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The $W^{k,p}$***-continuity of wave operators for Schrödinger operators III, even dimensional cases*** $m \geq 4$

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Abstract. Let $H = -\Delta + V(x)$ be the Schrödinger operator on $\mathbf{R}^m$, $m \geq 3$. We show that the wave operators $W_\pm = \lim_{t\to\pm\infty} e^{itH} \cdot e^{-itH_0}$, $H_0 = -\Delta$, are bounded in Sobolev spaces $W^{k,p}(\mathbf{R}^m)$, $1 \leq p \leq \infty$, $k = 0, 1, \ldots, \ell$, if V satisfies $\|D^\alpha V(y)\|_{L^{p_0}(|x-y|\leq 1)} \leq C(1 + |x|)^{-\delta}$ for $\delta > (3m/2) + 1$, $p_0 > m/2$ and $|\alpha| \leq \ell + \ell_0$, where $\ell_0 = 0$ if $m = 3$ and $\ell_0 = [(m-1)/2]$ if $m \geq 4$, $[\sigma]$ is the integral part of σ. This result generalizes the author's previous result which appears in J. Math. Soc. Japan 47, where the theorem is proved for the odd dimensional cases $m \geq 3$ and several applications such as L^p-decay of solutions of the Cauchy problems for time-dependent Schrödinger equations and wave equations with potentials, and the L^p-boundedness of Fourier multiplier in generalized eigenfunction expansions are given.

1. Introduction

Let $H_0 = D_1^2 + \cdots + D_m^2$, $D_j = -i\partial/\partial x_j$, be the free Schrödinger operator on $L^2(\mathbf{R}^m)$ and $H = H_0 + V$ its perturbation by the multiplication operator V with a real valued function $V(x)$. It is well known in the scattering theory (cf. [1], [3], [9]) that, if V is of short range in the sense that $\int_1^\infty \|F_R V(H_0+1)^{-1}\| dR < \infty$, where F_R is the multiplication with the characteristic function of $\{x \in \mathbf{R}^m : |x| \geq R\}$, then the wave operators $W_\pm$ defined by

$$W_\pm u = \lim_{t\to\pm\infty} e^{itH} e^{-itH_0} u\,, \quad u \in L^2(\mathbf{R}^m)$$

exist and they are isometries on $L^2(\mathbf{R}^m)$ with the final set $L_c^2(H)$, the continuous spectral subspace for H. The wave operators satisfy the intertwining property: $f(H)W_\pm = W_\pm f(H_0)$ for Borel functions f and they play important roles in the perturbation theory of continuous spectra as well as in the scattering theory ([14]).

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In [21] and [22], we showed that $W_\pm$ are in fact bounded in Sobolev spaces $W^{\ell,p}(\mathbf{R}^m)$:

$$W^{\ell,p}(\mathbf{R}^m) = \{f \in L^p(\mathbf{R}^m) : \sum_{|\alpha|\le\ell} \|D^\alpha f\|_{L^p}^p \equiv \|f\|_{W^{\ell,p}}^p < \infty\},$$

if either (1) the spatial dimension $m \ge 3$ is odd, or (2) $m \ge 4$ is even and V is small or $V(x) \ge 0$, where for $\alpha = (\alpha_1, \dots, \alpha_m)$, $D^\alpha = D_1^{\alpha_1} \cdots D_m^{\alpha_m}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_m$. More precisely, we proved the following theorem, where $\ell \ge 0$ is an integer and $m_* = (m-1)/(m-2)$. $\mathcal{F}$ is the Fourier transform, $\langle x\rangle = (1+|x|^2)^{1/2}$ and $H^s(\mathbf{R}^m) = W^{s,2}(\mathbf{R}^m)$.

Theorem 1.1 ([21], [22]). *Let $m \ge 3$. Let V be a real valued function such that, for some $\sigma > 2/m_*$, $\mathcal{F}(\langle x\rangle^\sigma D^\alpha V) \in L^{m_*}(\mathbf{R}^m)$ for $|\alpha| \le \ell$, and satisfy one of the following conditions:*

1. *$\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m_*}(\mathbf{R}^m)}$ is sufficiently small;*
2. *$m = 2m' - 1$ is odd and, with $\delta > \max(m+2, 3m/2 - 2)$, $|D^\alpha V(x)| \le C_\alpha \langle x\rangle^{-\delta}$ for $|\alpha| \le \max\{\ell, \ell + m' - 4\}$;*
3. *m is even, $V(x) \ge 0$ and, with $\delta > 3m/2 + 1$, $|D^\alpha V(x)| \le C_\alpha \langle x\rangle^{-\delta}$ for $|\alpha| \le m + \ell$.*

Suppose in addition that zero is neither eigenvalue nor resonance of H. Then, the wave operators $W_\pm$ are bounded in $W^{k,p}(\mathbf{R}^m)$ for any $k = 0, \dots, \ell$ and $1 \le p \le \infty$,

Remark 1. Zero is said to be resonance of H if the equation $-\triangle u(x) + V(x)u(x) = 0$ has a solution $u \notin L^2(\mathbf{R}^m)$ such that $(1+|x|)^{-1-\varepsilon} u \in L^2(\mathbf{R}^m)$ for any $\varepsilon > 0$. If zero is resonance or eigenvalue of H, $W_\pm$ can not be bounded in L^p for all $1 \le p \le \infty$ (cf. [21]). It is known that H does not admit zero resonance if $m \ge 5$ or $V(x) \ge 0$.

Theorem 1.1, however, does not cover the case that the spatial dimension m is even and $V(x)$ can be large negative. The main purpose of this paper is to fill this gap and prove the following theorem, where $\ell \ge 0$ is an arbitrarily fixed integer; $p_0 > m/2$ and $\ell_0 = [(m-1)/2]$ if $m \ge 4$; and $p_0 = 2$ and $\ell_0 = 0$ if $m = 3$. $[\sigma]$ is the integral part of σ.

Theorem 1.2. *Let $m \geq 3$. Suppose that $V(x)$ is real valued and, with $\delta > (3m/2) + 1$,*

$$\sup_{x \in \mathbf{R}^m} \langle x \rangle^\delta \left(\int_{|x-y| \leq 1} |D^\alpha V(y)|^{p_0} dy \right)^{1/p_0} < \infty \tag{1.1}$$

for $|\alpha| \leq \ell + \ell_0$. Suppose further that zero is neither eigenvalue nor resonance of H. Then, $W_\pm$ are bounded in $W^{k,p}(\mathbf{R}^m)$ for any $k = 0, \ldots, \ell$ and $1 \leq p \leq \infty$.

Remark 2. Theorem 1.2 is a generalization of Theorem 1.1 when m is even and V is large, however, none of them is stronger than the other otherwise. We remark that under the condition of Theorem 1.2 it is possible to find $\sigma > 2/m_*$ such that $\mathcal{F}(\langle x \rangle^\sigma D^\alpha V) \in L^{m_*}(\mathbf{R}^m)$ for $|\alpha| \leq \ell$.

We refer to [21] for various applications of Theorems and the related reference, and shall be devoted to the proof of Theorem 1.2 in this paper. We shall only prove the L^p boundedness of W_+ assuming $\ell = 0$ and m is even ≥ 4. The odd dimensional cases may be proved by slightly modifying the following argument or by the method of [21]; the proof for W_- is similar; and the extension to general ℓ may be done by estimating the multiple commutators $[D_{j_1}, [D_{j_2}, \cdots [D_{j_\ell}, W_+] \cdots]]$ as in section 5 of [21].

We outline the proof here, displaying the plan of this paper and introducing some notations. $B(X, Y)$ is the Banach space of bounded operators from Banach space X to Y and $B(X) = B(X, X)$. $R(z) = (H - z)^{-1}$, $R_0(z) = (H_0 - z)^{-1}$ are resolvents and $R^\pm(\lambda) = R(\lambda \pm i0)$, $R_0^\pm(\lambda) = R_0(\lambda \pm i0)$ are their boundary values on the upper and lower banks of $\mathbf{C} \setminus [0, \infty)$. By using the stationary representation formula ([9], [14]):

$$W_+ u = u - \frac{1}{2\pi i} \int_0^\infty R^-(\lambda) V \{R_0^+(\lambda) - R_0^-(\lambda)\} u d\lambda$$

and the identity $R^-(\lambda) = R_0^-(\lambda) - R_0^-(\lambda) V R^-(\lambda)$, we write $W_+ u = u + W_1 u + W_2 u$, where

$$W_1 u = -\frac{1}{2\pi i} \int_0^\infty R_0^-(\lambda) V \{R_0^+(\lambda) - R_0^-(\lambda)\} u d\lambda, \tag{1.2}$$

$$W_2 u = \frac{1}{2\pi i} \int_0^\infty R_0^-(\lambda) V R^-(\lambda) V \{R_0^+(\lambda) - R_0^-(\lambda)\} u d\lambda. \tag{1.3}$$

In the first half of section 2, we study the mapping property of $R_0^{\pm}(\lambda)$ and the decay and smoothness properties of the integral kernels of $R(0)$ and $\phi(H)$ for $\phi \in C_0^\infty(\mathbf{R})$. As we think them of independent interest, these properties will be stated and proved under much weaker assumptions on V than necessary in what follows. We then recall from [21] the argument that proves W_1 is bounded in L^p: Express W_1 explicitly in the form

$$W_1 u(x) = \int_\Sigma d\omega \int_{2x\omega}^\infty \widehat{K}_V(t,\omega) u(t\omega + x_\omega) dt\,, \tag{1.4}$$

where Σ is the unit sphere, $x_\omega = x - 2(x\omega)\omega$ is the reflection of x along the ω-axis and

$$\widehat{K}_V(t,\omega) = \frac{i}{2(2\pi)^{m/2}} \int_0^\infty \hat{V}(r\omega) r^{m-2} e^{itr/2} dr\,;$$

it follows by Minkowski inequality and the fact that $x \to x_\omega$ is measure preserving that for any $\sigma > 1/2$,

$$\begin{aligned} \|W_1 u\|_{L^p} &\le 2\|\widehat{K}_V\|_{L^1([0,\infty)\times\Sigma)} \|u\|_{L^p} \\ &\le C\|\langle x\rangle^\sigma V\|_{H^{(m-3)/2}} \|u\|_{L^p} \le C'\|u\|_{L^p}. \end{aligned} \tag{1.5}$$

We wish to show that W_2 is bounded in L^p by proving the well known criterion:

$$\max\left\{ \sup_{x\in\mathbf{R}^m} \int_{\mathbf{R}^m} |W_2(x,y)| dy, \quad \sup_{y\in\mathbf{R}^m} \int_{\mathbf{R}^m} |W_2(x,y)| dx \right\} < \infty \tag{1.6}$$

for its integral kernel $W_2(x,y)$. It can be written as

$$W_2(x,y) = \frac{1}{2\pi i} \int_0^\infty \langle R^-(k^2) V(G_{+,y,k} - G_{-,y,k}), VG_{+,x,k} \rangle dk^2, \tag{1.7}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ is a coupling between suitable function spaces and $G_{\pm,y,k}(x) = G_\pm(x-y,k)$ are the kernels of $R_0^\pm(k^2)$ or the incoming-outgoing fundamental solutions of $-\triangle - k^2$. They satisfy $G_\pm(x,k) \sim Ce^{\pm ik|x|}|x|^{-(m-1)/2} k^{(m-3)/2}$ as $|x| \to \infty$ and crude estimations would only yield

$$|\text{the integrand of (1.7)}| \le Ck^{m-3} \langle x\rangle^{-(m-1)/2} \langle y\rangle^{-(m-1)/2}. \tag{1.8}$$

Thus we are faced with the two difficulties:

(1) **High energy difficulty**: The integral (1.7) does not converge absolutely at $k = \infty$;

(2) **Low energy difficulty**: If we restrict the integral (1.7) to finite intervals, (1.8) produces only $|W_2(x,y)| \le C\langle x\rangle^{-(m-1)/2}\langle y\rangle^{-(m-1)/2}$ which is insufficient for (1.6). For obtaining improved decay property, we exploit the oscillation property of $G_\pm(x,k)$ and apply integration by parts with respect to the variable k. However, the singularity at $k=0$ of $G_\pm(x,k)$ prevents us from doing this as many times as necessary if m is even.

To separate two difficulties, we decompose W_2 into the low and the high energy parts and consider $W_{2,low} = \phi_1(H)W_2\phi_1(H_0)$ and $W_{2,high} = \phi_2(H)W_2\phi_2(H_0)$, where cut off functions $\phi_1 \in C_0^\infty(R^1)$ and $\phi_2 \in C^\infty(R^1)$ are such that $\phi_1(\lambda)^2+\phi_2(\lambda)^2=1$, and $\phi_1(\lambda)=1$ for $|\lambda|\le 1$ and $\phi_1(\lambda)=0$ for $|\lambda|\ge 2$. Note that $W_\pm = \sum_{j=1}^2 \phi_j(H)W_\pm\phi_j(H_0)$ thanks to the intertwining property of $W_\pm$ and $\phi_j(H_0)$ and $\phi_j(H)$, $j=1,2$, are bounded in L^p as proved in section 2. We show $W_{2,low}$ and $W_{2,high}$ are bounded in L^p separately.

In section 3, we treat the low energy part $W_{2,low}$. We split $R^-(\lambda) = R^-(0)+\tilde R^-(\lambda)$ to single out the contribution of $R^-(0)$ and decompose as $W_{2,low} = W_{2,low}^{(1)} + W_{2,low}^{(2)}$ accordingly. In virtue of the orthogonality of Hardy functions in the upper and the lower half planes, we have

$$(1.9)\quad W_{2,low}^{(1)}u = \phi_1(H)\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} R_0^-(\lambda)VR^-(0)VR_0^+(\lambda)d\lambda\right\}\phi_1(H_0)u;$$

using the identity $(R_0^+(\lambda)-R_0^-(\lambda))\phi_1(H_0) = (R_0^+(\lambda)-R_0^-(\lambda))\phi_1(\lambda)$, we write

$$(1.10)\quad \begin{aligned} W_{2,low}^{(2)}u = \frac{1}{2\pi i}\int_0^\infty & \phi_1(H)R_0^-(\lambda)V\tilde R^-(\lambda)V(R_0^+(\lambda)-R_0^-(\lambda)) \\ &\times\tilde\phi_1(\lambda)\phi_1(H_0)u\,d\lambda, \end{aligned}$$

where $\tilde\phi_1 \in C_0^\infty(\mathbf{R})$ is such that $\tilde\phi_1(\lambda)\phi_1(\lambda) = \phi_1(\lambda)$. For dealing with $W_{2,low}^{(1)}$ it is important to observe the following: If we write the integral kernel of $R^-(0)$ by $K(x,y)$ and set $M_y(x) = V(x)K(x,x-y)V(x-y)$, then $W_{2,low}^{(1)}$ can be expressed as a superposition

$$(1.11)\qquad W_{2,low}^{(1)}u = -\int_{R^m}\phi_1(H)W_1(M_y)\phi_1(H_0)u_y\,dy,$$

where $u_y(x) = u(x-y)$ and $W_1(M_y)$ is defined by (1.2) with M_y in place of V. We show in section 2 that

$$(1.12)\qquad \int_{\mathbf{R}^m}\|\langle x\rangle^\sigma M_y\|_{H^{(m-3)/2}(\mathbf{R}^m)}dy < \infty$$

for some $\sigma > 1/2$. Since (1.5) and (1.11) imply that $\|W_{2,low}^{(1)}u\|_{L^p}$ is bounded by a constant times

$$\int_{R^m} \|W_1(M_y)\|_{B(L^p)}\|u_y\|_{L^p}dy \le C\int_{R^m} \|\langle x\rangle^\sigma M_y\|_{H^{(m-3)/2}(\mathbf{R}^m)}dy \cdot \|u\|_{L^p},$$

$W_{2,low}^{(1)}$ is bounded in L^p.

We treat $W_{2,low}^{(2)}$ as follows. Set $G_{\pm,x,k}(y) = e^{\pm ik|x|}\tilde{G}_{\pm,x,k}(y)$ to make oscillation property explicit and write its integral kernel in the form $W_{2,low}^{(2)}(x,y) = W_{2,low}^{(2),+}(x,y) - W_{2,low}^{(2),-}(x,y)$:

$$\text{(1.13)} \quad W_{2,low}^{(2),\pm}(x,y) = \frac{1}{2\pi i}\int_0^\infty e^{-ik(|x|\mp|y|)}\langle \tilde{R}^-(k^2)V\tilde{G}_{\pm,y,k}, V\tilde{G}_{+,x,k}\rangle \times\tilde{\phi}_1(k^2)dk^2,$$

where we ignored the harmless factors $\phi_1(H_0)$ and $\phi_1(H)$. We then apply integration by parts with respect to k variable $\ell = (m+2)/2$ times (when m is even):

$$\text{(1.14)} \quad \begin{aligned} &= \frac{1}{2\pi i}\int_0^\infty \frac{D_k^\ell e^{-ik(|x|\mp|y|)}}{(|y|\mp|x|)^\ell}\langle \tilde{R}^-(k^2)V\tilde{G}_{\pm,y,k}, V\tilde{G}_{+,x,k}\rangle\tilde{\phi}_1(k^2)dk^2 \\ &= \frac{1}{\pi i}\int_0^\infty \frac{e^{-ik(|x|\mp|y|)}}{(|x|\mp|y|)^\ell}D_k^\ell\{k\langle \tilde{R}^-(k^2)V\tilde{G}_{\pm,y,k}, V\tilde{G}_{+,x,k}\rangle\tilde{\phi}_1(k^2)\}dk, \end{aligned}$$

and gain the addition decay factor $(|x|\mp|y|)^{-\ell}$. Here the boundary terms do not appear and the integral converges absolutely because $\tilde{R}^-(k^2)$ vanishes at $k=0$. (Actually we apply the integration by parts in a little more elaborate way. See the text for the details.) In this way we arrive at the estimate

$$\text{(1.15)} \quad |W_{2,low}^{(2),\pm}(x,y)| \le C(1+||x|\mp|y||)^{-(m+2)/2}\langle x\rangle^{-(m-1)/2}\langle y\rangle^{-(m-1)/2}$$

and $W_{2,low}^{(2)}(x,y)$ indeed satisfies the criterion (1.6). Though the splitting of $R^-(\lambda)$ as above is unnecessary when m is odd because of simpler structure of $G_\pm(x,k)$, it makes the proof of the theorem simpler even in that case.

In section 4, we prove that the high energy part $W_{2,high} = \phi_2(H)W_2\phi_2(H_0)$ is also bounded in L^p, overcoming the high energy difficulty by the method similar to one that was employed in section 4 of [21]:

We decompose W_2 into $2N+1$ summands: $W_2 = \sum_{n=2}^{2N+2}(-1)^n W^{(n)}$ by expanding $R^-(k^2)$ as

$$(1.16) \qquad R^-(k^2) = \sum_{n=0}^{2N-1}(-1)^n R_0^-(k^2)(VR_0^-(k^2))^n \\ +(R^-(k^2)V)^N R^-(k^2)(VR_0^-(k^2))^N$$

and inserting (1.16) into (1.3). A repeated application of the argument leading to (1.4) shows that $W^{(2)}, \ldots, W^{(2N+1)}$ have expressions similar to (1.4), and the estimate similar to the one used for proving (1.5) implies that they are all bounded in L^p.

To prove $W^{(2N+2)}$ is bounded in L^p, we let $F_N(k^2) = (R^-(k^2)V)^N R^-(k^2)(VR_0^-(k^2))^N$ and define the integral operator $W_{high}^{(2N+2)}$ with the integral kernel $W_{high}^{(2N+2)}(x,y) = W_{high}^{(2N+2),+}(x,y) - W_{high}^{(2N+2),-}(x,y)$:

$$(1.17) \quad W_{high}^{(2N+2),\pm}(x,y) = \frac{1}{2\pi i}\int_0^\infty e^{-ik(|x|\pm|y|)} \\ \times\langle F_N(k^2)V\tilde{G}_{\pm,y,k}, V\tilde{G}_{+,x,k}\rangle\tilde{\phi}_2(k^2)dk^2,$$

where $\tilde{\phi}_2 \in C^\infty(\mathbf{R})$ is such that $\tilde{\phi}_2(\lambda) = 0$ near $\lambda = 0$ and $\tilde{\phi}_2(\lambda)\phi_2(\lambda) = \phi_2(\lambda)$. Then we have $\phi_2(H)W^{(2N+2)}\phi_2(H_0) = \phi_2(H)W_{high}^{(2N+2)}\phi_2(H_0)$. If N is sufficiently large $F_N(k^2)$, as an operator valued function between suitable function spaces, decays rapidly as $k \to \infty$ and the integrals (1.17) converge absolutely. Moreover, integration parts with respect to k variable as in the proof of (1.15) yields

$$|W_{high}^{(2N+2),\pm}(x,y)| \le C(1+||x|\mp|y||)^{-(m+2)/2}\langle x\rangle^{-(m-1)/2}\langle y\rangle^{-(m-1)/2},$$

which shows that $W_{high}^{(2N+2)}(x,y)$ satisfies the criterion (1.6). In this way the argument is very much similar to that of the previous section and of section 4 of [21], and therefore, we shall be very sketchy in section 4.

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2. Preliminaries

In this section we first study the mapping property of $R_0^\pm(\lambda)$, $\lambda \ge 0$, and the decay and smoothness properties of the integral kernels of $R^\pm(0)$ and

$\phi(H)$, $\phi \in C_0^\infty(\mathbf{R})$, under the conditions which are more general than in 1.2. We then recall from [21] the argument for proving the L^p boundedness of W_1. For $1 \le p, q \le \infty$ and $\delta, \ell \in \mathbf{R}$, $L^p_\delta(\mathbf{R}^m)$ is the weighted L^p-space:

$$L^p_\delta(\mathbf{R}^m) = \{f \in L^p_{loc}(\mathbf{R}^m) : \|f\|_{L^p_\delta} \equiv \|\langle x\rangle^\delta f\|_{L^p} < \infty\}\,;$$

$H^\ell_\delta(\mathbf{R}^m)$ is the weighted Sobolev space:

$$H^\ell_\delta(\mathbf{R}^m) = \{f \in \mathcal{S}'(\mathbf{R}^m) \ : \|(1+|x|^2)^{\delta/2}(1-\triangle)^{\ell/2} f\|_{L^2} \equiv \|f\|_{H^\ell_\delta} < \infty\}\,;$$

and $\ell^p_\delta(L^q)$ is the amalgam space:

$$\ell^p_\delta(L^q) = \{f \in L^q_{loc}(\mathbf{R}^m) : \|f\|_{\ell^p_\delta(L^q)} \equiv \left(\sum_{n \in Z^m} \|f\|^p_{L^q(Q_n)} \langle n\rangle^{\delta p}\right)^{1/p} < \infty\},$$

where for $n = (n_1, \dots, n_m)$, $Q_n = [n_1, n_1+1) \times \cdots [n_m, n_m+1)$ is a unit cube.

2.1 Resolvent estimate for H_0

If $s > 1$ and $t \in \mathbf{R}$, the resolvent $R_0(z) = (H_0 - z)^{-1}$, which is originally defined as a $B(L^2)$-valued analytic function of $z \in \mathbf{C} \setminus [0, \infty)$, can be extended continuously to the closure $\overline{\mathbf{C} \setminus [0,\infty)}$ (in the Riemann surface of $\log z$) when considered as a $B(H^t_s, H^{t+2}_{-s})$-valued function ([9]). We denote the boundary values on the upper and lower edges by $\lim_{\epsilon \to +0} R_0(\lambda \pm i\epsilon) \equiv R_0^\pm(\lambda)$, $\lambda \in [0, \infty)$. The following mapping property of $R_0^\pm(\lambda)$ is well known (cf. Murata [12] and Jensen [4]). In what follows, D_k will denote $-i\partial/\partial k$ and should not be confused with $-i\partial/\partial x_k$. $[\sigma]$ is the largest integer not greater than $\sigma \in \mathbf{R}$.

LEMMA 2.1. *Let $\ell = 0, 1, 2, \cdots$, $t \in \mathbf{R}$ and $s > \ell + 1/2$. Then, as a $B(H^t_s, H^{t+2}_{-s})$-valued function of k, $R_0^\pm(k^2)$ is C^ℓ in $k \in (0, \infty)$. Moreover:*

1. *For $j = 0, 1, \cdots, \ell$ and $0 \le i \le 2 + [(j+1)/2]$, $\|D_k^j R_0^\pm(k^2)\|_{B(H^t_s, H^{t+i}_{-s})} \le Ck^{-1+i}$, $k \ge 1$.*
2. *If $\ell \ge 2$, then $R_0^\pm(k^2)$ has the following expansion in $B(H^t_s, H^{t+2}_{-s})$ valid for $k \to 0$:*

$$(2.1)\quad R_0^{\pm}(k^2) = \begin{cases} \displaystyle\sum_{j=0}^{2} G_j k^j + K_2(k), & \textit{when } \ m = 3; \\ \displaystyle\sum_{j=0}^{1} G_j k^{2j} + F_1 k^2 \log k^2 + K_2(k), & \textit{when } \ m = 4; \\ \displaystyle\sum_{j=0}^{1} G_j k^{2j} + K_2(k), & \textit{when } \ m \geq 5. \end{cases}$$

Here $F_1, G_j \in B(H_s^t, H_{-s}^{t+2})$, and $K_2(k)$ stands for a $B(H_s^t, H_{-s}^{t+2})$-valued C^ℓ-function of k such that, for $0 \leq j \leq \ell$, $\|D_k^j K_2\| = o(k^{2-j})$ as $k \to 0$. Relation (2.1) *remains valid if the boundary values $R_0^{\pm}(k^2)$ are replaced by $R_0(k^2)$,* $\operatorname{Im} k > 0$.

In section 4, we shall also use the following mapping property of $D_k^j R_0^{\pm}(k^2)$ between L^p type spaces. For $0 \leq \ell < (m-1)/2$, $\mathbf{P}_\ell^m$ is the pentagon in the (x,y)-plane surrounded by five lines $x = 1, x = 1/2 + (2\ell+1)/2m, y = 0, y = 1/2 - (2\ell+1)/2m$ and $y = x - 2(\ell+1)/(m+1)$, where the segments $\{(x,0) : 1/2 + (2\ell+1)/2m < x \leq 1\}$ and $\{(1,y) : 0 \leq y < 1/2 - (2\ell+1)/2m\}$ are included. Note that $(1/2 + (\ell+1)/m, 1/2 - (\ell+1)/m) \in \mathbf{P}_\ell^m$ as long as $\ell + 1 < m/2$.

Lemma 2.2. *Let $j = 0, 1, \ldots$ and let $1 \leq p \leq q \leq \infty$ and $1 \leq r \leq \rho \leq \infty$ be such that $1/r \geq 1/q - (j+2)/m$, where the equality is inclusive only when $1/q - (j+2)/m > 0$. Then, $D_k^j R_0^{\pm}(k^2)$ satisfies the following mapping property:*

(a) *The case m is odd ≥ 3:*

1. *If $0 \leq j < (m-1)/2$, $D_k^j R_0^{\pm}(k^2) \in B(\ell^p(L^q), \ell^\rho(L^r))$ for $(1/p, 1/\rho) \in \mathbf{P}_j^m$ and*

$$\|D_k^j R_0^{\pm}(k^2)\|_{B(\ell^p(L^q), \ell^\rho(L^r))} \leq C_j k^{m(1/p - 1/\rho) - 2 - j}, \qquad k \geq 1.$$

2. *If $(m-1)/2 \leq j < m-2$, $D_k^j R_0^{\pm}(k^2) \in B(\ell^1_{j-(m-1)/2}(L^q), \ell^\infty_{-j+(m-1)/2}(L^r))$ and*

$$\|D_k^j R_0^{\pm}(k^2)\|_{B(\ell^1_{j-(m-1)/2}(L^q), \ell^\infty_{-j+(m-1)/2}(L^r))} \leq C_j k^{(m-3)/2}, \qquad k \geq 1.$$

3. *If* $j \ge m-2$, $D_k^j R_0^\pm(k^2) \in B(L^1_{j-(m-1)/2}, L^\infty_{-j+(m-1)/2})$ *and*

$$\|D_k^j R_0^\pm(k^2)\|_{B(L^1_{j-(m-1)/2}, L^\infty_{-j+(m-1)/2})} \le C_j k^{(m-3)/2}, \qquad k \ge 1.$$

(b) *The case* m *is even* ≥ 4*:*

1. *If* $0 \le j \le (m-2)/2$, $D_k^j R_0^\pm(k^2) \in B(\ell^p(L^q), \ell^\rho(L^r))$ *for* $(1/p, 1/\rho) \in \mathbf{P}^m_j$ *and*

$$\|D_k^j R_0^\pm(k^2)\|_{B(\ell^p(L^q), \ell^\rho(L^r))} \le C_j k^{m(1/p-1/\rho)-2-j}, \qquad k \ge 1.$$

2. *If* $m/2 \le j \le m-3$, $D_k^j R_0^\pm(k^2) \in B(\ell^1_{j-(m-1)/2}(L^q), \ell^\infty_{-j+(m-1)/2}(L^r))$ *and*

$$\|D_k^j R_0^\pm(k^2)\|_{B(\ell^1_{j-(m-1)/2}(L^q), \ell^\infty_{-j+(m-1)/2}(L^r))} \le C_j k^{(m-3)/2}, \qquad k \ge 1.$$

3. *If* $j = m-2$, $D_k^j R_0^\pm(k^2) \in B(\ell^1_{j-(m-1)/2}(L^q), L^\infty_{-j+(m-1)/2})$ *for any* $1 < q \le \infty$.

$$\|D_k^j R_0^\pm(k^2)\|_{B(\ell^1_{j-(m-1)/2}(L^q), L^\infty_{-j+(m-1)/2})} \le C_j k^{(m-3)/2}, \qquad k \ge 1.$$

4. *If* $j \ge m-1$, $D_k^j R_0^\pm(k^2) \in B(L^1_{j-(m-1)/2}, L^\infty_{-j+(m-1)/2})$ *and*

$$\|D_k^j R_0^\pm(k^2)\|_{B(L^1_{j-(m-1)/2}, L^\infty_{-j+(m-1)/2})} \le C_j k^{(m-3)/2}, \qquad k \ge 1.$$

For proving Lemma 2.2, we use the following lemma. We write $u_k(x) = u(x/k)$.

LEMMA 2.3. (1) *If* $1 \le p \le q \le \infty$, $\delta \ge 0$ *and* $k \ge 1$, *then* $\|u_k\|_{\ell^p_\delta(L^q)} \le Ck^{m/p+\delta}\|u\|_{\ell^p_\delta(L^q)}$
(2) *If* $1 \le r \le \rho \le \infty$, $\delta \ge 0$ *and* $k \ge 1$, *then* $\|u_{1/k}\|_{\ell^\rho_{-\delta}(L^r)} \le Ck^{-m/\rho+\delta}\|u\|_{\ell^\rho_{-\delta}(L^r)}$.

PROOF. We only prove the first statement for integral $k \ge 1$. General case may be proved by a slight modification of the following argument. The second statement follows from the first by the duality. If $k \geq 1$ is integral, we have by Hölder's inequality:

$$
\begin{aligned}
\|f_k\|^p_{\ell^p_\delta(L^q)} &= \sum_{n\in Z^m} \langle n\rangle^{p\delta} \left(\int_{Q_n} |f(x/k)|^q dx\right)^{p/q} \\
&= \sum_{n\in Z^m} k^{mp/q}\langle n\rangle^{p\delta} \left(\int_{Q_{n/k}} |f(x)|^q dx\right)^{p/q} \\
&= k^{mp/q} \sum_{j\in Z^m} \left\{ \sum_{Q_{n/k}\subset Q_j} \left(\int_{Q_{n/k}} |f(x)|^q dx\right)^{p/q} \langle n\rangle^{p\delta} \right\} \\
&\leq k^{mp/q} \sum_{j\in Z^m} (k^m)^{1-p/q} \left(\sum_{Q_{n/k}\subset Q_j} \int_{Q_{n/k}} |f(x)|^q dx\right)^{p/q} (Ck\langle j\rangle)^{p\delta} \\
&= C^{p\delta} k^{m+p\delta} \sum_{j\in Z^m} \left(\int_{Q_j} |f(x)|^q dx\right)^{p/q} \langle j\rangle^{p\delta} = C^{p\delta}k^{m+p\delta}\|f\|^p_{\ell^p_\delta(L^q)},
\end{aligned}
$$

where the constant C depends only on the spatial dimension m. □

Proof of Lemma 2.2. We prove the lemma when $m \geq 3$ is even. The proof for the other case is similar. It is well known that $R_0^\pm(k^2)$, $k \geq 0$, are convolution operators with the outgoing $(+)$ or incoming $(-)$ fundamental solutions $G_\pm(x, k)$ of $-\triangle - k^2$ ([15]):

$$
G_\pm(x,k) = \frac{\pm i}{4(2\pi)^\nu |x|^{m-2}} (k|x|)^\nu H_\nu^{(\pm)}(k|x|), \quad \nu = \frac{m-2}{2} \tag{2.2}
$$

where $H_\nu^{(\pm)}(z)$ is the Hankel function and by Hankel's formula ([20])

$$
z^\nu H_\nu^{(\pm)}(z) = \frac{\sqrt{2}e^{\mp i(2\nu+1)\pi/4}e^{\pm iz}}{\sqrt{\pi}\Gamma(\nu+1/2)} \int_0^\infty e^{-t}t^{\nu-1/2}\left(z \pm \frac{it}{2}\right)^{\nu-1/2} dt. \tag{2.3}
$$

Here and hereafter we use the superscript $\pm$ in stead of the traditional $1, 2$ for Hankel functions and $\nu = (m-2)/2$. A simple computation shows that $D_k^j R_0^\pm(k^2)$ enjoys the homogeneity property

$$
\begin{aligned}
&[D_k^j R_0^\pm(k^2)u](x) = k^{-j-2}\{D_k^j R_0^\pm(k^2)|_{k=1} u_k\}(kx), \\
&u_k(x) = u(x/k).
\end{aligned} \tag{2.4}
$$

We prove the lemma for the case $k=1$ first. Let $\phi \in C_0^\infty(\mathbf{R}^m)$ be such that $\phi(x)=1$ for $|x|\le 1$ and $\phi(x)=0$ for $|x|\ge 2$. Write $G_{\pm}^{(j)}(x)$ for the convolution kernel of $D_k^j R_0^{\pm}(k^2)|_{k=1}$ and set $G_{1,\pm}^{(j)}(x)=G_{\pm}^{(j)}(x)\phi(x)$ and $G_{2,\pm}^{(j)}(x)=G_{\pm}^{(j)}(x)(1-\phi(x))$. Differentiating (2.2) and (2.3) by k shows that $G_{1,\pm}^{(j)}(x)$ satisfies the following estimate:

$$|G_{1,\pm}^{(j)}(x)| \le \begin{cases} C_j(1+|x|^{2-m+j}), & \text{if } m \text{ is odd;} \\ C_j(\langle \log|x| \rangle + |x|^{2-m+j}), & \text{if } m \text{ is even and } j \le m-2; \\ C_j, & \text{if } m \text{ is even and } j \ge m-1, \end{cases}$$

and that $G_{2,\pm}^{(j)}(x)$ can be written as

$$G_{2,\pm}^{(j)}(x) = e^{\pm i|x|} a_{j,\pm}(x)|x|^{(2j-m+1)/2}, \tag{2.5}$$

where $a_{j,\pm}(x) \in C^\infty(\mathbf{R}^m)$ is supported by $\{|x|\ge 1\}$ and satisfies for any α

$$|D^\alpha a_{j,\pm}(x)| \le C_{j\alpha}|x|^{-|\alpha|}.$$

Since $G_{1,\pm}^{(j)}(x)$ is supported by the compact set $\{|x|\le 2\}$, the convolution operator $G_{1,\pm}^{(j)}$ with $G_{1,\pm}^{(j)}(x)$ can be easily estimated by using the fractional integration theory and Young's inequality:
(i) If $0\le j\le m-3$, $G_{1,\pm}^{(j)} \in B(\ell^p(L^q),\ell^p(L^r))$ for any $1\le p\le\infty$ and $1\le r\le\infty$ if $1/q<(j+2)/m$; $1\le r<\infty$ if $1/q=(j+2)/m$; and $1/q-(j+2)/m\le 1/r\le 1$ if $1/q>(j+2)/m$.
(ii) If $j=m-2$, $G_{1,\pm}^{(j)} \in B(\ell^p(L^q),\ell^p(L^\infty))$ for any $1\le p\le\infty$, and $1<q\le\infty$ (if m is odd $q=1$ can be included);
(iii) If $j\ge m-1$, $G_{1,\pm}^{(j)} \in B(\ell^p(L^1),\ell^p(L^\infty))$ for any $1\le p\le\infty$.

On the other hand $G_{2,\pm}^{(j)}(x)$ contains the oscillating factor $e^{\pm i|x|}$ and we estimate the convolution operator $G_{2,\pm}^{(j)}$ with the kernel (2.5) by a theorem of Sogge (cf. [19], Lemma 5.4). We combine the result with the fact $G_{2,\pm}^{(j)} \in B(L^p, L^\infty)$, $1\le p<2m/(m+2j+1)$, which follows from Young's inequality, by using the interpolation theorem and the duality. We obtain the followings:
(iv) If $j\le (m-2)/2$, then $G_{2,\pm}^{(j)} \in B(L^p,L^\rho)$ for any p and ρ such that $(1/p,1/\rho)\in \mathbf{P}_j^m$ where $\mathbf{P}_j^m$ is the pentagon defined as above.
(v) If $j\ge m/2$, then $2j-m+1>0$ and $G_{2,\pm}^{(j)} \in B(L^1_{j-(m-1)/2}, L^\infty_{-j+(m-1)/2})$.

Note here that $\ell_\delta^{p_1}(L^{q_1}) \subset \ell_\delta^{p_2}(L^{q_2})$ whenever $p_1 \le p_2$ and $q_1 \ge q_2$. Thus, combing estimates (i) $\sim$ (v), we obtain the lemma for the case $k = 1$.

It remains to estimate the operator norm for $k \ge 1$. When $j \le (m-2)/2$ the estimates in the lemma immediately follow from (2.4) and Lemma 2.3. When $j \ge m/2$, the direct application of Lemma 2.3 would produce the superfluous power k^{j-1}. Note, however, that in this case $G_{2,\pm}^{(j)}(x-y)$ satisfies

$$|G_{2,\pm}^{(j)}(x-y)| \le C(|x|^{(2j-m+1)/2} + |y|^{(2j-m+1)/2} + 1),$$

and $G_{2,\pm}^{(j)}$ is in fact a sum of two operators, one in $B(L^1_{j-(m-1)/2}, L^\infty)$ and the other in $B(L^1, L^\infty_{-j+(m-1)/2})$. Hence, say in the case (b.2), $D_k^j R_0^\pm(k^2)$ may be written as a sum of two operators, one in $B(\ell^1_{j-(m-1)/2}(L^q), \ell^\infty(L^r))$ and the other in $B(\ell^1(L^q), \ell^\infty_{-j+(m-1)/2}(L^r))$. Applying Lema 2.3 to each summand separately and combining the results, we obtain the desired estimates. □

2.2 Integral kernels of $\phi(H)$ and $R(0)$

In this subsection, we study the integral kernel of $\phi(H)$ (resp. $R(0)$) assuming that V is of Kato class (resp. very short range). A real valued function $V(x)$ is said to be of Kato-class if

$$\lim_{\epsilon\to 0} \sup_{x\in\mathbf{R}^m} \int_{|x-y|\le\epsilon} \frac{|V(y)|}{|x-y|^{m-2}} dy = 0 \tag{2.6}$$

and to be **very short range** if, for some $\gamma > 0$, $\langle x\rangle^{2+\gamma} V(x)$ satisfies (2.6). In particular, we have for very short range potential that

$$\|V\|_{(\gamma)} \equiv \sup_{x\in\mathbf{R}^m} \langle x\rangle^{2+\gamma} \int_{|x-y|<1} \frac{|V(y)|}{|x-y|^{m-2}} dy < \infty. \tag{2.7}$$

We note that V which satisfies the assumption of Theorem 1.2 is very short range.

If V is of Kato class, then, the multiplication operator V with $V(x)$ is H_0-form bounded with relative bound zero and $H = H_0 + V$ defined via the form sum is self-adjoint([13]). If we write $A(x) = |V(x)|^{1/2}$ and $B(x) = V(x)^{1/2} \equiv |V(x)|^{1/2}\text{sign } V(x)$ and A and B for the multiplications by $A(x)$ and $B(x)$, respectively, then

$$R(z) = R_0(z) - R_0(z)B(1 + AR_0(z)B)^{-1}AR_0(z), \qquad z \in \mathbf{C}\setminus\mathbf{R}. \tag{2.8}$$

The following lemma solves an open problem in Simon ([17]):

LEMMA 2.4. *Let V be of Kato-class and $\phi(\lambda) \in C_0^\infty(\mathbf{R})$. Then, the integral kernel $\Phi(x,y)$ of $\phi(H)$ satisfies $|\Phi(x,y)| \le C_\delta(1+|x-y|)^{-\delta}$ for any $\delta \ge 0$. In particular, $\phi(H)$ is bounded in L^p for any $1 \le p \le \infty$.*

PROOF. The following argument which has simplified the original proof is due to Shu Nakamura (private communication). If we set $V_a(x) = V(x+a)$ and $H(a) = H_0+V_a$, $\Phi(x+a, y+a)$ is the integral kernel of $\phi(H(a))$. Hence, it suffices to show

$$\sup_{|y|\le 1} |\Phi(x,y)| \le C_\delta(1+|x|)^{-\delta} \tag{2.9}$$

with constants C_δ which is independent of a if H is replaced by $H(a)$. (We say that an estimate holds uniformly in a if it does with the same constant when H is replaced by $H(a)$, $a \in \mathbf{R}^m$). Write $\phi(\lambda) = (\lambda - z)^{-N}\psi(\lambda)(\lambda - z)^{-N}$ so that $\phi(H) = R(z)^N\psi(H)R(z)^N$. By Theorem B.6.3 of [17], $R(z)^N$ is bounded uniformly in a from L^1_δ to L^2_δ and from L^2_δ to L^∞_δ for any $\delta \in \mathbf{R}$, if N and real $-z$ are large enough. On the other hand $\psi(H)$ is bounded in L^2_δ uniformly in a as will be shown below. Hence, $\phi(H)$ is bounded from L^1_δ to L^∞_δ uniformly in a and

$$\begin{aligned}\sup_{x\in\mathbf{R},|y|\le 1} &\langle x\rangle^\delta |\Phi(x,y)| \\ &\le C_\delta \sup\{\|\phi(H)u\|_{L^\infty_\delta} : \|u\|_{L^1_\delta} = 1,\ \mathrm{supp}\, u \subset B(O,1)\} \\ &\le C_\delta \|\phi(H)\|_{B(L^1_\delta, L^\infty_\delta)} < \infty.\end{aligned}$$

It remains to show that $\psi(H)$ is bounded in L^2_δ for any $\delta > 0$ uniformly in a. It suffices to show that for any choice of $1 \le j_k \le m, k = 1,\dots,\ell$ and $\ell = 1,2,\dots$

$$\|[x_{j_1}, [x_{j_2}, \cdots, [x_{j_\ell}, \psi(H)] \cdots]]\|_{B(L^2)} \le C_\ell \tag{2.10}$$

uniformly in a. Let $\psi(z)$ be an almost analytic extension of $\psi(\lambda)$ which satisfies for any n and $N \ge 0$,

$$|(\partial\psi/\partial\overline{z})(z)| \le C_{nN}|\mathrm{Im}\ z|^n(1+|z|)^{-n-N}, \qquad z \in \mathbf{C}$$

and write

$$\psi(H) = \frac{-1}{2\pi i}\int_{\mathbf{C}} \frac{\partial\psi}{\partial\overline{z}}(z)(H-z)^{-1} d\overline{z}\wedge dz \tag{2.11}$$

(cf. [5]). Then, using inductively the obvious identity $i[x_j, R(z)] = R(z)p_jR(z)$ and using the fact that $\|R(z)\| \le |\text{Im } z|^{-1}$ and $\|p_jR(z)\| \le C|\text{Im } z|^{-1}$, where the constant C is independent of a (cf. [17]), we immediately obtain the desired boundedness (2.10). $\square$

If V is very short range, then V is form compact with respect to H_0; and in virtue of Lemma 2.1, the boundary values

$$\lim_{\epsilon\to+0} AR_0(\lambda \pm i\epsilon)B \equiv Q_0^{\pm}(\lambda)$$

exist in the operator norm of L^2 and are locally Hölder continuous in $\lambda \in [0,\infty)$. Moreover, $1 + Q_0^{\pm}(\lambda)$ is an isomorphism of $L^2(\mathbf{R}^m)$ if and only if λ is not an eigenvalue of H (λ is not the eigenvalue or resonance of H if $\lambda = 0$). Thus, if non-negative eigenvalues and zero resonance are absent from H, then the boundary values of the resolvent

$$\begin{aligned}\lim_{\epsilon\to+0} R(\lambda \pm i\epsilon) &\equiv R^{\pm}(\lambda) \\ &= R_0^{\pm}(\lambda) - R_0^{\pm}(\lambda)B(1 + Q_0^{\pm}(\lambda))^{-1}AR_0^{\pm}(\lambda)\end{aligned} \tag{2.12}$$

exist for all $\lambda \in [0,\infty)$ in the operator norm of $B(L^2_\delta, L^2_{-\delta})$ and are locally Hölder continuous in $\lambda \in [0,\infty)$ as well. Note that $R_0^{\pm}(0)$ is independent of the sign $\pm$ and so is $R^{\pm}(0)$. We write $R_0^{\pm}(0) = R_0(0) = G_0$ and $R^{\pm}(0) = R(0)$. We have the following lemma on the integral kernel of $R(0)$.

Theorem 2.5. *Let $V(x)$ be very short range. Suppose that zero is not an eigenvalue nor resonance of $H = H_0 + V$. Then, $R(0)$ has the integral kernel $K(x, y)$ which is jointly continuous for $x \ne y$ and satisfies $|K(x, y)| \le C|x - y|^{2-m}$.*

We begin the proof of Theorem 2.5 with the following elementary lemma. In what follows we assume that $\langle x\rangle^{2+\gamma}V(x)$ satisfies (2.6) for some $0 < \gamma < 1$.

Lemma 2.6.*Let $0 \le \rho < \gamma < 1$. Then, with a constant C_1 depending only on m, ρ and γ,*

$$\int_{\mathbf{R}^m} \frac{\langle y\rangle^{\rho}|V(y)|dy}{|x - y|^{m-2}} \le C_1\|V\|_{(\gamma)}\langle x\rangle^{\rho-\gamma}; \tag{2.13}$$

$$\text{(2.14)} \qquad \int_{\mathbf{R}^m} \frac{|V(z)|dz}{|x-z|^{m-2}|z-y|^{m-2}} \le \frac{C_1(\langle x\rangle^{-\gamma} + \langle y\rangle^{-\gamma})\|V\|_{(\gamma)}}{|x-y|^{m-2}}.$$

PROOF. Take $\phi \in C_0^\infty(\mathbf{R}^m)$ such that $\phi(x) = 0$ for $|x| \ge 1/2$ and $\int_{\mathbf{R}^m} \phi(z)dz = 1$. We estimate the integral over $|x-y| \ge 1$ as follows:

$$\begin{aligned}
\int_{|x-y|\ge 1} \frac{\langle y\rangle^\rho |V(y)|dy}{|x-y|^{m-2}} &= \int_{\mathbf{R}^m} dz \left\{ \int_{|x-y|\ge 1} \frac{\langle y\rangle^\rho |V(y)|\phi(y-z)dy}{|x-y|^{m-2}} \right\} \\
&\le 2^{m-2} \int_{\mathbf{R}^m} dz \left\{ \int_{\mathbf{R}^m} \frac{\langle y\rangle^\rho |V(y)|\phi(y-z)dy}{(1+|x-z|)^{m-2}} \right\} \\
&\le C_2 \|V\|_{(\gamma)} \|\phi\|_{L^\infty} \int_{\mathbf{R}^m} \frac{dz}{(1+|x-z|)^{m-2}\langle z\rangle^{2+\gamma-\rho}} \le C_3 \|V\|_{(\gamma)} \langle x\rangle^{\rho-\gamma}.
\end{aligned}$$

Since the integral over $|x-y| \le 1$ is obviously bounded by a constant times $\|V\|_{(\gamma)}\langle x\rangle^{\rho-2-\gamma}$, we obtain (2.13).

Write $w = x - y$ and change the variable z by $z + y$. Let $\Omega_1 = \{z : |w|/2 \le |z|\}$ and $\Omega_2 = \{z : |w|/2 \le |z-w|\}$. It is clear that $\mathbf{R}^m = \Omega_1 \cup \Omega_2$ and by using (2.13) with $\rho = 0$,

$$\begin{aligned}
\int_{\Omega_1} \frac{|V(z+y)|dz}{|w-z|^{m-2}|z|^{m-2}} &\le \frac{2^{m-2}}{|w|^{m-2}} \int_{\mathbf{R}^m} \frac{|V(z+y)|dz}{|w-z|^{m-2}} \\
&\le C_1 \langle x\rangle^{-\gamma} |w|^{2-m} \|V\|_{(\gamma)};
\end{aligned}$$

$$\begin{aligned}
\int_{\Omega_2} \frac{|V(z+y)|dz}{|w-z|^{m-2}|z|^{m-2}} &\le \frac{2^{m-2}}{|w|^{m-2}} \int_{\mathbf{R}^m} \frac{|V(z+y)|dz}{|z|^{m-2}} \\
&\le C_1 \langle y\rangle^{-\gamma} |w|^{2-m} \|V\|_{(\gamma)}.
\end{aligned}$$

Adding these up, we obtain (2.14). □

The following is a corollary of Lemma 2.6 and proves Theorem 2.5 when V is small.

LEMMA 2.7. *There exists a constant $C_0 > 0$ such that, if $\|V\|_{(\gamma)} < C_0$, then the integral kernel $K(x,y)$ of $R(0)$ is continuous for $x \ne y$ and satisfies $|K(x,y)| \le C|x-y|^{2-m}$.*

PROOF. The integral kernel of $G_0 = R_0^\pm(0)$ is given by the Newton potential $G_0(x-y) = c_m|x-y|^{2-m}$, $c_m = \Gamma(m-2/2)/4\pi^{m/2}$. By Schwarz inequality and (2.13) with $\rho = 0$,

$$
\begin{aligned}
|(Q_0^{\pm}(0)u, v)| &\le c_m \int_{\mathbf{R}^m} \frac{|A(x)||v(x)||B(y)||u(y)|}{|x-y|^{m-2}} dydx \\
&\le c_m \left(\int_{\mathbf{R}^m} \frac{|A(x)|^2 |u(y)|^2}{|x-y|^{m-2}} dxdy \right)^{1/2} \left(\int_{\mathbf{R}^m} \frac{|B(y)|^2 |v(x)|^2}{|x-y|^{m-2}} dydx \right)^{1/2} \\
&\le c_m C_1 \|V\|_{(\gamma)} \|u\| \|v\|.
\end{aligned}
$$

Hence, $1 + Q_0^{\pm}(0)$ is invertible in $B(L^2)$ if $\|V\|_{(\gamma)} < (c_m C_1)^{-1}$, and we may expand $(1 + Q_0^{\pm}(0))^{-1}$ into the Neumann series in (2.12) with $\lambda = 0$ to obtain

$$
R(0) = G_0 - G_0 V G_0 + G_0 V G_0 V G_0 - \cdots .
$$

Since any V with $\|V\|_{(\gamma)} < \infty$ may be approximated arbitrarily close by C_0^∞ functions in the norm $\| \cdot \|_{(\gamma')}$, $\gamma' < \gamma$, it is easy to see that the integral kernels of the summands of the series are continuous for $x \neq y$. Moreover estimating them inductively by using (2.14), we obtain a majorant series $\sum_{n=0}^{\infty} c_m^{n+1} (2C_1 \|V\|_{(\gamma)})^n |x-y|^{2-m}$ for $K(x,y)$. The latter series converges uniformly on every compact subset of $\{(x,y) : x \neq y\}$ and produces the bound $|K(x,y)| \le C_2 |x-y|^{2-m}$ if $2c_m C_1 \|V\|_{(\gamma)} < 1$. This proves the Lemma. □

For proving Theorem 2.5 for general potentials, we shall use the following lemma. For $0 < \rho < \min(1, \gamma)$, $\mathcal{X}_\rho$ is the Banach space defined by

$$
\mathcal{X}_\rho = \{u \in C(\mathbf{R}^m \setminus \{0\}) : \|u\|_{\mathcal{X}_\rho} = \sup_{x \in \mathbf{R}^m \setminus \{0\}} \langle x \rangle^{-\rho} |x|^{m-2} |u(x)| < \infty\}. \tag{2.15}
$$

We remark here that if $K(x,y)$ is as in Lemma 2.7, then $K_y(x) \equiv K(x+y, y)$ belongs to $\mathcal{X}_\rho$ and $y \to K_y$ is an $\mathcal{X}_\rho$ valued continuous function. This can be easily seen by the proof of the lemma (note that $K_y(x)$ is $K_0(x)$ corresponding to the potential $V_y(x) = V(x+y)$ and $y \to V_y$ is continuous in the $\| \cdot \|_{(\gamma')}$ norm, $\gamma' < \gamma$).

Lemma 2.8. *Let $V_1 \in C_0^\infty(\mathbf{R}^m)$. Let $K_0(x,y)$ be continuous for $x \neq y$ and satisfy $|K_0(x,y)| \le C|x-y|^{2-m}$. Define the integral operator Z_y for $y \in \mathbf{R}^m$ by*

$$
Z_y u(x) = \int_{\mathbf{R}^m} K_0(x+y, z+y) V_1(z+y) u(z) dz. \tag{2.16}
$$

Then, Z_y is a compact operator in $\mathcal{X}_\rho$ and is norm continuous with respect to $y \in \mathbf{R}^m$.

PROOF. We prove the lemma for $m \geq 5$. The proof for $m = 3, 4$ may be given by slightly modifying the following argument. Let S be the unit ball of $\mathcal{X}_\rho$. Then for $u \in S$, we have as in (2.14)

$$
|Z_y u(x)| \leq C \int_{\mathbf{R}^m} \frac{|V_1(z+y)|\langle z\rangle^\rho dz}{|x-z|^{m-2}|z|^{m-2}} \leq \begin{cases} C|x|^{4-m}, & |x| \leq 1; \\ C_y|x|^{2-m}, & |x| \geq 1, \end{cases} \tag{2.17}
$$

where C_y is a constant bounded for bounded y. Let $\psi \in C_0^\infty(\mathbf{R}^m)$ be such that $\psi(x) = 1$ for $|x| \geq 2$ and $\psi(x) = 0$ for $|x| \leq 1$. Set, for $\epsilon > 0$, $\psi_\epsilon(x) = \psi(x/\epsilon)$ and let $Z_{y,\epsilon}$ be the integral operator defined by (2.16) with $K_{0\epsilon}(x,y) = \psi_\epsilon(x-y)K_0(x,y)$ in place of $K_0(x,y)$. Because of the estimate (2.17) and the fact that $K_{0\epsilon}(x,y)$ is jointly continuous with respect to (x,y), it can be easily seen via Ascoli-Arzela's lemma that $Z_{y,\epsilon}$ is a compact operator in $\mathcal{X}_\rho$ and is norm continuous with respect to y. On the other hand, for y in a compact subset of $\mathbf{R}^m$, $Z_{y,\epsilon}u(x) = Z_y u(x)$ for $|x| \geq C_0$ and we have for $u \in S$ and $\epsilon \to 0$

$$
\begin{aligned}
&\sup_{x\in\mathbf{R}^m} |x|^{m-2}|Z_{y,\epsilon}u(x) - Z_y u(x)| \\
&\quad \leq c_m \sup_{|x|\leq C_0} |x|^{m-2} \int_{|x-z|<2\epsilon} \frac{\langle z\rangle^\rho |V_1(z+y)|dz}{|x-z|^{m-2}|z|^{m-2}} \\
&\quad \leq \sup_{|x|\leq C_0} C \int_{|x-z|<2\epsilon} \frac{|x|^{m-2}dz}{|x-z|^{m-2}|z|^{m-2}} \\
&\quad \leq C\epsilon^2 \sup_{x\in\mathbf{R}^m} \int_{|z|<2/|x|} \frac{|x|^2 dz}{|\hat{x}-z|^{m-2}|z|^{m-2}} \to 0
\end{aligned}
$$

uniformly with respect to y, where $\hat{x} = x/|x|$. This shows that $Z_{y,\epsilon}$ converges to Z_y in the operator norm of $\mathcal{X}_\rho$ locally uniformly with respect to y. Hence Z_y is compact and is norm continuous. □

PROOF OF THEOREM 2.5. Decompose $V(x) = V_0(x) + V_1(x)$ in such a way that $\|V_0\|_{(\gamma)} < C_0$ and $V_1 \in C_0^\infty(\mathbf{R}^m)$, where C_0 is the constant appeared in Lemma 2.7. Denote by $K_0(x, y)$ the integral kernel of $K_0 \equiv \lim_{\epsilon \to 0}(H_0 + V_0 \pm i0)^{-1}$. In virtue of Lemma 2.7, $K_0(x, y)$ is continuous for $x \neq y$ and satisfies $|K_0(x, y)| \le C|x - y|^{2-m}$. Thus, by Lemma 2.8, the integral operator Z_y defined in $\mathcal{X}_\rho$ by (2.16) with this $K_0(x, y)$ and $V_1(x)$ is compact and is norm continuous with respect to y.

We show that $1 + Z_y$ is an isomorphism of $\mathcal{X}_\rho$. Suppose that $u(x) + Z_y u(x) = 0$, $u \in \mathcal{X}_\rho$. Then $|u(x)|$ is bounded by a constant times the RHS of (2.17) and repeating the similar estimate implies that $u(x)$ is continuous and satisfies $|u(x)| \le C\langle x\rangle^{2-m}$. (This may also be seen by the elliptic regularity theorem for Schrödinger operators with Kato class potentials, see e.g. [16].) Set $u_y(x) = u(x - y)$. u_y is continuous, $|u_y(x)| \le \langle x - y\rangle^{2-m}$, and it satisfies the integral equation

$$u_y(x) + \int_{\mathbf{R}^m} K_0(x, z)V_1(z)u_y(z)dz = 0. \tag{2.18}$$

By applying $-\triangle + V_0(x)$ to (2.18), we see $-\triangle u_y(x) + V(x)u_y(x) = 0$. It follows that $u(x) \equiv 0$, since $u_y \in L^2_{-1-\epsilon}(\mathbf{R}^m)$ (or $u_y \in L^2(\mathbf{R}^m)$ if $m \ge 5$), and since we are assuming that zero is not resonance nor eigenvalue of $H = H_0 + V$. Thus $1 + Z_y$ is an isomorphism of $\mathcal{X}_\rho$.

Set $K_{0y}(x) = K_0(x + y, y)$. By the remark after the definition (2.15) of $\mathcal{X}_\rho$, K_{0y} is an $\mathcal{X}_\rho$ valued continuous function. Hence, $K_y = (1 + Z_y)^{-1}K_{0y}$ is well defined and is also an $\mathcal{X}_\rho$ valued continuous function. Set $K(x, y) = K_y(x - y)$. $K(x, y)$ is jointly continuous for $x \neq y$; $|K(x, y)| \le C_y\langle x - y\rangle^\rho |x - y|^{2-m}$ with C_y bounded for bounded y; and it satisfies the integral equation

$$K(x, y) = K_0(x, y) - \int_{\mathbf{R}^m} K_0(x, z)V_1(z)K(z, y)dz. \tag{2.19}$$

Note that (2.19) and (2.17) imply that $K(x, y)$ in fact satisfies the estimate $|K(x, y)| \le C_y|x - y|^{2-m}$, where C_y is again bounded for bounded y.

We show that $K(x, y)$ is the integral kernel of $R(0)$ and it satisfies the estimate mentioned in the theorem. Denote by K the integral operator with the integral kernel $K(x, y)$. Then, for $u \in C_0^\infty(\mathbf{R})$, $Ku(x)$ is continuous, $|Ku(x)| \le C\langle x\rangle^{2-m}$ and, in virtue of (2.19), $Ku = K_0u - K_0V_1Ku$. Subtract $R(0)u = K_0u - K_0V_1R(0)u$ from this equation side by side and write $v = R(0)u - Ku$. Then $v \in L^2_{-1-\epsilon}$, $\epsilon > 0$, and it satisfies $v + K_0V_1v = 0$. Applying $H_0 + V_0$ to both sides of this equation implies $-\triangle v(x) + V(x)v(x) = 0$ and we conclude $v = 0$ because zero is not a resonance or an eigenvalue of H. Hence $Ku = R(0)u$ for any $u \in C_0^\infty$ and $R(0) = K$. Since $R(0)^* = R(0)$, we have $K(x,y) = K(y,x)$ and $|K(x,y)| \le C_x|x-y|^{2-m}$ with C_x bounded for bounded x. Going back to (2.19), we conclude $|K(x,y)| \le C|x-y|^{2-m}$. This completes the proof of Theorem 2.5. □

Since $K(x,y)$ satisfies $-\triangle_x K(x,y) + V(x)K(x,y) = \delta(x-y)$, we expect from the elliptic regularity that $K(x,y)$ is smooth where V is. We prove the following result.

LEMMA 2.9. *Suppose V is as in Theorem 2.5 and, in addition, $D^\alpha V(x)$ satisfies (2.7) for $|\alpha| \le \ell$. Let $K(x,y)$ be the integral kernel of $R(0)$. Then, for $y \ne 0$, $K(x, x-y)$ is C^ℓ with respect to $x \in \mathbf{R}^m$ and $|D_x^\alpha K(x,x-y)| \le C_\alpha |y|^{2-m}$, $|\alpha| \le \ell$.*

PROOF. Let τ_h be the translation by h and $V_h(x) = V(x+h)$. Then $K(x+h, y+h)$ is the integral kernel of $\tau_h R(0) \tau_h^{-1} = (-\triangle + V_h)^{-1} \equiv R_h(0)$ and the resolvent equation $R_h(0) - R(0) = -R_h(0)(V_h - V)R(0)$ implies that

$$K(x+h,y+h) - K(x,y) = -\int_{\mathbf{R}^m} K(x+h,z+h)(V(z+h)-V(z))K(z,y)dz.$$

Hence Theorem 2.5, Lemma 2.6 and the assumption on DV together imply

$$(\partial/\partial h_j)K(x+h,y+h)|_{h=0} = -\int_{\mathbf{R}^m} K(x,z)(\partial V/\partial z_j)(z)K(z,y)dz.$$

Repeating this argument, we obtain

$$D_h^\alpha K(x+h,y+h)|_{h=0} = \sum_{\ell=1}^{|\alpha|} \sum_{\alpha_1+\cdots+\alpha_\ell=\alpha} C_{\alpha_1,\dots,\alpha_\ell} G_{\alpha_1,\dots,\alpha_\ell}(x,y),$$

where $G_{\alpha_1,\dots,\alpha_\ell}(x,y)$ is the integral kernel of $R(0)V^{(\alpha_1)}R(0)\cdots V^{(\alpha_\ell)}R(0)$. Applying Theorem 2.5 and Lemma 2.6 and using the assumptions on $D^\alpha V$ for estimating $G_{\alpha_1,\dots,\alpha_\ell}(x,y)$, we obtain the lemma immediately. □

We need the following lemma.

LEMMA 2.10. *Let $1 \le p,q,r \le \infty$ satisfy $r^{-1} \ge p^{-1} + q^{-1} - 1$. Then:*
(1) *If $\rho, \sigma < m$ and $\rho + \sigma > m$. Then $\|f * g\|_{\ell^\infty_{\rho+\sigma-m}(L^r)} \le C\|f\|_{\ell^\infty_\rho(L^p)} \cdot$*

$\|g\|_{\ell^\infty_\sigma(L^q)}$.

(2) *If* ρ *or* $\sigma > m$, *then* $\|f * g\|_{\ell^\infty_{\min(\rho,\sigma)}(L^r)} \le C\|f\|_{\ell^\infty_\rho(L^p)} \cdot \|g\|_{\ell^\infty_\sigma(L^q)}$.

Proof. Take $\phi \in C_0^\infty(|x| < 1/2)$ such that $\int \phi(x)dx = 1$ and set $f_y(x) = \phi(x-y)f(x)$ and etc. Clearly f_y is supported by $y + B(O, 1/2)$, $f(x) = \int f_y(x)dy$ and we may write

$$(f * g)(x) = \int (f_y * g_z)(x)dydz.$$

Note that $f_y * g_z$ is supported by $y + z + B(O,1)$. It follows by Young's inequality that, if Q^* is the cube of side 4 with center at the origin,

$$\begin{aligned}\|f * g\|_{L^r(Q_n)} &\le C \int_{y+z-n\in Q^*} \|f_y\|_{L^p(\mathbf{R}^m)}\|g_z\|_{L^q(\mathbf{R}^m)}dydz \\ &\le C\|f\|_{\ell^\infty_\rho(L^p)}\|g\|_{\ell^\infty_\sigma(L^q)} \int_{y+z-n\in Q^*} \langle y\rangle^{-\rho}\langle z\rangle^{-\sigma}dydz.\end{aligned}$$

Estimating the last integral in a standard fashion, we obtain the lemma. □

The following lemma implies the estimate (1.12) in the introduction.

Lemma 2.11. *Let* V *satisfy* (1.1) *for* $|\alpha| \le [(m-2)/2]$ *and* $\delta > (m+3)/2$. *Then:*

$$\int_{\mathbf{R}^m} \left\{\int \langle x\rangle^{2\sigma}|D^\alpha V(x)D_x^\beta K(x, x-y)D^\gamma V(x-y)|^2dx\right\}^{1/2} dy < \infty, \tag{2.20}$$

for $|\alpha+\beta+\gamma| \le [(m-2)/2]$ *and* $\sigma < \delta - 2$.

Proof. In virtue of Lemma 2.9, the left hand side of (2.20) is bounded by a constant times

$$\int_{\mathbf{R}^m} \left\{\int \langle x\rangle^{2\sigma}|D^\alpha V(x)D^\gamma V(x-y)|^2dx\right\}^{1/2} \frac{dy}{|y|^{m-2}}. \tag{2.21}$$

We estimate (2.21) by applying Lemma 2.10. We denote the function $\{\cdots\}^{1/2}$ in (2.21) by $W_{\alpha\gamma}(y)$. If $m = 3$, we have only the case $\alpha = \beta = \gamma = 0$. By using Lemma 2.10, (2), we have

$$W_{00}(y) = \left\{\int \langle x\rangle^{2\sigma}|V(x)V(x-y)|^2dx\right\}^{1/2} \in \ell^\infty_{\delta-\sigma}(L^2).$$

Hence, if $\sigma < \delta - 2$, we have (2.21) $\leq \int_{\mathbf{R}^m}(|W_{\alpha\gamma}(y)|/|y|)dy < \infty$.

When $m = 4$ or $= 5$, we only prove (2.20) for the case $|\alpha| = 1$ and $\beta = \gamma = 0$. We may assume $p_0(> m/2)$ is close to $m/2$. We have $|V|^2 \in \ell^\infty_{2\delta}(L^{q_0/2})$, $1/q_0 = 1/p_0 - 1/m$, by Sobolev's lemma. Thus Lemma 2.10 implies $W_{\alpha\gamma} \in \ell^\infty_{\delta-\sigma}(L^r)$, $1/r = 2/p_0 - 1/m - 1/2 < 2/m$, and $\int_{\mathbf{R}^m}(|W_{\alpha\gamma}(y)|/|y|^{m-2})dy < \infty$, if $\sigma < \delta - 2$. The proof for $m \geq 6$ is similar (in fact easier) and we omit the details. □

2.3 L^p boundedness of W_1

We close this section by recalling the argument in [21] that shows that W_1 defined by (1.2):

$$W_1 u(x) \equiv -\frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty} R_0(\lambda - i\varepsilon)VR_0(\lambda + i\varepsilon)u(x)d\lambda$$

is bounded in L^p. We begin with the following lemma (Lemma 2.3 of [21]), which may be proved by computing the inverse Fourier transform of essentially one dimensional function $\xi \to (2\eta\xi - \eta^2 + i\varepsilon)^{-1}$.

LEMMA 2.12. *Let $\eta \in \mathbf{R}^m \setminus \{0\}$ and $\widehat{\eta} = \eta/|\eta|$. Then*

$$\lim_{\varepsilon\downarrow 0}\frac{1}{(2\pi)^{m/2}}\int_{\mathbf{R}^m}\frac{e^{ix\xi}\widehat{f}(\xi)}{2\eta\xi - \eta^2 + i\varepsilon}d\xi = \frac{1}{2i|\eta|}\int_0^\infty e^{-it|\eta|/2}f(x + t\widehat{\eta})dt. \tag{2.22}$$

The following proposition proves that W_1 is bounded in L^p under a rather mild condition on $V(x)$. Σ is the unit sphere of $\mathbf{R}^m$ and $d\omega$ is its surface element.

PROPOSITION 2.13. *Set for $t \in \mathbf{R}$ and $\omega \in \Sigma$*

$$\widehat{K_V}(t,\omega) = \frac{i}{2(2\pi)^{m/2}}\int_0^\infty \widehat{V}(r\omega)r^{m-2}e^{itr/2}dr\,. \tag{2.23}$$

We write $x_\omega = x - 2(x\omega)\omega$ for the reflection of x along the ω-axis. Then:

1. *The operator W_1 can be expressed as follows:*

$$W_1 u(x) = \int_\Sigma d\omega\int_{2x\omega}^\infty \widehat{K_V}(t,\omega)u(t\omega + x_\omega)dt. \tag{2.24}$$

2. For any $1 \leq p \leq \infty$, we have

$$\|W_1 u\|_{L^p(\mathbf{R}^m)} \leq 2\|\widehat{K}_V\|_{L^1([0,\infty)\times\Sigma)}\|u\|_{L^p(\mathbf{R}^m)}\,. \tag{2.25}$$

3. Let $\sigma > 1/2$ and $\rho > m/2 + \sigma$. Then, there exist constants C_1, C_2 such that

$$\|\widehat{K}_V\|_{L^1([0,\infty)\times\Sigma)} \leq C_1\|\langle x\rangle^\sigma V\|_{H^{(m-3)/2}} \leq C_2 \sum_{|\alpha|\leq \ell_0} \|D^\alpha V\|_{\ell^\infty_\rho(L^{p_0})}\,, \tag{2.26}$$

where p_0, ℓ_0 are as in Theorem 1.2.

Proof. We compute the Fourier transform of $W_1 u$. Performing the λ-integration first via the residue theorem, we see that it is equal to

$$\begin{aligned} &\frac{-1}{(2\pi i)}\frac{1}{(2\pi)^{m/2}}\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\left\{\int_{\mathbf{R}^m}\frac{\widehat{V}(\eta)\widehat{u}(\xi-\eta)d\eta}{(\xi^2-\lambda+i\varepsilon)((\xi-\eta)^2-\lambda-i\varepsilon)}\right\}d\lambda \\ &\qquad = \lim_{\varepsilon\downarrow 0}\frac{-1}{(2\pi)^{m/2}}\int_{\mathbf{R}^m}\frac{\widehat{V}(\eta)\widehat{u}(\xi-\eta)}{2\xi\eta-\eta^2+i\varepsilon}d\eta\,. \end{aligned} \tag{2.27}$$

We then invert the Fourier transform. Applying (2.22), we deduce

$$\begin{aligned} W_1 u(x) = &\frac{-1}{(2\pi)^{m/2}} \\ &\times\int_{\mathbf{R}^m}\frac{\widehat{V}(\eta)}{2i|\eta|}\left\{\int_0^\infty e^{-it|\eta|/2+i\eta(x+t\widehat{\eta})}u(x+t\widehat{\eta})dt\right\}d\eta\,. \end{aligned} \tag{2.28}$$

Introducing the polar coordinates $\eta = r\omega$, $r > 0$, $\omega \in \Sigma$, and changing the order of integration, we obtain

$$W_1 u(x) = \int_\Sigma d\omega\int_0^\infty dt\left\{\frac{i}{2(2\pi)^m}\int_0^\infty \widehat{V}(r\omega)e^{i(t+2x\omega)r/2}r^{m-2}dr\right\}u(x+t\omega)\,.$$

The identity (2.24) follows from this by the change of variable $t \to t - 2(x\omega)$. Observing that $x \to x_\omega$ is measure preserving, we apply Minkowski's inequality to (2.24) and obtain (2.25).

By Parseval-Plancherel formula we have

$$\int_0^\infty |\widehat{K}_V(t,\omega)|^2 dt = \frac{1}{2(2\pi)^{m-1}}\int_0^\infty |\widehat{V}(r\omega)|^2 r^{2m-4}dr.$$

Integrating both sides with respect to ω over Σ gives

$$\|\widehat{K}_V\|^2_{L^2([0,\infty)\times\Sigma)} = \frac{1}{2(2\pi)^{m-1}} \int_{\mathbf{R}^m} |\xi|^{m-3}|\widehat{V}(\xi)|^2 d\xi \le C\|V\|^2_{H^{(m-3)/2}}\,.$$

Similarly we have

$$\begin{aligned}\|t\widehat{K}_V\|^2_{L^2([0,\infty)\times\Sigma)} &\le C\int_{\mathbf{R}^m} |\xi|^{m-3}(|\nabla_\xi \widehat{V}(\xi)|^2 + |\xi|^{-2}|\widehat{V}(\xi)|^2)d\xi \\ &\le C\|\langle x\rangle V\|^2_{H^{(m-3)/2}}\,.\end{aligned}$$

Interpolating these two estimates by the complex interpolation method, we deduce that for any $\sigma > 1/2$,

$$\|\widehat{K}_V\|_{L^1([0,\infty)\times\Sigma)} \le C_\sigma\|\langle t\rangle^\sigma \widehat{K}_V\|_{L^2([0,\infty)\times\Sigma)} \le C_\sigma\|\langle x\rangle^\sigma V\|_{H^{(m-3)/2}}.$$

The second inequality of (2.26) is obvious since $p_0 \ge 2$. $\square$

3. Estimate at low energy

In what follows we assume that V satisfies the condition of Theorem 1.2 with $\ell = 0$. In this section, we prove that the low energy part $W_\pm \phi_1(H_0)^2 = \phi_1(H)W_\pm\phi_1(H_0)$ of $W_\pm$ is bounded in L^p, where $\phi_1 \in C_0^\infty(R^1)$ is such that $\phi_1(\lambda) = 1$ for $|\lambda| \le 1$ and $\phi_1(\lambda) = 0$ for $|\lambda| \ge 2$. We prove this for the case $m \ge 4$ is even only. Nevertheless, we state some results for the case $m \ge 3$ is odd as well when we think them of independent interest.

Since V is clearly very short range and $H = H_0 + V$ admits no positive eigenvalues ([2]), all statements in the previous section hold. Moreover, writing $V(x) = A(x)B(x)$ as before, we have the following properties which are all well known in scattering theory (cf. [1], [7], [14]):

1. $AR_0(\lambda \pm i0)B \equiv Q_0^\pm(\lambda) \in B(L^2)$ is uniformly bounded on $[0,\infty)$ and $1 + Q_0^\pm(\lambda)$ has a bounded inverse in $B(L^2)$ for all $\lambda \in [0,\infty)$. We have the resolvent equation (2.12):

$$R^\pm(\lambda) = R_0^\pm(\lambda) - R_0^\pm(\lambda)B(1 + Q_0^\pm(\lambda))^{-1}AR_0^\pm(\lambda). \tag{3.29}$$

2. $AR^\pm(\lambda)B$ are uniformly bounded in $B(L^2)$ and locally Hölder continuous on $[0,\infty)$.

3. A and B are H_0- as well as H-smooth in the sense of Kato:

$$\sup_{\epsilon>0}\int_{-\infty}^{\infty}\|AR_0(\lambda\pm i\epsilon)u\|^2 d\lambda \le C\|u\|^2; \tag{3.30}$$
$$\sup_{\epsilon>0}\int_{0}^{\infty}\|AR(\lambda\pm i\epsilon)u\|^2 d\lambda \le C\|u\|^2.$$

4. The wave operators $W_\pm$ exist and have the stationary expression (1.2) $\sim$ (1.3).

In virtue of Proposition 2.13 the L^p boundedness of $\phi_1(H)W_\pm\phi_1(H_0)$ is equivalent to that of $W_{2,low} = \phi_1(H)W_2\phi_1(H_0)$. We decompose $W_{2,low} = W^{(1)}_{2,low} + W^{(2)}_{2,low}$ by splitting the resolvent as $R^-(\lambda) = \tilde{R}^-(\lambda) + R(0)$ in the formula (1.3):

$$W^{(1)}_{2,low}u = \phi_1(H) \times \left\{\frac{1}{2\pi i}\int_0^\infty R_0^-(\lambda)VR(0)V(R_0^+(\lambda) - R_0^-(\lambda))d\lambda\right\}\phi_1(H_0)u, \tag{3.31}$$

$$W^{(2)}_{2,low}u = \phi_1(H) \times \left\{\frac{1}{2\pi i}\int_0^\infty R_0^-(\lambda)V\tilde{R}^-(\lambda)V(R_0^+(\lambda) - R_0^-(\lambda))d\lambda\right\}\phi_1(H_0)u. \tag{3.32}$$

We prove that $W^{(1)}_{2,low}$ and $W^{(2)}_{2,low}$ are both bounded in L^p separately.

We rewrite (3.31) as follows. By using that $R_0^+(\lambda) = R_0^-(\lambda)$ for $\lambda \le 0$, we extend the region of integration to the whole line and write

$$(W^{(1)}_{2,low}u, v) = \frac{1}{2\pi i}\int_0^\infty (AR(0)B\cdot A(R_0^+(\lambda) - R_0^-(\lambda))\phi_1(H_0)u, BR_0^+(\lambda)\phi_1(H)v)d\lambda.$$

Here, in virtue of (3.30), $AR_0^-(\lambda)\phi_1(H_0)u$ and $BR_0^+(\lambda)\phi_1(H)v$ are boundary values of L^2-valued Hardy functions in the lower and upper half planes respectively. Hence they are orthogonal to each other and we obtain

$$(W^{(1)}_{2,low}u, v) = \frac{1}{2\pi i}\int_0^\infty \langle VR(0)VR_0^+(\lambda)\phi_1(H_0)u, R_0^+(\lambda)\phi_1(H)v\rangle d\lambda. \tag{3.33}$$

Recall that $\phi_1(H_0), \phi_1(H)$ are bounded in L^p as shown in section 2. Denote the integral kernel of $R(0)$ by $K(x,y)$, the multiplication with the function

$M_y(x) = V(x)K(x, x-y)V(x-y)$ by M_y, and the translation by $y \in \mathbf{R}^m$ by τ_y. Then we write $VR(0)V$ in the form

$$\begin{aligned}(3.34)\qquad VR(0)Vu(x) &= \int_{\mathbf{R}^m} V(x)K(x, x-y)V(x-y)u(x-y)dy \\ &= \int_{\mathbf{R}^m} M_y\tau_y u(x)dy,\end{aligned}$$

and inserting (3.34) into (3.33), we obtain

$$\begin{aligned}(3.35)\quad &(W^{(1)}_{2,low}u, v) \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\int_{\mathbf{R}^m} \langle M_y R_0^+(\lambda)\phi_1(H_0)\tau_y u, R_0^+(\lambda)\phi_1(H)v\rangle dyd\lambda.\end{aligned}$$

Here the integral is absolutely convergent with respect to $dyd\lambda$. Indeed, for $\sigma > 1/2$ we have $\langle x\rangle^\sigma M_y(x) \in H^{(m-3)/2}(\mathbf{R}^m_x)$ for some $\sigma > 1/2$ in virtue of Lemma 2.11 and $\|M_y\|_{L^{m/2}(\mathbf{R}^m_x)} \le C\|\langle x\rangle^\sigma M_y(x)\|_{H^{(m-3)/2}(\mathbf{R}^m_x)}$ by Sobolev's lemma. Hence $|M_y|^{1/2}$ is H_0-smooth for every $y \in \mathbf{R}^m$ ([7]):

$$\int_{\mathbf{R}} \||M_y|^{1/2}R_0^\pm(\lambda)u\|^2 d\lambda \le C\|\langle x\rangle^\sigma M_y(x)\|_{H^{(m-3)/2}(\mathbf{R}^m_x)}\|u\|^2_{L^2}$$

and, thanks to (2.20) we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{\mathbf{R}^m} |\langle M_y R_0^+(\lambda)\phi_1(H_0)\tau_y u, R_0^+(\lambda)\phi_1(H)v\rangle| d\lambda dy \\ &\qquad \le C\|\phi_1(H_0)u\|_{L^2}\|\phi_1(H)v\|_{L^2}\int_{\mathbf{R}^m}\|\langle x\rangle^\sigma M_y\|_{H^{(m-3)/2}}dy < \infty.\end{aligned}$$

It follows by changing the order of integration in (3.35) that

$$\begin{aligned}(3.36)\quad &(W^{(1)}_{2,low}u, v) \\ &= \int_{\mathbf{R}^m}\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\langle R_0^-(\lambda)M_y R_0^+(\lambda)\phi(H_0)\tau_y u, \phi_1(H)v\rangle d\lambda\right\} dy\end{aligned}$$

and the application of Proposition 2.13 and (2.20) to (3.36) yields, with $\sigma > 1/2$ and $1/p + 1/q = 1$ that

$$|(W^{(1)}_{2,low}u, v)| \le C\int_{\mathbf{R}^m}\|\langle x\rangle^\sigma M_y\|_{H^{(m-3)/2}}dy \cdot \|u\|_{L^p}\|v\|_{L^q} \le C_1\|u\|_{L^p}\|v\|_{L^q}$$

Thus, we have proved the following lemma.

LEMMA 3.14. *$W^{(1)}_{2,low}$ is bounded in L^p for any $1 \le p \le \infty$.*

Before starting the proof of the L^p boundedness of $W^{(2)}_{2,low}$, we record some results about the differentiability of $R^{\pm}(\lambda)$ that are necessary in what follows. They are simple consequences of the resolvent equation (3.29), Lemma 2.1 and the decay property of the potential $D^\alpha V \in \ell^\infty_\delta(L^{p_0})$, and we omit the proof.

LEMMA 3.15. *Let $0 \le j \le (m+2)/2$ and $\epsilon > 0$. Then $R^{\pm}(\lambda)$ is j times differentiable as a $B(L^2_{j+1/2+\epsilon}, L^2_{-j-1/2-\epsilon})$ valued function of $\lambda \in (0, \infty)$.*

LEMMA 3.16. *Let $2 \le \rho \le (m+2)/2$ and $s > \rho + 1/2$. Then, for $0 < k < 1$,*

$$\text{(3.37)} \quad \|(d/dk)^j \tilde{R}^{\pm}(k^2)\|_{B(L^2_s, L^2_{-s})} \le \begin{cases} C_j k^{2-j}\langle \log k\rangle, & \text{if } \ m \ge 4; \\ C_j k^{1-j}, & \text{if } \ m = 3, \end{cases}$$

for $0 \le j \le \rho$.

We show that the integral kernel $W^{(2)}_{2,low}(x,y)$ of $W^{(2)}_{2,low}$ satisfies the criterion (1.6). Using the identity $(R_0^+(\lambda) - R_0^-(\lambda))\phi_1(H_0) = (R_0^+(\lambda) - R_0^-(\lambda))\phi_1(\lambda)$ and changing the variable $\lambda = k^2$, we write

$$\text{(3.38)} \quad W^{(2)}_{2,low} = \frac{1}{\pi i}\int_0^\infty \phi_1(H) R_0^-(k^2) V \tilde{R}^-(k^2) V (R_0^+(k^2) - R_0^-(k^2)) \\ \times \phi_1(H_0)\tilde{\phi}_1(k^2) k dk\,,$$

where $\tilde{\phi}_1 \in C_0^\infty(\mathbf{R})$ is such that $\tilde{\phi}_1(\lambda)\phi_1(\lambda) = \phi_1(\lambda)$, Hence, if we denote the integral kernels of $R_0^{\pm}(k^2)\phi_1(H_0)$ and $R_0^{\pm}(k^2)\phi_1(H)$ respectively by $G^{(*)}_{\pm}(x,y,k)$ and $G^{(**)}_{\pm}(x,y,k)$, and if we set $G^{(*)}_{\pm,k,y}(x) = G^{(*)}_{\pm}(x,y,k)$ and $G^{(**)}_{\pm,k,y}(x) = G^{(**)}_{\pm}(x,y,k)$, then $W^{(2)}_{2,low}(x,y)$ is given by $W^{(2)}_{2,low}(x,y) = W^{(2),+}_{2,low}(x,y) - W^{(2),-}_{2,low}(x,y)$, where

$$\text{(3.39)} \quad W^{(2),\pm}_{2,low}(x,y) = \frac{1}{\pi i}\int_0^\infty \tilde{\phi}(k^2)\langle \tilde{R}^-(k^2) V G^{(*)}_{\pm,k,y}, V G^{(**)}_{+,k,x}\rangle k dk,$$

Recall that the integral kernel of $R_0^{\pm}(k^2)$ is given by $G_{\pm}(x-y,k)$ (see (2.2)) and that we are assuming m is even. Expanding $(z \pm (it/2))^\nu$ in the

Hankel formula (2.3):

$$\tag{3.40} \begin{aligned} &\pm i\tfrac{z^\nu H_\nu^{(j)}(z)}{4(2\pi)^\nu} = \textstyle\sum_{s=0}^\nu C_{\nu s}^\pm e^{\pm iz} z^s H_{\nu s}^\pm(z), \\ &H_{\nu s}^\pm(z) = \textstyle\int_0^\infty e^{-t} t^{2\nu-s-1/2}\left(z \pm \frac{it}{2}\right)^{-1/2} dt. \end{aligned}$$

and introducing $\varphi(x,y) = |x-y| - |x|$, we decompose

$$\tag{3.41} \begin{aligned} G_{\pm,x,k}(y) &= e^{\pm ik|x|} \sum_{s=0}^{\nu} k^s C_{\nu s}^\pm \frac{e^{\pm ik\varphi(x,y)} H_{\nu s}^\pm(k|x-y|)}{|x-y|^{m-2-s}} \\ &\equiv e^{\pm ik|x|} \sum_{s=0}^{\nu} k^s G_{\pm,x,k,s}(y)\,, \end{aligned}$$

where $C_{\nu s}^\pm$ are constants and the definition of $G_{\pm,x,k,s}(y)$ should be obvious. We have obvious inequality $|\varphi(x,y)| \le |y|$. We decompose $G_\pm^{(*)}(x,y,k)$ and $G_\pm^{(**)}(x,y,k)$ accordingly: Write $\Phi_0(x,y)$ and $\Phi(x,y)$ for the kernels of $\phi(H_0)$ and $\phi(H)$ respectively, and define

$$\tag{3.42} \begin{aligned} G_{\pm,x,k,s}^{(*)}(y) &= \int_{\mathbf{R}^m} e^{\pm ik(|z|-|x|)} G_{\pm,z,k,s}(y)\Phi_0(z,x)dz\,; \\ G_{\pm,x,k,s}^{(**)}(y) &= \int_{\mathbf{R}^m} e^{\pm ik(|z|-|x|)} G_{\pm,z,k,s}(y)\Phi(z,x)dz\,. \end{aligned}$$

We have

$$\tag{3.43} \begin{aligned} G_{\pm,x,k}^{(*)}(y) &= e^{\pm ik|x|} \textstyle\sum_{s=0}^\nu k^s G_{\pm,x,k,s}^{(*)}(y)\,, \\ G_{\pm,x,k}^{(**)}(y) &= e^{\pm ik|x|} \textstyle\sum_{s=0}^\nu k^s G_{\pm,x,k,s}^{(**)}(y)\,, \end{aligned}$$

and inserting (3.43) into (3.39) yields

$$\tag{3.44} \begin{aligned} W_{2,low}^{(2),\pm}(x,y) = \sum_{s,s'=0}^{\nu} \frac{1}{\pi i} \int_0^\infty & e^{-ik(|x|\mp|y|)} \\ & \times\tilde\phi_1(k^2)\langle \widetilde{R}^-(k^2) V G_{\pm,y,k,s}^{(*)}, V G_{+,x,k,s'}^{(**)}\rangle k^{s+s'+1} dk\,. \end{aligned}$$

We write each summand in the RHS of (3.44)

$$\tag{3.45} T_{ss'}^\pm(x,y) = \int_0^\infty e^{-ik(|x|\mp|y|)}\tilde\phi_1(k^2) L_{ss'}^\pm(x,y,k) k^{s+s'+1} dk,$$

$$\tag{3.46} L_{ss'}^\pm(x,y,k) = (1/\pi i)\langle \widetilde{R}^-(k^2) V G_{\pm,y,k,s}^{(*)}, V G_{+,x,k,s'}^{(**)}\rangle\,.$$

Lemma 3.17. *Let $\alpha+\beta = 0, 1, \ldots, (m+2)/2$ and $s = 0, \ldots, (m-2)/2$. Then, for some $\epsilon > 0$,*

$$\|VD_k^\beta G^{(*)}_{\pm,x,k,s}\|_{L^2_{\alpha+1+\epsilon}} \le \begin{cases} C\langle x\rangle^{-m+s+3/2}k^{-1/2-\beta}, & \text{if } m \text{ is even;} \\ C\langle x\rangle^{-m+2+s}, & \text{if } m \text{ is odd,} \end{cases} \tag{3.47}$$

for $0 < k \le 2$. The estimate (3.47) remains true if $G^{(*)}_{\pm,x,k,s}$ is replaced by $G^{(**)}_{\pm,x,k,s}$.

Proof. We prove only the case m is even. We have $|k|x|(k|x| \pm (it/2))^{-1}| \le 1$ and

$$\begin{aligned} |D_k^\beta H^{\pm}_{\nu s}(k|x|)| &\le C|x|^\beta \left|\int_0^\infty e^{-t}t^{2\nu-s-1/2}(k|x| \pm (it/2))^{-1/2-\beta}dt\right| \\ &\le C|x|^\beta (k|x|)^{-1/2-\beta} = Ck^{-1/2-\beta}|x|^{-1/2} \end{aligned}$$

It follows that $|D_k^\beta G_{\pm,x,k,s}(y)| \le Ck^{-1/2-\beta}|x-y|^{3/2-m+s}\langle y\rangle^\beta$. On the other hand we know from Lemma 2.4 that $|\Phi_0(z,x)| \le C_N\langle z-x\rangle^{-N}$ for any N. Using these, we deduce from (3.42) that

$$|D_k^\beta G^{(*)}_{\pm,x,k,s}(y)| \le Ck^{-1/2-\beta}\langle x-y\rangle^{3/2-m+s}\langle y\rangle^\beta.$$

Since $\|V(y)\langle y\rangle^\beta\langle y\rangle^{\alpha+1+\epsilon}\|_{L^2(Q_n)} \le C\langle n\rangle^{\alpha+\beta+1+\epsilon-\delta}$ and $\delta-(\alpha+\beta+1+\epsilon) > m-1$ for sufficiently small $\epsilon > 0$, the estimate (3.47) for $G^{(*)}_{\pm,x,k,s}$ follows. The proof for $G^{(**)}_{\pm,x,k,s}$ is similar. □

Applying Lemma 2.1 and Lemma 3.17 with $\beta = 0$, we obtain that

$$|L^{\pm}_{ss'}(x,y,k)| \le Ck^{-1}\langle x\rangle^{-m+s'+3/2}\langle y\rangle^{-m+s+3/2}$$

and by integration

$$|T^{\pm}_{ss'}(x,y)| \le C\langle x\rangle^{-m+s'+3/2}\langle y\rangle^{-m+s+3/2}. \tag{3.48}$$

For improving the decay estimate of (3.48), we apply integrations by parts with respect to the variable k $\mu_{ss'} = \max\{s, s'\} + 2$ times in (3.45). A computation with Leibniz' formula shows that

$$
\begin{aligned}
&D_k^{\mu_{ss'}}(\tilde{\phi}(k^2)k^{s+s'+1}L^{\pm}_{ss'}(x,y,k)) \\
&= \textstyle\sum_{\alpha+\beta+\gamma=\mu_{ss'}} \\
&\times C_{\alpha\beta\gamma}\langle D_k^{\alpha}(\tilde{\phi}(k^2)k^{s+s'+1}\widetilde{R}^{-}(k^2))VD_k^{\beta}G^{(*)}_{\pm,y,k,s}, VD_k^{\gamma}G^{(**)}_{+,x,k,s'}\rangle
\end{aligned} \tag{3.49}
$$

and applying Lemma 3.17 and Lemma 3.16, we see that each summand in (3.49) is bounded in modulus by a constant times

$$
\begin{aligned}
&k^{s+s'+3-\alpha}\langle \log k\rangle k^{-1/2-\beta}\langle y\rangle^{-m+s+3/2}k^{-1/2-\gamma}\langle x\rangle^{-m+s'+3/2} \\
&\quad \le C\langle \log k\rangle \langle x\rangle^{-m+s'+3/2}\langle y\rangle^{-m+s+3/2}, \quad 0 \le k \le 2\,.
\end{aligned} \tag{3.50}
$$

It follows that no boundary terms appear in the following integration by parts:

$$
\begin{aligned}
T^{\pm}_{ss'}(x,y) &= \int_0^{\infty} \frac{(-D_k)^{\mu_{ss'}}(e^{-ik(|x|\mp|y|)})}{(|x|\mp|y|)^{\mu_{ss'}}}\tilde{\phi}(k^2)L^{\pm}_{ss'}(x,y,k)k^{s+s'+1}dk \\
&= \frac{1}{(|x|\mp|y|)^{\mu_{ss'}}} \\
&\quad \times \int_0^{\infty} e^{-ik(|x|\mp|y|)}D_k^{\mu_{ss'}}(\tilde{\phi}(k^2)L^{\pm}_{ss'}(x,y,k)k^{s+s'+1})dk
\end{aligned}
$$

and, in virtue of (3.49) $\sim$ (3.50),

$$
|T^{\pm}_{ss'}(x,y)| \le C_{s,s'}\langle x\rangle^{-m+s'+3/2}\langle y\rangle^{-m+s+3/2}||x|\mp|y||^{-\mu_{ss'}}
$$

Combining this with (3.48) and summing up for $0 \le s, s' \le \nu = (m-2)/2$, we obtain

$$
|W^{(2),\pm}_{2,low}(x,y)| \le \sum_{s,s'=0}^{\nu} C_{s,s'}\frac{\langle x\rangle^{-m+s'+3/2}\langle y\rangle^{-m+s+3/2}}{\langle |x|\mp|y|\rangle^{\mu_{ss'}}}\,. \tag{3.51}
$$

Now we can complete the proof of the following

LEMMA 3.18. *The functions $W^{(2),\pm}_{2,low}(x,y)$ satisfy the estimates* (1.6) *and the operator $W^{(2)}_{2,low}$ is bounded in L^p for any $1 \le p \le \infty$.*

PROOF. We integrate (3.51) with respect to the variable x by using the polar coordinates: The (s,s')-summand in the RHS produces a constant times

$$
\begin{aligned}
&\int_{\mathbf{R}^m} \frac{\langle x\rangle^{-m+s'+3/2}\langle y\rangle^{-m+s+3/2}}{\langle |x|\mp|y|\rangle^{\mu_{ss'}}}dx \\
&\leq C\int_0^\infty \frac{\langle r\rangle^{s'+1/2}dr}{\langle r-|y|\rangle^{\mu_{ss'}}\langle y\rangle^{m-s-3/2}} \qquad (3.52)\\
&\leq C\int_{-\infty}^\infty \frac{\langle r\rangle^{s'+1/2}+\langle y\rangle^{s'+1/2}}{\langle r\rangle^{\mu_{ss'}}\langle y\rangle^{m-s-3/2}}dr\,.
\end{aligned}
$$

Here $s'+1/2\leq m-s-3/2$, since $s+s'\leq m-2$, and the $\sup_{y\in\mathbf{R}^m}$ of the RHS is finite. Hence,

$$
\sup_{y\in\mathbf{R}^m}\int_{\mathbf{R}^m}|W^{\pm}_{2,low}(x,y)|dx<\infty\,.
$$

We may likewise prove the other relation of (1.6) and the lemma follows. □

4. Estimate at high energy

In this section we prove that the high energy part $\phi_2(H)W_2\phi_2(H_0)u$ of W_2 is also bounded in L^p. Recall that W_2 is given by (1.3):

$$
W_2u=\frac{1}{2\pi i}\int_0^\infty R_0^-(\lambda)VR^-(\lambda)V\{R_0^+(\lambda)-R_0^-(\lambda)\}ud\lambda
$$

and that $\phi_2\in C^\infty(\mathbf{R})$ is such that $\phi_2(\lambda)=1$ for $\lambda\geq 2$ and $\phi_2(\lambda)=0$ for $\lambda\leq 1$. As the argument in this section is very much similar to that of the previous section as well as of section 4 of [21], we shall be rather sketchy here.

Expand $R^-(\lambda)$ via the repeated use of the resolvent equation (3.29):

$$
R^-(\lambda)=\sum_{n=0}^{2N-1}(-1)^nR_0^-(\lambda)(VR_0^-(\lambda))^n+(R_0^-(\lambda)V)^NR^-(\lambda)(VR_0^-(\lambda))^N,
$$

and decompose $W_2=\sum_{n=2}^{2N+2}(-1)^nW^{(n)}$ accordingly, where $W^{(n)}$ is given by

$$
\begin{aligned}
W^{(n)}u&=\frac{1}{2\pi i}\int_0^\infty R_0^-(\lambda)(VR_0^-(\lambda))^{n-1}V\{R_0^+(\lambda)-R_0^-(\lambda)\}ud\lambda,\\
&\qquad n=2,\dots,2N+1;\\
W^{(2N+2)}u&=\frac{1}{2\pi i}\int_0^\infty R_0^-(\lambda)VF_N(\lambda)V\{R_0^+(\lambda)-R_0^-(\lambda)\}ud\lambda.
\end{aligned}
$$

Here we wrote $F_N(\lambda) = (R_0^-(\lambda)V)^N R^-(\lambda)(VR_0^-(\lambda))^N$. It is shown in section 2 of [21] by repeated application of the argument similar to the one used in the proof of Proposition 2.13 that $W^{(n)}u$, $n = 2, \ldots, 2N+1$, has the following expression: Set for $s_1, \ldots, s_n \in \mathbf{R}^1$ and $\omega_1, \ldots, \omega_n \in \Sigma$, Σ being the unit sphere of $\mathbf{R}^m$,

$$K_n(s_1, \cdots, s_n, \omega_1, \cdots, \omega_n) = C^n (s_1 \cdots s_n)^{m-2} \prod_{j=1}^n \widehat{V}(s_j\omega_j - s_{j-1}\omega_{j-1}),$$

where C is an absolute constant, whose precise value is not important here, and $s_j\omega_j = 0$ if $j = 0$; and denote its "Fourier transform" with respect to the radial variables $(s_1, \cdots, s_n)$ by

$$\begin{aligned}&\widehat{K}_n(t_1, \ldots, t_n, \omega_1, \ldots, \omega_n)\\ &\quad = \int_{[0,\infty)^n} e^{i\sum_{j=1}^n t_j s_j/2} K_n(s_1, \ldots, s_n, \omega_1, \ldots, \omega_n) ds_1 \cdots ds_n.\end{aligned}$$

Then $W^{(n)}u$, $n = 2, \ldots, 2N+1$, can be written in the form

$$\begin{aligned}W^{(n)}u(x) = &\int_{[0,\infty)^{n-1}\times I \times \Sigma^n}\\ &\times \widehat{K}_n(t_1, \ldots, t_{n-1}, \tau, \omega_1, \ldots, \omega_n) u(x_{\omega_n} + \rho) dt_1 \cdots dt_{n-1} d\tau d\omega_1 \cdots d\omega_n\end{aligned}$$

where $I = (2x \cdot \omega_n, \infty)$ is the range of the integration by the variable τ, $x_{\omega_n} = x - 2(\omega_n \cdot x)\omega_n$, is the reflection of x along ω_n, and $\rho = t_1\omega_1 + \cdots + t_{n-1}\omega_{n-1} + \tau\omega_n$. Since $x \to x_{\omega_n}$ is measure preserving and ρ is independent of x, Minkowski's inequality implies as in section 2 that

$$\|W^{(n)}u\|_{L^p} \le 2\|\widehat{K}_n\|_{L^1([0,\infty)^n \times \Sigma^n)} \|f\|_{L^p}, \qquad 1 \le p \le \infty. \tag{4.53}$$

We showed in Lemma 2.5 of [21] that for any $\sigma > 1$

$$\|\widehat{K}_n\|_{L^1([0,\infty)^n\times\Sigma^n)} \le C^n \|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m*}}^n.$$

Set $\rho = (m-2)/2$ if $m \ge 4$, $\rho = 0$ if $m = 3$ and $t = 2(m-1)/(m-3)$. If $m \ge 4$, we have $t\rho > m$ and, by Hölder's inequality,

$$\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m*}} \le \|\langle\xi\rangle^{-\rho}\|_{L^t} \|\langle\xi\rangle^\rho \mathcal{F}(\langle x\rangle^\sigma V)\|_{L^2} \le C\|\langle x\rangle^\sigma V\|_{H^\rho}$$

for any σ and this holds obviously if $m = 3$. On the other hand it is clearly possible to find $1 < \sigma < \delta$ such that

$$\|\langle x\rangle^\sigma V\|_{H^\rho} \le C_1 \sum_{|\alpha|\le \ell_0} \|D^\alpha V\|_{\ell^\infty_\delta(L^{p_0})}.$$

This proves that $W^{(n)}$ hence $\phi_2(H)W^{(n)}\phi_2(H_0)$ are bounded in L^p if $n = 2, \dots, 2N+1$.

For completing the proof of Theorem 1.2, it remains only to prove that the operator $\phi_2(H)W^{(2N+2)}\phi_2(H_0)$ is bounded in L^p. We write it in the following form:

$$\phi_2(H)\frac{1}{2\pi i}\left(\int_0^\infty R_0^-(\lambda)VF_N(\lambda)V\{R_0^+(\lambda) - R_0^-(\lambda)\}\tilde\phi_2(\lambda)d\lambda\right)\phi_2(H_0)\,.$$

Here $\tilde\phi_2 \in C^\infty(\mathbf{R})$ is such that $\tilde\phi_2(\lambda)\phi_2(\lambda) = \phi_2(\lambda)$ and $\tilde\phi_2(\lambda) = 0$ for $\lambda \le 1/2$. We need only prove that the operator inside the parenthesis

$$T_\pm = \int_0^\infty R_0^-(k^2)VF_N(k^2)VR_0^\pm(k^2)\tilde\phi_2(k^2)kdk$$

is bounded in L^p. The integral kernel $T_\pm(x,y)$ of $T_\pm$ can be computed as in the previous section and are given by

$$\begin{aligned}(4.54)\quad T_\pm(x,y) &= \int_0^\infty (F_N(k^2)VG_{\pm,y,k}, VG_{+,x,k})\tilde\phi_2(k^2)kdk \\ &= \int_0^\infty e^{-ik(|x|\mp|y|)}(F_N(k^2)V\tilde G_{\pm,y,k}, V\tilde G_{+,x,k})\tilde\phi_2(k^2)kdk,\end{aligned}$$

where we wrote as in (3.41):

$$G_{\pm,x,k}(y) = e^{\pm ik|x|}\sum_{s=0}^{\nu} k^s G_{\pm,x,k,s}(y) \equiv e^{\pm ik|x|}\tilde G_{\pm,x,k}(y). \tag{4.55}$$

Here, as can be easily see from (2.2) and (2.3), we have for $k \ge 1/4$:

$$|D_k^\rho \tilde G_{\pm,x,k}(y)| \le C_\rho\langle y\rangle^\rho|x-y|^{2-m}(1+k|x-y|)^{(m-3)/2}. \tag{4.56}$$

Using Lemma 2.1 and Lemma 2.2 for the mapping property and the decay of the resolvent in the k variable, we obtain as in section 4 of [21] that, for sufficiently large N,

$$|\tilde\phi_2(k^2)(F_N(k^2)VG_{\pm,y,k}, VG_{+,x,k})| \le C\langle k\rangle^{-3}\langle x\rangle^{-(m-1)/2}\langle y\rangle^{-(m-1)/2}.$$

Integrating with respect to the variable k gives

$$|T_\pm(x,y)| \le C\langle x\rangle^{-(m-1)/2}\langle y\rangle^{-(m-1)/2}. \tag{4.57}$$

which is, however, is not sufficient for $T_\pm(x,y)$ to satisfy the criterion (1.6). For proving that $T_\pm(x,y)$ enjoys better decay property, we perform integrations by parts $\mu = (m+2)/2$ times in (4.54) as in the previous section:

$$\begin{aligned} T_\pm(x,y) &= \int_0^\infty (|y| \mp |x|)^{-\mu}(D_k^\mu e^{-ik(|x|\pm|y|)}) \\ &\quad \cdot(F_N(k^2)V\tilde{G}_{\pm,y,k}, V\tilde{G}_{+,x,k})\tilde{\phi}_2(k^2)kdk \\ &= \sum_{\alpha+\beta+\gamma+\delta=\mu}\int_0^\infty \frac{e^{-ik(|x|-|y|)}}{(|x|\mp|y|)^\mu} \\ &\quad \times(D_k^\alpha F_N(k^2)VD_k^\beta\tilde{G}_{\pm,y,k}, VD_k^\gamma\tilde{G}_{+,x,k})D_k^\delta(\tilde{\phi}_2(k^2)k)dk. \end{aligned} \tag{4.58}$$

Note that we do not have to worry about singularities at $k=0$ because $\tilde{\phi}_2(k^2) = 0$ for $0 \le k \le 1/4$. By using again Lemma 2.1 and Lemma 2.2, we see that

$$\begin{aligned} &|(D_k^\alpha F_N(k^2)VD_k^\beta\tilde{G}_{\pm,y,k}, VD_k^\gamma\tilde{G}_{+,x,k})| \\ &\qquad \le C\langle k\rangle^{-3}\langle x\rangle^{-(m-1)/2}\langle y\rangle^{-(m-1)/2}. \end{aligned} \tag{4.59}$$

Thus applying (4.59) to (4.58), and combining the result with (4.57), we obtain

$$|T_\pm(x,y)| \le C\langle x\rangle^{-(m-1)/2}\langle y\rangle^{-(m-1)/2}\langle |x| \mp |y|\rangle^{-(m+2)/2}.$$

Thus the estimation as in the final paragraph of section 3 implies that $T_\pm(x,y)$ satisfies (1.6). Thus $\phi_2(H)W^{(2N+2)}\phi_2(H_0)$ is also bounded in L^p. This completes the proof of Theorem 1.2.

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