J. Math. Sci. Univ. Tokyo
2 (1995), 77–116.

Asymptotic completeness for long-range many-particle systems with Stark effect

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Abstract. We prove the existence and the asymptotic completeness of the Graf-type modified wave operators for many-particle Stark Hamiltonians with long-range potentials.

1. Introduction

The problem of the asymptotic completeness for many–particle quantum systems has made a major progress for these several years. This problem was first solved by Sigal–Soffer [16] for a large class of short–range pair potentials. After that work, alternative proofs have been given by several authors [6, 11, 18, 23]. On the other hand, for the long–range case, Enss [4] first proved the completeness for three–particle systems with pair potentials decaying like $O(|x|^{-\nu})$ at infinity for some $\nu > \sqrt{3}-1$. This result has been extended by Dereziński [3] to N–particle systems and also the case of potentials decaying more slowly has been dealt with by [5, 21] for three–particle systems. We here study the problem of the asymptotic completeness for many–particle Stark Hamiltonians with long–range potentials. Recently the first author (T. Adachi) has proved the completeness of the modified wave operators for three–particle systems under the conditions that a uniform electric field is sufficiently strong and that any two–particle subsystem has a non–zero reduced charge ([1]). The second condition implies that no two–particle subsystems have bound states, so that scattering states have only a single channel. We extend this result to N–particle systems without assuming the two additional conditions above.

1991 *Mathematics Subject Classification.* Primary 81U10; Secondary 81V70.

We consider a system of N particles moving in a given constant electric field $\mathcal{E} \in \mathbf{R}^3$, $\mathcal{E} \neq 0$. Let m_j, e_j and $r_j \in \mathbf{R}^3$, $1 \le j \le N$, denote the mass, charge and position vector of the j–th particle, respectively. The N particles under consideration are supposed to interact with one another through the pair potentials $V_{jk}(r_j - r_k)$, $1 \le j < k \le N$. Then the total Hamiltonian for such a system is described by

$$\tilde{H} = \sum_{1\le j\le N} \left\{ -\frac{1}{2m_j}\Delta_{r_j} - e_j\mathcal{E}\cdot r_j \right\} + V,$$

where $\xi\cdot\eta = \sum_{j=1}^{3} \xi_j\eta_j$ for ξ, $\eta \in \mathbf{R}^3$ and the interaction V is given as the sum of the pair potentials

$$V = \sum_{1\le j<k\le N} V_{jk}(r_j - r_k).$$

As usual, we consider the Hamiltonian $\tilde{H}$ in the center–of–mass frame. We introduce the metric $\langle r, \tilde{r}\rangle = \sum_{j=1}^{N} m_j r_j \cdot \tilde{r}_j$ for $r = (r_1, \dots, r_N)$ and $\tilde{r} = (\tilde{r}_1, \dots, \tilde{r}_N) \in \mathbf{R}^{3\times N}$. We use the notations $r^2 = \langle r, r\rangle$ and $|r| = \langle r, r\rangle^{1/2}$. Let X and $X_\perp$ be the configuration spaces equipped with the metric $\langle\cdot,\cdot\rangle$, which are defined by

$$X = \{r \in \mathbf{R}^{3\times N} : \sum_{1\le j\le N} m_j r_j = 0\},$$
$$X_\perp = \{r \in \mathbf{R}^{3\times N} : r_j = r_k \text{ for } 1 \le j < k \le N\}.$$

These two subspaces are mutually orthogonal. We denote by $\pi : \mathbf{R}^{3\times N} \to X$ and $\pi_\perp : \mathbf{R}^{3\times N} \to X_\perp$ the orthogonal projections onto X and $X_\perp$, respectively. For $r \in \mathbf{R}^{3\times N}$, we write $x = \pi r$ and $x_\perp = \pi_\perp r$, respectively. Let $E \in X$ and $E_\perp \in X_\perp$ be defined by

$$E = \pi\left(\frac{e_1}{m_1}\mathcal{E}, \dots, \frac{e_N}{m_N}\mathcal{E}\right), \quad E_\perp = \pi_\perp\left(\frac{e_1}{m_1}\mathcal{E}, \dots, \frac{e_N}{m_N}\mathcal{E}\right),$$

respectively. Then the total energy Hamiltonian $\tilde{H}$ is decomposed into $\tilde{H} = H \otimes Id + Id \otimes T_\perp$, where Id is the identity operator, H is defined by

$$H = -\Delta/2 - \langle E, x\rangle + V \qquad \text{on } L^2(X),$$

$T_\perp$ denotes the free Hamiltonian $T_\perp = -\Delta_\perp/2 - \langle E_\perp, x_\perp \rangle$ acting on $L^2(X_\perp)$, and Δ (resp. $\Delta_\perp$) is the Laplace–Beltrami operator on X (resp. $X_\perp$). We assume that $|E| \neq 0$. This is equivalent to saying that $e_j/m_j \neq e_k/m_k$ for at least one pair (j,k). Then H is called an N–particle Stark Hamiltonian in the center–of–mass frame.

The problem of the asymptotic completeness is to determine completely the asymptotic states as time $t \to \pm\infty$ of the solutions $\psi(t) = \exp(-itH)\psi$ to the Schrödinger equation for all initial states $\psi \in L^2(X)$. The asymptotic behavior of the solutions depends on the values of e_j/m_j. If $e_j/m_j \neq e_k/m_k$ for any $j \neq k$, then each pair cluster has a non–zero reduced charge. Hence the N particles are expected to be scattered along the direction of $\mathcal{E}$ without forming bound states and also the solution $\psi(t)$ has a single channel as the asymptotic state. On the other hand, if, for example, $e_j/m_j = e_k/m_k$ for some pair (j,k), then the pair cluster (j,k) has a zero reduced charge and these particles may escape to infinity, forming a bound state at some energy. Therefore the solution $\psi(t)$ has scattering channels associated with such bound states as the asymptotic state. Thus the asymptotic behavior of the solutions is different according to the values of e_j/m_j. We shall discuss the matter more precisely. To do this, we require several basic notations in many–particle scattering theory.

A non–empty subset of the set $\{1,\dots,N\}$ is called a cluster. Let C_j, $1 \le j \le m$, be clusters. If $\cup_{1\le j\le m} C_j = \{1,\dots,N\}$ and $C_j \cap C_k = \emptyset$ for $1 \le j < k \le m$, $a = \{C_1,\dots,C_m\}$ is called a cluster decomposition. We denote by $\#(a)$ the number of clusters in a. We denote by $\tilde{\mathcal{A}}$ the set of cluster decompositions and set $\mathcal{A} = \{a \in \tilde{\mathcal{A}} : \#(a) \ge 2\}$. We let a, $b \in \tilde{\mathcal{A}}$. If b is obtained as a refinement of a, that is, if each cluster in b is a subset of a cluster in a, we say $b \subset a$, and its negation is denoted by $b \not\subset a$. We note that $a \subset a$ is regarded as a refinement of a itself. If, in particular, b is a strict refinement of a, that is, if $b \subset a$ and $b \neq a$, this relation is denoted by $b \subsetneq a$. We denote by $\alpha = (j,k)$ the $(N-1)$–cluster decomposition $\{(j,k),(1),\dots,\widehat{(j)},\dots,\widehat{(k)},\dots,(N)\}$.

Next we define the two subspaces X^a and X_a of X as

$$X^a = \{r \in X : \sum_{j\in C} m_j r_j = 0 \ \text{ for each cluster } \ C \ \text{ in } \ a\},$$
$$X_a = \{r \in X : r_j = r_k \ \text{ for each pair } \ \alpha = (j,k) \subset a\}.$$

We note that X^α is the configuration space for the relative position of j–th and k–th particles. Hence we can write $V_\alpha(x^\alpha) = V_{jk}(r_j - r_k)$. These spaces are mutually orthogonal and span the total space $X = X^a \oplus X_a$, so that $L^2(X)$ is decomposed as the tensor product $L^2(X) = L^2(X^a) \otimes L^2(X_a)$. We also denote by $\pi^a : X \to X^a$ and $\pi_a : X \to X_a$ the orthogonal projections onto X^a and X_a, respectively, and write $x^a = \pi^a x$ and $x_a = \pi_a x$ for a generic point $x \in X$. The intercluster interaction I_a is defined by

$$I_a(x) = \sum_{\alpha \not\subset a} V_\alpha(x^\alpha),$$

and the cluster Hamiltonian

$$H_a = H - I_a = -\Delta/2 - \langle E, x\rangle + V^a, \quad V^a(x^a) = \sum_{\alpha \subset a} V_\alpha(x^\alpha),$$

governs the motion of the system broken into non–interacting clusters of particles. Let $E^a = \pi^a E$ and $E_a = \pi_a E$. Then the operator H_a acting on $L^2(X)$ is decomposed into

$$H_a = H^a \otimes Id + Id \otimes T_a \qquad \text{on } L^2(X^a) \otimes L^2(X_a),$$

where H^a is the subsystem Hamiltonian defined by

$$H^a = -\Delta^a/2 - \langle E^a, x^a\rangle + V^a \qquad \text{on } L^2(X^a),$$

T_a is the free Hamiltonian defined by

$$T_a = -\Delta_a/2 - \langle E_a, x_a\rangle \qquad \text{on } L^2(X_a),$$

and Δ^a (resp. Δ_a) is the Laplace–Beltrami operator on X^a (resp. X_a). By choosing the coordinates system of X, which is denoted by $x = (x^a, x_a)$, appropriately, we can write $\Delta^a = (\nabla^a)^2$ and $\Delta_a = (\nabla_a)^2$, where $\nabla^a = \partial_{x^a} = \partial/\partial x^a$ and $\nabla_a = \partial_{x_a} = \partial/\partial x_a$ are the gradients on X^a and X_a, respectively. We note that we denote by x^a (resp. x_a) a vector in X^a (resp. X_a) as well as the coordinates system of X^a (resp. X_a).

We now state the precise assumption on the pair potentials. Let c be a maximal element of the set $\{a \in \mathcal{A} : E^a = 0\}$ with respect to the relation

$\subset$. As is easily seen, such a cluster decomposition uniquely exists and has the following property : $E \in X_c$ and $E \notin X_b$ for any $b \supsetneq c$. If, in particular, $e_j/m_j \neq e_k/m_k$ for any $j \neq k$, then c becomes the N–cluster decomposition. The potential V_α with $\alpha \not\subset c$ (resp. $\alpha \subset c$) describes the pair interaction between two particles with $e_j/m_j \neq e_k/m_k$ (resp. $e_j/m_j = e_k/m_k$). We make different assumptions on V_α according as $\alpha \not\subset c$ or $\alpha \subset c$. We assume that :

(V) $V_\alpha(x^\alpha) \in C^\infty(X^\alpha)$ is a real–valued function and has the decay property

$$\partial_{x^\alpha}^\beta V_\alpha(x^\alpha) = O(|x^\alpha|^{-\rho-|\beta|}), \qquad |x^\alpha| \to \infty,$$

with $\rho > 0$ for $\alpha \not\subset c$ and with $\rho > \sqrt{3} - 1$ for $\alpha \subset c$.

Under this assumption, all the Hamiltonians defined above are essentially self–adjoint on C_0^∞. We denote their closures by the same notations. Throughout the whole exposition, the notations c and ρ are used with the meanings described above. Without loss of generality, we may assume that $0 < \rho \leq 1/2$ for $\alpha \not\subset c$ and $\sqrt{3} - 1 < \rho \leq 1$ for $\alpha \subset c$. If V_α satisfies this decay assumption, then V_α is called a long–range potential.

To formulate the obtained result precisely, we define the modified wave operators. The definition requires several new notations. We assume that $a \subset c$. Then the subsystem operator H^a does not have a uniform electric field ($E^a = 0$). Hence it may have bound states in $L^2(X^a)$. We denote by $P^a : L^2(X^a) \to L^2(X^a)$ the eigenprojection associated with H^a. We also write p_a for the coordinates dual to x_a and denote by $D_a = -i\nabla_a$ the corresponding velocity operator. Let I_a^c be the intercluster interaction obtained from H^c:

$$I_a^c(x) = I_a^c(x^c) = \sum_{\alpha \subset c,\, \alpha \not\subset a} V_\alpha(x^\alpha).$$

We consider the time–dependent Hamiltonian

$$H_{aG}(t) = H_a + I_a^c(tD_a) + I_c(t^2E/2) \qquad \text{on } L^2(X). \tag{1.1}$$

The three operators on the right side commute with one another. This can be easily seen, if we take account of the fact that $I_a^c(tp_a) = I_a^c(t\pi^c p_a)$. Thus the propagator $U_{aG}(t)$ generated by $H_{aG}(t)$, that is, $\{U_{aG}(t)\}_{t\in\mathbf{R}}$ is a family of unitary operators such that for $\psi \in D(H_{aG}(0))$, $\psi_t = U_{aG}(t)\psi$ is a strong solution of $id\psi_t/dt = H_{aG}(t)\psi_t$, $\psi_0 = \psi$, is represented by

$$U_{aG}(t) = \exp(-itH_a)\exp\left(-i\int_0^t \{I_a^c(sD_a) + I_c(s^2E/2)\}\,ds\right).$$

With these notations, the modified wave operators in question are now defined by

$$W_{aG}^{\pm} = s - \lim_{t\to\pm\infty} \exp(itH)U_{aG}(t)(P^a \otimes Id), \quad a \subset c. \tag{1.2}$$

We can easily prove that if these wave operators exist, their ranges are all closed and mutually orthogonal

$$Range\, W_{aG}^{\pm} \perp Range\, W_{bG}^{\pm}, \quad a \neq b.$$

If $V_\alpha(x^\alpha)$ decays like $V_\alpha(x^\alpha) = O(|x^\alpha|^{-\nu})$, $\nu > 1/2$, for $\alpha \not\subset c$ and like $V_\alpha(x^\alpha) = O(|x^\alpha|^{-\nu})$, $\nu > 1$, for $\alpha \subset c$, V_α is called a short–range potential. For the class of short–range pair potentials, the ordinary wave operators

$$W_a^{\pm} = s - \lim_{t\to\pm\infty} \exp(itH)\exp(-itH_a)(P^a \otimes Id)$$

exist without the modifiers $I_a^c(tD_a)$ and $I_c(t^2E/2)$. The asymptotic completeness has been also proved by [12] for three–particle systems and by [19] for N–particle systems. However it is known ([9, 14]) that such wave operators do not generally exist for the class of long–range potentials which we consider here.

The main result of this paper is the following theorem.

THEOREM 1.1. *Assume that (V) is fulfilled. Let c be as above. Then the N–particle Stark Hamiltonian H has no bound states, and the wave operators $W_{aG}^{\pm}$, $a \subset c$, exist and are asymptotically complete*

$$L^2(X) = \sum_{a\subset c} \oplus\, Range\, W_{aG}^{\pm}.$$

If, in particular, c is the N–cluster decomposition, then the solution $\psi(t) = \exp(-itH)\psi$ has a single scattering channel as the asymptotic state for any initial state $\psi \in L^2(X)$ and behaves like

$$\psi(t) = \exp\left(-i\int_0^t V(s^2E/2)\,ds\right)\exp(-itH_0)\psi^{\pm} + o(1), \quad t \to \pm\infty,$$

for some $\psi^{\pm} \in L^2(X)$, where $o(1)$ denotes terms converging to 0 strongly and $H_0 = -\Delta/2 - \langle E, x\rangle$ is the free Hamiltonian.

We conclude the section by making a brief review on the results related to the main theorem above.

Remark 1.2. The long–range scattering problem for two–particle Stark Hamiltonians has been studied by several authors [7, 10, 22]. The above type of modified wave operators was first introduced by Graf [7], and, also for three–particle case, was used in [1]. They assumed that the pair potentials $V_\alpha(x^\alpha)$ satisfy the decay properties $\partial^\beta_{x^\alpha} V_\alpha(x^\alpha) = O(|x^\alpha|^{-\rho-\mu|\beta|})$ for some ρ, $\mu > 0$ with $\rho + \mu > 1$. We can also modify the assumption for the pair potentials $V_\alpha(x^\alpha)$ with $\alpha \not\subset c$ analogously in our case.

On the other hand, the works [10, 22] have dealt with another class of long–range potentials: $\partial^\beta_{x^\alpha} V_\alpha(x^\alpha) = O(|x^\alpha|^{-\nu-|\beta|/2})$ for some $\nu > 0$. In this case, the modified wave operators may be defined with $I_c(t^2E/2)$ replaced by $I_c(t^2E/2 + tD_c)$, if we take account of classical Stark trajectories. The completeness of such wave operators is much more difficult to prove and the argument here does not directly extend to this problem.

Remark 1.3. As previously stated, the completeness of long–range many–particle systems without uniform electric fields has been proved by [3]. The proof of the main theorem uses this result applied to the subsystem Hamiltonian H^c without uniform electric fields ($E^c = 0$). To this end, the decay assumption with $\rho > \sqrt{3} - 1$ is required for V_α, $\alpha \subset c$.

Remark 1.4. The non–existence of bound states has been already proved by [20] under the assumption that $|V_\alpha(x^\alpha)| + |\nabla V_\alpha(x^\alpha)| = O(|x^\alpha|^{-\nu})$, $\nu > 1/2$. See also Sigal [15] for a certain class of singular potentials. The argument developed in [20] extends to the class of potentials satisfying (V)

without any essential changes. We omit the proof of the non–existence of bound states.

Acknowledgements. T. Adachi thanks Professor Kenji Yajima for a lot of support and encouragement.

2. Conjugate operators

Throughout the discussion below, we always assume (V) to be fulfilled. The proof of the main theorem is based on the conjugate operator method initiated by Mourre [13]. We fix an energy $\lambda \in R$ arbitrarily. Let $f \in C_0^\infty(\mathbf{R})$ be a non–negative smooth function with support in a small interval around λ. Then a self–adjoint operator A is said to be a conjugate operator of H, if both $f(H)i[H,A]f(H)$ and $f(H)i[i[H,A],A]f(H)$ are bounded operators on $L^2(X)$ and if

$$f(H)i[H,A]f(H) \geq \sigma\, f(H)^2, \quad \sigma > 0, \tag{2.1}$$

where [,] denotes the commutator relation and the inequality is understood in the form sense over $L^2(X)$. Such a conjugate operator has been constructed in [20]. For later references, we here make a brief review on its construction.

We start by introducing some new notations. Let S_X be the unit ball in X. We define $S_0(X)$ by

$$S_0(X) = \{q \in C^\infty(X) : q(x) \text{ is homogeneous of degree zero outside } S_X\}.$$

We introduce a smooth non-negative partition of unity $\{j_a(x)\}_{a\in\mathcal{A}}$ over X such that :
$(j.1)$ $j_a(x) \in S_0(X)$ and $\sum_{a\in\mathcal{A}} j_a(x)^2 = 1$ over X.
$(j.2)$ $j_a(x)V_\alpha(x^\alpha) = O(|x|^{-\rho})$, $|x| \to \infty$, for any $\alpha \not\subset a$.
With these notations, the conjugate operator A in question is constructed in the form

$$A = \sum_{a\in\mathcal{A}} j_a B_a j_a + M\,\gamma, \quad M \gg 1,$$

where γ is defined by

$$\gamma = -(i/2)\left(\langle x/\langle x\rangle, \nabla\rangle + \langle \nabla, x/\langle x\rangle\rangle\right), \qquad \langle x\rangle = (1+|x|^2)^{1/2}, \tag{2.2}$$

and ∇ is the gradient on X. Below we summarize the properties of operators B_a.

(1) Let Λ and Λ_0 be the sets of cluster decompositions defined by

$$\Lambda = \{a \in \mathcal{A} : E_a \neq 0\}, \qquad \Lambda_0 = \{a \in \mathcal{A} : E_a = 0\}. \tag{2.3}$$

The operator B_a with $a \in \Lambda$ is defined as

$$B_a = -i\langle \widehat{\omega}_a, \nabla \rangle, \qquad \widehat{\omega}_a = E_a/|E_a| \in S_X \cap X_a. \tag{2.4}$$

By (j.2), H is approximated by the cluster Hamiltonian $H_a = H^a + T_a$ on the support of j_a. In fact, we can easily show that $j_a(f(H) - f(H_a)) : L^2(X) \to L^2(X)$ is a compact operator. Since H^a and B_a commute with each other, we have

$$f(H_a)i[H_a, B_a]f(H_a) = |E_a|\, f(H_a)^2, \quad |E_a| > 0. \tag{2.5}$$

(2) The operator B_a with $a \in \Lambda_0$ is also constructed to satisfy

$$f(H_a)i[H_a, B_a]f(H_a) \geq \sigma_a f(H_a)^2, \quad \sigma_a > 0. \tag{2.6}$$

The construction is done by induction on $a \in \Lambda_0$. We first consider the case $\#(a) = N-1$, assuming that such a cluster decomposition belongs to Λ_0. In this case, the cluster Hamiltonian H_a takes the form $H_a = H^a + T_a$, where $T_a = -\Delta_a/2$ does not have a uniform electric field ($E_a = 0$) and

$$H^a = -\Delta^a/2 - \langle E^a, x^a \rangle + V_\alpha, \quad \alpha = a,$$

with $E^a = E \neq 0$. We define B_a as $B_a = -i\langle \widehat{\omega}^a, \nabla \rangle$ with $\widehat{\omega}^a = E^a/|E^a| \in S_X \cap X^a$ and calculate

$$f(H^a)i[H^a, B_a]f(H^a) = |E^a|\, f(H^a)^2 + K_\alpha^a,$$

where $K_\alpha^a = f(H^a)i[V_\alpha, B_a]f(H^a) : L^2(X^a) \to L^2(X^a)$ is a compact operator by the assumption (V). Since H^a has no bound states, we can take the support of f so small that

$$f(H^a)i[H^a, B_a]f(H^a) \geq \sigma_a\, f(H^a)^2, \quad \sigma_a > 0, \tag{2.7}$$

in the form sense over $L^2(X^a)$. If H^a takes energy in a small interval around $\lambda - \theta$, $\theta \geq 0$, we can prove that the form inequality

$$f(H^a+\theta)i[H^a, B_a]f(H^a+\theta) \geq \sigma_a\, f(H^a+\theta)^2$$

is still valid with $\sigma_a > 0$ independent of θ. Since B_a acts on $L^2(X^a)$ and commutes with $T_a = -\Delta_a/2 \geq 0$, we obtain (2.6) for $a \in \Lambda_0$ with $\#(a) = N-1$ by making use of the direct integral representation.

(3) Before constructing B_a, $a \in \Lambda_0$, inductively, we shall explain the role of the operator γ. This operator is introduced to control error terms which arise from the commutators between H and the partition $\{j_a\}_{a\in\mathcal{A}}$. All the operators B_a are constructed as first order differential operators with smooth bounded coefficients. Hence these error terms take the form

$$\langle x\rangle^{-1/2}\nabla c_2 \nabla \langle x\rangle^{-1/2} + c_1\langle x\rangle^{-2}\nabla + c_0\langle x\rangle^{-3},$$

where c_j, $0 \leq j \leq 2$, are bounded smooth functions. As is easily seen, $f(H)(c_1\langle x\rangle^{-2}\nabla + c_0\langle x\rangle^{-3})f(H)$ is a compact operator on $L^2(X)$. But

$$f(H)\langle x\rangle^{-1/2}\nabla c_2 \nabla \langle x\rangle^{-1/2} f(H) : L^2(X) \to L^2(X)$$

is not necessarily compact for the Stark Hamiltonian H. We can show by a simple calculation that

$$f(H)i[H,\gamma]f(H) \geq f(H)\langle x\rangle^{-1/2}(-\Delta/2)\langle x\rangle^{-1/2}f(H) + K$$

for some compact operator K. Thus such error terms are controlled by choosing $M \gg 1$ large enough.

(4) We now construct B_a, $a \in \Lambda_0$, inductively. These operators are constructed as first order differential operators acting on X^a as well as on X. We introduce a partition of unity $\{j_b^a(x^a)\}_{b\subsetneq a}$ over X^a, which has the same properties as $(j.1)$ and $(j.2)$ with natural modifications. If $b \in \Lambda_0$, then we may assume by induction that B_b has been constructed so as to fulfill the form inequality (2.7) for H^b. We note that this B_b can be regarded as an operator acting on X^a. For $b \in \Lambda$, we define B_b by (2.4). Since

$$E_b = E - E^b = E^a - E^b \in X^a$$

for $a \in \Lambda_0$, B_b can be regarded as an operator acting on X^a. The operator B_a in question is now defined as

$$B_a = \sum_{b \subsetneq a} j_b^a B_b j_b^a + M_a \gamma_a, \quad M_a \gg 1,$$

where $\gamma_a = -(i/2)\left(\langle x^a/\langle x^a\rangle, \nabla\rangle + \langle \nabla, x^a/\langle x^a\rangle\rangle\right)$ plays the same role as γ. The subsystem operator H^a has non–zero uniform electric field $E^a = E \neq 0$ and hence it has no bound states. If we take account of this fact, then we can prove that B_a defined above satisfies (2.6). The basic form inequality (2.1) follows from (2.5) and (2.6).

Remark 2.1. In the work [20], the conjugate operator has been constructed under the same assumption as in Remark 1.4. However the argument there extends to the class of pair potentials satisfying (V) without any essential changes.

3. Commutator calculus

We always denote by A the conjugate operator constructed in the previous section and by Q the multiplication operator with $\langle x\rangle^{1/2}$. The commutator calculus for the operators H, A and Q is used as a basic tool in studying the propagation properties of $\exp(-itH)$. We here summarize some basic results on these commutators, which are often used without further references throughout the future discussion.

We start by making a brief review on the almost analytic extension method due to Helffer–Sjöstrand [8], which is useful in analyzing operators given by functions of self–adjoint operators. For two operators B_1 and B_2, we define

$$ad_{B_1}^0(B_2) = B_2, \quad ad_{B_1}^n(B_2) = [ad_{B_1}^{n-1}(B_2), B_1], \quad n \geq 1.$$

For $m \in \mathbf{R}$, let $\mathcal{F}^m$ be the set of functions $f \in C^\infty(\mathbf{R})$ such that

$$|f^{(k)}(s)| \leq C_k \langle s\rangle^{m-k}, \quad k \geq 0.$$

If $f \in \mathcal{F}^m$ with $m \in \mathbf{R}$, then there exists $F \in C^\infty(\mathbf{C})$ such that $F(s) = f(s)$ for $s \in \mathbf{R}$, $\operatorname{supp} F(\zeta) \subset \{\zeta \in \mathbf{C} : |Im\,\zeta| \leq d(1 + |Re\,\zeta|)\}$ for some $d > 0$ and

$$|\overline{\partial}_\zeta F(\zeta)| \leq C_M \langle\zeta\rangle^{m-1-M} |Im\,\zeta|^M, \quad M \geq 0.$$

Such a function $F(\zeta)$ is called an almost analytic extension of f. Let B be a self–adjoint operator. If $f \in \mathcal{F}^{-m}$ with $m > 0$, then $f(B)$ is represented by

$$f(B) = \frac{i}{2\pi} \int_{\mathbf{C}} \overline{\partial}_\zeta F(\zeta)(B-\zeta)^{-1}\, d\zeta \wedge d\overline{\zeta}. \tag{3.1}$$

For $f \in \mathcal{F}^m$ with $m \in \mathbf{R}$, we have the following formulas of the asymptotic expansion of the commutator:

$$[B_1, f(B)] = \sum_{n=1}^{M-1} \frac{(-1)^{n-1}}{n!} ad_B^n(B_1) f^{(n)}(B) + R_M, \tag{3.2}$$

$$R_M = \frac{1}{2\pi i} \int_{\mathbf{C}} \overline{\partial}_\zeta F(\zeta)(B-\zeta)^{-1} ad_B^M(B_1)(B-\zeta)^{-M}\, d\zeta \wedge d\overline{\zeta}. \tag{3.3}$$

R_M is bounded if there exists k such that $m+k < M$ and $ad_B^M(B_1)(B+i)^{-k}$ is bounded. For the proof, see [5].

LEMMA 3.1. *Let $f_j \in C_0^\infty(\mathbf{R})$, $1 \le j \le 2$, and let $g \in \mathcal{F}^0$. Assume that B is a self–adjoint operator such that $ad_B^j(H)(H+i)^{-1}$, $1 \le j \le 2$, are bounded from $L^2(X)$ into itself. Then one has :*

(1) $[f_1(H), g(B/t)] = [f_1(H), B/t]g'(B/t) + O(t^{-2})$.

(2) $[[f_1(H), B], f_2(B/t)] = O(t^{-1})$.

Here $O(t^{-\nu})$ denote bounded operators with their norm estimated by $C\, t^{-\nu}$ as $t \to \infty$.

PROOF. The lemma is more or less known (see, for example, Lemma 3.2 of [17]).

(1) Set $B_t = B/t$, $t \ge 1$. By assumption, we have

$$ad_B^2((H-\zeta)^{-1}) = O(\langle\zeta\rangle |Im\,\zeta|^{-2}) + O(\langle\zeta\rangle^2 |Im\,\zeta|^{-3}).$$

Hence it follows from (3.1) that $ad_{B_t}^2(f_1(H)) = O(t^{-2})$. Let $G \in C^\infty(\mathbf{C})$ be an almost analytic extension of g. Then, by the formulas (3.2) and (3.3), the commutator $[f_1(H), g(B_t)]$ under consideration is written as

$$[f_1(H), B_t]g'(B_t) + \frac{1}{2\pi i} \int_C \overline{\partial}_\zeta G(\zeta)(B_t-\zeta)^{-1} ad_{B_t}^2(f_1(H))(B_t-\zeta)^{-2}\, d\zeta \wedge d\overline{\zeta}.$$

This proves (1).

(2) This can be proved in the same way as above. By assumption, we have

$$[[f_1(H), B], (B_t - \zeta)^{-1}] = O(t^{-1})\, O(|Im\, \zeta|^{-2}).$$

Hence we can obtain (2) again by the almost analytic extension method. □

The lemma above is used with $B = A$ or Q. It is easy to see that these operators fulfill the assumption in the lemma. In fact, this is verified as follows. Let $\omega = E/|E|$ be the direction of E. We denote the coordinate $z \in \mathbf{R}$ by $z = \langle x, \omega \rangle$, so that H is written as $H = -\Delta/2 - |E|\, z + V$. Note that $\langle z \rangle \le \langle x^a \rangle$ for $a \in \Lambda_0$, Λ_0 being defined by (2.3). By construction, A takes the form

$$A = -i\,(\langle a(x), \nabla \rangle + \langle \nabla, a(x) \rangle),$$

where the coefficient $a(x)$ is a smooth function which values in X and obeys the estimates $|\partial_x^\beta a(x)| \le C_\beta \langle z \rangle^{-|\beta|}$. Hence A admits a unique self–adjoint realization on its natural domain in $L^2(X)$ and we denote its self–adjoint realization by the same notation A. As is easily seen,

$$\langle z \rangle^{-1/2} \nabla (H + i)^{-1}, \;\; \langle z \rangle^{-1} \nabla\nabla (H + i)^{-1} : L^2(X) \to L^2(X)$$

are bounded. Hence it follows from the assumption (V) that A satisfies the assumption in Lemma 3.1. This is verified also for Q in a similar way.

The next lemma can be also proved in almost the same way as Lemma 3.1. As for (3), we should note that $ad_Q^2(H)$ is bounded. We skip the proof.

Lemma 3.2. *Let $f_j \in C_0^\infty(\mathbf{R})$, $1 \le j \le 2$. Then one has :*

(1) $[(H+i)^{-1}, f_1(A/t)] = O(t^{-1})$, $\quad [(H+i)^{-1}, f_1(Q/t)] = O(t^{-1})$.

(2) $[Q, f_1(A/t)] = O(t^{-1})$, $\quad [f_2(Q/t), f_1(A/t)] = O(t^{-2})$.

(3) $(H + i)[f_1(H), f_2(Q/t)] = O(t^{-1})$.

We end the section by introducing the following convention for smooth cut–off functions F with $0 \le F \le 1$, which is often used throughout the discussion below. For $\delta > 0$ small enough, we define :

$$F(s < d) = 1 \;\text{ for }\; s \le d - \delta, \quad = 0 \;\text{ for }\; s \ge d + \delta,$$
$$F(s > d) = 1 \;\text{ for }\; s \ge d + \delta, \quad = 0 \;\text{ for }\; s \le d - \delta,$$
$$F(s = d) = 1 \;\text{ for }\; |s - d| \le \delta, \quad = 0 \;\text{ for }\; |s - d| \ge 2\,\delta$$

and $F(d_1 < s < d_2) = F(s > d_1)\, F(s < d_2)$.

4. Propagation properties

We prove the main theorem for the + case only. The important step toward the proof is to show that the solution $\exp(-itH)\psi$ concentrates asymptotically on classical Stark trajectories as $t \to \infty$. We define

$$\Gamma(\theta; d, \delta) = \{x \in X : |x| > d, \ \ |x/|x| - \theta| < \delta\}$$

for $\theta \in S_X$, S_X being again the unit sphere in X.

THEOREM 4.1. *Fix $0 < \epsilon \ll 1$ arbitrarily. Let $\omega = E/|E| \in S_X$ be again the direction of E and let $q_0 \in S_0(X)$ be a non–negative cut–off function such that q_0 is supported in $\Gamma(\omega; |E|/4, 2\epsilon)$ and $q_0 = 1$ on $\Gamma(\omega; |E|/3, \epsilon)$. Then*

$$s - \lim_{t\to\infty} \{1 - F(Q/t = (|E|/2)^{1/2}) q_0\} \exp(-itH) = 0.$$

The theorem above implies that the N particles escape to infinity along the direction of uniform electric field $\mathcal{E}$ as $t \to \infty$, which are accelerated in the configuration space X with the acceleration that is about E. This propagation property, which has played an important role in [1], has been proved under the assumption that $\mathcal{E}$ is sufficiently strong and any pair cluster has a non–zero reduced charge. We here prove this without such an additional assumption. The main body of the present work is occupied by the proof of Theorem 4.1.

We begin by fixing some new notations used in the proof of Theorem 4.1. Throughout the discussion below, we always denote by $f \in C_0^\infty(\mathbf{R})$ a non–negative function supported in a small interval around $\lambda \in \mathbf{R}$, λ being fixed arbitrarily. We use the notations $\| \ \|$ and $(\ , \ )$ for the L^2 norm and scalar product in $L^2(X)$, respectively. We also denote by $B(t)$, $t \ge 0$, operators having the following properties : (1) $f(H)B(t)f(H) : L^2(X) \to L^2(X)$ is bounded ; (2)

$$\int_1^\infty |(B(t)f(H)\exp(-itH)\psi, f(H)\exp(-itH)\psi)|\, dt \le C\, \|\psi\|^2, \qquad \psi \in L^2(X).$$

PROPOSITION 4.1. *There exists $M \gg 1$ such that*

$$\int_1^\infty \frac{dt}{t} \|F(Q/t < M^{-1}) f(H) \exp(-itH)\psi\|^2 \le C\,\|\psi\|^2$$

for $\psi \in L^2(X)$, where the choice of M depends on λ.

This proposition is obtained as an immediate consequence of the two lemmas below.

LEMMA 4.1. *There exists $d > 0$ such that*

$$\int_1^\infty \frac{dt}{t} \|F(|A|/t < d) f(H) \exp(-itH)\psi\|^2 \le C\,\|\psi\|^2.$$

LEMMA 4.2. *Let $d > 0$ be as above. Then one can take $M \gg 1$ so large that*

$$F(|A|/t > d) F(Q/t < M^{-1})(H + i)^{-1} = O(t^{-1}).$$

PROOF OF LEMMA 4.1. The proof is done in exactly the same way as in the proof of Lemma 4.1 of [17]. Let $G \in \mathcal{F}^0$ be defined by

$$G(s) = \int_{-\infty}^s F(|u| < d)^2\, du,$$

so that $G'(s) = F(|s| < d)^2 \in C_0^\infty(\mathbf{R})$. To prove the lemma, we use

$$\Phi_1(t) = G(A/t)$$

as a propagation observable. The Heisenberg derivative of this observable is calculated as

$$D\,\Phi_1(t) = i[H, \Phi_1(t)] + \Phi_1'(t). \tag{4.1}$$

If we take $g \in C_0^\infty(\mathbf{R})$ such that $g = 1$ on the support of f, then

$$f(H) i[H, \Phi_1(t)] f(H) = f(H) i[g(H)H, \Phi_1(t)] f(H).$$

By repeated use of Lemma 3.1, we have

$$f(H)i[H, \Phi_1(t)]f(H) = Ff(H)i[H, A/t]f(H)F + B(t)$$

with $F = F(|A|/t < d)$. Hence it follows from (2.1) that

$$f(H)i[H, \Phi_1(t)]f(H) \geq \sigma\, t^{-1} F f(H)^2 F + B(t)$$

for some $\sigma > 0$. On the other hand, the second term on the right side of (4.1) is evaluated as

$$f(H)\Phi_1'(t)f(H) \geq -2\, dt^{-1} f(H)F^2 f(H).$$

Thus we obtain

$$f(H)D\, \Phi_1(t)f(H) \geq (\sigma - 2d)t^{-1} f(H)F^2 f(H) + B(t)$$

by Lemma 3.1 again. This proves the lemma. □

PROOF OF LEMMA 4.2. We set $F_d = F(|A|/t > d)$ and $F_\kappa = F(Q/t < \kappa)$. Then $u = (H - i)^{-1} F_d F_\kappa w$ solves the equation

$$(H - i)u = F_d F_\kappa w, \qquad w \in L^2(X).$$

Recall that the conjugate operator A is a first order differential operator with smooth bounded coefficients. Hence it follows that $A^2 \leq C(-\Delta + 1)$ for some $C > 0$ and, by the boundedness of V and the fact that u is the solution of the above equation, we also have $\|Au\| \leq C(\|Qu\| + \|w\|)$ with another $C > 0$. Since $|A|F_d \geq (dt/2)F_d$ and $QF_\kappa \leq 2\kappa t F_\kappa$, we make repeated use of Lemma 3.2 to obtain that

$$\|Au\| \geq (dt/2)\|(H - i)^{-1} F_d F_\kappa w\| - C\,\|w\|,$$
$$\|Qu\| \leq 2\kappa t\|(H - i)^{-1} F_d F_\kappa w\| + C\,\|w\|$$

for some $C > 0$ independent of $t \gg 1$. Hence we can take $\kappa > 0$ so small that $F_d F_\kappa (H + i)^{-1} = O(t^{-1})$. This completes the proof. □

PROPOSITION 4.2. *There exists $M \gg 1$ dependent on λ such that :*

(1) *For* $\psi \in L^2(X)$,

$$\int_1^\infty \frac{dt}{t} \|F(Q/t = M) f(H) \exp(-itH)\psi\|^2 \leq C \|\psi\|^2.$$

(2) *For* $\psi \in \mathcal{S}(X)$, $\mathcal{S}(X)$ *being the Schwartz space over* X,

$$\int_1^\infty \frac{dt}{t} \|F(Q/t > M) f(H) \exp(-itH)\psi\|^2 < \infty.$$

PROOF. This proposition can be also proved in the same way as in the proof of Theorem 4.3 of [17] (also see [1]). We take the propagation observables

$$\Phi_2(t) = -F(Q/t > M) \;\text{ and }\; \Phi_3(t) = -(Q/t - M)F(Q/t > M)$$

to prove (1) and (2), respectively. The detailed proof is omitted. □

PROPOSITION 4.3. *Let* Λ *be defined by* (2.3). *Assume that* $q \in S_0(X)$ *vanishes in a small conical neighborhood of* $\widehat{\omega}_a = E_a/|E_a|$ *for all* $a \in \Lambda$. *Then there exists* $M \gg 1$ *large enough such that*

$$\int_1^\infty \frac{dt}{t} \|F(M^{-1} < Q/t < M) q f(H) \exp(-itH)\psi\|^2 \leq C \|\psi\|^2.$$

To prove the proposition above, we prepare two lemmas. We prove the first lemma only. The second lemma is obtained as an immediate consequence of the first one. We should note that the operator $D_\perp$ in the first lemma was first introduced by Yafaev [23], who has derived the radiation conditions–estimates for many–particle short–range systems without electric fields, and the second author (H. Tamura [19]) used it in order to get the same property as in Proposition 4.3 for many–particle short–range systems with constant electric fields.

LEMMA 4.3. *Denote by* Q_σ *the multiplication operator with* $\langle x\rangle^{\sigma/2}$*. Let* $D_\perp$ *be defined by* $D_\perp = -i\,\nabla - Q_{-1}\gamma Q_{-1}\,x$*, where* γ *is given by* (2.2)*. Then one has*

$$\int_1^\infty \frac{dt}{t}\|F(M^{-1} < Q/t < M)Q_{-1}D_\perp f(H)\exp(-itH)\psi\|^2 \le C\,\|\psi\|^2.$$

LEMMA 4.4. *If* $q \in S_0(X)$*, then one has*

$$\int_1^\infty \frac{dt}{t}\|F(M^{-1} < Q/t < M)Q[q,\Delta]f(H)\exp(-itH)\psi\|^2 \le C\,\|\psi\|^2.$$

PROOF OF LEMMA 4.3. We write $F_M = F_M(t)$ for $F(M^{-1} < Q/t < M)$ and take

$$\Phi_4(t) = F_M\Phi F_M, \qquad \Phi = Q_{-1/2}\gamma Q_{-1/2},$$

as a propagation observable. By Lemma 3.2 (3), we see that $f(H)\Phi_4(t)f(H)$ is uniformly bounded in $t \ge 1$. We calculate the Heisenberg derivative of $\Phi_4(t)$. Propositions 4.1 and 4.2 enable us to take $\tilde F_M \in C_0^\infty(\mathbf{R})$ such that $\tilde F_M = 1$ on the support of $F'(M^{-1} < s < M)$ and

$$\int_1^\infty \frac{dt}{t}\|\tilde F_M(Q/t)f(H)\exp(-itH)\psi\|^2 \le C\,\|\psi\|^2.$$

We write $\tilde F_{M,t} = \tilde F_M(Q/t)$. Since the support of ∇F_M or $\partial_t F_M$ lies in the support of $\tilde F_{M,t}$ which is the forbidden region of the propagator $\exp(-itH)$, it follows from Lemma 3.2 (3) that $f_1(H)\{D\,\Phi_4(t) - F_M i[H,\Phi]F_M\}f_1(H) = t^{-1}\tilde F_{M,t}C(t)\tilde F_{M,t} + O(t^{-2})$ for some $f_1 \in C_0^\infty(\mathbf{R})$ such that $f_1 f = f$ and for some $C(t)$ which is uniformly bounded in $t \ge 1$. Hence the Heisenberg derivative $D\,\Phi_4(t)$ takes the form

$$D\,\Phi_4(t) = F_M i[H,\Phi]F_M + B(t).$$

We now assert that

$$F_M i[H,\Phi]F_M = (3/2)\,F_M Q_{-3/2}(-\Delta - \gamma^2)Q_{-3/2}F_M + B(t). \tag{4.2}$$

If this is verified, then the lemma immediately follows, because a simple computation yields

$$F_M Q_{-3/2}(-\Delta - \gamma^2) Q_{-3/2} F_M = F_M Q_{-3/2} \langle D_\perp, D_\perp \rangle Q_{-3/2} F_M + B(t).$$

We calculate the commutator on the left side of (4.2) as

$$i[H, \Phi] = i[-\Delta/2, \Phi] + i[-|E|\, z, \Phi] + i[V, \Phi], \tag{4.3}$$

where $z \in \mathbf{R}$ again denotes $z = \langle x, \omega \rangle$. By the assumption (V), it follows that $F_M i[V, \Phi]\; F_M = O(t^{-3})$. The second operator on the right side takes the form

$$i[-|E|z, \Phi] = Q_{-3/2} |E| z Q_{-3/2} = Q_{-3/2}(-\Delta/2 - H + V) Q_{-3/2}$$

and hence

$$F_M i[-|E|z, \Phi] F_M = F_M Q_{-3/2}(-\Delta/2) Q_{-3/2} F_M + B(t).$$

Finally we look at the first operator on the right side of (4.3). Since

$$i[-\Delta/2, Q_{-1/2}] = -(1/4) Q_{-5/2} \gamma + O(\langle x \rangle^{-9/4}), \tag{4.4}$$

it follows that

$$F_M i[-\Delta/2, Q_{-1/2}] \gamma Q_{-1/2} F_M = -(1/4) F_M Q_{-3/2} \gamma^2 Q_{-3/2} F_M + B(t)$$

and also we have

$$F_M Q_{-1/2} i[-\Delta/2, \gamma] Q_{-1/2} F_M = F_M Q_{-3/2}(-\Delta - \gamma^2) Q_{-3/2} F_M + B(t)$$

by a direct calculation. Thus we combine the two relations above to obtain that

$$F_M i[-\Delta/2, \Phi] F_M = F_M Q_{-3/2}(-\Delta - 3\gamma^2/2) Q_{-3/2} F_M + B(t).$$

This yields (4.2) and the proof is complete. □

PROOF OF PROPOSITION 4.3. The proof is long and is divided into several steps. Throughout the proof, we use the notation Q_σ and F_M with the same meanings as above. We also use the notation

$$con\, supp\, q = \overline{\{\theta \in S_X : q(x) = q(|x|\theta) \neq 0,\ |x| > 1\}}$$

for $q \in S_0(X)$. Let q be as in the proposition. By assumption, there exists $d_0 > 0$ such that

$$1 - |\langle \widehat{\omega}_a, \theta\rangle| \geq d_0, \quad \theta \in con\, supp\, q, \tag{4.5}$$

for all $a \in \Lambda$.

(1) We define the subset S_a, $a \in \mathcal{A}$, of S_X by

$$S_a = \{\theta = (\theta_1, \dots, \theta_N) \in S_X : \theta_j = \theta_k \ \text{ for } \ \alpha \subset a,\ \ \theta_j \neq \theta_k \ \text{ for } \ \alpha \not\subset a\}.$$

By definition, $\{S_a\}_{a\in\mathcal{A}}$ becomes a family of disjoint subsets and $S_X = \cup_{a\in\mathcal{A}} S_a$. The first step toward the proof is to construct a smooth non–negative partition of unity $\{k_a\}_{a\in\mathcal{A}}$ over X with the following properties :

(k.1) $k_a \in S_0(X)$ and $\sum_{a\in\mathcal{A}} k_a(x)^2 = 1$ over X.

(k.2) $con\, supp\, k_a \cap con\, supp\, k_b \neq \emptyset \Longrightarrow a \subset b$ or $b \subset a$.

(k.3) $S_a \cap con\, supp\, k_b \neq \emptyset \Longrightarrow a \subset b$ and hence $S_a \subset \cup_{a\subset b} con\, supp\, k_b$.

(k.4) $con\, supp\, k_a \subset \{\theta \in S_X : |\theta^a| < \delta\}$ for $\delta > 0$ small enough.

Here $\theta^a = \pi^a\theta$ is the projection onto X^a of θ and the choice of $\delta > 0$ depends on the value of d_0 in (4.5). Such a partition of unity can be easily constructed by use of a simple geometrical properties of $\{S_a\}$ ([18]). By construction, it follows from (k.2) and (k.3) that

$$k_a(x) V_\alpha(x^\alpha) = O(|x|^{-\rho}), \qquad |x| \to \infty, \tag{4.6}$$

for $\alpha \not\subset a$ and also we have by (k.1) that

$$q(x)^2 = \sum_a q_a(x)^2, \qquad q_a = k_a q \in S_0(X). \tag{4.7}$$

(2) We recall the properties of the operators B_a, $a \in \mathcal{A}$, constructed in section 2. These are first order differential operators with smooth bounded

coefficients. In particular, B_a, $a \in \Lambda$, is defined by $B_a = -i \langle \widehat{\omega}_a, \nabla \rangle$, so that $i[H_a, B_a] = |E_a| > 0$. The operator B_a, $a \in \Lambda_0$, also has the property

$$f(H_a) i\, [H_a, B_a] f(H_a) \geq \sigma_a\, f(H_a)^2, \quad \sigma_a > 0. \tag{4.8}$$

With these operators, we now define the observable $\Phi_5(t)$ by

$$\Phi_5(t) = \sum_a Y_a(t) = \sum_a F_M \Phi_a F_M, \quad \Phi_a = Q_{-1/2} q_a B_a q_a Q_{-1/2}.$$

We see that $f(H)\Phi_5(t) f(H)$ is uniformly bounded in $t \geq 1$. We assert that the Heisenberg derivative of $Y_a(t)$ is evaluated as

$$f(H) D\, Y_a(t) f(H) \geq d_a\, f(H) F_M Q_{-1/2} q_a^2 Q_{-1/2} F_M f(H) + B(t) \tag{4.9}$$

for some $d_a > 0$. If this is proved, then the proposition follows from (4.7) at once.

(3) We first consider the case $a \in \Lambda$. By Propositions 4.1 and 4.2, we have

$$D\, Y_a(t) = F_M i[H, \Phi_a] F_M + B(t).$$

The commutator on the right side is calculated as

$$i[H, \Phi_a] = Q_{-1/2} q_a i[H, B_a] q_a Q_{-1/2} + G_{1a} + G_{2a},$$

where

$$G_{1a} = i[-\Delta/2, Q_{-1/2}] q_a B_a q_a Q_{-1/2} + \{\text{adjoint}\},$$
$$G_{2a} = Q_{-1/2} i[-\Delta/2, q_a] B_a q_a Q_{-1/2} + \{\text{adjoint}\}.$$

Since $\gamma B_a + B_a \gamma \leq -2\,\Delta + d$ for some $d > 0$, it follows from (4.4) that

$$F_M G_{1a} F_M \geq F_M Q_{-3/2} q_a (\Delta/2) q_a Q_{-3/2} F_M + B(t).$$

We put $G = Q i[-\Delta/2, q_a] Q_{-1/2}$. Then we have by Lemma 4.4

$$\int_1^\infty \|G F_M f(H) \exp(-itH)\psi\|^2\, dt \leq C \|\psi\|^2.$$

Since $B_a^2 \leq -\Delta + d$ for some $d > 0$ and since

$$G_{2a} \geq -\epsilon Q_{-1/2} q_a B_a^2 q_a Q_{-1/2} - \epsilon^{-1} G^* G$$

for any $\epsilon > 0$, we have

$$F_M G_{2a} F_M \geq \epsilon\, F_M Q_{-3/2} q_a (\Delta/2) q_a Q_{-3/2} F_M + B(t)$$

for any $\epsilon > 0$ small enough. Thus, if we make use of the relation $\Delta/2 = -|E|z - H + V$, then we obtain

$$D\,Y_a(t) \geq F_M Q_{-1/2} q_a (i[H, B_a] - (1+\epsilon)|E|z/\langle x \rangle) q_a Q_{-1/2} F_M + B(t).$$

(4) We continue to consider the case $a \in \Lambda$. We write

$$i[H, B_a] = |E_a| + \sum_{\alpha \not\subset a} i[V_\alpha, B_a].$$

By (4.6), the second operator on the right side obeys

$$F_M Q_{-1/2} q_a [V_\alpha, B_a] q_a Q_{-1/2} F_M = O(t^{-3-2\rho}).$$

Hence (4.9) is obtained for $a \in \Lambda$, if it can be shown that

$$|E_a| - (1+\epsilon)\langle E, x/\langle x \rangle \rangle > 0$$

strictly for $|x| \gg 1$ with $x \in \mathit{supp}\, q_a$. Let $d_0 > 0$ be as in (4.5). Since

$$\langle E, x/\langle x \rangle \rangle = \langle E^a, x^a/\langle x \rangle \rangle + \langle E_a, x_a/\langle x \rangle \rangle,$$

we can take δ in $(k.4)$ so small that

$$|\langle E, x/\langle x \rangle \rangle| \leq |E_a|(|\langle \widehat{\omega}_a, \theta \rangle| + d_0/2) \leq |E_a|(1 - d_0/2)$$

for $|x| \gg 1$ with $\theta = x/|x| \in \mathit{con\, supp}\, k_a$. If ϵ is further chosen small enough, then we can make the quantity in question strictly positive and hence (4.9) is obtained for $a \in \Lambda$.

(5) The proof is completed in this step. We prove (4.9) also for $a \in \Lambda_0$. The operator B_a, $a \in \Lambda_0$, satisfies $\gamma B_a + B_a \gamma \le d(-\Delta + 1)$ for some $d > 0$. Hence we make use of the same argument as in step (3) to obtain that

$$D\,Y_a(t) \ge F_M Q_{-1/2} q_a (i[H_a, B_a] - d\,\langle E, x/\langle x \rangle\rangle) q_a Q_{-1/2} F_M + B(t)$$

with another $d > 0$. The Hamiltonian H can be approximated by H_a on the support of q_a. Indeed, we can show that

$$F_M q_a f(H) - f(H_a) F_M q_a = O(t^{-\min(1,2\rho)}).$$

Hence it follows from (4.8) that

$$\begin{aligned} & f(H) D\,Y_a(t) f(H) \\ & \qquad \ge f(H) F_M Q_{-1/2} q_a \{\sigma_a - d\,\langle E, x/\langle x \rangle\rangle\} q_a Q_{-1/2} F_M f(H) + B(t) \end{aligned}$$

for some $\sigma_a > 0$. Since $E_a = 0$ for $a \in \Lambda_0$, we can show, repeating the same argument as in step (4), that the quantity in brackets can be made strictly positive for $|x| \gg 1$ with $x \in \mathit{supp}\, q_a$. This proves (4.9) for $a \in \Lambda_0$ and the proof is complete. □

Lemma 4.5. *Let $q \in S_0(X)$ be as in Proposition 4.3 and let $\Phi(t)$ denote one of the following three operators*

$$F(Q/t < M^{-1}), \quad F(Q/t > M), \quad F(M^{-1} < Q/t < M)q$$

with $M \gg 1$. Then one has

$$s - \lim_{t \to \infty} \Phi(t) f(H) \exp(-itH) = 0.$$

Proof. We calculate the Heisenberg derivative of $\Phi(t)$. Then the support of $\nabla \Phi$ or $\partial_t \Phi$ lies in the forbidden region of the propagator $\exp(-itH)$ in the sense of Propositions 4.1 $\sim$ 4.3. Hence these propositions imply the existence of the strong limit

$$s - \lim_{t \to \infty} \exp(itH) \Phi(t) f(H) \exp(-itH). \tag{4.10}$$

In fact, taking $f_1 \in C_0^\infty(\mathbf{R})$ such that $f_1 f = f$ and noting that $Q[q, f_1(H)]$ is bounded, we have $[\Phi(t), f_1(H)] = O(t^{-1})$ by Lemma 3.1. Hence, to prove (4.10), it suffices to show the existence of the strong limit $s - \lim_{t\to\infty} \tilde{W}(t)$, where

$$\tilde{W}(t) = \exp(itH) f_1(H)\Phi(t) f(H) \exp(-itH).$$

By Propositions 4.1 $\sim$ 4.3, we have

$$|(\varphi, \tilde{W}(s_1)\psi) - (\varphi, \tilde{W}(s_2)\psi)| = o(1)\|\varphi\|, \quad s_1,\ s_2 \to \infty,$$

for $\varphi,\ \psi \in L^2(X)$. This implies that $\{\tilde{W}(t)\psi\}_{t\geq 1}$ is a Cauchy sequence and hence the existence of (4.10) is proved. By Propositions 4.1 $\sim$ 4.3 again, we see that for $\psi \in \mathcal{S}(X)$, there exists a subsequence $\{t_n\}_{n\in\mathbf{N}}$ with $t_n \to \infty$ such that

$$\lim_{n\to\infty} \Phi(t_n) f(H) \exp(-it_n H)\psi = 0, \tag{4.11}$$

where the choice of subsequence $\{t_n\}_{n\in\mathbf{N}}$ depends on ψ. By (4.10) and (4.11), we have for $\psi \in \mathcal{S}(X)$,

$$\lim_{t\to\infty} \Phi(t) f(H) \exp(-itH)\psi = 0.$$

Thus the lemma follows by density argument. □

In general, E_a and E_b can equal each other, even if $a \neq b$. We now set $\tilde{\mathcal{E}} = \{E_a : a \in \Lambda\}$ and let $\tilde{E} \in \tilde{\mathcal{E}}$. We define $a(\tilde{E})$ to be the maximal element of the set $\{b \in \Lambda : \tilde{E}^b = 0\}$ with respect to the relation $\subset$, and set $\Sigma = \{a(\tilde{E}) : \tilde{E} \in \tilde{\mathcal{E}}\}$. We should note $c \in \Sigma$.

We now introduce a non–negative cut–off function $q_a \in S_0(X)$ with conical support in a small neighborhood of $\widehat{\omega}_a$ for $a \in \Sigma$. The function $q_a(x)$ has the following property : q_a is supported in $\Gamma(\widehat{\omega}_a; |E|/4, 2\epsilon)$ and $q_a = 1$ on $\Gamma(\widehat{\omega}_a; |E|/3, \epsilon)$. We should note $q_a(x)I_a(x) = O(|x|^{-\rho})$ for $a \in \Sigma$. We also define

$$\varphi_a(t, x) = F(M^{-1} < \langle x\rangle^{1/2}/t < M) q_a(x), \quad a \in \Sigma, \tag{4.12}$$

with $M \gg 1$. Then we have the following proposition as a consequence of Lemma 4.5:

PROPOSITION 4.4. *Let φ_a be as above. Then one has*

$$s - \lim_{t\to\infty} \left(1 - \sum_{a\in\Sigma} \varphi_a\right) f(H)\exp(-itH) = 0.$$

To prove Theorem 4.1, it suffices by Proposition 4.4 to show that

$$s - \lim_{t\to\infty} \{1 - F(Q/t = (|E|/2)^{1/2})q_0\}\varphi_a f(H)\exp(-itH) = 0, \quad a \in \Sigma. \tag{4.13}$$

It should be noted that $(|E|/2)^{1/2}$ is independent of λ, while the choice of $M \gg 1$ depends on λ. The following two sections are devoted to the proof of (4.13).

5. Time–dependent Hamiltonians

The result obtained above reduces the proof of Theorem 4.1 to the propagation analysis in a conical neighborhood of $\widehat{\omega}_a$, $a \in \Sigma$. To this analysis, it is convenient to introduce an auxiliary time–dependent Hamiltonian which approximates the full Hamiltonian H. Our choice of it is inspired by the argument in [1]. In this section, we study the relation between $\exp(-itH)$ and the propagator generated by such a time–dependent Hamiltonian.

Let $a \in \Sigma$ and $\tilde{\varphi}_a$ be another cut–off function such that $\tilde{\varphi}_a$ takes a form similar to (4.12) and $\tilde{\varphi}_a = 1$ on the support of φ_a. We define

$$W_a(t,x) = W_a(t,x^a,x_a) = \tilde{\varphi}_a(t,x)I_a(x).$$

By the assumption (V), W_a obeys the estimate

$$|\partial_t^m \partial_x^\beta W_a(t,x)| \le C_{m\beta}\langle t\rangle^{-m}(\langle t\rangle^2 + \langle x\rangle)^{-\rho-|\beta|}. \tag{5.1}$$

We now consider the time–dependent Hamiltonian

$$H_{a0}(t) = H_a + W_{a0}(t),$$

where $W_{a0}(t)$ is defined by

$$W_{a0}(t) = \begin{cases} W_a(t, x^a, t^2E_a/2), & E^a \neq 0 \\ W_a(t, 0, t^2E/2) = I_a(t^2E/2), & E^a = 0. \end{cases}$$

We should note that $a = c$ if $E^a = 0$. Let $U_{a0}(t)$ be the propagator generated by $H_{a0}(t)$, that is, $\{U_{a0}(t)\}_{t\geq 1}$ is a family of unitary operators such that for $\psi \in D(H_{a0}(1))$, $\psi_t = U_{a0}(t)\psi$ is a strong solution of $id\psi_t/dt = H_{a0}(t)\psi_t$, $\psi_1 = \psi$. By definition, $H_{a0}(t)$ is decomposed into

$$H_{a0}(t) = H^a(t) \otimes Id + Id \otimes T_a \quad \text{on} \;\; L^2(X^a) \otimes L^2(X_a),$$

where $H^a(t) = H^a + W_{a0}(t)$ acts on $L^2(X^a)$. Hence $U_{a0}(t)$ is represented as

$$U_{a0}(t) = U^a(t) \otimes \exp(-i(t-1)T_a),$$

where $U^a(t)$ denotes the propagator generated by $H^a(t)$. The aim here is to prove the following

PROPOSITION 5.1. *Let the notations be as above. Then there exists the strong limit*

$$s - \lim_{t\to\infty} U_{a0}(t)^* \varphi_a f(H) \exp(-itH).$$

To prove this proposition, we follow the argument in [1], and further introduce an auxiliary Hamiltonian

$$H_a(t) = H_a + W_a(t), \quad W_a(t) = W_a(t, x),$$

and denote by $U_a(t)$ the propagator generated by $H_a(t)$. For this propagator, the Heisenberg derivative of $f(H_a)\Phi(t)f(H_a)$ is calculated as

$$f(H_a)\{i[H_a, \Phi(t)] + \Phi'(t)\}f(H_a) + i[W_a(t), f(H_a)\Phi(t)f(H_a)].$$

Since $[W_a(t), (H_a - \zeta)^{-1}] = O(t^{-1-2\rho})O(\langle\zeta\rangle|Im\,\zeta|^{-2})$ by (5.1), we have $[W_a(t), f(H_a)] = O(t^{-1-2\rho})$. By (5.1) again and the above fact, it follows that

$$i[W_a(t), f(H_a)\Phi_j(t)f(H_a)] = O(t^{-\nu}) \quad \text{for some } \nu > 1, \tag{5.2}$$

for all the propagation observables $\Phi_j(t)$, $1 \le j \le 5$, which are used in the proofs of Propositions 4.1 $\sim$ 4.3. Thus the contributions from these commutators can be dealt with as integrable terms and hence $U_a(t)$ is proved to preserve the same propagation properties as in these propositions :

LEMMA 5.1. *Let the notations be as above. Then one has for* $\psi \in L^2(X)$,

$$\int_1^\infty \frac{dt}{t}\|F(Q/t < M^{-1})f(H_a)U_a(t)\psi\|^2 \le C\,\|\psi\|^2,$$
$$\int_1^\infty \frac{dt}{t}\|F(Q/t = M)f(H_a)U_a(t)\psi\|^2 \le C\,\|\psi\|^2,$$
$$\int_1^\infty \frac{dt}{t}\|F(M^{-1} < Q/t < M)qf(H_a)U_a(t)\psi\|^2 \le C\,\|\psi\|^2,$$

where $q \in S_0(X)$ *as in Proposition 4.3. Moreover, for* $\psi \in \mathcal{S}(X)$

$$\int_1^\infty \frac{dt}{t}\|F(Q/t > M)f(H_a)U_a(t)\psi\|^2 < \infty.$$

LEMMA 5.2. *There exists the strong limit*

$$s - \lim_{t\to\infty} U_a^*(t)\varphi_a f(H)\exp(-itH).$$

PROOF. Taking $f_1 \in C_0^\infty(\mathbf{R})$ such that $f_1 f = f$ and noting $\varphi_a(W_a(t,x) - I_a(x)) = 0$ and $f_1(H_a)\varphi_a - \varphi_a f_1(H) = O(t^{-\min(1,2\rho)})$, in virtue of Lemma 5.1, the existence of the limit can be proved as the existence of (4.10). □

PROOF OF PROPOSITION 5.1. By Lemma 5.2, we have only to show the existence of the strong limit

$$s - \lim_{t\to\infty} U_{a0}(t)^* U_a(t). \tag{5.3}$$

It follows from (5.1) that

$$|W_a(t,x) - W_{a0}(t)| \le \begin{cases} O(t^{-2-2\rho})|x_a - t^2E_a/2|, & E^a \ne 0 \\ O(t^{-2-2\rho})(|x_a - t^2E_a/2| + |x^a|), & E^a = 0. \end{cases}$$

Hence (5.3) is obtained as an immediate consequence of the lemma below. □

LEMMA 5.3. *Assume that $\psi \in \mathcal{S}(X)$. Then one has :*
(1) $\|(x_a - t^2E_a/2)U_a(t)\psi\| = O(t)$.
(2) *If, in particular, $E^a = 0$, then $\|x^a U_a(t)\psi\| = O(t)$.*

PROOF. (1) Recall that $D_a = -i\nabla_a$ denotes the velocity operator corresponding to p_a. For the propagator $U_a(t)$, the Heisenberg derivative of $D_a - tE_a$ satisfies

$$i[H_a(t), D_a - tE_a] + \frac{d}{dt}(D_a - tE_a) = i[W_a(t), D_a] = O(t^{-2\rho-2}),$$

so that we have

$$\|(D_a - tE_a)U_a(t)\psi\| = O(1). \tag{5.4}$$

Similarly the Heisenberg derivative of $x_a - t^2E_a/2$ is calculated as

$$i[H_a(t), x_a - t^2E_a/2] - tE_a = D_a - tE_a$$

and hence (1) follows from (5.4) at once.

(2) We first note that the Heisenberg derivative of x^a is $D^a = -i\nabla^a$, where ∇^a is the gradient on X^a. By assumption, H^a does not have a uniform electric field and hence we see that for $f \in \mathcal{F}^{-1/2}$, $D^a f(H^a)$ is bounded. In order to prove (2), it suffices to show that for $g \in \mathcal{F}^{1/2}$,

$$\|g(H^a)U_a(t)\psi\| = O(1). \tag{5.5}$$

Let $G \in C^\infty(\mathbf{C})$ be an almost analytic extension of g. Since the Heisenberg derivative of $g(H^a)$ is $i[W_a(t), g(H^a)]$, by the formulas (3.2) and (3.3), we have

$$i[W_a(t), g(H^a)]$$

$$
\begin{aligned}
=& i[W_a(t), H^a]g'(H^a) \\
&+ \frac{1}{2\pi i}\int_{\mathbf{C}} \overline{\partial}_\zeta G(\zeta)(H^a-\zeta)^{-1} ad^2_{H^a}(W_a(t))(H^a-\zeta)^{-2}\, d\zeta\wedge d\overline{\zeta}.
\end{aligned}
$$

Since $g' \in \mathcal{F}^{-1/2}$, we see that the first term is $O(t^{-2\rho-2})$. Since

$$
\begin{aligned}
&(H^a-\zeta)^{-1} ad^2_{H^a}(W_a(t))(H^a-\zeta)^{-2} \\
=& O(t^{-2\rho-2})O(|Im\,\zeta|^{-3}) + O(t^{-2\rho-4})O(\langle\zeta\rangle|Im\,\zeta|^{-3}) \\
&+ O(t^{-2\rho-6})O(\langle\zeta\rangle|Im\,\zeta|^{-3}),
\end{aligned}
$$

we also have the second term is $O(t^{-2\rho-2})$, and hence

$$
i[W_a(t), g(H^a)] = O(t^{-2\rho-2}). \tag{5.6}
$$

(5.6) proves (5.5) and the proof is complete. □

We have the following as a consequence of Proposition 5.1:

PROPOSITION 5.2. *Let the notations be as above. Then there exists* $\psi_a \in L^2(X)$ *such that*

$$
\varphi_a \exp(-itH)f(H)\psi = U_{a0}(t)\psi_a + o(1), \quad t\to\infty.
$$

The following can be proved as Lemma 5.3:

LEMMA 5.4. *For* $\psi\in\mathcal{S}(X)$,
(1) $\|(x_a - t^2E_a/2)U_{a0}(t)\psi\| = O(t)$.
(2) *If, in particular,* $E^a = 0$, *then* $\|x^a\, U_{a0}(t)\psi\| = O(t)$.

LEMMA 5.5. *Let* $q_0 \in S_0(X)$ *be as in Theorem 4.1. If* $E^a=0$, *then*

$$
s-\lim_{t\to\infty}\{1 - F(Q/t = (|E|/2)^{1/2})q_0\}U_{a0}(t) = 0.
$$

PROOF. We follow the argument in [1]. We put $\phi_t(x) = F(Q/t = (|E|/2)^{1/2})q_0(x)$ and note $\phi_t(t^2E/2) = \phi_t(t^2E_a/2) = 1$ for $t \gg 1$. Since $|1-\phi_t(x)| = |\phi_t(t^2E_a/2)-\phi_t(x)| \le O(t^{-2})(|x_a - t^2E_a/2| + |x^a|)$, by Lemma 5.4, we have for $\psi \in \mathcal{S}(X)$

$$\lim_{t\to\infty}\{1-\phi_t(x)\}U_{a0}(t)\psi = 0.$$

By density argument, the lemma follows. □

The lemma above, together with Proposition 5.2, implies (4.13) when $E^a = 0$, that is in the case $a = c$. Thus it remains to prove (4.13) also in the case $E^a \ne 0$ in order to complete the proof of Theorem 4.1. To do this, we further continue the analysis on the propagation properties of $U^a(t)$ in the space $L^2(X^a)$.

6. Propagation properties of subsystem operators

The proof of Theorem 4.1 is completed in this section. Throughout the section, we fix $a \in \Sigma \setminus \{c\}$, so that $E^a \ne 0$. We also work in the space $L^2(X^a)$ and use the notation Q^a for the multiplication operator with $\langle x^a\rangle^{1/2}$.

LEMMA 6.1. *Let $h \in C_0^\infty(\mathbf{R})$ be such that $h = 1$ on the interval $[-1,1]$. Then one has*

$$(1-h(H^a/R))U^a(t) = o(1), \quad R \to \infty,$$

uniformly in $t \ge 1$ in the strong topology.

PROOF. Recall that $U^a(t)$ is the propagator generated by $H^a(t) = H^a + W_{a0}(t)$, $W_{a0}(t) = W_{a0}(t,x^a)$. To prove the lemma, it suffices to show that the Heisenberg derivative of $1-h(H^a/R)$ is

$$-i[W_{a0}(t), h(H^a/R)] = o(1)O(\langle t\rangle^{-1-2\rho}), \quad R \to \infty,$$

uniformly in t. This is verified by the almost analytic extension method. Let $u \in L^2(X^a)$ be the solution to the equation $(H^a-\zeta)u = w$, $Im\,\zeta \ne 0$, with $w \in L^2(X^a)$. Then u satisfies

$$\|\langle x^a\rangle^{-1/2}\nabla u\|^2 \le C(\langle\zeta\rangle\|u\|^2 + \|w\|\,\|u\|)$$

for $C > 0$ independent of ζ. Hence it follows that

$$\langle x^a \rangle^{-1/2} \nabla (H^a - \zeta)^{-1} = O(\langle \zeta \rangle^{1/2} |Im\, \zeta|^{-1}) + O(|Im\, \zeta|^{-1/2})$$

and also we have

$$\langle x^a \rangle^{-1/2} \nabla (H^a/R - \zeta)^{-1} = O(R^{1/2}) O(\langle \zeta \rangle^{1/2} |Im\, \zeta|^{-1} + |Im\, \zeta|^{-1/2}).$$

This, together with (5.1), shows that

$$\begin{aligned} &[W_{a0}(t), (H^a/R - \zeta)^{-1}] \\ &= O(R^{-1/2}) O(\langle \zeta \rangle^{1/2} |Im\, \zeta|^{-2} + |Im\, \zeta|^{-3/2}) O(\langle t \rangle^{-1-2\rho}). \end{aligned}$$

Thus the desired result is obtained from (3.1) and the proof is complete. □

By Lemma 6.1 and Proposition 5.2, we obtain the following: For any $\epsilon > 0$ small enough, there exists $g_\epsilon \in C_0^\infty(\mathbf{R})$ such that

$$\varphi_a f(H) \exp(-itH)\psi = g_\epsilon(H^a) U_{a0}(t) \psi_a + O(\epsilon) + o(1), \quad t \to \infty, \tag{6.1}$$

where the norm of remainder term $O(\epsilon)$ is estimated by $C\epsilon$ uniformly in $t \ge 1$. Let $b \in \mathcal{A}$ be such that $b \subset a$. Then we can write

$$X^a = X^b \oplus X^a_b, \qquad X^a_b = X_b \cap X^a,$$

so that E^a has the orthogonal decomposition $E^a = E^b + E^a_b$ with $E^a_b \in X^a_b$. Similarly we write $x^a = x^b + x^a_b$ for $x^a \in X^a$. Let $\Lambda_a = \{b \in \mathcal{A} : b \subsetneq a,\ \ E^a_b \ne 0\}$. We set $\tilde{\mathcal{E}}^a = \{E^a_b : b \in \Lambda_a\}$ and let $\tilde{E} \in \tilde{\mathcal{E}}^a$. We define $b(\tilde{E})$ to be the maximal element of the set $\{b' \in \Lambda_a : \tilde{E}^{b'} = 0\}$ with respect to $\subset$, and set $\Sigma_a = \{b(\tilde{E}) : \tilde{E} \in \tilde{\mathcal{E}}^a\}$. The Hamiltonian H^a still has a non–zero uniform electric field $E^a \ne 0$. Hence we can construct a conjugate operator A^a of H^a which satisfies

$$f(H^a) i[H^a, A^a] f(H^a) \ge \sigma_a f(H^a)^2, \quad \sigma_a > 0. \tag{6.2}$$

Making use of this form inequality, we can show the following propagation properties of $U^a(t)$ as Propositions 4.1 $\sim$ 4.3:

LEMMA 6.2. *Let the notations be as above. Then one has for* $\psi \in L^2(X^a)$,

$$\int_1^\infty \frac{dt}{t} \|F(Q^a/t < M^{-1}) f(H^a) U^a(t)\psi\|^2 \le C\,\|\psi\|^2,$$
$$\int_1^\infty \frac{dt}{t} \|F(Q^a/t = M) f(H^a) U^a(t)\psi\|^2 \le C\,\|\psi\|^2,$$
$$\int_1^\infty \frac{dt}{t} \|F(M^{-1} < Q^a/t < M) q^a f(H^a) U^a(t)\psi\|^2 \le C\,\|\psi\|^2,$$

where $q^a \in S_0(X^a)$ *vanishes in a small conical neighborhood of* $\widehat{\omega}_b^a = E_b^a/|E_b^a|$ *for all* $b \in \Lambda_a$. *Moreover, for* $\psi \in \mathcal{S}(X^a)$

$$\int_1^\infty \frac{dt}{t} \|F(Q^a/t > M) f(H^a) U^a(t)\psi\|^2 < \infty.$$

In virtue of Lemma 6.2, we can prove the following as Lemma 4.5:

LEMMA 6.3. *Let* $\Phi^a(t)$ *denote one of the following three operators*

$$F(Q^a/t < M^{-1}), \quad F(Q^a/t > M), \quad F(M^{-1} < Q^a/t < M) q^a,$$

where $q^a \in S_0(X^a)$ *is as in Lemma 6.2. Then one has*

$$s - \lim_{t\to\infty} \Phi^a(t) f(H^a) U^a(t) = 0.$$

Let $b \in \Sigma_a$ and $q_b^a \in S_0(X^a)$ be a non–negative cut–off function supported in a small conical neighborhood of $\widehat{\omega}_b^a = E_b^a/|E_b^a|$. The function $q_b^a(x^a)$ has a property similar to $q_a \in S_0(X)$ in (4.12). We define

$$\varphi_b^a(t, x^a) = F(M^{-1} < \langle x^a\rangle^{1/2}/t < M) q_b^a(x^a), \quad b \in \Sigma_a.$$

Then, in virtue of Lemma 6.3, we can prove the following as Proposition 4.4:

Proposition 6.1. *Let φ_b^a be as above. Then one has*

$$s - \lim_{t\to\infty} \left(1 - \sum_{b\in\Sigma_a} \varphi_b^a\right) f(H^a)U_{a0}(t) = 0.$$

To prove (4.13) in the case $E^a \neq 0$, it suffices by (6.1) and Proposition 6.1 to show that

$$(6.3)\quad s - \lim_{t\to\infty} \{1 - F(Q/t = (|E|/2)^{1/2})q_0\}\varphi_b^a f(H^a)U_{a0}(t) = 0, \quad b \in \Sigma_a.$$

We repeat the same argument as in the previous section to prove (6.3). Let $I_b^a = I_b^a(x^a)$ be the intercluster potential of H^a. Then the cluster Hamiltonian H_b^a obtained from H^a takes the form

$$H_b^a = H^a - I_b^a = H^b \otimes Id + Id \otimes T_b^a \qquad \text{on } L^2(X^b) \otimes L^2(X_b^a),$$

where $T_b^a = T_b - T_a$ acts on $L^2(X_b^a)$. We introduce a time–dependent Hamiltonian which approximate the generator $H^a(t)$ of the propagator $U^a(t)$. Let $\tilde{\varphi}_b^a(t, x^a)$ be another cut–off function which takes a form similar to φ_b^a and satisfies $\tilde{\varphi}_b^a = 1$ on the support of φ_b^a. We define

$$W_b^a(t, x^a) = W_b^a(t, x^b, x_b^a) = \tilde{\varphi}_a(t, x^a)(W_{a0}(t, x^a) + I_b^a(x^a)).$$

Since $E_b^a \neq 0$, we may assume that if $E^b \neq 0$, $W_{a0}(t, x^b, x_b^a) = 0$ at $x_b^a = t^2E_b^a/2$. If, in particular, $E^b = 0$, then $E_b^a = E^a$. Taking account of these facts, we set

$$W_{b0}^a(t) = \begin{cases} W_b^a(t, x^b, t^2E_b^a/2) = \tilde{\varphi}_a(t, x^b, t^2E_b^a/2)I_b^a(x^b, t^2E_b^a/2), & E^b \neq 0 \\ W_b^a(t, 0, t^2E^a/2) = I_b^a(t^2E^a/2), & E^b = 0. \end{cases}$$

Then the time–dependent Hamiltonian in question is defined by

$$H_{b0}^a(t) = H_b^a + W_{b0}^a(t).$$

This Hamiltonian has the decomposition

$$H_{b0}^a(t) = H^{ba}(t) \otimes Id + Id \otimes T_b^a \qquad \text{on } L^2(X^b) \otimes L^2(X_b^a),$$

where $H^{ba}(t) = H^b + W^a_{b0}(t)$ acts on $L^2(X^b)$. If we denote by $U^{ba}(t)$ the propagator generated by $H^{ba}(t)$, then the propagator $U^a_{b0}(t)$ generated by $H^a_{b0}(t)$ is represented by

$$U^a_{b0}(t) = U^{ba}(t) \otimes \exp(-i(t-1)T^a_b).$$

We can prove the following by the same argument as in the proof of Proposition 5.1:

PROPOSITION 6.2. *There exists the strong limit*

$$s - \lim_{t\to\infty} U^a_{b0}(t)^* \varphi^a_b f(H^a) U^a(t).$$

Recall that $U_{a0}(t) = U^a(t) \otimes \exp(-i(t-1)T_a)$ and $T_b = T_a + T^a_b$. Hence we have the relation

$$U^a_{b0}(t) \otimes \exp(-i(t-1)T_a) = U^{ba}(t) \otimes \exp(-i(t-1)T_b).$$

If $E^b = 0$, then H^b has no electric fields, and the following can be proved as Lemma 5.5:

LEMMA 6.4. *Let the notations be as above. Then one has*

$$s - \lim_{t\to\infty} \{1 - F(Q/t = (|E|/2)^{1/2}) q_0\} U^a_{b0}(t) \otimes \exp(-i(t-1)T_a) = 0.$$

Hence (6.3) follows from this and Proposition 6.2. On the other hand, if $E^b \neq 0$, then $H^{ba}(t)$ still has a non–zero uniform electric field. To prove (6.3) for such $b \in \Sigma_a$, we work in the space $L^2(X^b)$ and repeat the same argument as applied to $U^a(t)$ to analyze the propagation properties of $U^{ba}(t)$. In any case, (6.3) is verified by repeated use of similar arguments and the proof of Theorem 4.1 is now complete.

7. Asymptotic completeness

The present section is devoted to proving the main theorem. The proof uses the asymptotic completeness of long–range many–particle systems without uniform electric fields. As stated in section 1, this result has been already established by [3].

Proof of Theorem 1.1. Let c be as in the theorem. Then $\omega = E/|E| = E_c/|E_c|$ and also H^c does not have a uniform electric field ($E^c = 0$). Let $q_0 \in S_0(X)$ be as in Theorem 4.1. Recall that $q_0(x)$ has a support in a small conical neighborhood of ω. We define :

$$\begin{aligned}
&\varphi_c(t,x) = F(\langle x\rangle^{1/2}/t = (|E|/2)^{1/2})q_0(x),\\
&W_{c0}(t) = I_c(t^2E/2).
\end{aligned}$$

To prove the theorem, we consider an auxiliary time–dependent Hamiltonian

$$H_{c0}(t) = H_c + W_{c0}(t).$$

Denote by $U_{c0}(t) = \Theta(t)\exp(-itH_c)$ the propagator generated by the above operator, where

$$\Theta(t) = \exp\left(-i\int_0^t I_c(s^2E/2)\,ds\right).$$

Then it follows from Theorem 4.1 that there exists the strong limit

$$s-\lim_{t\to\infty} U_{c0}(t)^*\exp(-itH). \tag{7.1}$$

In fact, by Theorem 4.1, we have only to prove the existence of the strong limit

$$s-\lim_{t\to\infty} U_{c0}(t)^*\varphi_c\exp(-itH).$$

Let $f \in C_0^\infty(\mathbf{R})$ and take $M \gg 1$ as in Propositions 4.1∼4.3, which depends on f. Let $q_1 \in S_0(X)$ be such that $q_1q_0 = q_0$ and q_1 has a support in a small conical neighborhood of ω, and $q_2 \in S_0(X)$ be such that $q_2q_1 = q_1$ and q_2 has a support in a small conical neighborhood of ω. We define

$$\tilde\varphi_c(t,x) = F(M^{-1} < \langle x\rangle^{1/2}/t < M)q_1(x),$$

$$H_c(t) = H_c + W_c(t, x),$$
$$W_c(t, x) = W_c(t, x^c, x_c) = F(\langle x\rangle^{1/2}/t > (2M)^{-1})q_2(x)I_c(x),$$

and denote by $U_c(t)$ the propagator generated by $H_c(t)$. We note that $W_c(t, t^2E/2) = W_{c0}(t)$ for $t \geq 1$. We also note that $W_c(t, x)$ satisfies (5.1), since c is the maximal element of the set $\{a \in \mathcal{A} : E^a = 0\}$. Then we have only to prove the existence of the strong limits

$$s - \lim_{t\to\infty} U_c(t)^* \tilde{\varphi}_c f(H) \exp(-itH), \qquad s - \lim_{t\to\infty} U_{c0}(t)^* U_c(t),$$

because $\tilde{\varphi}_c \varphi_c = \varphi_c$. The existence of these limits is proved in exactly the same way as Proposition 5.1 and Lemma 5.2. We can also show the existence of the strong limit

$$s - \lim_{t\to\infty} \exp(itH)U_{c0}(t). \tag{7.2}$$

This is also proved in almost the same way as Proposition 5.1 (see [1]). In fact, both the propagators $U_{c0}(t)$ and $U_c(t)$ have the same properties as in Lemmas 5.3 and 5.4. Hence, in particular, Theorem 4.1 remains true for $U_c(t)$:

PROPOSITION 7.1. *Let the notations be as above. Then*

$$s - \lim_{t\to\infty} \{1 - F(Q/t = (|E|/2)^{1/2})q_0\}U_c(t) = 0.$$

Thus we can prove the existence of (7.2) as that of (7.1).

It follows from the existence of (7.1) that for any $\psi \in L^2(X)$,

$$\exp(-itH)\psi = \Theta(t)\{\exp(-itH^c) \otimes \exp(-itT_c)\}\psi_c + o(1), \quad t \to \infty, \tag{7.3}$$

with some $\psi_c \in L^2(X)$. Thus the proof of the theorem is reduced to analyzing the asymptotic behavior as $t \to \infty$ of $\exp(-itH^c)$.

We now use the asymptotic completeness for the subsystem Hamiltonian H^c without uniform electric field (see [3]). For $a \subset c$, we define

$$H^c_{a0}(t) = H^c_a + I^c_a(tD_a) = H^a + T^c_a + I^c_a(tD_a) \qquad \text{on } L^2(X^c)$$

and denote the propagator $U^c_{a0}(t)$ generated by $H^c_{a0}(t)$ as

$$U^c_{a0}(t) = \exp(-itH^c_a)\exp\left(-i\int_0^t I^c_a(sD_a)\,ds\right),$$

where $H^c_a = H^c - I^c_a$ is the cluster Hamiltonian obtained from H^c and $T^c_a = T_a - T_c = -\Delta^c_a/2$ acts on $L^2(X^c_a)$. Then we have that : (1) There exists the strong limit

$$\Omega^c_a = s - \lim_{t\to\infty} \exp(itH^c)U^c_{a0}(t)(P^a \otimes Id) : L^2(X^c) \to L^2(X^c) \tag{7.4}$$

for the eigenprojection P^a associated with H^a. (2) The wave operators Ω^c_a defined above are asymptotically complete

$$L^2(X^c) = \sum_{a\subset c} \oplus Range\,\Omega^c_a. \tag{7.5}$$

Let $H_{aG}(t)$, $a \subset c$, be defined by (1.1). Then $H_{aG}(t)$ is decomposed into

$$H_{aG}(t) = H^c_{a0}(t) + T_c + W_{c0}(t).$$

The three operators on the right side commute with one another. The propagator $U_{aG}(t)$ generated by $H_{aG}(t)$ is also represented by

$$U_{aG}(t) = \Theta(t)(U^c_{a0}(t) \otimes \exp(-itT_c)).$$

In virtue of the existence of (7.2), the existence of W^+_{aG} defined by (1.2) can be proved by showing the existence of the strong limit

$$s - \lim_{t\to\infty} U_{c0}(t)^* U_{aG}(t)(P^a \otimes Id) = s - \lim_{t\to\infty} \{\exp(itH^c)U^c_{a0}(t)(P^a \otimes Id)\} \otimes Id,$$

which follows from the existence of (7.4). On the other hand, by (7.3) and (7.5), we have with $\psi_c = \sum_{j:\text{finite}} \psi^c_j \otimes \psi^j_c + O(\epsilon)$, $\psi^c_j \in L^2(X^c)$ and $\psi^j_c \in L^2(X_c)$,

$$\exp(-itH)\psi = \Theta(t) \sum_{j:\text{finite}} \exp(-itH^c)\psi^c_j \otimes \exp(-itT_c)\psi^j_c + O(\epsilon)$$

$$= \Theta(t) \sum_{j:\text{finite}} \sum_{a \subset c} \exp(-itH^c)(\Omega_a^c \tilde{\psi}_{aj}^c) \otimes \exp(-itT_c)\psi_c^j + O(\epsilon)$$
$$= \Theta(t) \sum_{j:\text{finite}} \sum_{a \subset c} U_{a0}^c(t)\tilde{\psi}_{aj}^c \otimes \exp(-itT_c)\psi_c^j + O(\epsilon) + o(1)$$

for some $\tilde{\psi}_{aj}^c \in L^2(X^c)$, which implies

$$\|\psi - \sum_{j:\text{finite}} \sum_{a \subset c} W_{aG}^+(\tilde{\psi}_{aj}^c \otimes \psi_c^j)\| \leq O(\epsilon).$$

Since $\epsilon > 0$ is arbitrary and $\sum_{a \subset c} \oplus Range W_{aG}^+$ is closed, we have

$$\psi \in \sum_{a \subset c} \oplus Range W_{aG}^+,$$

which implies the asymptotic completeness of the wave operators W_{aG}^+. The proof of Theorem 1.1 is now completed. □

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(Received March 29, 1994)
(Revised May 30, 1994)

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