compact support, it can be represented in the form

$$\tilde{v}(\xi)\tilde{f}(\xi) = \sum_{|\alpha|\leq m} D^\alpha h_\alpha(\xi),$$

where m is a nonnegative integer and $h_\alpha(\xi)$ are ordinary functions in G. Without loss of generality we may assume that $m \geq 2n$.

It is well - known that the Dirichlet problem for the elliptic differential equation

$$L_{2m}\tilde{z}(\xi) = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha(D^\alpha \tilde{z}(\xi)) = \tilde{v}(\xi)\tilde{f}(\xi)$$

has a (unique) solution $\tilde{z}(\xi) \in W^0_{m,2}(G)$ (see, for example, [7, p. 82]). Because of (2) we obtain

$$(3) \qquad < \tilde{z}(\xi), L_{2m}\tilde{w}(\xi) > = < f(x), \varphi(x) >$$

for all $\tilde{w}(\xi) \in C_0^\infty(G)$. The left side of (3) admits a closure up to an arbitrary function $\tilde{w}(\xi) \in W^0_{m,2}(G)$. Hence, replacing $\tilde{w}(\xi)$ by $\xi^\alpha\tilde{w}(\xi)$, we get

$$(4) \qquad < \tilde{z}(\xi), L_{2m}(\xi^\alpha\tilde{w}(\xi)) > = (-i)^{|\alpha|} < D^\alpha f(x), \varphi(x) >$$

for all $\tilde{w}(\xi) \in W^0_{m,2}(G)$.

Now let $\tilde{w}_0(\xi) \in W^0_{m,2}(G)$ be the solution of the equation $L_{2m}\tilde{w}_0(\xi) = \overline{\tilde{z}(\xi)}$. Since $0 \notin G$, we get

$$L_{2m}(\xi^\alpha\tilde{w}_\alpha(\xi)) = \prod_{j=1}^n (\xi_j^0 - 2\epsilon)^{\alpha_j}\overline{\tilde{z}(\xi)},$$

where $\tilde{w}_\alpha(\xi) = \prod_{j=1}^n (\xi_j^0 - 2\epsilon)^{\alpha_j}\xi^{-\alpha}\tilde{w}_0(\xi)$ and $\alpha \geq 0$. Therefore, it follows from (4) that

$$(5) \qquad \prod_{j=1}^n (\xi_j^0 - 2\epsilon)^{\alpha_j} < \tilde{z}(\xi), \overline{\tilde{z}(\xi)} > \leq ||D^\alpha f||_{p_\alpha}||v||_1||w_\alpha||_{q_\alpha} ,$$

where $1/p_\alpha + 1/q_\alpha = 1$.

On the other hand, there exists a constant $C > 0$ such that

$$(6) \qquad ||v||_1||w_\alpha||_{q_\alpha} \leq C, \ \alpha \geq 0.$$

Indeed, let $|\beta| \leq 2n$. Using

$$x^\beta w_\alpha(x) = (-i)^{|\beta|}\prod_{j=1}^n (\xi_j^0 - 2\epsilon)^{\alpha_j}\int_G e^{ix\xi}D^\beta(\xi^{-\alpha}\tilde{w}_0(\xi))d\xi,$$

the Leibniz formula and the definition of G, we get

$$\sup_{\mathbb{R}^n}|x^\beta w_\alpha(x)| \leq C_1\prod_{j=1}^n (\frac{\xi_j^0 - 2\epsilon}{\xi_j^0 - \epsilon})^{\alpha_j}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\prod_{k=1}^n \alpha_k...(\alpha_k + \gamma_k - 1),$$

where

$$C_1 = \max\{\int_G |\xi^{-\gamma} D^{\beta-\gamma}\tilde{w}_0(\xi)|d\xi : \gamma \leq \beta, |\beta| \leq 2n\}.$$

On the other hand, since

$$\prod_{k=1}^{n} \alpha_k \ldots\ (\alpha_k + \gamma_k - 1) < (|\alpha| + 2n)^{2n}$$

(because of $|\gamma| \leq |\beta| \leq 2n$),

$$2^{|\beta|} = \sum_{\gamma \leq \beta} \binom{\beta}{\gamma}$$

and

$$\lim_{|\alpha|\to\infty} (|\alpha| + 2n)^{2n} \prod_{j=1}^{n} \Big(\frac{\xi_j^0 - 2\epsilon}{\xi_j^0 - \epsilon}\Big)^{\alpha_j} = 0\ ,$$

we obtain

$$\sup_{x\in\mathbb{R}^n} |x^\beta \omega_\alpha(x)| \leq C_2$$

for all $|\beta| \leq 2n$ and $\alpha \geq 0$. Therefore, there is an absolute constant C_3 such that

$$\sup_{\mathbb{R}^n} (1 + x_1^2) \cdots (1 + x_n^2)|w_\alpha(x)| \leq C_3,\ \alpha \geq 0.$$

So we have proved (6) with $C = C_3\pi^n||v||_1$. Combining (5) and (6) we obtain

$$1 \leq \varliminf_{|\alpha|\to\infty} (||D^\alpha f||_{p_\alpha} \prod_{j=1}^{n} (\xi_j^0 - 2\epsilon)^{-\alpha_j})^{1/|\alpha|}.$$

Therefore, since $\epsilon > 0$ is arbitrarily chosen and

$$\Big[\prod_{j=1}^{n} \Big(\frac{\xi_j^0 - 2\epsilon}{\xi_j^0}\Big)^{-\alpha_j}\Big]^{1/|\alpha|} \leq \max_{1\leq j\leq n} \frac{\xi_j^0}{\xi_j^0 - 2\epsilon}$$

we obtain (1) (with $\xi = \xi^0$) by letting $\epsilon \to 0$.

Now we prove (1) for "zero points": Let $\xi^0 \in \mathrm{sp}(f), \xi^0 \neq 0$ and $\xi_1^0 ... \xi_n^0 = 0$. For the sake of convenience, we assume that $\xi_j^0 > 0, j = 1, ..., k$ and $\xi_{k+1}^0 = \cdots = \xi_n^0 = 0 (1 \leq k < n)$. Then it is sufficient to show (1) only

for indices α such that $\alpha_{k+1} = \cdots = \alpha_n = 0$. Then the proof is analogous to the above one after the following modification of choosing ϵ: We fix a number $0 < \epsilon < \frac{1}{2} \min\limits_{1 \le j \le k} \xi_j^0$.
The proof is complete. □

If $\mathrm{sp}(f)$ is bounded, we have the following more exact result:

THEOREM 2. *Let I be an unbounded set of integral multi-indices $\alpha = (\alpha_1, ..., \alpha_n), \alpha_j \ge 0,\ j = 1, ..., n,\ 1 \le p_\alpha \le \infty$, let $f(x)$ be a measurable function such that its generalized derivative $D^\alpha f(x)$ belongs to $L_{p_\alpha}(\mathbb{R}^n), \alpha \in I$ and $\mathrm{sp}(f)$ be bounded. Then*

(7)
$$(I) \varliminf_{|\alpha|\to\infty} (||D^\alpha f||_{p_\alpha} / \sup_{\mathrm{sp}(f)} |\xi^\alpha|)^{1/|\alpha|} \ge 1.$$

PROOF. It is sufficient to show that

(8)
$$(P) \varliminf_{|\alpha|\to\infty} (||D^\alpha f||_{p_\alpha} / \sup_{\mathrm{sp}(f)} |\xi^\alpha|)^{1/|\alpha|} \ge 1,$$

where P is the set of all $\alpha \in I$ such that $\sup\limits_{\mathrm{sp}(f)} |\xi^\alpha| > 0$. Assume the contrary, that there exist a subsequence $I_1 \subset P$, a number $\lambda < 1$ and a vector $\beta \ge 0, |\beta| = 1$ such that

(9)
$$(I_1) \lim_{|\alpha|\to\infty} (||D^\alpha f||_{p_\alpha} / \sup_{\mathrm{sp}(f)} |\xi^\alpha|)^{1/|\alpha|} < \lambda\ ,$$

(10)
$$(I_1) \lim_{|\alpha|\to\infty} \frac{\alpha}{|\alpha|} = \beta.$$

Note that

(11)
$$(I_1) \varliminf_{|\alpha|\to\infty} \sup_{\mathrm{sp}(f)} |\xi^\alpha|^{1/|\alpha|} > 0\ .$$

Indeed, assume the contrary, that there exists a subsequence $J \subset I_1$ such that

(12)
$$(J) \lim_{|\alpha|\to\infty} \sup_{\mathrm{sp}(f)} |\xi^\alpha|^{1/|\alpha|} = 0.$$

For any $1 \le k \le n$ and $i_1, \ldots, i_k \in \{1, \ldots, n\}$ we put

$$T_{i_1 \ldots i_k} = \{\alpha \ge 0 : \alpha_{i_1} \ne 0, \ldots, \alpha_{i_k} \ne 0 \ \text{ and } \ \alpha_j = 0 \ \text{ if } \ j \notin \{i_1, \ldots, i_k\}\} \ .$$

Then there exist $1 \le k \le n$ and $i_1, \ldots, i_k \in \{1, \ldots, n\}$ such that $J_{i_1 \ldots i_k} = J \cap T_{i_1 \ldots i_k}$ is unbounded. Therefore, we get

$$(J_{i_1 \ldots i_k}) \varliminf_{|\alpha| \to \infty} \sup_{\mathrm{sp}(f)} |\xi^\alpha|^{1/|\alpha|} \ge (J_{i_1 \ldots i_k}) \varliminf_{|\alpha| \to \infty} |\eta^\alpha|^{1/|\alpha|} > 0,$$

where η is any point of $\mathrm{sp}(f)$ such that $\eta_{i_1} \ne 0, \ldots, \eta_{i_k} \ne 0$. This contradicts (12). So we have proved (11).

Further, let ${}_\alpha\xi \in \mathrm{sp}(f) : |{}_\alpha\xi^\alpha| = \sup_{\mathrm{sp}(f)} |\xi^\alpha|$. Then ${}_\alpha\xi_{i_1} \ne 0, \ldots, {}_\alpha\xi_{i_k} \ne 0$ for any $\alpha \in J_{i_1 \ldots i_k}$ and, by taking a subsequence, without loss of generality we may assume that for some $\xi^* \in \mathrm{sp}(f)$

$$(13) \qquad (J_{i_1 \ldots i_k}) \lim_{|\alpha| \to \infty} {}_\alpha\xi = \xi^* \ .$$

Now we consider two cases of ξ^*:

If $\xi^*_{i_j} \ne 0, j = 1, \ldots, k$. Then, obviously,

$$(J_{i_1 \ldots i_k}) \lim_{|\alpha| \to \infty} |{}_\alpha\xi^\alpha|^{1/|\alpha|} = |\xi^{*\beta}| = (J_{i_1 \ldots i_k}) \lim_{|\alpha| \to \infty} |\xi^{*\alpha}|^{1/|\alpha|}$$

which together with $\xi^* \in \mathrm{sp}(f)$, (1) and (9) implies

$$1 \le (J_{i_1 \ldots i_k}) \lim_{|\alpha| \to \infty} (||D^\alpha f||_{p_\alpha} / |\xi^{*\alpha}|)^{1/|\alpha|} =$$
$$= (J_{i_1 \ldots i_k}) \lim_{|\alpha| \to \infty} (||D^\alpha f||_{p_\alpha} / \sup_{\mathrm{sp}(f)} |\xi^\alpha|)^{1/|\alpha|} < \lambda < 1 \ ,$$

which is impossible.

Otherwise, without loss of generality we may assume that $\xi^*_{i_1} = \cdots = \xi^*_{i_m} = 0$ and $\xi^*_{i_{m+1}} \ne 0, \ldots, \xi^*_{i_k} \ne 0$ for some $1 \le m \le k$.

From (11) and (13), it follows that $\xi^* \neq 0$, Therefore, $m < k$. Further, by virtue of (10) - (11), (13), the definition of ${}_\alpha\xi$ and $\xi^*_{i_1} = \cdots = \xi^*_{i_m} = 0$, we obtain $\beta_{i_1} = \cdots = \beta_{i_m} = 0$. Since, clearly,

$$(J_{i_1\ldots i_k}) \lim_{|\alpha|\to\infty} |{}_\alpha\xi_{i_{m+1}}^{\alpha_{i_{m+1}}} \ldots {}_\alpha\xi_{i_k}^{\alpha_{i_k}}|^{1/|\alpha|} = |{\xi^*_{i_{m+1}}}^{\beta_{i_{m+1}}} \ldots {\xi^*_{i_k}}^{\beta_{i_k}}|$$
$$= (J_{i_1\ldots i_k}) \lim_{|\alpha|\to\infty} |{\xi^*_{i_{m+1}}}^{\alpha_{i_{m+1}}} \ldots {\xi^*_{i_k}}^{\alpha_{i_k}}|^{1/|\alpha|} \quad ,$$

there exist $\nu \in J_{i_1\ldots i_k}$ and $N > 0$ such that

$$|{}_\alpha\xi_{i_\ell}| \leq \lambda^{-1}|{}_\nu\xi_{i_\ell}|, \ell = m+1, \ldots, k \tag{14}$$

for all $|\alpha| \geq N, \alpha \in J_{i_1\ldots i_k}$.

On the other hand, it follows from ${}_\nu\xi_{i_1} \neq 0, \ldots, {}_\nu\xi_{i_k} \neq 0$ and

$$(J_{i_1\ldots i_k}) \lim_{|\alpha|\to\infty} {}_\alpha\xi_{i_j} = \xi^*_{i_j} = 0, j = 1, \ldots, m$$

that there exists $M > 0$ such that

$$|{}_\alpha\xi_{i_j}| \leq |{}_\nu\xi_{i_j}|, j = 1, \ldots, m$$

for all $|\alpha| \geq M, \alpha \in J_{i_1\ldots i_k}$. This together with (14) implies

$$|{}_\alpha\xi_{i_j}| \leq \lambda^{-1}|{}_\nu\xi_{i_j}|, j = 1, \ldots, k$$

for all $|\alpha| \geq \max\{M, N\}, \alpha \in J_{i_1\ldots i_k}$. Therefore,

$$\sup_{\mathrm{sp}(f)} |\xi^\alpha|^{1/|\alpha|} = |{}_\alpha\xi^\alpha|^{1/|\alpha|} \leq \lambda^{-1}|{}_\nu\xi^\alpha|^{1/|\alpha|}$$

which together with (1) and (9) implies

$$1 \leq (J_{i_1\ldots i_k}) \varliminf_{|\alpha|\to\infty} (||D^\alpha f||_{p_\alpha}/|{}_\nu\xi^\alpha|)^{1/|\alpha|} \leq$$
$$\leq (J_{i_1\ldots i_k})\lambda^{-1} \lim_{|\alpha|\to\infty} (||D^\alpha f||_{p_\alpha}/ \sup_{\mathrm{sp}(f)} |\xi^\alpha|)^{1/|\alpha|} < 1.$$

We thus arrive at a contradiction. So we have proved (8) and then Theorem 2. □

Remark 1. Let $\sigma = (\sigma_1, \ldots, \sigma_n), 0 < \sigma_j < \infty, j = 1, \ldots, n, \Delta_\sigma = \{\xi \in \mathbb{R}^n : |\xi_j| \leq \sigma_j, j = 1, \ldots, n\}, 1 \leq p_\alpha \leq \infty, f \in L_1(\mathbb{R}^n)$ and $\mathrm{sp}(f) \subset \Delta_\sigma$. Then it follows from the Nikolskii inequality [9 - 10] and the Bernstein - Nikolskii inequality [10, p. 114] that $D^\alpha f \in L_1(\mathbb{R}^n)$ for all $\alpha \geq 0$ and

$$||D^\alpha f||_{p_\alpha} \leq \sigma^\alpha ||f||_{p_\alpha} \leq 2^n \sigma^\alpha (\sigma_1 \cdots \sigma_n)^{1-1/p_\alpha} \|f\|_1, \ \alpha \geq 0.$$

Therefore, if $\mathrm{sp}(f)$ contains at least one vertex of the parallelepiped Δ_σ (such a function f exists), then because of (1), we get

$$\lim_{|\alpha|\to\infty} (||D^\alpha f||_{p_\alpha} / \sup_{\mathrm{sp}(f)} |\xi^\alpha|)^{1/|\alpha|} = \lim_{|\alpha|\to\infty} (||D^\alpha f||_{p_\alpha} / \sigma^\alpha)^{1/|\alpha|} = 1, \tag{15}$$

which means that inequalities (1) and (7) hold with equality.

Remark 2. Because of (15), it is natural to ask whether we always have

$$(I) \ \lim_{|\alpha|\to\infty} (||D^\alpha f||_{p_\alpha} / \sigma^\alpha)^{1/|\alpha|} = 1$$

if $\mathrm{sp}(f) \subset \Delta_\sigma$ and $\mathrm{sp}(f)$ contains at least one vertex of Δ_σ. Unfortunately, this fact is false. For simplicity we will construct a counterexample for the case $n = 1$: Let $f(x) = \frac{\sin x}{x}$. Then $f(x) \in L_p(\mathbb{R})$ for any $p > 1, f(x) \notin L_1(\mathbb{R})$ and $\sup_{\mathrm{sp}(f)} |\xi^m| = 1$ for all $m \geq 1$ because of

$$\tilde{f}(\xi) = \begin{cases} 1, & |\xi| \leq 1, \\ 0, & |\xi| > 1. \end{cases}$$

We first observe that

$$\lim_{p\to 1+} ||D^m f||_p = \infty \tag{16}$$

for any m = 0,1, ... Actually, case $m = 0$ is easy to show. Let $m \geq 1$. Then

$$D^m f(x) = f(x) + \sum_{k=1}^{m} (-1)^k k! C_m^k x^{-k-1} D^{m-k} \sin x.$$

Therefore, since

$$\int_1^\infty \Big|\frac{D^{m-k}\sin x}{x^{k+1}}\Big|^p dx \le \int_1^\infty \frac{dx}{x^2} = 1,\ \ k = 1, 2\ldots, m,$$

we get

$$\begin{aligned} ||D^m f||_p &> \Big(\int_1^\infty |D^m f(x)|^p dx\Big)^{1/p} \ge \\ &\ge \Big(\int_1^\infty |f(x)|^p dx\Big)^{1/p} - \sum_{k=1}^m k! C_m^k. \end{aligned} \tag{17}$$

On the other hand, we have

$$\lim_{p\to 1+} \int_1^\infty |f(x)|^p dx = \infty$$

which together with (17) imply (16).

In view of (16), there are $p_m > 1, m = 1, 2, \ldots$ such that

$$||D^m f||_{p_m} \ge m^m, m = 1, 2\ldots\ .$$

Therefore,

$$\lim_{m\to\infty} ||D^m f||_{p_m}^{1/m} = \lim_{m\to\infty} (||D^m f||_{p_m} / \sup_{\mathrm{sp}(f)} |\xi^m|)^{1/m} = \infty.$$

Remark 3. Theorems 1 - 2 still hold for functions defined on the torus $\mathbb{T}^n$.

Remark 4. Theorems 1 - 2 still hold for $0 < p_\alpha \le \infty$.

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