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# *Solutions of heteroclinic type for some classes of semilinear elliptic partial differential equations*

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**Abstract.** This paper studies certain classes of equations of the form $-\Delta u = g(x, y, u)$ in an infinite strip (if $n = 2$) or cylinder (if $n > 2$). Variational arguments are used to establish the existence of solutions asymptotic to a pair of $x$-periodic states.

## §1. Introduction

This paper studies the existence of solutions of heteroclinic type for 3 families of semi-linear elliptic partial differential equations. For $n \geq 2$, let $\mathcal{D}$ denote a bounded domain in $\mathbb{R}^{n-1}$ having a smooth boundary, $\partial\mathcal{D}$. Let $\Omega = \mathbb{R} \times \mathcal{D}$, an infinite cylinder if $n > 2$ and an infinite strip if $n = 2$. Points in $\Omega$ will be denoted by $(x, y)$, $x \in \mathbb{R}$, $y \in \mathcal{D}$. Consider the partial differential equation

$$-\Delta u = g(x, y, u) \qquad (x, y) \in \Omega \tag{PDE}$$

together with the boundary conditions

$$u(x, y) = 0, \qquad x \in \mathbb{R}, y \in \partial\mathcal{D} \tag{1.1}$$

or

$$\frac{\partial u}{\partial \nu}(x, y) = 0, \qquad x \in \mathbb{R}, y \in \partial\mathcal{D}. \tag{1.2}$$

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In (1.2), $\nu = \nu(y)$ denotes the outward pointing normal to $\partial\mathcal{D}$.

Suppose that $g$ is periodic in $x$ and there is a corresponding family, $\mathcal{M}$, of solutions of (PDE) and (1.1) or (1.2) which are periodic in $x$ and are minimizers of the associated variational problem. The main question of interest here is in the existence of solutions of (PDE) together with (1.1) or (1.2) which approach different members of $\mathcal{M}$ as $x \to \pm\infty$. If $n = 1$, (PDE) reduces to an ordinary differential equation in $x$ and such solutions are then heteroclinic to a pair of different elements of $\mathcal{M}$.

The existence of solutions of heteroclinic type will be established for three classes of problems. The first is for the Neumann problem when $g$ is also periodic in $u$. The second is again for (1.2) for a family of $g$'s where

$$G(x,y,u) = \int_0^u g(x,y,t)dt \tag{1.3}$$

has a finite number of global maxima $u_1, \dots, u_m$ independently of $(x,y)$ and $G(x,y,u_i) = 0$, $1 \le i \le m$. The third case is for the Dirichlet problem (1.1) when $G$ is appropriately coercive in $u$ as $|u| \to \infty$.

The first type of problem was suggested by some recent work on heteroclinic solutions of reversible Hamiltonian systems. See [1–2] and also the papers [3–4] of Bolotin. We mention also the somewhat related work of Moser [5] on minimal solutions of a variational problem on a torus. The third class of problems was motivated by earlier research of Kirchgässner [6] who studied a problem arising in the theory of water waves—see also Turner [7]:

$$-\Delta u = \lambda a(y)u - f(x,y,u,\lambda) \tag{1.4}$$

for $x, y \in \mathbb{R}$ and $|y| < 1$ with the boundary conditions

$$u(x, \pm 1) = 0 \tag{1.5}$$

where $a(y) > 0$, $f(x,y,0,\lambda) = 0 = f_u(x,y,0,\lambda)$, and $\lambda$ is near $\lambda_1$, the smallest eigenvalue of the related linearized eigenvalue problem

$$\begin{cases} \dfrac{-d^2v}{dy^2} = \lambda a(y)v, \qquad |y| < 1 \\ v(\pm 1) = 0 \end{cases}.$$

Kirchgässner used the Center Manifold Theorem to prove the existence of a small amplitude solution of (1.4)–(1.8) which tends to $x$-independent states as $|x| \to \infty$. Further studies of such problems were also made by Kirchgässner and his collaborators. See e.g. [8–9].

The approach taken here is a global variational one that is the analogue for (PDE) and (1.1) or (1.2) of the arguments used in [1–2]. The more complicated cases of $g$ periodic in $u$ will be treated in §2. Then the remaining cases will be sketched in §3.

## §2. The periodic case

Periodic in $u$ nonlinearities $g$ for (1.1) will be considered in this section. Suppose that $g$ satisfies

($g_1$) $g \in C^1(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$,

($g_2$) $g(x, y, u)$ is even and 1-periodic in $x$,

($g_3$) $G(x, y, u)$ is 1-periodic in $u$.

It is not important that the periods of $g$ in $x$ and $u$ are the same. This assumption is made merely for notational convenience.

The first step in finding heteroclinic type solutions of (PDE) and (1.2) is to establish the existence of a large class, $\mathcal{M}$, of solutions of (PDE) and (1.2) which are 1-periodic in $x$. They will be obtained as minima of a corresponding variational problem. Thus let $G$ be as in (1.3). Let $\Omega_1 = [0, 1] \times \mathcal{D}$ and $|\Omega_1| =$ volume of $\Omega_1$. Let $E_1 = \{u \in W^{1,2}(\Omega_1) \mid u \text{ is 1-periodic in } x\}$. For $u \in E_1$, set

$$I_1(u) = \int_{\Omega_1} \left( \frac{1}{2} |\nabla u|^2 - G(x, y, u) \right) dxdy. \tag{2.1}$$

Then by ($g_1$), ($g_3$), and standard results [10], $I \in C^1(E_1, \mathbb{R})$. Note that by ($g_3$), for all $k \in \mathbb{Z}$ and $u \in E_1$,

$$I_1(u + k) = I_1(u), \tag{2.2}$$

i.e. $I_1$ has a natural $\mathbb{Z}$ symmetry.

Let

$$c_1 = \inf_{u \in E_1} I_1(u). \tag{2.3}$$

Proposition 2.4. *There is a* $\bar{u} \in E_1$ *such that* $I_1(\bar{u}) = c_1$.

Proof. Let $(u_n)$ be a minimizing sequence for (2.3) with

$$I_1(u_m) \le K. \tag{2.5}$$

By (2.2), $(u_m + k_m)$ is also a minimizing sequence for (2.3) for any $(k_m) \subset \mathbb{Z}$. Hence it can be assumed that

$$0 \le [u_m] \equiv \frac{1}{|\Omega_1|} \int_{\Omega_1} u_m dxdy < 1. \tag{2.6}$$

We claim that $(u_m)$ is bounded in $E_1$. The weak lower semicontinuity of $I_1$, then implies that along a subsequence, $u_m \to \bar{u} \in E_1$ weakly in $E_1$ and $I_1(\bar{u}) = c_1$.

By (2.5) and $(g_1)$, $(g_3)$, for some constant $K_1$,

$$\|\nabla u_m\|^2_{L^2(\Omega_1)} \le K_1 \|u_m\|_{L^1(\Omega_1)} + 2K \tag{2.7}$$

for all $m \in \mathbb{N}$. Elementary calculus arguments show

$$\|u_m\|_{L^1(\Omega_1)} \le |\Omega_1| |[u_m]| + K_2 \|\nabla u_m\|_{L^2(\Omega_1)} \tag{2.8}$$

and

$$\|u_m\|_{L^2(\Omega_1)} \le K_3(|[u_m]| + \|\nabla u\|_{L^2(\Omega_1)}). \tag{2.9}$$

Combining (2.7)–(2.9) shows $(u_m)$ is bounded in $E_1$ and Proposition 2.4 is proved. □

Corollary 2.10. $\bar{u}$ *is a classical solution of* (PDE) *and* (1.2) *with* $\bar{u}(x+1, y) = \bar{u}(x, y)$ *for all* $(x, y) \in \Omega$.

Proof. Since $\bar{u}$ minimizes $I_1$ on $E_1$, $I_1 \in C^1(E_1, \mathbb{R})$, and $(g_1)$, $(g_3)$ are satisfied, standard arguments imply $\bar{u}$ in a classical solution of (PDE) and (1.2) holds. The periodicity in $x$ follows since $\bar{u} \in E_1$. □

By (2.2), whenever $\bar{u}$ minimizes $I_1$ on $E_1$, so does $\bar{u}+k$ for all $k \in \mathbb{Z}$. Let

$$\mathcal{M} = \{u \in E_1 \mid I_1(u) = c_1\}. \tag{2.11}$$

Before constructing solutions of (PDE) of heteroclinic type, some further properties of $\mathcal{M}$ must be obtained. Let

$$\overline{c_1} = \inf_{u \in W^{1,2}(\Omega_1)} I_1(u). \tag{2.12}$$

PROPOSITION 2.13. $\overline{c_1} = c_1$.

PROOF. Since $E_1 \subset W^{1,2}(\Omega_1)$, $\overline{c_1} \le c_1$. To prove equality, suppose $\overline{c_1} < c_1$. Then there is a $u \in W^{1,2}(\Omega_1)$ such that $I_1(u) < c_1$. Writing

$$\begin{aligned} I_1(u) &= \int_0^{\frac{1}{2}} \int_{\mathcal{D}} \mathcal{L}(u)\,dx\,dy + \int_{\frac{1}{2}}^1 \int_{\mathcal{D}} \mathcal{L}(u)\,dx\,dy \\ &\equiv \alpha + \beta \end{aligned} \tag{2.14}$$

where

$$\mathcal{L}(u) = \frac{1}{2}|\nabla u|^2 - G(x,y,u), \tag{2.15}$$

it follows that either $\alpha$ or $\beta < c_1/2$. Suppose e.g. $\alpha < c_1/2$. Define

$$\begin{aligned} v(x,y) &= u(x,y) \qquad 0 \le x \le \frac{1}{2}, y \in \mathcal{D} \\ &= u(1-x,y) \qquad \frac{1}{2} \le x \le 1, y \in \mathcal{D} \end{aligned} \tag{2.16}$$

and extend $v$ to $\mathbb{R} \times \mathcal{D}$ as a 1-periodic function in $x$. Then $v \in E_1$. Since $G$ is even and 1-periodic in $x$, by $(g_2)$,

$$I_1(v) = 2\alpha < c_1 \tag{2.17}$$

contrary to (2.3) and Proposition 2.4. Thus $\overline{c_1} = c_1$. □

Proposition 2.18. *If $u \in W^{1,2}(\Omega_1)$ and $I_1(u) = c_1$, then $u \in \mathcal{M}$ and $u$ is even in $x$.*

Proof. Suppose $u \in W^{1,2}(\Omega_1)$ with $I_1(u) = c_1$. Define $v$ as in (2.16) so $v \in E_1$ and is even in $x$. By (2.3), (2.12), and Proposition 2.13, $v$ and $u$ are critical points of $I_1$ and therefore both are classical solutions of (PDE) and (1.2) with $v = u$ on $[0, \frac{1}{2}] \times \mathcal{D}$. Let $w = u - v$.

Then $w$ satisfies

$$-\Delta w = g(x, y, u) - g(x, y, v) \equiv b(x, y)w \tag{2.19}$$

where

$$\begin{aligned} b(x, y) &= \frac{g(x, y, u) - g(x, y, v)}{u - v} \qquad \text{if} \qquad u(x, y) \neq v(x, y) \\ &= g_u(x, y, u) \qquad \text{if} \qquad u(x, y) = v(x, y) \end{aligned}$$

and

$$\frac{\partial w}{\partial \nu}(x, \partial \mathcal{D}) = 0. \tag{2.20}$$

Note that $b$ is continuous and $w(x, y) = 0$ for $0 \le x \le \frac{1}{2}$ and $y \in \mathcal{D}$. But then a local unique continuation theorem for elliptic equations—see e.g. Nirenberg [11]—and a continuation argument imply $w(x, y) \equiv 0$, $x \in [0, 1]$, $y \in \bar{\mathcal{D}}$, i.e. $u \equiv v$ in $\Omega_1$. □

A special case of interest is when $G$ is independent of $x$. Then the elements of $\mathcal{M}$ also possess this property:

Proposition 2.21. *If $G$ is independent of $x$ and $w \in \mathcal{M}$, then $w$ is independent of $x$.*

Proof. For each $\theta \in \mathbb{R}$, set

$$I_1^\theta(u) = \int_{\theta - \frac{1}{2}}^{\theta + \frac{1}{2}} \int_{\mathcal{D}} \left( \frac{1}{2} |\nabla u|^2 - G(y, u) \right) dx dy. \tag{2.22}$$

If $u \in E_1$, then $I_1^\theta(u) = I_1(u)$. Therefore

$$c_1 = \inf_{E_1} I_1^\theta(u). \tag{2.23}$$

Since $G$ is independent of $x$, the argument of Proposition 2.13 shows any minimizer $w_1$ of $I_1^\theta$ is even about $x = \theta$ for each $\theta \in \mathbb{R}$. Hence $w$ is independent of $x$. But the minimizers of $I_1^\theta$ are the minimizers of $I_1$. $\square$

For later purposes, for $k \in \mathbb{N}$, we must also study solutions of (PDE) in

$$E_k = \{u \in W^{1,2}(\mathbb{R} \times \mathcal{D}) \mid u \text{ is } k \text{ periodic in } x\}. \tag{2.24}$$

Define

$$I_k(u) = \int_0^k \int_{\mathcal{D}} \mathcal{L}(u) dx dy \tag{2.25}$$

and

$$c_k = \inf_{E_k} I_k(u). \tag{2.26}$$

Proposition 2.27. *$c_k = kc_1$ and if $u$ is a critical point of $I_k$ on $E_k$, then $u \in \mathcal{M}$.*

Proof. Any $u \in E_1$ can be considered to be an element of $E_k$ with $I_k(u) = kI_1(u)$. Therefore $c_k \le kc_1$. Let $u_k$ be a critical point of $I_k$ corresponding to $c_k$. (It exists via the argument of Proposition 2.4.) Suppose $c_k < kc_1$. As in Proposition 2.4, consider the restrictions of $u_k$ to the cylinders $[0, \frac{1}{2}] \times \mathcal{D}, [\frac{1}{2}, 1] \times \mathcal{D}, \dots, [k - \frac{1}{2}, k] \times \mathcal{D}$. Choose the restriction which makes the smallest contribution to $I_k(u_k)$. This contribution must be less than $c_1/2$. Extending this restriction of $u_k$ evenly and then 1-periodically about an endpoint yields $v \in E_1$ with $I_1(v) < c_1$, a contradiction. Therefore $c_k = kc_1$. The argument of Proposition 2.18 shows $u_k \in \mathcal{M}$. $\square$

Proposition 2.4 and $(g_3)$ show $\mathcal{M}$ is a sizable set. We assume it is not too large in the following sense:

$(\mathcal{M})$ $\mathcal{M}$ consists of isolated points.

If $(\mathcal{M})$ is not satisfied, perturbing the problem slightly produces a new functional for which $(\mathcal{M})$ holds. E.g. if $\bar{u} \in \mathcal{M}$, replace $G(x,y,u)$ by $\hat{G}(x,y,u) = G(x,y,u) - \epsilon(u-\bar{u}(x,y))^2$ if $(x,y) \in \mathbb{R}\times\bar{\mathcal{D}}$ and $|u-\bar{u}(x,y)| \le \frac{1}{4}$; $\hat{G}(x,y,u) = G(x,y,u) - \epsilon(u-(\bar{u}(x,y)+1))^2$ if $(x,y) \in \mathbb{R}\times\bar{\mathcal{D}}$ and $|u-(\bar{u}(x,y)+1)| \le \frac{1}{4}$, etc. and extend $\hat{G}$ to the rest of $\mathbb{R}\times\bar{\mathcal{D}}\times\mathbb{R}$ appropriately so that $\hat{G}$ satisfies $(g_1)$–$(g_3)$. Then for the associated functional, $\hat{I}_1, \hat{I}_1(\bar{u}) = I_1(\bar{u}) = c_1$ but if $u \notin \bar{u}+\mathbb{Z}$, $\hat{G}(u) < G(u)$ and therefore $\hat{I}_1(u) > I_1(u)$. Therefore the only minima of $\hat{I}_1$ are $\{\bar{u}+k \mid k \in \mathbb{Z}\}$ and $\mathcal{M}$ is satisfied for the $\hat{I}_1$ problem.

Our goal now is to show that for any $v \in \mathcal{M}$, there is a solution $U$ of (PDE) such that $\|U-v\|_{L^\infty([n,n+1]\times\mathcal{D})} \to 0$ as $n \to -\infty$, i.e. $U \to v$ uniformly in this sense as $x \to -\infty$ and similarly $U \to \mathcal{M}\backslash\{v\}$ as $x \to \infty$. By $(\mathcal{M})$, this means there is a $w \in \mathcal{M}\backslash\{v\}$ such that $U \to w$ uniformly as $x \to \infty$. Thus $U$ is a solution of (PDE) of heteroclinic type. A variational argument will be used to prove this result after some further technical preliminaries. Let $B_\rho(v) = \{u \in W^{1,2}(\Omega_1) \mid \|u-v\|_{W^{1,2}(\Omega_1)} < \rho\}$ and $N_\rho(S) = \{u \in W^{1,2}(\Omega_1) \mid \|u-S\|_{W^{1,2}(\Omega_1)} \le \rho\}$.

Proposition 2.28. *Suppose that $g$ satisfies $(g_1)$–$(g_3)$ and $(\mathcal{M})$ holds. Then there is a constant $\rho_0 > 0$ such that if $0 < \rho < \rho_0$,*
(i) *$B_\rho(v) \cap B_\rho(w) = \phi$ for all $v \ne w \in \mathcal{M}$.*
(ii) *$I_1(u) > c_1$ for all $u \in W^{1,2}(\Omega_1)\backslash\mathcal{M}$.*
(iii) *There is an $\alpha(\rho) > 0$ such that*

$$I_1(u) \ge c + \alpha(\rho) \qquad \text{for all} \qquad u \in W^{1,2}(\Omega_1)\backslash N_\rho(\mathcal{M}).$$

Proof. Let

$$\gamma = \inf\{\|v-w\|_{W^{1,2}(\Omega_1)} \mid v \ne w \in \mathcal{M}\}. \tag{2.29}$$

We claim $\gamma > 0$ and therefore (i) follows with e.g. $\rho_0 = \gamma/2$. To see that $\gamma > 0$, let $u \in \mathcal{M}$ with

$$-1 \le [u] < 2. \tag{2.30}$$

Then as in the proof of Proposition 2.4, there is a $K = K(c_1)$ such that $\|u\|_{W^{1,2}(\Omega_1)} \le K$. Since $u \in \mathcal{M}$, it is a classical solution of (PDE) and

(1.2). Therefore by standard elliptic regularity theory arguments, for any $\beta \in (0,1)$, there is a $K_\beta = K_\beta(c_1)$ such that $\|u\|_{C^{2,\beta}} \le K_\beta$ where $C^{2,\beta}$ denotes the set of $u$ on $[0,1] \times \mathcal{D}$ which are 1-periodic in $x$, $C^2$, and whose second derivatives are Hölder continuous of order $\beta$. Now if $\gamma = 0$, there are sequences $(v_j) \neq (w_j) \subset \mathcal{M}$ such that

$$\|v_j - w_j\|_{W^{1,2}(\Omega_1)} \to 0 \tag{2.31}$$

as $j \to \infty$. As in (2.6), it can be assumed that

$$0 \le [w_j] < 1 \tag{2.32}$$

for all $j \in \mathbb{N}$. By (2.31), $[v_j]$ satisfies (2.30) for large $j$. Consequently both $v_j$ and $w_j$ converge in $C^2$ along a subsequence to $u \in \mathcal{M}$. But then $u$ is not an isolated solution of (PDE) and (1.2), contrary to $(\mathcal{M})$. Thus $\gamma > 0$ and (i) holds.

Property (ii) follows from the definition of $c_1$ and Propositions 2.13 and 2.18.

To prove (iii), an indirect argument will again be employed. If (iii) is false, there is a sequence $(u_m) \subset W^{1,2}(\Omega_1) \backslash N_\rho(\mathcal{M})$ such that $0 \le [u_m] < 1$ and

$$I_1(u_m) \to c_1. \tag{2.33}$$

As for (i), (2.33) implies $(u_m)$ is bounded in $W^{1,2}(\Omega_1)$. Hence there is a $u \in W^{1,2}(\Omega_1)$ such that, along a subsequence, $u_m \to u$ weakly in $W^{1,2}(\Omega_1)$. Since $I_1$ is weakly lower semicontinuous, $I_1(u) = c_1$. Consequently $u \in \mathcal{M}$. Set $\varphi_m = u - u_m$. Note that $\|\varphi_m\|_{W^{1,2}(\Omega_1)} \ge \rho$ and $\varphi_m \to 0$ weakly in $W^{1,2}(\Omega_1)$ along the subsequence. Therefore

$$\begin{aligned} I_1(u_m) = I_1(u) + \int_{\Omega_1} \Big[ &\frac{1}{2}(|\nabla \varphi_m|^2 + \varphi_m^2) - \nabla u \cdot \nabla \varphi_m \\ &-G(x,y,u-\varphi_m) + G(x,y,u) - \frac{1}{2}\varphi_m^2 \Big] dxdy \ge c_1 + \frac{1}{2}\rho^2 \\ &- \int_{\Omega_1} \Big[ \nabla u \cdot \nabla \varphi_m + G(x,y,u-\varphi_m) \\ &-G(x,y,u) + \frac{1}{2}\varphi_m^2 \Big] dxdy. \end{aligned} \tag{2.34}$$

Since $\varphi_m \to 0$ weakly in $W^{1,2}(\Omega_1)$, $\varphi_m \to 0$ in $L^2(\Omega_1)$ along the subsequence and

$$\int_{\Omega_1} G(x,y,u-\varphi_m)dxdy \to \int_{\Omega_1} G(x,y,u)dxdy. \tag{2.35}$$

Thus the right hand side of (2.34) tends to 0 as $m \to \infty$, along the subsequence, i.e.

$$\lim_{m\to\infty} I_1(u_m) \geq c_1 + \frac{1}{2}\rho^2 \tag{2.36}$$

contrary to (2.33). Thus there is an $\alpha(\rho) > 0$ as claimed and (iii) holds. □

Remark 2.37.

(i) The argument of (i) of Proposition 2.28 implies that $\mathcal{M}/\mathbb{Z}$ is compact. This fact together with $(\mathcal{M})$ shows that $\mathcal{M}/\mathbb{Z}$ is a finite set. Therefore setting

$$\bar{\gamma} = \inf\{\|u-v\|_{L^2(\Omega_1)} \mid u = v \in \mathcal{M}\}, \tag{2.38}$$

it follows that $\bar{\gamma} > 0$.

(ii) Choosing $\alpha(\rho)$ still smaller if necessary, the argument of (iii) of Proposition 2.28 shows that $I_2(u) \geq 2c_1 + \alpha(\rho)$ for all $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \bar{\mathcal{D}}) \backslash N^2_\rho(\mathcal{M})$ where $N^2_\rho(\mathcal{M})$ denotes a uniform neighborhood of $\mathcal{M}$ in $W^{1,2}([0,2] \times \bar{\mathcal{D}})$.

Let $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R} \times \bar{\mathcal{D}})$ and $k \in \mathbb{Z}$. Set

$$P_k u = u \mid_{[k,k+1]\times\bar{\mathcal{D}}} . \tag{2.39}$$

Then $P_k u$ can be identified with

$$w(x,y) = P_k u(x+k,y) \in W^{1,2}(\Omega_1).$$

This identification will be clear from the context and will not be made explicitly in what follows.

Now the variational problem which will be used to find solutions of (PDE) of heteroclinic type can be formulated. For $v \in \mathcal{M}$, set

$$(2.40)\quad \begin{aligned}\Gamma^-(v) = \{U \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathcal{D}) \mid \|P_kU - v\|_{L^2(\Omega_1)} \to 0 \\ \text{as}\quad k\to -\infty \qquad \text{and} \qquad \|P_kU - (\mathcal{M}\backslash\{v\})\|_{L^2(\Omega_1)} \to 0 \\ \text{as}\quad k\to\infty\}.\end{aligned}$$

It is clear from the definition that $\Gamma^-(v) \neq \phi$. For $k \in \mathbb{Z}$ and $U \in \Gamma^-(v)$, set

$$(2.41)\qquad a_k(U) = \int_k^{k+1}\int_{\mathcal{D}} (\mathcal{L}(U) - \mathcal{L}(v))dxdy.$$

By the definition of $\mathcal{M}$ and Proposition 2.13,

$$(2.42)\qquad a_k(U) \geq 0$$

for all such $k$ and $U$ and $a_k(U) = 0$ if and only if $P_kU \in \mathcal{M}$.

Finally define

$$(2.43)\qquad J(U) = \sum_{k\in\mathbb{Z}} a_k(U)$$

and set

$$(2.44)\qquad c = \inf_{U\in\Gamma^-(v)} J(U).$$

At first glance, it may seem that $J(U)$ equals

$$(2.45)\qquad \int_{\mathbb{R}\times\mathcal{D}} (\mathcal{L}(U) - \mathcal{L}(v))dxdy.$$

However there are functions $U \in \Gamma^-(v)$ such that $J(U) < \infty$ but for which the integral in (2.45) is not conditionally convergent.

Now the main results for this section can be stated

THEOREM 2.46. *Let* $(g_1)$–$(g_3)$ *and* $(\mathcal{M})$ *be satisfied. Then for each* $v \in \mathcal{M}$*, there is a* $U \in \Gamma^-(v)$ *such that* $J(U) = c$.

Corollary 2.47. *$U$ is a classical solution of* (PDE) *with $P_kU \to v$ in $C^2(\Omega_1)$ as $k \to -\infty$ and $P_kU \to \mathcal{M}\backslash\{v\}$ in $C^2(\Omega_1)$ as $k \to \infty$.*

Proof of Theorem 2.46. The proof will be divided into four main steps: (A) Construction of a minimizing sequence for (2.44) which converges weakly to some $U \in W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \bar{\mathcal{D}})$; (B) Obtaining the asymptotic behavior for $U$ as $x \to -\infty$; (C) Obtaining the asymptotic behavior for $U$ as $x \to \infty$; (D) Showing that $U$ minimizes $J$ on $\Gamma^-(v)$.

**(A)** A convergent minimizing sequence

For $k \in \mathbb{Z}$ and $W \in \Gamma^-(v)$, set

$$\tau_k W(x,y) = W(x-k,y). \tag{2.48}$$

Then $\tau_k W \in \Gamma^-(v)$ and by $(g_1)$–$(g_2)$,

$$J(\tau_k W) = J(W). \tag{2.49}$$

Now let $(W_m)$ be a minimizing sequence for (2.44). By (2.49), $(\tau_{k(m)} W_m)$ is also a minimizing sequence for any $(k(m)) \subset \mathbb{Z}$. Let

$$\bar{\rho} \in (0, \bar{\gamma}/3) \tag{2.50}$$

where $\bar{\gamma}$ is defined in (2.38). For each $m$, choose $k(m)$ so that

$$\|P_j \tau_{k(m)} W_m - v\|_{L^2(\Omega_1)} \le \bar{\rho} \tag{2.51}$$

for all $j < 0$ and

$$\|P_0 \tau_{k(m)} W_m - v\|_{L^2(\Omega_1)} > \bar{\rho}. \tag{2.52}$$

This is possible via the definition of $\Gamma^-(v)$ and (2.38). Without loss of generality, it can be assumed that $k(m) = 0$ for all $m$. Since $J(W_m) \to c$, there is an $M > 0$ such that

$$J(W_m) \le M \tag{2.53}$$

for all $m \in \mathbb{N}$. By (2.51), for any $\ell \in \mathbb{N}$,

$$\|W_m\|_{L^2([-\ell,0]\times\mathcal{D})} \le \|v\|_{L^2([-\ell,0]\times\mathcal{D})} + \bar{\rho}\ell. \tag{2.54}$$

Also by (2.42)–(2.43) and (2.53),

$$(2.55)\ \frac{1}{2}\|\nabla W_m\|^2_{L^2([-\ell,0]\times\mathcal{D})} \le \max_{(x,y,u)\in[0,1]\times\bar{\mathcal{D}}\times[0,1]} |g(x,y,u)|\|W_m\|_{L^1([-\ell,0]\times\mathcal{D})} + \ell c_1 + M.$$

Now (2.54)–(2.55) and $(g_1)$, $(g_3)$ yield bounds for $(W_m)$ in $W^{1,2}([-\ell,0]\times\bar{\mathcal{D}})$. As in (2.55),

$$(2.56)\ \frac{1}{2}\|\nabla W_m\|^2_{L^2([0,\ell]\times\mathcal{D})} \le \max_{(x,y,u)\in[0,1]\times\bar{\mathcal{D}}\times[0,1]} |g(x,y,u)|\|W_m\|_{L^1([0,\ell]\times\mathcal{D})} + \ell c_1 + M.$$

By elementary calculus estimates,

$$(2.57)\qquad \|W_m\|_{L^1([0,\ell]\times\mathcal{D})} \le \ell\int_{\mathcal{D}} |W_m(0,y)|dy + \ell\|\nabla W_m\|_{L^1([0,\ell)\times\mathcal{D})}$$

and similarly

$$(2.58)\qquad \ell\int_{\mathcal{D}} |W_m(0,y)|dy \le \|W_m\|_{L^1([-\ell,0]\times\mathcal{D})} + \ell\|\nabla W_m\|_{L^1([-\ell,0]\times\mathcal{D})}.$$

Therefore combining (2.56)–(2.58) and the bounds already obtained for $(W_m)$ in $W^{1,2}([-\ell,0]\times\mathcal{D})$ provides bounds for $W_m$ in $W^{1,2}([-\ell,\ell]\times\mathcal{D})$. Since $\ell$ is arbitrary, $W_m$ is bounded in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathcal{D})$. Consequently there is a $U\in W^{1,2}_{\mathrm{loc}(\mathbb{R}\times\mathcal{D})}$ such that along a subsequence, $W_m\to U$ weakly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathcal{D})$ and strongly in $L^2_{\mathrm{loc}}(\mathbb{R}\times\mathcal{D})$. Note that by (2.52),

$$(2.59)\qquad \|P_kU-v\|_{L^2(\Omega_1)}\le\bar{\rho},\qquad k<0$$

and

$$(2.60)\qquad \|P_0U-v\|_{L^2(\Omega_1)}\ge\bar{\rho}.$$

**(B)** The asymptotic behavior of $U$ as $x\to-\infty$.

By (2.53), for each $\ell \in \mathbb{N}$,

$$\sum_{-\ell}^{\ell} a_p(W_m) \le M. \tag{2.61}$$

Therefore by the weak lower semicontinuity of $a_p(\cdot)$,

$$\sum_{-\ell}^{\ell} a_p(U) \le M \tag{2.62}$$

for each $\ell \in \mathbb{N}$. Hence

$$J(U) \le M. \tag{2.63}$$

Now (2.63) implies

$$a_p(U) \to 0 \qquad \text{as} \qquad |p| \to \infty. \tag{2.64}$$

It will be shown next that (2.64) leads to (B) and (C), i.e. $U \in \Gamma^-(v)$. Let $\rho$ be positive and together with $\bar\rho$ satisfy

$$\rho + \bar\rho < \bar\gamma/2 \tag{2.65}$$

where $\bar\gamma$ is as in (2.38) and $\bar\rho$ also satisfies (2.50). Choose $p_0$ so that

$$a_p(U) < \frac{\alpha(\rho)}{2} \tag{2.66}$$

for $|p| > p_0$ where $\alpha(\rho)$ is given by Proposition 2.28. Thus if $|p| > p_0$, $P_pU \in N_\rho(\mathcal{M})$. Therefore there is a $u_p \in \mathcal{M}$ such that

$$\|P_pU - u_p\|_{W^{1,2}(\Omega_1)} \le \rho. \tag{2.67}$$

Set

$$Q_pU = U\,|_{[p,p+2]\times\mathcal{D}}\ .$$

Since by (2.66) again,

$$a_p(U) + a_{p+1}(U) < \alpha(\rho), \tag{2.68}$$

Remark 2.37 (ii) implies that

$$\|Q_p U - \mathcal{M}\|_{W^{1,2}([0,2]\times\mathcal{D})} \le \rho, \tag{2.69}$$

$\mathcal{M}$ being interpreted as a subset of $E_2$. Therefore there is a $\bar{u}_p \in \mathcal{M}$ such that

$$\|Q_p U - \bar{u}_p\|_{W^{1,2}([0,2]\times\mathcal{D})} \le \rho. \tag{2.70}$$

But

$$\begin{aligned} \|Q_p U - \bar{u}_p\|^2_{W^{1,2}([0,2]\times\mathcal{D})} &= \|P_p U - \bar{u}_p\|^2_{W^{1,2}(\Omega_1)} \\ &\quad + \|P_{p+1} U - \bar{u}_p\|^2_{W^{1,2}(\Omega_1)} \end{aligned} \tag{2.71}$$

so by (2.67) and (2.71),

$$\begin{aligned} \|u_p - \bar{u}_p\|_{W^{1,2}(\Omega_1)} &\le \|u_p - P_p U\|_{W^{1,2}(\Omega_1)} \\ &\quad + \|P_p U - \bar{u}_p\|_{W^{1,2}(\Omega_1)} \le 2\rho. \end{aligned} \tag{2.72}$$

Similarly

$$\|u_{p+1} - \bar{u}_p\|_{W^{1,2}(\Omega_1)} \le 2\rho < r. \tag{2.73}$$

Since $\bar{u}_p \in \mathcal{M}$, the definition of $\gamma$ implies $u_p = u_{p+1} = \bar{u}_p$ for all $|p| > p_0$. Hence by (2.59), (2.67), and (2.65), for $p < -p_0$,

$$\begin{aligned} \|v - u_p\|_{L^2(\Omega_1)} &\le \|v - P_p U\|_{L^2(\Omega_1)} + \|P_p U - u_p\|_{L^2(\Omega_1)} \\ &\le \bar{\rho} + \rho < \bar{\gamma}. \end{aligned} \tag{2.74}$$

Therefore $u_p = v$ for $p < -p_0$ and

$$\|P_p U - v\|_{W^{1,2}(\Omega_1)} \le \rho. \tag{2.75}$$

To complete (B) requires showing

$$\|P_p U - v\|_{W^{1,2}(\Omega_1)} \to 0 \tag{2.76}$$

as $p \to -\infty$. But that is now immediate from (2.64) and (iii) of Proposition 2.28.

**(C)** The asymptotic behavior of $U$ as $x \to \infty$.

The arguments of (B) have already established the existence of $w \in \mathcal{M}$ such that

$$\|P_p U - w\|_{W^{1,2}(\Omega_1)} \le \rho \tag{2.77}$$

for all $p > p_0$. Hence (2.64) and (iii) of Proposition 2.28 imply $P_p U \to w$ as $p \to \infty$. It remains only to prove that $w \ne v$. A comparison argument will be employed to do so.

Suppose that $w = v$. We claim that there is a $j \ge 0$ and $\beta > 0$ such that

$$\|P_j W_m - u\|_{W^{1,2}(\Omega_1)} > \beta \tag{2.78}$$

for all $u \in \mathcal{M}$ and for all $m$ sufficiently large. Otherwise for each $j \ge 0$, there is a sequence $k_i(j) \to \infty$ as $i \to \infty$ and $v_{k_i(j)} \in \mathcal{M}$ such that

$$\|P_j W_{k_i(j)} - v_{k_i(j)}\|_{W^{1,2}(\Omega_1)} \to 0 \tag{2.79}$$

as $i \to \infty$. Since $(W_m)$ converges weakly to $U$ in $W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathcal{D})$ along a subsequence as $m \to \infty$, $(P_j W_m)$ is bounded for each $j$ along this subsequence. Therefore only finitely many functions in $\mathcal{M}$ are possible candidates for $v_{k_i(j)}$ in (2.79) via Remark 2.37 (i). Thus without loss of generality, $v_{k_i(j)} = v_j$ independently of $i$. Again invoking the weak convergence of $W_m$ to $U$ in $W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathcal{D})$ shows

$$v_j = P_j U \tag{2.80}$$

for each $j \ge 0$. By (2.77),

$$v_j = w = P_j U \tag{2.81}$$

for large $j$. Since $v_j \in \mathcal{M}$, for all $j \ge 0$,

$$v_j(0, y) = v_j(1, y) = v_{j+1}(0, y) \tag{2.82}$$

via (2.80) and

$$\frac{\partial v_j}{\partial x}(0,y) = 0 = \frac{\partial v_{j+1}}{\partial x}(0,y). \tag{2.83}$$

These observations and the unique continuation result used in Proposition 2.18 imply $v_j = v_{j+1}$ for all $j \geq 0$. Hence by (2.81), $v_j = w$ for all $j \geq 0$. In particular

$$v_0 = w = P_0 U. \tag{2.84}$$

But since $v = w$, (2.84) is contrary to (2.60). Thus (2.78) is valid.

Combining (2.78) with (iii) of Proposition 2.28 shows

$$a_j(W_m) \geq \alpha(\beta) \tag{2.85}$$

for all large $m$. Since $J(W_m) \to c$, it can be assumed that

$$J(W_m) \leq c + \frac{1}{3}\alpha(\beta) \tag{2.86}$$

for large $m$. These inequalities will be used to prove that $w \neq v$. Choose

$$\delta \in (0, \bar{\gamma}/2) \tag{2.87}$$

so that

$$\max_{u \in \overline{B_{3\delta}(w)}} \int_{\Omega_1} (\mathcal{L}(u) - \mathcal{L}(w))dxdy \leq \frac{\alpha(\beta)}{3}. \tag{2.88}$$

With $j$ given via (2.78), since $P_p U \to w$ in $W^{1,2}(\Omega_1)$ as $p \to \infty$, $\bar{p} = \bar{p}(\delta) > j$ can be chosen so that for $p \geq \bar{p}$,

$$\|P_p U - w\|_{W^{1,2}(\Omega_1)} \leq \frac{\delta}{4}. \tag{2.89}$$

We claim there is a large $m = m(\delta)$ so that for some $p = p(m) \geq \bar{p}$,

$$\|P_p W_m - w\|_{W^{1,2}(\Omega_1)} \leq \delta. \tag{2.90}$$

Indeed otherwise, for each $p \geq \bar{p}$ and all large $m$,

$$\|P_p W_m - w\|_{W^{1,2}(\Omega_1)} > \delta. \tag{2.91}$$

By (2.89),

$$\|P_p W_m - w\|_{L^2(\Omega_1)} \leq \|P_p W_m - P_p U\|_{L^2(\Omega_1)} + \frac{\delta}{4}. \tag{2.92}$$

The first term on the right hand side of (2.92) can be assumed to converge to 0 as $m \to \infty$. Hence there is an $\bar{m} = \bar{m}(p, \delta)$ such that for $m \geq \bar{m}$,

$$\|P_p W_m - w\|_{L^2(\Omega_1)} \leq \frac{\delta}{2}. \tag{2.93}$$

If $u \in \mathcal{M} \backslash \{w\}$, by (2.87) and (2.93),

$$\|P_p W_m - u\|_{W^{1,2}(\Omega_1)} \geq \frac{\delta}{2}. \tag{2.94}$$

Consequently by (2.91), (2.94), and (iii) of Proposition 2.28,

$$a_p(W_m) \geq \alpha\left(\frac{\delta}{2}\right) \tag{2.95}$$

for all $m \geq \bar{m}$. Choose $\ell = \ell(\delta) \in \mathbb{N}$ so that

$$(\ell + 1)\alpha\left(\frac{\delta}{2}\right) > c + \frac{\alpha(\beta)}{3}. \tag{2.96}$$

Now (2.95) holds for each $p \geq \bar{p}(\delta)$ and $m \geq \bar{m}(p, \delta)$. Hence for $m \geq \max_{\bar{p} \leq p \leq \bar{p}+\ell} \bar{m}(p, \delta)$, by (2.95)–(2.96),

$$J(W_m) > c + \frac{\alpha(\beta)}{3}, \tag{2.97}$$

contrary to (2.86). Thus (2.90) holds.

Now finally to complete the proof of (c), for $m$ and $p$ given by (2.90) define

$$(2.98)\qquad \begin{aligned} U_m(x,y) &= v(x,y) \quad x \le p, y \in \bar{\mathcal{D}} \\ &= ((p+1)-x)v(x,y) + (x-p)W_m(x,y) \\ &\qquad p < x \le p+1, y \in \bar{\mathcal{D}} \\ &= W_m(x,y) \quad p+1 < x, y \in \bar{\mathcal{D}}. \end{aligned}$$

Then $U_m \in \Gamma^-(v)$. It will be shown that $J(U_m) < c$ which violates (2.44) and therefore $v \ne w$. Note that

$$(2.99)\qquad \begin{aligned} \|P_p U_m - w\|_{W^{1,2}(\Omega_1)} &= \|(p-x)(W_m - w)\|_{W^{1,2}(\Omega_1)} \\ &\le 2\|P_p W_m - w\|_{W^{1,2}(\Omega_1)} \le 2\delta. \end{aligned}$$

Hence by (2.88) and (2.99).

$$(2.100)\qquad a_p(U_m) \le \frac{\alpha(\beta)}{3}.$$

Therefore by (2.98), (2.100), (2.85)–(2.86),

$$(2.101)\qquad \begin{aligned} J(U_m) &= a_p(U_m) + \sum_{p+1}^{\infty} a_k(W_m) \\ &\le a_p(U_m) + J(W_m) - a_p(W_m) \\ &\le \frac{\alpha(\beta)}{3} + c + \frac{\alpha(\beta)}{3} - \alpha(\beta) = c - \frac{\alpha(\beta)}{3} < c \end{aligned}$$

which is impossible. Thus (C) is proved.

**(D)** $J(U) = c$.

By (B) and (C), $U \in \Gamma^-(v)$. Therefore

$$(2.102)\qquad J(U) \ge c.$$

Let $\epsilon > 0$. Then for $m$ sufficiently large and any $\ell \in \mathbb{N}$,

$$(2.103)\qquad c + \epsilon \ge J(W_m) \ge \sum_{-\ell}^{\ell} a_p(W_m).$$

Letting $m \to \infty$, (2.103) shows

$$c + \epsilon \geq \sum_{-\ell}^{\ell} a_p(U). \tag{2.104}$$

Since $\ell$ is arbitrary, by (2.104),

$$c + \epsilon \geq J(U). \tag{2.105}$$

Hence

$$c \geq J(U) \tag{2.106}$$

and equality must hold.

The proof of Theorem 2.46 is complete. □

Proof of Corollary 2.47. Let $\varphi \in C^\infty(\mathbb{R} \times \mathcal{D})$ with 'compact support', i.e. $\varphi$ vanishes for large $x$ and near $\partial\mathcal{D}$. Let $\delta \in \mathbb{R}$. Then $U + \delta\varphi \in \Gamma^-(v)$ and $J(U + \delta\varphi)$ is a $C^1$ function of $\delta$ with a local minimum at $\delta = 0$. Therefore

$$J'(U)\varphi \equiv \int_{\mathbb{R}\times\mathcal{D}} (\nabla U \cdot \nabla\varphi - g(x, y, U)\varphi)dxdy \tag{2.107}$$

for all such $\varphi$, i.e. $U$ is a weak solution of (PDE). Standard regularity arguments then show $U$ is a classical solution of (PDE) and satisfies the boundary conditions (1.2). In fact, since $g \in C^1$, $U \in C^{2,\beta}(\mathbb{R} \times \bar{\mathcal{D}})$ for any $\beta \in (0, 1)$. Moreover since $P_kU$ and $v$ are bounded in $C^{2,\beta}([0, 1] \times \bar{\mathcal{D}})$ and $P_kU - v \to 0$ as $k \to -\infty$ in $W^{1,2}(\Omega_1)$, by standard interpolation inequalities $P_kU \to v$ in $C^2(\Omega_1)$ as $k \to -\infty$. Similarly $P_kU \to w$ in $C^2(\Omega_1)$ as $k \to \infty$. □

Remark 2.108. (i) It is natural to conjecture at least for the case when $\mathcal{M} = v + \mathbb{Z}$, i.e. $\mathcal{M}/\mathbb{Z}$ is a singleton, that $P_kU \to v + 1$ or $v - 1$ as $k \to \infty$. Indeed this has been shown for $n = 1$, the O.D.E. case, in [2]. (ii) The arguments given here extend to the case when $\partial\mathcal{D}$ is a 1-periodic function of $x$ and even in $x$ and in (1.2), $\nu$ is the normal to $\partial\Omega$. (iii) If $(g_2)$ is weakened to permit $g$ to behave differently for e.g. $x > 0$ and $x < 0$, one still gets an analogue of Theorem 2.46. See e.g. [2] for an O.D.E. case.

## §3. Some related problems

In this section, it will be shown how the results of §2 carry over to some other situations. Since the ideas are so close to those of §2, the exposition will be brief. The first variant of §2 involves (PDE) on a semi-infinite domain. To describe it, consider (PDE) for e.g. $x \le 0$ with the boundary conditions:

$$u(0, y) = \psi(y), \qquad y \in \mathcal{D} \tag{3.1}$$

and

$$\frac{\partial u}{\partial \nu}(x, y) = 0 \qquad x \le 0, y \in \partial\mathcal{D} \tag{3.2}$$

where e.g. $\psi$ is smooth and

$$\frac{\partial \psi}{\partial \nu}(y) = 0, \qquad y \in \partial\mathcal{D} \tag{3.3}$$

so (3.1) and (3.2) are compatible.

Let

$$\begin{aligned} \Gamma^- = \{u \in W^{1,2}_{\text{loc}}(\mathbb{R}^- \times \mathcal{D}) \mid u\,|_{x=0} = \psi \qquad & \text{and} \\ \|P_k u - \mathcal{M}\|_{W^{1,2}(\Omega_1)} \to 0 \qquad \text{as} \qquad & k \to -\infty\}. \end{aligned} \tag{3.4}$$

For $u \in \Gamma^-$ set

$$J(u) = \sum_{-\infty}^{0} a_p(u) \tag{3.5}$$

where the $a_p$ are as in (2.41) with any $v \in \mathcal{M}$. Let

$$c = \inf_{u \in \Gamma^-} J(u). \tag{3.6}$$

The analogue of Theorem 2.46 here is:

Theorem 3.7. *Let* $(g_1)$–$(g_3)$ *and* $(\mathcal{M})$ *be satisfied. Let* $\psi$ *be smooth and satisfy* (3.3). *Then there is a* $U \in \Gamma^-$ *such that* $J(U) = c$. *Moreover* $U$ *is a classical solution of* (PDE), (3.1)–(3.2) *and for some* $v \in \mathcal{M}$,

$$\|P_k U - v\|_{W^{1,2}(\Omega_1)} \to 0 \qquad as \qquad k \to -\infty.$$

Proof. The argument follows the same lines as the proof of Theorem 2.46. If $(W_m)$ is a minimizing sequence for (3.6), the estimates (2.55) and

$$\|W_m\|_{L^1([-\ell,0]\times\mathcal{D})} \le \ell(\|\psi\|_{L^1(\mathcal{D})} + \|\nabla W_m\|_{L^1([-\ell,0]\times\mathcal{D})}) \tag{3.8}$$

lead to $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^- \times \mathcal{D})$ bounds for $(W_m)$. Continuing to argue essentially as in (A), (C), (D) and Corollary 2.47 yields Theorem 3.7.

To prepare for the next two applications, we reexamine what was done in §2 a bit more abstractly. Thus consider (PDE) on $\mathbb{R} \times \mathcal{D}$ under either the boundary conditions (1.1) or (1.2). Assume that $g$ satisfies $(g_1)$ and a Sobolev growth condition:

$(g_4)$ there are constants $a_1, a_2 \ge 0$ and $s \in [1, \frac{n+2}{n-2})$ such that $|g(x,y,z)| \le a_1 + a_2|z|^s$.

If $n = 2$, $(g_4)$ can be weakened.

Let $I_1$ be as in (2.1) and let $E_1$ denote the class of $W^{1,2}$ functions on $\Omega_1$ which are 1-periodic in $x$ and also satisfy (1.1) if that boundary condition is operative. By $(g_1)$ and $(g_4)$, $I_1$ is bounded on bounded sets and is weakly lower semicontinuous. Define $c_1$ via (2.3). *Assume*: $(a_1)$ any minimizing sequence for $c_1$ is bounded in $E_1$. Note that this is *not* true for the setting of §2 unless $(g_3)$ is employed to divide out the $\mathbb{Z}$ symmetry. Given $(a_1)$, the arguments of §2 show $\mathcal{M}$ defined in (2.11) is nonempty. If $(g_2)$ is also satisfied, then the conclusions of Propositions 2.13, 2.18, and 2.21 are also valid. The same is true of Proposition 2.27 if $(a_k)$, the analogue of $(a_1)$ for $k \in \mathbb{N}$, holds. Further assuming $(\mathcal{M})$, Proposition 2.28 obtains. Suppose: (b) the cardinality of $\mathcal{M}$ is at least 2. Then by Remark 2.37 (i), $\mathcal{M}$ is finite.

For any $v \in \mathcal{M}$, the set $\Gamma^-(v)$ can be introduced as in (2.40), building in the boundary conditions (2.1) if necessary. Define $J$ and $c$ as in (2.43)–(2.44). Then an examination of the proofs of Theorem 2.46 and Corollary 2.47 show they carry over if: (c) a minimizing sequence for $c$ is bounded in $W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathcal{D})$.

To recapitulate, given $(g_1)$, $(g_2)$ and $(g_4)$ there is an analogue of Theorem 2.46 and Corollary 2.47 provided that $(a_k)$, (b) (c) and $(\mathcal{M})$ are satisfied. Next these conditions will be verified for two examples. For the first, suppose that we are dealing with (1.2) and $g$ satisfies

- $(g_5)$ $G(x,y,z) \leq 0$ and $=0$ if and only if $z$ belongs to a finite set $\mathcal{F}$ of cardinality at least two.
- $(g_6)$ There are constants $\beta > 0$, $R \geq 0$ such that $G(x,y,z) \geq \beta |z|^2$ for $|z| \geq R$.

Note that by $(g_5)$ and $(g_1)$, $G$ has local maxima for $x \in \mathbb{R}$, $y \in \bar{\mathcal{D}}$, and $z \in \mathcal{F}$. Therefore $g(x,y,z) = 0$ at such points. $\square$

Proposition 3.9. *If $g$ satisfies $(g_1)$–$(g_2)$, $(g_4)$–$(g_6)$, then $(a_k)$, (b), (c), and $(\mathcal{M})$ are satisfied.*

Proof. By $(g_1)$ and $(g_5)$, $\mathcal{M} = \mathcal{F}$ and $c_1 = 0$. Therefore (b) and $(\mathcal{M})$ hold. To verify $(a_1)$ (and similarly $(a_k)$), note that if $(u_m)$ is a minimizing sequence for $c_1$, there is a $K > 0$ such that

$$\int_{\Omega_1} \left( \frac{1}{2} |\nabla u_m|^2 - G(x,y,u_m) \right) dxdy \leq K \tag{3.10}$$

so by $(g_5)$,

$$\|\nabla u_m\|^2_{L^2(\Omega_1)} \leq 2K. \tag{3.11}$$

Moreover

$$\begin{aligned} \int_{\Omega_1} u_m^2 dxdy = & \int_{\{(x,y) \mid |u_m(x,y)| \leq R\}} u_m^2 dxdy \\ & + \int_{\{(x,y) \mid |u_m(x,y)| > R\}} u_m^2 dxdy \\ \leq & \, |\Omega_1| R^2 + \beta^{-1} K \end{aligned} \tag{3.12}$$

via $(g_6)$ and (3.10). Thus $(u_m)$ is bounded in $W^{1,2}(\Omega_1)$. Likewise to verify (c), if $(u_m)$ is a minimizing sequence for $c$, for any $\ell \in \mathbb{N}$,

$$\int_{-\ell}^{\ell} \int_{\mathcal{D}} \left( \frac{1}{2} |\nabla u_m|^2 - G(x,y,u_m) \right) dxdy \leq M + \ell c, \tag{3.13}$$

and as in (3.10)–(3.12), this leads to a bound for $(u_m)$ in $W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathcal{D})$.

For the final application of these ideas, consider (PDE) under (1.1). Then there is a constant $\alpha_1 > 0$ such that

$$\|u\|_{L^2(\Omega_1)} \le \alpha_1 \|\nabla u\|_{L^2(\Omega_1)} \tag{3.14}$$

for all $u \in E_1$. Suppose that $g$ satisfies $(g_1)$–$(g_2)$ and

$(g_7)$ There is an $\bar{M} > 0$ such that

$$|g(x,y,z)| \le \bar{M} \qquad \text{for all} \qquad (x,y,z) \in \mathbb{R} \times \bar{\mathcal{D}} \times \mathbb{R}.$$

Then a fortiori $(g_4)$ is verified and by (2.7) and (3.14), $(a_1)$ and similarly $(a_k)$ and (c) hold. If also

$(g_8)$ $g(x,y,0) = 0$

and

$(g_9)$ there is a $\varphi \in E$, such that $I_1(\varphi) < 0$,

then $I_1(0) = 0 > c_1$. Suppose also that

$(g_{10})$ $g(x,y,z) = -g(x,y,-z)$.

Then $v \in \mathcal{M}$ implies $-v \in \mathcal{M}$ and $-v \ne v$. Thus (b) is satisfied. In general, $(\mathcal{M})$ may not hold but a modification in the spirit of earlier remarks can be made to replace $g$ by a new function for which $(\mathcal{M})$ obtains.

Condition $(g_9)$ can be verified for problems of the type considered by Kirchgässner [6] if $\lambda > \lambda_1$. More precisely consider (1.4) together with (1.5) or its higher dimensional analogue (1.1). It is assumed that $a(y) > 0$ on $\bar{\mathcal{D}}$ and is in $C^1$. The function $f$ is also assumed to be in $C^1$ and $f(x,y,z,\lambda) = o(|z|)$ as $|z| \to 0$ uniformly on bounded $\lambda$ intervals. Suppose $\lambda > \lambda_1$, the smallest eigenvalue of (1.6) (or its higher dimensional variant). Then it is not difficult to show—see e.g. [10]—that

$$\begin{aligned} I_1(u) = {} & \int_{\Omega_1} \left( \frac{1}{2}|\nabla u|^2 - \frac{\lambda}{2} a u^2 \right) dx dy \\ & + o(\|u\|^2_{W^{1,2}(\Omega_1)}) \qquad \text{as} \qquad u \to 0. \end{aligned} \tag{3.15}$$

In particular for $u = \psi$, a small multiple of the eigenfunction of (1.6) corresponding to $\lambda_1$,

$$I_1(u) = \frac{(\lambda_1 - \lambda)}{2} \int_{\Omega_1} |\nabla u|^2 dx dy + o(\|\nabla u\|^2_{L^2(\Omega_1)}) < 0 \tag{3.16}$$

and $(g_9)$ is verified.

As an example of a class of functions for which $(g_1)$–$(g_2)$, $(g_7)$–$(g_{10})$ all are satisfied, consider $g(x,y,z) = \lambda a(y)z - h(x,y,z,\lambda)z$ where $\lambda > \lambda_1$, $a$ is as earlier, and $h$ is $C^1$ in its arguments, even and 1-periodic in $x$, $h(x,y,0) = 0$, $h(x,y,-z) = h(x,y,z)$, and there is an $\hat{R} > 0$ such that

$$h(x,y,\hat{R}) > \lambda a(y). \tag{3.17}$$

It is easy to check that $(g_1)$–$(g_2)$ and $(g_8)$–$(g_{10})$ obtain but a priori, $(g_7)$ is not satisfied. Set

$$\begin{aligned} \hat{g}(x,y,z) &= g(x,y,z) && |z| \le \hat{R} \\ &= g(x,y,\hat{R}) && z > \hat{R} \\ &= -g(x,y,-\hat{R}) && z < -\hat{R}. \end{aligned} \tag{3.18}$$

Then $\hat{g}$ satisfies $(g_1)$–$(g_2)$ and $(g_7)$–$(g_{10})$. (Actually $\hat{g}$ is Lipschitz continuous rather than $C^1$ but that is sufficient for the earlier arguments to work.) Therefore there is a solution of heteroclinic type for

$$-\Delta u = \hat{g}(x,y,w) \qquad x \in \mathbb{R}, y \in \mathcal{D} \tag{3.19}$$

and (1.1). If this solution satisfies

$$\|u\|_{L^\infty(\mathbb{R}\times\mathcal{D})} \le \hat{R} \tag{3.20}$$

then $u$ is also a solution of (PDE) via (3.18).

To verify (3.20), let $\hat{\mathcal{M}}$ denote the set of minimizing periodic solutions for (3.19) and (1.1). Then $\hat{\mathcal{M}} \subset \mathcal{M}$. Indeed if $v \in \hat{\mathcal{M}}$, since $v \not\equiv 0$, it has either a positive maximum or negative minimum in $\Omega_1$, e.g. at $(\hat{x},\hat{y})$. Assuming the former case, if $v(\hat{x},\hat{y}) \ge \hat{R}$,

$$0 \le -\Delta v(\hat{x},\hat{y}) = \hat{g}(\hat{x},\hat{y},v) < 0 \tag{3.21}$$

by (3.17)–(3.18). Hence $\hat{R} > v$ in $\Omega_1$ and similarly $v > -\hat{R}$, i.e.

$$\|v\|_{L^\infty(\Omega_1)} < \hat{R}. \tag{3.22}$$

Therefore $v$ satisfies (PDE) and $v \in \mathcal{M}$. Finally if $u$ is a solution of (3.19), since $u \to \hat{\mathcal{M}}$ as $|x| \to \infty$, $|u(x,y)| < \hat{R}$ for large $|x|$ via (3.22). If $u$ does not satisfy (3.20), it has a positive maximum or a negative minimum at some $(\hat{x},\hat{y}) \in \mathbb{R}\times\mathcal{D}$. Then the argument of (3.21) again leads to (3.20) so $u$ is a solution of (PDE).

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