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π_1 *of smooth points of a log del Pezzo surface is finite : II*

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Abstract. Let S be a normal projective algebraic surface with at worst log terminal singularities (i.e., quotient singularities) and ample anti-canonical divisor $-K_S$. In this Part II, we shall give a structure theorem (Theorem 1.1) for S and complete the proof of the following result stated in the Part I: The smooth part of S has finite fundamental group.

Introduction

A normal projective surface S over $\mathbf{C}$ is called a log del Pezzo surface if S has at most quotient singularities and $-K_S$ is ample, where K_S denotes the canonical divisor of S. In Part **I** (cf. [2]) of this paper we set out to prove the following :

Main Theorem. *The fundamental group of the space of smooth points of a log del Pezzo surface is finite.*

In this part **II**, we will complete the proof of this result. We will use the notations and results from Part **I** freely. Recall from Part **I** that if $\widetilde{S}$ is a minimal resolution of singularities of S, then we can find a "minimal" (-1)-curve C on $\widetilde{S}$ (cf. Lemma 3.1 and Prop. 3.6 of Part **I**). In §3, §4, §5 of Part **I**, we reduced to consider the cases (II-3) and (II-4) there. As remarked in the Introduction of Part **I**, it suffices to consider the case (II-4) (the "2-component case"), to complete the proof of our Main Theorem. This will

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be done in this part **II** of our paper. As in Part **I**, our proof for the case (II-4) gives quite precise information about the configuration of $C + D$. After the results of parts **I** and **II** of our paper were announced in a conference in Kinosaki, Japan, A. Fujiki, R. Kobayashi and S. Lu have found another proof of our Main Theorem using differential geometric methods (cf. [1]). Their proof of the Main Theorem is short, but it does not seem to give as precise information about the singular locus of S as our proof.

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1. The proof of the Main Theorem in the case (II-4)

In this section, we consider the case(II-4) in Remark 3.11 of Part **I.** We employ the notations there.

Recall that $f : \widetilde{S} \to S$ is a minimal resolution of singularities of S and $D = f^{-1}$ (Sing S). We can also write

$$f^*K_S = K_{\widetilde{S}} + D^*$$

where D^* is an effective **Q**-divisor with support contained by D (cf. Lemma 1.1 of Part **I**).

The (-1)-curve C, used in the case(II-4) of Remark 3.11 in Part **I**, now meets exactly a (-2)-curve D_1 and a $(-n)$-curve D_2 with $n \geq 3$. Let Δ_i be the connected component of D containing D_i. Let $C + T_i$ $(i = 1, 2)$ be the maximal twig of $C + \Delta_i$ such that $T_i = 0$ if D_i is not a tip component of Δ_i and T_i is the maximal twig of Δ_i containing D_i otherwise.

Our aim is to prove the following Theorem 1.1, which will imply the Main Theorem in the case (II-4).

THEOREM 1.1. *Suppose that the case (II-4) in Remark 3.11 occurs. Then one of the following five cases occurs :*

(1) Δ_i *is a linear chain with* D_i *as a tip for* $i = 1$ *or* 2. *Hence* $\pi_1(S^o)$ *is finite (cf. Lemma 1.2 below).*

(2) *There are irreducible components* A_i $(i = 1, \cdots, a)$, B_j $(j = 1, \cdots, b)$ *of* $\Delta_1 + \Delta_2$ *and there is a* $\mathbf{P}^1$*-fibration* $\varphi : \widetilde{S} \to \mathbf{P}^1$ *such that*

(2-1) *a singular fiber of* φ *has support equal to Supp*$(C + \sum_i A_i)$,

(2-2) *every component of* $D - \sum_j B_j$ *is contained in a singular fiber of* φ, *and*

(2-3) $F.\sum_j B_j \leq 2$ *for a general fiber* F *of* φ.

In particular, there is a $\mathbf{C}^*$*-fibration on* S^o *and hence* $\pi_1(S^o)$ *is finite (cf. Lemma 2.2 of Part* **I***).*

(3) *For* $i = 1$ *and* $j = 2$, *or* $i = 2$ *and* $j = 1$, *the intersection matrix of* $C + T_i + \Delta_j$ *has a positive eigenvalue and hence* $\kappa(\widetilde{S}, C + T_i + \Delta_j) = 2$.

In particular, $\pi_1(S^o)$ *is finite (cf. Lemma 1.12).*

(4) $C + \Delta_1 + \Delta_2$ *is described in Figure 1, 2, 3 or 4 below. Moreover, there is a* $\mathbf{P}^1$*-fibration* $\varphi : \widetilde{S} \to \mathbf{P}^1$ *such that* $C + D$ *and all singular fibers of* φ *are precisely described in the proof of Lemma 1.10. (We shall call them Case (4-1), (4-2), (4-3) or (4-4) of Theorem 1.1.)*

Hence $\pi_1(S^o)$ *is finite (cf. Lemma 1.13).*

(5) $C + \Delta_1 + \Delta_2$ *is described in Figure 5 or 6 below, where the divisor* H *in Figure 5 might be a zero divisor. (We shall call them Case (5-5) or (5-6) of Theorem 1.1.)*

Hence $\pi_1(S^o)$ *is finite (cf. Lemma 1.13).*

Theorem 1.1 is a consequence of Lemmas 1.3, 1.8, 1.9, 1.10 and 1.11 below.

LEMMA 1.2. *Suppose that* Δ_i *is a linear chain with* D_i *as a tip for* $i = 1$ *or* 2. *Then* $\pi_1(S^o)$ *is a finite group.*

PROOF. Suppose Δ_1 is a linear chain with D_1 as a tip. As the Picard number $\rho(S) = 1$, we see that $C + \Delta_1 + \Delta_2$ supports a divisor with strictly positive self-intersection. By Lemma 1.10 of Part **I**, we have a surjection $\pi_1(U - \Delta_1 - \Delta_2) \to \pi_1(\widetilde{S} - D)$, where U is a small tubular neighborhood of $C \cup \Delta_1 \cup \Delta_2$. We can write $U = U_1 \cup U_2$, where U_i is a small neighborhood of $C \cup \Delta_i$. It is easy to see that $U_i - \Delta_1 - \Delta_2$ contains a small neighborhood N_i of Δ_i as a strong deformation retract for $i = 1, 2$. By assumption, $\pi_1(N_i - \Delta_i)$ is finite for $i = 1, 2$ and by Mumford's presentation (cf. [3]), $\pi_1(N_1 - \Delta_1)$ is a cyclic group generated by "the" loop γ_1 in $C - \Delta_1 - \Delta_2$ around the point $C \cap \Delta_1$. Now an easy application of Van-Kampen's theorem for the covering $U_1 - \Delta_1 - \Delta_2$ and $U_2 - \Delta_1 - \Delta_2$ of $U - \Delta_1 - \Delta_2$ shows that $\pi_1(U - \Delta_1 - \Delta_2)$ is finite and hence so is $\pi_1(\widetilde{S} - D)$. □

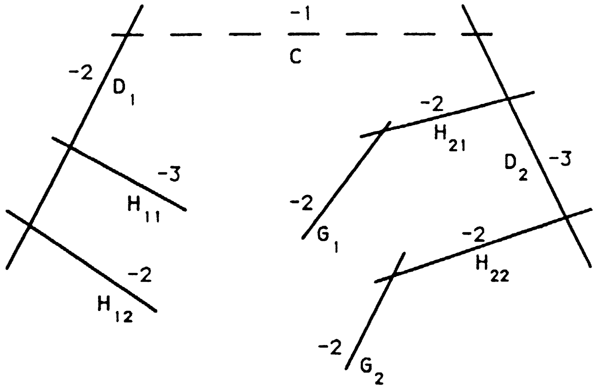


Fig. 1

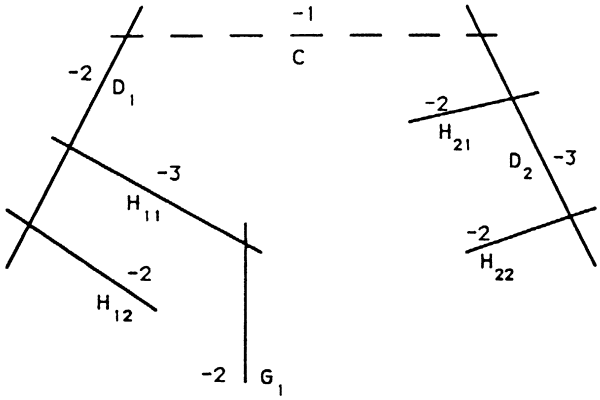
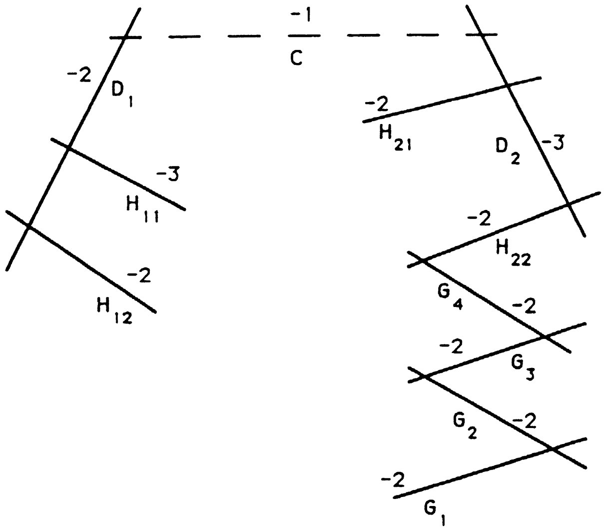


Fig. 2

LEMMA 1.3. (1) *Suppose that* Δ_1 *contains* $G_i\,(i = 1, \cdots, s; s \geq 3)$ *such that* $G_i^2 = -2$, $G_1 = D_1$, $G_j.G_{j+1} = G_{s-2}.G_s = 1\,(j = 1, \cdots, s-2)$. *(This is the case if* Δ_1 *consists of only* (-2)*-curves but* D_1 *is not a tip of* Δ_1*.) Then Theorem 1.1 (2) or (3) occurs.*

(2) *Suppose that* Δ_1 *is a fork with* D_1 *as its central component. Then Theorem 1.1 (3) occurs.*

PROOF. (1) Let $S_0 = 2(C + G_1 + \cdots + G_{s-2}) + G_{s-1} + G_s$ and let φ :


-1
C
-2
D1
-3
H11
-2
H12
-2
H21
D2
-3
-2
H22
G4
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G3
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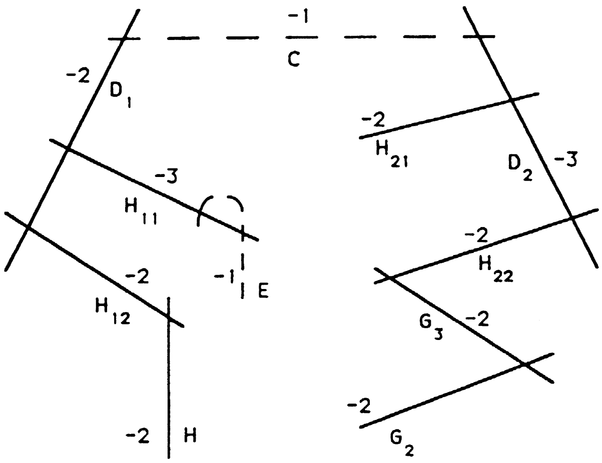
-2
-2
G1

Fig. 3


-1
C
-2
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-3
H11
-2
H12
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E
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H22
G3
-2
-2
G2

Fig. 4

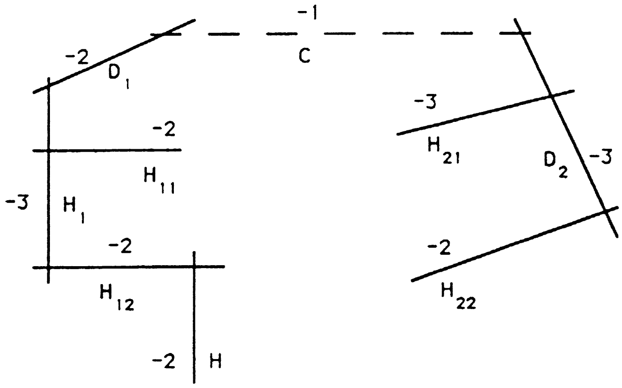

Fig. 5

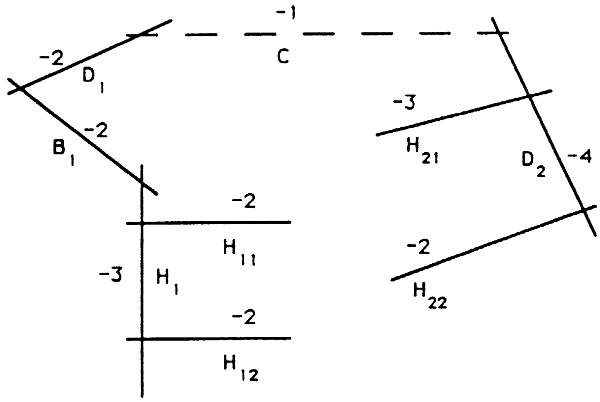

Fig. 6

$\widetilde{S} \to \mathbf{P}^1$ be the $\mathbf{P}^1$-fibration with S_0 as a singular fiber. If $\Delta_1 = \sum_i G_i$, then Theorem 1.1 (2) occurs with $\sum_i A_i = \sum G_i$, $\sum_i B_i = B_1 = D_2$. Otherwise, Theorem 1.1 (3) occurs. Indeed, the intersection matrix of $C + \Delta_1$ then has a positive eigenvalue.

(2) If the central component D_1 meets two (-2)-components of $\Delta_1 - D_1$, then we are reduced to the previous case. So we may assume that D_1 meets $\Delta_1 - D_1$ in one (-2)-component and two components of self intersections ≤ -3. But then $D^* \geq 4/5 D_1 + 1/3 D_2$ (cf. Lemma 1.5 below) and $0 <$

$-C.(K_{\widetilde{S}}+D^*) \leq 1-C.(4/5D_1+1/3D_2) = 1-4/5-1/3 < 0$, a contradiction (cf. Lemma 1.4 below).

This proves Lemma 1.3. □

From now on till the end of the section, we shall assume the following hypothesis:

($*$) *neither the case of Lemma 1.2 nor the cases of Lemma 1.3 occur.*

By the maximality of the twig $C + T_i$ and by the hypothesis($*$), if D_i is a tip component of Δ_i (Δ_i is a fork in this case) then there are irreducible components H_i, H_{i1}, H_{i2} in $\Delta_i - T_i$ such that

$$T_i.(\Delta_i - T_i) = T_i.H_i = 1, \ H_i.H_{i1} = H_i.H_{i2} = 1.$$

If D_i is not a tip component of Δ_i, then $T_i = 0$ and we let $H_i := D_i$ and H_{21}, H_{22} two components in $\Delta_i - D_i$ adjacent to D_i.

Let $\sigma : \widetilde{S} \to \widetilde{T}$ be the smooth blowing-down of curves in $C + T_1 + T_2$ such that

(1) $\sigma(C + \Delta_1 + \Delta_2)$ *consists of exactly one* (-1)*-curve* $\widetilde{C}$, *with* $\widetilde{C} \leq \sigma(C + T_1 + T_2)$, *and several* $(-n_i)$*-curves with* $n_i \geq 2$, *and*

(2) *the condition(1) will not be satisfied if* σ *is replaced by the composite of* σ *and the blowing-down of* $\widetilde{C}$.

Thus, $\sigma = id$ if and only if D_1 is not a tip of Δ_1. If $\sigma \neq id$, then C is contracted by σ and $\sigma'(\widetilde{C}) \leq D$.

Let $\widetilde{D} = D$ (resp. $\widetilde{\Delta}_i := \sigma(\Delta_i)$) if $\sigma = id$, and $\widetilde{D} = \sigma(D) - \widetilde{C}$ (resp. $\widetilde{\Delta}_i := \sigma(\Delta_i)$ with $\widetilde{C}$ deleted if any) otherwise. Let $\widetilde{H}_i = \sigma(H_i)$, $\widetilde{H}_{ij} = \sigma(H_{ij})$, etc. By the definition of σ there is an irreducible component J_i in $T_i + H_i$ such that

$$\widetilde{C}.\widetilde{D} = \widetilde{C}.(\widetilde{J}_1 + \widetilde{J}_2) = 2, \ \ \widetilde{C}.\widetilde{J}_i = 1, \ \text{where } \widetilde{J}_i := \sigma(J_i).$$

The divisor $\widetilde{D}$ on $\widetilde{T}$ is contractible to quotient singularities with, say $g : \widetilde{T} \to T$ the contraction morphism. T is again a log del Pezzo surface of rank one with g as a minimal desingularization (cf. [4, Lemma 4.3]). So we can apply Lemma 1.1 of Part **I** for T. In particular, we have

$$g^* K_T = K_{\widetilde{T}} + \widetilde{D}^*, \ -R.(K_{\widetilde{T}} + \widetilde{D}^*) > 0$$

for every curve R on $\widetilde{T}$ which is not contractible by g. Here $\widetilde{D}^*$ is an effective $\mathbf{Q}$-divisor with support in $\widetilde{D}$.

Suppose that there are two smooth blowing-downs $\sigma_1 : \widetilde{S} \to \widetilde{S}_1, \ \sigma_2 : \widetilde{S}_1 \to \widetilde{T}$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Let E be the unique (-1)-curve in $\sigma_1(C + \Delta_1 + \Delta_2)$. Let $M := D$ if $\sigma_1 = id$ and $M := \sigma_1(D) - E$ otherwise. By [4, Lemma 4.3] and Lemma 1.1 in Part **I**, we have the following :

Lemma 1.4. *M is contractible to quotient singularities with, say $f_1 : \widetilde{S}_1 \to S_1$ the contraction morphism, and S_1 is again a log del Pezzo surface of rank one with f_1 as a minimal desingularization. In particular, we have*

$$f_1^* K_{S_1} = K_{\widetilde{S}_1} + M^*, \quad -R.(K_{\widetilde{S}_1} + M^*) > 0,$$

where M^* is an effective $\mathbf{Q}$-divisor with support contained in M and R is an arbitrary curve on $\widetilde{S}_1$ not contracible by f_1. (We can take $R = E$.)

Throughout the proof of Theorem 1.1, we shall frequently make use of Lemma 1.5 below to estimate the coefficients of the effective $\mathbf{Q}$-divisor M^* in Lemma 1.4. For instance, we often combine Lemmas 1.4 and 1.5 to rule out certain cases.

To state Lemma 1.5 in a general setting, we need some preparation. Let X be a log del Pezzo surface. Let $h : \widetilde{X} \to X$ be a minimal resolution of singularities of X and let $P = h^{-1}$ (Sing X). We decompose P into irreducible components : $P = \sum_{i=1}^n P_i$. By Lemma 1.1 in Part **I**, we can write

$$h^* K_X = K_{\widetilde{X}} + P^*$$

where $P^* = \sum_{i=1}^n \alpha_i P_i$ for some non-negative rational number α_i. Let $\{Q_1, \cdots, Q_r\}$ be a subset of $\{P_1, \cdots, P_n\}$, say $Q_i = P_i$ for $1 \le i \le r$. We formally assign an integer Q_i^2 to Q_i so that $P_i^2 \le Q_i^2 \le -2$. Now we define rational numbers $\beta_i (1 \le i \le r)$ by the condition :

$$Q_j.(K_{\widetilde{X}} + \sum_{i=1}^r \beta_i Q_i) = 0 \quad (j = 1, \cdots, r),$$

where we set $Q_i.Q_j := P_i.P_j$ if $i \neq j$ and $Q_i.K_{\widetilde{X}} := -2 - Q_i^2$. Then we have the following (cf. [4, Lemma 1.7]) :

Lemma 1.5. *We have $\alpha_i \geq \beta_i$ for $1 \leq i \leq r$ and $\alpha_i \geq 1 + 2/P_i^2$ for $1 \leq i \leq n$.*

Let us continue the proof of Theorem 1.1. Suppose that for $a = 1$ or 2, we have $J_a = H_a$ and $\widetilde{H}_a^2 = -2$. Let $\widetilde{G} \sim K_{\widetilde{T}} + 2(\widetilde{C} + \widetilde{H}_a) + \widetilde{H}_{a1} + \widetilde{H}_{a2} + \widetilde{J}_b$ where $\{a, b\} = \{1, 2\}$ as sets. Note that $H^2(\widetilde{T}, \widetilde{G}) \cong H^0(\widetilde{T}, -(2(\widetilde{C} + \widetilde{H}_a) + \widetilde{H}_{a1} + \widetilde{H}_{a2} + \widetilde{J}_b)) = 0$. Note also that $\widetilde{G}.B = 0$ for $B = \widetilde{C}, \widetilde{H}_a, \widetilde{H}_{a1}, \widetilde{H}_{a2}, \widetilde{J}_b$. Hence $\widetilde{G}^2 = \widetilde{G}.K_{\widetilde{T}}$. Now the Riemann-Roch theorem implies that

$$h^0(\widetilde{T}, \widetilde{G}) \geq \frac{1}{2}\widetilde{G}.(\widetilde{G} - K_{\widetilde{T}}) + 1 = 1.$$

So we may assume that $\widetilde{G} \geq 0$.

Lemma 1.6. *Assume the above conditions. Then we have :*

(1) $\widetilde{G}$ is a nonzero effective divisor.

(2) $\widetilde{G} \cap (\widetilde{C} + \widetilde{H}_a + \widetilde{H}_{a1} + \widetilde{H}_{a2} + \widetilde{J}_b) = \phi$. In particular, $\widetilde{G}_1.\widetilde{G} = \widetilde{G}_1.K_{\widetilde{T}}$ for every irreducible component $\widetilde{G}_1$ of $\widetilde{G}$.

(3) We can decompose $\widetilde{G}$ into $\widetilde{G} = \widetilde{\Sigma} + \widetilde{\Delta}$ such that Supp $\widetilde{\Delta}$ is contained in Supp $\widetilde{D}$ and $\widetilde{\Sigma} = \sum_{i=1}^{r} \widetilde{\Sigma}_i$ $(r \geq 1)$ where $\widetilde{\Sigma}_i$ is a (-1)-curve.

(4) Write $\sigma^*\widetilde{G} \sim \sigma^*(K_{\widetilde{T}} + 2(\widetilde{C} + \widetilde{H}_a) + \widetilde{H}_{a1} + \widetilde{H}_{a2} + \widetilde{J}_b) = K_{\widetilde{S}} + sC+$ (an effective divisor with support in D). Then $r \leq s - 1$.

(5) Let $\widetilde{B}$ be an irreducible component of $\widetilde{D} - (\widetilde{H}_a + \widetilde{H}_{a1} + \widetilde{H}_{a2} + \widetilde{J}_b)$. Then $\widetilde{B}.\widetilde{G} > 0$ if and only if $\widetilde{B}^2 \leq -3$ or $\widetilde{B}.(\widetilde{H}_a + \widetilde{H}_{a1} + \widetilde{H}_{a2} + \widetilde{J}_b) > 0$.

(6) If $\widetilde{\Sigma}$ is a reduced divisor, then $\widetilde{G} = \widetilde{\Sigma}$ and $\widetilde{\Sigma}$ is a disjoint union of $\widetilde{\Sigma}_i$'s.

Proof. From the definition of $\widetilde{G}$, one can calculate that :

Claim(1). $\widetilde{G}.\widetilde{B} = 0$ if $\widetilde{B}$ is one of $\widetilde{C}, \widetilde{H}_a, \widetilde{H}_{a1}, \widetilde{H}_{a2}$ and $\widetilde{J}_b$. Moreover, $\widetilde{G}.\widetilde{B} \geq 0$ for every irreducible component B of $\widetilde{D}$.

By the fact that $|K_{\widetilde{S}} + C + D| = \phi$ and the definition of σ, we get :

CLAIM(2). $|K_{\widetilde{T}}+\widetilde{C}+\widetilde{D}|=\phi$.

(1) By the hypothesis(*) which is stated after Lemma 1.3, $\widetilde{J}_b$ meets an irreducible component $\widetilde{B}$ of $\widetilde{\Delta}_b$. So, $\widetilde{G}.\widetilde{B}=(K_{\widetilde{T}}+\widetilde{J}_b).\widetilde{B}\geq 1$. Hence $\widetilde{G}>0$.

(2) Suppose $\widetilde{G}\cap\widetilde{C}\neq\phi$. Then $\widetilde{C}\leq\widetilde{G}$ by Claim(1). Now, $\widetilde{H}_a\leq\widetilde{G}-\widetilde{C}$ because $\widetilde{H}_a.(\widetilde{G}-\widetilde{C})=-\widetilde{H}_a.\widetilde{C}=-1<0$. This leads to $0\leq\widetilde{G}-\widetilde{C}-\widetilde{H}_a\in|K_{\widetilde{T}}+\widetilde{C}+\widetilde{H}_a+\widetilde{H}_{a1}+\widetilde{H}_{a2}+\widetilde{J}_b|\subseteq|K_{\widetilde{T}}+\widetilde{C}+\widetilde{D}|$, a contradiction to Claim(2). So, $\widetilde{G}\cap\widetilde{C}=\phi$. One iterates this argument and can prove (2).

(3) Decompose $\widetilde{G}$ into $\widetilde{G}=\widetilde{\Sigma}+\widetilde{\Delta}$ where Supp $\widetilde{\Delta}\subseteq$ Supp $\widetilde{D}$ and $\widetilde{\Sigma}$ contains no irreducible components of $\widetilde{D}$. First, by Claim(1), we have $\widetilde{G}.\widetilde{\Delta}_i\geq 0$ for every irreducible component $\widetilde{\Delta}_i$ of $\widetilde{\Delta}$. Hence $0\leq\widetilde{G}.\widetilde{\Delta}=\widetilde{\Sigma}.\widetilde{\Delta}+\widetilde{\Delta}^2<\widetilde{\Sigma}.\widetilde{\Delta}$ when $\widetilde{\Delta}\neq 0$, because Supp $\widetilde{\Delta}\subseteq$ Supp $\widetilde{D}$ and $\widetilde{D}$ is negative definite. This proves that $\widetilde{\Sigma}\neq 0$.

Let $\widetilde{\Sigma}_i$ be an irreducible component of $\widetilde{\Sigma}$. Note that $\widetilde{\Sigma}_i.K_{\widetilde{T}}\leq\widetilde{\Sigma}_i.(K_{\widetilde{T}}+\widetilde{D}^*)<0$ (cf. Lemma 1.4). So, if $\widetilde{\Sigma}_i^2<0$, then $\widetilde{\Sigma}_i$ is a (-1)-curve. Suppose that $\widetilde{\Sigma}_i^2\geq 0$. Then, by (2), $\widetilde{\Sigma}_i^2\leq\widetilde{\Sigma}_i.\widetilde{G}=\widetilde{\Sigma}_i.K_{\widetilde{T}}<0$. We reach a contradiction. This proves (3).

(4) By (2), $\sigma^*\widetilde{\Sigma}_i$ is again a (-1)-curve and $\sigma^*(\widetilde{\Delta})\subseteq D$. Write $f(C)\equiv c(-K_S)$, $f(\sigma^*\widetilde{\Sigma}_i)\equiv e_i(-K_S)$, where $c>0, e_i>0$. Then $(sc-1)(-K_S)\equiv f(\sigma^*\widetilde{G})\equiv\sum_{i=1}^r e_i(-K_S)$. Since $K_S^2>0$, we have

$$sc-1=\sum_i e_i\geq rc$$

by the minimality of $-C.(K_{\widetilde{S}}+D^*)=c(K_{\widetilde{S}}+D^*)^2=c(K_S)^2$ (cf. the choice of C in Part **I**). Hence $(s-r)c\geq 1>0$. (4) then follows.

(5) follows from the equality : $\widetilde{B}.\widetilde{G}=\widetilde{B}.(K_{\widetilde{T}}+\widetilde{H}_{a1}+\widetilde{H}_{a2}+\widetilde{J}_b)$.

(6) By the condition, $\widetilde{\Sigma}_i\neq\widetilde{\Sigma}_j$ if $i\neq j$. So,

$$-1=\widetilde{\Sigma}_i^2=\widetilde{\Sigma}_i.\widetilde{G}-\widetilde{\Sigma}_i.(\widetilde{\Delta}+\sum_{j\neq i}\widetilde{\Sigma}_j)\leq\widetilde{\Sigma}_i.\widetilde{G}=\widetilde{\Sigma}.K_{\widetilde{T}}=-1.$$

Thus, $\widetilde{\Sigma}_i.(\widetilde{\Delta}+\sum_{j\neq i}\widetilde{\Sigma}_j)=0$ for every i. So, $\widetilde{\Sigma}$ is a disjoint union of $\widetilde{\Sigma}_i$'s and $\widetilde{\Sigma}\cap\widetilde{\Delta}=\phi$. In particular, $\widetilde{G}.\widetilde{\Delta}=\widetilde{\Delta}^2$. By Claim(1), we have $\widetilde{G}.\widetilde{\Delta}\geq 0$. So, $\widetilde{\Delta}^2\geq 0$. Since $\widetilde{\Delta}$ is contained in $\widetilde{D}$ and $\widetilde{D}$ is negative definite, we have $\widetilde{\Delta}=0$. This proves (6) and Lemma 1.6 is proved. □

Corollary 1.7. *Assume that σ is a contraction of curves in $C + T_1$. Assume further that $J_1 = H_1$ and $\widetilde{H}_1^2 = -2$ (hence $J_2 = D_2$ and the hypothesis in Lemma 1.6 is satisfied with $a = 1$). Then $K_{\widetilde{T}} + 2(\widetilde{C} + \widetilde{H}_1) + \widetilde{H}_{11} + \widetilde{H}_{12} + \widetilde{J}_2 \sim \widetilde{G} = \widetilde{\Sigma} = \widetilde{\Sigma}_1$, i.e., $\widetilde{G}$ is reduced and a (-1)-curve.*

Proof. We apply Lemma 1.6 to $\widetilde{G} \sim K_{\widetilde{T}} + 2(\widetilde{C} + \widetilde{H}_1) + \widetilde{H}_{11} + \widetilde{H}_{12} + \widetilde{J}_2$. By the hypothesis, $\sigma^*\widetilde{G} \sim K_{\widetilde{S}} + 2C +$ (an effective divisor with support in D). Then Corollary 1.7 follows from Lemma 1.6. □

Lemma 1.8. *Suppose the case (II-4) in Remark 3.11 of Part* **I** *occurs. Then one of the following two cases occurs :*

(1) *Theorem 1.1, (2) or (3) occurs.*

(2) *$(\widetilde{J}_a^2, \widetilde{J}_b^2) = (-2,-2), (-2,-3)$ or $(-2,-4)$ where $\{a,b\} = \{1,2\}$ as sets. If $\widetilde{J}_k^2 = -2$ (this is the case if $k = a$), then $J_k = H_k$ and $H_{kj}^2 \leq -3$ for $j = 1$ or 2. Moreover, Δ_2 is not a fork with D_2 as its central component.*

Proof. By [4, Lemma 4.4], $\widetilde{J}_a^2 = -2$ for $a = 1$ or 2. Let $\{a,b\} = \{1,2\}$ as sets.

Case(1) $\widetilde{J}_b^2 = -2$. If $\widetilde{J}_s$ is a tip of $\widetilde{\Delta}_s$ for $s = a$ or b, say $s = b$, then $J_b \neq H_b$ and Theorem 1.1 (3) occurs. Indeed, the intersection matrix of $\widetilde{C} + \widetilde{J}_b + \widetilde{\Delta}_a$ has a positive eigenvalue and so does $C + T_b + \Delta_a$. Thus we may assume $J_a = H_a, J_b = H_b$.

Suppose $H_{s1}^2 = H_{s2}^2 = -2$ for $s = a$ or b, say $s = a$. Let $S_0 := 2(\widetilde{C} + \widetilde{H}_a) + \widetilde{H}_{a1} + \widetilde{H}_{a2}$ and let $\psi : \widetilde{S} \to \mathbf{P}^1$ be the $\mathbf{P}^1$-fibration with S_0 as a singular fiber. If $\widetilde{\Delta}_a = \widetilde{H}_a + \widetilde{H}_{a1} + \widetilde{H}_{a2}$, then Theorem 1.1 (2) occurs with $\varphi = \psi \cdot \sigma$, $\sum_i B_i = B_1 = H_b$. If $\widetilde{\Delta}_a > \widetilde{H}_a + \widetilde{H}_{a1} + \widetilde{H}_{a2}$, Theorem 1.1 (3) occurs. Indeed, the intersection matrix of $\widetilde{C} + \widetilde{\Delta}_a$ then has a positive eigenvalue and so does $C + T_b + \Delta_a$. Thus we may assume that $H_{aj}^2 \leq -3$ for $j = 1$ or 2. The same argument works for $s = b$.

To finish the proof of Lemma 1.8 in this case, we have to consider the case where $\widetilde{\Delta}_2$ is a fork with $\widetilde{D}_2$ as its central component. By the previous arguments, now we have $J_1 = H_1, J_2 = H_2 = D_2, H_{11}^2 \leq -3$ say, and $\widetilde{H}_2$ meets $\widetilde{\Delta}_2 - \widetilde{H}_2$ in one (-2)-component $\widetilde{H}_{23}$ and two components $\widetilde{H}_{21}, \widetilde{H}_{22}$ of self intersections ≤ -3.

Let $\sigma_2 : \widetilde{S}_1 \to \widetilde{T}$ be the blowing-up of the point $\widetilde{C} \cap \widetilde{D}_2$ and let L be the exceptional curve of σ_2. Note that $\sigma \neq id$ because $\widetilde{D}_2^2 = -2$ while $D_2^2 \leq -3$. So we have a smooth blowing-down $\sigma_1 : \widetilde{S} \to \widetilde{S}_1$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Applying Lemma 1.4, we get $-L.(K_{\widetilde{S}_1} + M^*) > 0$, where $M^* \geq 1/2\sigma_2'\widetilde{H}_{11} + 1/2\sigma_2'\widetilde{H}_1 + 1/4\sigma_2'\widetilde{H}_{12} + 1/4\sigma_2'\widetilde{C} + 5/11\sigma_2'\widetilde{H}_{23} + 10/11\sigma_2'\widetilde{D}_2 + 7/11\sigma_2'\widetilde{H}_{22} + 7/11\sigma_2'\widetilde{H}_{21}$ (cf. Lemma 1.5). This leads to $-L.(K_{\widetilde{S}_1} + M^*) \leq 1 - L.(1/4\sigma_2'\widetilde{C} + 10/11\sigma_2'\widetilde{D}_2) = 1 - 1/4 - 10/11 < 0$. We reach a contradiction. So it is impossible that $\widetilde{\Delta}_2$ is a fork with $\widetilde{D}_2$ as its central component. Lemma 1.8 is proved in the present case.

Case(2) $\widetilde{J}_b^2 \leq -3$. Then by the definition of σ (cf. the second condition), $J_a = H_a$, i.e., $\widetilde{J}_a$ is not a tip of $\widetilde{\Delta}_a$. If $H_{a1}^2 = H_{a2}^2 = -2$, then by the arguments in the above paragraph, Theorem 1.1, (2) or (3) occurs. So we may assume that $H_{aj}^2 \leq -3$ for $j = 1$ or 2, say $j = 1$.

We now prove that $d := -\widetilde{J}_b^2 \leq 4$. Since it is impossible that $\widetilde{\Delta}_b$ is a linear chain with $\widetilde{J}_b$ as a tip, we have $\widetilde{D}^* \geq (d-2)/(d-1)\widetilde{J}_b + 3/7\widetilde{H}_{a1} + 2/7\widetilde{H}_a + 1/7\widetilde{H}_{a1}$ (cf. Lemma 1.5). By Lemma 1.4, we have $0 < -\widetilde{C}.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - \widetilde{C}.((d-2)/(d-1)\widetilde{J}_b + 2/7\widetilde{H}_a) = 1/(d-1) - 2/7$. Hence $d \leq 4$.

To finish the proof of Lemma 1.8 in this case, we still have to consider the case where $\widetilde{\Delta}_2$ is a fork with $\widetilde{D}_2$ as its central component. Now $J_2 = H_2 = D_2$ and $J_a = H_a$. If $\widetilde{D}_2^2 = -2$, i.e., if $a = 2, b = 1$, then $\widetilde{J}_1^2 \leq -3$, and by the previous argument, $\widetilde{D}_2$ meets $\widetilde{\Delta}_2 - \widetilde{D}_2$ in one (-2)-component and two components of self intersections ≤ -3. This will lead to a contradiction to $0 < -\widetilde{C}.(K_{\widetilde{T}} + \widetilde{D}^*)$ as in Lemma 1.3 (2). So, we have $\widetilde{D}_2^2 \leq -3$, $H_{11}^2 \leq -3$ and $\widetilde{H}_1^2 = -2$, i.e., $a = 1, b = 2$.

If $\sigma \neq id$, then a contradiction is derived as in the case(1) above. If $\sigma = id$, then $J_1 = H_1 = D_1, K_{\widetilde{S}} + 2(C + D_1) + H_{11} + H_{12} + D_2 \sim \widetilde{G} = \widetilde{\Sigma}$, where $\widetilde{\Sigma}$ is a (-1)-curve (cf. (4) and (6) of Lemma 1.6). We have also $\widetilde{\Sigma}.\widetilde{H}_{2j} > 0$ for $j = 1, 2$ and 3, where $\widetilde{H}_{2j}$ are irreducible components of D adjacent to D_2 (cf. Lemma 1.6 (5)). Now applying Lemma 1.5, we get $D^* \geq 2/3D_2 + 1/3H_{21} + 1/3H_{22} + 1/3H_{23}$. Hence $-\widetilde{\Sigma}.(K_{\widetilde{S}} + D^*) \leq 1 - \widetilde{\Sigma}.(1/3H_{21} + 1/3H_{22} + 1/3H_{23}) \leq 0$, a contradiction to Lemma 1.4.

So it is impossible that $\widetilde{\Delta}_2$ is a fork with $\widetilde{D}_2$ as its central component. Lemma 1.8 is proved in the present case. □

LEMMA 1.9. *Suppose the case(2) in Lemma 1.8 occurs. Then it is* *impossible that* $\widetilde{J}_1^2 = \widetilde{J}_2^2 = -2$.

PROOF. We consider the case where $\widetilde{J}_1^2 = \widetilde{J}_2^2 = -2$. By the hypothesis, we have $J_i = H_i$, $\widetilde{H}_i^2 = -2$ for $i = 1, 2$ and we may assume that $H_{11}^2 \leq -3$, $H_{21}^2 \leq -3$.

Case(1) σ is a contraction of curves contained in $C + T_1$.

Then the conditions of Corollary 1.7 are satisfied. Hence $K_{\widetilde{T}} + 2(\widetilde{C} + \widetilde{H}_1) + \widetilde{H}_{11} + \widetilde{H}_{12} + \widetilde{H}_2 \sim \widetilde{G} = \widetilde{\Sigma}$ where $\widetilde{\Sigma}$ is a (-1)-curve. Note that $\widetilde{\Sigma}.\widetilde{H}_{21} = \widetilde{G}.\widetilde{H}_{21} = (K_{\widetilde{T}} + \widetilde{H}_2).\widetilde{H}_{21} \geq 1 + 1$ (cf. Lemma 1.6 (2)). Let $\Sigma := \sigma^*(\widetilde{\Sigma})$. Then Σ is again a (-1)-curve with $\Sigma.H_{21} \geq 2$ (cf. Lemma 1.6 (2)). On the other hand, $D^* \geq 1/2D_2 + 1/2H_{21}$ because $D_2^2 \leq -3, H_{21}^2 \leq -3$ (cf. Lemma 1.5). This leads to $-\Sigma.(K_{\widetilde{S}} + D^*) \leq 1 - 1/2\Sigma.H_{21} \leq 0$, a contradicion to Lemma 1.4. So Case(1) is impossible.

Case(2) σ contracts at least one irreducible component of the maximal twig T_2 of Δ_2.

By noting that $D_1^2 = -2, D_2^2 \leq -3$, there are two smooth blowing-downs $\sigma_1 : \widetilde{S} \to \widetilde{S}_1, \sigma_2 : \widetilde{S}_1 \to \widetilde{T}$ such that $\sigma = \sigma_2 \cdot \sigma_1$ and that :

(1) $\sigma_1(T_1 + C + T_2) = T_1' + E + T_2'$ where E is a (-1)-curve and $T_i' \leq \sigma_1(T_i)$,

(2) $T_1' + \sigma_1(H_1) = \sum_{i=1}^{s} L_i$, $E.L_1 = L_i.L_{i+1} = 1\,(i = 1, \cdots, s-1; s \geq 2)$, $L_s = \sigma_1(H_1)$, $L_1^2 = -2$, $L_2^2 = -(t+1)$, $L_j^2 = -2 (2 < j < s)$, and

(3) $T_2' + \sigma_1(H_2) = \sum_{i=1}^{t} M_i$, $E.M_1 = M_i.M_{i+1} = 1\,(i = 1, \cdots, t-1; t \geq 2)$, $M_t = \sigma_1(H_2)$, $M_1^2 = -3$, $M_t^2 = -s$, $M_j^2 = -2\,(2 \leq j < j \geq 2, j \neq t)$.

Since $\sigma_1(\Delta_1 + \Delta_2) - E$ is contractible to quotient singularities, we have $(s, t) = (2, 2), (2, 3)$ or $(3, 2)$. By Lemma 1.4, we have $1 - E.M^* = -E.(K_{\widetilde{S}_1} + M^*) > 0$. We can also get lower bounds for coefficients of M^* by applying Lemma 1.5 to $X = S_1$, $\sum_{i=1}^{s+t+4} Q_i = H_{11} + H_{21} + H_{12} + H_{22} + \sum_i L_i + \sum_j M_j$, $Q_1^2 = Q_2^2 = -3$, $Q_3^2 = Q_4^2 = -2$. Now the inequality $1 - E.M^* > 0$, together with these lower bounds, will deduce an inequality $(2t-1)(s-1) < 3$. This is impossible because $s \geq 2$ and $t \geq 2$.

This proves Lemma 1.9. □

In the proof of the following Lemmas 1.10 and 1.11, to rule out most of the cases, we shall frequently use Lemma 1.5 to get an estimate on the coefficients of M^* and then deduce a contradiction to Lemma 1.4.

LEMMA 1.10. *Suppose that the case in Corollary 1.7 occurs. Suppose further that the case(2) in Lemma 1.8 occurs with* $(\widetilde{J}_a^2, \widetilde{J}_b^2) = (-2, -3)$ *or* $(-2, -4)$ *(hence* $a = 1, b = 2, J_1 = H_1, J_2 = D_2$*). Then Theorem 1.1, (3) or (4) occurs.*

PROOF. By the hypothesis in the case(2) of Lemma 1.8, we may assume that $H_{11}^2 \le -3$. By Corollary 1.7, $K_{\widetilde{T}} + 2(\widetilde{C} + \widetilde{H}_1) + \widetilde{H}_{11} + \widetilde{H}_{12} + \widetilde{J}_2 \sim \widetilde{G} = \widetilde{\Sigma}$ where $\widetilde{\Sigma}$ is a (-1)-curve.

CLAIM(1). (1) $\widetilde{D}^* \ge 3/7\widetilde{H}_{11} + 2/7\widetilde{H}_1 + 1/7\widetilde{H}_{12} + (a-2)/(a-1)\widetilde{J}_2$. Here $a := -\widetilde{J}_2^2 \ge 3$ and hence $(a-2)/(a-1) \ge 1/2$.

(2) $\widetilde{\Delta}_1$ is a linear chain.

(3) Either $\widetilde{\Delta}_2$ is a linear chain with $\widetilde{J}_2 = \widetilde{H}_2$, or $\widetilde{\Delta}_2$ is a fork with $\widetilde{J}_2$ as a tip.

(4) $\widetilde{\Delta}_1 - \widetilde{H}_{11}$ consists of (-2)-curves.

(5) $\widetilde{\Delta}_2 - \widetilde{J}_2$ consists of (-2)-curves.

Since $\widetilde{H}_{11}^2 \le -3$ and since it is imposible that $\widetilde{\Delta}_2$ is a linear chain with $\widetilde{J}_2$ as a tip (cf. the hypothesis(∗) after Lemma 1.3), (1) follows.

If $\widetilde{\Delta}_1$ is not a linear chain, then $\widetilde{D}^* \ge 1/2\widetilde{H}_1 + 1/2\widetilde{H}_{11}$ (cf. Lemma 1.5). This leads to $-\widetilde{C}.(K_{\widetilde{S}} + \widetilde{D}^*) \le 1 - \widetilde{C}.(1/2\widetilde{H}_1 + 1/2\widetilde{J}_2) = 0$, a contradiction to Lemma 1.4. So, (2) of Claim(1) is true.

Suppose (3) of Claim(1) is false. Then $\widetilde{\Delta}_2$ contains $L_i (i = 1, \cdots, s; s \ge 4)$ such that $L_2 = \widetilde{J}_2$, $L_i.L_{i+1} = L_{s-2}.L_s = 1\,(i = 1, \cdots, s-2)$. So we have $\widetilde{D}^* \ge 1/3L_1 + 2/3\sum_{i=2}^{s-2} L_i + 1/3L_{s-1} + 1/3L_s$ (cf. Lemma 1.5). On the other hand, for $i = 1, 3$ (and also for $i = 4$ if $s = 4$), we have $L_i.\widetilde{\Sigma} = L_i.(K_{\widetilde{T}} + \widetilde{D}_2) \ge 1$ (cf. Lemma 1.6). This leads to $-\widetilde{\Sigma}.(K_{\widetilde{T}} + \widetilde{D}^*) \le 1 - \widetilde{\Sigma}.(1/3L_1 + 2/3\sum_{i=2}^{s-2} L_i + 1/3L_{s-1} + 1/3L_s) \le 0$. We reach a contradiction to Lemma 1.4. Thus, (3) of Claim(1) is true.

Suppose $\widetilde{\Delta}_1 - \widetilde{H}_{11}$ contains a $(-n)$-curve B with $n \ge 3$. If B and $\widetilde{H}_{12}$ are in the same connected component of $\widetilde{\Delta}_1 - \widetilde{H}_1$, then $\widetilde{D}^* \ge 1/2\widetilde{H}_1 + 1/2\widetilde{J}_2$ (cf. Lemma 1.5) and hence $-\widetilde{C}.(K_{\widetilde{T}} + \widetilde{D}^*) \le 1 - \widetilde{C}.(1/2\widetilde{H}_1 + 1/2\widetilde{J}_2) = 0$, a contradiction to Lemma 1.4. If B and $\widetilde{H}_{11}$ are in the same connected component of $\widetilde{\Delta}_1 - \widetilde{H}_1$, we let $L_1 + \cdots + L_s$ be a linear chain in $\widetilde{\Delta}_1$ such that $L_1 = \widetilde{H}_{11}, L_s = B, L_i.L_{i+1} = 1 (i = 1, \cdots, s-1)$. Then one has $D^* > 1/2\sum_i L_i$ (cf. Lemma 1.5). Moreover, $L_i.\widetilde{\Sigma} = L_i.(K_{\widetilde{T}} + \widetilde{H}_{11}) \ge 1$ for $i = 2, s$

and $L_2.\widetilde{\Sigma} \geq 2$ if $s = 2$. This leads to $-\widetilde{\Sigma}.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - \widetilde{\Sigma}.1/2 \sum_i L_i \leq 0$, a contradiction to Lemma 1.4. Therefore, (4) of Claim(1) is true.

Suppose that $\widetilde{\Delta}_2 - \widetilde{J}_2$ contains a $(-n)$-curve B with $n \geq 3$. Let $L_1 + \cdots + L_s$ be a linear chain contained in $\widetilde{\Delta}_2$ such that $L_1 = \widetilde{J}_2$, $L_s = B$, $L_i.L_{i+1} = 1\,(i = 1, \cdots, s-1)$. Then we have $\widetilde{D}^* \geq 1/2 \sum_i L_i$ (cf. Lemma 1.5). Note that for $i = 2, s$, we have $L_i.\widetilde{\Sigma} = L_i.(K_{\widetilde{T}} + \widetilde{J}_2) \geq 1$. Moreover, $L_2.\widetilde{\Sigma} \geq 2$ if $s = 2$. This leads to $-\widetilde{\Sigma}.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - \widetilde{\Sigma}.(1/2 \sum_i L_i) \leq 0$. We reach a contradiction to Lemma 1.4. Therefore, (5) of Claim(1) is true.

This proves Claim(1).

Claim(2). Suppose that $\widetilde{J}_2^2 = -4$. Then Theorem 1.1 (3) occurs.

We consider the case $\widetilde{J}_2^2 = -4$. Then $\widetilde{D}^* \geq 2/3\widetilde{J}_2$ by Claim(1). If $\widetilde{H}_{11}$ is not a tip of $\widetilde{\Delta}_1$ (resp. $\widetilde{H}_{12}$ is not a tip, or $H_{11}^2 \leq -4$), then by Lemma 1.5, $D^* \geq 6/11\widetilde{H}_{11} + 4/11\widetilde{H}_1 + 2/11\widetilde{H}_{12}$ (resp. $D^* \geq 4/9\widetilde{H}_{11} + 3/9\widetilde{H}_1 + 2/9\widetilde{H}_{12}$, or $D^* \geq 3/5\widetilde{H}_{11} + 2/5\widetilde{H}_1 + 1/5\widetilde{H}_{12}$). Any of the three cases implies that $-\widetilde{C}.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - \widetilde{C}.(1/3\widetilde{H}_1 + 2/3\widetilde{J}_2) = 0$, a contradiction to Lemma 1.4.

Thus, $\widetilde{\Delta}_1 = \widetilde{H}_1 + \widetilde{H}_{11} + \widetilde{H}_{12}$ and $\widetilde{H}_1^2 = -2, \widetilde{H}_{11}^2 = -3, \widetilde{H}_{12}^2 = -2$ (cf. Claim(1)). If $\widetilde{J}_2$ is a tip of $\widetilde{\Delta}_2$, i.e., if $J_2 \neq H_2$, then Theorem 1.1 (3) occurs since the intersection matrix of $\widetilde{C} + \widetilde{J}_2 + \widetilde{\Delta}_1$ and hence that of $C + T_2 + \Delta_1$ have a positive eigenvalue.

We may now assume that $J_2 = H_2$. Then $\widetilde{D}^* \geq 2/3\widetilde{H}_2 + 1/3\widetilde{H}_{21} + 1/3\widetilde{H}_{22}$ (cf. Claim(1)). We shall show that this leads to a contradiction. By Claim(1), $\widetilde{\Delta}_2$ is now a linear chain. If H_{2j} is not tip of $\widetilde{\Delta}_2$ for $j = 1$ and 2, then $D^* \geq 2/4\widetilde{H}_{21} + 3/4\widetilde{H}_2 + 2/4\widetilde{H}_{22}$. This leads to $0 < -\widetilde{C}.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - \widetilde{C}.(2/7\widetilde{H}_1 + 3/4\widetilde{H}_2) = 1 - 2/7 - 3/4 < 0$, a contradiction.

So we may assume that H_{21} is a tip of $\widetilde{\Delta}_2$. If $\widetilde{\Delta}_2$ has more than four irreducible components, then $D^* \geq 4/11\widetilde{H}_{21} + 8/11\widetilde{H}_2 + 6/11\widetilde{H}_{22}$. This leads to $0 < -\widetilde{C}.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - \widetilde{C}.(2/7\widetilde{H}_1 + 8/11\widetilde{H}_2) = 1 - 2/7 - 8/11 < 0$, a contradiction. Therefore, $H := \widetilde{\Delta}_2 - (\widetilde{H}_{21} + \widetilde{H}_2 + \widetilde{H}_{22})$ is zero or a (-2)-curve adjacent to $\widetilde{H}_{22}$ (cf. Claim(1)).

Note that $\widetilde{\Sigma}.\widetilde{H}_{2j} = (K_{\widetilde{T}} + \widetilde{H}_2).\widetilde{H}_{2j} = 1$ for $j = 1$ and 2 (cf. Lemma 1.6). If $B.\widetilde{\Sigma} > 0$ for some irreducible component B of $\widetilde{D} - (\widetilde{H}_{21} + \widetilde{H}_{22})$, then B is not contained in $\widetilde{\Delta}_1$ nor $\widetilde{\Delta}_2$, $B^2 \leq -3$ and $B.\widetilde{\Sigma} = B.K_{\widetilde{T}}$ (cf. Lemma 1.6

(5)). Hence $\widetilde{D}^* \geq 1/3B$. This leads to $0 < -\widetilde{\Sigma}.(K_{\widetilde{T}} + D^*) \leq 1 - \widetilde{\Sigma}.(1/3B + 1/3\widetilde{H}_{21} + 1/3\widetilde{H}_{22}) = 0$, a contradiction. So, $\widetilde{\Sigma}$ meets transversally only $\widetilde{H}_{21}$ and $\widetilde{H}_{22}$ in $\widetilde{D}$.

Let $S_0' := 2\widetilde{\Sigma} + \widetilde{H}_{21} + \widetilde{H}_{22}$ and let $\psi : \widetilde{T} \to \mathbf{P}^1$ be the $\mathbf{P}^1$-fibration with S_0' as a singular fiber. Let S_1' be the singualr fiber containing $\widetilde{C} + \widetilde{\Delta}_1$. Then there is a (-1)-curve E such that $E.\widetilde{H}_{11} = 1$ and $S_1' = 2(\widetilde{C} + \widetilde{H}_1) + \widetilde{H}_{11} + \widetilde{H}_{12} + E$. Since $\rho(\widetilde{T}) = 1$ and since every irreducible component of $\widetilde{D} - (H + \widetilde{H}_2)$ is contained in singular fibers of ψ, every singular fiber S_2' other than S_1' consists of one (-1)-curve and several irreducible components of $\widetilde{D}$ (cf. Lemma 1.1 (4) of Part **I**). Here we set $H := \widetilde{\Delta}_2 - (\widetilde{H}_{21} + \widetilde{H}_2 + \widetilde{H}_{22})$. Moreover, $H \neq 0$ because $\rho(T) = 1$. So, H is a (-2)-curve adjacent to $\widetilde{H}_{22}$. Since H is a cross-section, $H.E = 1$ and S_0', S_1' are the only singular fibers of ψ for otherwise H would meet a (-1)-curve F in some singular fiber S_2' and F has multiplicity at least two.

Let $\tau : \widetilde{T} \to \Sigma_2$ be the smooth blowing-down of curves in singular fibers of ψ such that $\tau(H)^2 = -2$. On the one hand, $\widetilde{H}_2$ is a 2-section with $\widetilde{H}_2 \cap H = \phi$ and hence $\tau(\widetilde{H}_2)^2 = 8$. On the other hand, a calculation shows that $\tau(\widetilde{H}_2)^2 = \widetilde{H}_2^2 + 1 + 4 = 1$. We reach a contradiction.

This proves Claim(2).

In view of Claim(2), we may assume that $\widetilde{J}_2^2 = -3$. If $\widetilde{J}_2$ is a tip of $\widetilde{\Delta}_2$, i.e., if $J_2 \neq H_2$, then Theorem 1.1 (3) occurs. Indeed, the intersection matrix of $\widetilde{C} + \widetilde{J}_2 + \widetilde{\Delta}_1$ and hence that of $C + T_2 + \Delta_1$ then have a positive eigenvalue.

Thus we may assume that $J_2 = H_2$. Then $\widetilde{\Delta}_2$ is a linear chain (cf. Claim(1)). We have also

$$\widetilde{D}^* \geq 3/7\widetilde{H}_{11} + 2/7\widetilde{H}_1 + 1/7\widetilde{H}_{12} + 1/4\widetilde{H}_{21} + 2/4\widetilde{H}_2 + 1/4\widetilde{H}_{22}.$$

Note that $H.\widetilde{\Sigma} = H.(K_{\widetilde{T}} + \widetilde{H}_{11} + \widetilde{H}_{12} + \widetilde{H}_2) = 1$ (cf. Lemma 1.6) if H is an irreducible component of $\widetilde{D} - \widetilde{H}_1$ adjacent to one of $\widetilde{H}_{11}, \widetilde{H}_{12}, \widetilde{H}_2$. In particular, $\widetilde{\Sigma}.\widetilde{H}_{21} = \widetilde{\Sigma}.\widetilde{H}_{22} = 1$.

Claim(3). $\widetilde{D} - (\widetilde{H}_{11} + \widetilde{H}_2)$ consists of (-2)-curves.

Suppose to the contrary that Claim(3) is false. Then $\widetilde{D} - (\widetilde{\Delta}_1 + \widetilde{\Delta}_2)$ contains a $(-n)$-curve B with $n \geq 3$ (cf. Claim(1)). By Lemma 1.6, we have

$B.\widetilde{\Sigma} = B.K_{\widetilde{T}} = n-2$. Note that $\widetilde{D}^* \geq (n-2)/nB$ and $0 < -\widetilde{\Sigma}.(K_{\widetilde{T}}+\widetilde{D}^*) \leq 1 - \widetilde{\Sigma}.(n-2)/nB = 1-(n-2)^2/n$. So, $n = 3$ and $B.\widetilde{\Sigma} = 1$.

If $\widetilde{D} - \widetilde{H}_1$ has an irreducible component H adjacent to $\widetilde{H}_{11}$, then $\widetilde{D}^* \geq 3/11H + 6/11\widetilde{H}_{11} + 4/11\widetilde{H}_1 + 2/11\widetilde{H}_{12}$. This leads to $0 < -\widetilde{\Sigma}.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1-\widetilde{\Sigma}.(1/3B+3/11H+1/4\widetilde{H}_{21}+1/4\widetilde{H}_{22}) = 1-1/3-3/11-1/4-1/4 < 0$. We reach a contradiction. So, $\widetilde{H}_{11}$ is a tip of $\widetilde{\Delta}_1$.

If $\widetilde{D} - \widetilde{H}_1$ has an irreducible component H adjacent to $\widetilde{H}_{12}$ but H is not a tip of $\widetilde{\Delta}_1$, then $\widetilde{D}^* \geq 2/11H + 3/11\widetilde{H}_{12} + 4/11\widetilde{H}_1 + 5/11\widetilde{H}_{11}$. This leads to $0 < -\widetilde{\Sigma}.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - \widetilde{\Sigma}.(1/3B + 2/11H + 1/4\widetilde{H}_{21} + 1/4\widetilde{H}_{22}) = 1 - 1/3 - 2/11 - 1/4 - 1/4 < 0$. We reach again a contradiction. Thus, $H := \widetilde{\Delta}_1 - (\widetilde{H}_{11} + \widetilde{H}_1 + \widetilde{H}_{12})$ is zero or a (-2)-curve adjacent to $\widetilde{H}_{12}$ (cf. Claim(1)).

Let $S'_0 := 2\widetilde{\Sigma}+\widetilde{H}_{21}+\widetilde{H}_{22}$ and let $\psi : \widetilde{T} \to \mathbf{P}^1$ be the $\mathbf{P}^1$- fibration with S'_0 as a singular fiber. Let S'_1 be the singular fiber containing $\widetilde{C}+\widetilde{H}_1+\widetilde{H}_{11}+\widetilde{H}_{12}$.

Suppose $H_{11}^2 = -3$. Then there is a (-1)-curve E such that $E.\widetilde{H}_{11} = 1$ and $S'_1 = 2(\widetilde{C} + \widetilde{H}_1) + \widetilde{H}_{12} + \widetilde{H}_{11} + E$. Since B is a 2-section, we have $B.E = 2$. This leads to $0 < -E.(K_{\widetilde{T}} + D^*) \leq 1 - E.(1/3B + 3/7\widetilde{H}_{11}) = 1 - (1/3) \cdot 2 - 3/7 < 0$, a contradiction. So, $H_{11}^2 \leq -4$.

Suppose $\sigma \neq id$. Let $\sigma_2 : \widetilde{S}_1 \to \widetilde{T}$ be the blowing-up of the point $P_2 := \widetilde{C} \cap \widetilde{H}_2$ and set $L := \sigma_2^{-1}(P_2)$. Then by the hypothesis in Corollary 1.7, there is a smooth blowing-down $\sigma_1 : \widetilde{S} \to \widetilde{S}_1$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Applying Lemma 1.4, we have $-L.(K_{\widetilde{S}_1}+M^*) > 0$, where $M^* \geq 2/3\sigma'_2\widetilde{H}_{11}+2/3\sigma'_2\widetilde{H}_1+1/3\sigma'_2\widetilde{H}_{12}+1/3\sigma'_2\widetilde{C}+1/3\sigma'_2\widetilde{H}_{21}+2/3\sigma'_2\widetilde{H}_2+1/3\sigma'_2\widetilde{H}_{22}$. This leads to $0 < -L.(K_{\widetilde{S}_1} + M^*) \leq 1 - L.(1/3\sigma'_2\widetilde{C} + 2/3\sigma'_2\widetilde{H}_2) = 0$, a contradiction. So, $\sigma = id$. Hence $\widetilde{T} = \widetilde{S}, H_i = D_i (i = 1, 2)$.

Let $S_0 := 3C + 2D_1 + H_{12} + D_2$ and let $\varphi : \widetilde{S} \to \mathbf{P}^1$ be the $\mathbf{P}^1$- fibration with S_0 as a singular fiber. Then $\widetilde{\Sigma}$ and the (-3)-curve B are contained in the same singular fiber of φ, say S_1. By the minimality of $-C.(K_{\widetilde{S}} + D^*)$ and by noting that C has multiplicity 3 in S_0 and the summation of the multiplicities of (-1)-curves in S_1 is at least 3 (cf. [4, Lemma 1.6]), every (-1)-curve F in S_1, especially $\widetilde{\Sigma}$, satisfies $-F.(K_{\widetilde{S}} + D^*) = -C.(K_{\widetilde{S}} + D^*)$. So, every singular fiber of the previous fibration ψ defined by $|2\widetilde{\Sigma}+\widetilde{H}_{21}+\widetilde{H}_{22}|$ has one of two types in Lemma 6.11 of Part **I**. However, S'_1 above contains a curve H_{11} with $H_{11}^2 \leq -4$. We reach a contradiction.

This proves Claim(3).

Let

$$S_0 := 3\widetilde{C} + 2\widetilde{H}_1 + \widetilde{H}_{12} + \widetilde{H}_2$$

and let $\varphi : \widetilde{T} \to \mathbf{P}^1$ be the $\mathbf{P}^1$-fibration with S_0 as a singular fiber. $\widetilde{H}_{21}, \widetilde{H}_{22}$ (resp. $\widetilde{H}_{11}$) is a cross-section (resp. 2-section). Denote by S_1 the singular fiber containing $\widetilde{\Sigma}$. Let

$$S_i \ (i = 0, 1, \cdots, r)$$

be all singular fibers of φ. By Claim(3), every singular fiber S_i $(i \geq 1)$ consists of only (-1) or (-2)-curves. So, S_i has one of two types in Lemma 6.11 of Part **I**.

CLAIM(4). Suppose that S_k has the first type in Lemma 6.11 of Part **I** for some $k \geq 1$. Then Case(4-1) of Theorem 1.1 occurs.

Suppose S_1 has the first type in Lemma 6.11 of Part **I**. Namely, $\widetilde{\Sigma}$ is the unique (-1)-curve in S_1. Then the 2-section $\widetilde{H}_{11}$ meets two multiplicity-one or one multiplicity-two irreducible component(s) other than $\widetilde{\Sigma}$ in S_1. This implies that $\widetilde{\Delta}_1$ is a fork (cf. Lemma 1.1 (4) of Part **I**), a contradiction to Claim(1). So, S_1 consists of two (-1)-curves $\widetilde{\Sigma}, E$ and several (-2)-curves.

Suppose that S_k has the first type in Lemma 6.11 of Part **I** for some $k \geq 2$, say $k = 2$. Namely, there is a unique (-1)-curve F in S_2. Since $\widetilde{H}_{2j}.S_2 = 1$ $(j = 1, 2)$, there are two (-2)-curves $G_j (j = 1, 2)$ such that $F.G_j = 1, \widetilde{H}_{2j}.G_j = \widetilde{H}_{11}.F = 1$ and

$$S_2 = 2F + G_1 + G_2.$$

Now we have (cf. Claim(1)) :

$$\widetilde{\Delta}_2 = G_1 + \widetilde{H}_{21} + \widetilde{H}_2 + \widetilde{H}_{22} + G_2.$$

We have also $\widetilde{D}^* \geq 1/5G_1 + 2/5\widetilde{H}_{21} + 3/5\widetilde{H}_2 + 2/5\widetilde{H}_{22} + 1/5G_2$.

If H is an irreducible component of $\widetilde{\Delta}_1 - \widetilde{H}_1$ adjacent to $\widetilde{H}_{12}$, then H is a cross-section and $H.G_j = 1$ for $j = 1$ or 2. This leads to $\widetilde{\Delta}_1 = \widetilde{\Delta}_2$, a contradiction. So, $\widetilde{H}_{12}$ is a tip of $\widetilde{\Delta}_1$.

If H is an irreducible component of $\widetilde{\Delta}_1 - \widetilde{H}_1$ adjacent to $\widetilde{H}_{11}$, then $\widetilde{D}^* \geq 3/11H + 6/11\widetilde{H}_{11} + 4/11\widetilde{H}_1 + 2/11\widetilde{H}_{12}$. This leads to $0 < -\widetilde{\Sigma}.(K_{\widetilde{T}} +$

$D^*) \le 1 - \widetilde{\Sigma}.(3/11H + 2/5\widetilde{H}_{21} + 2/5\widetilde{H}_{22}) = 1 - 3/11 - 2/5 - 2/5 < 0$, a contradiction. So, $\widetilde{H}_{11}$ is tip of $\widetilde{H}_1$.

Therefore,

$$\widetilde{\Delta}_1 = \widetilde{H}_1 + \widetilde{H}_{11} + \widetilde{H}_{12}.$$

In particular, $\widetilde{\Sigma}$ meets only $\widetilde{H}_{2j} (j = 1, 2)$ in $\widetilde{D}$ (cf. Lemma 1.6 and Claim(3)). So,

$$S_1 = \widetilde{\Sigma} + E$$

with $\widetilde{\Sigma}.E = 1$ and $\widetilde{H}_{11}.E = 2$.

If $\widetilde{H}_{11}^2 \le -4$, then $D^* > 1/2\widetilde{H}_{11}$ and $0 < -E.(K_{\widetilde{T}} + \widetilde{D}^*) \le 1 - E.1/2\widetilde{H}_{11} = 0$, a contradiction. So, $\widetilde{H}_{11}^2 = -3$.

For every $i \ge 3$, since $\widetilde{H}_{21}$ meets a (-1)-curve of multiplicity one in S_i, the fiber S_i has the second type in Lemma 6.11 of part **I**. Since $\widetilde{D} - (\widetilde{H}_{21} + \widetilde{H}_{22} + \widetilde{H}_{11})$ are contained in singular fibers of φ and since $\rho(T) = 1$, we see that $r = 3$ and

$$S_i \ (i = 0, 1, 2, 3)$$

are all singular fibers of φ (cf. [4, Lemma 1.5 (1)]). Let $E_j (j = 1, 2)$ be the two (-1)-curves in S_3.

Let $\tau : \widetilde{T} \to \Sigma_2$ be the smooth blowing-down of curves in singular fibers such that $\tau(\widetilde{H}_{21})^2 = -2$. Then $\tau(\widetilde{H}_{22}) \sim \tau(\widetilde{H}_{21}) + 2\tau(S_0)$ and $\tau(\widetilde{H}_{11}) \sim 2\tau(\widetilde{H}_{21}) + 4\tau(S_0)$. In particular, $\tau(\widetilde{H}_{22})^2 = 2$ and $\tau(\widetilde{H}_{11})^2 = 8$. So we may assume that $\widetilde{H}_{2j}.E_j = \widetilde{H}_{11}.E_j = 1 \ (j = 1, 2)$. Moreover,

$$S_3 = E_1 + G_3 + G_4 + E_2$$

where $G_3 + G_4$ is a connected component of $\widetilde{D}$ with two (-2)-curves (cf. Lemma 1.1 (4) of Part **I**) and with $E_j.G_{j+2} = 1$.

Now $\widetilde{H}_{11}^2 = -3$, and

$$\widetilde{\Delta}_1, \ \widetilde{\Delta}_2, \ G_3 + G_4$$

are all connected components of $\widetilde{D}$ (cf. Lemma 1.1, (4) of Part **I**). To show that Case(4-1) of Theorem 1.1 occurs, it suffices to show that $\sigma = id$. Let $\sigma_2 : \widetilde{S}_1 \to \widetilde{T}$ be the blowing-up of the point $P_2 := \widetilde{C} \cap \widetilde{H}_2$ and let $L := \sigma_2^{-1}(P_2)$. Suppose to the contrary that $\sigma \ne id$. Then by the hypothesis in Corollary 1.7, there is a smooth blowing-down $\sigma_1 : \widetilde{S} \to \widetilde{S}_1$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Now applying Lemma 1.4, we get $-L.(K_{\widetilde{S}_1} + M^*) > 0$, where $M^* =$

$1/2\sigma_2'\widetilde{H}_{11}+1/2\sigma_2'\widetilde{H}_1+1/4\sigma_2'\widetilde{H}_{12}+1/4\sigma_2'\widetilde{C}+1/4\sigma_2'G_1+2/4\sigma_2'\widetilde{H}_{21}+3/4\sigma_2'\widetilde{H}_2+2/4\sigma_2'\widetilde{H}_{22}+1/4\sigma_2'G_2$. This leads to $-L.(K_{\widetilde{S}_1}+M^*)=1-L.(1/4\sigma_2'\widetilde{C}+3/4\sigma_2'\widetilde{H}_2)=0$. We reach a contradiction. So, $\sigma = id$ and Case(4-1) of Theorem 1.1 occurs.

This proves Claim(4).

In view of Claim(4), we may assume that each singular fiber S_i ($i = 1, \cdots, r$) has the second type in Lemma 6.11 of Part **I**. Then the number of singular fibers containing two (-1)-curves is one less than the number of sectional-components of $\widetilde{D}$ because $\rho(T) = 1$. So, $r = 2$ and S_0, S_1, S_2 are all singular fibers if $\widetilde{H}_{12}$ is a tip of $\widetilde{\Delta}_1$, or $r = 3$ and S_0, S_1, S_2, S_3 are all singular fibers otherwise. Let

$$\mu : \widetilde{T} \to \Sigma_2$$

be the smooth blowing-down of curves in singular fibers of φ such that $\mu(\widetilde{H}_{21})^2 = -2$. Write $\mu(\widetilde{H}_{ij}) = \overline{H}_{ij}$, $\mu(S_i) = \overline{S}_i$, etc. Then $\overline{H}_{22} \sim \overline{H}_{21} + 2\overline{S}_0$ and $\overline{H}_{11} \sim 2\overline{H}_{21} + 4\overline{S}_0$. In particular, $\overline{H}_{22}^2 = 2$, $\overline{H}_{11}^2 = 8$, $\overline{H}_{22}.\overline{H}_{11} = 4$.

CLAIM(5). Suppose that $\widetilde{H}_{11}$ is not a tip. Then Case(4-2) of Theorem 1.1 occurs.

One can see that $\widetilde{H}_{11}$ is a $(-3)-$curve, as in the proof of Claim (4) above. Note that $r \geq 2$ and we can write

$$S_1 = \widetilde{\Sigma} + \sum_{i=1}^{s} G_i + E$$

such that $E^2 = -1$, $G_i^2 = -2$, $\widetilde{H}_{11}.G_1 = \widetilde{\Sigma}.G_1 = G_j.G_{j+1} = G_s.E = 1$ ($j = 1, \cdots, s-1$) (cf. Lemma 1.6), and

$$S_2 = E_1 + \sum_{i=s+1}^{s+t} G_i + E_2$$

such that $E_i^2 = -1$, $G_j^2 = -2, E_1.G_{s+1} = G_j.G_{j+1} = G_{s+t}.E_2 = 1\,(j \leq s+t-1)$. Note that $\widetilde{H}_{11}.E = 1$ for $\widetilde{H}_{11}.S_1 = 2$ and $\widetilde{H}_{11}.G_1 = 1$.

Note that $\widetilde{D}^* \geq 2/11\widetilde{H}_{12} + 4/11\widetilde{H}_1 + 6/11\widetilde{H}_{11} + 3/11G_1$. If $F.\widetilde{H}_{11} \geq 2$ for some (-1)-curve F, then $0 < -F.(K_{\widetilde{T}} + D^*) \leq 1 - 6/11F.\widetilde{H}_{11} \leq 1 - 2 \cdot (6/11) < 0$, a contradiction. So, $F.\widetilde{H}_{11} \leq 1$ for every (-1)-curve F and the equality holds if F is in S_i $(i \geq 2)$ because $\widetilde{H}_{11}.S_i = 2$ (cf. (2) of Claim(1)).

Case(5.1) $\widetilde{H}_{12}$ is a tip of $\widetilde{\Delta}_1$, while $\widetilde{H}_{2j}$ is not a tip of $\widetilde{\Delta}_2$ for $j = 1$ or 2, say $j = 1$. Then $r = 2$. We may assume $\widetilde{H}_{21}.G_{s+1} = 1$. Since $\overline{H}_{22}^2 = 2$, one gets $\widetilde{H}_{22}.E_2 = 1$ and $t = 4$. This leads to $\widetilde{D}^* \geq 1/10G_{s+4} + 2/10G_{s+3} + 3/10G_{s+2} + 4/10G_{s+1} + 5/10\widetilde{H}_{21} + 6/10\widetilde{H}_2 + 3/10\widetilde{H}_{22}$ and $0 < -\widetilde{\Sigma}.(K_{\widetilde{T}} + D^*) \leq 1 - \widetilde{\Sigma}.(5/10\widetilde{H}_{21} + 3/10\widetilde{H}_{22} + 3/11G_1) = 1 - 5/10 - 3/10 - 3/11 < 0$, a contradiction. So, Case(5.1) is impossible.

Case(5.2). $\widetilde{H}_{12}$ is a tip of $\widetilde{\Delta}_1$ and both $\widetilde{H}_{21}$ and $\widetilde{H}_{22}$ are tips of $\widetilde{\Delta}_2$. Then $r = 2$, i.e.,

$$S_i \ (i = 0, 1, 2)$$

are all singular fibers of φ, and

$$\widetilde{\Delta}_1 = \widetilde{H}_{12} + \widetilde{H}_1 + \widetilde{H}_{11} + \sum_{i=1}^{s} G_i, \quad \widetilde{\Delta}_2 = \widetilde{H}_{21} + \widetilde{H}_2 + \widetilde{H}_{22},$$

because $\widetilde{\Delta}_i$'s are linear chains. Moreover,

$$\widetilde{\Delta}_1, \quad \widetilde{\Delta}_2, \quad \sum_{i=s+1}^{s+t} G_i$$

are all connected components of $\widetilde{D}$ (cf. Lemma 1.1 (4) of Part **I**). We shall show that Case(4-2) of Theorem 1.1 occurs. We may assume that $\widetilde{H}_{21}.E_1 = 1$. By the same reasoning as in the previous case, we have $\widetilde{H}_{22}.E_2 = 1$ and $t = 3$. Then $8 = \overline{H}_{11}^2 = \widetilde{H}_{11}^2 + 2 + (s+4) + 4$. Hence $s = -\widetilde{H}_{11}^2 - 2$. If $s \geq 2$, then $\widetilde{H}_{11}^2 \leq -4$ and $\widetilde{D}^* \geq 1/4\widetilde{H}_{12} + 2/4\widetilde{H}_1 + 3/4\widetilde{H}_{11} + 2/4G_1 + 1/4G_2$. This leads to $0 < -\widetilde{C}.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - \widetilde{C}.(1/2\widetilde{H}_1 + 1/2\widetilde{H}_2) = 0$, a contradiction. So, $s = 1, \widetilde{H}_{11}^2 = -3$.

Now $s = 1, t = 3, \widetilde{H}_{11}^2 = -3$. To show that Case(4-2) of Theorem 1.1 occurs, it is sufficient to show that $\sigma = id$. Let $\sigma_2 : \widetilde{S}_1 \to \widetilde{T}$ be the blowing-up of the point $P_2 := \widetilde{C} \cap \widetilde{H}_2$ and let $L := \sigma_2^{-1}(P_2)$. Suppose to the contrary that $\sigma \neq id$. Then by the hypothesis in Corollary 1.7, there is a smooth blowing-down $\sigma_1 : \widetilde{S} \to \widetilde{S}_1$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Applying Lemma 1.4, we get $-L.(K_{\widetilde{S}_1} + M^*) > 0$, where $M^* = 1/3\sigma_2' G_1 + 2/3\sigma_2' \widetilde{H}_{11} + 2/3\sigma_2' \widetilde{H}_1 + 1/3\sigma_2' \widetilde{H}_{12} + 1/3\sigma_2' \widetilde{C} + 1/3\sigma_2' \widetilde{H}_{21} + 2/3\sigma_2' \widetilde{H}_2 + 1/3\sigma_2' \widetilde{H}_{22}$. Hence $0 < -L.(K_{\widetilde{S}_1} + M^*) = 1 - L.(1/3\sigma_2' \widetilde{C} + 2/3\sigma_2' \widetilde{H}_2) = 0$. We reach a contradiction. Therefore, $\sigma = id$ and Case(4-2) of Theorem 1.1 occurs.

Case(5.3). $\widetilde{H}_{12}$ is not a tip of $\widetilde{\Delta}_1$. Let H be the irreducible component of $\widetilde{D} - \widetilde{H}_1$ adjacent to $\widetilde{H}_{12}$. Then $\widetilde{D}^* \geq 1/7H + 2/7\widetilde{H}_{12} + 3/7\widetilde{H}_1 + 4/7\widetilde{H}_{11} + 2/7G_1$. Note that H is a cross-section and $H.\widetilde{\Sigma} = H.(K_{\widetilde{T}} + \widetilde{H}_{12}) = 1$ (cf. Lemma 1.6).

If $\widetilde{H}_{2j}$ is not a tip of $\widetilde{\Delta}_2$ for $j = 1$ or 2, say $j = 1$, then $\widetilde{D}^* \geq 4/11\widetilde{H}_{21} + 6/11\widetilde{H}_2 + 3/11\widetilde{H}_{22}$, and this leads to $0 < -\widetilde{\Sigma}.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - \widetilde{\Sigma}.(4/11\widetilde{H}_{21} + 3/11\widetilde{H}_{22} + 1/7H + 2/7G_1) = 1 - 4/11 - 3/11 - 1/7 - 2/7 < 0$, a contradiction. So, $\widetilde{H}_{2j}$'s are tips of $\widetilde{\Delta}_2$ and hence $\widetilde{\Delta}_2 = \widetilde{H}_{21} + \widetilde{H}_2 + \widetilde{H}_{22}$.

If G_1 or H is not a tip of $\widetilde{\Delta}_1$ (resp. if $\widetilde{H}_{11}^2 \leq -4$), then $\widetilde{D}^* \geq 3/19H + 6/19\widetilde{H}_{12} + 9/19\widetilde{H}_1 + 12/19\widetilde{H}_{11} + 8/19G_1$ or $\widetilde{D}^* \geq 4/17H + 6/17\widetilde{H}_{12} + 8/17\widetilde{H}_1 + 10/17\widetilde{H}_{11} + 5/17G_1$ (resp. $\widetilde{D}^* \geq 2/11H + 4/11\widetilde{H}_{12} + 6/11\widetilde{H}_1 + 8/11\widetilde{H}_{11} + 4/11G_1$) and hence $-\widetilde{\Sigma}.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - \widetilde{\Sigma}.(3/19H + 8/19G_1 + 1/4\widetilde{H}_{21} + 1/4\widetilde{H}_{22}) = 1 - 3/19 - 8/19 - 1/4 - 1/4 < 0$, or $\leq 1 - \widetilde{\Sigma}.(4/17H + 5/17G_1 + 1/4\widetilde{H}_{21} + 1/4\widetilde{H}_{22}) = 1 - 4/17 - 5/17 - 1/4 - 1/4 < 0$ (resp. $\leq 1 - \widetilde{\Sigma}.(2/11H + 4/11G_1 + 1/4\widetilde{H}_{21} + 1/4\widetilde{H}_{22}) = 1 - 2/11 - 4/11 - 1/4 - 1/4 < 0$). We reach a contradiction in any of the cases. So, $s = 1, \widetilde{\Delta}_1 = H + \widetilde{H}_{12} + \widetilde{H}_1 + \widetilde{H}_{11} + G_1, \widetilde{H}_{11}^2 = -3$.

Note that $r = 3$. Let E_1, E_2 (resp. E_3, E_4) be the (-1)-curves in S_2 (resp. S_3). Let $t_i + 2$ be the number of irreducible components of S_i. We may assume that $\widetilde{H}_{21}.E_j = 1$ for $j = 1$ and 3. Note that $8 = \overline{H}_{11}^2 = \widetilde{H}_{11}^2 + 2 + (1 + 4) + (t_1 + 1) + (t_2 + 1)$. So, $t_1 + t_2 = 2$. Now $\overline{H}_{22}^2 = 2$ implies that $\widetilde{H}_{22}.E_j = 1$ for $j = 2, 4$. But then it is impossible that $\overline{H}^2 = \overline{H}.\overline{H}_{22} = 2$. So, Case(5-3) is impossible.

This proves Claim(5).

In view of Claim(5), we may assume that

$$\widetilde{H}_{11} \text{ is a tip of } \widetilde{\Delta}_1.$$

Thus,

$$S_1 = \widetilde{\Sigma} + E$$

where E is a (-1)-curve such that $E.\widetilde{\Sigma} = 1$ and $E.\widetilde{H}_{11} = S_1.\widetilde{H}_{11} = 2$ (cf. Lemma 1.6 (5)). If $\widetilde{H}_{11}^2 \leq -4$, then $\widetilde{D}^* \geq 1/2\widetilde{H}_{11}$ and $0 < -E.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - E.1/2\widetilde{H}_{11} = 0$, a contradiction. So,

$$\widetilde{H}_{11}^2 = -3.$$

Claim(6). Suppose that $\widetilde{H}_{12}$ is a tip. Then Case(4-3) of Theorem 1.1 occurs.

In this case, we have $r = 2$, i.e.,

$$S_i \ (i = 0, 1, 2)$$

are all singular fibers of φ and

$$\widetilde{\Delta}_1 = \widetilde{H}_1 + \widetilde{H}_{11} + \widetilde{H}_{12}.$$

Hence $\widetilde{\Sigma}$ meets only $\widetilde{H}_{2j}$ $(j = 1, 2)$ in $\widetilde{D}$ (cf. Lemma 1.6 (5) and Claim(3)).

Write

$$S_2 = E_1 + \sum_{i=1}^{t} G_i + E_2$$

such that $E_1.G_1 = G_i.G_{i+1} = G_t.E_2 = 1\,(i = 1, \cdots, t-1)$. We may assume that $\widetilde{H}_{2j}$ does not meet $\sum_i G_i$ for $j = 1$ or 2, say $j = 1$. We may assume also that $\widetilde{H}_{21}.E_1 = 1$. Then $\overline{H}_{22}^2 = 2$ implies that either $t = 3$ and $\widetilde{H}_{22}.E_2 = 1$, or $t = 4$ and $\widetilde{H}_{22}.G_4 = 1$. Since $\overline{H}_{11}^2 = 8$, we must have $t = 4$ and $\widetilde{H}_{11}.E_j = 1$ for $j = 1$ and 2. Now $\widetilde{H}_{11}^2 = -3$,

$$\widetilde{\Delta}_2 = \widetilde{H}_{21} + \widetilde{H}_2 + \widetilde{H}_{22} + G_4 + G_3 + G_2 + G_1, \quad \text{and}$$

$$\widetilde{\Delta}_1, \quad \widetilde{\Delta}_2$$

are all connected components of $\widetilde{D}$ (cf. Lemma 1.1 (4) of Part **I**).

To prove that Case(4-3) of Theorem 1.1 occurs, it is sufficent to show that $\sigma = id$. Let $\sigma_2 : \widetilde{S}_1 \to \widetilde{T}$ be the blowing-up of the point $P_2 := \widetilde{C} \cap \widetilde{H}_2$

and set $L := \sigma_2^{-1}(P_2)$. Suppose to the contrary that $\sigma \neq id$. Then by the hypothesis in Corollary 1.7, there is a smooth blowing-down $\sigma_1 : \widetilde{S} \to \widetilde{S}_1$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Applying Lemma 1.4, we get $-L.(K_{\widetilde{S}_1} + M^*) > 0$, where $M^* = 1/2\sigma_2'\widetilde{H}_{11} + 1/2\sigma_2'\widetilde{H}_1 + 1/4\sigma_2'\widetilde{H}_{12} + 1/4\sigma_2'\widetilde{C} + 1/8\sigma_2'G_1 + 2/8\sigma_2'G_2 + 3/8\sigma_2'G_3 + 4/8\sigma_2'G_4 + 5/8\sigma_2'\widetilde{H}_{22} + 6/8\sigma_2'\widetilde{H}_2 + 3/8\sigma_2'\widetilde{H}_{21}$. This leads to $-L.(K_{\widetilde{S}_1} + M^*) = 1 - L.(1/4\sigma_2'\widetilde{C} + 3/4\sigma_2'\widetilde{H}_2) = 0$, a contradiction. Therefore $\sigma = id$ and Case(4-3) of Theorem 1.1 occurs.

This proves Claim(6).

CLAIM(7). Suppose that $\widetilde{H}_{12}$ is not a tip. Then either Theorem 1.1 (3) occurs or Case(4-4) of Theorem 1.1 occurs.

Then $r = 3$, i.e.,

$$S_i \ (i = 0, 1, 2, 3)$$

are all singular fibers of φ. Write

$$S_2 = E_1 + \sum_{i=1}^{t_1} G_i + E_2,$$

$$S_3 = E_3 + \sum_{i=t_1+1}^{t_1+t_2} G_i + E_4$$

such that $E_j^2 = -1$, $G_i^2 = -2$, $E_1.G_1 = G_{t_1}.E_2 = E_3.G_{t_1+1} = G_{t_1+t_2}.E_4 = G_i.G_{i+1} = 1$.

Let H be an irreducible component of $\widetilde{\Delta}_1 - \widetilde{H}_1$ adjacent to $\widetilde{H}_{12}$. If H is not a tip of $\widetilde{\Delta}_1$ then the intersection matrix of $\widetilde{C} + \widetilde{\Delta}_1$ and hence that of $C + T_2 + \Delta_1$ have a positive eigenvalue. So Theorem 1.1 (3) occurs. Thus we may assume that H is a tip of $\widetilde{\Delta}_1$. Hence $\widetilde{\Sigma}.H = 1$ and

$$\widetilde{\Delta}_1 = \widetilde{H}_1 + \widetilde{H}_{11} + \widetilde{H}_{12} + H.$$

Note that

$$\widetilde{D}^* = 1/9H + 2/9\widetilde{H}_{12} + 3/9\widetilde{H}_1 + 4/9\widetilde{H}_{11} + \text{(other terms)}.$$

Now one may assume that $E_j.H = 1$ for $j = 2, 4$. Let $\varepsilon : \widetilde{T} \to \Sigma_2$ be the smooth blowing-down of curves in the singular fibers of φ such that $\varepsilon(H)^2 = -2$. Then $\varepsilon(H_{2j})^2 = 2$ $(j = 1, 2)$ and $\varepsilon(H_{11})^2 = 8$.

If $\widetilde{H}_{11}.E_i = 2$ for $i = 1$ or 3, say $i = 1$, then $S_2 = E_1 + E_2, S_3 = E_3 + E_4$ and $\widetilde{H}_{11}.E_k = 1$ for $k = 3$ and 4 because $\varepsilon(H_{11})^2 = 8$. But then $\varepsilon(H_{2j})^2 \le -2 + 3$ $(j = 1, 2)$, a contradiction. If $\widetilde{H}_{11}.E_i = 2$ for $i = 2$ or 4, then $-E_i.(K_{\widetilde{T}} + \widetilde{D}^*) \le 1 - E_i.(1/9H + 4/9\widetilde{H}_{11}) = 1 - 1/9 - (4/9) \times 2 = 0$, a contradiction. So, $\widetilde{H}_{11}.E_j = 1$ for $j = 1, 2, 3$ and 4. Now $\varepsilon(H_{11})^2 = 8$ implies that $t_1 + t_2 = 3$.

If $\widetilde{H}_{2j}$ is not a tip of $\widetilde{\Delta}_2$ for both $j = 1$ and 2, then one may assume that $(t_1, t_2) = (1, 2)$ and $\widetilde{H}_{21}.G_1 = 1$. Then it is impossible that $\varepsilon(H_{21})^2 = 2$. So, one may assume that $\widetilde{H}_{21}$ is a tip of $\widetilde{\Delta}_2$.

Since $\varepsilon(H_{21})^2 = 2$, one may assume that $(t_1, t_2) = (1, 2)$ and $\widetilde{H}_{21}.E_j = 1$ for $j = 2$ and 3. Now $\varepsilon(H_{22}).\varepsilon(H_{21}) = 2$ implies that $\widetilde{H}_{22}.E_1 = \widetilde{H}_{22}.G_3 = 1$. So,

$$\widetilde{\Delta}_2 = \widetilde{H}_{21} + \widetilde{H}_2 + \widetilde{H}_{22} + G_3 + G_2,$$

and

$$\widetilde{\Delta}_1, \quad \widetilde{\Delta}_2, \quad G_1$$

are all connected components of $\widetilde{D}$ (cf. Lemma 1.1 (4), Part **I**).

Now $(t_1, t_2) = (1, 2)$ and $\widetilde{H}_{11}^2 = -3$. To prove that Case(4-4) takes place, we have only to show that $\sigma = id$. Let $\sigma_2 : \widetilde{S}_1 \to \widetilde{T}$ be the blowing-up of the point $\widetilde{C} \cap \widetilde{H}_2$ and set $L := \sigma_2^{-1}(P_2)$. Suppose to the contrary that $\sigma \neq id$. Then by the hypothesis in Corollary 1.7, there is a smooth blowing-down $\sigma_1 : \widetilde{S} \to \widetilde{S}_1$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Applying Lemma 1.4, we get $-L.(K_{\widetilde{S}_1} + M^*) > 0$, where $M^* = 2/9\sigma_2'H + 4/9\sigma_2'\widetilde{H}_{12} + 6/9\sigma_2'\widetilde{H}_1 + 5/9\sigma_2'\widetilde{H}_{11} + 3/9\sigma_2'\widetilde{C} + {}$ $+4/11\sigma_2'\widetilde{H}_{21} + 8/11\sigma_2'\widetilde{H}_2 + 6/11\sigma_2'\widetilde{H}_{22} + 4/11\sigma_2'G_3 + 2/11\sigma_2'G_2$. This leads to $-L.(K_{\widetilde{S}_1} + M^*) = 1 - L.(1/3\sigma_2'\widetilde{C} + 8/11\sigma_2'\widetilde{H}_2) = 1 - 1/3 - 8/11 < 0$, a contradiction. Therefore, $\sigma = id$ and Case(4-4) of Theorem 1.1 occurs.

This proves Claim(7) and also Lemma 1.10. □

Lemma 1.11. *Suppose that the case (2) in Lemma 1.8 occurs with* $(\widetilde{J}_a^2, \widetilde{J}_b^2) = (-2, -3)$ *or* $(-2, -4)$ *but the case in Corollary 1.7 does not occur. Then Theorem 1.1, (3) or (5) occurs.*

Proof. By the hypothesis, $J_a = H_a$ and we may assume that $\widetilde{H}_{a1}^2 \le$

-3.

Claim(1). It is impossible that $\widetilde{J}_b^2 = -4$.

We consider the case $\widetilde{J}_b^2 = -4$. Since the case in Corollary 1.7 does not occur, we have $\sigma \neq id$. Let $\tau_i : \widetilde{S}_i \to \widetilde{T}$ be the blowing-up of the point $P_i := \widetilde{C} \cap \widetilde{J}_i$. Let $E_i := \tau_i^{-1}(P_i)$. Then for $t = a$ or b, there is a smooth blowing-down $\sigma_t : \widetilde{S} \to \widetilde{S}_t$ such that $\sigma = \tau_t \cdot \sigma_t$. Now we apply Lemma 1.4. In particular, we have $-E_t.(K_{\widetilde{S}_t} + M^*) > 0$.

Case $t = a$. Then $M^* \geq 8/13\tau_a'\widetilde{H}_a + 7/13\tau_a'\widetilde{H}_{a1} + 4/13\tau_a'\widetilde{H}_{a2} + 2/5\tau_a'\widetilde{C} + 4/5\tau_a'\widetilde{J}_b$. This leads to $0 < -E_a.(K_{\widetilde{S}_t} + M^*) \leq 1 - E_a.(8/13\tau_a'\widetilde{H}_a + 2/5\tau_a'\widetilde{C}) = 1 - 8/13 - 2/5 < 0$, a contradiction. So this case is impossible.

Case $t = b$. Then $M^* \geq 1/4\tau_b'\widetilde{C} + 1/2\tau_b'\widetilde{H}_a + 1/2\tau_b'\widetilde{H}_{a1} + 1/4\tau_b'\widetilde{H}_{a2} + 3/4\tau_b'\widetilde{J}_b$. This leads to $0 < -E_b.(K_{\widetilde{S}_t} + M^*) \leq 1 - E_b.(1/4\tau_b'\widetilde{C} + 3/4\tau_b'\widetilde{J}_b) = 0$, a contradiction. So this case is also impossible.

This proves Claim(1).

Therefore, $\widetilde{J}_b^2 = -3$.

Claim(2). $\widetilde{\Delta}_a$ is a linear chain and the connected component of $\widetilde{\Delta}_a - \widetilde{H}_a$ containing $\widetilde{H}_{a2}$ is a (-2)-chain.

Since it is impossible that $\widetilde{\Delta}_b$ is a linear chain with $\widetilde{J}_b$ as a tip (cf. the hypopthesis $(*)$ after Lemma 1.3), we have $\widetilde{D}^* \geq 1/2\widetilde{J}_b$. We shall also show that if Claim(2) is false then $\widetilde{D}^* \geq 1/2\widetilde{H}_a$.

In fact, if $\widetilde{\Delta}_a$ is a fork, such that either $\widetilde{H}_a$ is the central component or $\widetilde{H}_{a2}$ and the central componet are contained in the same connected component of $\widetilde{\Delta}_a - \widetilde{H}_a$, then $\widetilde{D}^* \geq 1/2\widetilde{H}_{a1} + 1/2\widetilde{H}_a + 1/4\widetilde{H}_{a2}$. If $\widetilde{\Delta}_a$ is a fork such that $\widetilde{H}_{a1}$ and the central componet are contained in the same connected component of $\widetilde{\Delta}_a - \widetilde{H}_a$, then $\widetilde{D}^* \geq 1/4\widetilde{H}_{a2} + 2/4\widetilde{H}_a + 3/4\widetilde{H}_{a1}$. If $\widetilde{\Delta}_a$ is a linear chain but the connected component of $\widetilde{\Delta}_a - \widetilde{H}_a$ containing $\widetilde{H}_{a2}$ is not a (-2)-chain, then $\widetilde{D}^* \geq 1/2\widetilde{H}_{a1} + 1/2\widetilde{H}_a + 1/2\widetilde{H}_{a2}$.

Now suppose Claim(2) is false. Then we have $\widetilde{D}^* \geq 1/2\widetilde{H}_a$ by the above arguments. This leads to $0 < -\widetilde{C}.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - \widetilde{C}.(1/2\widetilde{H}_a + 1/2\widetilde{J}_b) = 0$, a contradiction. So Claim(2) is true.

Thus, $\widetilde{H}_{a2}^2 = -2$. If $\widetilde{J}_b$ is a tip of $\widetilde{\Delta}_b$, i.e., if $J_b \neq H_b$, then Theorem 1.1 (3) occurs. Indeed, $\widetilde{C} + \widetilde{J}_b + \widetilde{H}_a + \widetilde{H}_{a2}$ is a support of a singular fiber of a $\mathbf{P}^1$-fibration; hence the intersection matrices of $\widetilde{C}+\widetilde{J}_b+\widetilde{\Delta}_a$ and $C+T_b+\Delta_a$ have a positive eigenvalue.

Therefore, we may assume that $J_b = H_b$. Since the case in Corollary 1.7 does not occur, there are two smooth blowing-downs $\sigma_1 : \widetilde{S} \to \widetilde{S}_1$, $\sigma_2 : \widetilde{S}_1 \to \widetilde{T}$ such that $\sigma = \sigma_2 \cdot \sigma_1$ and that :

(1) $\sigma_1(T_a+C+T_b) = T'_a+E+T'_b$ where E is a (-1)-curve and $T'_i \leq \sigma_1(T_i)$,

(2) $T'_a + \sigma_1(H_a) = \sum_{i=1}^{s} L_i$, $E.L_1 = L_i.L_{i+1} = 1\,(i = 1, \cdots, s-1; s \geq 1)$, $L_s = \sigma_1(H_a)$, $L_1^2 = -t-1 \leq -3$, $L_j^2 = -2(j > 1)$,

(3) $T'_b + \sigma_1(H_b) = \sum_{i=1}^{t} M_i$, $E.M_1 = M_i.M_{i+1} = 1\,(i = 1, \cdots, t-1; t \geq 2)$, $M_t = \sigma_1(H_b)$, $M_j^2 = -2(j < t)$, $M_t^2 = -s-2 \leq -3$, and

(4) σ_1 does not factorize through the blowing-up of the point $P_a := E \cap L_1$.

In particular, we see that $\sigma_1(\Delta_b)$ is a fork and hence $\widetilde{\Delta}_b$ is a linear chain. Now we apply Lemma 1.4. In particular, we have $-E.(K_{\widetilde{S}_1} + M^*) > 0$.

Claim(3). $\sigma_1 = id$. Hence $a = 2$, $b = 1$, $C = E$, $D_1 = M_1$, $D_2 = L_1$, $D_2^2 = -t-1 \leq -3$, $H_{21}^2 \leq -3$ and $T_1 = \sum_{i=1}^{t-1} M_i$ is a (-2)-twig.

Let $\tau_2 : \widetilde{X} \to \widetilde{S}_1$ be the blowing-up of the point $P_b := E \cap M_1$ and set $F := \tau_2^{-1}(P_b)$. Suppose that Claim(3) is false. Then by the definition of σ_1 (cf. the above condition(4)), there is a smooth blowing-down $\tau_1 : \widetilde{S} \to \widetilde{X}$ such that $\sigma_1 = \tau_2 \cdot \tau_1$. Now we apply Lemma 1.4. In particular, we have $-F.(K_{\widetilde{X}} + N^*) > 0$, where $N = D$ if $\tau_1 = id$ and $N = \tau_1(D) - F$ otherwise.

Since $\tau_1(C + \Delta_1 + \Delta_2) - F$ can be contractible to quotient singularities (cf. Lemma 1.4), we have $s = 1$ or 2, and if $s = 2$ then $t = 2$, $\widetilde{H}_{a1}^2 = -3$ and $\tau_1(\Delta_a) = \tau'_2(E + \sum_i L_i) + \tau_1(H_{a1} + H_{a2})$.

Suppose $s = 1$. Then $N^* \geq (3t-2)/(6t-2)\tau'_2 E + 2(3t-2)/(6t-2)\tau_1(H_a) + (4t-2)/(6t-2)\tau_1(H_{a1}) + (3t-2)/(6t-2)\tau_1(H_{a2}) + \sum_i (t+i)/(2t+1)\tau'_2(M_i) + t/(2t+1)\tau_1(H_{b1}) + t/(2t+1)\tau_1(H_{b2})$. This leads to

$$0 < -F.(K_{\widetilde{X}} + N^*) \leq 1 - F.((3t-2)/(6t-2)\tau'_2 E + (t+1)/(2t+1)\tau'_2 M_1) =$$

$$1 - (3t-2)/(6t-2) - (t+1)/(2t+1) = (-t+2)/(6t-2)(2t+1) \leq 0,$$

because $t \geq 2$. We reach a contradiction.

Suppose that $s = 2$. Then $N^* \geq 9/23\tau_2' E + 18/23\tau_2'(L_1) + 22/23\tau_1(H_a) + 15/23\tau_1(H_{a1}) + 11/23\tau_1(H_{a2}) + 10/16\tau_2'(M_1) + 14/16\tau_1(H_b) + 7/16\tau_1(H_{b1}) + 7/16\tau_1\ (H_{b2})$. This leads to

$$0 < -F.(K_{\widetilde{X}} + N^*) \leq 1 - F.(9/23\tau_2' E + 10/16\tau_2' M_1) = 1 - 9/23 - 10/16 < 0.$$

We reach a contradiction.

So Claim(3) is true.

CLAIM(4). $s = 1$. Hence Δ_2 is a linear chain, $H_2 = D_2$ and $H_1^2 = -s - 2 = -3$.

Suppose $s \geq 3$. Then $s = 3$, $t = 2$, $H_{21}^2 = -3$, $D_2 = L_1$, $H_2 = L_3, D_1 = M_1, H_1 = M_2, H_2 = L_3$, $\Delta_2 = D_2 + L_2 + H_2 + H_{21} + H_{22}$ because Δ_2 is contractible to a quotient singularity. So, we have $D^* \geq 3/7D_1 + 6/7H_1 + 3/7H_{11} + 3/7H_{12} + 10/17D_2 + 13/17L_2 + 16/17H_2 + 11/17H_{21} + 8/17H_{22}$. This leads to $0 < -C.(K_{\widetilde{S}} + D^*) \leq 1 - C.(3/7D_1 + 10/17D_2) = 1 - 3/7 - 10/17 < 0$, a contradiction.

Suppose $s = 2$. Then $D_2 = L_1$, $H_2 = L_2$, $D^* \geq \sum_i 2i/(2t+1)M_i + t/(2t+1)H_{11} + t/(2t+1)H_{12} + (7t-5)/(7t+1)D_2 + 4(2t-1)/(7t+1)H_2 + (5t-1)/(7t+1)H_{21} + 2(2t-1)/(7t+1)H_{22}$. This leads to $0 < -C.(K_{\widetilde{S}} + D^*) \leq 1 - C.(2/(2t+1)D_1 + (7t-5)/(7t+1)D_2) = 1 - 2/(2t+1) - (7t-5)/(7t+1) = (4-2t)/(2t+1)(7t+1) \leq 0$, because $t \geq 2$. We reach a contradiction.

This proves Claim(4).

CLAIM(5). $t = 2, 3$. Hence $D_2^2 = -t - 1 = -3, -4$.

Note that $D^* \geq \sum_i i/(t+1)M_i + t/2(t+1)H_{11} + t/2(t+1)H_{12} + (6t-4)/(6t+1)D_2 + (4t-1)/(6t+1)H_{21} + (3t-2)/(6t+1)H_{22}$, where $D_2 = L_1$. So, $0 < -C.(K_{\widetilde{S}} + D^*) \leq 1 - C.(1/(t+1)D_1 + (6t-4)/(6t+1)D_2) = 1 - 1/(t+1) - (6t-4)/(6t+1) = (4-t)/(t+1)(6t+1)$. Hence $t \leq 3$. This proves Claim(5).

CLAIM(6). Theorem 1.1 (5) occurs.

Consider first the case $D_2^2 = -t - 1 = -3$. Then $D_1 = M_1, H_1 = M_2, D_2 = H_2, D^* \geq 1/3D_1 + 2/3H_1 + 1/3H_{11} + 1/3H_{12} + 7/13H_{21} + 8/13D_2 +$

$4/13H_{22}$. If H_{21} is not a tip (resp. H_{22} is not a tip, or $H_{21}^2 \le -4$), then $D^* \ge 2/3H_{21} + 2/3D_2 + 1/3H_{22}$ (resp. $D^* \ge 5/9H_{21} + 6/9D_2 + 4/9H_{22}$, or $D^* \ge 2/3H_{21} + 2/3D_2 + 1/3H_{22}$). Any of the three cases leads to $0 < -C.(K_{\widetilde{S}} + D^*) \le 1 - C.(1/3D_1 + 2/3D_2) = 0$, a contradiction. Thus, $\Delta_2 = D_2 + H_{21} + H_{22}$ and $H_{21}^2 = -3$. So, Δ_2 is as described in Figure 5 or 6.

Let T_1', T_1'' be twigs of Δ_1 containing H_{11}, H_{12}, respectively. If both T_1' and T_1'' have more than one irreducible components (resp. T_1' or T_1'', say T_1' has more than two irreducible components), then $D^* \ge 3/7D_1 + 6/7H_1 + 4/7H_{11} + 4/7H_{12}$ (resp. $D^* \ge 2/5D_1 + 4/5H_1 + 3/5H_{11} + 2/5H_{12}$). Any of the two cases leads to $0 < -C.(K_{\widetilde{S}} + D^*) \le 1 - C.(2/5D_1 + 8/13D_2) = 1 - 2/5 - 8/13 < 0$, a contradiction.

To show that $C + \Delta_1 + \Delta_2$ is as described in Figure 5 or 6, it remains to show that $\Delta_1 - H_1$ consists of only (-2)-curves. Indeed, if $H_{1j}^2 \le -3$ for $j = 1$ or 2, say $j = 1$, then $D^* \ge 2/5D_1 + 4/5H_1 + 3/5H_{11} + 2/5H_{12}$ and a contradiction is derived as in the above paragraph. Note that $H := \Delta_1 - (D_1 + H_1 + H_{11} + H_{12})$ is zero or a single curve. It remains to show that $H^2 = -2$ if $H \neq 0$. Indeed, suppose $H^2 \le -3$ and suppose, without loss of generality, $H \le T_1'$. Then $D^* \ge 3/7D_1 + 6/7H_1 + 4/7H + 5/7H_{11} + 3/7H_{12}$, and we reach again a contradiction as in the above paragraph.

We have proved that $C + \Delta_1 + \Delta_2$ is as described in Figure 5 if $D_2^2 = -3$.

Now we consider the case $D_2^2 = -4$. Let $\gamma_1 : \widetilde{S} \to \widetilde{X}$ be the blowing-down of C. Let $\gamma_2 : \widetilde{X} \to \widetilde{T}$ be the smooth blowing-down such that $\sigma = \gamma_2 \cdot \gamma_1$. Now we apply Lemma 1.4. In particular, we have $-F.(K_{\widetilde{X}} + N^*) > 0$ where $F = \gamma_1(D_1)$ is a (-1)-curve and $N = \gamma_1(D) - F$.

Now F meets a (-2)-curve $\gamma(M_2)$ and a (-3)-curve $\gamma(D_2)$. By making use of the latter inequality for F and by the arguments for the case $D_2^2 = -3$, we can also prove that $\gamma(\Delta_1 - D_1)$, $\gamma(\Delta_2)$ have the same weighted dual graphs as Δ_1, Δ_2, respectively in Figure 5. To verify that $C + \Delta_1 + \Delta_2$ is as described in Figure 6, it remains to show that $H := \Delta_1 - (D_1 + M_2 + H_1 + H_{11} + H_{12}) = 0$. Suppose $H \neq 0$, say H is adjacent to H_{11}. Then $D^* \ge 2/7D_1 + 4/7M_2 + 6/7H_1 + 2/7H + 4/7H_{11} + 3/7H_{12} + 11/19H_{21} + 14/19D_2 + 7/19H_{21}$. This leads to $0 < -C.(K_{\widetilde{S}} + D^*) \le 1 - C.(2/7D_1 + 14/19D_2) = 1 - 2/7 - 14/19 < 0$, a contradiction.

This proves Claim(6) and hence Lemma 1.11. $\square$

Lemma 1.12. *In the Case (3) of Theorem 1.1, $\pi_1(S^o)$ is finite.*

Proof. The argument in this case is similar to the proof of Lemma 6.24 at the end of Part **I.** We can assume that the intersection matrix of $C + T_1 + \Delta_2$ has a positive eigenvalue. Let $T_1 = B_1 + L_2 + \cdots + L_r$ be the twig. If U is a nice tubular neighborhood of $C + T_1 + \Delta_2$, then it is easy to see that $U - D$ has $N - D$ as a strong deformation retract, where N is a tubular neighborhood of $C + \Delta_2$. Now the rest of the argument is exactly as in the proof of Lemma 6.24 in Part **I.** □

Lemma 1.13. *Suppose that Theorem 1.1 (4) occurs. Then $\pi_1(S^o)$ is finite.*

Proof. We will use the description of $C+\Delta_1+\Delta_2$ in Figure 1, 2, 3 or 4. As before, the intersection matrix of $C + \Delta_1 + \Delta_2$ has a positive eigenvalue and by Lemma 1.10 of Part **I** we have a surjection $\pi_1(U-\Delta_1-\Delta_2) \to \pi_1(S^o)$, where U is a small neighborhhod of $C\cup\Delta_1\cup\Delta_2$. We will use the presentation of $\pi_1(U - \Delta_1 - \Delta_2)$ given by Mumford in [3].

Case (4-1). Then $\pi_1(\partial U)$ is given by generators $e_0, e_1, e_{11}, e_{12}, e_2, e_{21}, e_{22}, g_1, g_2$ corresponding to $C, H_1, H_{11}, H_{12}, H_2, H_{21}, H_{22}, G_1, G_2$ respectively and the following relations (cf. Figure 1) :

$$1 = e_{11}^{-3}e_1 = e_{12}^{-2}e_1 = e_{11}e_{12}e_1^{-2}e_0 = e_1e_0^{-1}e_2$$

$$= e_0e_2^{-3}e_{21}e_{22} = g_1e_{21}^{-2}e_2 = g_1^{-2}e_{21} = g_2e_{22}^{-2}e_2 = g_2^{-2}e_{22}.$$

Now $\pi_1(U - D)$ is obtained by putting $e_0 = 1$ in the relations above. Hence in $\pi_1(U - D)$, we have

$$e_1 = e_{12}^2,\ e_2 = e_1^{-1} = e_{12}^{-2},\ e_{11} = e_1^2e_{12}^{-1} = e_{12}^3,\ e_{22} = g_2^2,$$

$$e_{21} = e_2^3e_{22}^{-1} = e_{12}^{-6}g_2^{-2},\ g_1 = e_2^{-1}e_{21}^2 = e_{12}^2(e_{12}^{-6}g_2^{-2})^2,$$

$$e_{12}^2 = e_1 = e_{11}^3 = e_{12}^9,\ e_{12}^7 = 1,$$

$$e_{12}^2 = e_2^{-1} = g_2e_{22}^{-2} = g_2^{-3},\ e_{12} = e_{12}^{-6} = g_2^9.$$

Here 7 is the absolute value of the determinant of the intersection matrix of Δ_1. The above relation shows that all the generators of $\pi_1(U - D)$ can be expressed in terms of g_2 and $g_2^{63} = e_{12}^7 = 1$. Hence $\pi_1(U - D)$ is a finite cyclic group generated by g_2. Thus $\pi_1(S^o)$ is a finite cyclic group in this case.

Case (4-2). We argue exactly as above. The determinant of $\Delta_1 = \pm 11$ and $\pi_1(U - D)$ is generated by e_{21} (corresponding to H_{21}) (cf. Figure 2). Again $\pi_1(U - D)$ is a finite cyclic group.

Case (4-3). Then the determinant of $\Delta_1 = \pm 7$ (cf. Figure 3). In this case $\pi_1(U - D)$ is a finite group generated by g_1 (corresponding to G_1).

In the above cases, the crucial fact used was the linearity of Δ_1, Δ_2.

Case (4-4). Then the determinants of Δ_1, Δ_2 are $\pm 9, \pm 14$ respectively (both non-primes) (cf. Figure 4). In this case we use the (-1)-curve E in the singular fiber S_1. Now $E + \Delta_1$ supports a divisor with a positive self-intersection. E intersects only the curve H_{11} from Δ_1 ($E.H_{11} = 2$) which is a tip of the linear chain Δ_1. Now the argument used for the case $|K + C + D| \neq \phi$ in Part **I**, using Lemma 1.14 in Part **I**, proves that $\pi_1(S^o)$ is a finite group.

This proves Lemma 1.13. □

LEMMA 1.14. *Suppose that Theorem 1.1 (5) occurs. Then $\pi_1(S^o)$ is finite.*

PROOF. In Case (5-5) of Theorem 1.1, the determinant of $\Delta_2 = \pm 13$ and Δ_2 is linear (whether $H = \phi$ or $\neq \phi$). In Case (5-6) of Theorem 1.1, the determinant of $\Delta_2 = \pm 19$ and Δ_2 is linear (cf. Figures 5, 6).

If U is a tubular neighborhood of $C \cup \Delta_1 \cup \Delta_2$, then using Mumford's presentation we see that $\pi_1(U - D)$ is a homomorphic image of $\pi_1(U_1 - \Delta_1)$, where U_1 is a small tubular neighborhood of Δ_1. Since Δ_1 defines a quotient singular point, we deduce the finiteness of $\pi_1(S^o)$. Lemma 1.14 is proved. □

Thus we have proved Theorem 1.1 and also the Main Theorem.

REMARK. By [6, 7, 8], we see that our main theorem is still true with the ampleness of the anti-canonical divisor $-K_S$ replaced by the weaker nef and bigness, but it is not true any more if either one replaces the ampleness of $-K_S$ by that the anti-Kodaira dimension equals two, or one lets S have worse log canonical singularities.

References

[1] Fujiki, A., Kobayashi, R. and S. Lu, On the Fundamental group of Certain Open normal Surfaces, Saitama Math. J. **11** (1993), 15–20.

[2] Gurjar, R. V. and D.-Q. Zhang, π_1 of Smooth Points of a Log Del Pezzo Surface is Finite : **I**, J. Math. Sci. Univ. Tokyo **1** (1994), 137–180.

[3] Mumford, D., The topology of normal singularities of an algebraic surface, Publ. Math. I.H.E.S. no. 9 (1961), 5–22.

[4] Zhang, D.-Q., Logarithmic del Pezzo surfaces of rank one with contractible boundries, Osaka J. Math. **25** (1988), 461–497.

[5] Gurjar, R. V. and D.-Q. Zhang, On the fundamental groups of some open rational surfaces, Preprint 1994.

[6] Zhang, D.-Q., Algebraic surfaces with nef and big anti-canonical divisor, Math. Proc. Camb. Phil. Soc. **117** (1995), 161–163.

[7] Zhang, D.-Q., Normal algebraic surfaces of anti-Kodaira dimension one or two, Intern. J. Math. **6** (1995), 329–336.

[8] Zhang, D.-Q., Algebraic surfaces with log canonical singularities and the fundamental groups of their smooth parts, to appear.

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