

The Andreotti Grauert vanishing theorem for dihedrons of \mathbb{C}^n

By Giuseppe ZAMPIERI

Abstract. Let X be a complex manifold, \mathcal{O}_X the sheaf of analytic functions on X , W an open set of X with C^2 -boundary $M = \partial W$ (W locally on one side of M), z_o a point of M , p_o the exterior conormal to W at z_o . If the number of negative eigenvalues for the Levi form of M in a neighborhood of p_o is $\geq s^-$ (resp. $\equiv s^-$), then vanishing of local cohomology groups of \mathcal{O}_X over W in degree $< s^-$ and $\neq 0$ (resp. $\neq 0, s^-$) is stated in [1], [2], [14] (resp. [10]). Let now W be an open asymptotically convex dihedron with C^2 (transversal) faces M_1, M_2 and with “generic” edge M_3 . If the numbers of negative eigenvalues for the Levi forms of M_1, M_2 , and for the “microlocal” Levi form of M_3 are $\geq s^-$ along the exterior conormal cone to W at z_o , then vanishing of cohomology $< s^-$ and $\neq 0$ is proved here in the line of [1]. Under the additional assumption that M_3 is real analytic and that $T_{M_3}^*X$ contains no germ of complex curve, the same result as by [10] is also stated. The content of this paper was exposed at the University Paris XIII, June 21 1991.

1. Notations

Let X be a complex manifold, M a C^2 -submanifold, $\pi : T^*X \rightarrow X$ (resp. $\pi : T_M^*X \rightarrow M$, resp. $\tau : TM \rightarrow M$) the cotangent bundle to X (resp. the conormal bundle to M , resp. the tangent bundle to M). Let or_X (resp. $or_{M/X}$) be the orientation (resp. the relative orientation) sheaf. Let $z_o \in M$ and $p \in \dot{T}_M^*X (= T_M^*X \setminus \{0\})$ with $\pi(p) = z_o$. We shall use the notations

$$e(p) = T_p T^*X$$

1991 *Mathematics Subject Classification.* 58G, 32F.

$$\begin{aligned}
 T_{z_o}^{\mathbf{C}}M &= T_{z_o}M \cap \sqrt{-1}T_{z_o}M \\
 \lambda_M(p) &= T_p T_M^* X \\
 \lambda_o(p) &= T_p \pi^{-1} \pi(p) \\
 \gamma_M(p) &= \dim_{\mathbf{C}}(\lambda_M(p) \cap \sqrt{-1}\lambda_M(p) \cap \lambda_o(p)) \\
 \nu(p) &= \text{the complex line spanned by the} \\
 &\quad \text{radial vector field.}
 \end{aligned}$$

We shall drop sometimes z_o and p in the above notations.

Let $\alpha = \alpha^{\mathbf{R}} + \sqrt{-1}\alpha^{\mathbf{I}}$ be the canonical 1-form on T^*X and $\sigma = \sigma^{\mathbf{R}} + \sqrt{-1}\sigma^{\mathbf{I}}$ ($=d\alpha$) the canonical 2-form. Let $X^{\mathbf{R}}$ (resp. $(T^*X)^{\mathbf{R}}$) denote the real underlying manifold to X (resp. T^*X). We shall identify $T^*(X^{\mathbf{R}}) \simeq (T^*X)^{\mathbf{R}}$ by endowing $T^*(X^{\mathbf{R}})$ with the 1-form $\alpha^{\mathbf{R}}$.

For an isotropic plane $\mu \subset e$ and for a subspace λ of e , we shall set $\lambda^\mu = ((\lambda \cap \mu^\perp) + \mu)/\mu$. For λ (resp. l) \mathbf{R} - (resp. \mathbf{C} -) Lagrangian, we shall set $\mu = \lambda \cap \sqrt{-1}\lambda$ and define $L_{l/\lambda}(u, v) \stackrel{\text{def.}}{=} \sigma^\mu(u, \bar{v}) \forall u, v \in l^\mu$ where σ^μ is the form induced by σ on e^μ , and \bar{v} is the conjugate to v , modulo μ , in the sum $(\lambda + \sqrt{-1}\lambda)/\mu$. Let ϕ be a C^2 -function on $X^{\mathbf{R}}$ at z_o with $\phi|_M \equiv 0$ and $(z_o; d\phi(z_o)) = p$. We define the Levi form $L_\phi(z_o)$ in a local system of coordinates $z \in \mathbf{C}^n \simeq X$, as the Hermitian form with matrix $(\partial_{z_i} \partial_{\bar{z}_j} \phi(z_o))_{1 \leq i, j \leq n}$. According to [15], one may see that

$$(1.1) \quad L_\phi(z_o)|_{T_{z_o}^{\mathbf{C}}M} \sim L_{\lambda_o/\lambda_M}$$

where “ \sim ” means equivalence in signature and rank. (Cf. also [5] as for the case of higher codimension.) We shall also write $L_M(p)$ instead of $L_\phi(z_o)|_{T_{z_o}^{\mathbf{C}}M}$. We shall define $s_M^{+,-,0}(p)$ to be the numbers of eigenvalues of $L_\phi(z_o)|_{T_{z_o}^{\mathbf{C}}M}$ which are respectively $> 0, < 0, = 0$; by (1.1) they do not depend on the choice of ϕ .

We shall denote by $D^b(X)$ the derived category of the category of complexes of sheaves with bounded cohomology, and by $D^b(X; p)$ the localized category with respect to the null-system $\{\mathcal{F} \in D^b(X); p \notin \text{SS}(\mathcal{F})\}$. (Here $\text{SS}(\mathcal{F})$ is the microsupport in the sense of [10] and [11], a closed conic subset of T^*X .) We recall that for a complex \mathcal{F} which verifies $\text{SS}(\mathcal{F}) \subset T_M^*X$ in a neighborhood of $p, p \in T_M^*X$, one may find a complex M of \mathbf{Z} -modules such that \mathcal{F} is microlocally isomorphic to the constant complex M_M at p

(i.e. isomorphic in $D^b(X; p)$). This criterion, stated in [10, §6], for M being a C^2 -submanifold, easily extends to the case C^1 (see [6]).

We shall denote by \mathcal{O}_X the sheaf of holomorphic functions on X and by \mathbf{Z}_W ($W \subset X$ locally closed) the sheaf which is 0 on $X \setminus W$ and the constant sheaf with stalk \mathbf{Z} on W . We shall deal with the complex $\mu_W(\mathcal{O}_X) \stackrel{\text{def.}}{=} \mu\text{hom}(\mathbf{Z}_W, \mathcal{O}_X)$ of microfunctions along W (in particular with the complex $\mu_M(\mathcal{O}_X)$), where $\mu\text{hom}(\cdot, \cdot)$ is the bifunctor of [10].

Finally we shall denote by $N(W)$ (resp. $N(W)^{oa}$) the normal cone to W (resp. the polar antipodal cone to $N(W)$); this is an open cone of TX (resp. a closed cone of T^*X).

2. Statement of the Vanishing Theorems

Let X be a complex manifold, W an open set of X with C^2 -boundary $M = \partial W$ (W locally on one side of M), z_o a point of M , p_o the exterior conormal to W at z_o .

THEOREM A. (i) ([1]) *In the above situation we have*

$$(2.1) \quad \lim_{\substack{\longrightarrow \\ B}} H^j(W \cap B, \mathcal{O}_X) = 0 \quad 0 < \forall j < s_M^-(p_o),$$

for B describing a system of neighborhoods of z_o .

(ii) ([10]) *If moreover $s_M^-(p) \equiv \text{const } \forall p \in T_M^*X$ near p_o , then (2.1) holds $\forall j \neq 0, s_M^-(p_o)$.*

Refinements of (2.1) are given by several authors ([14], [2], [10]); in particular vanishing of (2.1) for $j > s_M^- + s_M^o$ is also stated. As for Th. A (ii) the result was already known before [10] under the stronger assumption of constant rank for $L_M(p)$ (cf. e.g. [14]).

The purpose of the present note is to generalize Theorem A to the case of a dihedron. Let M_1, M_2 be C^2 -hypersurfaces of $X^{\mathbf{R}}$ through z_o , $M_i^+, i = 1, 2$ open half-spaces with boundary M_i , and let W be the dihedron defined by $W = M_1^+ \cap M_2^+$. We put $M_3 = M_1 \cap M_2$, and assume that M_3 is a manifold which verify the “genericity” condition:

$$(2.2) \quad T_{M_3}^*X_{z_o} \cap \sqrt{-1}T_{M_3}^*X_{z_o} = \{0\}.$$

which is equivalent to $\gamma_{M_3}(p) = 0$ for $p \in \pi^{-1}(z_o)$. We denote $p_i, i = 1, 2$ the exterior conormal to M_i , and denote s^- the minimum between $s_{M_1}^-(p_1), s_{M_2}^-(p_2)$ and $s_{M_3}^-(p) \forall p \in N_{z_o}(W)^{oa}$.

THEOREM 2.1. *Let M_1, M_2 intersect transversally, and let the intersection M_3 be generic. Then*

$$(2.4) \quad \lim_{\xrightarrow{B}} H^j(W \cap B, \mathcal{O}_X) = 0 \quad 0 < \forall j < s^-$$

(and

$$(2.5) \quad \lim_{\xrightarrow{B}} \Gamma(B, \mathcal{O}_X) \xrightarrow{\sim} \lim_{\xrightarrow{B}} \Gamma(W \cap B, \mathcal{O}_X) \text{ if } s^- \geq 1).$$

In [13, Prop. 11 p. 148] vanishing of (2.4) for $j < s^- - 1, j \neq 0$ is proved (in a different frame); in particular the case $j = s^- - 1$ seems not to be treated.

THEOREM 2.2. *Let M_1, M_2 intersect transversally, let (2.2) be satisfied, and suppose moreover M_3 real analytic with $T_{M_3}^*X$ containing no germ of complex curve. Assume that for a constant s^- and for an open set B_o :*

$$(2.6) \quad s_{M_i}^-(p) \equiv s^- \quad \forall p \in T_{M_i}^*X \cap N(W)^{oa} \cap \pi^{-1}(B_o), \text{ and } \forall i = 1, 2, 3.$$

Then (2.4) holds $\forall j \neq 0, s^-$ (and (2.5) is still fulfilled when $s^- \geq 1$).

(By (2.2), if M_3 contains no germ of complex curve, the same is also true for $T_{M_3}^*X$.)

3. Proofs

Proof of Theorem 2.1

We shall use the notations:

$$\begin{aligned} \Lambda_i &= T_{M_i}^*X \cap N(W)^{oa}, \quad N_i = \pi(\Lambda_i), \\ \overset{o}{\Lambda}_i &= \Lambda_i \setminus (\Lambda_j \cup \Lambda_k), \quad \overset{o}{N}_i = N_i \setminus (N_j \cup N_k), \quad j, k \neq i. \end{aligned}$$

(a) We first deal with a neighborhood of the conormal p_1 to M_1 ; since $p_1 \in (\Lambda_1 \cap \Lambda_3) \setminus \Lambda_2$, we shall forget Λ_2 for a while.

LEMMA 3.1. *We may find a (complex) contact transformation χ between neighborhoods of p_1 and $q_1 \stackrel{\text{def.}}{=} \chi(p_1)$ which interchanges:*

$$(3.1) \quad \begin{cases} T_{\tilde{M}_3}^* X \rightarrow T_{\tilde{M}_3}^* X, & \text{cod } \tilde{M}_3 = 1, s_{\tilde{M}_3}^-(q_1) = 0 \\ T_{\tilde{M}_1}^* X \rightarrow T_{\tilde{M}_1}^* X, & \text{cod } \tilde{M}_1 = 1. \end{cases}$$

PROOF. We shall deal in the linear space $e' = e^\nu$ (due to $\dim_{\mathbf{R}}(\lambda_{M_i} \cap \nu) = 1$). According to [17], to solve (3.1) is equivalent as to find a \mathbf{C} -Lagrangian plane $l'_o \subset e'$ such that (with $\lambda'_i \stackrel{\text{def.}}{=} \lambda_{M_i}^\nu$):

$$(3.2) \quad l'_o \cap \lambda'_3 = \{0\},$$

$$(3.3) \quad L_{l'_o/\lambda'_3} \geq 0,$$

$$(3.4) \quad l'_o \cap \lambda'_1 = \{0\}.$$

Now the complex planes l'_o which verify (3.4) are an open dense set in the Lagrangian Grassmannian of e' (obvious). The ones which verify (3.2), (3.3) are an open non-empty set. In fact this is clearly non-empty (cf. e.g. the proof of [10, Prop. 11.3.1]); moreover the rank of $L_{l'_o/\lambda'_3}$ is constant ($= n - 1 - \dim(\lambda'_3 \cap \sqrt{-1}\lambda'_3)$) for l'_o varying in the open set defined by $l'_o \cap \lambda'_3 = \{0\}$ and hence the signature is constant in each component of the above set. Thus the problem of finding l'_o as in (3.2)–(3.4) can be solved. \square

(This was already remarked in [7].)

REMARK 3.2. In the same way as before one may construct a contact transformation satisfying the requirements (3.1) but with $s_{\tilde{M}_1}^-(q_1) = 0$ instead of $s_{\tilde{M}_3}^-(q_1) = 0$.

We shall put in what follows:

$$\tilde{\Lambda}_i = \chi(\Lambda_i), \quad \overset{\circ}{\tilde{\Lambda}}_i = \chi(\overset{\circ}{\Lambda}_i), \quad \tilde{N}_i = \pi(\tilde{\Lambda}_i), \quad \overset{\circ}{N}_i = \pi(\overset{\circ}{\tilde{\Lambda}}_i).$$

(b) We recall from [5] that

$$s_{M_3}^-(p_1) \leq s_{M_1}^-(p_1) \leq s_{M_3}^-(p_1) + 1.$$

We also choose a quantization Φ_K of χ by a kernel K according to [10, ch. 11].

LEMMA 3.3. *Let $s_{M_1}^-(p_1) = s_{M_3}^-(p_1)$. Then, if χ is a contact transformation which satisfies (3.1), we have:*

$$(3.5) \quad \begin{cases} s_{M_1}^-(q_1) = 0, \\ \tilde{\Lambda}_1 \cup \tilde{\Lambda}_3 = T_Y^* X \text{ for a } C^1\text{-hypersurface } Y, \\ \Phi_K(\mathbf{Z}_{N_1}^{\circ}) \simeq \mathbf{Z}_Y[s_{M_1}^-(p_1)] \text{ in } D^b(X; q_1). \end{cases}$$

PROOF. We remark that $T_{M_1}^* X \cap T_{M_3}^* X$ is clean of codimension 1, whence \tilde{M}_1, \tilde{M}_3 intersect at the order 2 along a submanifold \tilde{R} of codimension 2. In particular $T\tilde{M}_1|_{\tilde{R}} = T\tilde{M}_3|_{\tilde{R}}$ and hence either $\tilde{N}_1 \cup \tilde{N}_3$ or $\tilde{N}_1 \cup (\tilde{M}_3 \setminus \tilde{N}_3)$ is a C^1 -hypersurface Y . To make the right choice we exploit Φ_K and remark that

$$\begin{aligned} \text{SS}(\Phi_K(\mathbf{Z}_{N_1}^{\circ})) &\subset \tilde{\Lambda}_1 \cup \tilde{\Lambda}_3 \subset \pi^{-1}(\tilde{N}_1 \cup \tilde{N}_3), \\ \text{SS}(\Phi_K(\mathbf{Z}_{N_1})) &\subset \tilde{\Lambda}_1 \cup (T_{\tilde{M}_3}^* X \setminus \tilde{\Lambda}_3) \subset \pi^{-1}(\tilde{N}_1 \cup (\tilde{M}_3 \setminus \tilde{N}_3)). \end{aligned}$$

Hence one of the above complexes has its microsupport contained in $T_Y^* X$; and this complex must then be constant due to [10] and [6]. But we have

$$(3.6) \quad \begin{aligned} \Phi_K(\mathbf{Z}_{N_1}^{\circ}) &= \begin{cases} \mathbf{Z}_{\tilde{M}_1}[s_{M_1}^-(p_1) - s_{M_1}^-(q_1)] & \text{in } \tilde{\Lambda}_1^{\circ}, \\ \mathbf{Z}_{\tilde{M}_3}[s_{M_3}^-(p_1)] & \text{in } \tilde{\Lambda}_3^{\circ}, \end{cases} \\ \Phi_K(\mathbf{Z}_{N_1}) &= \begin{cases} \mathbf{Z}_{\tilde{M}_1}[s_{M_1}^-(p_1) - s_{M_1}^-(q_1)] & \text{in } \tilde{\Lambda}_1^{\circ}, \\ \mathbf{Z}_{\tilde{M}_3}[s_{M_3}^-(p_1) + 1] & \text{in } T_{\tilde{M}_3}^* X \setminus \tilde{\Lambda}_3. \end{cases} \end{aligned}$$

Recall that $s_{\tilde{M}_3}^-(p_1) = s_{\tilde{M}_1}^-(p_1)$; hence constancy may hold only for $\Phi_K(\mathbf{Z}_{N_1}^{\circ})$ (which implies $\tilde{N}_1 \cup \tilde{N}_3 = Y \in C^1$) and under the additional condition $s_{\tilde{M}_1}^-(p_1) = 0$.

We also observe that $\Phi_K(\mathbf{Z}_{N_1}^{\circ})$ turns out to be a constant sheaf along Y and by (3.6) this is simple with shift $\frac{1}{2} + s_{\tilde{M}_1}^-(p_1)$ along $\overset{\circ}{\tilde{\Lambda}}_1$ and $\overset{\circ}{\tilde{\Lambda}}_3$. This completes the proof of (3.5). \square

Denote by $\Sigma_{\tilde{M}_i}, i = 1, 2$ the closed half-spaces with boundary \tilde{M}_i and inward conormal q_1 .

LEMMA 3.4. *Let $s_{\tilde{M}_1}^-(p_1) = s_{\tilde{M}_3}^-(p_1) + 1$. Then by the contact transformation of Remark 3.2, we have*

$$(3.5') \quad \begin{cases} s_{\tilde{M}_3}^-(q_1) = 0, \Sigma_{\tilde{M}_1} \supset \Sigma_{\tilde{M}_3}, \\ \pi(\tilde{\Lambda}_1 \cup \tilde{\Lambda}_3) = \partial(\Sigma_{\tilde{M}_1} \setminus \Sigma_{\tilde{M}_3})^+, \\ \Phi_K(\mathbf{Z}_{N_1}^{\circ}) \simeq \mathbf{Z}_{(\Sigma_{\tilde{M}_1} \setminus \Sigma_{\tilde{M}_3})^+}, \end{cases}$$

where $(\cdot)^+$ is a component of \cdot .

PROOF. The proof is the same as in Lemma 3.3. \square

We should prove Lemma 3.4 (and similarly Lemma 3.3) also by another argument. We recall that \tilde{M}_1, \tilde{M}_3 intersect at the order 2; thus either $\Sigma_{\tilde{M}_1} \supset \Sigma_{\tilde{M}_3}$ or $\Sigma_{\tilde{M}_1} \subset \Sigma_{\tilde{M}_3}$. But

$$\begin{aligned} \text{Hom}_{D^b(X;p_1)}(\mathbf{Z}_{M_1}, \mathbf{Z}_{M_3}) &\stackrel{\Phi_K}{\simeq} \text{Hom}_{D^b(X;q_1)}(\mathbf{Z}_{\Sigma_{\tilde{M}_1}}, \mathbf{Z}_{\Sigma_{\tilde{M}_3}}[-s_{\tilde{M}_3}^-(q_1)]) \\ &\cong H^{-s_{\tilde{M}_3}^-(q_1)} \mathbf{R}\Gamma_{\Sigma_{\tilde{M}_1}}(\mathbf{Z}_{\Sigma_{\tilde{M}_3}})_{z'_0}, (z'_0 = \pi(q_1)). \end{aligned}$$

This gives the first of (3.5'). The proof of the remaining statements is then easy.

(c) We have

$$(3.7) \quad \mathbf{Z}_W \xleftarrow{\sim} \mathbf{Z}_{N_1}^{\circ}[-1] \otimes \text{or}_{M_1/X} \quad \text{in } D^b(X; p_1).$$

In fact one uses the distinguished triangle

$$\mathbf{Z}_{\tilde{W} \setminus N_1} \rightarrow \mathbf{Z}_{\tilde{W}} \rightarrow \mathbf{Z}_{N_1} \xrightarrow{+1},$$

applies the functor $\cdot^* \stackrel{\text{def.}}{=} \mathbf{R}\mathcal{H}\text{om}(\cdot, \mathbf{Z}_X)$, and gets

$$\mathbf{Z}_{N_1}^o[-1] \otimes \text{or}_{M_1/X} \rightarrow \mathbf{Z}_W \rightarrow \mathbf{Z}_{\tilde{W} \setminus N_1}^* \xrightarrow{+1}.$$

But

$$\begin{aligned} \text{SS}(\mathbf{Z}_{\tilde{W} \setminus N_1}^*)_{z_o} &= \text{SS}(\mathbf{Z}_{\tilde{W} \setminus N_1})_{z_o}^a \\ &= \text{the convex cone of the plane } T_{M_3}^* X_{z_o} \\ &\quad \text{bounded by } \mathbf{R}^+ p_2, \mathbf{R}^- p_1. \end{aligned}$$

Thus $p_1 \notin \text{SS}(\mathbf{Z}_{\tilde{W} \setminus N_1}^*)_{z_o}$ and (3.7) follows. We remark that along with (3.7) we have

$$(3.8) \quad \mathbf{Z}_W \xleftarrow{\sim} \mathbf{Z}_{M_3}[-2] \otimes \text{or}_{M_3/X} \text{ in } D^b(X; \mathring{\Lambda}_3)$$

$$(3.9) \quad \mathbf{Z}_W \xleftarrow{\sim} \mathbf{Z}_{M_1}[-1] \otimes \text{or}_{M_1/X} \text{ in } D^b(X; \mathring{\Lambda}_1).$$

(d) We first assume $s_{M_1}^-(p_1) = s_{M_3}^-(p_1)$, put $s_1^- \stackrel{\text{def.}}{=} s_{M_1}^-(p_1)$, and denote by \tilde{W} the open subset of X with boundary Y and exterior conormal q_1 (cf. Lemma 3.2). By (3.5), (3.7) and by the analogous for \tilde{W} , we get

$$(3.10) \quad \begin{aligned} \Phi_K(\mathbf{Z}_W) &\simeq \Phi_K(\mathbf{Z}_{N_1}^o[-1] \otimes \text{or}_{M_1/X}) \\ &\simeq \mathbf{Z}_Y[s_1^- - 1] \otimes \text{or}_{Y/X} \simeq \mathbf{Z}_{\tilde{W}}[s_1^-] \end{aligned}$$

Let $\mu_{\tilde{W}}(\cdot) = \mu\text{hom}(\mathbf{Z}_{\tilde{W}}, \cdot)$ be the functor defined in [K-S 1, ch. 5], and recall the triangle

$$(3.11) \quad \left(\mathcal{O}_X \Big|_{\overline{\tilde{W}}}\right)_{z'_o} \rightarrow \mathbf{R}\Gamma_{\tilde{W}}(\mathcal{O}_X)_{z'_o} \rightarrow \mu_{\tilde{W}}(\mathcal{O}_X)_{q_o} \xrightarrow{+1} (z'_o = \pi(q_1)).$$

Since $\mathcal{O}_X z'_o \hookrightarrow \mathcal{H}_{\bar{W}}^0(\mathcal{O}_X) z'_o$ is injective by analytic continuation, then: $H^j \mu_{\bar{W}}(\mathcal{O}_X)_{q_1} = 0 \forall j < 0$. By (3.10), by the fact that $\Phi_K(\mathcal{O}_X) = \mathcal{O}_X$, and by [10, Th. 7.4.1], one concludes

$$(3.12) \quad H^j \mu_W(\mathcal{O}_X)_{p_1} = 0 \quad \forall j < s_1^-.$$

Let now $s_{M_3}^-(p_1) = s_{M_1}^-(p_1) + 1$; in this case we have

$$\Phi_K(\mathbf{Z}_W) \simeq \mathbf{Z}_{(\Sigma_{\bar{M}_1} \setminus \Sigma_{\bar{M}_3})^+} [s_1^- - 1].$$

We also remark that

$$\mathcal{H}_{\Sigma_{\bar{M}_3}}^1(\mathcal{O}_X)_{z'_o} \hookrightarrow \mathcal{H}_{\Sigma_{\bar{M}_1}}^1(\mathcal{O}_X)_{z'_o}$$

is injective. Then (3.12) is still fulfilled in this case

(e) One may repeat the same argument as in (a)–(d) and get (3.12) with p_1, s_1^- replaced by p_2, s_2^- . One may also use (3.10) and get in similar (and simpler) way:

$$(3.13) \quad H^j \mu_W(\mathcal{O}_X)_p = 0 \quad \forall p \in \overset{\circ}{\Lambda}_3, \forall j < s_{M_3}^-(p)$$

Finally by the triangle

$$(3.14) \quad (\mathcal{O}_X)_{\bar{W}} \rightarrow \mathbf{R}\Gamma_W(\mathcal{O}_X) \rightarrow \mathbf{R}\dot{\pi}_* \mu_W(\mathcal{O}_X) \xrightarrow{+1},$$

we get $H^j \mathbf{R}\Gamma_W(\mathcal{O}_X)_{z_o} = 0$ if $j \neq 0$, and $j < s_1^-, s_2^-, \inf_{p \in \overset{\circ}{\Lambda}_3} s_{M_3}^-(p)$.

From (3.14), one also gets $\mathcal{O}_X z_o \simeq \mathcal{H}_W^0(\mathcal{O}_X) z_o$ when the infimum of the above s^- 's is ≥ 1 .

Proof of Theorem 2.2

(a) We first focus our attention in a neighborhood of p_1 . Observing that $s_{M_1}^-(p_1) = s_{M_3}^-(p_1)$ in the present case, then by the proof of Theorem 2.1 we get $\Phi_K(\mathbf{Z}_W) = \mathbf{Z}_{\tilde{W}}[s^-]$, ($s^- = s_1^-$) at q_1 for \tilde{W} open with C^1 -boundary. According to [10, Th. 11.2.8] $s_{M_1}^-(q) - s_{M_1}^-(p) \equiv \text{const}$; but $s_{M_i}^-(p) \equiv \text{const}$, $s_{M_i}^-(q_1) = 0$ whence $s_{M_i}^-(q) \equiv 0$. We also observe that in the present situation $\Sigma_{\tilde{M}_3} \supset \Sigma_{\tilde{M}_1}$.

LEMMA 3.5. \tilde{W} is pseudoconvex in a neighborhood of $z'_o \stackrel{\text{def.}}{=} \pi(q_1)$.

PROOF. We identify $X \simeq \mathbf{C}^n$ and recall from [9, ch. 2, 4] that \tilde{W} is pseudoconvex at z'_o if and only if $-\log \delta$, is plurisubharmonic (with $\delta(z) = \text{dist}(z, X \setminus \tilde{W})$). We remark that $\forall z \in \tilde{W}$ there exists a unique $z^* \in \partial\tilde{W}$ such that $\delta(z) = |z - z^*|$ (due to $\tilde{W} \supset X \setminus \Sigma_{\tilde{M}_3}$). We then let $S = \{z \in \tilde{W}; z^* \in \tilde{M}_1 \cap \tilde{M}_3\}$ (which is a C^2 -hypersurface), and denote by \tilde{W}_h , $h = 1, 2$ the two components of $\tilde{W} \setminus S$. We also define δ_h to be the distance to \tilde{M}_h ; clearly $\forall z \in \tilde{W} \setminus S$ we have $\delta = \delta_1$ or δ_3 at z whence $\delta \in C^2(\tilde{W} \setminus S) \cap C^1(\tilde{W})$. It is also easy to see (by adapting the proof by [9, Th. 2.6.12]) that $-\log \delta|_{\tilde{W} \setminus S}$ is plurisubharmonic if and only if $s_{M_h}^-(q') \equiv 0 \forall q' \in \tilde{N}_h \times_{\tilde{M}_h} T_{\tilde{M}_h}^* X$. Thus $\forall w \in \mathbf{C}^n$ and $\forall g \in C_c^\infty(\tilde{W})$, $g \geq 0$:

$$\int_{\tilde{W}} \sum (-\log \delta) w_i \bar{w}_j \partial_i \bar{\partial}_j g d\lambda = \sum_{h=1,2} \int_{\tilde{W}_h} \sum \partial_i \bar{\partial}_j (-\log \delta) w_i \bar{w}_j g d\lambda \geq 0$$

(which holds, by Stokes formula, because $\delta \in C^2(\tilde{W} \setminus S) \cap C^1(\tilde{W})$ with bounded second derivatives). The lemma immediately follows. \square

By the above lemma we have $H^j \mathbf{R}\Gamma_{\tilde{W}}(\mathcal{O}_X)_{z'} = 0 \forall j \neq 0$, ($z' = \pi(q)$) due to [9, Corollary 7.4.2]. By the triangle (3.11) and by (3.10), we then conclude

$$(3.15) \quad H^j \mu_W(\mathcal{O}_X)_{p_1} = 0 \quad \forall j \neq s^-.$$

(b) The same as (3.15) also holds for p_2 and $\forall p \in \overset{o}{\Lambda}_3$. We need now:

LEMMA 3.6. *We have*

$$(3.16) \quad H^j \mathbf{R}\pi_* \mu_W(\mathcal{O}_X)_{z_o} = 0 \quad \forall j \neq s^-.$$

PROOF. We set $\mathcal{F} = H^{s^-} \mu_W(\mathcal{O}_X)$, $Z = \dot{N}_{z_o}(W)^{oa} = \mathbf{R}^+ p_1 \cup \mathbf{R}^+ p_2 \cup \text{int } N_{z_o}(W)^{oa}$ and denote B_i , $i = 1, 2$ conic neighborhoods of p_i in Z . Recall that M_3 is real analytic and that $T_{M_3}^* X$ contains no germ of complex curve. Then [16] applies and gives that $\mathcal{F}|_{\Lambda_3}^o = H^{s^-+2} \mu_{M_3}(\mathcal{O}_X)|_{\Lambda_3}^o$ is conically flabby. In particular

$$(3.17) \quad H^j(\overset{o}{\Lambda}_3, \mathcal{F}) = 0 \forall j \neq 0; \quad \Gamma(\overset{o}{\Lambda}_3, \mathcal{F}) \twoheadrightarrow \Gamma(\overset{o}{\Lambda}_3 \cap B_i, \mathcal{F}), \quad i = 1, 2$$

(where “ \twoheadrightarrow ” stands for surjectivity). By an elementary application of the Mayer-Vietoris long exact sequence (and by taking the B_i ’s small enough) one gets

$$H^j(B_1 \cup B_2 \cup \overset{o}{\Lambda}_3, \mathcal{F}) = 0 \quad \forall j \neq 0,$$

which implies immediately (3.16). \square

By the triangle (3.14) and by Lemma 3.6 we get at once the conclusion of the proof of Theorem 2.2.

4. Remarks and Examples

REMARK 4.1. Let M_1, M_2 be C^2 -hypersurfaces, M_1^+, M_2^+ open half-spaces with boundary M_1, M_2 , and set $M_3 = M_1 \cap M_2$. We do not assume that M_1, M_2 are transversal but still suppose that M_3 is a manifold and that there is an inclusion e.g. $M_2^+ \supset M_1^+$. We denote by W a domain whose boundary is the union of one component of $M_1 \setminus M_3$ and one of $M_2 \setminus M_3$ and which has proper conormal cone. We also assume that $\partial W \in C^1$ (or equivalently that $T_{M_1}^* X \cap T_{M_2}^* X = M_3 \times_{M_1} T_{M_i}^* X$, $i = 1, 2$). Then we get (2.4) $0 < \forall j < s^-$ (resp. $\forall j \neq 0, s^-$ if $s_{M_i}^-(p) \equiv s^- \forall p \in \Lambda_i \stackrel{\text{def.}}{:=} T_{M_i}^* X \cap N(W)^{oa}$, $i = 1, 2$); we also get (2.5) if $s^- \geq 1$. In fact we first remark that $s_{M_1}^-(p_1) \leq s_{M_2}^-(p_1)$ due to $M_2^+ \supset M_1^+$. Assume first $s_{M_1}^-(p_1) = s_{M_2}^-(p_1)$.

We then perform a contact transformation χ similar as in §3 but satisfying in the present case:

$$\begin{aligned} \chi(T_{\tilde{M}_1}^* X) &= T_{\tilde{M}_1}^* X, \text{codim } \tilde{M}_1 = 1, s_{\tilde{M}_1}^-(q_1) = 0; \\ \chi(T_{\tilde{M}_2}^* X) &= T_{\tilde{M}_2}^* X, \text{codim } \tilde{M}_2 = 1. \end{aligned}$$

By quantization we get accordingly (with $s_i^- = s_{M_i}^-(p_1)$ and $\tilde{s}_i^- = s_{\tilde{M}_i}^-(q_1)$):

$$\begin{aligned} \Phi_K(\mathbf{Z}_{\partial W}) &= \begin{cases} \mathbf{Z}_{\tilde{M}_1}[s_1^-] & \text{in } \overset{o}{\tilde{\Lambda}}_1 \\ \mathbf{Z}_{\tilde{M}_2}[s_2^- - \tilde{s}_2^-] & \text{in } \overset{o}{\tilde{\Lambda}}_2 \end{cases} \\ \Phi_K(\mathbf{Z}_{M_2^+ \setminus M_1^+}) &= \begin{cases} \mathbf{Z}_{\tilde{M}_1}[s_1^-] & \text{in } \overset{o}{\tilde{\Lambda}}_1 \\ \mathbf{Z}_{\tilde{M}_2}[s_2^- - \tilde{s}_2^- - 1] & \text{in } T_{\tilde{M}_2}^* X \setminus \overset{o}{\tilde{\Lambda}}_2. \end{cases} \end{aligned}$$

Thus constancy may hold only for the first sheaf and provided that $\tilde{s}_2^- = 0$. The proof then goes in the same line as in §3. The case $s_2^- = s_1^- + 1$ is similar.

REMARK 4.2. It seems that we may extend Theorem 2.2 to polyhedrons. Let $M_i^+, i = 1, \dots, m$ be half-spaces with C^2 transversal boundaries and let $W = \bigcap_i M_i^+$. Let $\beta = (\beta_1, \dots, \beta_m), \beta_i \in \{0, 1\}$, define $M_\beta = \bigcap_{\{i:\beta_i=1\}} M_i$ and assume that $M_\beta, |\beta| = m$ is generic at $z_o, z_o \in M_\beta$ (i.e. $\gamma_{M_\beta}(p) = 0, p \in \pi^{-1}(z_o)$). We remark that if $\gamma_{M_\beta}(p) = 0$ holds for $|\beta| = m$, then it also holds $\forall |\beta| \leq m$. If we then suppose that

$$s_{M_\beta}^-(p) \equiv s^- \forall p \in \dot{T}_{M_\beta}^* X \cap N(W)^{oa} \cap \pi^{-1}(B_o), \forall \beta,$$

we get (2.4) $\forall j \neq 0, s^-$ (and (2.5) when $s^- \geq 1$). We shall prove it in our forthcoming papers. It seems that the above conclusions also hold for polyhedrons with non-transversal faces (but with C^1 -boundary) in the frame of Remark 4.1.

Example 4.3. (Dihedron with transversal faces.) Let $X = \mathbf{C}^n$ with coordinates $z = (z_1, z_2, z'), z = x + \sqrt{-1}y$, let $M_1 = \{z \in X; y_1 = y_2 +$

$|y'|^2\}$, $M_2 = \{y_1 = -y_2 - |y'|^2\}$, $M_3 = \{y_1 = 0, y_2 = -|y'|^2\}$, $W = \{y_2 > -|y'|^2, -(y_2 + |y'|^2) < y_1 < y_2 + |y'|^2\}$. In this case $s_{M_i}^-(p) \equiv n - 2 \forall p \in T_{M_i}^*X \cap N(W)^{oa}$, $i = 1, 2, 3$. It follows (2.4) $\forall j \neq 0, n - 2$ (and (2.5) when $n > 2$).

Example 4.4. (Dihedron with C^1 -boundary.) Let $W = \{z \in X; y_1 > -|y'|^2 + c_1 y_2^2 \text{ for } y_2 \geq 0; y_1 > -|y'|^2 + c_2 y_2^2 \text{ for } y_2 \leq 0\}$ (c_1 and $c_2 \geq 0$, $c_1 \neq c_2$). We still have (2.4) $\forall j \neq 0, n - 2$ (and (2.5) for $n > 2$).

References

- [1] Andreotti, A. and H. Grauert, Théorèmes de finitude pour la cohomologie des espaces complexes, Bull. Soc. Math. France **90** (1962), 193–259.
- [2] Andreotti, A. and C. D. Hill, E.E. Levi convexity and the Hans Lewy problem. Part II: Vanishing theorems, Ann. Sc. Norm. Sup. Pisa (1972), 747–806.
- [3] Baouendi, M. S., Chang, C. H. and F. Trèves, Microlocal hypo-analyticity and extension of CR-functions, J. Differential Geom. **18** (1983), 331–391.
- [4] Baouendi, M. S., Rothschild, L. P. and F. Trèves, CR Structures with group action and extendability of CR functions, Invent. Math. **82** (1985), 359–396.
- [5] D'Agnolo, A. and G. Zampieri, Levi's forms for higher codimensional submanifolds, Rend. Acc. Naz. Lincei **9(2)** (1991), 29–33.
- [6] D'Agnolo, A. and G. Zampieri, A propagation theorem for a class of microfunctions at the boundary, Rend. Acc. Naz. Lincei **9(1)** (1990), 54–58.
- [7] D'Agnolo, A. and G. Zampieri, Generalized Levi's form for microdifferential systems, \mathcal{D} -modules and microlocal geometry, Walter de Gruyter and Co., Berlin, New-York (1992), 25–35.
- [8] D'Agnolo, A. and G. Zampieri, Vanishing theorem for sheaves of microfunctions at the boundary on CR manifolds, Comm. in P.D.E. **17 (5, 6)** (1992), 989–999.
- [9] Hörmander, L., An introduction to complex analysis in several complex variables, Van Nostrand, Princeton N.J. (1966).
- [10] Kashiwara, M. and P. Schapira, Microlocal study of sheaves, Astérisque **128** (1985).
- [11] Kashiwara, M. and P. Schapira, Sheaves on manifolds, Springer Grundlehren der Math. **292** (1990).
- [12] Kashiwara, M. and P. Schapira, A vanishing theorem for a class of systems with simple characteristics, Invent. Math. **82** (1985), 579–592.
- [13] Nacinovich, M. and G. Valli, Tangential Cauchy–Riemann complexes on distributions, Ann. Mat. Pura e Appl. **146** (1987), 123–160.

- [14] Sato, M., Kashiwara, M. and T. Kawai, Hyperfunctions and pseudodifferential equations, Springer Lecture Notes in Math. **287** (1973), 265–529.
- [15] Schapira, P., Condition de positivité dans une variété symplectique complexe. Applications à l'étude des microfonctions, Ann. Sci. Ec. Norm. Sup. **14** (1981), 121–139.
- [16] Schapira, P. and J.-M. Trépreau, Microlocal pseudoconvexity and “edge of the wedge” theorem, Duke Math. J. **61** (1) (1990), 105–118.
- [17] Trépreau, J. M., Systèmes différentiels à caractéristiques simples et structures réelles-complexes (d'après Baouendi-Trèves et Sato-Kashiwara-Kawai), Sémin. Bourbaki **595** (1981–82).
- [18] Zampieri, G., Tuboids of \mathbf{C}^n with cone property and domains of holomorphy, Proc. Japan Academy **67** (A) **6** (1991), 217–222.

(Received October 18, 1994)

Dipartimento di Matematica
Pura ed Applicata
Università Degli Studi di Padova
v. Belzoni 7, 35131 Padova
Italy