Nevanlinna theory for holomorphic mappings and related problems

正則写像のネヴァンリンナ理論と関連する問題

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Contents

Introduction

This thesis deals with the Nevanlinna theory, which was started by R. Nevanlinna [N] in 1925. After his paper was published, the theory was extended and deepened by contributions of many authors. Fisrtly, the theory was developed into several complex variables by a number of mathematician, e.g., A. Bloch, H. Cartan, H. J. Weyles, L. Ahlfors and then it was developed to the theory of the meromorphic mappings from parabolic space into complex projective space by W. Stoll. On the other hand, it was also developed to the function fields case by the works of D. Masson, W. D. Brownawell-D. Masser, J.F. Voloch, J. Noguchi, J. Wang, and some others. In the recent years, J. Noguchi, J. Winkelmann, K. Yamanoi have developed this theory for holomorphic curve into semi-Abelian varieties which implies stronger results on the Second Main Theorem than those already known. The development of Nevanlinna theory has been applied to a lot of related problems such as the unicity problem of meromorphic mappings, the big Picard theorem and the normality for meromorphic mapping with the number paper was published by many authors, L. Smiley, H. Fujimoto, S. Mori, Z. H. Tzu, Y. Aihara and the others.

The purpose of this thesis is two folds. Our first aim is to study the Nevanlinna theory for meromorphic mappings. We establish some Second Main Theorems in three cases: (i) meromorphic mappings into complex projective spaces, (ii) holomorphic curves over function fields and (iii) holomorphic curves from punctured disk into semi-Abelian varieties. Our second aim is to study the related problems of the Nevanlinna theory, such as a unicity problem of meromorphic mappings into complex projective spaces and the norAbelianmality problem for meromorphic mappings.

Here we introduce the necessary notation to state the results. Let M be a compact complex manifold and let (L, H) be a Hermitian line bundle over M with the Chern form $c_1(L)$. Let D be a divisor determined by a global section of the line bundle L. Let $\mathbf{C}^m$ (resp. Δ^*) be the complex space of dimension m (resp. the punctured disk $\Delta^* = \{z \in \mathbf{C} : \ |z| \geqslant 1\}$). Let $f : \mathbf{C}^m \to M$ (resp. $f : \Delta^* \to M$) be a meromorphic mapping (resp. holomorphic curve). Denote by $T_f(r; L)$ the characteristic function of f with respective to L, $m_f(r; D)$ and $N(r, f^*D)$ respectively the proximity function and the counting function of f with respective to D. Then, the First Main Theorem of Nevanlinna theory is stated as follow;

$$T_f(r; c_1(L)) = T_f(r; L) = m_f(r; D) + N(r, f^*D) + C(r),$$

where $C(r) = O(1)$ (resp. $C(r) = O(\log r)$). We note here that, in the Nevanlinna theory, the First Main Theorem for divisors is well established, and the question is how to establish the Second Main Theorem, which is an estimate of the characteristic function $T_f(r; L)$ by one or several counting functions with or without truncating the multiplicities of the intersection of f with the divisors. After establishing the Second Main Theorem, we can apply it to the uniqueness problem of meromorphic mappings by pulling back divisors or to the normality problem of meromorphic mappings, etc. This thesis will also deal with these problems.

We shall sketch the content of each chapter of the present thesis.

• In chapter 1, we give some brief outline of the Nevanlinna theory for meromorphic functions and meromorphic mappings into compact complex manifolds. We introduce the basic notation and necessary notion, recall the classical First and Second Main Theorems for meromorphic functions and meromorphic mappings into complex projective spaces. We also state the First Main Theorem in general case for meromorphic mappings into a compact complex manifold.

• In chapter 2, we study the Second Main Theorem for meromorphic mappings into complex projective space. If $\{H_j\}_{j=1}^{q}$ is a family of hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in N-subgeneral position, then we have the following Second Main Theorem (cf. [Noc82A], [Noc82B], [Noc83], [Nog05], [Ch90]), usually called the Cartan-Nochka theorem for a linearly non-degenerate meromorphic mapping f from $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$,

$$|| \quad (q - 2N + n - 1)T_f(r) \leqslant \sum_{i=1}^{q} N_n(r, \operatorname{div}(f, H_i)) + S(r).$$

Here $T_f(r)$ denotes the characteristic function of f with respect to the hyperplane bundle over $\mathbf{P}^n(\mathbf{C})$. In this estimate, the characteristic function $T_f(r)$ is bounded by the counting functions of the divisors $\operatorname{div}(f, H_j)$ with truncation to level n. The natural arises here: *how to generalize the Cartan-Nochka theorem to the case of moving hyperplanes (i.e, moving targets)?*

Concerning to this question, M. Ru and W. Stoll ([RS91A], [RS91B]) and M. Shirosaki [Sh90] proved that for a meromorphic mapping f from $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ and a family of moving hyperplanes $\{a_i\}_{i=1}^{q}$ of $\mathbf{P}^n(\mathbf{C})$ in general position which have the small growth of the characteristic functions with respect to f,

$$|| \quad (q - n - 1)T_f(r) \leqslant \sum_{i=1}^{q} N(r, \operatorname{div}(f, a_i)) + \epsilon T_f(r), \ \forall \epsilon > 0,$$

provided that f is linearly nondegenerate over the field $\mathcal{R}\{a_i\}$ (see subsection 2.2.1 for the definition of $\mathcal{R}\{a_i\}$). It is noticed that there is no truncation for the counting functions $N(r, \operatorname{div}(f, a_i))$ in the above estimate.

In [RW] M. Ru and J. Wang obtained an estimate of the Second Main Theorem type for moving targets with truncated multiplicities (Theorem 15). With the assumption in Theorem 15, they proved that

$$|| \quad T_f(r) \leqslant n(2n-1) \sum_{i=0}^{q-1} N_n(r, \operatorname{div}(f, a_i)) + O\big(\max_{0 \leqslant i \leqslant q-1} T_{a_i}(r)\big) + O\big(\log^+ T_f(r)\big).$$

Although the type of the estimate is weaker than the Cartan-Nochka theorem, the counting functions are truncated by the dimension of the complex projective space. Our first aim of this chapter is to improve their result (Theorem 16 and Theorem 19) as follow:

$$|| \quad T_f(r) \leqslant \sum_{i=0}^{q-1} N_n(r, \operatorname{div}(f, a_i)) + O\big(\max_{0\leqslant i\leqslant q-1} T_{a_i}(r)\big) + O\big(\log^+ T_f(r)\big).$$

Our result improves the coefficient $n(2n-1)$ in the above estimate of M. Ru and J. Wang to one. We would like to note that in our result (Theorem 16, Theorem 19) we do not need the condition on the nondegeneracy of f.

Our second aim of this chapter is to show some truncation of the counting functions in the Cartan-Nochka theorem for moving hyperplanes. The level of the truncation is unfortunately very large, but can be estimated.

• In chapter 3, we study the Cartan-Nochka theorem over function fields which is analogous to Cartan-Nochka's Second Main Theorem with truncated counting functions. Let k be an algebraically closed field of characteristic 0 (for simplicity we assume that $k = \mathbf{C}$), let R be a smooth projective algebraic variety of dimension N over k with a Hodge metric form ω, and let K denote the rational function field of R. Let $a_0, \cdots, a_m$ be elements of K. We denote by $\operatorname{ht}((a_j); \omega)$ the height function of the family (a_j). In [Nog96] and [Nog97], J. Noguchi obtained an analogue of Cartan-Nochka theorem with truncated counting functions over K, which is an estimate of $\operatorname{ht}((a_j); \omega)$ by a sum of several truncated counting functions for some linearly forms in $m+1$ variables over k in general position. Independently, J. Wang [W96] also obtained the similar result in the case where k has the positive characteristic.

Our purpose in this chapter is to prove an analogue of the Cartan-Nochka theorem over function fields (Theorem 31), where the condition of *being in general position* is replace by a weaker one (cf. definition 3.3).

• In chapter 4, we consider a holomophic curve into a semi-Abelian variety M. Let D be a divisor on M and let $f : \mathbf{C} \to M$ be an algebraically nondegenerate holomorphic curve. In 2002, J. Noguchi, J. Winkelmann and K. Yamanoi [NWY02] proved that if D satisfies the *boundary condition* (cf. (4.11), [NWY02]), then there exists a number $k_0 = k_0(f, D)$ such that

$$|| \; T_f(r, c_1(\overline{D})) \leqslant N_{k_0}(r, f^*D) + S_f(r, c_1(\overline{D})),$$

where $\overline{D}$ is the closure of D in a special compactification $\overline{M}$ of M, $c_1(\overline{D})$ denotes the Chern form of the line bundle $[\overline{D}]$ over $\overline{M}$, and $T_f(r, c_1(\overline{D}))$ is the characteristic function of f (regarded as a holomorphic curve into $\overline{M}$) with respect to that Chern form $c_1(\overline{D})$. Here $S_f(r, c_1(\overline{D}))$ denotes a remainder term such that

$$|| \quad S_f(r, c_1(\overline{D})) = O(\log T_f(r, c_1(\overline{D}))) + O(logr)$$

Recently, in 2008, J. Noguchi, J. Winkelmann and K. Yamanoi have improved their above theorem, decreasing the number k_0 to 1, and for an arbitrary divisor D without condition by constracting a good compactification of M.

Our purpose of this chapter is to establish a Second Main Theorem for holomorphic curves from the punctured disk Δ^* into a semi-Abelian variety M. For an arbitrary divisor D on M and an algebraically nondegenerate holomorphic curve f of Δ^* into M, we show that (Theorem 42) there exists a compactification $\overline{M}$ of M and a natural number k_0 such that

$$|| \ T_f(r, c_1(\overline{D})) \leqslant N_{k_0}(r, f^*D) + S_f(r, c_1(\overline{D})).$$

Since the mapping f is from the punctured disk, applying Theorem 42 we can obtain the results on extension of meromorphic mappings into semi-Abelian variety and the results on normality of family of meromorphic mappings into a semi-Abelian variety.

• In chapter 5, we apply the Nevanlinna theory to some related problems, which are the unicity of meromophic mappings and the normality for a family of meromorphic mappings. In the first part of this chapter we present our results on the unicity problem and on the normality one in the the second part.

Let f and g be two linearly nondegenerate meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ and let $\{H_i\}_{i=1}^q$ be q hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in general position. We assume that $f^*H_i = g^*H_i$, $(1 \leqslant i \leqslant q)$. In 1975 [Fu75], H. Fujimoto prove that if $q = 3n+2$ then $f \equiv g$, and if $q = 3n+1$ then there exists a projective linear transformation L of $\mathbf{P}^n(\mathbf{C})$ such that $g = L \circ f$. After that, this problem has been studied very intensively by many authors; e.g., H. Fujimoto ([Fu98], [Fu99]), W. Stoll ([St82]), L. Smiley ([S]), S. Ji ([J]), M. Ru ([MR01]), Z. Ye ([Ye]), and so on.

We need some more notation in this chapter. Let f be a linearly nondegenerate meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Take q hyperplanes $H_1, \cdots, H_q$ of $\mathbf{P}^n(\mathbf{C})$ in general position satisfying the condition

(a) $\dim\big(f^{-1}(H_i) \cap f^{-1}(H_j)\big) \leqslant m-2$ for all $1 \leqslant i < j \leqslant q$.

For each positive integer M (or $+\infty$), we denote by $\mathcal{G}\big(\{H_j\}_{j=1}^q, f, M\big)$ the set of all linearly nondegenerate meromorphic mappings g of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ such that

(b) $\min\{\mathrm{div}(g, H_j), M\} = \min\{\mathrm{div}(f, H_j), M\}, \quad j \in \{1, \cdots, q\}$ and

(c) $g \equiv f$ on $\cup_{j=1}^q f^{-1}(H_j)$.

With the above notation, H. Fujimoto [Fu98] and L. Smiley [S] showed that if $q \geqslant 3n+2$ then $g_1 = g_2$ for any $g_1, g_2 \in \mathcal{G}\big(\{H_j\}_{j=1}^q, f, 1\big)$. In [Fu98], H. Fujimoto also showed that if $q \geqslant 3n+1$ then $\mathcal{G}\big(\{H_j\}_{j=1}^q, f, 2\big)$ contains at most two mappings.

We are here interested *to obtain unicity theorem with the smaller number of hyperplanes.*

To do so, we introduce another condition for f and g. In condition (c) we assume that the diffirentials df and dg of f and g on the inverse images of hyperplanes are equal to each other. Then we get an answer for this problem with $\max\{\frac{7(n+1)}{4}, \frac{\sqrt{17n^2+16n}+3n+4}{4}\}$ hyperplanes (Theorems 48, 54).

In the second part of this charpter we will use the Second Main Theorem to estabish some normality criteria for families of meromorphic mappings from a domain in $\mathbf{C}^m$ into complex projective space $\mathbf{P}^n(\mathbf{C})$. This problem was studied by H. Fujimoto in 1974, where he introduced the notion of a meromorphically normal family of meromorphic mappings and gave a criterion for a meromorphically normal family by the condition on the inverse images of $2n+1$ fixed hyperplanes (cf. [Fu75]). In 2005, D. D. Thai, P. N. Mai, P. N. T. Trang (cf. [TMT]) obtained some normality criteria for fixed hyperplanes. Also in 2005, Z. H. Tu and J. Li (cf. [TL]) extended the result of H. Fujimoto to the case of $2n+1$ moving hyperplanes. In this part, the first aim of ours is to generalize these results to the case of moving hypersurfaces (Theorem 68, Theorem 71), and obtain a better result on counting multiplicities (Theorem 70). The second aim of ours is to find a normality criterion for only n moving hypersurfaces of some special type (Theorem 71).

The results of section 2.1 in chapter 2 are joint works with D. D. Thai, which was published in [ThQ08]. The results of section 2.2 in Chapter 2 and those in charpter 3 are joint works with D. D. Thai, which has been accepted for publication in Acta Mathematica Vietnamica, 2010 [ThQ10]. The results in chapter 5 which are joint works with T. V. Tan, were published in [QT08A] and [QT08B].

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Chapter 1

Preliminaries

1.1 Nevanlinna theory of meromorphic function

(a) In this section, we give some brief outline about the value distribution theory of meromorphic functions on $\mathbf{C}$, which was initiated by R. Nevanlinna in 1925.

Firstly, we introduce some notation. We write the complex coordinate $z = x+iy$, $(x, y \in \mathbf{R})$ on $\mathbf{C}$. For $a \in \mathbf{C}$ and $r > 0$ we set

$$\Delta(a,r) = \{z \in \mathbf{C} : \ |z-a| < r\}, \qquad \Delta(r) = \Delta(0,r).$$

Let f be a meromorphic function defined on a domain U of $\mathbf{C}$. If $f(z) \not\equiv 0$, there is a unique expression $f(z) = (z-a)^m g(z)$ in a neighborhood of every point $a \in U$, where $m \in \mathbf{Z}$, $g(z)$ is a holomophic function and $g(a) \neq 0$. When $m > 0$, a is called a zero of order m of $f(z)$, and when $m < 0$, a is called a pole of order $|m|$ of $f(z)$. The set of zeros and poles of $f(z)$ is discrete.

Let $\{a_\nu\}_{\nu=0}^{\infty}$ be the union of zeros and poles of $f(z)$. Then in a neighborhood of every point a_ν, there is a holomorphic function $g(z)$ such that

$$f(z) = (z-a_\nu)^{\lambda_\nu} g(z), \qquad g(a_\nu) \neq 0.$$

Then the zero (resp. polar) divisor of $f(z)$ is defined by

$$\mathrm{div}_0(f) = \sum_{\lambda_\nu > 0} \lambda_\nu a_\nu \left(\text{resp. } \mathrm{div}_\infty(f) = \sum_{\lambda_\nu < 0} -\lambda_\nu a_\nu \right)$$

and the divisor of function $f(z)$ is defined by

$$\mathrm{div}(f) = \mathrm{div}_0(f) - \mathrm{div}_\infty(f).$$

A divisor on $\mathbf{C}$ is a formal sum $E = \sum_{\nu=1}^{\infty} \mu_\nu z_\nu$ of discrete points $z_\nu \in \mathbf{C}$ with coefficients $\mu_\nu \in \mathbf{Z}$. Then the *truncated counting function* to level k of E is defined by

$$n_k(t,E) = \sum_{|z_\nu|<t} \min\{k, \mu_\nu\}, \quad t>0, \tag{1.1.1}$$

$$N_k(t,E) = \int_1^r n_k(t,E), \qquad r>1.$$

In particular, we write $n(t,E) = n_\infty(t,E)$ and $N(t,E) = N_\infty(t,E)$, which are simply called the counting functions of E.

Let $f(z)$ be non zero meromorphic function on $\mathbf{C}$. The proximity function of $f(z)$ is defined by

$$m(r,f) = \frac{1}{2\pi}\int_{|z|=r} \log^+ |f(z)| d\theta. \tag{1.1.2}$$

Here we set $A^+ = \max\{0, A\}$ for $A \in [-\infty, \infty]$.

Nevanlinna's *characteristic function* (called also *order function*) of f is defined by

$$T(r,f) = N(r, \mathrm{div}_\infty(f)) + m(r,f), \quad r>0.. \tag{1.1.3}$$

Theorem 1. (Nevanlinna's First Main Theorem) *Let $f(z)$ be a meromorphic function and $a \in \mathbf{C}$. Then*

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1),$$

$$|O(1)| \leqslant \log^+|a| + \log 2 + \left|\frac{1}{2\pi}\int_{|z|=1} \log|f(z)-a| d\theta\right|.$$

Proof. The proof can be found in [NO].

In the next theorem, we shall recall the second main theorem for meromorphic function. We use a convention: $\frac{1}{f-\infty} = f$. We will denote the so-called small term by $S(r,f)$ satisfying

$$|| \ S(r,f) = O(\log T(r,f) + \log r),$$

where the symbol "||" stands for the estimate to hold as $r \to \infty$ possibly outside a Borel subset of finite measure.

Theorem 2. (Nevanlinna's Second Main Theorem). *Let $f(z)$ be a meromorphic function, and let $a_1, \cdots, a_q$ be distinct points ($q>2$) of the Riemann sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Then*

$$(q-2)T(r,f) \leqslant \sum_{i=1}^{q} N_1(r, \mathrm{div}_0(f-a_i)) + S(r,f).$$

Proof. The proof can be found in [Nog2]

(b) (over a punctured disk) Let $\Delta^* = \{z \in \mathbf{C} \ : \ |z| \geqslant 1\}$ be the punctured disk and let f be a meromorphic function on Δ^*. Then we define the *characteristic function* $T(r, f)$, *proximity function* $m(r, f)$, *counting function* $N(r, E)$ and $N_k(r, E)$ in the same way as for meromorphic functions on $\mathbf{C}$. In this case the Fisrt Main Theorem is stated as follow:

$$T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(\log r),$$

and the Second Main Theorem is stated same as in the case meromorphic function on $\mathbf{C}$.

1.2 First Main Theorem for meromorphic mappings into a compact manifold

In this section, we give an extension of Nevanlinna's First Main Theorem to meromorphic mappings in the higher dimensions.

1.2.1 Basic notation

We set $||z|| = \left(|z^1|^2 + \cdots + |z^m|^2\right)^{1/2}$ for $z = (z^1, \ldots, z^m) \in \mathbf{C}^m$ and define

$$B(r) := \{z \in \mathbf{C}^m : ||z|| < r\}, \quad \Gamma(r) := \{z \in \mathbf{C}^m : ||z|| = r\} \ (0 < r < \infty).$$

For the differential we have the natural decomposition $d = \partial + \bar{\partial}$, and set

$$d^c = \frac{i}{4\pi}(\bar{\partial} - \partial).$$

Define

$$\begin{aligned} \alpha &:= dd^c||z||^2 = \frac{i}{2\pi}\sum_{j=1}^{m} dz^j \wedge d\bar{z}^j, \\ \beta &:= dd^c \log||z||^2 \quad (z \neq 0), \\ \eta &:= d^c \log||z||^2 \wedge \beta^{m-1} \quad (z \neq 0). \end{aligned}$$

Let M be a complex manifold of dimension n. Denote by $\mathrm{Mer}(M)$ the set of all meromorphic functions on M. A divisor E on M is given by a formal sum $E = \sum \mu_\nu X_\nu$, where $\{X_\nu\}$ is a locally finite family of analytic hypersurfaces in M and $\mu_\nu \in \mathbf{Z}$. We may assume that X_ν are irreducible and distinct to each other, and every $\mu_\nu \neq 0$. Then the set $\mathrm{Supp}(E) = \cup_\nu X_\nu$ is called the support of E and each X_ν is called an irreducible component of E. We denote by $\mathrm{supp}(E)$ the support of E. If every $\mu_\nu \geqslant 0$, then E is said to be effective. Sometime, we identify the divisor E with a function $E(z)$ from M into $\mathbf{Z}$ as follows: For a point $p \in M$, there exist a neighborhood U of p in M with a local coordinate $(\omega^1, \cdots, \omega^n)$ and two holomorphic functions f and g on U such that $\mathrm{div}(f) - \mathrm{div}(g) = E|_U$. Then we define

$$E(p) := \max\{d : \mathcal{D}^\alpha f(p) = 0 \text{ for all } \alpha \text{ with } |\alpha| < d\} \\ - \max\{d : \mathcal{D}^\alpha g(p) = 0 \text{ for all } \alpha \text{ with } |\alpha| < d\},$$

where $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbf{Z}_+^n$, $|\alpha| = \sum_{i=1}^n \alpha_i$, and $\mathcal{D}^\alpha \varphi = \dfrac{\partial^{|\alpha|}\varphi}{\partial^{\alpha_1}\omega^1 \cdots \partial^{\alpha_n}\omega^n}$ for a holomorphic function φ. We note that the above definition of function $E(z)$ is independent from the choices of neighborhood U, local coordinate on U and holomorphic functions f and g.

We denote by $D^k(M)$ the vector space of all differential k-forms with coefficients of C^∞-functions with compact support, The current $[u]$ defined by a locally integrable function $u \in L_{\mathrm{loc}}(M)$ is given by

$$[u](\phi) = \int_M u\phi, \quad \phi \in D^{2n}(M),$$

and a divisor $E = \sum \mu_\nu X_\nu$ defines a current

$$E(\phi) = < E, \phi > = \sum \mu_\nu \int_{R(X_\nu)} \phi, \quad \phi \in D^{2n-2}(M),$$

where $R(X_\nu)$ is the set of regular points of X_ν. If E is effective, then the current $E(\cdot)$ is positive and d-closed.

The following is called **Poincare-Lelong's formula.**

Theorem 3. *Let $f \in \mathrm{Mer}(M)$ with $f \not\equiv 0$. Then we have*

$$dd^c[\log |f|^2] = \mathrm{div}(f). \tag{1.2.1}$$

Here the both sides of (1.2.1) are taken in the sense of currents. For the proof of this theorem, we refer to Theorem (5.1.13) in [NO]

The following is called **Jensen's formula**, which has various applicatios.

Lemma 4. *Let u be a C^2- or subharmonic function in a neighborhood of $\overline{B(R)} \subset \mathbf{C}^m$ $(0 \leqslant R < +\infty)$. Then we have*

$$\int_{\Gamma(R)} u\eta - \int_{\Gamma(r)} u\eta = 2\int_r^R < dd^c[u] \wedge \alpha^{m-1}, \chi_{B(t)} > \frac{dt}{t},$$

for $0 \leqslant r \leqslant R$, where $\chi_{B(t)}$ is the characteristic function of the subset $B(t)$ and $\int_{\Gamma(0)} u\eta = u(0)$ $(r = 0)$.

For the proof of this lemma, we refer to Lemma (3.3.39) in [NO].

Let T be a current of type $(1, 1)$ of order 0 on $\mathbf{C}^m$, we set

$$n(t, T) = \frac{< T \wedge \alpha^{m-1}, \chi_{B(t)} >}{t^{2m-2}}.$$

Since $n(t, T)$ is left continuous, the integral

$$N(r,T) = \int_1^r n(r,T)\frac{dt}{t}$$

is defined and continuos in r.

For an effective divisor E on $\mathbf{C}^m$, we set

$$N(r,E) = \int_1^r n(r,E)\frac{dt}{t}, \tag{1.2.2}$$

which is called the *counting function* or *order function* of the divisor E. Let $k > 0$ be an interger and let $E = \sum \mu_\nu X_\nu$ with irreducible components X_ν of E. We define the truncated divisor E_k by

$$E_k = \sum \min\{k, \mu_\nu\} X_\nu.$$

Then we define the *truncated counting functions* to level k of E as follows:

$$n_k(r,E) = \frac{< E_k, \chi_{B(t)}\alpha^{m-1} >}{t^{2m-2}},$$
$$N_k(r,E) = \int_1^r n_k(r,E)\frac{dt}{t}.$$

1.2.2 First Main Theorem for meromorphic mapping

Let $f : \mathbf{C}^m \to M$ be a meromorphic mapping from C^m into a complex manifold M and let ω be a real continuous form of type $(1,1)$ on M. Then we have the following definition, which is called the *characteristic function* or *oder function* of f with respect to ω:

$$T_f(r;\omega) = \int_1^r n(r,[f^*\omega])\frac{dt}{t}. \tag{1.2.3}$$

Let (L, H) be a Hermitian line bundle over M with the Chern class $c_1(L)$, we denote $||.||$ the norm on the fibres L_x defined by H, ω the Chern form and $|L|$ the set of all divisors determined by global sections of line bundle L. Let $D \in |L|$. We assume that

$$\text{(i) } \omega \geqslant 0, \tag{1.2.4}$$
$$\text{(ii) } f(\mathbf{C}^m) \not\subset \operatorname{supp}(D).$$

With above assumtions, we set $f_0 = f|_{\mathbf{C}^m \setminus I(f)}$, then have the following lemma.

Lemma 5. (Lemma 5.2.6, [NO]) Let $\sigma \in \Gamma(M, L)$ with $\operatorname{div}(\sigma) = D$. Then we have the followings.
(i) $\log ||\sigma \circ f_0||^2$ is written as the difference of two plurisubharmonic functions on $\mathbf{C}^m$,

then $\log ||\sigma \circ f_0||^2 \in L_{\rm loc}(\mathbf{C}^m)$ and $\log ||\sigma \circ f_0||^2 \in L_{\rm loc}(\Gamma(r)),\ (r > 0)$
(ii) The current $[f^*\omega]$ is closed and positive.
(iii) $dd^c[\log ||\sigma \circ f||^2] = f^*D - [f^*\omega]$ as currents

Moreover, we assume that M is compact. We may choose $\sigma \in \Gamma(M, L)$ with $\mathrm{div}(\sigma) = D$ which satisfies

$$||\sigma(x)|| < 1,\ (x \in M).$$

By Lemma 5 (i), we define

$$m_f(r; D) = \int_{\Gamma(r)} \log \frac{1}{||\sigma \circ f||} \eta \geqslant 0.$$

Take another $\sigma' \in \Gamma(M, L)$ such that $\mathrm{div}(\sigma') = D$ and $||\sigma'(x)|| < 1$ for every $x \in M$. Then there is a constant $a \in \mathbf{C}^*$ such that $\sigma' = a\sigma$. Therefor $m_f(r; D)$ is defined up to a constant term and called the *proximiry function* of f with respect to $D \in |L|$.

From Lemma 5 (iii), we obtain that

$$T_f(r; \omega) - N(r, f^*D) = -\int_1^r \frac{< dd^c \log ||\sigma \circ f||^2 \wedge \alpha^{m-1}, \chi_{B(t)} >}{t^{2m-1}} dt. \tag{1.2.5}$$

By Lemma 5 (i), $\log ||\sigma \circ f_0||^2$ is written as the difference of two plurisubharmonic functions on $\mathbf{C}^n$. Then we can apply Jensen's formula to the righ-hand side of (1.2.5), so that

$$T_f(r; \omega) - N(r, f^*D) = m_f(r; D) - m_f(1; D). \tag{1.2.6}$$

Moreover, we assume that M is Kähler. Let ω' be a real closed $\mathbf{C}^\infty$-differential form of type $(1, 1)$ such that $[\omega'] = c_1(L)$ and $\omega' \geqslant 0$. Since M is Kähler and compact, there is another Hermitian metric H' such that ω' is the Chern form of (L, H'). Then by the definition of Hermittian metrics, there is a positive-valued $\mathbf{C}^\infty$-function $b > 0$ such that $H' = bH$. It follows from (1.2.6) that

$$T_f(r; \omega) - T_f(r; \omega') = \int_{\Gamma(r)} \log(b \circ f)\eta - \int_{\Gamma(1)} \log(b \circ f)\eta = O(1)\ (r \to \infty).$$

Under the assumption (1.2.4), the *characteristic function* $T_f(r; L)$ of f with respect to L is defined by

$$T_f(r; L) := T_f(r; \omega). \tag{1.2.7}$$

where $\omega \geqslant 0$ and $[\omega] = c_1(L)$. By (1.2.5), we have the *First Main Theorem:*

Theorem 6. *Let L be a holomorphic line bundle over a compact Kähler manifold such that the Chern class $c_1(L) \geqslant 0$. Let $f : \mathbf{C}^m \to M$ be a meromorphic mapping and $D \in |L|$ such that $f(\mathbf{C}^n) \not\subset \operatorname{Supp} D$. Then*

$$T_f(r; L) = N(r, f^*D) + m_f(r; D) + O(1), \ \ (r \geqslant 1).$$

Since $m_f(r; f^*D) \geqslant 0$, we have immediately the following which is called the *Nevanlinna inequality*

Corollary 7. (Corollary 5.2.16, [NO]) Assume the same conditions as in the Theorem 6. Then

$$N(r, f^*D) \leqslant T_f(r; L) + O(1).$$

Chapter 2

Nevanlinna theory for meromorphic mappings from $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$

The theory of the Nevanlinna's second main theorem for meromorphic mappings of $\mathbf{C}^m$ into the complex projective space $\mathbf{P}^n(\mathbf{C})$ intersecting a finite set of moving or fixed hyperplanes in $\mathbf{P}^n(\mathbf{C})$ began about 70 years ago and has grown into a huge theory. Unfortunately, so far there has been a few Second Main Theorems with truncated counting function for moving targets, in particular only general position or subgeneral position of moving targets was considered. Our main purpose of this chapter is to show some truncated Second Main Theorems of meromorphic mappings from $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ for moving targets.

2.1 Second Main Theorem for moving target with truncated multiplicities

Recently, M. Ru and J. Wang [RW] showed some truncated Second Main Theorems for holomorphic curves intersecting a finite set of moving hyperplanes (i.e, moving targets) in $\mathbf{P}^n(\mathbf{C})$, where the set of moving targets is assumed to be nondegenerate (see the definition below).

With the same assumption on the nondegeneracy of moving targets, in this section we will improve their result with better estimates (Theorems 16 and 19).

First of all, we need some preparations

2.1.1 Baisc notation in Nevalinna theory for meromophic mappings into complex projective space

We keep the same notation as in the chapter 1. Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \cdots : w_n)$ on $\mathbf{P}^n(\mathbf{C})$, we take a reduced representation $f = (f_0 : \cdots : f_n)$, which means that each f_i is a holomorphic function on $\mathbf{C}^m$ and $f(z) = \big(f_0(z) : \cdots : f_n(z)\big)$ outside the analytic set $\{f_0 = \cdots = f_n = 0\}$ of codimension $\geqslant 2$. Set $\|f\| = \big(|f_0|^2 + \cdots + |f_n|^2\big)^{1/2}$.

We denote by $L \to \mathbf{P}^n(\mathbf{C})$ the hyperplane line bundle over $\mathbf{P}^n(\mathbf{C})$, ω_{FS} the Fubini Study form on $\mathbf{P}^n(\mathbf{C})$, which is a Chern form of line bundle L. If there is no confusion, in the case the meromorphic mapping f of complex projective space, we always write $T_f(r)$ for $T_f(r;\omega_{FS})$ and call it the *characteristic function* of f. Then, by Jensen's formula, we have

$$T_f(r) = T_f(r;\omega_{FS}) = \int_{\Gamma(r)} \log ||f||\eta + O(1),$$

where $(f_0 : \cdots : f_n)$ is a reduce representation of f.

Let H be a hyperplane or hypersurface of $\mathbf{P}^n(\mathbf{C})$, we regard H as a divisor on $\mathbf{P}^n(\mathbf{C})$. For convenience, we write $m_{f,H}(r)$ for the proximity function of f with respective to H, i.e $m_{f,H}(r) = m_f(r;H)$.

Let φ be a nonzero meromorphic function on $\mathbf{C}^m$. The proximity function of φ is defined by

$$m(r,\varphi) := \int_{\Gamma(r)} \log^+ |\varphi|\eta.$$

The following act essential roles in Nevanlinna theory (cf. [NO], [St82], [St85]).

Theorem 8. (First Main Theorem for hyperplanes) *Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a linearly nondegenerate meromorphic mapping and H be a hyperplane in $\mathbf{P}^n(\mathbf{C})$. Then*

$$N(r, \operatorname{div}(f,H)) + m_{f,H}(r) = T_f(r) \ (r > 1).$$

Theorem 9. (Second Main Theorem for hyperplanes) *Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a linearly nondegenerate meromorphic mapping and $H_1, ..., H_q$ be q hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in general position,* (i.e $\cap_{j=0}^n H_{i_j} = \emptyset$ *for every subset* $\{i_0, \cdots, i_n\} \subset \{1, \cdots, q\}$). *Then*

$$|| \ (q-n-1)T_f(r) \leqslant \sum_{i=1}^{q} N_n(r, \operatorname{div}(f, H_i)) + o(T_f(r)).$$

Let a be a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ with reduced representation $a = (a_0 : \cdots : a_n)$. We define

$$m_{f,a}(r) = \int\limits_{\Gamma(r)} \log \frac{||f|| \cdot ||a||}{|(f,a)|}\eta - \int\limits_{\Gamma(1)} \log \frac{||f|| \cdot ||a||}{|(f,a)|}\eta,$$

where $\|a\| = \big(|a_0|^2 + \cdots + |a_n|^2\big)^{1/2}$ and $(f,a) = \sum_{i=0}^n f_i \cdot a_i$.

If $f, a : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ are meromorphic mappings such that $(f,a) \not\equiv 0$, then the First Main Theorem for moving targets in value distribution theory (cf. [RS91A], [RS91B]) states

$$T_f(r) + T_a(r) = m_{f,a}(r) + N(r, \operatorname{div}(f,a)) \ (r > 1).$$

Denote by $\mathcal{M}$ the field of meromorphic functions on $\mathbf{C}^m$. Let $\mathcal{A} = \{a_0, ..., a_{q-1}\}$ be a set of q meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ with reduced representations

$$a_i = (a_{i0} : ... : a_{in}) \ (0 \leqslant i \leqslant q-1).$$

We denote by $\mathcal{R}\{a_i\}$ the smallest subfield of $\mathcal{M}$ which contains $\mathbf{C}$ and all $\frac{a_{ij}}{a_{ik}}$ for all i, j such that $a_{ik} \not\equiv 0$. We say that $\mathcal{A}$ is located in general posittion if for any arbitrary $n+1$ mappings $(a_{i_0}, \cdots, a_{i_n})$ then $\det\{a_{i_jk}\}_{0 \leqslant j,k \leqslant n} \not\equiv 0$.

The Second Main Theorem for moving targets (cf. [RS91A], [RS91B], [Sh90]) is stated as follow:

Theorem 10. *Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping and $\{a_1, ..., a_q\}$ be the family of q- meromorphic mappings located in general position of $\mathbf{P}^n(\mathbf{C})$, which are small respect to f. Then*

$$|| \ (q-n-1)T_f(r) \leqslant \sum_{i=1}^{q} N_n(r, \operatorname{div}(f, a_i)) + \epsilon T_f(r).$$

Let F be a nonzero holomorphic function on a domain Ω in $\mathbf{C}^m$. For a set $\alpha = (\alpha_1, ..., \alpha_m)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + ... + \alpha_m$ and $\mathcal{D}^\alpha F = \frac{\partial^{|\alpha|} F}{\partial^{\alpha_1} z^1 ... \partial^{\alpha_n} z^m}$. We have the following lemma, which is called *Lemma on logarithmic derivative.*

Lemma 11. ([Vitter], [NO], [Nog81]) *Let f be a nonzero meromorphic function on $\mathbf{C}^m$. Then*

$$\Big|\Big| \quad m\Big(r, \frac{\mathcal{D}^\alpha(f)}{f}\Big) = O(\log^+ T_f(r)) \ (\forall \alpha \in \mathbf{Z}^m_+).$$

Let $\mathcal{A} = \{a_0, ..., a_{q-1}\}$ be a set of q meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ with reduced representations $a_i = (a_{i0} : ... : a_{in})$ $(0 \leqslant i \leqslant q-1)$. In [RW], M. Ru and J. Wang introduced a notion *nondegenerated* family of meromorphic mappings as following, which is a genelization of notion located in general posittion.

Definition 12. *The family $\mathcal{A}$ is said to be nondegenerate over $\mathcal{M}$ if* $\dim(\mathcal{A})_{\mathcal{M}} = n+1$ *and for each nonempty proper subset $\mathcal{A}_1$ of $\mathcal{A}$*

$$(\mathcal{A}_1)_{\mathcal{M}} \cap (\mathcal{A} \setminus \mathcal{A}_1)_{\mathcal{M}} \cap \mathcal{A} \neq \emptyset,$$

where $(\mathcal{A})_{\mathcal{M}}$ is the linear span of $\mathcal{A}$ over the field $\mathcal{M}$.

By changing the homogeneous coordinate system of $\mathbf{P}^n(\mathbf{C})$ if necessary, without loss of generality, we may assume that $a_{i0} \not\equiv 0$ $(0 \leqslant i \leqslant q-1)$. Put $\tilde{a}_{ij} = \frac{a_{ij}}{a_{i0}}$ $(0 \leqslant j \leqslant n)$ and $\tilde{a}_i = (\tilde{a}_{i0}, ..., \tilde{a}_{in})$ $(0 \leqslant i \leqslant q-1)$ and $\tilde{\mathcal{A}} = \{\tilde{a}_0, ..., \tilde{a}_{q-1}\}$. Denote by $\mathcal{R}\{a_i\}$ the smallest subfield of $\mathcal{M}$ which contains $\mathbf{C}$ and all $\tilde{a}_{ij}$ for all i, j, and set $\mathcal{R}^*\{a_i\} = \mathcal{R}\{a_i\} \setminus \{0\}$.

We say that the family $\tilde{\mathcal{A}}$ is *nondegenerate over* $\mathcal{R}\{a_i\}$ if $\dim(\tilde{\mathcal{A}})_{\mathcal{R}\{a_i\}} = n+1$ and for each nonempty proper subset $\tilde{\mathcal{A}}_1$ of $\tilde{\mathcal{A}}$

$$(\tilde{\mathcal{A}}_1)_{\mathcal{R}\{a_i\}} \cap (\tilde{\mathcal{A}} \setminus \tilde{\mathcal{A}}_1)_{\mathcal{R}\{a_i\}} \cap \tilde{\mathcal{A}} \neq \emptyset,$$

where $(\tilde{\mathcal{A}})_{\mathcal{R}\{a_i\}}$ is the linear span of $\tilde{\mathcal{A}}$ over the field $\mathcal{R}\{a_i\}$.

Lemma 13. *If the family $\mathcal{A}$ is nondegennerate over $\mathcal{M}$, then $\tilde{\mathcal{A}}$ is nondegenerate over $\mathcal{R}\{a_i\}$.*

Proof. Since $\dim\,(\mathcal{A})_{\mathcal{M}} = n+1$, there exists a family $\{a_{i_0},...,a_{i_n}\} \subset \mathcal{A}$ such that the family $\{a_{i_0},...,a_{i_n}\}$ is linearly independent over $\mathcal{M}$. This implies that the family $\{\tilde{a}_{i_0},...,\tilde{a}_{i_n}\}$ is linearly independent over $\mathcal{R}\{a_i\}$. Hence $\dim\,(\tilde{\mathcal{A}})_{\mathcal{R}\{a_i\}} = n+1$.

Assume that $\tilde{\mathcal{A}}_1$ is a proper subset of $\tilde{\mathcal{A}}$. Since $\mathcal{A}_1$ is a proper subset of $\mathcal{A}$, there are the following two possibilities:

i) There exists $a_t \in (\mathcal{A}_1)_{\mathcal{M}} \bigcap (\mathcal{A} \setminus \mathcal{A}_1)_{\mathcal{M}} \bigcap (\mathcal{A} \setminus \mathcal{A}_1)$:

Then there exist $b_1,...,b_k \in \mathcal{A}_1$, $c_1,...,c_k \in \mathcal{M} \setminus 0$ such that $\{b_1,...,b_k\}$ is linearly independent over $\mathcal{M}$ and $a_t = \sum_{i=1}^k c_i b_i$.

We have the family $\{\tilde{b}_1,...,\tilde{b}_k\}$ is linearly independent over $\mathcal{M}$ and $\tilde{a}_t = \sum_{i=1}^k \left(\dfrac{c_i b_{i0}}{a_{t0}}\right) \cdot \tilde{b}_i$,

i.e, $\tilde{a}_{tj} = \sum_{i=1}^k \left(\dfrac{c_i b_{i0}}{a_{t0}}\right) \cdot \tilde{b}_{ij}\ (j = 0,...,n)$.

Since $\{\tilde{b}_1,...,\tilde{b}_k\}$ is linearly independent over $\mathcal{M}$, without loss of generality we may assume that $\det\left(\tilde{b}_{st}\right)_{1\leqslant s,t\leqslant k} \not\equiv 0$.

By solving the linear equation system $\sum_{i=1}^k \tilde{b}_{ij} \cdot \left(\dfrac{c_i b_{i0}}{a_{t0}}\right) = \tilde{a}_{tj}\ (1 \leqslant j \leqslant k)$, we have $\dfrac{c_i b_{i0}}{a_{t0}} \in \mathcal{R}\ (1 \leqslant j \leqslant k)$, and hence, $\tilde{a}_t \in (\tilde{\mathcal{A}}_1)_{\mathcal{R}}$.

Since $\tilde{a}_t \in \tilde{\mathcal{A}} \setminus \tilde{\mathcal{A}}_1$, $\tilde{a}_t \in (\tilde{\mathcal{A}})_{\mathcal{R}} \bigcap (\tilde{\mathcal{A}} \setminus \tilde{\mathcal{A}}_1)_{\mathcal{R}} \bigcap \tilde{\mathcal{A}}$.

ii) There exists $a_t \in (\mathcal{A}_1)_{\mathcal{M}} \bigcap (\mathcal{A} \setminus \mathcal{A}_1)_{\mathcal{M}} \bigcap \mathcal{A}_1$: The proof is similar to the above case. ■

Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping with a reduced representation $f := (f_0 : ... : f_n)$. We put $(f, \tilde{a}_i) := \sum_{j=0}^n f_j \tilde{a}_{ij}$.

For convenience, in this section we will use notation $\mathcal{R}$ for $\mathcal{R}\{a_i\}$ and $\mathcal{R}^*$ for $\mathcal{R}^*\{a_i\}$. Assume that $\mathcal{L}$ is a subset of $\mathcal{M}$ or $\mathcal{L}$ is a subset of $\mathcal{M}^{n+1}$. The set $\mathcal{L}$ is said to be *minimal* if it is linearly dependent over $\mathcal{R}$ and each proper subset of $\mathcal{L}$ is linearly independent over $\mathcal{R}$.

Repeating the argument in (Prop. 4.5 [Fu85]), we have the following:

Proposition 14. *Let $\Phi_0,...,\Phi_k$ be meromorphic functions on $\mathbf{C}^m$ such that $\{\Phi_0,...,\Phi_k\}$ are linearly independent over $\mathbf{C}$. Then there exist a family $\{\alpha_i = (\alpha_{i1},...,\alpha_{im})\}_{i=0}^k \subset \mathbf{Z}_+^m$ with $|\alpha_i| = \sum_{j=1}^m |\alpha_{ij}| \leqslant k\ (0 \leqslant i \leqslant k)$ such that the following are satisfied:*

(i) $\{\mathcal{D}^{\alpha_i}\Phi_0, ..., \mathcal{D}^{\alpha_i}\Phi_k\}_{i=0}^k$ *is linearly independent over* $\mathcal{M}$, *i.e,* $\det(\mathcal{D}^{\alpha_i}\Phi_j) \not\equiv 0$.

(ii) $\det(\mathcal{D}^{\alpha_i}(h\Phi_j)) = h^{k+1} \cdot \det(\mathcal{D}^{\alpha_i}\Phi_j)$ *for any nonzero meromorphic function* h *on* $\mathbf{C}^m$.

Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping with reduced representation $f := (f_0 : \dots : f_n)$. The mapping f is said to be *nondegenerate of rank* d if $\operatorname{rank}(f_0, ..., f_n) = d + 1$ over $\mathbf{C}$, i.e $f(\mathbf{C}^m)$ is contained in a linear subspace of $\mathbf{P}^n(\mathbf{C})$ of dimension d. It is clear that $d \leqslant n$ for each meromorphic mapping f from $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$.

2.2 Second Main Theorem with truncated multiplicities

In this section, we will show our results on Second Main Theorem for meromorphic mappings into complex projective space $\mathbf{P}^n(\mathbf{C})$ and moving targets, where the characteristic function is estimated by sum of several counting functions with truncation level to the dimension of complex projective space.

Recently, M. Ru and J. Wang [RW] proved a Second Main Theorem for moving targets with truncated multiplicities as follow:

Theorem 15. (Theorem 1.1, [RW]) *Let* $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ *be a meromorphic mapping. Let* $\mathcal{A} = \{a_0, ..., a_{q-1}\}$ *be a set of* q *meromorphic mappings of* $\mathbf{C}^m$ *into* $\mathbf{P}^n(\mathbf{C})$ *such that* $(f, a_i) \not\equiv 0$ $(0 \leqslant i \leqslant q-1)$. *Assume that* $\mathcal{A}$ *is nondegenerate over* $\mathcal{M}$. *Then*

$$|| \quad T_f(r) \leqslant n(2n-1)\sum_{i=0}^{q-1} N_n(r, \operatorname{div}(f, a_i)) + O\big(\max_{0\leqslant i\leqslant q-1} T_{a_i}(r)\big) + O\big(\log^+ T_f(r)\big).$$

We shall improve the coefficient $n(2n-1)$ in the above estimate of M. Ru and J. Wang to one. Namely, we prove the following

Theorem 16. *Let* $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ *be a meromorphic mapping. Let* $\mathcal{A} = \{a_0, ..., a_{q-1}\}$ *be a set of* q *meromorphic mappings of* $\mathbf{C}^m$ *into* $\mathbf{P}^n(\mathbf{C})$ *such that* $(f, a_i) \not\equiv 0$ $(0 \leqslant i \leqslant q-1)$. *Assume that* $\mathcal{A}$ *is nondegenerate over* $\mathcal{M}$. *Then*

$$|| \quad T_f(r) \leqslant \sum_{i=0}^{q-1} N_n(r, \operatorname{div}(f, a_i)) + O\big(\max_{0\leqslant i\leqslant q-1} T_{a_i}(r)\big) + O\big(\log^+ T_f(r)\big).$$

Here we notice that, in Theorem 16, the counting functions are truncated to level n, moreover we do not need the condition on nongeneracy of the meromorphic mapping f.

In order to prove Theorem 16 we need the following lemma.

Lemma 17. *There exist the subsets* $I_1, ..., I_k$ *of* $\{(f, \tilde{a}_i)\}_{i=0}^{q-1}$ *such that the following are satisfied:*

(i) I_1 *is minimal and* I_i *is linearly independent over* $\mathcal{R}$ $(2 \leqslant i \leqslant k)$,

(ii) $\left(\bigcup_{j=1}^{k} I_j\right)_{\mathcal{R}} = \left(\{(f, \tilde{a}_i)\}_{i=0}^{q-1}\right)_{\mathcal{R}}$,

(iii) *For each* $2 \leqslant i \leqslant k$, *there exist meromorphic functions* $c_\alpha \in \mathcal{R}^*$ *such that*

$$\sum_{(f,\tilde{a}_\alpha)\in I_i} c_\alpha(f, \tilde{a}_\alpha) \in \left(\bigcup_{j=1}^{i-1} I_j\right)_{\mathcal{R}}.$$

Proof. Since $\mathcal{A}$ is nondegenerate over $\mathcal{M}$, $\tilde{\mathcal{A}}$ is nondegenerate over $\mathcal{R}$.

Thus $(\{\tilde{a}_0\})_{\mathcal{R}} \bigcap (\tilde{\mathcal{A}} \setminus \{\tilde{a}_0\})_{\mathcal{R}} \neq \emptyset$, and hence $\tilde{a}_0 \in (\tilde{\mathcal{A}} \setminus \{\tilde{a}_0\})_{\mathcal{R}}$. This implies that $(f, \tilde{a}_0) \in (\{(f, \tilde{a}_i)\}_{i=1}^{q-1})_{\mathcal{R}}$, and hence the family $\{(f, \tilde{a}_i)\}_{i=0}^{q-1}$ is linearly dependent over $\mathcal{R}$. We now choose a set I_1 such that I_1 is the minimal subset of $\{(f, \tilde{a}_i)\}_{i=0}^{q-1}$ containing $(f, \tilde{a}_0)$.

If $(I_1)_{\mathcal{R}} = (\{f, \tilde{a}_i)\}_{i=0}^{q-1})_{\mathcal{R}}$, then the proof is finished.

Otherwise, assume $I_1 = \{(f, \tilde{a}_0), ..., (f, \tilde{a}_{t_1})\}$ $(t_1 \leqslant n+1)$. Then $\{\tilde{a}_0, ..., \tilde{a}_{t_1}\}$ is a proper subset of $\tilde{\mathcal{A}}$. Since $\tilde{\mathcal{A}}$ is nondegenerate over $\mathcal{R}$, there exists an element $\tilde{a} \in \tilde{\mathcal{A}}$ such that $\tilde{a} \in (\{\tilde{a}_0, ..., \tilde{a}_{t_1}\})_{\mathcal{R}} \bigcap (\tilde{\mathcal{A}} \setminus \{\tilde{a}_0, ..., \tilde{a}_{t_1}\})_{\mathcal{R}}$ .Then

$$0 \not\equiv (f, \tilde{a}) \in \left(\{(f, \tilde{a}_i)\}_{i=0}^{t_1}\right)_{\mathcal{R}} \bigcap \left(\{(f, \tilde{a}_i)\}_{i=t_1+1}^{q-1}\right)_{\mathcal{R}}.$$

Assume $t_2 - t_1$ is the minimal positive integer which satisfies the following:

There exist elements, without loss of generality we may assume that they are

$$\{(f, \tilde{a}_{t_1+1}), ..., (f, \tilde{a}_{t_2})\},$$

and nonzero meromorphic functions $\{c_i\}_{i=t_1+1}^{t_2} \subset \mathcal{R}$ such that

$$\sum_{i=1}^{t_2-t_1} c_{t_1+i} \cdot (f, \tilde{a}_{t_1+i}) \in \left(\{(f, \tilde{a}_i)\}_{i=0}^{t_1}\right)_{\mathcal{R}} \setminus \{0\}.$$

By the minimality of $(t_2 - t_1)$, $\{(f, \tilde{a}_i)\}_{i=t_1+1}^{t_2}$ is linearly independent on $\mathcal{R}$.

Put $I_2 = \{(f, \tilde{a}_i)\}_{i=t_1+1}^{t_2}$ $(t_2 - t_1 \leqslant n+1)$.

If $(I_1 \cup I_2)_{\mathcal{R}} = (\{(f, \tilde{a}_i)\}_{i=0}^{q-1})_{\mathcal{R}}$, then the proof is finished.

Otherwise, by repeating the above argument, we have the subset I_3.

Continuiting this process, since $\dim (\{(f, \tilde{a}_i)\}_{i=0}^{q-1})_{\mathcal{R}}$ is finite , there exist the subsets $I_1, ..., I_k$ satisfying the assertions of Lemma 17. ■

Proof of Theorem 16. Let $f := (f_0 : ... : f_n)$ be a reduced representation of f. By changing the homogeneous coordinate system of $\mathbf{P}^n(\mathbf{C})$ if necessary, we may assume that $f_0 \not\equiv 0$.

By Lemma 17, we may assume that there exist the subsets $I_i = \{(f, \tilde{a}_{t_{i-1}+1}), ..., (f, \tilde{a}_{t_i})\}$ $(1 \leqslant i \leqslant k)$, where $t_0 = -1$, which satisfy the assertions of Lemma 17.

Put

$$p := \max\{t_1, \max_{1\leqslant i\leqslant k}(t_i - t_{i-1})\} - 1, \tag{2.2.1}$$

then it easy to see that $p \leqslant n$

Since I_1 is minimal, there exists a linear relation among I_1. That is, there exist $c_{1j} \in \mathcal{R}^*$ such that

$$\sum_{j=0}^{t_1} c_{1j} \cdot (f, \tilde{a}_j) = 0.$$

Define $c_{1j} = 0$ for all $j > t_1$. Then $\sum_{j=0}^{t_k} c_{1j} \cdot (f, \tilde{a}_j) = 0$.

Since $\{c_{1j}(f, \tilde{a}_j)\}_{j=1}^{t_1}$ is linearly independent over $\mathcal{R}$, there exists $\{\alpha_{11}, ..., \alpha_{1t_1}\} \subset \mathbf{Z}_+^m$ ($|\alpha_{1j}| \leqslant t_1 - 1 \leqslant p$) such that

$$A_1 \equiv \begin{vmatrix} \mathcal{D}^{\alpha_{11}}(c_{11}(f, \tilde{a}_1)) & \cdots & \mathcal{D}^{\alpha_{11}}(c_{1t_1}(f, \tilde{a}_{t_1})) \\ \mathcal{D}^{\alpha_{12}}(c_{11}(f, \tilde{a}_1)) & \cdots & \mathcal{D}^{\alpha_{12}}(c_{1t_1}(f, \tilde{a}_{t_1})) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{1t_1}}(c_{11}(f, \tilde{a}_1)) & \cdots & \mathcal{D}^{\alpha_{1t_1}}(c_{1t_1}(f, \tilde{a}_{t_1})) \end{vmatrix}$$

$$\equiv f_0^{t_1} \cdot \begin{vmatrix} \mathcal{D}^{\alpha_{11}}\left(\dfrac{c_{11}(f, \tilde{a}_1)}{f_0}\right) & \cdots & \mathcal{D}^{\alpha_{11}}\left(\dfrac{c_{1t_1}(f, \tilde{a}_{t_1})}{f_0}\right) \\ \mathcal{D}^{\alpha_{12}}\left(\dfrac{c_{11}(f, \tilde{a}_1)}{f_0}\right) & \cdots & \mathcal{D}^{\alpha_{12}}\left(\dfrac{c_{1t_1}(f, \tilde{a}_{t_1})}{f_0}\right) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{1t_1}}(\dfrac{c_{11}(f, \tilde{a}_1)}{f_0}) & \cdots & \mathcal{D}^{\alpha_{1t_1}}(\dfrac{c_{1t_1}(f, \tilde{a}_{t_1})}{f_0}) \end{vmatrix} \equiv f_0^{t_1} \cdot \tilde{A}_1 \not\equiv 0.$$

Now consider $i \geqslant 2$. By constructing the set I_i, there exist meromorphic mappings $c_{ij} \not\equiv 0$ $(t_{i-1}+1 \leqslant j \leqslant t_i)$ such that $\sum_{j=t_{i-1}+1}^{t_i} c_{ij} \cdot (f, \tilde{a}_j) \in \left(\bigcup_{j=1}^{i-1} I_j\right)_{\mathcal{R}}$. On the other hand, there exist meromorphic mappings $c_{ij} \in \mathcal{R}$ $(0 \leqslant j \leqslant t_i)$ such that $c_{ij} \not\equiv 0$ $(t_{i-1}+1 \leqslant j \leqslant t_i)$ and $\sum_{j=0}^{t_i} c_{ij} \cdot (f, \tilde{a}_j) = 0$.

Define $c_{ij} = 0$ for all $j > t_i$. Then $\sum_{j=0}^{t_k} c_{ij} \cdot (f, \tilde{a}_j) = 0$.

Since $\{c_{ij}(f, \tilde{a}_j)\}_{j=t_{i-1}+1}^{t_i}$ is $\mathcal{R}-$ linearly independent, there exists $\{\alpha_{ij}\}_{j=t_{i-1}+1}^{t_i} \subset \mathbf{Z}_+^m$ ($|\alpha_{ij}| \leqslant t_i - t_{i-1} - 1 \leqslant p$) such that

$$A_i = \det\left(\mathcal{D}^{\alpha_{ij}}\left(c_{is}(f, \tilde{a}_s)\right)\right)_{j,s=t_{i-1}+1}^{t_i} = f_0^{t_i - t_{i-1}} \cdot \det\left(\mathcal{D}^{\alpha_{ij}}\left(\frac{c_{is}(f, \tilde{a}_s)}{f_0}\right)\right)_{j,s=t_{i-1}+1}^{t_i}$$

$$= f_0^{t_i - t_{i-1}} \cdot \tilde{A}_i \not\equiv 0.$$

Consider an $t_k \times (t_k + 1)$ minor matrixes $\mathcal{T}$ and $\tilde{\mathcal{T}}$ given by

$$\mathcal{T} = \begin{bmatrix} \mathcal{D}^{\alpha_{11}}(c_{10}(f,\tilde{a}_0)) & \cdots & \mathcal{D}^{\alpha_{11}}(c_{1t_k}(f,\tilde{a}_{t_k})) \\ \mathcal{D}^{\alpha_{12}}(c_{10}(f,\tilde{a}_0)) & \cdots & \mathcal{D}^{\alpha_{12}}(c_{1t_k}(f,\tilde{a}_{t_k})) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{1t_1}}(c_{10}(f,\tilde{a}_0)) & \cdots & \mathcal{D}^{\alpha_{1t_1}}(c_{1t_k}(f,\tilde{a}_{t_k})) \\ \mathcal{D}^{\alpha_{2t_1+1}}(c_{20}(f,\tilde{a}_0)) & \cdots & \mathcal{D}^{\alpha_{2t_1+1}}(c_{2t_k}(f,\tilde{a}_{t_k})) \\ \mathcal{D}^{\alpha_{2t_1+2}}(c_{20}(f,\tilde{a}_0)) & \cdots & \mathcal{D}^{\alpha_{2t_1+2}}(c_{2t_k}(f,\tilde{a}_{t_k})) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{2t_2}}(c_{20}(f,\tilde{a}_0)) & \cdots & \mathcal{D}^{\alpha_{2t_2}}(c_{2t_k}(f,\tilde{a}_{t_k})) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{kt_{k-1}+1}}(c_{k0}(f,\tilde{a}_0)) & \cdots & \mathcal{D}^{\alpha_{kt_{k-1}+1}}(c_{kt_k}(f,\tilde{a}_{t_k})) \\ \mathcal{D}^{\alpha_{kt_{k-1}+2}}(c_{k0}(f,\tilde{a}_0)) & \cdots & \mathcal{D}^{\alpha_{kt_{k-1}+2}}(c_{kt_k}(f,\tilde{a}_{t_k})) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{kt_k}}(c_{k0}(f,\tilde{a}_0)) & \cdots & \mathcal{D}^{\alpha_{kt_k}}(c_{kt_k}(f,\tilde{a}_{t_k})) \end{bmatrix}$$

$$\tilde{\mathcal{T}} = \begin{bmatrix} \mathcal{D}^{\alpha_{11}}\left(\frac{c_{10}(f,\tilde{a}_0)}{f_0}\right) & \cdots & \mathcal{D}^{\alpha_{11}}\left(\frac{c_{1t_k}(f,\tilde{a}_{t_k})}{f_0}\right) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{1t_1}}\left(\frac{c_{10}(f,\tilde{a}_0)}{f_0}\right) & \cdots & \mathcal{D}^{\alpha_{1t_1}}\left(\frac{c_{1t_k}(f,\tilde{a}_{t_k})}{f_0}\right) \\ \mathcal{D}^{\alpha_{2t_1+1}}\left(\frac{c_{20}(f,\tilde{a}_0)}{f_0}\right) & \cdots & \mathcal{D}^{\alpha_{2t_1+1}}\left(\frac{c_{2t_k}(f,\tilde{a}_{t_k})}{f_0}\right) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{2t_2}}\left(\frac{c_{20}(f,\tilde{a}_0)}{f_0}\right) & \cdots & \mathcal{D}^{\alpha_{2t_2}}\left(\frac{c_{2t_k}(f,\tilde{a}_{t_k})}{f_0}\right) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{kt_{k-1}+1}}\left(\frac{c_{k0}(f,\tilde{a}_0)}{f_0}\right) & \cdots & \mathcal{D}^{\alpha_{kt_{k-1}+1}}\left(\frac{c_{kt_k}(f,\tilde{a}_{t_k})}{f_0}\right) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{kt_k}}\left(\frac{c_{k0}(f,\tilde{a}_0)}{f_0}\right) & \cdots & \mathcal{D}^{\alpha_{kt_k}}\left(\frac{c_{kt_k}(f,\tilde{a}_{t_k})}{f_0}\right) \end{bmatrix}$$

Denote by $\mathcal{D}_i$ (res. $\tilde{\mathcal{D}}_i$) the determinant of the matrix obtained by deleting the $i-th$ column of the minor matrix $\mathcal{T}$ (res. $\tilde{\mathcal{T}}$). Since the sum of each row of $\mathcal{T}$ (res.$\tilde{\mathcal{T}}$) is zero, we actually have

$$\mathcal{D}_i = (-1)^i \mathcal{D}_0 = (-1)^i \prod_{i=1}^{k} A_i = (-1)^i f_0^{t_k} \prod_{i=1}^{k} \tilde{A}_i = (-1)^i f_0^{t_k} \tilde{\mathcal{D}}_0 = f_0^{t_k} \tilde{\mathcal{D}}_i.$$

Since $\dim(\tilde{\mathcal{A}})_{\mathcal{R}} = n+1$, without loss of generality, we assume that $\{\tilde{a}_{i_1}, ..., \tilde{a}_{i_{n+1}}\}$ is a basic of $(\tilde{\mathcal{A}})_{\mathcal{R}}$ over $\mathcal{R}$. Then $\det(\tilde{a}_{i_t s},\ 1 \leqslant t \leqslant n+1,\ 0 \leqslant s \leqslant n) \not\equiv 0$. By solving the linear equation system $(f, \tilde{a}_{i_t}) = \tilde{a}_{i_t 0} \cdot f_0 + ... + \tilde{a}_{i_t n} \cdot f_n$ $(1 \leqslant t \leqslant n+1)$, it implies that $f_v = \sum_{t=1}^{n+1} A_{vt}(f, \tilde{a}_{i_t})$ $(A_{vt} \in \mathcal{R})$ for each $0 \leqslant v \leqslant n$.

Choose a maximal linearly independent family $\{(f,\tilde{a}_{j_1}),...,(f,\tilde{a}_{j_d})\}$ of $\bigcup_{i=1}^{k} I_i$ over $\mathcal{R}$. Then $\left(\{(f,\tilde{a}_i)\}_{i=0}^{q-1}\right)_{\mathcal{R}} = \left(\bigcup_{i=1}^{k} I_i\right)_{\mathcal{R}} = \left(\{(f,\tilde{a}_{j_1}),...,(f,\tilde{a}_{j_d})\}\right)_{\mathcal{R}}$.

This implies that $f_v = \sum_{t=1}^{d} B_{vt}(f,\tilde{a}_{j_t})$ $(B_{vt}\in\mathcal{R})$ for each $0\leqslant v\leqslant n$. Hence

$$|f_v(z)| \leqslant (\sum_{t=1}^{d}|B_{vt}(z)|)\cdot \max_{0\leqslant i\leqslant t_k}\big(|(f,\tilde{a}_i)(z)|\big)\ (z\in\mathbf{C}^m).$$

Put $\Psi(z)=\sum_{t=1}^{d}\sum_{v=0}^{n}|B_{vt}(z)|$ $(z\in\mathbf{C}^m)$. Then

$$||f(z)||\leqslant \Psi(z)\cdot \max_{0\leqslant i\leqslant t_k}\big(|(f,\tilde{a}_i)(z)|\big)\ (z\in\mathbf{C}^m),\quad \text{and}$$

$$\int\limits_{\Gamma(r)}\log^+\Psi(z)\eta\leqslant\sum_{t=1}^{d}\sum_{v=0}^{n}\int\limits_{\Gamma(r)}\log^+|B_{vt}|\eta\leqslant\sum_{t=1}^{d}\sum_{v=0}^{n}T_{B_{vt}}(r)=O(\max_{0\leqslant i\leqslant q-1}T_{a_i}(r)).$$

Fix $z_0\in\mathbf{C}^m$. Without loss of generality, we may assume that

$$|(f,\tilde{a}_i)(z_0)|=\max_{0\leqslant j\leqslant t_k}(|f,\tilde{a}_j)(z_0)|.$$

Then

$$\begin{aligned}\frac{|\mathcal{D}_0(z_0)|\cdot||f(z_0)||}{\prod_{i=0}^{t_k}|(f,\tilde{a}_i)(z_0)|}&=\frac{|\mathcal{D}_i(z_0)|}{\prod_{\substack{j=0\\ j\neq i}}^{t_k}|(f,\tilde{a}_j)(z_0)|}\cdot\left(\frac{||f(z_0)||}{|(f,\tilde{a}_i)(z_0)}\right)\\ &\leqslant\Psi(z_0)\cdot\frac{|\mathcal{D}_i(z_0)|}{\prod_{j=0,j\neq i}^{t_k}|(f,\tilde{a}_j)(z_0)|}\\ &\leqslant\max\left(1,\Psi(z_0)\cdot\left(\frac{|\mathcal{D}_i(z_0)|}{\prod_{j=0,j\neq i}^{t_k}|(f,\tilde{a}_j)(z_0)|}\right)\right).\end{aligned}$$

This implies that

$$\begin{aligned}\log\frac{|\mathcal{D}_0(z_0)|.||f(z_0)||}{\prod_{j=0}^{t_k}|(f,\tilde{a}_j)(z_0)|}&\leqslant\log^+\left(\Psi(z_0)\cdot\left(\frac{|\mathcal{D}_i(z_0)|}{\prod_{j=0,j\neq i}^{t_k}|(f,\tilde{a}_j)(z_0)|}\right)\right)\\ &\leqslant\log^+\left(\frac{|\mathcal{D}_i(z_0)|}{\prod_{j=0,j\neq i}^{t_k}|(f,\tilde{a}_j)(z_0)|}\right)+\log^+\Psi(z_0).\end{aligned}$$

Thus, for each $z \in \mathbf{C}^m$, we have

$$\log \frac{|\mathcal{D}_0(z)|.||f(z)||}{\prod_{i=0}^{t_k}|(f,\tilde{a}_i)(z)|} \leqslant \sum_{i=0}^{t_k} \log^+\left(\frac{|\mathcal{D}_i(z)|}{\prod_{j=0,j\neq i}^{t_k}|(f,\tilde{a}_j)(z)|}\right) + \log^+\Psi(z)$$
$$= \sum_{i=0}^{t_k} \log^+\left(\frac{|\tilde{\mathcal{D}}_i(z)|}{\prod_{j=0,j\neq i}^{t_k}\left|\frac{(f,\tilde{a}_j)(z)}{f_0(z)}\right|}\right) + \log^+\Psi(z).$$

Hence

$$\log||f(z)|| + \log \frac{|\mathcal{D}_0(z_0)|}{\prod_{i=0}^{t_k}|(f,\tilde{a}_i)(z_0)|} \leqslant \sum_{i=0}^{t_k} \log^+\left(\frac{|\tilde{\mathcal{D}}_i(z)|}{\prod_{j=0,j\neq i}^{t_k}\left|\frac{(f,\tilde{a}_j)(z)}{f_0(z)}\right|}\right) + \log^+\Psi(z). \quad (2.2.2)$$

Note that $$\frac{\tilde{\mathcal{D}}_i}{\prod_{j=0,j\neq i}^{t_k}\frac{(f,\tilde{a}_j)}{f_0}} = \det \begin{bmatrix} \frac{\mathcal{D}^{\alpha_{11}}\left(\frac{c_{10}(f,\tilde{a}_0)}{f_0}\right)}{\frac{(f,\tilde{a}_0)}{f_0}} & \cdots & \frac{\mathcal{D}^{\alpha_{11}}\left(\frac{c_{1t_k}(f,\tilde{a}_{t_k})}{f_0}\right)}{\frac{(f,\tilde{a}_{t_k})}{f_0}} \\ \vdots & \vdots & \vdots \\ \frac{\mathcal{D}^{\alpha_{kt_k}}\left(\frac{c_{k0}(f,\tilde{a}_0)}{f_0}\right)}{\frac{(f,\tilde{a}_0)}{f_0}} & \cdots & \frac{\mathcal{D}^{\alpha_{kt_k}}\left(\frac{c_{kt_k}(f,\tilde{a}_{t_k})}{f_0}\right)}{\frac{(f,\tilde{a}_{t_k})}{f_0}} \end{bmatrix}$$

(The determinant is counted after deleting the $i-th$ column in the above matrix)

Each element of the above matrix has a form

$$\frac{\mathcal{D}^{\alpha}\left(\frac{c(f,\tilde{a}_j)}{f_0}\right)}{\frac{(f,\tilde{a}_j)}{f_0}} = \frac{\mathcal{D}^{\alpha}\left(\frac{c(f,\tilde{a}_j)}{f_0}\right)}{\frac{c(f,\tilde{a}_j)}{f_0}} \cdot c \ (c \in \mathcal{R}).$$

By the Lemma on logarithmic derivative, we have

$$m\left(r, \frac{\mathcal{D}^{\alpha}\left(\frac{c(f,\tilde{a}_j)}{f_0}\right)}{\frac{(f,\tilde{a}_j)}{f_0}}\right) \leqslant m\left(r, \frac{\mathcal{D}^{\alpha}\left(\frac{c(f,\tilde{a}_j)}{f_0}\right)}{\frac{c(f,\tilde{a}_j)}{f_0}}\right) + m(r,c)$$
$$\leqslant O\left(\log^+ T\left(r, \frac{c(f,\tilde{a}_j)}{f_0}\right)\right) + O(\max_{0\leqslant i\leqslant q-1} T_{a_i}(r))$$
$$\leqslant O(\log^+ T_f(r)) + O(\max_{0\leqslant i\leqslant q-1} T_{a_i}(r)).$$

This yields that

$$m\left(r, \frac{\tilde{\mathcal{D}}_i}{\prod_{j=0,j\neq i}^{t_k} \frac{(f,\tilde{a}_j)}{f_0}}\right) \leqslant O(\log^+ T_f(r)) + O(\max_{0\leqslant i\leqslant q-1} T_{a_i}(r)) \ (0 \leqslant i \leqslant t_k).$$

Hence

$$\sum_{i=0}^{t_k} m\left(r, \frac{\tilde{\mathcal{D}}_i}{\prod_{j=0,j\neq i}^{t_k} \frac{(f,\tilde{a}_j)}{f_0}}\right) \leqslant O(\log^+ T_f(r)) + O(\max_{0\leqslant i\leqslant q-1} T_{a_i}(r)).$$

Integrating both of sides of the inequality (2.2.2), we have

$$\begin{aligned}
&\int_{\Gamma(r)} \log||f||\eta + \int_{\Gamma(r)} \log\left(\frac{|\mathcal{D}_0|}{\prod_{i=0}^{t_k} |(f,\tilde{a}_i)|}\right)\eta \\
&\leqslant \sum_{i=0}^{t_k} \int_{\Gamma(r)} \log^+\left(\frac{|\tilde{\mathcal{D}}_i|}{\prod_{j=0,j\neq i}^{t_k} |\frac{(f,\tilde{a}_j)}{f_0}|}\right)\eta + \int_{\Gamma(r)} \log^+ \Psi(z)\eta \\
&\leqslant \sum_{i=0}^{t_k} \int_{\Gamma(r)} \log^+\left(\frac{|\mathcal{D}_i|}{\prod_{j=0,j\neq i}^{t_k} |(f,\tilde{a}_j)|}\right)\eta + O(\max_{0\leqslant i\leqslant q-1} T_{a_i}(r)) \\
&\leqslant \sum_{i=0}^{t_k} m\left(r, \frac{\tilde{\mathcal{D}}_i}{\prod_{j=0,j\neq i}^{t_k} \frac{(f,\tilde{a}_j)}{f_0}}\right) + O(\max_{0\leqslant i\leqslant q-1} T_{a_i}(r)) \\
&= O(\log^+ T_f(r)) + O(\max_{0\leqslant i\leqslant q-1} T_{a_i}(r)).
\end{aligned}$$

Hence

$$|| \quad T_f(r) + \int_{\Gamma(r)} \log\frac{|\mathcal{D}_0|}{\prod_{i=0}^{t_k} |(f,\tilde{a}_i)|}\eta \leqslant O(\log^+ T_f(r)) + O(\max_{0\leqslant i\leqslant q-1} T_{a_i}(r)), \text{ i.e,}$$

$$\begin{aligned}
|| \quad T_f(r) &\leqslant \int_{\Gamma(r)} \log\frac{\prod_{i=0}^{t_k} |(f,\tilde{a}_i)|}{|\mathcal{D}_0|}\eta + O(\log^+ T_f(r)) + O(\max_{0\leqslant i\leqslant q-1} T_{a_i}(r)) \\
&\leqslant N\big(r, \mathrm{div}_0(\frac{\prod_{i=0}^{t_k}(f,\tilde{a}_i)}{\mathcal{D}_0})\big) - N\big(r, \mathrm{div}_0(\frac{\mathcal{D}_0}{\prod_{i=0}^{t_k}(f,\tilde{a}_i)})\big) \\
&\quad + O(\log^+ T_f(r)) + O(\max_{0\leqslant i\leqslant q-1} T_{a_i}(r)) \\
&\leqslant N\big(r, \mathrm{div}_0(\frac{\prod_{i=0}^{t_k}(f,\tilde{a}_i)}{\mathcal{D}_0})\big) + O(\log^+ T_f(r)) + O(\max_{0\leqslant i\leqslant q-1} T_{a_i}(r)). \qquad (2.2.3)
\end{aligned}$$

Note that a zero of $\frac{\prod_{i=0}^{t_k}(f,\tilde{a}_i)}{\mathcal{D}_0}$ is a zero (or pole) of some $(f,\tilde{a}_i)$ or a pole of some c_{si}.

Case 1. Assume that z is a zero of $(f, \tilde{a}_i)$ with multiplicity $\nu_i > 0$ $(0 \leqslant i \leqslant t_k)$. Without loss of generality, we may assume that $\nu_0 \geqslant \nu_1 \geqslant \cdots \geqslant \nu_{t_k}$. Then

$$\begin{aligned}
\mathrm{div}\big(\mathcal{D}^{\alpha_{st_{s-1}+j}}(c_{si}(f, \tilde{a}_i))\big)(z) &\geqslant \min_{\beta \in \mathbf{Z}_+^m \text{ with } \alpha_{st_{s-1}+j}-\beta \in \mathbf{Z}_+^m} \{\mathrm{div}\big(\mathcal{D}^{\beta} c_{si} \mathcal{D}^{\alpha_{st_{s-1}+j}-\beta}(f, \tilde{a}_i)\big)(z)\} \\
&\geqslant \min_{\beta \in \mathbf{Z}_+^m \text{ with } \alpha_{st_{s-1}+j}-\beta \in \mathbf{Z}_+^m} \big\{\max\{0, \mathrm{div}_0(f, \tilde{a}_i)(z) - |\alpha_{st_{s-1}+j} - \beta|\} \\
&\qquad - (\beta+1)\mathrm{div}_{\infty}(c_{si})(z)\big\} \\
&\geqslant \max\{0, \mathrm{div}_0(f\tilde{a}_i)(z) - p\} - (p+1)\mathrm{div}_{\infty}(c_{si})(z) \\
&= \max\{0, \nu_i - p\} - (p+1)\mathrm{div}_{\infty}(c_{si})(z),
\end{aligned}$$

for each $0 \leqslant i \leqslant t_k, 1 \leqslant j \leqslant t_s - t_{s-1}, 1 \leqslant s \leqslant k$.

Put $\mathcal{I}(z) = (p+1)\sum_{s=1}^{k}\sum_{i=0}^{t_k}(t_s - t_{s-1})\mathrm{div}_{\infty}(c_{si})(z)$, where $t_0 = 0$. Then

$$\mathrm{div}(\mathcal{D}_0)(z) \geqslant \sum_{i=0}^{t_k-1} \max\{0, \nu_i - p\} - \mathcal{I}(z).$$

Assume that $\{a_{d_0}, \cdots, a_{d_n}\}$ is a base of $(\mathcal{A})_{\mathcal{M}}$. Since $\left(\{(f, \tilde{a}_i)\}_{i=0}^{t_k}\right)_{\mathcal{R}} = \left(\{(f, \tilde{a})\}_{a \in \mathcal{A}}\right)_{\mathcal{R}}$,

$$(f, \tilde{a}_{d_j}) = \sum_{i=0}^{t_k} \gamma_{ij} \cdot (f, \tilde{a}_i) \ (\gamma_{ij} \in \mathcal{R}) \ \text{ for each } 0 \leqslant j \leqslant n.$$

Put $I = \{\gamma_{ij}, 0 \leqslant i \leqslant t_k, 0 \leqslant j \leqslant n$ such that $\gamma_{ij} \neq 0\}$.

- If $\nu := \max_{\gamma_{ij} \in I} \mathrm{div}_{\infty}(\gamma_{ij})(z) \geqslant \nu_{t_k}$, then $\nu_{t_k}(z) \leqslant \sum_{\gamma_{ij} \in I} \mathrm{div}_{\infty}(\gamma_{ij})(z)$.
- Otherwise, we have z is a zero of $(f, \tilde{a}_{d_j})$ with multiplicity at least $\nu_{t_k} - \nu$ for each $0 \leqslant j \leqslant n$. Then z is a zero of (f, a_{d_j}) with multiplicity at least $\nu_{t_k} - \nu$ for each $0 \leqslant j \leqslant n$. If $z \notin I(f)$, then z is a zero with multiplicity at least $\nu_{t_k} - \nu$ of $\det(a_{d_j s})$. This implies that $\nu_{t_k}(z) \leqslant \sum_{\gamma_{ij} \in I} \mathrm{div}_{\infty}(\gamma_{ij})(z) + \mathrm{div}_0(\det(a_{d_j s}))(z)$ if $z \notin I(f)$. Hence

$$\begin{aligned}
\mathrm{div}_0\Big(\frac{\prod_{i=0}^{t_k}(f, \tilde{a}_i)}{\mathcal{D}_0}\Big)(z) &\leqslant \max\{0, \sum_{i=0}^{t_k} \nu_i - \mathrm{div}_0(\mathcal{D})(z)\} \leqslant p \cdot t_k + \mathcal{I}(z) + \nu_{t_k} \\
&\leqslant p \cdot t_k \cdot \nu_{t_k} + \mathcal{I}(z) + \nu_{t_k}(z) \leqslant (p \cdot t_k + 1) \cdot \nu_{t_k} + \mathcal{I}(z) \\
&\leqslant (pt_k + 1)\Big(\sum_{\gamma_{ij} \in I} \mathrm{div}_{\infty}(\gamma_{ij})(z) + \mathrm{div}_0(\det(a_{d_j s}))(z)\Big) + \mathcal{I}(z) \\
&= M_1(z) + \mathcal{I}(z),
\end{aligned}$$

outside an analytic set of dimension $\leqslant m-2$.

Case 2. Assume that z is a zero of $(f, \tilde{a}_i)$ with multiplicity $\nu_i > 0$ $(0 \leqslant i \leqslant t_q + j)$, z is not a zero of each $(f, \tilde{a}_i)$ $(t_q + j + 1 \leqslant i \leqslant t_k)$, and z is a pole of $(f, \tilde{a}_i)$ with multiplicity $\nu_i^- \geqslant 0$ $(t_q + j + 1 \leqslant i \leqslant t_k)$.

Repeating the above argument, we have

$$\mathrm{div}\big(\mathcal{D}^{\alpha_{st_{s-1}+l}}(c_{si}(f,\tilde{a}_i))\big)(z) \geqslant \max\{0, \nu_i - p\} - (p+1)\mathrm{div}_\infty(c_{si})(z)$$

for each $0 \leqslant i \leqslant t_q + j, 1 \leqslant l \leqslant t_s - t_{s-1}, 1 \leqslant s \leqslant k$, and

$$\mathrm{div}\big(\mathcal{D}^{\alpha_{st_{s-1}+l}}(c_{si}(f,\tilde{a}_i))\big)(z) \geqslant -(p+1)(\nu_i^-(z) + \mathrm{div}_\infty(c_{si})(z))$$

for each $t_q + j + 1 \leqslant i \leqslant t_k - 1, 1 \leqslant l \leqslant t_s - t_{s-1}, 1 \leqslant s \leqslant k$. Hence

$$\mathrm{div}(\mathcal{D}_0)(z) \geqslant \sum_{i=0}^{t_q+j} \max\{0, \nu_i - p\} - \sum_{i=t_q+j+1}^{t_k-1} (p+1)\nu_j^- - \mathcal{I}(z).$$

This yields that

$$\begin{aligned}
\mathrm{div}_0\Big(\frac{\prod_{i=0}^{t_k}(f,\tilde{a}_i)}{\mathcal{D}_0}\Big)(z) &\leqslant \max\{0, \sum_{i=0}^{t_q+j}\nu_i - \sum_{i=t_q+j+1}^{t_k}\nu_i^- - \mathrm{div}\mathcal{D}_0(z)\} \\
&\leqslant \sum_{i=0}^{t_q+j}\min\{p,\nu_i\} + p\sum_{i=t_q+j+1}^{t_k}\mathrm{div}_0(a_{i0})(z) + \mathcal{I}(z) \\
&\leqslant \sum_{i=0}^{t_k}\min\{p,\mathrm{div}_0(f,\tilde{a}_i)(z)\} + p\sum_{i=0}^{t_k}\mathrm{div}_0(a_{i0})(z) + \mathcal{I}(z).
\end{aligned}$$

Case 3. Assume that z is not a zero and is not a pole of any $(f,\tilde{a}_i)$ $(0 \leqslant i \leqslant t_k)$. Then $\mathrm{div}(\mathcal{D}_0)(z) \geqslant -\mathcal{I}(z)$, and hence $\mathrm{div}_0\big(\frac{\prod_{i=0}^{t_k}(f,\tilde{a}_i)}{\mathcal{D}_0}\big)(z) \leqslant \mathcal{I}(z)$. Roughly speaking, we have

$$\mathrm{div}_0\Big(\frac{\prod_{i=0}^{t_k}(f,\tilde{a}_i)}{\mathcal{D}_0}\Big)(z) \leqslant \sum_{i=0}^{t_k}\min\{p,\mathrm{div}_0(f,\tilde{a}_i)(z)\} + p\sum_{i=0}^{t_k}\mathrm{div}_0(a_{i0})(z) + M_1(z) + \mathcal{I}(z),$$

outside an analytic set of dimension $\leqslant m-2$.

Hence

$$\begin{aligned}
N\Big(r, \mathrm{div}_0\Big(\frac{\prod_{i=0}^{t_k}(f,\tilde{a}_i)}{\mathcal{D}_0}\Big)\Big) &\leqslant \sum_{i=0}^{t_k} N_p(r, \mathrm{div}(f,\tilde{a}_i))(r) + O(\max_{0\leqslant i\leqslant q-1} T_{a_i}(r)) \\
&\leqslant \sum_{i=0}^{t_k} N_p(r, \mathrm{div}(f,a_i)(r) + O(\max_{0\leqslant i\leqslant q-1} T_{a_i}(r)) \\
&\leqslant \sum_{i=0}^{q-1} N_p(r, \mathrm{div}(f,a_i)) + O(\max_{0\leqslant i\leqslant q-1} T_{a_i}(r)).
\end{aligned}$$

Since $p \leqslant n$, we obtain that

$$N\Big(r, \mathrm{div}_0\Big(\frac{\prod_{i=0}^{t_k}(f,\tilde{a}_i)}{\mathcal{D}_0}\Big)\Big) \leqslant \sum_{i=0}^{q-1} N_n(r, \mathrm{div}(f,a_i)) + O(\max_{0\leqslant i\leqslant q-1} T_{a_i}(r)).$$

Combining this and (2.2.3), we have

$$|| \, T_f(r) \leqslant \sum_{i=0}^{q-1} N_n(r, \mathrm{div}(f, a_i)) + O(\max_{0 \leqslant i \leqslant q-1} T_{a_i}(r)) + O(\log^+ T_f(r)). \quad \blacksquare$$

Remark. If f is nondegenerate of rank d, then it easy to see that $p \leqslant d$. Then the estimate in the Theorem 16 will make the following form:

$$|| \, T_f(r) \leqslant \sum_{i=0}^{q-1} N_d(r, \mathrm{div}(f, a_i)) + O(\max_{0 \leqslant i \leqslant q-1} T_{a_i}(r)).$$

We say that the family of moving targets $\mathcal{A} = \{a_i\}_{i=0}^{q-1}$ is located in general position if for any subset $\{a_{i_0}, \cdots, a_{i_n}\}$ of $\mathcal{A}$, the determinant $\det\big(a_{i_j k}\big)_{0 \leqslant j,k \leqslant n} \not\equiv 0$. From the Theorem 16, we will get the following corollary in the case family of moving targets is located in general position:

Corollary 18. *Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping. Let $\mathcal{A} = \{a_0, ..., a_{q-1}\}$ ($q \geqslant 2n+1$) be a set of q meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ located in general position such that $(f, a_i) \not\equiv 0$ ($0 \leqslant i \leqslant q-1$). Then*

$$|| \quad \frac{q}{2n+1} \cdot T_f(r) \leqslant \sum_{i=0}^{q-1} N_n(r, \mathrm{div}(f, a_i))(r) + O\big(\max_{0 \leqslant i \leqslant q-1} T_{a_i}(r)\big) + O\big(\log^+ T_f(r)\big).$$

Proof. Consider 2n+1 moving targets $\{a_{i_1}, ..., a_{i_{2n+1}}\}$ ($0 \leqslant i_j \leqslant q-1$). It is easy to see that this family is nondegennerate over $\mathcal{M}$. Hence

$$|| \quad T_f(r) \leqslant \sum_{j=0}^{2n+1} N_n(r, \mathrm{div}(f, a_{i_j})) + O\big(\log^+ T_f(r)\big) + O\big(\max_{0 \leqslant i \leqslant q-1} T_{a_i}(r)\big).$$

Taking summing-up of both sides of this inequality over all combinations $\{j_1, ..., j_{2n+1}\}$ with $0 \leqslant j_1 < ... < j_{2n+1} \leqslant q-1$, we have

$$\Big|\Big| \quad \frac{q}{2n+1} \cdot T_f(r) \leqslant \sum_{j=0}^{q-1} N_n(r, \mathrm{div}(f, a_j))(r) + O\big(\log^+ T_f(r)\big) + O\big(\max_{0 \leqslant i \leqslant q-1} T_{a_i}(r)\big). \qquad \blacksquare.$$

Here we note that, in the assumption of Theorem 16, we only require the family of moving target $\{a_i\}$ is *non degenerated* and we do not need any condition for the meromorphic mapping f. On the other hand, for an arbitrary family of moving targets with a curtain condition for the meromorphic mapping, we have the following theorem.

Theorem 19. . *Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping. Let $\mathcal{A} = \{a_0, ..., a_{q-1}\}$ be a set of q meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ such that $(f, a_i) \not\equiv 0$ ($0 \leqslant i \leqslant$*

$q-1$). *Assume that* $\dim(\mathcal{A})_{\mathcal{R}} = n+1$ *and* $\left(\{(f,a_i)\}_{a_i\in\mathcal{A}_1}\right)_{\mathcal{R}} \bigcap \left(\{(f,a_i)\}_{a_i\in\mathcal{A}\setminus\mathcal{A}_1}\right)_{\mathcal{R}} \neq \{0\}$ *for any nonempty proper subset* $\mathcal{A}_1$ *of* $\mathcal{A}$. *Then*

$$|| \quad T_f(r) \leqslant \sum_{i=0}^{q-1} N_n(r, \operatorname{div}(f,a_i))(r) + O\big(\max_{0\leqslant i\leqslant q-1} T_{a_i}(r)\big) + O\big(\log^+ T_f(r)\big).$$

Proof. By repeating the argument as in the proof of Theorem 16, it suffices to prove Lemma 17.

Indeed, since $0 \not\equiv (f,\tilde{a}_0) \in \big(\{(f,\tilde{a}_i)\}_{i=1}^{q-1}\big)_{\mathcal{R}}$, we firstly choose a set I_1 such that I_1 is the minimal subset of $\{(f,\tilde{a}_i)\}_{i=0}^{q-1}$ containing $(f,\tilde{a}_0)$.

If $(I_1)_{\mathcal{R}} = \big(\{(f,\tilde{a}_i)\}_{i=0}^{q-1}\big)_{\mathcal{R}}$, then the proof is finished.

Otherwise, assume $I_1 = \{(f,\tilde{a}_0),...,(f,\tilde{a}_{t_1})\}$. It is easy to see that $(I_1)_{\mathcal{R}} \not\equiv \big(\{(f,\tilde{a}_i)\}_{i>t_1}\big)_{\mathcal{R}}$. Define $J_1 = \{a \in \mathcal{A} : (f,\tilde{a}) \in (I_1)_{\mathcal{R}}\}$. Then J_1 must be proper subset of $\mathcal{A}$. Hence

$$\left(\{(f,\tilde{a}_i)\}_{i=0}^{t_1}\right)_{\mathcal{R}} \bigcap \left(\{(f,\tilde{a}_i)\}_{i=t_1+1}^{q-1}\right)_{\mathcal{R}} \neq \{0\}.$$

Assume $t_2 - t_1$ is the minimal positive integer which satisfies the following:

There exist elements, without loss of generality we may assumme that they are

$$\{(f,\tilde{a}_{t_1+1}),...,(f,\tilde{a}_{t_2})\}$$

and nonzero meromorphic functions $c_i \in \mathcal{R}$ $(t_1+1 \leqslant i \leqslant t_2)$ such that

$$0 \not\equiv \sum_{i=1}^{t_2-t_1} c_{t_1+i}\cdot(f,\tilde{a}_{t_1+i}) \in \left(\{(f,\tilde{a}_i)\}_{i=0}^{t_1}\right)_{\mathcal{R}}.$$

Since $t_2 - t_1$ is minimal, $\{(f,\tilde{a}_i)\}_{i=t_1+1}^{t_2}$ is linearly independent over $\mathcal{R}$.

Put $I_2 = \{(f,\tilde{a}_i)\}_{i=t_1+1}^{t_2}$. If $(I_1 \bigcup I_2)_{\mathcal{R}} = \big(\{(f,\tilde{a}_i)\}_{i=0}^{q-1}\big)_{\mathcal{R}}$, the proof is completed. Otherwise, by repeating the above argument, we have the subset I_3.

Continuiting this process, since $\sharp\{(f,\tilde{a}_i)\}_{i=0}^{q-1} \leqslant q$, there exist the subsets $I_1,...,I_k$ satisfying the assertions of the Lemma 1. $\blacksquare$.

2.2.1 Cartan - Nochka theorem over complex projective spaces

Let $\{H_j\}_{j=1}^q$ be hyperplanes of $\mathbf{P}^n(\mathbf{C})$. Set the index set $Q = \{1,2,\cdots,q\}$. Let $N \geqslant n$ and $q \geqslant N+1$. We say that the family $\{H_j\}_{j=1}^q$ are in *N-subgeneral position* if for every subset $R \subset Q$ with the cardinality $\sharp R = N+1$

$$\bigcap_{j\in R} H_j = \emptyset.$$

If they are in n-subgeneral position, we simply say that they are in *general position.*

Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a linearly nondegenerate meromorphic mapping and $\{H_j\}_{j=1}^q$ be hyperplanes in N-subgeneral position in $\mathbf{P}^n(\mathbf{C})$. Then the Cartan-Nochka theorem (cf. [Noc82A], [Noc82B], [Noc83], [Nog05]) is stated as follow

$$|| \ (q-2N+n-1)T(r,f) \leqslant \sum_{i=1}^{q} N_n(r, \operatorname{div}(f,H_i)) + o(T(r,f)).$$

The above mentioned theorem plays an essential role in Nevanlinna theory (see [Nog05], Theorem 3.1 for a complete proof). Recently, motivated by the accomplishment of the second main theorem of meromorphic function for moving targets, M. Ru and W. Stoll ([RS91A], [RS91B]) gave a generalization of the Cartan-Nochka theorem to a finite set of moving moving targets in $\mathbf{P}^n(\mathbf{C})$ as follow

Theorem 20. *Let V be a Hermittian vector space of dimension $n+1>1$. Let $f : \mathbf{C} \to \mathbf{P}(V)$ be a trunscendental holomorphic map and $\mathcal{G}$ be a finite set of holomorphic maps $g : \mathbf{C} \to \mathbf{P}(V)$ small respect to f with $k = \sharp\mathcal{G} \geqslant n+1$. Assume that $\mathcal{G}$ is in general position and f is linnearly degenerate over the field $\mathcal{R}_{\mathcal{G}}$. Take $s>0$, then*

$$\sum_{g\in\mathcal{G}} m_{f,g}(r) \leqslant (n+1+\epsilon)(T_f(r)-T_f(s))$$

holds for all r exept for a finite measure subset $E_s \subset \mathbf{R}^+$.

In their result, however the counting functions are not truncated. Our purpose of this section is to show some theorems of Cartan-Nochka type with truncated counting function for moving targets (Theorems 25 and 27).

Firtly, we need some definitions and preparations. Denote by $\mathcal{M}$ be the field of all meromorphic functions on $\mathbf{C}^m$ as in the section 2.1. Let $a_1,\dots,a_q$ $(q \geqslant n+1)$ be q meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})^*$ with reduced representations $a_j = (a_{j0} : \cdots : a_{jn})$ $(1 \leqslant j \leqslant q)$.

We say that that the family $\{a_j\}_{j=1}^q$ is in N-subgeneral position if and only if for every subset $R \subset Q$ with $|R| = N+1$ and for an arbitrary $(N+1,n+1)$-matrix $(a_{jk})_{j\in R, 0\leqslant k\leqslant n}$

$$\operatorname{rank}_{\mathcal{M}} \ (a_{jk})_{j\in R,0\leqslant k\leqslant n} = n+1.$$

We also denote the rank of the index subset R by

$$\operatorname{rank} R = \operatorname{rank}_{\mathcal{M}} \ (a_{jk})_{j\in R,0\leqslant k\leqslant n} \tag{2.2.4}$$

We define the field $\mathcal{R}\{a_j\}$ as the same in the case when each a_j is meromorphic mapping into $\mathbf{P}^n(\mathbf{C})$.

Let f be a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ with reduced representation $f = (f_0 : \cdots : f_n)$. Then $f := (f_0 : \cdots : f_n) : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ is said to be k-nondegenerate over $\mathcal{R}\{a_j\}$ if there exist exactly $k+1$ linearly independent elements in $\{f_0,\dots,f_n\}$

over the field $\mathcal{R}\{a_j\}$. We define the divisor $\mathrm{div}(f, a_j)$ and proximity function of f for a_j as the same in the case when a_j is meromorphic mapping into $\mathbf{P}^n(\mathbf{C})$ with reduced representation $(a_{j0} : \cdots : a_{jn})$. Then we get the First Main Theorem similar to the case when a_j is meromorphic mapping into $\mathbf{P}^n(\mathbf{C})$ in the previuos sections.

We put $Q = \{1, \dots, q\}$ $(q \geqslant 1)$. For a finite set R, $\sharp R$ denotes the cardinality of R. By Nochka (cf. [Noc83], [Ch90], [Nog05]) we have the following.

Lemma 21. *Let $\{a_i\}_{i \in Q}$ be q moving targets in $\mathbf{P}^n(\mathbf{C})^*$ in N-subgeneral position, and assume that $q > 2N - n + 1$. Then there are positive rational constants $\omega_j, j \in Q$ satisfying the following:*

1. $0 < \omega_j \leqslant 1, \ \forall j \in Q,$
2. *Setting $\tilde{\omega} = \max_{j \in Q} \omega_j$, one gets*
$$\sum_{j=1}^{q} \omega_j = \tilde{\omega}(q - 2N + n - 1) + n + 1.$$
3. $\dfrac{n+1}{2N - n + 1} \leqslant \tilde{\omega} \leqslant \dfrac{n}{N}.$
4. *For $R \subset Q$ with $0 < \sharp R \leqslant N + 1$, $\sum_{j \in R} \omega_j \leqslant \mathrm{rank}_{\mathcal{R}\{a_i\}} \{a_i\}_{i \in R}$.*

The above ω_j are called *Nochka weights*, and $\tilde{\omega}$ the *Nochka constant.*

Lemma 22. *Let the notation be as above. Let $E_j \geqslant 0, j \in Q$ be arbitrarily given numbers. Then for every subset $R \subset Q$ with $0 < \sharp R \leqslant N + 1$, there is a subset $R^\circ \subset R$ such that $\sharp R^\circ = \mathrm{rank}\, R^\circ = \mathrm{rank}\, R$ and*
$$\sum_{i \in R} \omega_i E_i \leqslant \sum_{i \in R^\circ} E_i.$$

For a subset $\Phi \subset \mathcal{M}(\mathbf{C}^m)$ we denote by $\mathcal{L}(\Phi)$ the $\mathbf{C}$-vector space spanned by Φ over $\mathbf{C}$. Assume that $q := \sharp\Phi < \infty$, and $1 \in \Phi$. Then for a positive integer p, we set $\Phi(p) = \{\varphi_1\varphi_2 \cdots \varphi_k | \varphi_j \in \Phi; j = 1, \dots, p\}$. Then
$$1 \in \Phi(p), \quad \Phi(p) \subset \Phi(p+1), \quad \sharp\Phi(p) = \binom{p+q-1}{p} = \binom{p+q-1}{q-1}.$$

Let $0 < \epsilon < 1$ be arbitrarily given. Then we denote by $p(\epsilon, q)$ the smallest positive integer p such that $\binom{p+q-1}{q-1} \leqslant (1+\epsilon)^p$. Set
$$\sharp\Phi(p(\epsilon, q) + 1) = P(\epsilon, q) = \binom{p(\epsilon, q) + q - 1}{q - 1} \left(\leqslant (1+\epsilon)^{p(\epsilon,q)}\right).$$

Lemma 23. *Let the notation be as above. Then there exists an integer $p'(\epsilon, q) \leqslant p(\epsilon, q)$ such that*
$$\frac{\dim \mathcal{L}(\Phi(p'(\epsilon, q) + 1))}{\dim \mathcal{L}(\Phi(p'(\epsilon, q)))} \leqslant (1 + \epsilon), \quad \dim \mathcal{L}(\Phi(p'(\epsilon, q) + 1)) \leqslant P(\epsilon, q).$$

Proof. Suppose that $\dim \mathcal{L}(\Phi(p+1))/\dim \mathcal{L}(\Phi(p)) > (1+\epsilon)$ for all $1 \leqslant p \leqslant p(\epsilon, q)$. Then

$$P(\epsilon, q) \geqslant \dim \mathcal{L}(\Phi(p(\epsilon, q)+1)) \geqslant \prod_{i=1}^{p(\epsilon,q)} \frac{\dim \mathcal{L}(\Phi(i+1))}{\dim \mathcal{L}(\Phi(i))} > (1+\epsilon)^{p(\epsilon,q)}.$$

Hence

$$P(\epsilon, q) > (1+\epsilon)^{p(\epsilon,q)}.$$

This is a contradiction. Thus, there exists a positive integer $p'(\epsilon, q) \leqslant p(\epsilon, q)$ such that

$$\frac{\dim \mathcal{L}(\Phi(p'(\epsilon, q)+1))}{\dim \mathcal{L}(\Phi(p'(\epsilon, q)))} \leqslant 1+\epsilon.$$

Moreover, we have

$$\dim \mathcal{L}(\Phi(p'(\epsilon, q)+1)) \leqslant \dim \mathcal{L}(\Phi(p(\epsilon, q)+1)) \leqslant P(\epsilon, q) \leqslant (1+\epsilon)^{p(\epsilon,q)}. \qquad \blacksquare$$

Remark 24. We give an evaluation of $p(\epsilon, q)$ for some small $\epsilon > 0$. We are going to show that $\binom{p+q-1}{q-1} \leqslant (1+\epsilon)^p$ for $0 < \epsilon < \sqrt{e}-1$ and every integer $p \geqslant \frac{q}{\log^2(1+\epsilon)}$, so that

$$P(\epsilon, q) \leqslant \left[(1+\epsilon)^{\frac{q}{\log^2(1+\epsilon)}}\right]+1 \quad (0 < \epsilon < \sqrt{e}-1),$$

where $[\bullet]$ stands for Gauss's symbol. Since $0 < \epsilon < \sqrt{e}-1$, $p/q > 16$. Note that for $x > 16$

$$\sqrt{x} - 1 - \log(1+x) > 0.$$

By condition

$$p\log(1+\epsilon) \geqslant q\sqrt{\frac{p}{q}},$$

and hence

$$\begin{aligned}
p\log(1+\epsilon) &\geqslant q\left(1+\log(1+\frac{p}{q})\right) = q + q\log(p+q) - q\log q \\
&> p\log(1+\frac{q}{p}) + q\log(p+q) - q\log q \\
&= (p+q)\log(p+q) - (p+q) - p\log p + p - q\log q + q \\
&= \int_p^{p+q} (\log x - \log(x-p))dx > \sum_{i=1}^{p+q}(\log(p+i) - \log i) \\
&> \sum_{i=1}^{p+q-1} (\log(p+i) - \log i) = \log\binom{p+q-1}{q-1}.
\end{aligned}$$

Thus, we have

$$\binom{p+q-1}{q-1} \leqslant (1+\epsilon)^p.$$

By the definition of $p(\epsilon, q)$, we have $p(\epsilon, q) \leqslant [\frac{q}{\log^2(1+\epsilon)}]+1$, and hence

$$P(\epsilon, q) = \binom{p(\epsilon, q)+q-1}{q-1} \leqslant \left[(1+\epsilon)^{[\frac{q}{\log^2(1+\epsilon)}]+1}\right].$$

Now, we will prove a Second Main Theorem in the type of Cartan-Nochka with truncated multiplicities for the case moving targets.

Theorem 25. *Let $f : \mathbf{C}^m \longrightarrow \mathbf{P}^n(\mathbf{C})$ be a nonconstant meromorphic mapping. Let $\{a_i\}_{i=1}^q$ be meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})^*$ in N-subgeneral position such that a_i are small with respect to f and f is linearly nondegenerate over $\mathcal{R}(\{a_i\}_{i=1}^q)$. Then for an arbitrary $0 < \epsilon < 1$*

$$|| \quad (q - 2N + n - 1 - \epsilon)T_f(r) \leqslant \sum_{i=1}^{q} N_{(n+1)P(\epsilon,qN)-1}(r, \operatorname{div}(f, a_i)) + o(T_f(r)).$$

Proof. Without loss of generality we may assume that $a_{i0} \not\equiv 0$ $(1 \leqslant i \leqslant q)$. Set $\tilde{a}_{ij} := a_{ij}/a_{i0}$, $||\tilde{a}_i|| := \sum_{j=0}^n |\tilde{a}_{ij}|$, $F_i := \sum_{j=0}^n f_j\tilde{a}_{ij}$. We put $\Phi = \{\tilde{a}_{ij}\}$. Then $\sharp\Phi = qN + 1$, and $\sharp\Phi(p) = \binom{p+qN}{qN}$. Take arbitrarily $0 < \epsilon < 1$. By Lemma 23 there exists a positive integer $p'(\epsilon, qN + 1) \leqslant p(\epsilon, qN + 1)$ such that

$$\frac{\dim \mathcal{L}(\Phi(p'(\epsilon, qN + 1) + 1))}{\dim \mathcal{L}(\Phi(p'(\epsilon, qN + 1)))} \leqslant (1 + \epsilon),$$
$$\dim \mathcal{L}(\Phi(p'(\epsilon, qN + 1) + 1)) \leqslant P(\epsilon, qN + 1).$$

We put

$$s = \dim \mathcal{L}(\Phi(p'(\epsilon, qN + 1))), \qquad t = \dim \mathcal{L}(\Phi(p'(\epsilon, qN + 1) + 1)).$$

Let $\{b_1, \ldots, b_s\}$ be a base of $\mathcal{L}(\Phi(p'(\epsilon, qN+1)))$ and $\{b_1, \ldots, b_t\}$ be a base of $\mathcal{L}(\Phi(p'(\epsilon, qN+1) + 1))$. The followings are satisfied:

1. $\dfrac{t}{s} \leqslant 1 + \epsilon, \quad t \leqslant P(\epsilon, qN + 1)$.
2. $\{b_j f_k (1 \leqslant i \leqslant t, 0 \leqslant k \leqslant n)\}$ is linearly independent over $\mathbf{C}$.

Claim. If $\{a_{i_1}, \ldots, a_{i_l}\}$ is linearly independent over $\mathcal{M}(\mathbf{C}^m)$, then $\{b_j F_{i_k} (1 \leqslant j \leqslant s, 1 \leqslant k \leqslant l)\}$ is linearly independent over $\mathbf{C}$.

Indeed, assume that $\sum_{1\leqslant j\leqslant s, 1\leqslant k\leqslant l} c_{jk} b_j F_{i_k} \equiv 0$, where $c_{jk} \in \mathbf{C}$. Then

$$\sum_{v=0}^{n} (\sum_{k=1}^{l} (\sum_{j=1}^{s} c_{jk} b_j) \tilde{a}_{i_k v}) f_v \equiv 0.$$

Since f is linearly nondegenerate over $\mathcal{R}\{a_i\}$, it implies that

$$\sum_{k=1}^{l} (\sum_{j=1}^{s} c_{jk} b_j) \tilde{a}_{i_k v} \equiv 0 \quad (0 \leqslant v \leqslant n).$$

Hence

$$\sum_{j=1}^{s} c_{jk} b_j \equiv 0 \quad (1 \leqslant k \leqslant l).$$

This yields that

$$c_{jk} = 0 \quad (1 \leqslant \ k \leqslant l, 1 \leqslant j \leqslant s).$$

Claim is proved.

Set $Q := \{1, \dots, q\}$. Let $R \subset Q$ be such that $\sharp R = N + 1$. Choose $R^\circ \subset R$ such that $\{a_i\}_{i \in R^\circ}$ is linearly independent over $\mathcal{M}(\mathbf{C}^m)$ and R° satisfies Lemma 22. Then $\{F_i\}_{i \in R^\circ}$ is linearly independent over $\mathcal{R}\{a_j\}$. Assume that $R := \{r_1, \dots, r_{N+1}\}$ and $R^\circ := \{r_1^\circ, \dots, r_{n+1}^\circ\}$.

Since $b_j F_{r_k^\circ} (1 \leqslant j \leqslant s, 1 \leqslant k \leqslant n+1)$ is linearly independent over $\mathbf{C}$, we can choose $\beta_{pj}^{kl} \in \mathbf{C}$ such that there is $C_{R^\circ} \in GL((n+1)t; \mathbf{C})$ satisfying

$$\begin{aligned}
&\det(b_j F_{r_k^\circ}(1 \leqslant j \leqslant s, 1 \leqslant k \leqslant n+1), h_{pl}(s+1 \leqslant l \leqslant t, 0 \leqslant p \leqslant n)) \\
&= C_{R^\circ} \det(b_j f_k (1 \leqslant j \leqslant t, 0 \leqslant k \leqslant n)),
\end{aligned}$$

where $h_{pj} = \sum_{1 \leqslant k \leqslant t, 0 \leqslant l \leqslant n} \beta_{pj}^{kl} b_k f_l (s+1 \leqslant j \leqslant t, 0 \leqslant p \leqslant n)$, and C_{R° is a constant.

Let $\alpha := (\alpha_1, \dots, \alpha_{(n+1)t}) \in (\mathbf{Z}_+^m)^{(n+1)t}$ be a minimal multi-index in the lexicographical order such that

$$W \equiv \det\big(\mathcal{D}^{\alpha_w} b_j f_k (1 \leqslant j \leqslant t, 0 \leqslant k \leqslant n)\big)_{1 \leqslant w \leqslant (n+1)t} \not\equiv 0.$$

By [Fu85], Proposition 4.5, we have $|\alpha_i| \leqslant (n+1)t - 1, \forall 1 \leqslant i \leqslant (n+1)t$. Set

$$W_{R^\circ} \equiv \det\big(\mathcal{D}^{\alpha_w} b_j F_{r_k^\circ}, \mathcal{D}^{\alpha_w} h_{vl}\big),$$

where $1 \leqslant j \leqslant t, 1 \leqslant k \leqslant n+1, s+1 \leqslant l \leqslant t, 0 \leqslant v \leqslant n$, and $1 \leqslant w \leqslant (n+1)t$. It is easy to see that $W_{R^\circ} = W \cdot \det C_{R^\circ}$.

Let z be a fixed point. Then there exists $R \subset Q$ with $\sharp R = N+1$ such that $|F_i(z)| \leqslant |F_j(z)|$, $\forall i \in R, j \notin R$. On the other hand, we have

$$F_{r_k^\circ} := \sum_{j=0}^{n} \tilde{a}_{r_k^\circ j} f_j.$$

This implies that

$$f_k := \sum_{j=1}^{n+1} A_{kj} F_{r_j^\circ},$$

where $A_{kj} \in \mathcal{R}(\{a_i\})$. We put $A_R := \sum_{j=1}^{n+1} \sum_{k=0}^{n} |A_{kj}|$. Then

$$||f(z)|| \leqslant A_R(z) |F_j(z)|, \quad \forall j \notin R.$$

Set $A := \sum_{R \subset Q} A_R$. Then

$$|| \int_{\Gamma(r)} \log^+ A(z) \eta = o(T_f(r)).$$

We also have

$$
\begin{aligned}
&\frac{||f(z)||^{\tilde{\omega}\cdot s(q-2N+n-1)}|W(z)|}{|F_1(z)|^{\omega_1 s}\cdots|F_q(z)|^{\omega_q s}\cdot||f(z)||^{(n+1)(t-s)}} \\
&= \frac{||f(z)||^{s(\sum_{i=1}^{q}\omega_i-n-1)}|W(z)|}{|F_1(z)|^{\omega_1 s}\cdots|F_q(z)|^{\omega_q s}\cdot||f(z)||^{(n+1)(t-s)}} \\
&\leqslant \frac{A^{s\sum_{i\notin R}\omega_i}(z)||f(z)||^{s\sum_{i\in R}\omega_i}|W(z)|}{\prod_{i\in R}|F_i(z)|^{\omega_i s}||f(z)||^{(n+1)t}} \\
&= \left(\prod_{i\in R}\left(\frac{||f(z)||\cdot||\tilde{a}_i(z)||}{|F_i(z)|}\right)^{\omega_i}\right)^s \frac{A^{s\sum_{i\notin R}\omega_i}(z)|W(z)|}{\prod_{i\in R}||\tilde{a}_i(z)||^{s\omega_i}\cdot||f(z)||^{(n+1)t}} \\
&\leqslant \left(\prod_{i\in R^\circ}\frac{||f(z)||\cdot||\tilde{a}_i(z)||}{|F_i(z)|}\right)^s \frac{A^{s\sum_{i\notin R}\omega_i}(z)|W_{R^\circ}(z)\det C_{R^\circ}|}{\prod_{i\in R}||\tilde{a}_i(z)||^{s\omega_i}\cdot||f(z)||^{(n+1)t}} \\
&= \frac{\prod_{i\in R^\circ}||\tilde{a}_i(z)||\cdot|\det C_{R^\circ}|}{\prod_{i\in R}||\tilde{a}_i(z)||^{s\omega_i}}\cdot\frac{A^{s\sum_{i\notin R}\omega_i}(z)|W_{R^\circ}(z)|}{\prod_{i\in R^\circ}|F_i(z)|^s\cdot||f(z)||^{(n+1)(t-s)}}. \qquad (2.2.5)
\end{aligned}
$$

We put

$$B_R := \frac{\prod_{i\in R^\circ}||\tilde{a}_i||\cdot\det C_{R^\circ}}{\prod_{i\in R}||\tilde{a}_i||^{s\omega_i}}\cdot\frac{A^{s\sum_{i\notin R}\omega_i}(z)|W_{R^\circ}|}{\prod_{i\in R^\circ}|F_i|^s\cdot||f||^{(n+1)(t-s)}}.$$

It easily follows that

$$|| \int_{\Gamma(r)} \log^+ B_R(z)\eta = o(T_f(r)).$$

By (2.2.5) we have

$$\log\left(\frac{||f(z)||^{\tilde{\omega}\cdot s(q-2N+n-1)}|W(z)|}{|F_1(z)|^{\omega_1 s}\cdots|F_q(z)|^{\omega_q s}\cdot||f(z)||^{(n+1)(t-s)}}\right)\leqslant \sum_{R\subset Q}\log^+ B_R$$

for $z \in \mathbf{C}^m$. Integrating both sides of the above inequality over $\Gamma(r)$, we have

$$
\begin{aligned}
|| \ (q-2N+n-1)T_f(r) \leqslant \sum_{i=1}^{q}\frac{\omega_i}{\tilde{\omega}}N(r,\mathrm{div}(f,a_i))(r) + \frac{n+1}{\tilde{\omega}}\left(\frac{t}{s}-1\right)T_f(r) \\
-\frac{1}{\tilde{\omega}s}N(r,\mathrm{div}_0 W) + \frac{1}{\tilde{\omega}s}N(r,\mathrm{div}_\infty W) + o(T_f(r)). \qquad (2.2.6)
\end{aligned}
$$

Claim 26. $|| \ N(r,\mathrm{div}_\infty W) = o(T_f(r))$.

First of all we see that if f and g are nonzero meromorphic functions on $\mathbf{C}^m$, then the followings are satisfied for $\alpha \in \mathbf{Z}_+^m$ and $z \in \mathbf{C}^m$ outside an analytic subset of dimension $\leqslant n-2$:

(i). $\mathrm{div}(fg)(z) = \mathrm{div}(f)(z) + \mathrm{div}(g)(z)$.

(ii). $\mathrm{div}(\mathcal{D}^\alpha(fg))(z) \geqslant \mathrm{div}(\mathcal{D}^\alpha f)(z) - \mathrm{div}_\infty(\mathcal{D}^\alpha g)(z)$.

(iii). $\mathrm{div}_\infty(\mathcal{D}^\alpha f)(z) \leqslant (|\alpha|+1)\mathrm{div}_\infty(f)(z)$.

(iv). $\mathrm{div}_0(f)(z) \leqslant \mathrm{div}_0(\mathcal{D}^\alpha f)(z) + |\alpha|$.

Denote by $I(f)$ (resp. $I(a_i)$) the indeternacy locus of f (resp. a_i). Put $\mathcal{I} = \bigcup_{i=1}^q I(a_i) \cup I(f)$, and

$$\lambda = \sum_{\substack{1\leqslant j\leqslant t \\ 1\leqslant \omega\leqslant (n+1)t}} (n+1)\mathrm{div}_\infty(\mathcal{D}^{\alpha_\omega} b_j) + \sum_{\substack{s+1\leqslant j\leqslant t, 0\leqslant v\leqslant n \\ \sharp R=N+1, 1\leqslant\omega\leqslant(n+1)t}} \mathrm{div}_\infty(\mathcal{D}^{\alpha_\omega} h_{vj}^R).$$

By the above (i)~(iii) we have that $\| \ N(r,\lambda) = o(T_f(r))$. Since $N(r, \mathrm{div}_\infty W) \leqslant N(r,\lambda)$, Claim 26 follows.

We are going to show

$$\sum_{i=1}^q \omega_i N(r, \mathrm{div}(f,a_i)) - \frac{1}{s} N(r, \mathrm{div}_0 W)$$
$$\leqslant \sum_{i=1}^q \omega_i N_{(n+1)t-1}(r, \mathrm{div}(f,a_i)) + o(T_f(r)). \tag{2.2.7}$$

Assume that z is a zero of some (f,a_i). We consider two cases.

Case 1. z is a common zero of at least $N+2$ functions in the family $\{(f,a_i)\}$.

Suppose that $(f,a_i)(z) = 0$ for $1\leqslant i\leqslant p$ with $p > N+1$, and that $(f,a_i)(z)\neq 0$ for $i>p$. Without loss of generality one may assume that

$$\mathrm{div}(f,a_1)(z) \geqslant \mathrm{div}(f,a_2)(z) \geqslant \cdots \geqslant \mathrm{div}(f,a_p)(z).$$

Put $R := \{1,2,\ldots,N+1\}$, Choose $R^\circ := \{r_1^\circ,\ldots,r_{n+1}^\circ\} \subset R$ such that $\{a_i\}_{i\in R^\circ}$ is linearly independent over $\mathcal{M}$ and R° satisfies Lemma 22. Then z either is a zero with multiplicity at least $\mathrm{div}(f, a_{r_{n+1}^\circ})$ of $\det(a_{r_i^\circ,j})_{1\leqslant i\leqslant n+1, 0\leqslant j\leqslant n}$, or z is in $I(f)$.

In fact, for $z\notin I(f)$, we may assume that $f_0(z)\neq 0$. Then, there exists a neighbourhood U of z such that f_0 is non-vanishing on U. We have

$$\det(a_{r_i^\circ,j})_{1\leqslant i\leqslant n+1,0\leqslant j\leqslant n} = \det\begin{vmatrix} a_{r_1^\circ,0} + \frac{f_1}{f_0}a_{r_1^\circ,1} + \cdots + \frac{f_n}{f_0}a_{r_1^\circ,n} & a_{r_1^\circ,1} & \cdots & a_{r_1^\circ,n} \\ \vdots & \vdots & \vdots & \\ a_{r_{n+1}^\circ,0} + \frac{f_1}{f_0}a_{r_{n+1}^\circ,1} + \cdots + \frac{f_n}{f_0}a_{r_1^\circ,n} & a_{r_{n+1}^\circ,1} & \cdots & a_{r_{n+1}^\circ,n} \end{vmatrix}$$
$$= \frac{1}{f_0}\det\begin{vmatrix} (f,a_{r_1^\circ}) & a_{r_1^\circ,1} & \cdots & a_{r_1^\circ,n} \\ \vdots & \vdots & \vdots & \\ (f,a_{r_{n+1}^\circ}) & a_{r_{n+1}^\circ,1} & \cdots & a_{r_{n+1}^\circ,n} \end{vmatrix}$$

on the open subset U. Hence

$$\mathrm{div}_0(\det(a_{r_i^\circ,j})_{1\leqslant i\leqslant n+1,0\leqslant j\leqslant n})(z) \geqslant \mathrm{div}(f, a_{r_{n+1}^\circ})(z).$$

This implies that

$$\sum_{i=N+2}^p \mathrm{div}(f,a_i)(z) \leqslant (p-N-1)\mathrm{div}(f,a_{r_{n+1}^\circ})(z)$$
$$\leqslant (q-N-1)\mathrm{div}_0(\det(a_{r_i^\circ,j})_{1\leqslant i\leqslant n+1,0\leqslant j\leqslant n})(z).$$

Moreover, we have

$$\sum_{i=1}^{N+1}\omega_i(\mathrm{div}(f,a_i)(z)-\min\{\mathrm{div}(f,a_i)(z),(n+1)t-1\}$$
$$\leqslant\sum_{i=1}^{n+1}(\mathrm{div}(f,a_{r_i^\circ})(z)-\min\{\mathrm{div}(f,a_{r_i^\circ})(z),(n+1)t-1\})$$

and

$$\begin{aligned}
&\mathrm{div}_0 W(z)=\mathrm{div}_0 W_{R^\circ}(z)\\
&\geqslant\min_{\sigma\in S_{(n+1)t}}\mathrm{div}\left(\prod_{\substack{1\leqslant j\leqslant s\\1\leqslant k\leqslant n+1}}\mathcal{D}^{\alpha_{\sigma((j-1)(n+1)+k)}}b_jF_{r_k^\circ}\times\prod_{\substack{s+1\leqslant j\leqslant t\\0\leqslant v\leqslant n}}\mathcal{D}^{\alpha_{\sigma((j-1)(n+1)+v+1)}}h_{vj}^R\right)(z)\\
&\geqslant\min_{\sigma\in S_{(n+1)t}}\left(\sum_{\substack{1\leqslant j\leqslant s\\1\leqslant k\leqslant n+1}}\mathrm{div}\mathcal{D}^{\alpha_{\sigma((j-1)(n+1)+k)}}F_{r_k^\circ}(z)-\sum_{\substack{1\leqslant j\leqslant s\\1\leqslant k\leqslant n+1}}\mathrm{div}_\infty\mathcal{D}^{\alpha_{\sigma((j-1)(n+1)+k)}}b_j(z)\right.\\
&\qquad\left.-\sum_{s+1\leqslant j\leqslant t,0\leqslant v\leqslant n}\mathrm{div}_\infty\mathcal{D}^{\alpha_{\sigma((j-1)(n+1)+v+1)}}h_{vj}^R(z)\right)\\
&\geqslant\min_{\sigma\in S_{(n+1)t}}\left(\sum_{\substack{1\leqslant j\leqslant s\\1\leqslant k\leqslant n+1}}\left(\mathrm{div}\mathcal{D}^{\alpha_{\sigma((j-1)(n+1)+k)}}(f,a_{r_k^\circ})(z)-\mathrm{div}_\infty\mathcal{D}^{\alpha_{\sigma((j-1)(n+1)+k)}}(\frac{1}{a_{r_k^\circ 0}})(z)\right)\right.\\
&\qquad\left.-\lambda(z)\right)\\
&\geqslant\min_{\sigma\in S_{(n+1)t}}\left(\sum_{\substack{1\leqslant j\leqslant s\\1\leqslant k\leqslant n+1}}\left(\mathrm{div}(f,a_{r_k^\circ})(z)-\min\{\mathrm{div}(f,a_{r_k^\circ})(z),|\alpha_{\sigma((j-1)(n+1)+k)}|\}\right.\right.\\
&\qquad\left.\left.-(|\alpha_{\sigma((j-1)(n+1)+k)}|+1)\mathrm{div}_\infty(\frac{1}{a_{r_k^\circ 0}})(z)\right)-\lambda(z)\right)\\
&\geqslant s\sum_{1\leqslant k\leqslant n+1}\left(\mathrm{div}(f,a_{r_k^\circ})(z)-\min\{\mathrm{div}(f,a_{r_k^\circ})(z),(n+1)t-1\}\right)\\
&\qquad-\sum_{1\leqslant k\leqslant n+1}(n+1)ts\,\mathrm{div}_\infty(\frac{1}{a_{r_k^\circ 0}})(z)-\lambda(z),
\end{aligned}$$

where $S_{(n+1)t}$ is the symmetric group of degree $(n+1)t$.

This implies that

$$\sum_{i=1}^{N+1}\omega_i\left(\mathrm{div}(f,a_i)(z)-\min\{\mathrm{div}(f,a_i)(z),(n+1)t-1\}\right)$$

$$\leqslant \sum_{i=1}^{n+1}\Big(\mathrm{div}(f,a_{r_i^\circ})(z)-\min\{\mathrm{div}(f,a_{r_i^\circ})(z),(n+1)t-1\}\Big)$$
$$\leqslant \mathrm{div}_0 W(z)+\sum_{1\leqslant k\leqslant n+1} n(n+1)ts\mathrm{div}_\infty a_{r_k^\circ 0}(z)+\lambda(z).$$

Thus, we have either

$$\sum_{i=1}^{q}\omega_i\Big(\mathrm{div}(f,a_i)(z)-\min\{\mathrm{div}(f,a_i)(z),(n+1)t-1\}\Big)$$
$$\leqslant \frac{1}{s}\mathrm{div}_0 W(z)+\sum_{1\leqslant k\leqslant n+1}(n+1)t\mathrm{div}_\infty(\frac{1}{a_{r_k^\circ 0}})(z)$$
$$+\frac{1}{s}\lambda(z)+(q-N-1)\mathrm{div}_0\det(a_{r_i^\circ j})_{1\leqslant i\leqslant n+1,0\leqslant j\leqslant n}(z)$$

or $z\in I(f)$.

Case 2. z is a common zero of at most $N+1$ functions in the family $\{(f,a_i)\}$. Suppose that $(f,a_i)(z)=0$ for $1\leqslant i\leqslant p$ with $p\leqslant N+1$, and that $(f,a_i)(z)\neq 0$ for $i>p$. Consider the set $R=\{1,\dots,N+1\}$. Repeating the argument in Case 1, we have

$$\sum_{i=1}^{q}\omega_i\Big(\mathrm{div}(f,a_i)(z)-\min\{\mathrm{div}(f,a_i)(z),(n+1)t-1\}\Big)$$
$$\leqslant \frac{1}{s}\mathrm{div}_0 W(z)+\sum_{1\leqslant k\leqslant n+1}(n+1)t\mathrm{div}_\infty(\frac{1}{a_{r_k^\circ 0}})(z)+\frac{1}{s}\lambda(z).$$

From the consequence of the above two cases we infer that for $z\in\mathbf{C}^m$ outside an analytic subset of dimension $\leqslant n-2$

$$\sum_{i=1}^{q}\omega_i\Big(\mathrm{div}(f,a_i)(z)-\min\{\mathrm{div}(f,a_i)(z),(n+1)t-1\}\Big)$$
$$\leqslant \frac{1}{s}\mathrm{div}_0 W(z)+\sum_{1\leqslant k\leqslant q}(n+1)t\mathrm{div}_\infty(\frac{1}{a_{r_k^\circ 0}})(z)+\frac{1}{s}\lambda(z)$$
$$+(q-N-1)\sum_{\sharp R=N+1}\mathrm{div}_0\det(a_{r_i^\circ j})_{1\leqslant i\leqslant n+1,0\leqslant j\leqslant n}(z).$$

Integrating on both sides, we have

$$\sum_{i=1}^{q}\omega_i\left(N(r,\mathrm{div}(f,a_i))-N_{(n+1)t-1}(r,\mathrm{div}(f,a_i))\right)$$
$$\leqslant \frac{1}{s}N(r,\mathrm{div}_0 W)+\sum_{1\leqslant k\leqslant q}(n+1)tN(r,\mathrm{div}_\infty(\frac{1}{a_{k0}}))+\frac{1}{s}N(r,\lambda)$$
$$+(q-N-1)\sum_{\sharp R=N+1}N(r,\mathrm{div}_0\det(a_{r_i^\circ,j})_{1\leqslant i\leqslant n+1,0\leqslant j\leqslant n})$$
$$=\frac{1}{s}N(r,\mathrm{div}_0 W)+o(T_f(r)).$$

Hence 2.2.7 was proved. Thus, we have

$$|| \ (q-2N+n-1-\epsilon)T_f(r) \leqslant \sum_{i=1}^{q} N_{(n+1)P(\epsilon,qN)-1}(r, \operatorname{div}(f,a_i)) + o(T_f(r)). \qquad \blacksquare$$

The following is a reformulation of Theorem 25:

Theorem 27. *Let $f : \mathbf{C}^m \longrightarrow \mathbf{P}^n(\mathbf{C})$ be a nonconstant meromorphic mapping, and let $\{a_i\}_{i=1}^q$ be small (with respect to f) meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})^*$ in general position such that f is k-nondegenerate over $\mathcal{R}(\{a_i\}$. Let $0 < \epsilon < 1$ be an arbitrary number. Then the following holds*

$$|| \quad (q-2n+k-1-\epsilon)T_f(r) \leqslant \sum_{i=1}^{q} N_{(k+1)P(\epsilon,qn)-1}(r, \operatorname{div}(f,a_i)) + o(T_f(r)).$$

Chapter 3

Cartan-Nochka theorem over function fields

In this section, we study the Nevanlinna theory over function fields. We will prove an analogous of the Cartan-Nochka theorem (Theorem 31) with truncated multiplicities in the case function fileds, where the *Height function* of a family of rational functions is estimated by sum of some truncation counting functions of divisors.

The basic notation we use in this chapter is dues to the paper [Nog97] of J. Noguchi.

3.1 First Main Theorem over function fields

Let $\mathbf{k}$ be an algebraically closed field of characteristic 0 (for simplicity we assume that $\mathbf{k} = \mathbf{C}$), let R be a smooth projective algebraic variety of dimension N over $\mathbf{k}$, and let K denote the rational function field of R.

We fix a Hodge metric form ω on R. For a divisor D on R, we define the counting function of D with respect to ω by

$$N(D;\omega) = \int_D \omega^{N-1}.$$

Let $a_j \in K, j = 0, \ldots, m$, be not all zero; so say, $a_0 \neq 0$. We define a divisor on R by

$$((a_j))_\infty = -\min\left\{\operatorname{div}\frac{a_j}{a_0}; 0 \leqslant j \leqslant m\right\}.$$

Then the (projective) height $\mathrm{ht}((a_j);\omega)$ of $(a_0, \ldots, a_m)$ with respect to ω is defined by

$$\mathrm{ht}((a_j);\omega) = N((a_j))_\infty;\omega).$$

By [Nog97], Section 2.1, we have

$$\mathrm{ht}((a_j);\omega) = \int_R dd^c \log\big(\sum_{j=0}^{m} |a_j|^2\big) \wedge \omega^{N-1}.$$

There is another interpretation of $\mathrm{ht}((a_j);\omega)$. Let $L \to R$ be a line bundle determined by the divisor $((a_j))_\infty$, and $\sigma_0 \in \Gamma(R, L)$ be a global holomorphic section determining the divisor $\mathrm{div}(\sigma_0) = ((a_j))_\infty$. Then $N(\mathrm{div}(\sigma_0);\omega)$ is considered as a *counting function.*

Setting $\sigma_j = (a_j/a_0)\sigma_0 \in \Gamma(R, L), 0 \leqslant j \leqslant m$, one gets the following reduced representation of a rational mapping f from R into $\mathbf{P}^m(\mathbf{C})$:

$$f = (\sigma_0 : \cdots : \sigma_m) : R \to \mathbf{P}^m(\mathbf{C}).$$

Let Ω denote the Fubini-Study form on $\mathbf{P}^m(\mathbf{C})$. We define the *characteristic* or *orderfunction* of f by

$$T(f;\omega) = \int_R f^*\Omega \wedge \omega^{N-1}.$$

Then we have the following *First Main Theorem* over function fields:

$$T(f;\omega) = \mathrm{ht}((a_j);\omega) = N(\mathrm{div}\sigma_j;\omega).$$

Thus we write $\mathrm{ht}(f;\omega) = T(f;\omega)$.

3.2 Wronskian

We use the same notations as in subsection 3.1. Let $z_1, \ldots, z_N$ be a transcendental base of K. Then there exists a *Zariski* open subset U of R such that the holomorphic vector fields $\partial/\partial_j, 1 \leqslant j \leqslant N$ are defined on U and

$$\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_N} \neq 0$$

at every point of U. Without loss of generality we may assume that the bundle $L \to R$ is trivial on U. Then the restriction to U of every holomorphic section in $\Gamma(R, L)$ can be considered as a holomorphic function on U.

Let $\{a_1, \ldots, a_t\}$ be a subset of $\Gamma(R, L)$ such that the family $\{a_1, \ldots, a_t\}$ is linearly independent over $\mathbf{C}$. We set $g = (a_1 : \cdots : a_t) : R \to \mathbf{P}^{t-1}(\mathbf{C})$. Then g is a linearly non-degenerate rational mapping. Let r be the rank of the differential dg at a general point. Then, by [Fu85] and the construction of the Wronskian in [Nog97], Section 2, we have a generalized Wronskian $W((a_i)) = W(a_1, \ldots, a_t)$ as described below.

For $x \in U$ we consider the vectors

$$(\mathcal{D}^\alpha a_1(x), \ldots, \mathcal{D}^\alpha a_t(x)) \in \mathbf{C}^t,$$

where $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbf{Z}_+^N$ are non-negative multi-indices and $\mathcal{D}^\alpha = \partial^{|\alpha|}/\partial z_1^{\alpha_1} \cdots z_N^{\alpha_N}$ with $|\alpha| = \alpha_1 + \cdots + \alpha_N$. We write $\mathrm{ord}\mathcal{D}^\alpha = |\alpha|$.

We denote $V_l(x)$ by the linear subspace of $\mathbf{C}^t$ spanned by $(\mathcal{D}^\alpha a_1(x), \ldots, \mathcal{D}^\alpha a_t(x))$ with $|\alpha| \leqslant l$, and set

$$\lambda_l = \max_{x \in U} \dim V_l(x).$$

Starting from $(a_1(x), \dots, a_t(x))$, we can take $(\mathcal{D}_i^{\alpha} a_1(x), \dots, \mathcal{D}_i^{\alpha} a_t(x)), 1 \leqslant i \leqslant \lambda_1 - 1$, with $|\alpha_i| = l$ such that for all $x \in U$ outside a thin analytic subset of U, the vectors $(a_1(x), \dots, a_t(x))$ and $(\mathcal{D}_i^{\alpha} a_1(x), \dots, \mathcal{D}_i^{\alpha} a_t(x)), 1 \leqslant i \leqslant \lambda_1 - 1$ are maximal linearly independent subset of $\{(\mathcal{D}^{\alpha} a_1(x), \dots, \mathcal{D}^{\alpha} a_t(x)), |\alpha| \leqslant 1,\}$. Here one notes that $\lambda_1 - 1 \geqslant r$. Similarly to the above, we take $(\mathcal{D}_i^{\alpha} a_1(x), \dots, \mathcal{D}_i^{\alpha} a_t(x)), \lambda_1 \leqslant i \leqslant \lambda_2 - 1$ with $|\alpha_i| = 2$. Thus, we inductively find the family $\{\alpha_i\}, 1 \leqslant i \leqslant t-1$, and obtain the generalized Wronskian

$$W((a_j))(x) = W(a_1, \dots, a_t)(x) = \begin{vmatrix} a_1(x) & \cdots & a_t(x) \\ \mathcal{D}^{\alpha_1} a_1(x) & \cdots & \mathcal{D}^{\alpha_1} a_t(x) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{t-1}} a_1(x) & \cdots & \mathcal{D}^{\alpha_{t-1}} a_t(x) \end{vmatrix} \not\equiv 0$$

such that

$$|\alpha_i| = 1, \qquad 1 \leqslant i \leqslant r,$$
$$|\alpha_i| \leqslant i - r + 1, \ r+1 \leqslant i \leqslant t-1.$$

By the contruction we have $W((\zeta a_j)) = \zeta^t W((a_j))$ for all $\zeta \in K$.

There exists an effective divisor J on R given by the family of local trivializations $\{(R_\nu, \xi_\nu)\}$ such that $R = \cup R_\nu$ is a Zariski open covering and ξ_ν are holomorphic functions on R_ν with $(\xi_\nu) = J|R_\nu, L|R_\nu \equiv R_\nu \times \mathbf{C}$, and $\xi_\nu \partial/\partial z_i, 1 \leqslant i \leqslant N$, are holomophic on R_ν. By [Nog97], §2 we have

$$W((a_j)) \in \Gamma(R, [pJ] \otimes L^t),$$
$$\triangle((a_j)) = \frac{W((a_j))}{a_1 \cdots a_t} \in \Gamma_{\mathrm{rat}}(R, [pJ]),$$

where $\Gamma_{\mathrm{rat}}(\cdot)$ denotes the space of rational sections and $p = \sum_{i=1}^t |\alpha_i| \leqslant \dfrac{m(m+1)}{2}$.

3.3 Nondegenerace family

Being given a family $\mathcal{B} = \{b_0, \dots, b_{q-1}\} \subset (K^*)^{m+1}$, where $K^* = K \setminus \{0\}$ and $b_i = (b_{i0} : \cdots : b_{im})$ $(0 \leqslant i \leqslant q-1)$, we say that the family $\mathcal{B}$ is *nondegenerate over* K if $\dim(\mathcal{B})_K = m+1$ and for every nonempty proper subset $\mathcal{B}_1$ of $\mathcal{B}$

$$(\mathcal{B}_1)_K \cap (\mathcal{B} \setminus \mathcal{B}_1)_K \cap \mathcal{A} \neq \emptyset,$$

where $(\mathcal{B})_K$ is the linear span of a subset $\mathcal{B}$ of K^{m+1} over the field K.

The set $\mathcal{L} \subset K^{m+1}$ is said to be *minimal (over K)* if it is linearly dependent over K and every proper subset of $\mathcal{L}$ is linearly independent over K.

Lemma 28. *Assume that the family $\mathcal{B} = \{b_0, \dots, b_{q-1}\} \subset (K^*)^{m+1}$ is nondegenerate over K. Then there exist subsets $I_1, \dots, I_k$ of $\mathcal{B}$ such that*

1. *I_1 is minimal and I_i is linearly independent over K $(2 \leqslant i \leqslant k)$,*

2. *for each $2 \leqslant i \leqslant k$, there exist a meromorphic function $c_\alpha \in (K^*)$ satisfying*

$$\sum_{\alpha \in I_i} c_\alpha b_\alpha \in \left(\bigcup_{j=1}^{i-1} I_j\right)_K \quad \text{and} \quad \left(\bigcup_{j=1}^{k} I_j\right)_K = (\mathcal{B})_K.$$

Proof. Since $b_0 \in (\mathcal{B} \setminus \{b_0\})_K$, we can choose a set I_1 such that I_1 is the minimal subset of $\mathcal{B}$ containing $\{b_0\}$. Assume that $I_1 = \{b_0, \ldots, b_{t_1}\}$. Then there exist meromorphic functions c_i, $1 \leqslant i \leqslant t_1$, and $c_0 = 1$ such that $\sum_{i=0}^{t_1} c_i b_i = 0$.

If $I_1 = \mathcal{B}$, then the proof is finished.

Otherwise, one of the following two cases holds:

i) There exists $b \in \mathcal{B} \setminus \{I_1\}$. We may assume that $b = b_{t_1+1}$ and $b \in (I_1)_K$. Put $I_2 = \{b_{t_1+1}\}$ and $c_{t_1+1} = 1$. Then there exist $c_{2j}, b_j \in I_1$ and $c_{2t_1+1} = c_{t_1+1}$ such that $\sum_{j=0}^{t_1+1} c_{2j} b_j = 0$. Moreover, we also may assume that $\{b_j \mid b_j \in I_1,\ c_{2j} \neq 0\}$ is independent over K.

ii) There exists $b \in \{I_1\}$. We may assume that $b = b_{t_1}$ and $b \in (\mathcal{B} \setminus I_1)_K$. Then there exists a subset of $\mathcal{B} \setminus I_1$ which is independent over K. We may assume that this subset is $\{b_{t_1+1}, \ldots, b_{t_2}\}$. On the other hand, there are c_i $(i \leqslant t_1 + 1 \leqslant t_2)$ such that $b_{t_1} + \sum_{j=t_1+1}^{t_2} c_j b_j = 0$. Set $I_2 = \{b_{t_1+1}, \ldots, b_{t_2}\}$.

If $I_1 \cup I_2 = \mathcal{B}$, then the proof is finished; otherwise, by repeating the above argument, we have the subset I_3.

Continuiting this process, there exist the subsets $I_1, \ldots, I_k$ satisfying the assertions of Lemma 28. ■

Remark. Set

$$\nu_1 = \max\{\operatorname{div}\det\big(a_{i_j t_j}\big)_{1 \leqslant j,h \leqslant m+1} \mid \det\big(a_{i_j t_j}\big)_{1 \leqslant j,h \leqslant m+1} \not\equiv 0\},$$

$$\nu_2 = \min\{\operatorname{div}\det\big(a_{i_j t_j}\big)_{1 \leqslant j,h \leqslant m+1} \mid \det\big(a_{i_j t_j}\big)_{1 \leqslant j,h \leqslant m+1} \not\equiv 0\}.$$

By solving linear equations, for c_i and c_{ij} as in the above we see that $\operatorname{div} c_i \leqslant \nu_1 - \nu_2$ and $\nu_{c_{ij}} \leqslant \nu_1 - \nu_2$.

3.4 Second Main Theorem over function fields

Keep the same notation as in the previous subsections. In [Nog96] and [Nog97], J. Noguchi proved that

Theorem 29. *If $a_j \in K, 0 \leqslant j \leqslant m$, are linearly independent over k, we have*

$$(q - m - 1)\operatorname{ht}((a_j)) \leqslant \sum_{i=1}^{q} N_m(\operatorname{div}(H_i(a_0, \cdots, a_m)), \omega) + m(m+1)(g-1),$$

where H_i are linear forms in general position, and g is the genus of R.

For the case k has characteristic p and R has dimension 1 with genus g, J. Wang in [W96] also independently obtained the similar result as follows:

Theorem 30. *Suppose that $a_0, ..., a_m$ are elements of K, and are linearly independent over kK^{pn} with interger number n. Suppose that $H_1, ..., H_q$ are linear forms in $m+1$ variables with coefficients in k, and are in general position, i.e. any $m+1$ elements of $\{H_i\}$ are linearly independent over k. Then*

$$(q-n-1)\mathrm{ht}((a_j)) \leqslant \sum_{i=1}^{q}\sum_{p\notin S} \min\{mc(k), \mathrm{div}(H_i(a))(p) - \min_{0\leqslant i\leqslant m} \mathrm{div}(a_i)(p)\}$$
$$+ \frac{m(m+1)}{2}c(k)\max\{0, 2g-2+|S|\},$$

where $a=(a_0,\cdots,a_m)$, $c(k)=1$ if $p=0$ and $c(k)=p^{n-1}$ if $p>0$, and S is finite set of points of R.

In this charpter, we will prove a Second Main Theorem (Theorem 31) with truncated counting function, where the linealy forms H_j has coefficients in the function field K.

Denote by $\mathcal{R}(\mathcal{B})$ the smallest subfield of K containing $\mathbf{C}$ and all $\{\frac{b_{il}}{b_{ik}}\}$ with $b_{ik}\neq 0$, $0\leqslant i\leqslant q-1$, $0\leqslant k,l\leqslant m$. By solving linear equations $\sum_{j=0}^{t_i+1} c_{(i+1)j}b_j = 0$ or $b_{t_i} + \sum_{j=t_i+1}^{t_{i+1}} c_j b_j = 0$ over the field $\mathcal{R}(\mathcal{B})$, it is easy to see that all elements c_i and c_{ij} belong to $\mathcal{R}(\mathcal{B})$.

Theorem 31. *Let $f=(\sigma_0:\cdots:\sigma_m): R\to \mathbf{P}^m(\mathbf{C})$ be a rational mapping with $\sigma_j\in \Gamma(R,L)$. Let $\mathcal{B}=\{b_0,\ldots,b_{q-1}\}\subset (K^*)^{m+1}$ be a finite family which is nondegenerate. Assume that f is linearly nondegenerate over $\mathcal{R}(\mathcal{B})$, i.e. $(f,c)=\sum_{i=0}^m c_i\sigma_i\neq 0$ for all $c=(c_0,\ldots,c_m)\in(\mathcal{R}(\mathcal{B}))^{m+1}\setminus\{0\}$. Then*

$$\mathrm{ht}(f;\omega) \leqslant \sum_{i=0}^{q-1} N_m(\mathrm{div}(f,b_i);\omega) + \frac{m(m+1)}{2}N(J;\omega)$$
$$+ qN(\nu_1;\omega) + 2(q-1)N(\nu_2;\omega).$$

Proof. By Lemma 28, we may assume that there exist the subsets $I_i=\{b_{t_{i-1}+1},\ldots,b_{t_i}\}$ $(1\leqslant i\leqslant k)$, where $t_0=-1$, which satisfy the assertions of Lemma 28. Since I_1 is minimal, there exists a linear relation among I_1. That is, there exist $c_{1j}\in\mathcal{R}(\mathcal{B})$ such that

$$\sum_{j=0}^{t_1} c_{1j}\cdot b_j = 0.$$

Define $c_{1j}=0$ for all $j>t_1$. Then $\sum_{j=0}^{t_k} c_{1j}\cdot b_j=0$. Since f is linearly nondegenerate over $\mathcal{R}(\mathcal{B})$, it implies that $\{c_{1j}(f,b_j)\}_{j=1}^{t_1}$ is linearly independent over $\mathbf{C}$. Hence there exists $\{\alpha_{11},\ldots,\alpha_{1t_1}\}\subset \mathbf{Z}_+^{N+1}$ $(|\alpha_{1j}|\leqslant t_1-1\leqslant N)$ such that

$$A_1 \equiv \begin{vmatrix} \mathcal{D}^{\alpha_{11}}(c_{11}(f,b_1)) & \cdots & \mathcal{D}^{\alpha_{11}}(c_{1t_1}(f,b_{t_1})) \\ \mathcal{D}^{\alpha_{12}}(c_{11}(f,b_1)) & \cdots & \mathcal{D}^{\alpha_{12}}(c_{1t_1}(f,b_{t_1})) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{1t_1}}(c_{11}(f,b_1)) & \cdots & \mathcal{D}^{\alpha_{1t_1}}(c_{1t_1}(f,b_{t_1})) \end{vmatrix} \not\equiv 0.$$

Now consider $i \geqslant 2$. By the choice of the set I_i there exist meromorphic mappings $c_{ij} \not\equiv 0$, $c_{ij} \in \mathcal{R}(\mathcal{B})$ $(t_{i-1}+1 \leqslant j \leqslant t_i)$ such that $\sum_{j=t_{i-1}+1}^{t_i} c_{ij} \cdot b_j \in \left(\bigcup_{j=1}^{i-1} I_j\right)_K$. Then, there exist meromorphic mappings $c_{ij} \in K$ $(0 \leqslant j \leqslant t_{i-1})$ such that $\sum_{j=0}^{t_i} c_{ij} \cdot b_j = 0$. Define $c_{ij} = 0$ for all $j > t_i$. Then $\sum_{j=0}^{t_k} c_{ij} \cdot (f, b_j) = 0$. Since $\left\{c_{ij}(f, b_j)\right\}_{j=t_{i-1}+1}^{t_i}$ is $\mathbf{C}$-linearly independent, there exists $\{\alpha_{ij}\}_{j=t_{i-1}+1}^{t_i} \subset \mathbf{Z}_+^n$ $(|\alpha_{ij}| \leqslant t_i - t_{i-1} - 1 \leqslant N)$ such that

$$A_i = \det\left(\mathcal{D}^{\alpha_{ij}}\left(c_{is}(f, \tilde{a}_s)\right)\right)_{j,s=t_{i-1}+1}^{t_i} \not\equiv 0.$$

Consider an $t_k \times (t_k+1)$ minor matrixes $\mathcal{T}$ given by

$$\mathcal{T} = \begin{bmatrix} \mathcal{D}^{\alpha_{11}}(c_{10}(f,b_0)) & \cdots & \mathcal{D}^{\alpha_{11}}(c_{1t_k}(f,b_{t_k})) \\ \mathcal{D}^{\alpha_{12}}(c_{10}(f,b_0)) & \cdots & \mathcal{D}^{\alpha_{12}}(c_{1t_k}(f,b_{t_k})) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{1t_1}}(c_{10}(f,b_0)) & \cdots & \mathcal{D}^{\alpha_{1t_1}}(c_{1t_k}(f,b_{t_k})) \\ \mathcal{D}^{\alpha_{2t_1+1}}(c_{20}(f,b_0)) & \cdots & \mathcal{D}^{\alpha_{2t_1+1}}(c_{2t_k}(f,b_{t_k})) \\ \mathcal{D}^{\alpha_{2t_1+2}}(c_{20}(f,b_0)) & \cdots & \mathcal{D}^{\alpha_{2t_1+2}}(c_{2t_k}(f,b_{t_k})) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{2t_2}}(c_{20}(f,b_0)) & \cdots & \mathcal{D}^{\alpha_{2t_2}}(c_{2t_k}(f,b_{t_k})) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{kt_{k-1}+1}}(c_{k0}(f,b_0)) & \cdots & \mathcal{D}^{\alpha_{kt_{k-1}+1}}(c_{kt_k}(f,b_{t_k})) \\ \mathcal{D}^{\alpha_{kt_{k-1}+2}}(c_{k0}(f,b_0)) & \cdots & \mathcal{D}^{\alpha_{kt_{k-1}+2}}(c_{kt_k}(f,b_{t_k})) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{kt_k}}(c_{k0}(f,b_0)) & \cdots & \mathcal{D}^{\alpha_{kt_k}}(c_{kt_k}(f,b_{t_k})) \end{bmatrix}.$$

Denote by $\mathcal{D}_i$ the minor of the matrix obtained by deleting the i-th column of the minor matrix $\mathcal{T}$. Since the sum of each row of $\mathcal{T}$ is zero, we actually have

$$\mathcal{D}_i = (-1)^i \mathcal{D}_0 = (-1)^i \prod_{i=1}^{k} A_i \not\equiv 0.$$

Without loss of generality, we may assume that $t_k = q-1$. It is easy to see that $A_i \in \Gamma_{\mathrm{rat}}(R, [(\sum_{j=t_{i-1}+1}^{t_i} |\alpha_{ij}|)J] \otimes L^{t_i - t_{i-1}})$. Hence $D_i \in \Gamma_{\mathrm{rat}}(R, [(\sum_{ij} |\alpha_{ij}|)J] \otimes L^{q-1})$. This yields that

$$\frac{D_i}{\prod_{j \neq i}(f, b_j)} \in \Gamma_{\mathrm{rat}}(R, [(\sum_{i,j} |\alpha_{ij}|)J]), \quad 0 \leqslant \forall i \leqslant q-1. \tag{3.4.1}$$

On the other hand, we have

$$\sum_{i=0}^{q-1} |(f, b_i)|^2 = \left(\sum_{i=0}^{q-1} \left|\frac{D_i}{\prod_{j \neq i}(f, b_j)}\right|^2\right) \frac{\prod_{i=0}^{q-1} |(f, b_i)|^2}{|D_0|^2}.$$

By transmiting current equations, it implies that

$$dd^c\left[\log\left(\sum_{i=0}^{q-1}|(f,b_i)|^2\right)\right]$$

$$= dd^c\left[\log\left(\sum_{i=0}^{q-1}\left|\frac{D_i}{\prod_{j\neq i}(f,b_j)}\right|^2\right)\right] + dd^c\left[\log\left(\frac{\prod_{i=0}^{q-1}|(f,b_i)|^2}{|D_0|^2}\right)\right].$$

By (3.4.1) we also have

$$\int_R dd^c\left[\log\left(\sum_{i=0}^{q-1}\left|\frac{D_i}{\prod_{j\neq i}(f,b_j)}\right|^2\right)\right]\wedge\omega^{N-1} = \left(\sum_{i,j}|\alpha_{ij}|\right)N(J;\omega).$$

Since $(f,b_i)\in\Gamma_{\mathrm{rat}}(R,L)$, it implies that

$$\int_R dd^c\left[\log\left(\sum_{i=0}^{q-1}|(f,b_i)|^2\right)\right]\wedge\omega^{N-1} = N(\mathrm{div}\sigma_0;\omega) = T(f;\omega).$$

Denote by ν the divisor given by the section $\dfrac{\prod_{i=0}^{q-1}(f,b_i)}{D_0}$. Then

$$\int_R dd^c\left[\log\left(\frac{\prod_{i=0}^{q-1}|(f,b_i)|^2}{|D_0|^2}\right)\right]\wedge\omega^{N-1} = N(\nu;\omega).$$

This yields that

$$T(f;\omega) = N(\nu;\omega) + \left(\sum_{i,j}|\alpha_{ij}|\right)N(J;\omega).$$

We also see that

$$\sum_{i,j}|\alpha_{ij}| \leqslant \sum_{i=1}^{k}\frac{(t_i-t_{i-1})(t_i-t_{i-1}+1)}{2} \leqslant \frac{m(m+1)}{2}.$$

Now we compute ν. Let z be a fixed point of M. Then there exists a neighbourhood U of z in M such that the restriction to U of the section σ_0 can be viewed as a holomorphic function on U. We also assume that there is a meromorphic function h on U such that $\mathrm{div}(h) = -\min_{i,j}\{\mathrm{div}(c_{i,j})\}$ on U and there is an unique analytic subset S of pure codimension 1 such that $S = \bigcup_i \mathrm{supp}\ (\mathrm{div}(f,b_i))\bigcup\bigcup_{i,j}\mathrm{supp}\ (\mathrm{div}\, c_{i,j})$. Without loss of generality we may assume that z is a regular point of S.

Put $m_i = \mathrm{div}(f,b_i)(z)$ $(0\leqslant i\leqslant q-1)$. Without loss of generality we may assume that $m_0\leqslant m_1\leqslant\cdots\leqslant m_q - 1$. Then

$$\mathrm{div}\left(\mathcal{D}^{\alpha_{i,t_{i-1}+j}}\left(\frac{hc_{iv}(f,b_v)}{(f,b_0)}\right)\right)(z) \geqslant \max\{0, m_v - m_0 - |\alpha_{i,t_{i-1}+j}|\}$$
$$\geqslant \max\{0, m_v - m_0 - m\}.$$

On the other hand, we have

$$\frac{\prod_{i=1}^{q-1}(f,b_i)}{D_0} = h^{q-1}\frac{\prod_{i=1}^{q-1}((f,b_i)/(f,b_0))}{(\frac{h}{(f,b_0)})^{q-1}D_0}.$$

Hence

$$\begin{aligned}
&\operatorname{div}\left(\frac{\prod_{i=1}^{q-1}(f,b_i)}{D_0}\right)(z)\\
&\leqslant \sum_{i=1}^{q-1}(m_i - m_0 - \max\{0, m_i - m_0 - m\}) + (q-1)\operatorname{div} h(z)\\
&\leqslant \sum_{i=1}^{q-1}\min\{m_i - m_0, m\} + (q-1)\operatorname{div} h(z)\\
&\leqslant \begin{cases} \sum_{i=1}^{q-1} \min\{m_i, m\} + (q-1)\operatorname{div} h(z) & \text{if } m_0 \geqslant 0,\\ \sum_{i=1}^{q-1} \min\{m_i, m\} - (q-1)m_0 + (q-1)\operatorname{div} h(z) & \text{if } m_0 \leqslant 0.\end{cases}
\end{aligned}$$

Since (f,b_i) does have a multiplicity greater than m_0 at z, it implies that $m_0 \leqslant \nu_1(z) - \nu_2(z)$ if $m_0 \geqslant 0$ and $m_0 \geqslant \nu_2(z)$ if $m_0 \leqslant 0$. Moreover, since $\nu_{c_{ij}} \geqslant \nu_2 - \nu_1$, we have $\nu_h \leqslant \nu_1 - \nu_2$. This implies that

$$\begin{aligned}
\operatorname{div}\left(\frac{\prod_{i=0}^{q-1}(f,b_i)}{D_0}\right)(z) \leqslant &\sum_{i=0}^{q-1}\min\{m_i, m\} + \nu_1(z) - (q-1)\nu_2(z)\\
&+ (q-1)(\nu_1(z) - \nu_2(z)).
\end{aligned}$$

Hence

$$\nu \leqslant \sum_{i=0}^{q-1}\min\{\operatorname{div}(f,b_i), m\} + \nu_1 - (q-1)\nu_2 + (q-1)(\nu_1 - \nu_2).$$

Integrating both sides of the above inequality, we have

$$N(\nu;\omega) \leqslant \sum_{i=0}^{q-1} N_m(\operatorname{div}(f,b_i);\omega) + qN(\nu_1;\omega) + 2(q-1)N(\nu_2;\omega).$$

Combining the above assertions, we deduce that

$$\begin{aligned}
\operatorname{ht}(f;\omega) \leqslant &\sum_{i=0}^{q-1} N_m(\operatorname{div}(f,b_i);\omega) + \frac{m(m+1)}{2}N(J;\omega)\\
&+ qN(\nu_1;\omega) + 2(q-1)N(\nu_2;\omega). \qquad \blacksquare
\end{aligned}$$

Chapter 4

Nevanlinna theory for holomorphic curves from punctured disk into semi-Abelian varieties

In 2002, J. Noguchi, J. Wikelmann and K. Yamanoi [NWY02] proved that for a holomorphic curve $f : \mathbf{C} \to M$ into a semi-Abelian variety M and an algebraic divisor D on M, there exist an interger $k_0 = k_0(f, D)$ such that the following holds

$$|| \; T_f(r; c_1(\overline{D})) \leqslant N_{k_0}(r, f^*D) + O(\log r) + S_f(r; c_1(\overline{D}))). \tag{4.0.1}$$

where $S_f(r; c_1(\overline{D}))$ denotes a remainder term such that

$$|| \quad S_f(r, c_1(\overline{D})) = O(\log T_f(r, c_1(\overline{D}))) + O(logr).$$

They used a compactification $\overline{M}$ of M such that the maximal affine subgroup $(\mathbf{C}^*)^p$ of M was compactified by $(\mathbf{P}^1(\mathbf{C}))^p$, and they assumed that the closure $\overline{D}$ of D in $\overline{M}$ satisfies a boundary condition (cf. (4.11), [NWY02]).

Recently, in [NWY08] the same three authors proved that for the holomorphic curve f as above and an arbitrary divisor D on M, there exists a good compactification $\overline{M}$ of M such that

$$||_\epsilon \; T_f(r; c_1(\overline{D})) \leqslant N_1(r, f^*D) + \epsilon T_f(r; c_1(\overline{D})), \; \forall \epsilon > 0, \tag{4.0.2}$$

where $||_\epsilon$ stands for the inequality to hold for every $r > 1$ outside a Borel set of finite Lebesgue measure which is depend on ϵ.

In this section, our aim is to prove the Second Main Theorem which is similar to (4.0.1) for holomorphic curves from punctured disk into semi-Abelian variety, and apply it to give an altenative proof of a big Picard's theorem for nondegenerate $f : \Delta^* \to M \setminus D$.

The basic notation in this charpter is due to [Nog86], [NWY02] and [NWY08].

4.1 Complex semi-torus

Let M be a complex Lie group admitting the exact sequence

$$0 \to (\mathbf{C}^*)^p \to M \xrightarrow{\eta} M_0 \to 0, \tag{4.1.1}$$

where $\mathbf{C}^*$ is the multiplicative group of non zero complex numbers, and M_0 is a (compact) complex torus. Such an M is called a *complex semi-torus* or a *quasi torus.* If M_0 is algebraic, i.e, an Abelian variety, M is called a *semi-Abelian variety* or *quasi- Abelian variety.* In this chapter, we assume that M is a complex semi-Abelian variety.

Taking the universal covering of (4.1.1), one gets

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{C}^p & \to & \mathbf{C}^n & \to & \mathbf{C}^m & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & (\mathbf{C}^*)^p & \to & M & \to & M_0 & \to & 0, \end{array}$$

and an additive discrete subgroup Λ of $\mathbf{C}^n$ such that

$$\begin{aligned} \pi : \mathbf{C}^n \to M = \mathbf{C}^n/\Lambda, \\ \pi_0 : \mathbf{C}^m = (\mathbf{C}^n/C^p) \to M_0 = (\mathbf{C}^n/\mathbf{C}^p)/(\Lambda/\mathbf{C}^p), \\ (\mathbf{C}^*)^p = \mathbf{C}^p/(\Lambda \cap \mathbf{C}^p). \end{aligned}$$

4.2 Order function

We keep the same notation as in the previoous section. We set punctured disks on $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ about ∞ by

$$\begin{aligned} \Delta^* &= \{z \in \mathbf{C} \ : \ |z| \geqslant 1\}, \\ \Delta^*(t) &= \{z \in \mathbf{C} \ : \ |z| \geqslant t\}, \quad t \geqslant 1. \end{aligned}$$

In this chapter, we always assume that functions on Δ^* and mappings from Δ^* are defined on a neighborhood of Δ^* in $\mathbf{C}$. Let ξ be a function on Δ^* satisfying that

(i) ξ is differentiable outside a discrete set of point,

(ii) ξ is locally written as a difference of two subharmonic functions.

Then by [Nog81], section 1, we have

$$\int_1^t \frac{dt}{t} \int_{\Delta^*(t)} dd^c\xi = \frac{1}{4\pi}\int_{\Gamma(r)} \xi(re^{i\theta})d\theta - \frac{1}{4\pi}\int_{\Gamma(1)} \xi(re^{i\theta})d\theta - (\log r)\int_{\Gamma(1)} d^c\xi, \tag{4.2.1}$$

where $dd^c\xi$ is taken in the sense of current.

Let $f : \Delta^* \to M$ be a holomorphic curve and let $\overline{M}$ be a smooth compactification of M. We may regard f as a holomorphic curve from Δ^* into $\overline{M}$. Let Ω be a $(1, 1)$ form on $\overline{M}$. The *order function* or *characteristic function* of f with respect to Ω is defined by

$$T_f(r; \Omega) = \int_1^r \frac{dt}{t} \int_{\Delta^*(t)} f^*\Omega, \quad r > 1. \tag{4.2.2}$$

Let D be an effective divisor on M such that D is compactified to $\overline{D}$ in $\overline{M}$, i.e D is algebraic along the fibers of $\eta : M \to M_0$. We assume that $f(\Delta^*) \not\subset D$. We denote $L(\overline{D})$ the line bundle determined by $\overline{D}$. We fix a Hermittian fiber metric $|| \cdot ||$ in $L(\overline{D})$ with the curvature form ω representing the first Chern class $c_1(L(\overline{D}))$ of $L(\overline{D})$ (for the sake of simplicity, we write $c_1(\overline{D})$). We take a global section $\sigma \in H^0(\overline{M}, L(\overline{D}))$ with $\operatorname{div}(\sigma) = \overline{D}$ and $||\sigma|| \leqslant 1$. We set

$$T_f(r; c_1(\overline{D})) := T_f(r; \omega) = \int_1^r \frac{dt}{t} \int_{\Delta^*(t)} f^*\omega, \quad r > 1, \tag{4.2.3}$$

which is well-defined up to an $O(\log r)$-term. The *proximity function* of f with respect to $\overline{D}$ is defined by

$$m_f(r; \overline{D}) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{||\sigma(f(re^{i\theta}))||} d\theta. \tag{4.2.4}$$

Applying (4.2.1) to $\xi = f^* \log ||\sigma||$, we obtain the **First Main Theorem**:

$$\begin{aligned} T_f(r; c_1(\overline{D})) = N(r, f^*D) + m_f(r; \overline{D}) - m_f(1; \overline{D}) \\ + (\log r) \int_{\Gamma(1)} d^c \log ||\sigma \circ f||^2. \end{aligned} \tag{4.2.5}$$

Definition 32. *Let V be a complex algebraic variety. We say that a holomorphic curve $f : \Delta^* \to V$ is algebraically (resp. nondegenerate) degenerate if the image $f(\Delta^*)$ is (resp. not) contained in a proper algebraic subset of V.*

The following is a general property of the characteristic function which is due to [NO] (Lemma 6.1.5, [NO]).

Theorem 33. *Let $f : \Delta^* \to V$ be a holomorphic curve into a complex projective manifold V and let H be a line bundle on V. Assume that H is big, and that f is algebraically nondegenerate. Then*

$$T_f(r; c_1(L)) = O(T_f(r; c_1(H))) + O(\log r),$$

for every line bundle L on V.

4.3 General position

Let M be a semi-Abelian variety as above and let X be a complex algebraic variety, on which M acts:

$$(a, x) \in M \times X \to a \cdot x \in X$$

Let Y be a subvariety embedded into a Zariski open subset of X.

Definition 34. (Def. 3.2, [NWY02]) *We say that Y is generally positioned in X if the closure $\overline{Y}$ of Y in X contains no M-orbit. If the support of a divisor E on a Zariski open subset of X is generally positioned in X, then E is said to be generally positioned in X.*

Definition 35. *Let Z be a subset of M. We define the stabilizer of Z by*

$$\mathrm{St}(Z) = \{x \in M | x + Z = Z\}^0,$$

where $\{\cdot\}^0$ denotes the identity component.

With above definitions, we have the following two Lemmas due to [NWY08].

Lemma 36. (Prop. 3.9, [NWY08]) *Let $\overline{M}$ be a smooth equivariant compactification of a semi-Abelian variety M. Let D be an effective divisor on M and let $\overline{D}$ be its closure in M. Then the following properties hold.*

(i) *$\overline{M} \setminus M$ is a divisor with only simple normal crossings.*

(ii) *If $\mathrm{St}(D) = \{0\}$, then $\overline{D}$ is big on $\overline{M}$.*

Lemma 37. (Prop. 3.10, [NWY08]) *Let Z be a reduced subvariety of a semi-Abelian variety M and let $\overline{Z}$ be its closure in a smooth equivariant compactification $\overline{M}$ of M. If $\mathrm{St}(Z) = \{0\}$, then there is an equivariant blow-up $\hat{M} \to M$ such that the strict transform of Z is generally positioned in $\hat{M}$.*

In particular, there exists a smooth equivariant compactification of M in which Z is generally positioned.

Remark 38. If D be a divisor on the semi-Abelian variety M and $\mathrm{St}(D) = \{0\}$, then by Lemmas 36 and 37 there exists a compactification $\overline{M}$ of M such that the Zariski closure $\overline{D}$ of D in $\overline{M}$ is genarally positioned and big on $\overline{M}$. Apply Lemma 33, we have

$$T_f(r; c_1(L)) = O(T_f(r; c_1(\overline{D}))) + O(\log r),$$

for every line bundle L on M. In particular,

$$T_f(r; c_1(\partial M)) = O(T_f(r; c_1(\overline{D}))) + O(\log r).$$

where $\partial M = \overline{M} \setminus M$ is the boundary of M in $\overline{M}$, which is a divisor with only normal crossings on $\overline{M}$

The following Lemma is due to [NWY08].

Lemma 39. (Lemma 3.12, [NWY08]) *Let M be a semi-Abelian variety and let $M \hookrightarrow \overline{M}$ be a smooth equivariant algebraic compactification. Then there are only finitely many M-orbits in $\overline{M}$.*

N.B. From the proof of Lemma 39 (cf. Lemma 3.12, [NWY08]) of J. Noguchi, J. Winkelman and K. Yamanoi, we have the following property of the compactification $\overline{M}$:

Take an M-orbit A in $\overline{M}$. For every $p \in A$, we denote by $\mathrm{Iso}(p) = \{x \in M \ : \ x \cdot p = p$ the isotropy group at p. Then we have

$$A \cong M/\mathrm{Iso}(p). \tag{4.3.1}$$

With the help of Lemma 39 and Lemma 37, we have the following, which is a special case of Lemma 3.14 in [NWY08].

Lemma 40. *Let M be a semi-Abelian variety and let D be an algebraic divisor on M such that* $\mathrm{St}(D) = \{0\}$. *Then there exists a smooth equivariant compactification $\overline{M}$ of M such that the closure $\overline{D}$ of D in $\overline{M}$ is big, generally positioned and $\overline{M}$ has a stratification*

$$\overline{M} = \bigcup_{\text{finite}} \Gamma_\lambda,$$

satisfying the following properties:

(i) Γ_λ *is an M-orbit and* $\Gamma_\lambda \cong M/\mathrm{Iso}(x)$ *with* $x \in \Gamma_\lambda$.

(ii) $\overline{D}$ *does not contain any stratum* Γ_λ.

(iii) M *coincides with one of the strata* Γ_λ.

Proof. By Lemma 37, there exists a smooth equvariant compactification $\overline{M}$ of M in which $\overline{D}$ is generally positioned and big. By Lemma 39 the family $\{\Gamma_\lambda\}_\lambda$ of all M-orbits in $\overline{M}$ is finite. Then we have a stratafication $\overline{M} = \cup_\lambda \Gamma_\lambda$ by M-orbits. By (4.4.1), the stratification $\{\Gamma_\lambda\}_\lambda$ has the property required by (i).

Because $\overline{D}$ is generally positioned in $\overline{M}$, $\overline{D}$ contains none of the strata Γ_λ. Thus, (ii) follows.

By the definition of Γ_λ, the open set M of $\overline{M}$ natural concides with one of the strata Γ_λ. Hence (iii) follows. ■

4.4 Logarithmic Jet space

Let M be a semi-Abelian variety as above with an exact sequence

$$0 \to (\mathbf{C}^*)^p \to M \to M_0 \to 0.$$

We take a smooth equivariant compactification $\overline{M}$ of M. Then the boundary divisor ∂M has only simple normal crossings.

Let $\Omega^1_{\overline{M}}(\log \partial M)$ denote the sheaf of germs of logarithmic 1-forms over $\overline{M}$. We take a base $\{\omega^j\}_{j=1}^n$ of $H^0(\overline{M}, \Omega^1_{\overline{M}}(\log \partial M))$. Then ω^j are d-closed, invariant with respect to the action of M, and $(\omega^1 \wedge \cdots \wedge \omega^n)(x) \neq 0$ at all $x \in M$ (cf. [NWY02] §3). Therefore, by the pairing $\{\omega^j\}_{j=1}^n$ gives the trivialization of the logarithmic tangent bundle:

$$T(\overline{M}, \log \partial M) \cong \overline{M} \times \mathbf{C}^n. \tag{4.4.1}$$

Moreover, by Noguchi [Nog86] we have the logarithmic k-jet bundle $J_k(\overline{M}, \log \partial M)$ over $\overline{M}$ and a natural morphism

$$\psi : J_k(\overline{M}, \log \partial M) \to J_k(\overline{M}).$$

The trivializtion (4.4.1) gives

$$J_k(\overline{M}, \log \partial M) \cong \overline{M} \times \mathbf{C}^{nk}.$$

Let

$$\pi_1 : J_k(\overline{M}, \log \partial M) \cong \overline{M} \times \mathbf{C}^{nk} \to \overline{M},$$
$$\pi_2 : J_k(\overline{M}, \log \partial M) \cong \overline{M} \times \mathbf{C}^{nk} \to \mathbf{C}^{nk}$$

be the projections. For a k-jet $y \in J_k(\overline{M}, \log \partial M)$ we call $\pi_2(y)$ the jet part of y.

Let $x \in \overline{D}$ and let $\sigma = 0$ be a local defining equation of $\overline{D}$ about x. For a germ $g : (\mathbf{C}, 0) \to (M, 0)$ of holomorphic mappings, we denote its k-jet by $j_k(g)$, and write

$$d^j\sigma(g) = \left.\frac{d^J}{d\xi^j}\right|_{\xi=0} \sigma(g(\xi)).$$

We set

$$J_k(\overline{D})_x = \{j_k(g) \in J_k(\overline{M})_x \mid d^\sigma(g) = 0,\ 1 \leqslant j \leqslant k\},$$
$$J_k(\overline{D}) = \cup_{x \in \overline{D}} J_k(\overline{D})_x,$$
$$J_k(\overline{D}, \log \partial M) = \psi^{-1} J_k(\overline{D}).$$

Then $J_k(\overline{D}, \log \partial M)$ is a subspace of $J_k(\overline{M}, \log \partial M)$, which is depending in general on the embedding $\overline{D} \hookrightarrow \overline{M}$(cf. [Nog86]). Note that $\pi_2\big(J_k(\overline{D}, \log \partial M)\big)$ is an algebraic subset of $\mathbf{C}^{nk}$, since π_2 is proper.

4.5 Second Main Theorem

We keep the same notation as in the previous sections. Let $f : \Delta^* \to M$ be a holomorphic curve, which can be regarded as a holomorphic into a smooth equivariant compactification $\overline{M}$ of M.

Let $\mathcal{M}^*_{\overline{M}}$ be the sheaf of germs of meromorphic functions on $\overline{M}$ which do not identically vanish, and define a sheaf $\mathcal{U}^1_{\overline{M}}$ by

$$\begin{array}{ccccccc} 0 \to \mathbf{C}^* \to & \mathcal{M}^*_{\overline{M}} & \stackrel{d\log}{\to} & \mathcal{U}^1_{\overline{M}} & \to 0. \\ & \gamma & \mapsto & d\log\gamma & \end{array}$$

Note that $\mathcal{U}^1_{\overline{M}}$ is a sheaf of $\mathbf{Z}$-modules, and $H^0(\overline{M}, \mathcal{U}_{\overline{M}})$ is not a complex vector space. We see that $H^0(\overline{M}, \mathcal{U}_{\overline{M}})$ spans $H^0(\overline{M}, \Omega^1(\log \partial M)$ over $\mathbf{C}$ (cf. e.g., [Nog77]). Therefore we may assume that $\omega^i \in H^0(\overline{M}, \mathcal{U}_{\overline{M}})$ for all $1 \leqslant i \leqslant n$.

The next lemma is the *Lemma on logarithmic derivatives* for holomorphic curve from punctured disk into algebaric varieties, which was proved by J. Noguchi in [Nog81]

Lemma 41. (Lemma 2.2, [Nog81]) *Let $f : \Delta^* \to \overline{M}$ be a holomorphic curve with $f(\mathbf{C}) \not\subset \partial M$, and let $\omega \in H^0(\overline{M}, \mathcal{U}_{\overline{M}})$. Setting $f^*\omega = \xi(z)dz$, we have*

$$\| \quad m(r, \xi) \leqslant O(\log T_f(r, \Omega)) + O(\log r),$$

where Ω is a Hermitian metric form on $\overline{M}$.

Note: We define functions ξ^i by setting $f^*\omega^i = \xi^i dz$. Then

$$|| \; m(r, \xi^i) \leqslant O(\log T_f(r, \Omega)) + O(\log r), \; \forall 1 \leqslant i \leqslant n.$$

If D is a divisor on M such that $\overline{D}$ is generally positioned in $\overline{M}$, by Theorem 33 we obtain

$$|| \; m(r, \xi^i) \leqslant O(\log T_f(r, c_1(\overline{D}))) + O(\log r), \; \forall 1 \leqslant i \leqslant n. \tag{4.5.1}$$

Now we prove the main theorem of this chapter.

Theorem 42. *Let $f : \Delta^* \longrightarrow M$ be an algebraically nondegenerate holomorphic curve into a semi-Abelian variety M, and let D be a reduced divisor on M. Then there exists a smooth equivariant compactification $\overline{M}$ of M independent of f and a natural number k_0 such that*

$$m_f(r; \overline{D}) = S_f(r; L(\overline{D})), \tag{4.5.2}$$

$$T_f(r; c_1(\overline{D})) = N_{k_0}(r, f^*D) + S_f(r; L(\overline{D})). \tag{4.5.3}$$

Proof. Without loss of generality we may assume that D is irreducible. If $\mathrm{St}(D) \neq \{0\}$, we take the quotient $q : M \longrightarrow M/\mathrm{St}(D)$ and deal with the holomorphic curve $q \circ f : \Delta^* \longrightarrow M/\mathrm{St}(D)$ and the divisor $D/\mathrm{St}(D)$. In this way, it reduces to the case when D is irreducible and $\mathrm{St}(D) = \{0\}$. Thus we may assume that D is irreducible and $\mathrm{St}(D) = \{0\}$.

By Lemma 40, there exists a smooth equivariant compactification $\overline{M}$ of M, in which $\overline{D}$ is generally positioned and big. Let $\{\Gamma_\lambda\}_\lambda$ be the stratification of $\overline{M}$ as in Lemma 40.

Let $J_k(f) : \Delta^* \longrightarrow J_k(\overline{M}, \log \partial M) \cong \overline{M} \times \mathbf{C}^{nk}$ be the k-jet lifting of f. We prove the following claim

Claim 43. *There exists a number k_0 such that*

$$\pi_2\big(J_{k_0}(\overline{D}, \log \partial M)\big) \cap \pi_2\big(\overline{J_{k_0} f(\Delta^*)}^{\mathrm{Zar}}\big) \neq \pi_2\big(\overline{J_{k_0} f(\Delta^*)}^{\mathrm{Zar}}\big),$$

where $\overline{J_{k_0} f(\Delta^*)}^{\mathrm{Zar}}$ is the Zariski closure of $J_{k_0} f(\Delta^*)$ in $J_k(\overline{M}, \log \partial M)$.

Proof. Suppose that $\pi_2\big(J_k(\overline{D}, \log \partial M)\big) \cap \pi_2\big(\overline{J_k f(\Delta^*)}^{\mathrm{Zar}}\big) = \pi_2\big(\overline{J_k f(\Delta^*)}^{\mathrm{Zar}}\big)$ for all $k \geqslant 1$. Then for an arbitrary $z \in \Delta^*$, we have

$$\pi_2(j_k f(z)) \in \pi_2\big(J_k(\overline{D}, \log \partial M)\big), \quad \forall k \geqslant 1.$$

Fix z_0 in Δ^* and we set $\xi_k = \pi_2(j_k(f)(z_0))$ for all $k \geqslant 1$. We identify ξ_k with a logarithmic k-jet field on $\overline{M}$ along ∂M (cf. [Nog86]). Put $S_k = \pi_1\big(J_k(\overline{D}, \log \partial M) \cap \pi_2^{-1}(\xi_k)\big)$. Then

$$\overline{D} \supset S_1 \supset S_2 \supset \cdots$$

which stabilizes to $S_0 = \cap_{k=1}^{\infty} S_k$. By the Noetherian property of the Zariski topology, we have $S_0 \neq \emptyset$. Let x_0 be a point in S_0.

Case 1. If $x_0 \in D$, then there exist a point $y_0 \in M$ such that

$$\begin{aligned} &f(z_0) + y_0 = x_0 \in D, \\ &\left.\frac{d^k}{dz^k}\right|_{z=z_0} \sigma(f(z) + y_0) = 0 \text{ for all } k \geqslant 1, \end{aligned}$$

where σ is a local defining function of D around x_0. Therefor $y_0 + f(\Delta^*) \subset D$. By the Zariski denseness of $f(\Delta^*)$ in M, this is a contradition.

Case. 2. If $x_0 \in \overline{D} \setminus M$. Then there exists a stratum Γ_λ such that $x_0 \in \Gamma_\lambda \cap \overline{D}$. Then

$$(x_0, \xi_k) \in J_k(\overline{D})_{x_0}, \ k \geqslant 1.$$

Let $\alpha : M \to M/\mathrm{Iso}(x_0) \cong \Gamma_\lambda$ be the quotient map. Then there exists an element $y_0 \in M$ such that

$$\begin{aligned} &y_0 \cdot (\alpha \circ f(z_0)) = x_0, \text{ and} \\ &\left.\frac{d^k}{dz^k}\right|_{z=z_0} \sigma(y_0 \cdot (\alpha \circ f(z))) = 0 \text{ for all } k \geqslant 1, \end{aligned}$$

where σ is a local defining function of $\Gamma_\lambda \cap \overline{D}$ in Γ_λ about x_0. This implies that $y_0 \cdot (\alpha \circ f(z)) \in \Gamma_\lambda \cap \overline{D}$ for z in a neighborhood of z_0, and hence $\alpha \circ f(\Delta^*) \subset \Gamma_\lambda \cap \overline{D}$. Since $\overline{D}$ is generally positioned in $\overline{M}$, $\Gamma_\lambda \cap \overline{D} \not\supset \Gamma_\lambda$. It follows that f is denerate; this is a contradition.

From case 1 and case 2, we have the contraditions. Hence there exists a number $k_0 = k_0(f, D)$ such that

$$\pi_2\big(J_{k_0}(\overline{D}, \log \partial M)\big) \cap \pi_2\big(\overline{J_{k_0} f(\Delta^*)}^{\mathrm{Zar}}\big) \neq \pi_2\big(\overline{J_{k_0} f(\Delta^*)}^{\mathrm{Zar}}\big)$$

The claim is proved.

Now, we continue prove the main theorem.

Let $\{U_\alpha\}$ be an affine open covering of $\overline{M}$ such that

$$L(\overline{D})|_{U_\alpha} \cong U_\alpha \times \mathbf{C}. \tag{4.5.4}$$

We take $\sigma \in H^0(\overline{M}, L(\overline{D}))$ so that $\mathrm{div}(\sigma) = \overline{D}$. Then we have local holomorphic functions $\sigma_\alpha = \sigma|_{U_\alpha}$ given by the trivialization (4.5.4).

We fix a Hermittian metric $||\cdot||$ in $L(\overline{D})$ as in section 4.2. Then there exist positive smooth fucnctions h_α on U_α such that

$$\frac{1}{||\sigma(x)||} = \frac{h_\alpha(x)}{|\sigma_\alpha(x)|}, \qquad x \in U_\alpha.$$

By Claim 43, there exists a polynomial $R(w)$ in variable $w = (w_{lk}) \in \pi_2\big(\overline{J_{k_0}(f)(\Delta^*)}^{\mathrm{Zar}}\big) \cong \mathbf{C}^{nk_0}$ sucth that

$$\begin{aligned} \pi_2\big(J_{k_0}(\overline{D}, \log \partial M)\big) \cap \pi_2\big(\overline{J_{k_0}(f)(\Delta^*)}^{\mathrm{Zar}}\big) &\subset \{w \in \pi_2\big(\overline{J_{k_0}(f)(\Delta^*)}^{\mathrm{Zar}}\big) \mid R(w) = 0\} \\ &\neq \pi_2\big(\overline{J_{k_0}(f)(\Delta^*)}^{\mathrm{Zar}}\big)\}. \end{aligned}$$

Then we have the following equation on every $U_\alpha \times \pi_2\big(\overline{J_{k_0}(f)(\Delta^*)}^{\text{Zar}}\big)$:

$$b_{\alpha 0}\sigma_\alpha + b_{\alpha 1} d\sigma_\alpha + \cdots + b_{\alpha k_0} d^{k_0}\sigma_\alpha = R(w), \tag{4.5.5}$$

where $b_{\alpha i}$ are jet differentials on U_α. Therefore, in every $U_j \times \pi_2\big(\overline{J_{k_0}(f)(\Delta^*)}^{\text{Zar}}\big)$, we have

$$\frac{1}{||\sigma||} = \frac{1}{|R|}\frac{h_\alpha}{|\sigma_\alpha|} = \frac{1}{|R|}\left| h_\alpha b_{\alpha 0} + h_\alpha b_{\alpha 1}\frac{d\sigma_\alpha}{\sigma_\alpha} + \cdots + h_\alpha b_{\alpha k_0}\frac{d^{k_0}\sigma_\alpha}{\sigma_\alpha}\right|.$$

Take relatively compact open subsets U'_α of U_α so that $\cup_\alpha U'_\alpha = \overline{M}$. For every α, there exists positive constants C_α such that for all $x \in U'_\alpha$ we have

$$h_\alpha |b_{\alpha i}| \leqslant \sum_{\text{finite}} h_\alpha |b_{\alpha i l k \beta_{lk}(x)}| \cdot |w_{lk}|^{\beta_{lk}} \leqslant C_\alpha \sum_{\text{finite}} |w_{lk}|^{\beta_{lk}}.$$

After making C_α larger if necessary, there exists $d_\alpha > 0$ such that for $f(z) \in U'_\alpha$ we have

$$h_\alpha(f(z))|b_{\alpha i}(J_{k_0}(f)(z))| \leqslant C_\alpha (1 + \sum_{\substack{1\leqslant l\leqslant n\\ 1\leqslant k\leqslant k_0}} |w_{lk} \circ J_{k_0}(f)(z)|)^{d_\alpha}.$$

We set $\xi_{(k)} = (w_{1k}(J_k(f)), \cdots, w_{2k}(J_k(f)))$ and $\xi_{l(k)} = w_{lk}(J_k(f))$. We deduce that for $f(z) \in U'_\alpha$,

$$\frac{1}{||\sigma(f(z))||} \leqslant \frac{1}{|R(\xi_{(1)}(z), \cdots, \xi_{(k_0)}(z))|}\sum_{j=1}^{N} C_\alpha (1 + \sum_{\substack{1\leqslant l\leqslant n\\ 1\leqslant k\leqslant k_0}} |\xi_{l(k)}(z)|)^{d_\alpha}$$

$$\times \left(1 + \left|\frac{d\sigma_\alpha}{\sigma_\alpha}(J_1(f)(z))\right| + \cdots + \left|\frac{d^{k_0}\sigma_\alpha}{\sigma_\alpha}(J_{k_0}(f)(z))\right|\right).$$

Hence one gets

$$m_f(r; \overline{D}) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{||\sigma(f(re^{i\theta}))||} d\theta + O(1) \tag{4.5.6}$$

$$\leqslant m\big(r, \frac{1}{R(\xi_{(1)}(z), \cdots, \xi_{(k_0)}(z))}\big) + O\big(\sum_{\substack{1\leqslant l\leqslant n\\ 1\leqslant k\leqslant k_0}} \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\xi_{l(k)}(re^{i\theta})| d\theta$$

$$+ \sum_{\substack{1\leqslant \leqslant n\\ 1\leqslant k\leqslant k_0}} \frac{1}{2\pi}\int_0^{2\pi} \log^+ \big|\frac{d^k\sigma_\alpha}{\sigma_\alpha}(J_k(f)(re^{i\theta}))\big| d\theta\big) + O(\log r).$$

By Lemma 41 and (4.5.1) we have

$$\| \quad m\big(r, \frac{d^k\sigma_j \circ f}{\sigma_j \circ f}\big) = O(T_f(r; c_1(\overline{D}))) + O(\log r). \tag{4.5.7}$$

Combining Lemma 41, (4.5.6) and (4.5.7) we obtain

$$\| \quad m_f(r;\overline{D}) = O(T_f(r;c_1(\overline{D}))) + O(\log r). \tag{4.5.8}$$

We next estimate the counting function $N(r, f^*D)$. We see that for all $z \in \Delta^*$,

$$\mathrm{ord}_z f^*D > k \Leftrightarrow J_k(f)(z) \in J_k(\overline{D}, \log \partial M)$$

We infer from (4.5.5) that

$$\mathrm{ord}_z f^*D - \min\{\mathrm{ord}_z f^*D, k_0\} \leqslant \mathrm{ord}_z \mathrm{div}_0(R(\xi_{(1)}, \cdots, \xi_{(k_0)})).$$

Thus

$$N(r, f^*D) - N_{k_0}(r, f^*D) \leqslant N(r, \mathrm{div}_0 R(\xi_{(1)}, \cdots, \xi_{(k_0)})). \tag{4.5.9}$$

By Lemma 41, we have

$$\| \; N(r, \mathrm{div}_0 R(\xi_{(1)}, \cdots, \xi_{(k_0)})) \leqslant T(r, R(\xi_{(1)}, \cdots, \xi_{(k_0)})) + O(\log r) \tag{4.5.10}$$

$$\leqslant O(\sum_{\substack{1 \leqslant l \leqslant n \\ 1 \leqslant k \leqslant k_0}} T(r, \xi_{l(k)})) + O(\log r) \tag{4.5.11}$$

$$= O(\sum_{\substack{1 \leqslant l \leqslant n \\ 0 \leqslant k \leqslant k_0 - 1}} m(r, \xi_{l(k)})) + O(\log r) = S_f(r, c_1(\overline{D})). \tag{4.5.12}$$

By (4.5.8), we have

$$T_f(r; c_1(\overline{D})) \leqslant N(r, f^*D) + S_f(r, c_1(\overline{D})). \tag{4.5.13}$$

Combining (4.5.13), 4.5.9 and 4.5.10 we obtain

$$\| \; T_f(r; c_1(\overline{D})) \leqslant N_{k_0}(r, f^*D) + S_f(r, c_1(\overline{D})) \qquad \blacksquare$$

4.6 Big Picard's theorem

Let $f : \Delta^* \to V$ be a holomorphic curve into a complex projective algebraic variety V. We know the following characterization of a removable singularity (cf. [Nog81]):

Lemma 44. *Let $f : \Delta^* \to V$ be as above and let $T_f(r)$ be a characteristic function with respect an ample line bundle over V. Then f extends holomorphically in a neighborhood of ∞ if and only if $\liminf_{r \to \infty} T_f(r)/(\log r) < \infty$.*

Now we prove a big Picard's theorem:

Theorem 1. *Let $f : \Delta^* \to M$ be a nondegenerate holomorphic curve and let D be a reduced divisor with $\mathrm{St}(D) = \{0\}$. If f omits D (i.e., $f(\Delta^*) \cap \Delta^* = \emptyset$), then f extends to a holomorphic curve from a neighborhood of ∞ into a compactification of $\overline{M}$.*

Remark 45. This theorem was proved by Dethloff-Lu [DL01] by the negative curvature method of a Finsler metric. Here we give an altenative proof by the Nevanlinna thoery.

Proof. We may take $\overline{M}$ in which $\overline{D}$ is generally positioned. It follows from the Second Main Theorem 42 and the assumption that

$$T_f(r; c_1(\overline{D})) = S_f(r; c_1(\overline{D})).$$

If we denote by $T_f(r)$ the characteristic function of f with respect to an ample divisor on $\overline{M}$, then this is equivalent to

$$\| \quad T_f(r) = O(\log T_f(r)) + O(\log r).$$

Therefore, $\liminf_{r\to\infty} T_f(r)/(\log r) < \infty$. By Lemma 44 we have the required extension of f. ■

Chapter 5

Some related problems

5.1 Unicity problem for meromorphic mappings

Let f and g be two meromorphic functions on $\mathbf{C}$. Using the Second Main Theorem and Borel's lemma, R. Nevanlinna [N] proved that $f \equiv g$ if they have the same inverse images for five distinct values, and that g is a special type of linear fractional transformation of f if they have the same inverse images with counting multiplicities for four distinct values.

In 1975, H. Fujimoto [Fu75] generalized Nevalinna's result to the case of meromorphic mappings of $\mathbf{C}$ into $\mathbf{P}^n(\mathbf{C})$. Let $\{H_i\}_{i=1}^q$ be q hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in general position, let f and g be two linearly nondegenerate meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ satifying $f^*H_i = g^*H_i$ $(1 \leqslant i \leqslant q)$. He showed that if $q = 3n+2$ then $f \equiv g$, and if $q = 3n+1$ then there exists a projective linear transformation L of $\mathbf{P}^n(\mathbf{C})$ such that $g = L \cdot f$. Since that time, this problem has been studied very intensively for many authors, e.g., H. Fujimoto ([Fu98], [Fu99]), W. Stoll [St82], L. Smiley [S], S. Ji [J], M. Ru [MR01], Z. Ye [Ye], D. D. Thai - S. D. Quang ([ThQ05], [ThQ06]) and so on.

Let f be a linearly nondegenerate meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Take q hyperplanes $H_1, \cdots, H_q$ of $\mathbf{P}^n(\mathbf{C})$ located in general position with

(a) $\dim\big(f^{-1}(H_i) \cap f^{-1}(H_j)\big) \leqslant m-2$ for all $1 \leqslant i < j \leqslant q$.

For each positive integer (or $+\infty$) M, denote by $\mathcal{G}\big(\{H_j\}_{j=1}^q, f, M\big)$ the set of all linearly nondegenerate meromorphic mappings g of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ such that

(b) $\min\{\mathrm{div}(g, H_j), M\} = \min\{\mathrm{div}(f, H_j), M\}, \quad j \in \{1, \cdots, q\}$ and

(c) $g = f$ *on* $\cup_{j=1}^q f^{-1}(H_j)$.

In 1983, L. Smiley [S] showed that:

Theorem 46. *If $q \geqslant 3n+2$ then $g_1 = g_2$ for any $g_1, g_2 \in \mathcal{G}\big(\{H_j\}_{j=1}^q, f, 1\big)$.*

Recently, in 1998, H. Fujimoto [Fu98] obtained that:

Theorem 47. *If $q \geqslant 3n+1$ then $\mathcal{G}\big(\{H_j\}_{j=1}^q, f, 2\big)$ contains at most two mappings.*

It is an interesting problem to ask whether these results remain valid if the number of hyperplanes is replaced by a smaller one. In this charpter, we will introduce another condition for f and g and get an answer for this problem with only $\max\{\frac{7(n+1)}{4}, \frac{\sqrt{17n^2+16n}+3n+4}{4}\}$ hyperplanes (Theorems 48 and 54).

For a meromorphic mapping h of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$, we denote by dh the diffential of h and $I(h)$ the indeternacy locus of h. We give an extension of uniqueness theorem to the case of few hyperplanes as follow.

Theorem 48. *Let $f, g: \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be two linearly nondegenerate meromorphic mappings and let $\{H_i\}_{i=1}^q$ be q hyperplanes of $\mathbf{P}^n(\mathbf{C})$ located in general position with $\dim\big(f^{-1}(H_i)\cap f^{-1}(H_j)\big) \leqslant 2$ for all $1 \leqslant i < j \leqslant q$. Assume that $q > \dfrac{\sqrt{17n^2+16n}+3n+4}{4}$ and*

(i) $\min\{\mathrm{div}(f, H_i)(z), n\} = \min\{\mathrm{div}(g, H_i)(z), n\}$, *for all* $i \in \{1, \cdots, q\}$,

(ii) $df(z) = dg(z)$ *for all* $z \in \cup_{i=1}^q f^{-1}(H_i) \setminus (I(f) \cup I(g))$.

Then $f \equiv g$.

Proof. Asumme that that $g \not\equiv f$. We take $f = (f_0 : \cdots : f_n)$ and $g = (g_0 : \cdots : g_n)$ are reduced representations of f and g respectively. We write $H_i : \sum_{j=o}^n a_{ij}\omega_j = 0$ $(a_{ij} \in \mathbf{C})$ and set $(f, H_i) = \sum_{j=0}^n a_{ij} f_j$, $(g, H_i) = \sum_{j=0}^n a_{ij} g_j$.

For any fixed index i, $(1 \leqslant i \leqslant q)$, it is easy to see that there exists $j \in \{1, \cdots, q\}\backslash\{i\}$ (depending on i) such that

$$P_{ij} := \frac{(f, H_i)}{(f, H_j)} - \frac{(g, H_i)}{(g, H_j)} \not\equiv 0.$$

Set $I := I(f) \cup I(g) \cup_{1 \leqslant k < s \leqslant q} \{z \in \mathbf{C}^m : \mathrm{div}(f, H_k)(z) > 0 \text{ and } \mathrm{div}(f, H_s)(z) > 0\}$. Then I is an analytic subset of codimension 2 or emptyset.

Case 1. $n \geqslant 2$

Let t be an arbitrary index in $\{1, \cdots, q\}\backslash\{i, j\}$. For any fixed point $z_0 \notin I$ satisfying $\mathrm{div}(f, H_t)(z_0) > 0$, by condition (ii) in the theorem, it follows that

$$\begin{aligned}
\mathcal{D}^\alpha P_{ij}(z_0) &= \mathcal{D}^\alpha\Big(\frac{(f, H_i)}{(f, H_j)} - \frac{(g, H_i)}{(g, H_j)}\Big)(z_0) \\
&= \mathcal{D}^\alpha\Big(\frac{(f, H_i)}{(f, H_j)}\Big)(z_0) - \mathcal{D}^\alpha\Big(\frac{(g, H_i)}{(g, H_j)}\Big)(z_0) = 0, \text{ for all } \alpha \text{ with } |\alpha| < 2.
\end{aligned}$$

So

$$\mathrm{div}_0(P_{ij})(z_0) \geqslant 2. \qquad (5.1.1)$$

For any fixed point $z_1 \notin I$ satisfying $\mathrm{div}(f, H_i)(z_1) > 0$, we have

$$\mathrm{div}_0(P_{ij})(z_1) \geqslant \min\{\mathrm{div}(f, H_i)(z_1), \mathrm{div}(g, H_i)(z_1)\} \geqslant \min\{\mathrm{div}(f, H_i)(z_1), n\}. \qquad (5.1.2)$$

From (5.1.1) and (5.1.2), we have

$$\mathrm{div}_0(P_{ij}) \geqslant \min\{n, \mathrm{div}(f, H_i)\} + \sum_{t \in \{1, \cdots, q\}\backslash\{i, j\}} 2\min\{1,\ \mathrm{div}(f, H_t)\},$$

(outside an analytic subset of codimension two).

It yields that

$$N(r, \mathrm{div}_0(P_{ij})) \geqslant N_n(r, \mathrm{div}(f, H_i)) + \sum_{t \in \{1, \cdots, q\} \setminus \{i,j\}} 2N_1(r, \mathrm{div}(f, H_t)) \tag{5.1.3}$$

It is clear that

$$N(r, \mathrm{div}_\infty(P_{ij})) \leqslant N(r, \nu_j), \text{ where } \nu_j(z) = \max\{\mathrm{div}(f, H_j)(z), \mathrm{div}(g, H_j)(z)\}. \tag{5.1.4}$$

We have

$$\begin{aligned} m\left(r, \frac{(f, H_i)}{(f, H_j)}\right) &= T(r, \frac{(f, H_i)}{(f, H_j)}) - N(r, \mathrm{div}(f, H_j)) + O(1) \\ &\leqslant T_f(r) - N(r, \mathrm{div}(f, H_j)) + O(1), \text{ and} \\ m\left(r, \frac{(g, H_i)}{(g, H_j)}\right) &\leqslant T_g(r) - N(r, \mathrm{div}(g, H_j)) + O(1), \end{aligned}$$

This implies that

$$\begin{aligned} m(r, P_{ij}) &\leqslant m\left(r, \frac{(f, H_i)}{(f, H_j)}\right) + m\left(r, \frac{(g, H_i)}{(g, H_j)}\right) + o(T_f(r) + T_g(r)) \\ &= T_f(r) + T_g(r) - N(r, \mathrm{div}(f, H_j)) - N(r, \mathrm{div}(g, H_j)) + o(T_f(r) + T_g(r)). \end{aligned}$$

Combining with (5.1.3) and (5.1.4) we get

$$\begin{aligned} N_n(r, \mathrm{div}(f, H_i)) &+ \sum_{t \in \{1, \cdots, q\} \setminus \{i,j\}} 2N_1(r, \mathrm{div}(f, H_t)) \\ &\leqslant N(r, \mathrm{div}_0(P_{ij})) \leqslant T_{P_{ij}}(r) + O(1) \\ &= N(r, \mathrm{div}_\infty(P_{ij})) + m(r, P_{ij}) + O(1) \\ &\leqslant T_f(r) + T_g(r) + N(r, \nu_j) - N(r, \mathrm{div}(f, H_j)) \\ &\quad - N(r, \mathrm{div}(g, H_j)) + o(T_f(r) + T_g(r)). \end{aligned}$$

This gives

$$\begin{aligned} N(r, \mathrm{div}(f, H_j)) &+ N(r, \mathrm{div}(g, H_j)) - N(r, \nu_j) + N_n(r, \mathrm{div}(f, H_i)) \\ &+ \sum_{t \in \{1, \cdots, q\} \setminus \{i,j\}} 2N_1(r, \mathrm{div}(f, H_t)) \leqslant T_f(r) + T_g(r) + o(T_f(r) + T_g(r)). \end{aligned}$$

On the other hand, since

$$\nu_j(z) - \mathrm{div}(f, H_j) - \mathrm{div}(g, H_j) + \min\{n, \mathrm{div}(f, H_j)\} \leqslant 0$$

(outside an analytic subset of codimension two), we have

$$N(r, \nu_j) - N(r, \mathrm{div}(f, H_j)) - N(r, \mathrm{div}(g, H_j)) + N_n(r, \mathrm{div}(f, H_j)) \leqslant 0.$$

Hence

$$N_n(r,\mathrm{div}(f,H_i))+N_n(r,\mathrm{div}(f,H_j))+\sum_{t\in\{1,\cdots,q\}\setminus\{i,j\}}2N_1(r,\mathrm{div}(f,H_t))$$
$$\leqslant T_f(r)+T_g(r)+o(T_f(r)+T_g(r)).$$

It implies that

$$N_n(r,\mathrm{div}(f,H_i))+\frac{2}{n}\sum_{t\in\{1,\cdots,q\}\setminus\{i\}}N_n(r,\mathrm{div}(f,H_t))\leqslant T_f(r)+T_g(r)+o(T_f(r)+T_g(r)), \tag{5.1.5}$$

(note that $n\geqslant 2$).

Taking summing-up of both side of (5.1.5) over all $i\in\{1,\cdots q\}$, we obtain

$$(1+\frac{2(q-1)}{n})\sum_{i=1}^{q}N_n(r,\mathrm{div}(f,H_i))\leqslant q(T_f(r)+T_g(r))+o(T_ff(r)+T_g(r)). \tag{5.1.6}$$

On the other hand, by Second Main Theorem we have

$$||\ (q-n-1)(T_f(r)+T_g(r))\leqslant 2\sum_{i=1}^{q}N_n(r,\mathrm{div}(f,H_i))+o(T_f(r)+T_g(r)). \tag{5.1.7}$$

From (5.1.6) and (5.1.7), letting $r\longrightarrow\infty$ we have

$$1+\frac{2(q-1)}{n}\leqslant\frac{2q}{q-n-1}.$$

This contradicts to $q>\dfrac{\sqrt{17n^2+16n}+3n+4}{4}$. Thus $f\equiv g$.

Case 2. n=1.We have $q\geqslant 4$. If $\dfrac{(f,H_1)}{(f,H_4)}\equiv\dfrac{(g,H_1)}{(g,H_4)}$, then $f\equiv g$.

We now asumme that $P_{14}:=\dfrac{(f,H_1)}{(f,H_4)}-\dfrac{(g,H_1)}{(g,H_4)}\not\equiv 0$. Let t be an arbitrary index in $\{1,2,3\}$. For any fixed point $z_0\notin I$ satisfying $\mathrm{div}(f,H_t)(z_0)>0$, it follows that

$$\mathcal{D}^\alpha P_{14}(z_0)=\mathcal{D}^\alpha\Big(\frac{(f,H_1)}{(f,H_4)}-\frac{(g,H_1)}{(g,H_4)}\Big)(z_0)$$
$$=\mathcal{D}^\alpha\Big(\frac{(f,H_1)}{(f,H_4)}\Big)(z_0)-\mathcal{D}^\alpha\Big(\frac{(g,H_1)}{(g,H_4)}\Big)(z_0)=0,\quad\text{for all }\ \alpha\text{ with }|\alpha|<2.$$

It implies that $\nu_{P_{14}}(z_0)\geqslant 2$. Hence, we have

$$\mathrm{div}_0(P_{14})\geqslant 2\big(\min\{1,\mathrm{div}(f,H_1)\}+\min\{1,\mathrm{div}(f,H_2)\}+\min\{1,\mathrm{div}(f,H_3)\}\big),$$

(outside an analytic subset of codimension two). It implies that

$$N(r, \mathrm{div}_0(P_{14})) \geqslant 2(N_1(r, \mathrm{div}(f, H_1)) + N_1(r, \mathrm{div}(f, H_2)) + N_1(r, \mathrm{div}(f, H_3)). \quad (5.1.8)$$

Let z_1 be an arbitrary *pole* of P_{14} such that $z_1 \notin I$. Then z_1 is a *zero* of (f, H_4) and $(f, H_2)(g, H_2)(z_1) \neq 0$. Then

$$\mathcal{D}^\alpha\Big(\frac{(f, H_1)(g, H_4)}{(f, H_2)(g, H_2)}\Big)(z_1) - \mathcal{D}^\alpha\Big(\frac{(f, H_4)(g, H_1)}{(f, H_2)(g, H_2)}\Big)(z_1) = 0,$$

for all α with $|\alpha| < 2$. This implies that

$$\mathrm{div}((f, H_1)(g, H_4) - (f, H_4)(g, H_1))(z_1) \geqslant 2.$$

Then, we have

$$\mathrm{div}_\infty(P_{14})(z_1) \leqslant \mathrm{div}(f, H_4)(z_1) + \mathrm{div}(g, H_4)(z_1) - 2.$$

Hence we see

$$\mathrm{div}_\infty(P_{14}) \leqslant \mathrm{div}(f, H_4) + \mathrm{div}(g, H_4) - 2\min\{1, \mathrm{div}(f, H_4)\},$$

(outside an analytic subset of codimension two). This implies that

$$N(r, \mathrm{div}_\infty(P_{14})) \leqslant N(r, \mathrm{div}(f, H_4)) + N(r, \mathrm{div}(g, H_4)) - 2N_1(r, \mathrm{div}(f, H_4)).$$

Combining with (5.1.8) we have

$$\begin{aligned}
2\big(N_1(r, \mathrm{div}(f, H_1)) + N_1(r, \mathrm{div}(f, H_2)) + N_1(r, \mathrm{div}(f, H_3))\big) &\leqslant N(r, \mathrm{div}_0(P_{14})) \\
&\leqslant T(r, P_{14}) + O(1) = m(r, P_{14}) + N(r, \mathrm{div}_\infty(P_{14})) + O(1) \\
&\leqslant m\big(r, \frac{(f, H_1)}{(f, H_4)}\big) + m\big(r, \frac{(g, H_1)}{(g, H_4)}\big) \\
&\quad + N(r, \mathrm{div}(f, H_4)) + N(r, \mathrm{div}(g, H_4)) - 2N_1(r, \mathrm{div}(f, H_4)) + O(1) \\
&= T(r, \frac{(f, H_1)}{(f, H_4)}) + T(r, \frac{(g, H_1)}{(g, H_4)}) - 2N_1(r, \mathrm{div}(f, H_4)) + O(1) \\
&\leqslant T_f(r) + T_g(r) - 2N_1(r, \mathrm{div}(f, H_4)) + o(T_f(r) + T_g(r)).
\end{aligned}$$

It implies that

$$\begin{aligned}
2(N_1(r, \mathrm{div}(f, H_1)) + N_1(r, \mathrm{div}(f, H_2)) + N_1(r, \mathrm{div}(f, H_3)) + N_1(r, \mathrm{div}(f, H_4))) \\
\leqslant T_f(r) + T_g(r) + o(T_f(r) + T_g(r)).
\end{aligned}$$

On the other hand, by Second Main Theorem, we also have

$$|| \ 2T_f(r) \leqslant \sum_{i=1}^{4} N_1(r, \mathrm{div}(f, H_i)) + o(T_f(r))$$

and $$\| \; 2T_g(r) \leqslant \sum_{i=1}^{4} N_1(r, \mathrm{div}(g, H_i)) + o(T_g(r))$$
$$= \sum_{i=1}^{4} N_1(r, \mathrm{div}(f, H_i)) + o(T_f(r)) + o(T_g(r))$$

Hence, we have

$$\| \; 2(T_f(r) + T_g(r)) \leqslant T_f(r) + T_g(r) + o(T_f(r) + T_g(r)).$$

Letting $r \longrightarrow \infty$, we have $2 \leqslant 1$. This is a contradiction, hence $f \equiv g$. We have completed the proof of Lemma 48. ■

Let f be a linearly nondegenerate meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ and $H_1, \cdots, H_q$ be q hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in general position with

$$\dim \{z \in \mathbf{C}^m : \nu_{(f,H_i)}(z) > 0 \text{ and } \nu_{(f,H_j)}(z) > 0\} \leqslant m-2 \; (1 \leqslant i < j \leqslant q).$$

Consider the set $\mathcal{F}(f, \{H_j\}_{j=1}^q, d)$ of all linearly nondegenerate meromorphic mappings $g : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ satisfying the conditions:

(a) $\min\, (\mathrm{div}(f, H_i), d) = \min\, (\mathrm{div}(g, H_i), d) \; (1 \leqslant i \leqslant q)$,

(b) $df(z) = dg(z)$ for all $z \in \cup_{i=1}^q f^{-1}(H_i) \setminus (I(f) \cup I(g))$,

Take $M+1$ maps $f^0, ..., f^M \in \mathcal{F}(f, \{H_j\}_{j=1}^q, d)$ with reduced representations

$$f^k := (f_0^k : ... : f_n^k)$$

and set $T(r) := \sum_{k=0}^M T_{f^k}(r)$. For each $c = (c_0, ..., c_n) \in \mathbf{C}^{n+1} \setminus \{0\}$ we put

$$(f^k, c) := \sum_{i=0}^{n} c_i f_i^k \; (0 \leqslant k \leqslant M).$$

Denote by $\mathcal{C}$ the set of all $c \in \mathbf{C}^{n+1} \setminus \{0\}$ such that

$$\dim\{z \in \mathbf{C}^m : (f^k, H_j)(z) = (f^k, c)(z) = 0\} \leqslant m-2 \; (1 \leqslant j \leqslant q, \; 0 \leqslant k \leqslant M).$$

Lemma 49. (Lemma 5.1, [J]) *$\mathcal{C}$ is dense in $\mathbf{C}^{n+1}$.*

Lemma 50. ([Fu98], [NO]) *For each $c \in \mathcal{C}$, we put $F_c^{jk} = \dfrac{(f^k, H_j)}{(f^k, c)}$. Then*

$$T_{F_c^{jk}}(r) \leqslant T_{f^k}(r) + o(T(r)).$$

Definition 51. *Let $F_0, ..., F_M$ be meromorphic functions on $\mathbf{C}^m$, where $M \geqslant 1$. Take a set $\alpha := (\alpha^0, ..., \alpha^{M-1})$ whose components α^k are composed of m nonnegative integers, and set $|\alpha| = |\alpha^0| + ... + |\alpha^{M-1}|$. We define Cartan's auxiliary function by*

$$\Phi^\alpha(F_0, ..., F_M) := F_0 \cdot F_1 \cdots F_M \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \mathcal{D}^{\alpha^0}(\frac{1}{F_0}) & \mathcal{D}^{\alpha^0}(\frac{1}{F_1}) & \cdots & \mathcal{D}^{\alpha^0}(\frac{1}{F_M}) \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_0}) & \mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_1}) & \cdots & \mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_M}) \end{vmatrix}$$

Lemma 52. (Prop. 3.4, [Fu98]) *If* $\Phi^\alpha(F,G,H)=0$ *and* $\Phi^\alpha(\frac{1}{F},\frac{1}{G},\frac{1}{H})=0$ *for all* α *with* $|\alpha|\leqslant 1$*, then one of the following assertions holds :*

(i) $F=G,\ G=H$ *or* $H=F$.

(ii) $\frac{F}{G},\ \frac{G}{H}$ *and* $\frac{H}{F}$ *are all constant.*

Lemma 53. *Assume that there exists* $\Phi^\alpha := \Phi^\alpha(F_c^{j_00},...,F_c^{j_0M})\not\equiv 0$ *for some* $c\in\mathcal{C}, |\alpha|\leqslant \dfrac{M(M-1)}{2}, d\geqslant|\alpha|$*. Then, for each* $0\leqslant i\leqslant M$*, the following holds:*

$$|| \ N_{d-|\alpha|}(\mathrm{div}(f^i,H_{j_0}))+2M\sum_{j\neq j_0}N_1(r,\mathrm{div}(f^i,H_j))\leqslant N(r,\mathrm{div}_0\Phi^\alpha)\leqslant T(r)+o(T(r)).$$

Proof. Denote by $\mathbf{P}$ the set of all β with $|\beta|\leqslant\dfrac{M(M-1)}{2}, d\geqslant|\beta|$ such that $\Phi^\beta=\Phi^\beta(F_c^{j_00},...,F_c^{j_0M})\not\equiv 0$ for some $c\in\mathcal{C}$. Let α be the *minimal* - multy index in $\mathbf{P}$ (in the lexicograhic order). Set

$$I:=\cup_{t=0}^M I(f^t)\cup_{1\leqslant t<j\leqslant q}\big((f,H_t)^{-1}\{0\}\cap(f,H_j)^{-1}\{0\}\big)\cup_{t=1}^q\big((f,H_t)^{-1}\{0\}\cap(f,c)^{-1}\{0\}\big).$$

Then I is an analytic subset of codimension two or emptyset.

Assume that a is a *zero* of some (f^i,H_j), $j\neq j_0$ such that $a\notin I$. Let Γ be an irreducible component of the zero divisor of the function (f^i,H_j) which contains a. We take a holomophic function h on $\mathbf{C}^m$ satisfying : $\mathrm{div}(h)|_\Gamma=1$ and $\mathrm{div}(h)|_{C^m\setminus\Gamma}=0$.

By the condition (b), we have that $\varphi_i := (\dfrac{1}{h^2F^{j_0i}}-\dfrac{1}{h^2F^{j_0M}})$ is a holomorphic function on a neighborhood U of a for all $i\in\{0,\cdots,M-1\}$. Since $\alpha:=\min\mathbf{P}$, we have

$$\Phi^\alpha := h^{2M}F^{j_00}\cdots F^{j_0M}\times\begin{vmatrix}\mathcal{D}^{\alpha^0}\varphi_0 & \cdots & \mathcal{D}^{\alpha^0}\varphi_{M-1}\\ \vdots & \vdots & \vdots\\ \mathcal{D}^{\alpha^{M-1}}\varphi_0 & \cdots & \mathcal{D}^{\alpha^{M-1}}\varphi_{M-1}\end{vmatrix}$$

It implies that

$$\mathrm{div}_0(\Phi^\alpha)(a)\geqslant Md. \tag{5.1.9}$$

Asumme that b is a *zero* of (f^i,H_{j_0}) such that $b\notin I$.

If $\mathrm{div}(f^i,H_{j_0})(b)\geqslant d$, we write

$$\Phi^\alpha=\sum_{\sigma\in S_{M+1}}\mathrm{sign}(\sigma)F^{j_00}\cdots F^{j_0M}\times\mathcal{D}^{\alpha^0}(\frac{1}{F^{j_0(\sigma(2)-1)}})\cdots D^{\alpha^{M-1}}(\frac{1}{F^{j_0(\sigma(M+1)-1)}}).$$

Then

$$\mathrm{div}_0(\Phi^\alpha)(b)\geqslant d-|\alpha|. \tag{5.1.10}$$

If $\mathrm{div}(f^i, H_{j_0})(b) < d$, then $\mathrm{div}(f^0, H_{j_0})(b) = \cdots = \mathrm{div}(f^M, H_{j_0})(b) < d$. There exists a holomorphic function h on an open neighbourhood U of b such that $\mathrm{div}(h) = \mathrm{div}(f^i, H_{j_0})|_U$. We write

$$\Phi^\alpha = h^{-M} F_c^{j_0 0} \cdots F_c^{j_0 M}$$
$$\times \begin{vmatrix} \left(\mathcal{D}^{\alpha^0}(\frac{h}{F_c^{j_0 0}}) - D^{\alpha^0}(\frac{h}{F_c^{j_0 M}})\right) & \cdots & \left(\mathcal{D}^{\alpha^0}(\frac{h}{F_c^{j_0 M-1}}) - D^{\alpha^0}(\frac{h}{F_c^{j_0 M}})\right) \\ \vdots & \vdots & \vdots \\ \left(\mathcal{D}^{\alpha^{M-1}}(\frac{h}{F_c^{j_0 0}}) - D^{\alpha^{M-1}}(\frac{h}{F_c^{j_0 M}})\right) & \cdots & \left(\mathcal{D}^{\alpha^{M-1}}(\frac{h}{F_c^{j_0 M-1}}) - D^{\alpha^{M-1}}(\frac{h}{F_c^{j_0 M}})\right) \end{vmatrix}.$$

Then

$$\mathrm{div}_0 \Phi^\alpha(b) \geqslant \mathrm{div}(f^i, H_{j_0})(b). \tag{5.1.11}$$

From (5.1.9), (5.1.10) and (5.1.11), we have

$$\min\{d - |\alpha|, \mathrm{div}(f^i, H_{j_0})\} + 2M \sum_{j \in \{1, \cdots, q\} \setminus \{j_0\}} \min\{1, \nu_{(f^i, H_j)}\} \leqslant \mathrm{div}_0 \Phi^\alpha,$$

(outside an analytic subset of codimension two). It immediately follows the first inequality in the lemma.

It is easy to see that a *pole* of Φ^α is a *zero* or a *pole* of some $F_c^{j_0 k}$. By (5.1.9), (5.1.10) and (5.1.11) we have that Φ^α is holomorphic at all *zeros* of $F_c^{j_0 i}$, $(0 \leqslant i \leqslant M)$. Then

$$N(r, \mathrm{div}_\infty(\Phi^\alpha)) \leqslant \sum_{i=0}^{M} N(r, \mathrm{div}_\infty(F_c^{j_0 i})).$$

On the other hand, it is easy to see that

$$\begin{aligned} m(r, \Phi^\alpha) &\leqslant \sum_{i=0}^{M} m(r, F_c^{j_0 i}) + O\left(\sum m\left(r, \frac{\mathcal{D}^{\alpha^i}(\varphi_c^{j_0 k})}{\varphi_c^{j_0 k}}\right)\right) + O(1) \\ &\leqslant \sum_{i=0}^{M} m(r, F_c^{j_0 i}) + o(T(r)), \end{aligned}$$

where $\varphi_c^{j_0 k} = 1/F_c^{j_0 k}$. Hence, we have

$$\begin{aligned} N(r, \mathrm{div}_0(\Phi^\alpha)) &\leqslant T_{\Phi^\alpha}(r) + O(1) \leqslant m(r, \Phi^\alpha) + N(r, \mathrm{div}_\infty(\Phi^\alpha)) + O(1) \\ &\leqslant \sum_{i=0}^{M} \left(N(r, \mathrm{div}_\infty(F_c^{j_0 i})) + m(r, F_c^{j_0 i})\right) + o(T(r)) \\ &= \sum_{i=0}^{M} T_{F_c^{j_0 i}}(r) + o(T(r)) \leqslant T(r) + o(T(r)). \qquad \blacksquare \end{aligned}$$

Theorem 54. *If* $q > \max\{\frac{7(n+1)}{4}, \frac{\sqrt{17n^2+16n}+3n+4}{4}\}$ *then* $\mathcal{F}(f, \{H_i\}_{i=1}^q, 2)$ *contains at most two mappings.*

Proof. If $n = 1$, by Lemma 48 we have $\sharp(f, \{H_i\}_{i=1}^q, 1) = 1$.

We prove the theorem for the case of $n \geqslant 2$. Assume that there exist three distinct mappings $f^0, f^1, f^2 \in \mathcal{F}(f, \{H_i\}_{i=1}^q, 2)$.

Denote by $\mathcal{Q}$ the set of all indices $j \in \{1, 2, ..., q\}$ satisfying the following: There exist $c \in \mathcal{C}$ and $\alpha \in \mathbf{Z}_+^m$ with $|\alpha| \leqslant 1$ such that $\Phi^\alpha(F_c^{j0}, F_c^{j1}, F_c^{j2}) \not\equiv 0$.

Set $T(r) = T_{f^0}(r) + T_{f^1}(r) + T_{f^2}(r)$.

We now prove that $\mathcal{Q} = \emptyset$. Suppose that there exists $j_0 \in \mathcal{Q}$. By Lemma 52, we have

$$N_1(r, \operatorname{div}(f^i, H_{j_0})) + 4 \sum_{j \in \{1,\cdots,q\} \setminus \{j_0\}} N_1(r, \operatorname{div}(f^i, H_j)) \leqslant N(r, \operatorname{div}_0 \Phi^\alpha)$$
$$\leqslant T(r) + o(T(r)) \ (0 \leqslant i \leqslant 2). \tag{5.1.12}$$

By the Second Main Theorem, we have

$$|| \quad \sum_{j \neq j_0} N_1(r, \operatorname{div}(f^i, H_j)) \geqslant \frac{q-n-2}{3n} T(r) + o(T(r))$$

$$\text{and } || \quad \sum_{j=0}^{q} N_1(r, \operatorname{div}(f^i, H_j)) \geqslant \frac{q-n-1}{3n} T(r) + o(T(r)).$$

This implies that

$$|| \quad N_1(r, \operatorname{div}(f^i, H_{j_0})) + 4 \sum_{j \in \{1,\cdots,q\} \setminus \{j_0\}} N_1(r, \operatorname{div}(f^i, H_j))$$
$$\geqslant \frac{4(q-n-2)+1}{3n} T(r) + o(T(r)). \tag{5.1.13}$$

From (5.1.12) and (5.1.13), letting $r \to \infty$ we get $4(q-n-2)+1 \leqslant 3n \Leftrightarrow q \leqslant \frac{7(n+1)}{4}$. This is a contradiction. Hence $\mathcal{Q} = \emptyset$. Then for each $1 \leqslant j \leqslant q$, $c \in \mathcal{C}$, $\alpha \in \mathbf{Z}_+^m$, $|\alpha| < 2$ we have $\Phi^\alpha(F_c^{j0}, F_c^{j1}, F_c^{j2}) \equiv 0$. Since $\mathcal{C}$ is dense in $\mathbf{C}^{n+1}$, we have that

$$\Phi^\alpha(F_i^{j0}, F_i^{j1}, F_i^{j2}) \equiv 0 \ (1 \leqslant i, j \leqslant q), \text{ for all } |\alpha| < 2,$$

where $F_i^{jt} := \frac{(f^t, H_j)}{(f^t, H_i)}$, $0 \leqslant t \leqslant 2$. By Lemma 52, for each $1 \leqslant i, j \leqslant q$, there exists a nonzero constant χ_{ij} such that $F_i^{j0} = \chi_{ij} F_i^{j1}$, $F_i^{j1} = \chi_{ij} F_i^{j2}$ or $F_i^{j2} = \chi_{ij} F_i^{j0}$. We now show that $\chi_{ij} = 1$. Indeed, if $\chi_{ij} \neq 1$, without loss of generality we may assume that $F_i^{j0} = \chi_{ij} F_i^{j1}$. Then $\bigcup_{t \in \{1,\cdots,q\} \setminus \{i,j\}} f^{-1}(H_t) = \emptyset$. Thus, by Second Main Theorem, we have

$$|| \quad (q-n-3) T_f(r) \leqslant \sum_{t \in \{1,\cdots,q\} \setminus \{i,j\}} N_n(r, \operatorname{div}(f, H_t)) + o(T_f(r)) = o(T_f(r)).$$

Letting $r \longrightarrow +\infty$, we obtain $q - n - 3 \leqslant 0$. This contradicts to $n \geqslant 2$. Thus,

$$\chi_{ij} = 1 \ (1 \leqslant i, j \leqslant q).$$

We take an arbitrary element $k \in \{0, 1, 2\}$ and an index $i \in \{1, \cdots, q\}$. We will show that $\mathrm{div}(f^k, H_i) = \mathrm{div}(f^l, H_i)$ or $\mathrm{div}(f^k, H_i) = \mathrm{div}(f^t, H_i)$, where $\{k, l, t\} = \{1, 2, 3\}$. Infact, if there is no index $j \neq i$ such that $F_i^{jk} = F_i^{jl}$ or $F_i^{jk} = F_i^{jt}$, then since $\chi_{ij} = 1$ we have $F_i^{jl} = F_i^{jt}$ for all $j \neq i$. This implies that $f^k \equiv f^l$. This is a contradiction. Hence there exists $j \neq i$ such that $F_i^{jk} = F_i^{jl}$ or $F_i^{jk} = F_i^{jt}$. This yields that

$$\mathrm{div}(f^k, H_i) = \mathrm{div}(f^l, H_i) \text{ or } \mathrm{div}(f^k, H_i) = \mathrm{div}(f^t, H_i), \quad k \in \{0, 1, 2\},\ i \in \{1, \cdots, q\}. \tag{5.1.14}$$

For any fixed index $i \in \{1, \cdots, q\}$. By (5.1.14), we may assume that $\mathrm{div}(f^0, H_i) = \mathrm{div}((f^1, H_i)$. By again (5.1.14), we also have $\mathrm{div}(f^2, H_i) = \mathrm{div}(f^0, H_i)$ or $\mathrm{div}(f^2, H_i) = \mathrm{div}(f^1, H_i)$. This yields that $\mathrm{div}(f^0, H_i) = \mathrm{div}(f^1, H_i) = \mathrm{div}(f^2, H_i)$. By Theorem 48, we have $f^0 \equiv f^1$. This is a contradiction.

Thus, $\sharp\, \mathcal{F}(f, \{H_i\}_{i=1}^q, 2) \leqslant 2$ if $q > \max\{\dfrac{7(n+1)}{4}, \dfrac{\sqrt{17n^2 + 16n} + 3n + 4}{4}\}$. ■

5.2 Normal family of meromorphic mappings

Let $\mathcal{F}$ be a family of holomorphic mappings of a domain D in $\mathbf{C}^m$ into a compact complex manifold X. $\mathcal{F}$ is said to be a normal family on D if any sequence in $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of D to a holomorphic mapping of D into X.

Let D be a domain in $\mathbf{C}^m$. Let $f : D \to \mathbf{C}^m$ be a holomorphic mapping. Assume that in a open subset V of D, f has a *reduce representation* (resp. *presentation*) $f = (f_0 : \cdots : f_n)$, then we also call the mappings $\tilde{f} = (f_0, \cdots, f_n) : V \to \mathbf{C}^{n+1}$ a *reduce representation* (resp. *representation*) of f on V.

A sequence $\{f_k\}_{k=1}^{\infty}$ of meromorphic mappings of D into $\mathbf{P}^n(\mathbf{C})$ is said to meromorphically converge on D to f if and only if, for any $z \in D$, each f_k has a reduced representation $\widetilde{f_k} = (f_{k0}, \cdots, f_{kn})$ on some fixed neighborhood U of z such that $\{f_{ki}\}_{k=1}^{\infty}$ converges uniformly on compact subsets of U to a holomorphic function f_i $(0 \leqslant i \leqslant n)$ on U with the property that $(f_0, \cdots, f_n)$ is a representation of f in U.

Definition 55. *A family $\mathcal{F}$ of meromorphic mappings of a domain D in $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ is said to be meromorphically normal on D if any sequence in $\mathcal{F}$ has a meromorphically convergent subsequence on D.*

Denote by $\mathcal{H}_D$ the ring of all holomorphic functions on D. Let Q be a homogeneous polynomial in $\mathcal{H}_D[x_0, \ldots, x_n]$ of degree $d \geqslant 1$. Denote by $Q(z)$ the homogeneous polynomial over $\mathbf{C}$ obtained by substituting a specific point $z \in D$ into the coefficients of Q. We also call a moving hypersurface of $\mathbf{P}^n(\mathbf{C})$ each homogeneous polynomials $Q \in \mathcal{H}_D[x_0, \ldots, x_n]$

such that coefficients of Q have no common zero point. Let Ω be a subset of $\mathbf{C}^m$. We say that moving hypersurfaces $\{Q_j\}_{j=1}^q$ in $\mathbf{P}^n(\mathbf{C})$ are in general position (resp. in pointwise general position on Ω) if there exists $z \in \mathbf{C}^m$ (resp. for all $z \in \Omega$) such that for any $1 \leqslant j_0 < \cdots < j_n \leqslant q$ the system of equations

$$\begin{cases} Q_{j_i}(z)(w_0, \ldots, w_n) = 0 \\ \quad 0 \leqslant i \leqslant n \end{cases}$$

has only the trivial solution $w = (0, \ldots, 0)$ in $\mathbf{C}^{n+1}$.

Let f be a meromorphic mapping of a domain D in $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. For each moving hypersurface Q of $\mathbf{P}^n(\mathbf{C})$, we define the divisor $\mathrm{div}(f, Q)$ on D as follows: For each $a \in D$, let $\widetilde{f} = (f_0 : \cdots : f_n)$ be a reduced representation of f in a neighborhood U of a, then we put $\mathrm{div}(f, Q)(a) := \mathrm{div}(Q(\widetilde{f}))(a)$, where $Q(\widetilde{f}) := Q(f_0, \cdots, f_n)$. Somtimes we indentify $f^{-1}(Q)$ with the divisor $\mathrm{div}(f, Q)$. We say that f intersects Q on D with multiplicity at least k if $\mathrm{div}(f, Q)(z) \geqslant k$ for all $z \in \mathrm{supp}(\mathrm{div}(f, Q))$.

In 1974, H. Fujimoto introduced the concept of a meromorphically normal family into the complex projective space and gave the following result.

Theorem 56. *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain D in $\mathbf{C}^m$ into CP^n and let $\{H_j\}_{j=1}^{(2n+1)}$ be hyperplanes in $\mathbf{P}^n(\mathbf{C})$ in general position such that for each $f \in \mathcal{F}$, $f(D) \not\subset H_j$ $(j = 1, \cdots, 2n+1)$ and for any fixed compact subset K of D, the $2(m-1)$-dimensional Lebesque areas of $f^{-1}(H_j) \cap K$ $(j = 1, \cdots, 2n+1)$ with counting multiplicities for all f in $\mathcal{F}$ are bounded above. Then $\mathcal{F}$ is a meromorphically normal family on D.*

In 2005, Tu and Li [TL] extended the above theorem to the case of moving hyperplanes as follows:

Theorem 57. *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain D in $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ and let $\{H_j\}_{j=1}^q$ be q $(\geqslant 2n+1)$ moving hyperplanes in $\mathbf{P}^n(\mathbf{C})$ in pointwise general position on D such that each f in $\mathcal{F}$ intersects H_j on D with multiplicity at least m_j $(j = 1, \cdots q)$ where $m_1, \cdots, m_q$ are fixed positive integers and may be $+\infty$, with $\sum_{j=1}^q \frac{1}{m_j} < \dfrac{q-n-1}{n}$. Then $\mathcal{F}$ is a normal family on D.*

Theorem 58. *Let $\mathcal{F}$ be a family of meromorphic mappings of a domain D in $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ and let $\{H_j\}_{j=1}^{(2n+1)}$ be moving hyperplanes in $\mathbf{P}^n(\mathbf{C})$ in pointwise general position on D such that for any fixed compact subset K of D, the $2(m-1)$-dimensional Lebesque areas of $f^{-1}(H_j) \cap K$ $(j = 1, \cdots 2n+1)$ with counting multiplicities for all f in $\mathcal{F}$ are bounded above. Then $\mathcal{F}$ is a meromorphically normal family on D.*

A natural question arises here: *Are there normality criteria for families of meromorphic mappings for hypersufaces ?*

Concerning to this question, in 2005, D. D. Thai, P. N. Mai and P. N. T. Trang [TMT] had some nomarlity criteria for fixed hyperplanes with counting multiplicities, where they need atleast $2n+1$ hyperplanes for their purpose. We would like to note that to the present, all of researches about normality criteria for families of meromorphic mapings into $\mathbf{P}^n(\mathbf{C})$ have been still restricted to the case where the number of hyperplanes $q \geqslant (2n+1)$. In this section we will give some normality criteria for families of meromorphic mappings of a domain D in $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ with moving hypersurfaces. The first aim of ours is to generalize the above results to the case of moving hypersurfaces. Furthermore, we also obtain an improvement concerning counting multiplicities (Theorem 70). The second aim of ours is to find some normal criteria for the case of few moving hypersurfaces.

5.2.1 Some preparation theorems and lemmas.

Firstly, we need to state the First Main Theorem in Value Distribution Theory for hypersurface as follow:

Theorem 59. (First Main Theorem) *Let f be a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ and Q be a homogenous polynomial in $\mathbf{C}[x_0, \cdots, x_n]$ of degree $d \geqslant 1$, such that $Q(f) \not\equiv 0$. Then*

$$N(r, \operatorname{div}(f, Q)) \leqslant d \cdot T_f(r) + O(1) \quad \textit{for all } r > 1.$$

The next followings are due to W. Stoll [St84], Z. H. Tu - P. Li[TL] and H. Fujimoto [Fu74].

Definition 60. ([St84]) *Let M be a locally compact Hausdorff space. A point a of M is called a limit point of the sequence $\{E_k\}_{k=1}^{\infty}$ of closed subsets of M if there exist an positive integer k_0 and points $a_k \in E_k$ $(k \geqslant k_0)$ such that $a = \lim\ a_k$. A point of M is called a cluster point of $\{E_k\}_{k=1}^{\infty}$ if it is a limit point of some subsequence of $\{E_k\}_{k=1}^{\infty}$. If the set E of limit points coincides with the set of cluster points, $\{E_k\}_{k=1}^{\infty}$ is said to converge to E and write $\lim E_k = E$.*

Lemma 61. (Prop. 4.11, [St84]) *Let $\{N_i\}_{i=1}^{\infty}$ be a sequence of pure $(m-1)$-dimensional analytic subset of a domain D in $\mathbf{C}^m$. Asumme that the $2(m-1)$-dimensional Lebesgue areas of $N_i \cap K$ regardless of multiplicities $(i = 1, 2, \cdots)$ are bounded above for any fixed compact subset K of D and $\{N_i\}_{i=1}^{\infty}$ converges to N as a sequence of cosed subsets of D. Then N is either empty or a pure (m-1)-demensional analytic subset of D.*

Lemma 62. (Prop. 4.12, [St84]) *Let $\{N_i\}_{i=1}^{\infty}$ be a sequence of pure $(m-1)$-dimensional analytic subset of a domain D in $\mathbf{C}^m$. If the $2(m-1)$-dimensional Lebesgue areas of $N_i \cap K$ regardless of multiplicities $(i = 1, 2, \cdots)$ are bounded above for any fixed compact subset K of D, then $\{N_i\}$ is normal as a family of closed subsets of D.*

Definition 63. (Def. 4.4, [TL]) *Let $\{\nu_i\}_{i=1}^{\infty}$ be a sequence of non-negative divisors on a domain D in $\mathbf{C}^m$. It is said to converge to a non-negative divisor ν on D if and only if any $a \in D$ has a neighborhood U such that there exist nonzero holomorphic functions h and h_i on U with $\nu_i = \nu_{h_i}$ and $\nu = \nu_h$ on U such that $\{h_i\}_{i=1}^{\infty}$ converges to h uniformly on compact subsets of U.*

Lemma 64. (Theorem 2.24, [St84]) *A sequence* $\{\nu_i\}_{i=1}^{\infty}$ *of non-negative divisors on a domain* D *in* $\mathbf{C}^m$ *is normal in the sense of the convergece of divisors on* D *if and only if the* $2(m-1)$*-dimensional Lebesgue areas of* $\nu_i \cap E (i = 1, 2, \cdots)$ *with counting multiplicities are bounded above for any compact subset* E *of* D.

Lemma 65. (Prop. 3.5, [Fu74]) *Let* $\{f_i\}$ *be a sequence of meromorphic mappings of a domain* D *in* $\mathbf{C}^m$ *into* $\mathbf{P}^n(\mathbf{C})$ *and let* S *be a thin analytic subset of* D. *Suppose that* $\{f_i\}$ *meromorphically converges on* $D \setminus S$ *to a meromorphic mapping* f *of* $D \setminus S$ *into* $\mathbf{P}^n(\mathbf{C})$. *If there exists a hyperplane* H *in* $\mathbf{P}^n(\mathbf{C})$ *such that* $f(D \setminus S) \not\subset H$ *and* $\{\mathrm{div}(f_i, H)\}$ *is a convergent sequence of divisors on* D, *then* $\{f_i\}$ *is meromorphically convergent on* D

The next, we recall the Zalcman's lemma, which is due to G. Aladro - G. S. Krantz [AK] and D. D. Thai - P. N. T. Trang - P. D. Huong [TTH]. Fisrtly, we have the following definition:

Definition 66. (Def. 2.2, [TTH]) *Let* X, Y *be complex spaces and* $\mathcal{F} \subset Hol(X, Y)$.

i) *We say that a sequence* $\{f_j\} \subset \mathcal{F}$ *is compactly divergent if for every compact set* $K \subset X$ *and for every compact set* $L \subset Y$ *there is a number* $j_0(K, L)$ *such that* $f_j(K) \cap L =$ *for all* $j \geqslant j_0(K, L)$.

ii) *The family* $\mathcal{F}$ *is said to be not compactly divergent if* $\mathcal{F}$ *contains no compactly divergent subsequences.*

Lemma 67. (Theorem 2.5, [TTH]) *Let* D *be a domain in* $\mathbf{C}^m$ *and* M *be a complex Hermittian space. Let* $\mathcal{F} \subset Hol(D, M)$. *Then the family* $\mathcal{F}$ *is not normal if and only if there exist sequences* $\{p_j\} \subset D$ *with* $\{p_j\} \to p_0 \in D$, $\{f_j\} \subset \mathcal{F}$, $\{\rho_j\} \subset \mathbf{R}$ *with* $\rho_j > 0$ *and* $\{\rho_j\} \to 0$ *such that* $g_j(z) := f_j(p_j + \rho_j z), z \in \mathbf{C}^m$ *satisfies one of the following two assertions:*

i) *The sequence* $\{g_j\}$ *is compactly divergent on* $\mathbf{C}^m$.

ii) *The sequence* $\{g_j\}$ *converges uniformly on compact subsets of* $\mathbf{C}^m$ *to a nonconstant holomorphic map* $g : \mathbf{C}^m \to M$.

5.2.2 Nomaltity criteria for meromorphic mappings family

In this part, we will show our researches on normality criteria for family of meromorphic mappings from a domain in complex space $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ under condition on intesection of mappings with moving hypersurfaces.

Theorem 68. *Let* $\mathcal{F}$ *be a family of meromorphic mappings of a domain* D *in* C^m *into* $\mathbf{P}^n(\mathbf{C})$, *and let* $Q_1, \cdots, Q_q, (q \geqslant 2n+1)$ *be* q *moving hypersurfaces in* $\mathbf{P}^n(\mathbf{C})$ *in general position such that*

(i) *For any fixed compact subset* K *of* D, *the* $2(m-1)$*-dimensional Lebesgue areas of* $f^{-1}(Q_j) \cap K, (1 \leqslant j \leqslant n+1)$ *with counting multiplicities for all* f *in* $\mathcal{F}$ *are bounded above.*

(ii) *There exists a thin analytic subset* S *of* D, *such that for any fixed compact subset* K *of* D, *the* $2(m-1)$*-dimensional Lebesgue areas of* $f^{-1}(Q_j) \cap (K \setminus S)$, $(n+2 \leqslant j \leqslant q)$

regardless multiplicities for all f in $\mathcal{F}$ are bounded above. Then $\mathcal{F}$ is a meromorphically normal family on D.

Proof. Without lose of generaliy, we may assume that D is a polydisc in $\mathbf{C}^m$, $D = \Delta^m$. By replacing Q_i by $Q_i^{d_i}$ where d_i is a suitable positive integer for all $i = 1, \cdots, q$, we may assume that Q_i $(i = 1, \cdots, q)$ have the same degree d.

Let $\{f_k\}_{k=1}^{\infty} \subset \mathcal{F}$ be an arbitrary sequence. By Lemma 62, there exists a subsequence (again denoted by $\{f_k\}_{i=k}^{\infty}$) such that

$$\lim_{k\to\infty} f_k^{-1}(Q_i) = S_i \ (i = 1, \cdots q) \tag{5.2.1}$$

as a sequence of close subsets of D, where S_i are either empty or pure $(m-1)$-dimensional analytic sets of D. Set,

$$\mathcal{T}_d := \big\{(i_0, \ldots, i_n) \in \mathbf{N}_0^{n+1} : i_0 + \cdots + i_n = d\big\}.$$

Asumme that

$$Q_j = \sum_{I\in\mathcal{T}_d} a_{jI}x^I \quad (j = 1, \ldots, q)$$

where $a_{jI} \in \mathcal{H}_D$, $x^I = x_0^{i_0} \cdots x_n^{i_n}$ for $x = (x_0, \ldots, x_n)$ and $I = (i_0, \ldots, i_n)$.

Let $T = (\ldots, t_{kI}, \ldots)$ $(k \in \{1, \ldots, q\}$, $I \in \mathcal{T}_d)$ be a family of variables. Set

$$\widetilde{Q}_j = \sum_{I\in\mathcal{T}_d} t_{jI}x^I \in \mathbf{Z}[T, x], \quad j = 0, \ldots, n.$$

For each subset $L \subset \{1, \cdots, q\}$ with $\mid L \mid= n + 1$, take $\widetilde{R}_L \in \mathbf{Z}[T]$ is the resultant of $\widetilde{Q}_j (j \in L)$. Since $\{Q_j\}_{j\in L}$ are in general position, $\widetilde{R}_L(\ldots, a_{kI}, \ldots) \not\equiv 0$. Set $\widetilde{S} := \{z \in D : \widetilde{R}_L(\cdots, a_{kI}, \cdots) = 0$ for all $L \subset \{1, \cdots, q\}$ with$\mid L \mid= n+1\}$. Let $E := \bigcup_{i=1}^q S_i \bigcup S \bigcup \tilde{S}$. Then E is a thin analytic subset of D. For any fixed point $z_0 \in D \setminus E$, there exists a relatively compact Stain neighborhood U_{z_0} of z_0 in $D \setminus E$ and a positive integer k_0 such that for all $k \geqslant k_0$,

$$f_k^{-1}(Q_i) \cap U_{z_0} = \emptyset. \tag{5.2.2}$$

Then $\{f_k|_{U_{z_0}}\}_{k=k_0}^{\infty} \subset Hol(U_{z_0}, \mathbf{P}^n(\mathbf{C})^m)$. We now prove that $\{f_k|_{U_{z_0}}\}_{k=k_0}^{\infty}$ is a normal family on U_{z_0}. Indeed, suppose that $\{f_k|_{U_{z_0}}\}_{k=k_0}^{\infty}$ is not normal on U_{z_0}, then by Lemma 67 there exists a subsequence (again denoted by $\{f_k|_{U_{z_0}}\}_{k=k_0}^{\infty}$) and $p_0 \in U_{z_0}, \{p_k\}_{k\geqslant k_0}^{\infty} \in U_{z_0}$ with $p_k \to p_0, \{\rho_k\} \subset (0, +\infty)$ with $\rho_j \to 0$ such that the sequence of holomorphic maps

$$g_k(z) := f_k(p_k + \rho_k z) : \Delta_{r_j}^m :\to \mathbf{P}^n(\mathbf{C}), \ k \geqslant k_0 \ (r_k \to \infty)$$

converges uniformly on compact subsets of $\mathbf{C}^m$ to a nonconstant holomorphic map $g : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$. Then, there exists reduce representations $\tilde{g}_j = (g_{j0}, \cdots, g_{jn})$ of g_j and a representation $\tilde{g} = (g_0, \cdots, g_n)$ of g such that $\{\tilde{g}_j\}$ converges uniformly on compact subsets of $\mathbf{C}^m$ to $\tilde{g}$. This implies that $Q_j(p_k+\rho_k z)(\tilde{g}_k(z))$ converges uniformly on compact subsets of $\mathbf{C}^m$ to $\widetilde{Q}_j(z_0)(\tilde{g}(z))$. By (5.2.2) and Hurwitz's theorem, we have

i)$Q_j(z_0)(\tilde{g}) \neq 0$ on $\mathbf{C}^m$, i.e. $g(\mathbf{C}^m) \cap Q_j(z_0) = \emptyset$.

ii)$Q_j(z_0)(\tilde{g}) \equiv 0$ on $\mathbf{C}^m$, i.e. $g(\mathbf{C}^m) \subset Q_j(z_0)$

(we indentify the polynomial $Q_j(z_0) \in \mathbf{C}[x_0, \cdots, x_n]$ with the hypersurface in $\mathbf{P}^n(\mathbf{C})$ defined by $Q_j(z_0)$).

Denote by I the set of all indices $j \in \{1, \cdots, q\}$ such that $g(\mathbf{C}^m) \subset Q_j(p_0)$. Set $X = \cap_{j\in I} Q_j(p_0)$ if $I \neq \emptyset$, and $X = \mathbf{P}^n(\mathbf{C})$ if $I = \emptyset$. Since $\mathbf{C}^m$ is irreducible, there exists an irreducible component Z of x such that $g(\mathbf{C}) \subset Z \setminus \cup_{i\notin I} Q_j(z_0)$. Since $p_0 \in U_{z_0}$, we see that $\{Q_j(p_0)\}$ are in general position in $\mathbf{P}^n(\mathbf{C})$. This implies that $\{Q_j(p_0) \cap Z\}_{j\notin I}$ are in general position in Z. Since $q \geqslant 2n+1$, it easy to see that $\sharp\{1, \cdots, q\} \setminus I\} \geqslant 2\dim Z - 1$. From these facts, by corollary 1.4 in [NW02], we infer that $Z \setminus \cup_{i\notin I} Q_j(p_0)$ is hyperbolic. Hence, g is constant. This is a contracdiction, hence $\{f_k|_{U_{z_0}}\}_{k=k_0}^{\infty}$ is a normal family on U_{z_0}.

By the usual diagonal argument, we can find a subsequence (again denoted by $\{f_k\}_{k=1}^{\infty}$) which converges uniformly on compact subset of $D \setminus E$ to a holomorphic map f. Since $\{Q_j\}_{j=1}^{n+1}$ are in general position, there exists a fixed index j_0 $(1 \leqslant j_0 \leqslant n+1)$ such that $Q_{j_0}(\tilde{f}) \not\equiv 0$ on $D \setminus E$.

We define meromorphic mappings $\{F_k\}_{k=1}^{\infty}$ of D into $\mathbf{P}^{n+1}(\mathbf{C})$ as follows: for any $z \in D$, if f_k has a reduce representation $\tilde{f}_k = (f_{k0}, \cdots, f_{kn})$on a neighborhood $U_z \subset D$ then F_k has a reduce representation $\tilde{F}_k = (f_{k0}^d, \cdots, f_{kn}^d, Q_{j_0}(\tilde{f}_k))$ on U_z. Let $H_i (i = 0, \cdots, n)$ are hyperplanes in $\mathbf{P}^n(\mathbf{C})$ defined by

$$H = \{(w_0 : \cdots w_n) | w_i = 0\}$$

and let $\overline{H}_i (i = 0, \cdots, n+1)$ are hyperplanes in $\mathbf{P}^{n+1}(\mathbf{C})$ defined by

$$\overline{H}_i = \{(w_0 : \cdots w_{n+1}) | w_i = 0\}.$$

It is easy to see that $\{F_k\}$ converges uniformly on compact subset of $D \setminus E$ to a holomorphic map F of $D \setminus E$ into $\mathbf{P}^{n+1}(\mathbf{C})$, and if f has a reduce representation $\tilde{f} = (f_0, \cdots, f_n)$ on an open subset $U \subset D$ then F has reduce representation $\tilde{F} = (f_0^d, \cdots, f_n^d, Q_{j_0}(\tilde{f}))$ on U. Since f is holomorphic on $D \setminus E$, there exists i_0 $(0 \leqslant i_0 \leqslant n)$ such that $H_{i_0}(\tilde{f}) \not\equiv 0$ on $D \setminus E$, and hence $\overline{H}_{i_0}(\widetilde{F}) \not\equiv 0$ on $D \setminus E$. Then there exists $k_0 > 0$ such that $H_{i_0}(\tilde{f}_k) \not\equiv 0$, $\overline{H}_{i_0}(\widetilde{F}_k) \not\equiv 0$ on $D \setminus E$ for all $k \geqslant k_0$.

Since $Q_{j_0}(\widetilde{f}) \not\equiv 0$ on $D \setminus E$, we have $\overline{H}_{n+1}(\widetilde{F}) \not\equiv 0$ on $D \setminus E$. By Lemma 62 and Lemma 64, we may assume that $F_k^{-1}(\overline{H}_{n+1})$ converges in the sense of convergence of divisors on D to a divisor (note that $1 \leqslant j_0 \leqslant n+1$). By Lemma 65, $\{F_k\}$ is meromorphically convergent on D. It implies that $\{F_k^{-1}(\overline{H}_{i_0})\}_{k\geqslant k_0}$ converges, and hence $\{f_k^{-1}(H_{i_0})\}_{k\geqslant k_0}$ converges in the sense of convergent of divisors on D. By again Lemma 65, we get that $\{f_k\}_{k\geqslant k_0}$ is meromorphically convergent on D. Therefore $\{f_k\}$ has a meromorphically convergent subsequence on D. Then $\mathcal{F}$ is a meromorphically normal family on D. We have completed the proof of Theorem 68. ■

To continue, we need estabish a Second Main Theorem for a meromorphic mapping and hypersurfaces.

Lemma 69. *Let $f : \mathbf{C}^m :\longrightarrow \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping. Let $Q_0, \cdots, Q_n$ be $n+1$ homogeneous polynomials of $\mathbf{C}[x_0, \cdots, x_n]$ of common degree $d \geqslant 1$. Assume that hypersurfaces defined by $Q_0, \cdots, Q_n$ in $\mathbf{P}^n(\mathbf{C})$ have no common point. We defined homogeneous polynomials $L_1, \cdots, L_n$ by*

$$L_i = \sum_{j=0}^{n} \lambda_{ij} Q_j^p,$$

where λ_{ij} are constants such that all sub-matries of the matrix $(\lambda_{ij})_{\substack{0\leqslant j\leqslant n\\ 1\leqslant i\leqslant n}}$ are nondegenerate and p is a positive integer $(p > n(n+1))$. Denote by F the meromormorphic mapping $F = (Q_0(\widetilde{f}) : \cdots : Q_n(\widetilde{f})) : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$. Then, we have

1) $T_F(r) = dT_f(r) + O(1)$.
2) $\| \; T_f(r) \leqslant \dfrac{1}{(p-n(n+1))d} \displaystyle\sum_{L_i(f)\not\equiv 0} N_n(r, \operatorname{div}(f, L_i)) + o(T_f(r))$.

Proof. Let $\tilde{f} = (f_0, \cdots, f_n)$ be a reduce representation of f. It is clear that $(Q_0(f), \cdots, Q_n(f))$ is a reduced representation of F. Then we have

$$\|F\| = \max\{| \; Q_i(f) \; |, i = 0, \cdots, n\} \leqslant c_1 \cdot \|f\|^d. \tag{5.2.3}$$

where c_1 is a positive constant.
Since the hypersurfaces defined by $\{Q_i\}_{i=0}^n$ in $\mathbf{P}^n(\mathbf{C})$ have no common point, by Hilbert's Nullsellensatz there exists positive integer $s \geqslant d$ such that

$$x_i^s = \sum_{i=0}^{n} R_i \cdot Q_i, \quad i \in \{1, \cdots, n\} \tag{5.2.4}$$

where $R_0, \cdots, R_n$ are zero or homogenous polinomials with degree $s - d$.
Then

$$f_i^s = \sum_{i=0}^{n} R_i(f) Q_i(f) \text{ for all } \quad i \in \{0, \cdots, n\}.$$

Then there is a positive constant c_2 such that

$$| \; f_i \; |^s \leqslant \sum_{i=0}^{n} | \; R_i(f) \; | \cdot \max_{i=0,\cdots,n} | \; Q_i(f) \; | \leqslant c_2 \|f\|^{s-d} \cdot \max_{i=0,\cdots,n} | \; Q_i(f) \; |$$

for all $i \in \{0, \cdots, n\}$. This implies that

$$\|f\|^d \leqslant c_2 \|F\|. \tag{5.2.5}$$

By (5.2.4) and (5.2.5) we have

$$T_F(r) = dT_f(r) + O(1).$$

Let $F : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping which has a reduce representation

$$\widetilde{F} = (Q_0^p(\widetilde{f}), \cdots, Q_n^p(\widetilde{f})).$$

By the assertion 1) we have

$$pdT_f(r) = T_F(r) + O(1), \text{ where } d = \deg Q_i. \tag{5.2.6}$$

We define hyperplanes $\{H_i\}_{i=0}^{2n}$ in $\mathbf{P}^n(\mathbf{C})$ by

$$H_i := \{\sum_{j=0}^{n} a_{ij} w_j = 0\}, \ (i = 0, \cdots, 2n) \quad \text{where}$$

$$\begin{aligned} a_{ij} &= 0 && \text{if } i \leqslant n, \ i \neq j \\ a_{ij} &= 1 && \text{if } i \leqslant n, \ i = j \\ a_{ij} &= \lambda_{(i-n)j} && \text{if } i \geqslant n+1. \end{aligned}$$

Since, all sub-matries of the matrix (λ_{ij}) are nondegenerate, we have that hyperplanes $\{H_i\}_{i=0}^{2n}$ are in general position. It is easy to see that $H_j(\widetilde{F}) = Q_j^p(\widetilde{f})$, for all $j \in \{0, \cdots, n\}$ and $H_j(\widetilde{F}) = L_{j-n}(f)$, for all $j \in \{n+1, \cdots, 2n\}$. We assume that $H_{j_i}(\widetilde{F}) \equiv 0$ with $i \in \{0, \cdots, k\}$ and $H_{j_i}(\widetilde{F}) \not\equiv 0$ with $i \in \{k+1, \cdots, 2n\}$, where $\{j_0, \cdots, j_{2n}\} = \{0, \cdots, 2n\}$. Since $\{H_i\}_{i=0}^{2n}$ are in general position, we have that $k \leqslant n-1$.

If $k = n-1$ then F is constant map. By (5.2.6), we have

$$T_f(r) = \frac{1}{pd} T_F(r) = 0.$$

Then, the lemma is proved.

If $k < n-1$. Let $G : \mathbf{C}^m \to \mathbf{P}^{n-k-1}(\mathbf{C})$ be the meromorphic mapping which has a reduce representation

$$\widetilde{G} = (H_{j_{k+1}}(\widetilde{F}), \cdots, H_{j_n}(\widetilde{F})),$$

Since $\{H_i\}_{i=0}^{2n}$ are in general position and $H_{j_i}(F) \equiv 0$ with $i \in \{0, \cdots, k\}$, we have

$$T_G(r) = T_F(r) + O(1) \tag{5.2.7}$$

We define hyperplanes $\{\widetilde{H}_i\}_{i=k+1}^{2n}$ in $\mathbf{P}^{n-k-1}(\mathbf{C})$ by

$$\widetilde{H}_i = \{ \sum_{j=0}^{n-k-1} b_{(i+k+1)j} w_j = 0\}, \ i = k+1, \cdots, 2n \tag{5.2.8}$$

where b_{ij} are constants satisfying

$$\begin{pmatrix} a_{j_i0} \\ \vdots \\ a_{j_in} \end{pmatrix} = \begin{pmatrix} a_{j_00} & \cdots & a_{j_n0} \\ \vdots & \vdots & \vdots \\ a_{j_0n} & \cdots & a_{j_nn} \end{pmatrix} \begin{pmatrix} b_{i0} \\ \vdots \\ b_{in} \end{pmatrix}.$$

Then we have that $\{\widetilde{H}_i\}_{i=k+1}^{2n}$ are in general position. We note that $2n-k > 2(n-k-1)+1$, so by Corolarry 15 and by the First Main Theorem, we have

$$\begin{aligned}
\| \; T_G(r) &\leqslant \frac{2n-k}{2(n-k-1)+1} T_G(r) \leqslant \sum_{i=k+1}^{2n} N_{n-k-1}(r, \mathrm{div}(\widetilde{H}_i(\widetilde{G}))) + o(T_G(r)) \\
&= \sum_{i=k+1}^{2n} N_{n-k-1}(r, \mathrm{div}(H_{j_i}(\widetilde{F}))) + o(T_G(r)) \\
&\leqslant \sum_{Q_i(\widetilde{f})\not\equiv 0} N_{n-k-1}(r\mathrm{div}(Q_i^p(\widetilde{f}))) + \sum_{L_i(f)\not\equiv 0} N_{n-k-1}(r, \mathrm{div}(L_i(f))) + o(T_G(r)) \\
&\leqslant \sum_{Q_i(\widetilde{f})\not\equiv 0} \frac{n-k-1}{p} N(r, \mathrm{div}(Q_i^p(\widetilde{f}))) + \sum_{L_i(\widetilde{f})\not\equiv 0} N_{n-k-1}(r, \mathrm{div}(L_i(f))) + o(T_G(r)) \\
&\leqslant \frac{n(n+1)}{p} T_F(r) + \sum_{L_i(\widetilde{f})\not\equiv 0} N_n(r, \mathrm{div} L_i(\widetilde{f}))) + o(T_G(r)) \\
&\leqslant \frac{n(n+1)}{p} T_G(r) + \sum_{L_i(\widetilde{f})\not\equiv 0} N_n(r, \mathrm{div}(L_i(\widetilde{f}))) + o(T_F(r)).
\end{aligned}$$

Then

$$\| \; T_G(r) \leqslant \frac{p}{p-n(n+1)} \sum_{L_i(\widetilde{f})\not\equiv 0} N_n(r.\mathrm{div}(L_i(\widetilde{f}))) + o(T_G(r)). \tag{5.2.9}$$

By (5.2.6) and (5.2.7) and (5.2.9), we have

$$\begin{aligned}
\| \; T_f(r) &= \frac{1}{pd} T_F(r) + O(1) \\
&= \frac{1}{pd} T_G(r) + O(1) \\
&\leqslant \frac{1}{(p-n(n+1))d} \sum_{L_i(\widetilde{f})\not\equiv 0} N_n(r, \mathrm{div}(\widetilde{f}, L_i)) + o(T_f(r)).
\end{aligned}$$

We have completed the proof of Lemma 4.2. ■

Theorem 70. *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain D in C^m into $\mathbf{P}^n(\mathbf{C})$, and let $Q_0, \cdots, Q_n$ are $n+1$ are moving hypersurfaces in $\mathbf{P}^n(\mathbf{C})$ in pointwise general position on D of common degree $d \geqslant 1$. We define moving hypersurfaces $L_1, \cdots, L_n$ in $\mathbf{P}^n(\mathbf{C})$ by*

$$L_i = \sum_{j=0}^{n} a_{ij} Q_j^p,$$

where p is a fixed positive integer $(p > n(n+1))$ and $a_{ij} (0 \leqslant i, j \leqslant n)$ are holomorphic functions on D,such that for any square sub-matrix A of the matrix $(a_{ij})_{0 \leqslant i,j \leqslant n}$, $detA \neq 0$ on D.

Assume that each f in $\mathcal{F}$ intersects L_i on D with multiplicity at least m_i, where $m_1, \cdots, m_n$ are fixed positive integers and may be ∞, with

$$\sum_{i=1}^{n} \frac{1}{m_i} < \frac{(p - n(n+1))d}{np}.$$

Then $\mathcal{F}$ is a normal family.

Proof. Without lose of generality, we may assume that D is a polydisc in $\mathbf{C}^m$, $D = \Delta^m$. Suppose that $\mathcal{F}$ is not normal on D. Then, by Lemma 67 there exist a subsequence denoted by $\{f_k\}_{k=1}^{\infty}$ and $p_0 \in D, \{p_k\}_{k \geqslant 1}^{\infty} \in D$ with $p_k \to p_0, \{\rho_k\} \subset (0, +\infty)$ with $\rho_j \to 0$ such that the sequence of holomorphic maps

$$g_k(z) := f_k(p_k + \rho_k z) : \Delta_{r_k}^m \to \mathbf{P}^n(\mathbf{C}),\ k \geqslant k_0\ (r_k \uparrow \infty)$$

converges uniformly on compact subsets of $\mathbf{C}^m$ to a nonconstant holomorphic map $g : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$.

We call $(\omega_0 : \cdots : \omega_n)$ the homogeneous coordinate of $\mathbf{P}^n(\mathbf{C})$. For any fixed point $z_0 \in \mathbf{C}^m$, there exists a ball $B(z_0; r_0) = \{z \in \mathbf{C}^m \ :\ ||z - z_0|| \leqslant r_0\}$ in $\mathbf{C}^m$ and an index i such that $g(B(z_0; r_0) \subset \{\omega_i \neq 0\}$. We may assume that the index $i = 0$. There exists k_0 such that $B(z_0; r_0) \subset \Delta_{r_k}^m$, for all $k > k_0$. Therefor, there exists a reduced representations $\tilde{g}_k = (1, g_{k1}, \cdots, g_{kn})$ of g_k on $B(z_0; r_0)$ and a reduced representation $\tilde{g} = (1, g_1, \cdots, g_n)$ of g on $B(z_0; r_0)$. Because of the convergent of $\{g_k\}_{k > k_0}$ on $B(z_0; r_0)$, $\{g_{ki}\}$ converges uniformly on compact subsets of $\Delta_{r_k}^m$ to g_i for all $i \leqslant 1$. This implies that $Q_j(p_k + \rho_k z)(\tilde{g}_k(z))$ and $L_j(p_k + \rho_k z)(\tilde{g}_k(z))$ converge uniformly on compact subsets of $\Delta_{r_k}^m$ to $Q_j(p_0)(\tilde{g}(z))$ and $L_j(p_0)(\tilde{g}(z))$ respectively. On the other hand each f in $\mathcal{F}$ intersects L_i on D with multiplicity at least m_i. So, by Hurwitz's theorem, we have or $\widetilde{L}_j(p_0)(\tilde{g}) \equiv 0$ or all zero points of $\widetilde{L}_j(p_0)(\tilde{g})$ have multiplicity at least m_j $(j = 1, \cdots, n)$. By Lemma 69, we have

$$\begin{aligned} || \ T_g(r) &\leqslant \frac{1}{(p - n(n+1))d} \sum_{L_j(p_0)(\tilde{g}) \not\equiv 0} N_n(r, \mathrm{div}(\tilde{g}, L_j)) + o(T_g(r)) \\ &\leqslant \frac{n}{(p - n(n+1))d} \sum_{L_j(p_0)(\tilde{g}) \not\equiv 0} \frac{1}{m_j} N(r, \mathrm{div}(g, L_j)) + o(T_g(r)) \end{aligned}$$

On the other hand by the First Main Theorem, we have

$$N(r, \operatorname{div}(\tilde{g}, L_j(p_0))) \leqslant pdT_g(r) + o(T_g(r)).$$

Thus, we get

$$|| \; T_g(r) \leqslant \frac{np}{(p-n(n+1))d} \sum_{i=1}^{n} \frac{1}{m_i} T_g(r) + o(T_g(r))$$

Letting $r \longrightarrow +\infty$, we obtain

$$\sum_{i=1}^{n} \frac{1}{m_i} \geqslant \frac{(p-n(n+1))d}{np}$$

This is imposible. Hence $\mathcal{F}$ is normal on D. We have completed the proof of Theorem 70. ■

Theorem 71. *Let $\mathcal{F}$ be a family of meromorphic mappings of a domain D in C^m into $\mathbf{P}^n(\mathbf{C})$, and let $Q_0, \cdots, Q_n$ are $n+1$ are moving hypersurfaces in $\mathbf{P}^n(\mathbf{C})$ in general position of common degree $d \geqslant 1$. We define moving hypersurfaces $L_1, \cdots, L_n$ in $\mathbf{P}^n(\mathbf{C})$ by*

$$L_i = \sum_{j=0}^{n} a_{ij} Q_j^p,$$

where p is a fixed positive integer $(p > n(n+1))$ and $a_{ij}(0 \leqslant i, j \leqslant n)$ are holomorphic functions on D, such that for any square sub-matrix A of the matrix $(a_{ij})_{0 \leqslant i,j \leqslant n}$, $detA \not\equiv 0$. Assume that for any fixed compact subset K of D, the $2(m-1)$-dimensional Lebesgue areas of $f^{-1}(L_j) \cap K, (1 \leqslant i \leqslant n)$ and $f^{-1}(Q_0) \cap K$ with counting multiplicities for all f in $\mathcal{F}$ are bounded above. Then $\mathcal{F}$ is a meromorphically normal family.

Proof. Without lose of generaliy, we may assume D is a polydisc in $\mathbf{C}^m$, $D = \Delta^m$.

Let $\{f_k\}_{k=1}^{\infty} \subset \mathcal{F}$ be an arbitrary sequence. By Lemma 5.2.3, we can fine a subsequence (again denoted by $\{f_k\}_{k=1}^{\infty}$) such that

$$\lim_{k\to\infty} f_k^{-1}(L_i) = S_i \; (i = 1, \cdots, n) \text{ and } \lim_{k\to\infty} f_k^{-1}(Q_0) = S_0 \qquad (5.2.10)$$

as a sequence of close subsets of D, where $S_i (i = 0, \cdots, n)$ are either empty or pure $(m-1)$-dimensional analytic sets of D. Because all sub-matries of the matrix $(a_{ij})_{\substack{1 \leqslant j \leqslant n \\ 0 \leqslant i \leqslant n}}$ are non-degeneral, then $\{Q_0, \cdots, Q_n, L_1, \cdots, L_n\}$ are in general position. It is easy to see that there exist an analytic set S of codimension at least 1 such that $\{Q_0, \cdots, Q_n, L_1, \cdots, L_n\}$ are in pointwise general position on $D \setminus S$ and all sub-matries of the matrix $(a_{ij})_{\substack{1 \leqslant j \leqslant n \\ 0 \leqslant i \leqslant n}}$ are pointwise non-degeneral on $D \setminus S$. Set $E := \bigcup_{i=0}^{n} S_i \bigcup S$. Then E is a thin analytic subset of D.

For any fixed point $z_0 \in D \setminus E$, by (5.2.10) there exist a open ball $B(z_0, r_0) \subset D \setminus E$ and a positive integer k_0 such that $L_i(\tilde{f}_k)$ $(i = 1, \cdots, n)$ and $Q_0(\tilde{f}_k)$ have no zero point on $B(z_0, r_0)$ for all $k > k_0$. By Theorem 70, $\{f_k\}_{k>k_0}$ is holomorphically normal family on $B(z_0, r_0)$. Hence, we get that $\{f_k\}$ has a subsequence which converges uniformly on compact subset of $D \setminus E$ to a holomorphic map.

By the usual diagonal argument, we can find a subsequence (again denoted by $\{f_k\}_{k=1}^{\infty}$) which converges uniformly on compact subsets of $D \setminus E$ to a holomorphic map f. We denote L_{n+1} by the moving hypersurface Q_0. Because $\{L_i\}_{i=1}^{n+1}$ are in pointwise general position on $D \setminus E$, there exists $i_0 \in \{1, \cdots, n+1\}$ such that $L_{i_0}(\tilde{f}) \not\equiv 0$ on $D \setminus E$. We define the meromorphic mappings $\{F_k\}_{k=1}^{\infty}$ of D into $\mathbf{P}^{n+1}(\mathbf{C})$ as follows: for any $z \in D$, if f_k has a reduce representation $\tilde{f}_k = (f_{k0}, \cdots, f_{kn})$ on a neighbourhood $U_z \subset D$ then F_k has a reduce representation $\tilde{F}_k = (f_{k0}^d, \cdots, f_{kn}^d, L_{i_0}(\tilde{f}_k))$ on U_z. By the same argument in the proof of Theorem 68, we have $\{F_k\}_{k=1}^{\infty}$ is a meromorphically convergent sequence on D and $\{f_k\}_{k=1}^{\infty}$ has a meromorphically convergent subsequence on D.

Thus, $\mathcal{F}$ is meromorphicall normal family on D. We have completed the proof of Theorem 71. ■

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