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Gorenstein quotient singularities of monomial type in dimension three

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Abstract. In this paper we give an explicit description of the construction of 3-dimensional smooth varieties coming from a crepant resolution of the underlying spaces of quotient singularities $\mathbb{C}^3/G$, which are defined by certain monomial type finite subgroups G of $SL(3, \mathbb{C})$. Moreover, we prove that the topological Euler number of these varieties equals the number of conjugacy classes of the corresponding acting group. The latter constitutes the verification of a part of the physicist's conjecture concerning "the orbifold Euler characteristics".

§1. Introduction

The purpose of this paper is to describe an algorithmic strategy for the construction of crepant resolution of the underlying spaces of quotient singularities $\mathbb{C}^3/G$, with certain monomial type defining groups $G \subset SL(3, \mathbb{C})$, and to give a direct interpretation of their "physical Euler characteristic" as the number of the conjugacy classes of G.

The present work contains the results presented in an author's talk, which was given at Research Institute for Mathematical Sciences of Kyoto University on 13th May,1994, and can be basically regarded as the continuation of her previous works [5] and [6].

The problem of the geometrical investigations of various topological properties of 3-dimensional desingularized orbit spaces with trivial canonical sheaf was initially arisen in superstring theory (cf. [2],[3]). In particular, it was turned out that, if we consider a smooth compact Kähler complex

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threefold M being equipped with an action of a finite group G, such that M has a G-invariant holomorphic volume form, then the quotient space M/G is provided with a physical *orbifold theory* whose Witten-index can be computed to be:

$$\chi(M,G) = \frac{1}{|G|}\sum_{gh=hg}\chi(M^{\langle g,h\rangle}).$$

The above summation runs over all the pairs of commuting elements of G, and $M^{\langle g,h\rangle}$ denotes the common fixed set of g and h.

In fact, $\chi(M,G)$ corresponds to the definition of the "*orbifold Euler characteristic*" of this theory (see [2], p.684). Direct arithmetical checking for several examples, like those of suitable discrete group actions on 3-dimensional tori, showed that one should expect $\chi(M,G)$ to be the *usual* Euler characteristic of an *appropriate desingularization* of M/G. More precisely we formulate the following:

Conjecture I. (global version [2],[3])
If $\omega_{M/G} \simeq \mathcal{O}_{M/G}$, then there exists a resolution of singularities $\widetilde{M/G}$ of M/G such that $\omega_{\widetilde{M/G}} \simeq \mathcal{O}_{\widetilde{M/G}}$ and

$$\chi(\widetilde{M/G}) = \chi(M,G).$$

Since G is finite, this conjecture follows from its local version, if one takes into account the natural stratification of the quotient space M/G being determined by the orbit types, as well as the additive and multiplicative properties of the topological Euler characteristic:

Conjecture II. (local version)
Let $G \subset SL(3,\mathbb{C})$ be a finite group. Then there exists a resolution of singularities $\sigma : \widetilde{X} \longrightarrow \mathbb{C}^3/G$ with $\omega_{\widetilde{X}} \simeq \mathcal{O}_{\widetilde{X}}$ and

$$\chi(\widetilde{X}) = \sharp\{\textit{conjugacy classes of } G\}.$$

From the viewpoint of algebraic geometry, the above conjecture says, in particular, that the minimal model of $\mathbb{C}^3/G$ is expected to be smooth (cf.[10]). Moreover, it gives an enumerative characterization of a "McKay Correspondence" for dimension 3. (For more general discussion on this correspondence and further intrinsic group-theoretical and cohomological properties of $\widetilde{X}$, the reader is referred to [7].)

Conjecture II was proved for abelian groups by Roan ([10]), and independently by Markushevich, Olshanetsky and Perelomov ([9]) by using the toric method. It was also proved for 5 other groups, for which X's can be expressed as hypersurfaces in $\mathbb{C}^4$: (i) for $WA_3{}^+, WB_3{}^+, WC_3{}^+$, where WX^+ denotes the positive determinant part of the Weyl group WX of a root system X, by Bertin and Markushevich ([1]), (ii) for H_{168}, by Markushevich ([8]), and (iii) for the icosahedral group I_{60}, by Roan ([12]). Recently the author proved Conjecture II for trihedral groups [5,6]:

Definition. *A trihedral group* is defined to be a finite group of the form $G = \langle A, T \rangle \subset SL(3,\mathbb{C})$, where $A \subset SL(3,\mathbb{C})$ is a finite group generated by diagonal matrices and

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Definition. Quotient singularities being defined by trihedral groups will be called *trihedral singularities.*

Definition. A resolution of singularities $f : Y \longrightarrow X$ of a normal variety X, where K_X is $\mathbb{Q}$-Cartier, is called *crepant* if $K_Y = f^* K_X$.

Theorem 1.1[5,6].

Let $X = \mathbb{C}^3/G$ be a quotient space by a trihedral group G. Then there exists a crepant resolution of singularities

$$f : \widetilde{X} \longrightarrow X,$$

and

$$\chi(\widetilde{X}) = \sharp\{\textit{conjugacy classes of } G\}.$$

Notation. For a diagonal matrix $g = \mathrm{diag}(\exp(2\pi ia/r), \exp(2\pi ib/r), \exp(2\pi ic/r)) \in SL(3,\mathbb{C})$, we use the abbreviation $g = \frac{1}{r}(a,b,c)$. If g belong to $SL(3,\mathbb{C})$, then we have $0 \leq a,b,c \leq r$ and $a+b+c = 0, 1$ or 2.

THEOREM 1.2 (Main Theorem).
The conjecture II holds true for the group of the following types:

(1) $G_1 = \langle A, S\rangle$
(2) $G_2 = \langle A, A', S\rangle$
(3) $G_3 = \langle A, S, T\rangle$ $(r \not\equiv 0 (mod\ 3))$
(4) $G_4 = \langle G_3, C\rangle$
(5) $G_5 = \langle C, S\rangle$

where $A := \frac{1}{r}(0,1,-1), A' := \frac{1}{r}(1,-1,0), C := \frac{1}{3}(1,1,1)$ *and*

$$S := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

REMARK 1.3.

(1) Obviously, these singularities are different from the trihedral ones. The defining groups of them belong, in fact, to that of type (B) and (D) of the "big classification table" of finite G's in $SL(3,\mathbb{C})$ (cf. [13]). Nevertheless, they are all monomial and solvable, and therefore the desired crepant resolutions of the corresponding quotient singularities can be constructed and treated by making use of the method which was applied in the trihedral case (cf. [5], [6]), i.e., by dividing by normal subgroups of G, by extending (in an appropriate way) the action of the so arising quotient groups on the toroidally resolved parts and, finally, by resolving the new singularities coming from the new fixed loci by means of suitable crepant morphisms.
(2) We should mention here, that our proof of Theorem 1.2. remains valid, even if we replace the group $\langle H\rangle$ by an arbitrary finite abelian subgroup of $SL(3,\mathbb{C})$.
(3) Throughout §2-§5, G' will denote the abelian subgroup of G which is generated by the diagonal matrices within G. (Note that G' is a normal subgroup of G for any G among those being given in formulation of Theorem 1.2.)

The rest of this paper is devoted to the proof of the statements of the Main Theorem. The proofs for the groups of types (1) and (2) are given in sections 2 and 3. In sections 4 and 5 we deal with the remaining types (3), (4) and (5) by using our Theorem 1.1 and the results of §3. We give some examples in the last section.

REMARK. After having finished this paper, we received a new preprint of Prof. S.-S. Roan([14]) which deals with the same problem. Roan's results also cover all the "remaining" groups of classes (B) and (D) of the (up to conjugation) classification table of finite subgroups of $SL(3,\mathbb{C})$ (cf.[13]). Combining his and our results with those of [1], [5], [6], [8], [9], [11] and [12], one obtains the verification of Conjecture II (and subsequently that of Conjecture I) in all the possible cases. For "finer" versions of this correspondence of "McKay type" we refer to the recent preprint [7].

§2. Proof for acting groups of type (1)

PROPOSITION 2.1.
Let $X = \mathbb{C}^3/G$, and $Y = \mathbb{C}^3/G'$. We can construct a "two-storey desingularization diagram":

$$\begin{array}{ccccc} & & & & \widetilde{X} \\ & & & & \downarrow{\scriptstyle \tau} \\ & & \widetilde{Y} & \xrightarrow{\widetilde{\mu}} & \widetilde{Y}/\mathbb{Z}_2 \\ & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \widetilde{\pi}} \\ \mathbb{C}^3 & \longrightarrow & \mathbb{C}^3/G' = Y & \xrightarrow{\mu} & \mathbb{C}^3/G = X \end{array}$$

where π is a resolution of the singularities of Y, $\widetilde{\pi}$ the induced morphism, τ a resolution of the singularities by $\mathbb{Z}_2$, and $\tau \circ \widetilde{\pi}$ is a crepant resolution of the singularities of X.

PROOF. Let us first recall the method of how one constructs the toric resolution of $Y = \mathbb{C}^3/G'$ (cf. [6], [9], [11]).

Let $\mathbb{R}^3$ be the 3-dimensional real vector space, $\{e^i | i = 1, 2, 3\}$ its standard base, L the lattice generated by e^1, e^2 and e^3, $N := L + \sum \mathbb{Z}v$, where the summation runs over all the elements $v = \frac{1}{r}(a,b,c) \in G'$, and

$$\sigma := \left\{ \sum_{i=1}^{3} x_i e^i \in \mathbb{R}^3, \quad x_i \geq 0, \forall i, 1 \leq i \leq 3 \right\}$$

the naturally defined rational convex polyhedral cone in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. The corresponding affine torus embedding X_σ is defined as $\mathrm{Spec}(\mathbb{C}[\check{\sigma} \cap M])$, where M is the dual lattice of N and $\check{\sigma}$ the dual cone of σ in $M_{\mathbb{R}}$ defined as $\check{\sigma} := \{\xi \in M_{\mathbb{R}} | \xi(x) \geq 0, \forall x \in \sigma\}$.

We define:$\Delta :=$ the simplex in $N_{\mathbb{R}}$

$$= \left\{ \sum_{i=1}^{3} x_i e^i \quad ; x_i \geq 0, \quad \sum_{i=1}^{3} x_i = 1 \right\},$$

$$t : N_{\mathbb{R}} \longrightarrow \mathbb{R} \qquad \sum_{i=1}^{3} x_i e^i \longmapsto \sum_{i=1}^{3} x_i$$

and

$$\Phi := \left\{ \frac{1}{r}(a,b,c) \in G' \mid a+b+c = r \right\}.$$

Lemma 2.2.

$Y = \mathbb{C}^3/G'$ corresponds to the toric variety which is induced by the cone σ within the lattice $N = L + \sum_{v \in \Phi} \mathbb{Z}v$.

Proof. Since $Y = \mathrm{Spec}(\mathbb{C}[x,y,z]^{G'})$, $x^i y^j z^k$ is G'-invariant if and only if $\alpha i + \beta j + \gamma k \in \mathbb{Z}$ for all $(\alpha, \beta, \gamma) \in G'$. $\square$

Remark 2.3. Let $\frac{1}{r}(a,b,c) \neq (0,0,0)$ be an element of G'. We have to make a distinction between two cases:

(1) $abc \neq 0$

(2) $abc = 0$

We denote by G_1 (resp. G_2) the set of the elements of G' fulfilling the property (1) (resp. the property (2)). Note that $G' \backslash \{e\} = G_1 \amalg G_2$. Correspondingly, we denote by Φ_1 (resp. by Φ_2) the set of lattice points from N satisfying (1) (resp. (2)). Φ admits the following splitting: $\Phi = \Phi_1 \amalg \Phi_2$.

Moreover, for $i = 1, 2$, we define two functions

$$\lambda_i \;:\; G_i \;\longrightarrow\; \Phi_i,$$

where λ_1 maps $g = \frac{1}{r}(a, b, c)$ $(a + b + c = r)$ and $g^{-1} = \frac{1}{r}(r - a, r - b, r - c)$ to the lattice point $\frac{1}{r}(a, b, c)$, and λ_2 maps $g = \frac{1}{r}(a, b, c)$ to the lattice point $\frac{1}{r}(a, b, c)$.

$$\{G_1\} \xrightarrow{2:1} \{\Phi_1\}\ , \qquad \{G_2\} \xrightarrow{1:1} \{\Phi_2\}\,.$$

Obviously, there exists a correspondence between the sets of elements of $G'\backslash\{e\}$ and Φ, which is 2:1 on G_1 and 1:1 on G_2. The elements of Φ correspond to the exceptional (prime) divisors of the toric resolution given below.

Claim I. There exists a unique toric resolution of Y for which $\mathbb{Z}_2$ acts symmetrically on the exceptional divisors.

Proof. We can construct a unique simplicial decomposition S of the triangle determined by e^1, e^2, e^3 with $\Phi \cup \bigcup_{i=1}^{3}\{e^i\}$ as the set of its vertices. (cf.[6], [9], [11]). $\square$

Claim II. If $\widetilde{Y} := X_S$ is the corresponding torus embedding, then X_S is non-singular.

Proof. It is sufficient to show that the $\sigma(s)$ are basic. We choose three $w^1, w^2, w^3 \in \Phi \cup \bigcup_{i=1}^{3}\{e^i\}$ which are linearly independent over $\mathbb{R}$. Assume that the simplex

$$\left\{\sum_{i=1}^{3}\alpha_i w^i \mid \alpha_i \geq 0,\ \sum_{i=1}^{3}\alpha_i = 1\right\}$$

intersects $\Phi \cup \bigcup_{i=1}^{3}\{e^i\}$ only at $\{w^i\}_{i=1}^3$.

The lattice N_0, which is generated by $\{w^i\}_{i=1}^3$, is a sublattice of N. If we assume that $N \neq N_0$, then there exists $\beta = \beta_1 w^1 + \beta_2 w^2 + \beta_3 w^3 \ \in\ N\backslash N_0$ $(0 \leq \beta_i < 1,\ \beta_i \in \mathbb{R}$ and the strict inequality holds at least for one i, $1 \leq i \leq 3.\ )$

Since $t(\beta) = \sum_{i=1}^{3} \beta_i t(w^i) = \sum_{i=1}^{3} \beta_i$, $0 < \sum_{i=1}^{3} \beta_i < 3$ and $t(N) \in \mathbb{Z}$, we get either $t(\beta) = 1$ or 2. If $t(\beta) = 2$, then we can replace β by $\beta' = \sum_{i=1}^{3}(1-\beta_i)w^i$. So we can always assume that $t(\beta) = 1$.

Now, there exists an element β in $\{\sum \alpha_i w^i \mid \alpha_i \geq 0, \sum \alpha_i = 1\} \cap (N - N_0)$, which is contained in $\Delta \cap N$. Since

$$N = \left\{ \bigcup_{v \in \Phi} (v \oplus L) \right\} \bigcup L,$$

$\Delta \cap N = \Phi \cup \bigcup_{i=1}^{3} \{e^i\}$. From our assumption, we conclude that $\beta = w^i$ for some i, which contradicts $\beta \notin N_0$. Therefore $N = N_0$. $\square$

We obtain a crepant resolution $\pi = \pi_S : \widetilde{Y} = X_S \longrightarrow \mathbb{C}^3/G' = Y$, because X_S is non-singular and Gorenstein.

Definition. G' will be called *of type (I) (resp. (II))*, if $|G'|$=odd (resp. even).

Claim III. Let F be the fixed locus on $\widetilde{Y}$ under the natural action of $\mathbb{Z}_2 \cong G/G'$, and E be the set of the exceptional divisors of $\widetilde{Y} \longrightarrow Y$. Then

$$F_0 := F \cap E = \begin{cases} 1 \text{ point}, & \text{if } G' \text{ is type of (I)} \\ 2 \text{ points}, & \text{if } G' \text{ is type of (II)}. \end{cases}$$

Proof. Considering the dual graph of the exceptional divisors occurring by the toric resolution and making use of Remark 2.3, we can identify the two exceptional divisors by the action of $\mathbb{Z}_2$ away from the "central component," i.e., the component in the center of the exceptional locus. There are 2 possibilities for this central component;

For G' of type (I): It consists of one point.

For G' of type (II): It consists of a divisor which is isomorphic to $\mathbb{P}^1$.

If $(y:z)$ is a coordinate of this $\mathbb{P}^1$, then the action of $\mathbb{Z}_2$ is described by:

$$(y:z) \longmapsto (-z:-y).$$

There are two fixed points with coordinates $(1:1)$ and $(1:-1)$ respectively. $\square$

Furthermore, the $\mathbb{Z}_2$-action in a neighbourhood U of a fixed point is analytically isomorphic to some linear action. (In the forthcoming statements, and up to a concrete choice of local coordinates, we shall write $\mathbb{C}^3$ instead of U.)

Now, we consider the resolution of the singularity of $\mathbb{C}^3$ by the group $\mathbb{Z}_2$.

F_0 consists of 1 or 2 points, and $F = F_0 \cup C'$, where C' is the strict transform of the fixed locus C in Y under the action of $\mathbb{Z}_2$.

Claim IV. If $Z := \mathbb{C}^3/\mathbb{Z}_2$, then $\chi(\widetilde{Z}) = \chi(\mathbb{C}^3, \mathbb{Z}_2) = 2$.

Proof. There is a representation of $\mathbb{Z}_2$ in $SL(3,\mathbb{C})$:

$$S' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The quotient singularities being created by S' are of the form $A_1 \times \{x\text{–axis}\}$. They are not isolated and therefore the corresponding exceptional divisor of the resolution is nothing but a $\mathbb{P}^1$-bundle over the x-axis. $\square$

In this way we can construct a crepant desingularization $\tau : \widetilde{X} \longrightarrow \widetilde{Y}/\mathbb{Z}_2$ by resolving the points of F_0. To complete the proof of Proposition 2.1, let $\widetilde{\pi} : \widetilde{Y}/\mathbb{Z}_2 \longrightarrow X$ denote the corresponding quotient map.

Claim V. The resolution $\tau \circ \widetilde{\pi} : \widetilde{X} \longrightarrow X$ is a crepant resolution of the quotient $X = \mathbb{C}^3/G$.

Proof. Obvious from the fact that both τ and π are crepant morphisms. $\square$

Lemma 2.4.
Let $X = \mathbb{C}^3/\langle G', S\rangle$, and $f : \widetilde{X} \longrightarrow X$ the crepant resolution as above. Then the Euler number of $\widetilde{X}$ is given by

$$\chi(\widetilde{X}) = \frac{1}{2}(|G'| - k) + 2k$$

where

$$k := \begin{cases} 1, & \text{if } G' \text{ is of type (I)} \\ 2, & \text{if } G' \text{ is of type (II).} \end{cases}$$

PROOF. Since G' is an abelian group, we have for a crepant toric resolution

$$\pi : \widetilde{Y} \longrightarrow Y = \mathbb{C}^3/G',$$

with $\chi(\widetilde{Y}) = |G'|$ (cf. [9],[11]).

By the action of $\mathbb{Z}_2$, the number of fixed points in the exceptional divisor by σ is equal to k. Hence,

$$\chi(\widetilde{Y}/\mathbb{Z}_2) = \frac{1}{2}(|G'| - k) + k.$$

By the resolution of the fixed loci, the Euler characteristic of each exceptional locus is 2 (by Claim IV).

Therefore,

$$\chi(\widetilde{X}) = \frac{1}{2}(|G'| - k) + 2k. \square$$

THEOREM 2.5.

$$\chi(\widetilde{X}) = \sharp\{\textit{conjugacy classes of } G\}.$$

PROOF.

(1) Case (I) : $|G'| = 2m + 1, (0 < m \in \mathbb{Z})$

For a nontrivial element $g \in G'$, there are two conjugate elements: g itself and SgS. There are m couples of this type. On the other hand, there exist 2 additional conjugacy classes of G, namely $[e]$ and $[S]$. Therefore, there are altogether $m + 2$ conjugacy classes in G.

Thus,

$$\begin{aligned}\chi(\widetilde{X}) &= \frac{1}{2}(|G'| - 1) + 2 \\ &= m + 2 \\ &= \sharp\{ \text{ conjugacy classes of } G\}.\end{aligned}$$

(2) Case (II) : $|G'| = 2m, (0 < m \in \mathbb{Z})$

There are 2 elements in the center of G': $e, a = \frac{1}{2}(0,1,1)$. The remaining $2m-2$ elements in G' are divided into $m-1$ conjugacy classes of G as in (1). Finally, there are 4 more conjugacy classes, namely: $e, a, [S], [aS]$. Therefore, there are altogether $m+3$ conjugacy classes in G.

Consequently,

$$\begin{aligned}\chi(\widetilde{X}) &= \frac{1}{2}(|G'|-2)+4 \\ &= m+3 \\ &= \sharp\{\text{ conjugacy classes of } G\}. \ \square\end{aligned}$$

§3. Proof for acting groups of type (2)

In this case, we proceed similarly as for the type (1). We first check the corresponding toric resolution:

Proposition 3.1. *There exists a toric resolution of Y for which $\mathbb{Z}_2$ acts symmetrically on the exceptional divisors.*

Proof. We show how we can construct a simplicial decomposition $\{\sigma(s)\}_{s\in S}$ of the simplex $\varDelta$ which is $\mathbb{Z}_2$-invariant and whose set of vertices is exactly $\Phi \cup \{e^i\}_{i=1}^3$.

Let us consider the distance d between $\frac{1}{r}(a,b,c)$ and $\frac{1}{2}(a,h,h)$, ($h := (b+c)/2$), given by

$$d\left(\frac{1}{r}(a,b,c), \frac{1}{2}(a,h,h)\right) = \left|\frac{b}{r} - \frac{1}{h}\right| + \left|\frac{c}{r} - \frac{1}{h}\right|.$$

The proof will be completed after the following steps:

(1) Find those lattice points $P = \frac{1}{r}(a,b,c)$, whose distance d is the minimum among the points for each a in Φ in the domain $D := \{1/2 \geq y\}$.

(2) Draw a triangle whose vertices are P and $P' = \frac{1}{r}(a,c,b)$ symmetrically.

(3) Decompose D whose vertices are all the P, P', $(1,0,0)$ and $(0,0,1)$, into simplices by using the vertices in Φ. Call this decomposition S_1.

(4) By the action of $\mathbb{Z}_2$, we obtain an S_2 for the other triangle. Therefore we obtain a "symmetric" resolution. $\square$

Next, we consider the singularities of $\widetilde{Y}/\mathbb{Z}_2$:

Proposition 3.2. *Let F be the fixed locus on $\widetilde{Y}$ under the action of $\mathbb{Z}_2$, and E be the set of the exceptional divisors of $\widetilde{Y} \longrightarrow Y$. Then*

$$F_0 := F \cap E = \{r \textit{ points }\}$$

Proof. Considering the dual graph of the exceptional divisors arising by the toric resolution and using Remark 2.3, we can identify the two exceptional divisors by the action of $\mathbb{Z}_2$ away from the central components which lie near the central part (from (1,0,0) to $\frac{1}{2}(0,1,1)$). Then there are 2 possibilities for the central component locally;

For G' of type (I): It consists of one point.

For G' of type (II): It consists of a divisor which is isomorphic to $\mathbb{P}^1$.

Analogously to the Claim III in section 2, there are n points in F_0.

Furthermore, the $\mathbb{Z}_2$-action in a neighbourhood of a fixed point is analytically isomorphic to some linear action. $\square$

Lemma 3.3.

Let $X := \mathbb{C}^3/\langle G', S\rangle$, and $f : \widetilde{X} \longrightarrow X$ be the crepant resolution as above. Then the Euler number of $\widetilde{X}$ is given by

$$\chi(\widetilde{X}) = \frac{1}{2}(|G'| - r) + 2r$$

Theorem 3.4.

$$\chi(\widetilde{X}) = \sharp\{\textit{conjugacy classes of } G\}.$$

Proof. $|G'| = r^2$. There are r elements of type $\frac{1}{r}(a_i, h_i, h_i) =: A_i$. For another nontrivial element $g \in G'$, there are two conjugate elements: g and SgS. There exist $\frac{r^2-r}{2}$ couples of this type and r additional conjugacy classes, namely: $[S]$ and $[A_iS]$. Therefore, there are altogether $\frac{r^2-r}{2} + 2r$ conjugacy classes in G.

Thus

$$\chi(\widetilde{X}) = \sharp\{\text{ conjugacy classes of } G\}. \square$$

§4. Proof for acting groups of type (3)

In this section, we assume that $r \equiv 1$ or $2 \pmod 3$.

Proposition 4.1.
Let $X = \mathbb{C}^3/G$, and $Y = \mathbb{C}^3/G'$. Then we can construct the following diagram:

$$\begin{array}{ccccc} & & & & \widetilde{X} \\ & & & & \downarrow{\scriptstyle\tau} \\ & & \widetilde{Y} & \xrightarrow{\widetilde{\mu}} & \widetilde{Y}/\mathfrak{S}_3 \\ & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\widetilde{\pi}} \\ \mathbb{C}^3 & \longrightarrow & \mathbb{C}^3/G' = Y & \xrightarrow{\mu} & \mathbb{C}^3/G = X \end{array}$$

where π is a resolution of the singularities of Y, $\widetilde{\pi}$ the induced morphism, τ a resolution of the singularities created by $\mathfrak{S}_3$, and $\tau \circ \widetilde{\pi}$ a crepant resolution of the singularities of X.

(As in §2,3 we have:)

Claim I. There exists a toric resolution of Y with $\mathfrak{S}_3$ acting symmetrically on the corresponding exceptional divisors.

Proof. We construct a simplicial decomposition satisfying the conditions of trihedral case [5] and of Proposition 3.1 in §3. $\square$

Claim II. Let F be the fixed locus on $\widetilde{Y}$ under the action of $\mathfrak{S}_3$, and E be the exceptional set of divisors of $\widetilde{Y} \longrightarrow Y$. Then

$$F_0 := F \cap E = \{\text{ 3r-2 points }\},$$

where one of them is due to the action of $\mathfrak{S}_3$, and all the others to that of $\mathbb{Z}_2$.

Proof. Considering the dual graph of the exceptional divisors occurring by the toric resolution and using away from Remark 2.3, we can identify the two exceptional divisors by the action of $\mathbb{Z}_2$ except the "central component," i.e., the component in the center of the exceptional locus. This central component is a point, which is fixed by the action of $\mathfrak{S}_3$. $\square$

Claim III. If $Z := \mathbb{C}^3/\mathfrak{S}_3$, then $\chi(\widetilde{Z}) = \chi(\mathbb{C}^3, \mathfrak{S}_3) = 3$.

Proof. There is an equivalent representation of T in $SL(3, \mathbb{C})$:

$$T' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

The quotient singularities being created by $\langle S, T' \rangle$ are the same as in the case of Theorem 1.2 (2). $\square$

Lemma 4.2.
Let $X = \mathbb{C}^3/ < G', S >$, and $f : \widetilde{X} \longrightarrow X$ the crepant resolution as above. Then the Euler number of $\widetilde{X}$ is given by

$$\chi(\widetilde{X}) = \frac{|G'| - 3(r-1) - 1}{6} + 3 + 2(r-1) \quad (|G'| = r^2)$$

Proof. Since G' is an abelian group, we have for a toric crepant resolution

$$\pi : \widetilde{Y} \longrightarrow Y = \mathbb{C}^3/G',$$

and $\chi(\widetilde{Y}) = |G'|$ (cf.[9],[11]).

By the action of $\mathfrak{S}_3$, the number of fixed points in the exceptional divisor by σ is equal to r, hence

$$\chi(\widetilde{Y}/\mathfrak{S}_3) = \frac{|G'| - 3(r-1) - 1}{6} + r.$$

By the resolution of the fixed loci, Euler characteristic of the exceptional locus is 2 or 3. (see Claim IV of §2 and Claim III of §4)

Therefore,

$$\chi(\widetilde{X}) = \frac{|G'| - 3(r-1) - 1}{6} + 2(r-1) + 3. \ \Box$$

THEOREM 4.3.

$$\chi(\widetilde{X}) = \sharp\{\textit{conjugacy classes of } G\}.$$

PROOF. For the 3(r-1) elements of type $\frac{1}{r}(a_i, h_i, h_i) =: A_i$, there are three conjugate elements: A_i, TA_iT^{-1} and $T^{-1}A_iT$. For the remaining nontrivial elements g, there are six conjugate elements: g, TgT^{-1}, $T^{-1}gT$, SgS, $STgT^{-1}S$ and $ST^{-1}gTS$. In addition, there are $r-1$ conjugacy classes of type $[A_iS]$. Finally, there are 3 more conjugacy classes: $[e]$, $[S]$ and $[T]$. So the number of the conjugacy classes is:

$$(r-1) + \frac{1}{6}\{r^2 - 3(r-1) - 1\} + (r-1) + 3.$$

Thus,

$$\chi(\widetilde{X}) = \sharp\{\text{ conjugacy classes of } G\}. \ \Box$$

§5. Proof for acting groups of types (4) and (5)

Let now G be an acting group of type (4) or (5).

PROPOSITION 5.1.
Let $X = \mathbb{C}^3/G$, and $Y = \mathbb{C}^3/G'$. Then we can again construct the following diagram:

$$\begin{tikzcd}
 & & \widetilde{X} \arrow[d, "\tau"] \\
 & \widetilde{Y} \arrow[r, "\widetilde{\mu}"] \arrow[d, "\pi"] & \widetilde{Y}/\mathfrak{S}_3 \arrow[d, "\widetilde{\pi}"] \\
\mathbb{C}^3 \arrow[r] & \mathbb{C}^3/G' = Y \arrow[r, "\mu"] & \mathbb{C}^3/G = X
\end{tikzcd}$$

where π is a resolution of the singularities of Y, $\widetilde{\pi}$ the induced morphism, τ a resolution of the singularities created by $\mathfrak{S}_3$, and $\tau \circ \widetilde{\pi}$ a crepant resolution of the singularities of X.

There exists a toric resolution of Y with $\mathfrak{S}_3$ acting symmetrically on the exceptional divisors as in §4.

In the center of the triangle whose vertices are (1,0,0), (0,1,1) and (0,0,1), there exists one $\mathbb{P}^2$ as a single exceptional component. For this reason, it is sufficient to consider the case in which $G' = \langle \frac{1}{3}(1,1,1) \rangle$.

Claim I. Let F be the fixed locus on $\widetilde{Y}$ under the action of $\mathfrak{S}_3$, and E be the set of the exceptional divisors of $\widetilde{Y} \longrightarrow Y$. Then

$$F_0 := F \cap E = \{3 \text{ points}\}.$$

Proof. Considering the dual graph of the exceptional divisors occurring by the toric resolution, there is only one possibility for the central component; it has to be a divisor which is isomorphic to $\mathbb{P}^2$. The number of the fixed points under the action of $\mathfrak{S}_3$ is three as in §4. Finally, the $\mathfrak{S}_3$-action in a neighbourhood of a fixed point is analytically isomorphic to some linear action. □

Lemma 5.2.
Let $X = \mathbb{C}^3/\langle G', T, S \rangle$, and $f : \widetilde{X} \longrightarrow X$ the crepant resolution as above. Then the Euler number of $\widetilde{X}$ equals

$$\chi(\widetilde{X}) = 9$$

Proof. For an abelian group G', we have a toric resolution

$$\pi : \widetilde{Y} \longrightarrow Y = \mathbb{C}^3/G',$$

and $\chi(\widetilde{Y}) = |G'| = 3$ (cf.[9],[11]).

By the action of $\mathfrak{S}_3$, the number of fixed points in the exceptional divisor by σ is three. Hence,

$$\chi(\widetilde{Y}/\mathfrak{A}_3) = \frac{1}{6}(|G'| - 3) + 3 = 3.$$

By the resolution of the fixed loci, the Euler characteristic of each exceptional locus is 3 (by Claim III in §4).

Therefore,

$$\chi(\widetilde{X}) = \frac{1}{6}(|G'| - 3) + 3 \times 3 = 9. \ \square$$

THEOREM 5.3.

$$\chi(\widetilde{X}) = \sharp\{\textit{conjugacy classes of } G\}.$$

PROOF. There are nine conjugacy classes in G: the identity, $\frac{1}{3}(1,1,1)$, $\frac{1}{3}(2,2,2)$, S,$\frac{1}{3}(1,1,1)S$,$\frac{1}{3}(2,2,2)S$,T,$\frac{1}{3}(1,1,1)$and $\frac{1}{3}(2,2,2)$. $\square$

Before proceeding to the general case of type (4), we consider firstly the type (5) of Theorem 1.2.

THEOREM 5.4.
For $G = G_5$, the conjecture II holds true.

PROOF. In a similar manner as above we construct a crepant resolution:

$$\begin{tikzcd}
 & & \widetilde{X} \arrow[d, "\tau"] \\
 & \widetilde{Y} \arrow[r, "\widetilde{\mu}"] \arrow[d, "\pi"] & \widetilde{Y}/\mathbb{Z}_2 \arrow[d, "\widetilde{\pi}"] \\
\mathbb{C}^3 \arrow[r] & \mathbb{C}^3/\langle C\rangle = Y \arrow[r, "\mu"] & \mathbb{C}^3/G_5 = X
\end{tikzcd}$$

There is an exceptional divisor which is isomorphic to $\mathbb{P}^2$ in $\widetilde{Y}$. Furthermore, there exist three singularities in $\widetilde{Y}/\mathbb{Z}_2$, which become three $\mathbb{P}^1$-bundles in $\widetilde{X}$. The Euler number of $\widetilde{X}$ is 6.

On the other hand, the conjugacy classes in G are also 6: id, C, C^2, S, CS and C^2S. $\square$

In general, we get the following result:

THEOREM 5.5.

Conjecture II is also true for acting groups G of type (4). In particular, the desingularization space $\widetilde{X}$ of $X = \mathbb{C}^3/G$ has Euler number

$$\chi(\widetilde{X}) = \frac{1}{2}(r^2 - 3r + 2) + 6r + 3,$$

which equals the number of the conjugacy classes in G.

Proof. Obviously,

$$\chi(\widetilde{Y}) = 3r^2.$$

There are 3+9(r-1) fixed points on the exceptional divisors in $\widetilde{Y}$, leading to 3+3(r-1) singularities on $\widetilde{Y}/\mathfrak{S}_3$, which means that

$$\chi(\widetilde{Y}/\mathfrak{S}_3) = \frac{1}{6}\{3r^2 - 9(r-1) - 3\} + 3(r-1) + 3$$

Then

$$\chi(\widetilde{X}) = 16\{3r^2 - 9(r-1) - 3\} + 2 \times 3(r-1) + 3 \times 3,$$

and this number coincides with the number of conjugacy classes of G, because $G_4 = G_3 \amalg G_3C \amalg G_3C^2$. $\square$

Hence, Main theorem (Theorem 1.2) is completely proved!

§6. Examples

In this section, we give some examples by drawing pictures, by means of which one could visualize the toric-geometrical part of our construction.

Example 1. Firstly, we present a G of type (1)-(I). We take, for instance, $G = \langle \frac{1}{5}(0,1,4), S \rangle$. Then we get a unique toric crepant resolution $\widetilde{Y}$ for the abelian normal subgroup $G' = \langle \frac{1}{5}(0,1,4) \rangle$ of G as follows:

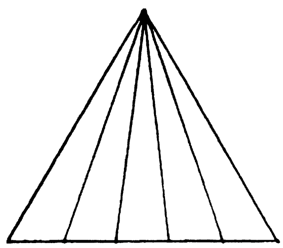

Fig. 1.1

By the action of $G/G' \cong \mathbb{Z}_2$, $\widetilde{Y}/\mathbb{Z}_2$ has one singularity.

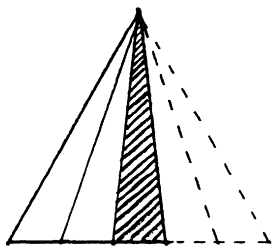

Fig. 1.2

Resolving it, we get $\widetilde{X}$. We can calculate the Euler number by using the following picture:

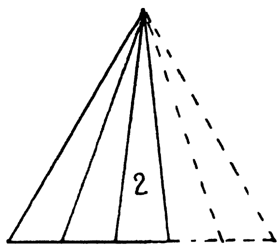

Fig. 1.3

We see that $\chi(\widetilde{X}) = 4$, which coincide with the number of the conjugacy classes of G.

For the rest of the section, let us draw similar pictures for some other cases.

Example 2. Let $G = \langle \frac{1}{4}(0,1,3), S\rangle$, whose type is (1)-(II). We can get $\chi(\widetilde{X}) = 5$ from the pictures below.

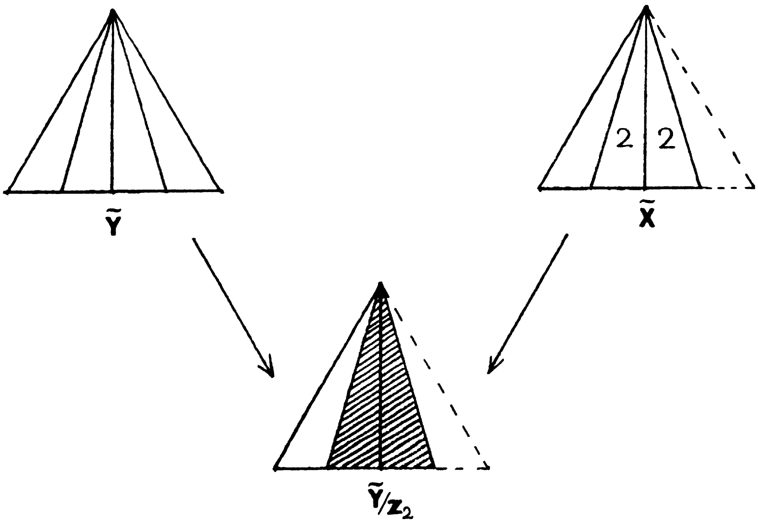

Fig. 2

Next example is one of most complicated in our construction.

Example 3. Let us consider $G = \langle \frac{1}{4}(0,1,3), S, T \rangle$ which is of type (3) with normal abelian subgroup $G' = \langle \frac{1}{4}(0,1,3), \frac{1}{4}(1,3,0) \rangle \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4$. Then we have two possibilities for the toric crepant resolution of $Y = \mathbb{C}^3/G'$ as follows:

Fig. 3.1

In each of these cases, we find the Euler number of $\widetilde{X}$ to be equal to 10. We draw the corresponding picture only for the second of them here.

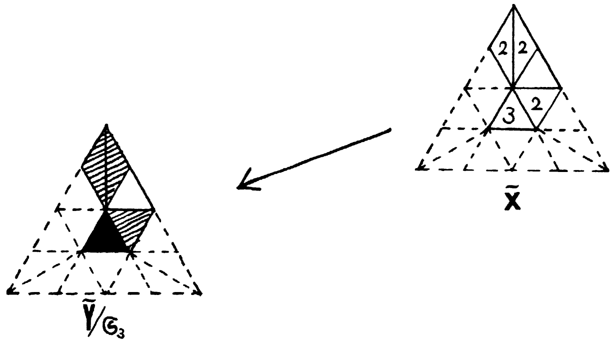

Fig. 3.2

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