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A bifurcation of multiple eigenvalues and eigenfunctions for boundary value problems in a domain with a small hole

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Dedicated to Professor Sigeru Mizohata

Abstract. In the present paper we study the asymptotic expansion of the multiple eigenvalues and eigenfunctions for boundary value problems in a domain with a small hole. We prove the bifurcation of these eigenvalues under certain conditions.

1. Introduction and main results

The purpose of this article is to study asymptotic formula of multiple eigenvalues and eigenfunctions for boundary value problems in a domain with a small hole. Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^3$ and $\{0\} \in \Omega$. Let B_1 be the unit ball in $\mathbb{R}^3$. We consider the following problem :

$$\Delta u(x,\varepsilon) + \lambda(\varepsilon)u(x,\varepsilon) = 0, \quad \text{in } \Omega_\varepsilon = \Omega \backslash \varepsilon B_1 \tag{1}$$

$$u(x,\varepsilon)\big|_{\partial\Omega_\varepsilon} = 0. \tag{2}$$

All the eigenvalues of (1)-(2) may be put in non-decreasing order $0 < \lambda_1(\varepsilon) < \lambda_2(\varepsilon) \leq \lambda_3(\varepsilon) \cdots$. The first eigenvalue is always simple (see [1]). The eigenvalue from $\lambda_2(\varepsilon)$ may be multiple. We shall study the behavior of

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the functions $\lambda_n(\varepsilon)$ when $\varepsilon \to 0 (n \geq 2)$. The problem (1)-(2) is connected closely with following one in the limit case :

$$\Delta u(x) + \lambda u(x) = 0, \quad \text{in } \Omega \tag{3}$$

$$u(x)\big|_{\partial\Omega} = 0. \tag{4}$$

All the eigenvalues of (3)-(4) may be also put in non-decreasing order $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \ldots.$ It is well-known that $\lim\limits_{\varepsilon\to 0} \lambda_j(\varepsilon) = \lambda_j$ (see[2]). Let λ_j be a simple eigenvalue. In the work [2], [3] Ozawa S. obtained the statement :

$$\lambda_j(\varepsilon) = \lambda_j + 4\pi u_j^2(0)\varepsilon + C_j\varepsilon^2 + 0(\varepsilon^{5/2}) \qquad (\varepsilon \to 0)$$

where $u_j(x)$ is the normed eigenfunction corresponding to λ_j and where C_j is a constant explicitly calculated.

We shall find a full asymptotic formula of $\lambda_j(\varepsilon)$ in a form $\lambda_j(\varepsilon) = \sum\limits_{i=0}^{\infty} \lambda_j^{<i>}\varepsilon^i$ and corresponding eigenfunctions $u_j(x,\varepsilon)$ in a form :

$$u_j(x,\varepsilon) = \sum_{k=0}^{\infty} \big(m_{kj}(x) + n_{kj}(\xi)\big)\varepsilon^k$$

where $\xi = x\varepsilon^{-1}$. The functions $m_{kj}(x)$ and $n_{kj}(\xi)$ have asymptotic expansions

$$m_{kj}(x) = \sum_{i=0}^{N} m_{kj}^{<i>}(\theta).|X|^i + \tilde{m}_{kj}^{<N>}(x) \tag{5}$$

$$n_{kj}(\xi) = \sum_{i=1}^{N} n_{kj}^{<i>}(\theta).|\xi|^{-i} + \tilde{n}_{kj}^{<N>}(\xi) \tag{6}$$

where $\left|D_x^\alpha \tilde{m}_{kj}^{<N>}(x)\right| \leq C_{N,k,\alpha,j}|x|^{-N+1-|\alpha|}$

$$|D_\xi^\alpha \tilde{n}_{kj}^{<N>}(\xi)| \leq C_{N,k,\alpha,j}|\xi|^{-N-1-|\alpha|}$$

$\theta = (\theta_1, \theta_2)$ denotes coordinates on S^2 and $m_{kj}^{<i>}(\theta), n_{kj}^{<i>}(\theta)$ are smooth functions on S^2. In the paper [4] Mazia V.G., Nazarov S.A., B.A.Plamenevskii found a full asymptotic formula for simple eigenvalues. Let λ_j be a simple eigenvalue of the problem (3)-(4). Then we have the following expansion for $\lambda_j(\varepsilon)$:

$$\lambda_j(\varepsilon) = \lambda_j + 4\pi u_j^2(0)\varepsilon + \lambda_j^{<2>}\varepsilon^2 + \ldots + \lambda_j^{<M>}\varepsilon^M + 0(\varepsilon^{M+1})$$

where M is any positive integer number. In the article [5] the author obtained the

THEOREM. *Let λ_j be a double eigenvalue of (3)-(4). It corresponds two orthonormal eigenfunctions $u_j(x), u_{j+1}(x)$. Assume that $u_j^2(0) + u_{j+1}^2(0) > 0$, then we have a formula for the eigenvalues $\lambda_j(\varepsilon) \le \lambda_{j+1}(\varepsilon)$ (respectively)*

$$\lambda_{j+k}(\varepsilon) = \sum_{i=0}^{M} \lambda_{j+k}^{<i>}\varepsilon^i + 0(\varepsilon^{M+1}) \qquad k = 0, 1.$$

Furthemore $\lambda_j^{<0>} = \lambda_{j+1}^{<0>} = \lambda_j, \lambda_j^{<1>} = 0,\ \lambda_{j+1}^{<1>} = 4\pi\big(u_j^2(0) + u_{j+1}^2(0)\big).$

REMARK. It is easy to see that the sum $(u_j^2(0) + u_{j+1}^2(0))$ is invariant under any orthogonal transformations in the plane (u_j, u_{j+1}).

COROLLARY. *Assume that $(u_j^2(0)+u_{j+1}^2(0)) > 0$. Then the eigenvalues $\lambda_j(\varepsilon), \lambda_{j+1}(\varepsilon)$ are simple and different as $\varepsilon \to 0$.*

In the present paper the author continue the studies in [2]-[5]. We shall consider the case when λ_j is a double or triple eigenvalues. Let λ_j be a double and $u_j(0) = u_{j+1}(0) = 0$. We expand $u_j(x), u_{j+1}(x)$ in series :

$$u_{j+k}(x) = u_{j+k}^{<1>}(\theta)r + u_{j+k}^{<2>}(\theta)r^2 + \ldots + u_{j+k}^{<M>}(\theta)r^M + 0(r^{M+1}) \quad (r \to 0)$$

where $k = 0, 1$ and $r = |x|$.

One can write the Laplace operator in the spherical coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\ \frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^2}$$

where-Δ_{S^2} is the Laplace - Beltrami operator on sphere. Since the functions $u_j(x), u_{j+1}(x)$ are the eigenfunctions, it follows that $u_j^{<1>}(\theta)$, $u_{j+1}^{<1>}(\theta)$ satisfy the equations (see [6]) :

$$\Delta_{S^2} u_{j+k}^{<1>}(\theta) + 2u_{j+k}^{<1>}(\theta) = 0 \qquad (k = 0, 1).$$

Therefore we have the indentities

$$u_{j+k}^{<1>}(\theta) = a_{j+k}^{<1>} A_1(\theta) + a_{j+k}^{<2>} A_2(\theta) + a_{j+k}^{<3>} A_3(\theta) \quad (k = 0, 1),$$

where $A_1(\theta), A_2(\theta), A_3(\theta)$ denote orthonormal eigenfunctions of Δ_{S^2} with the eigenvalue 2.

Theorem 1. *Let λ_j be a double eigenvalue and $u_j(0) = u_{j+1}(0) = 0$. Assume that*

$$T_j := \left| \sum_{i=1}^{3} [a_j^{(i)}]^2 - \sum_{i=1}^{3} [a_{j+1}^{(i)}]^2 \right| + \left| \sum_{i=1}^{3} a_j^{(i)} a_{j+1}^{(i)} \right| \neq 0.$$

Then we have the expansions for $\lambda_j(\varepsilon) \leq \lambda_{j+1}(\varepsilon)$ (resp.)

(7) $$\lambda_{j+k}(\varepsilon) = \lambda_j + \lambda_{j+k}^{<3>} \varepsilon^3 + \lambda_{j+k}^{<4>} \varepsilon^4 + \cdots + \lambda_{j+k}^{<M>} \varepsilon^M + 0(\varepsilon^{M+1}) \quad (\varepsilon \to 0)$$

where $k = 0, 1$ and $\lambda_j^{<3>} < \lambda_{j+1}^{<3>}$.

Corollary 1. *Assume that $u_j(0) = u_{j+1}(0) = 0, T_j \neq 0$, then $\lambda_j(\varepsilon)$, $\lambda_{j+1}(\varepsilon)$ are simple and different as $\varepsilon \to 0$.*

Remark. The condition $T_j \neq 0$ is equivalent to the following condition : the matrix

$$\begin{pmatrix} \left(u_j^{<1>}(\theta), u_j^{<1>}(\theta)\right)_{L^2(\partial B_1)} & \left(u_j^{<1>}(\theta), u_{j+1}^{<1>}(\theta)\right)_{L^2(\partial B_1)} \\ \left(u_j^{<1>}(\theta), u_{j+1}^{<1>}(\theta)\right)_{L^2(\partial B_1)} & \left(u_{j+1}^{<1>}(\theta), u_{j+1}^{<1>}(\theta)\right)_{L^2(\partial B_1)} \end{pmatrix} =: M$$

has two different eigenvalues. In the future it is easy to see that $3^{-1}\lambda_j^{<3>}$, $3^{-1}\lambda_{j+1}^{<3>}$ are the eigenvalues of the matrix M.

Now let λ_j are a triple eigenvalue of the problems (3)-(4). It corresponds three orthonormal functions $u_j(x), u_{j+1}(x), u_{j+2}(x)$. Assume that

$u_j^2(0) + u_{j+1}^2(0) + u_{j+2}^2(0) \neq 0$. Then we can always choose 3 functions $u_j^*(x), u_{j+1}^*(x), u_{j+2}^*(x)$ in the plane $(u_j(x), u_{j+1}(x), u_{j+2}(x))$ such that

$$u_{j+k}^*(x) = u_{j+k}^{*<1>}(\theta)|x| + u_{j+k}^{*<2>}(\theta)|x|^2 + \cdots + u_{j+k}^{*<M>}(\theta)|x|^M + 0(|x|^{M+1})$$

$$\big(u_{j+i}^*(x), u_{j+k}^*(x)\big)_{L^2(\Omega)} = \delta_{ik} \quad (i,k = 0,1,2), u_j^*(0) = u_{j+1}^*(0) = 0$$

$$u_{j+2}^{*2}(0) = u_j^2(0) + u_{j+1}^2(0) + u_{j+2}^2(0).$$

THEOREM 2. *Let λ_j be a triple eigenvalue of (3)-(4). Assume that $u_{j+2}^*(0) \neq 0$ and the matrix*

$$\begin{pmatrix} \big(u_j^{*<1>}(\theta), u_j^{*<1>}(\theta)\big)_{L^2(\partial B_1)} & \big(u_j^{*<1>}(\theta), u_{j+1}^{*<1>}(\theta)\big)_{L^2(\partial B_1)} \\ \big(u_j^{*<1>}(\theta), u_{j+1}^{*<1>}(\theta)\big)_{L^2(\partial B_1)} & \big(u_{j+1}^{*<1>}(\theta), u_{j+1}^{*<1>}(\theta)\big)_{L^2(\partial B_1)} \end{pmatrix} =: M^*$$

has two different eigenvalues. Then we have the asymptotic formula for $\lambda_j(\varepsilon) \leq \lambda_{j+1}(\varepsilon) \leq \lambda_{j+2}(\varepsilon)$

$$\lambda_{j+k}(\varepsilon) = \lambda_j + \lambda_{j+k}^{<3>}\varepsilon^3 + \lambda_{j+k}^{<4>}\varepsilon^4 + \cdots + \lambda_{j+k}^{<M>}\varepsilon^M + 0(\varepsilon^{M+1}) \quad (\varepsilon \to 0)$$
$$\lambda_{j+2}(\varepsilon) = \lambda_j + 4\pi[u_{j+2}^*(0)]^2\varepsilon + \lambda_{j+2}^{<2>}\varepsilon^2 + \cdots + \lambda_{j+2}^{<M>}\varepsilon^M + 0(\varepsilon^{M+1}) \quad (\varepsilon \to 0)$$

where $k = 0, 1$, and $\lambda_j^{<3>} < \lambda_{j+1}^{<3>}$.

COROLLARY 2. *If $u_{j+2}^*(0) \neq 0$ and the matrix M^* has two different eigenvalues, then the eigenvalues $\lambda_j(\varepsilon), \lambda_{j+1}(\varepsilon), \lambda_{j+2}(\varepsilon)$ are simple and different when $\varepsilon \to 0$.*

2. A process of finding the full asymptotic formula of the eigenvalues and the eigenfunctions A. The case of double eigenvalues :

Put $\lambda_{j+k}(\varepsilon)$ from (7) into (1) and (2) :

$$\begin{aligned} &[(\Delta + \lambda_j + \lambda_j^{<1>}\varepsilon + \lambda_j^{<2>}\varepsilon^2 + \lambda_j^{<3>}\varepsilon^3 + 0(\varepsilon^4)][(u_{jo} + v_{jo}) + \varepsilon(u_{j1} + v_{j1}) + \\ (8) \quad &+\varepsilon^2(u_{j2} + v_{j2}) + \varepsilon^3(u_{j3} + v_{j3}) + \varepsilon^4(u_{j4} + v_{j4}) + 0(\varepsilon^5)] = 0 \text{ in } \Omega_\varepsilon \end{aligned}$$

(9) $\quad [(u_{jo}+v_{jo})+\varepsilon(u_{j1}+v_{j1})+\varepsilon^2(u_{j2}+v_{j2})+\cdots+0(\varepsilon^5)]\big|_{\partial\Omega_\varepsilon}=0$

$[(\Delta+\lambda_j+\lambda_{j+1}^{<1>}\varepsilon+\lambda_{j+1}^{<2>}\varepsilon^2+\lambda_{j+1}^{<3>}\varepsilon^3+0(\varepsilon^4)][(p_{jo}+q_{jo})+\varepsilon(p_{j1}+q_{j1})+$

(10) $\quad +\varepsilon^2(p_{j2}+q_{j2})+\varepsilon^3(p_{j3}+q_{j3})+\varepsilon^4(p_{j4}+q_{j4})+0(\varepsilon^5)]=0 \text{ in } \Omega_\varepsilon$

(11) $\quad [(p_{jo}+q_{jo})+\varepsilon(p_{j1}+q_{j1})+\varepsilon^2(p_{j2}+q_{j2})+\cdots+0(\varepsilon^5)]\big|_{\partial\Omega_\varepsilon}=0$

where

$$u_j(x,\varepsilon)=[(u_{jo}+v_{jo})+\varepsilon(u_{j1}+v_{j1})+\varepsilon^2(u_{j2}+v_{j2})+\dots]$$
$$u_{j+1}(X,\varepsilon)=[(p_{j0}+q_{j0})+\varepsilon(p_{j1}+q_{j1})+\varepsilon^2(p_{j2}+q_{j2})+\dots]$$

denote eigenfunctions corresponding to $\lambda_j(\varepsilon),\lambda_{j+1}(\varepsilon)$. Functions $u_{j0}(x)$, $u_{j1}(x),\dots,p_{j0}(x),p_{j1}(x),\dots$ are defined in Ω and they keep an asymptotic expansion as the functions $m_{kj}(x)$ from (5). Functions $v_{j0}(\xi),v_{j1}(\xi)$, $q_{j0}(\xi),q_{j1}(\xi),\dots$ are defined in $\mathbb{R}^3\backslash B_1$ and they keep an asymptotic expansions as the function $n_{kj}(\xi)$ from (6). In the following we shall write $u_0(x),u_1(x),\dots,\ p_0(x),p_1(x),\dots,v_0(\xi),v_1(\xi),\dots,q_0(\xi),q_1(\xi),\dots$ for $u_{j0}(x),u_{j1}(x),\dots,\ \ p_{j0}(x),p_{j1}(x),\dots,\ \ v_{j0}(\xi),v_{j1}(\xi),\dots,q_{j0}(\xi),q_{j1}(\xi),\dots$. Comparing the coefficients in the identical orders of ε in (8)-(11) one obtain :

$$\varepsilon^0\begin{cases}\Delta u_0(x)+\lambda_j u_0(x)=0, & \text{in } \Omega\\ u_0(x)|_{\partial\Omega}=0\end{cases}$$
$$\varepsilon^0\begin{cases}\Delta p_0(x)+\lambda_j p_0(x)=0, & \text{in } \Omega\\ p_0(x)|_{\partial\Omega}=0.\end{cases}$$

Hence $u_0(x)=a_0^1u_j(x)+a_0^2u_{j+1}(x),p_0(x)=b_0^1u_j(x)+b_0^2u_{j+1}(x)$. Since $\Delta_\xi=\varepsilon^2\Delta_x$ then

$$\varepsilon^{-2}\begin{cases}\Delta v_0(\xi)=0, & \text{in } \mathbb{R}^3\backslash B_1\\ v_0(\xi)|_{\partial B_1}=0\\ \lim\limits_{|\xi|\to\infty} v_0(\xi)=0\end{cases}\qquad \varepsilon^{-2}\begin{cases}\Delta q_0(\xi)=0, & \text{in } \mathbb{R}^3\backslash B_1\\ q_0(\xi)|_{\partial B_1}=0\\ \lim\limits_{|\xi|\to\infty} q_0(\xi)=0.\end{cases}$$

Therefore $v_0(\xi)=q_0(\xi)=0$ and

(12) $$\Delta u_1(x)+\lambda_j u_1(x)+\lambda_j^{<1>}u_0(x)=0,\quad \text{in } \Omega$$

(13) $$u_1(X)|_{\partial\Omega}=0$$

(14) $$\Delta p_1(x) + \lambda_j p_1(x) + \lambda_{j+1}^{<1>} p_0(x) = 0, \quad \text{in } \Omega$$

(15) $$p_1(x)|_{\partial\Omega} = 0.$$

For the solvability of the problems (12)-(15) we have $\lambda_j^{<1>} = \lambda_{j+1}^{<1>} = 0$. Hence $u_1(x) = a_1^1 u_j(x) + a_1^2 u_{j+1}(x)$ and $p_1(x) = b_1^1 u_j(x) + b_1^2 u_{j+1}(x)$. Assume that under some conditions the eigenvalues $\lambda_j(\varepsilon) < \lambda_{j+1}(\varepsilon)$ for sufficiently small ε. In the process of finding the asymptotic formula that condition will be clear. If it happens, then we have $u_0 {}^\perp p_0$, i.e. if $u_0 = a_0^1 u_j + a_0^2 u_{j+1}$, so $p_0 = -a_0^2 u_j + a_0^1 u_{j+1}$. Hence one can choose $u_1(x) = c_1 p_0(x)$ and $p_1(x) = d_1 u_0(x)$. Suppose the functions $u_1(x)$ and $p_1(x)$ are found. Then the functions $v_1(\xi), q_1(\xi)$ satisfy :

$$\varepsilon^{-1}\begin{cases} \Delta v_1(\xi) = 0, \quad \text{in } \mathbb{R}^3\backslash B_1 \\ v_1(\xi)|_{\partial B_1} = -(\text{grad } u_0(0), \xi) =: -A_1(\theta) \\ \lim\limits_{|\xi|\to\infty} v_1(\xi) = 0 \end{cases}$$

$$\varepsilon^{-1}\begin{cases} \Delta q_1(\xi) = 0, \quad \text{in } \mathbb{R}^3\backslash B_1 \\ q_1(\xi)|_{\partial B_1} = -(\text{grad } p_0(0), \xi) =: -A_2(\theta) \\ \lim\limits_{|\xi|\to\infty} q_1(\xi) = 0. \end{cases}$$

If $v_1(\xi), q_1(\xi)$ are found we can find $u_2(x)$ and $p_2(x)$ from

(16) $$\Delta u_2(x) + \lambda_j u_2(x) + \lambda_j^{<2>} u_0(x) = 0, \quad \text{in } \Omega$$

(17) $$u_2(x)|_{\partial\Omega} = 0$$

(18) $$\Delta p_2(x) + \lambda_j p_2(x) + \lambda_{j+1}^{<2>} p_0(x) = 0, \quad \text{in } \Omega$$

(19) $$p_2(x)|_{\partial\Omega} = 0.$$

From the solvability of (16)-(19) we deduce that $\lambda_j^{<2>} = \lambda_{j+1}^{<2>} = 0$. Therefore one can choose $u_2(x) = c_2 p_0(x), p_2(x) = d_2 u_0(x)$. The functions $v_2(\xi)$ and $q_2(\xi)$ satisfy :

$$\begin{cases} \Delta v_2(\xi) = 0, \quad \text{in } \mathbb{R}^3\backslash B_1 \\ v_2(\xi)|_{\partial B_1} = -c_1 A_2(\theta) - B_1(\theta), \\ \lim\limits_{|\xi|\to\infty} v_2(\xi) = 0 \end{cases} \qquad \begin{cases} \Delta q_2(\xi) = 0, \quad \text{in } \mathbb{R}^3\backslash B_1 \\ q_2(\xi)|_{\partial B_1} = -d_1 A_1(\theta) - B_2(\theta) \\ \lim\limits_{|\xi|\to\infty} q_2(\xi) = 0, \end{cases}$$

where $B_1(\theta) = \sum\limits_{ik=1}^{3} \frac{\partial^2 u_0(0)}{\partial x_i \partial x_k} \xi_i \xi_k \Big|_{\partial B_1}$, $B_2(\theta) = \sum\limits_{ik=1}^{3} \frac{\partial^2 p_0(0)}{\partial x_i \partial x_k} \xi_i \xi_k \Big|_{\partial B_1}$.

Note that $\Delta_{s^2} B_1(\theta) + 6B_1(\theta) = 0$ and $\Delta_{s^2} B_2(\theta) + 6B_2(\theta) = 0$. It follows that

$$v_2(\xi) = -c_1 A_2(\theta)|\xi|^{-2} - B_1(\theta)|\xi|^{-3}, q_2(\xi) = -d_1 A_1(\theta)|\xi|^{-2} - B_2(\theta)|\xi|^{-3}.$$

Then $u_3(x), p_3(x)$ satisfy :

$$\Delta\{u_3 - A_1(\theta)|x|^{-2}\} + \lambda_j\{u_3 - A_1(\theta)|x|^{-2}\} + \lambda_j^{<3>} u_0 = 0 \tag{20}$$

$$\{u_3 - A_1(\theta)|x|^{-2}\}\big|_{\partial\Omega} = 0 \tag{21}$$

$$\Delta\{p_3 - A_2(\theta)|x|^{-2}\} + \lambda_j\{p_3 - A_2(\theta)|x|^{-2}\} + \lambda_{j+1}^{(3)} p_0 = 0 \tag{22}$$

$$\{p_3 - A_2(\theta)|x|^{-2}\}\big|_{\partial\Omega} = 0. \tag{23}$$

For solvability of (20)-(23) we have

$$\lambda_j^{<3>} = 3 \int\limits_{\partial B_1} A_1^2(\theta) d\theta,\ \lambda_{j+1}^{<3>} = 3 \int\limits_{\partial B_1} A_2^2(\theta) d\theta.$$

Note that $A_1(\theta) = a_0^1 u_j^{<1>}(\theta) + a_0^2 u_{j+1}^{<1>}(\theta)$ and

$$A_2(\theta) = -a_0^2 u_j^{<1>}(\theta) + a_0^1 u_{j+1}^{<1>}(\theta).$$

Multiplying (20) by $u_j(x), u_{j+1}(x)$ and integrating over Ω_ε then turning $\varepsilon \to 0$ one obtain :

$$(M - \frac{\lambda_j^{<3>}}{3} I) \begin{pmatrix} a_0^1 \\ a_0^2 \end{pmatrix} = 0 (\text{see the definition of } M \text{ in the introduction}).$$

It means that $3^{-1}\lambda_j^{<3>}$ is the eigenvalue of the matrix M and (a_0^1, a_0^2) is its eigenvector. By analogy we can prove $3^{-1}\lambda_{j+1}^{<3>}$ is also eigenvalue of M. Therefore if M has two different eigenvalues then $\lambda_j^{<3>}, \lambda_{j+1}^{<3>}$ and (a_0^1, a_0^2) are defined uniquely. So we found $\lambda_j^{<3>}, \lambda_{j+1}^{<3>}, u_0(x) p_0(x), v_0(\xi)$,

$q_0(\xi), v_1(\xi), q_1(\xi)$. Continuing this procedure we can find $\lambda_j^{<4>}, \lambda_{j+1}^{<4>}, u_1(x)$, $p_1(x), v_2(\xi), q_2(\xi)$.

A step of induction : Assume that $\lambda_j^{<n+3>}, \lambda_{j+1}^{<n+3>}, u_n(x), p_n(x)$, $v_{n+1}(\xi), q_{n+1}(\xi)$ are defined. We show how to find the functions $\lambda_j^{<n+4>}$, $\lambda_{j+1}^{<n+4>}, u_{n+1}(x), p_{n+1}(x), v_{n+2}(\xi), q_{n+2}(\xi)$. In previous steps we have already known the equations for $u_{n+1}(x), p_{n+1}(x)$ and found the condition for their solvability. However, the solutions are defined non-uniquely. Writing once again these equations :

$$\begin{cases} \Delta u_{n+1} + \sum\limits_{i=0}^{n+1} \lambda_j^{<i>} u_{n+1-i} + \sum\limits_{i=0}^{n} \lambda_j^{<i>} v_{n-i}^{<1>}(\theta)|x|^{-1} + \\ \qquad\qquad \sum\limits_{i=0}^{n-1} \lambda_j^{<i>} v_{n-1-i}^{<2>}(\theta)|x|^{-2} = 0 \\ \{u_{n+1} + \sum\limits_{i=1}^{n+1} v_{n+1-i}^{<i>}(\theta)|x|^{-i}\}\Big|_{\partial\Omega} = 0 \end{cases}$$

$$\begin{cases} \Delta p_{n+1} + \sum\limits_{i=0}^{n+1} \lambda_{j+1}^{<i>} p_{n+1-i} + \sum\limits_{i=0}^{n} \lambda_{j+1}^{<i>} q_{n-i}^{<1>}(\theta)|x|^{-1} + \\ \qquad\qquad \sum\limits_{i=0}^{n-1} \lambda_{j+1}^{<i>} q_{n-1-i}^{<2>}(\theta)|x|^{-2} = 0 \\ \{p_{n+1} + \sum\limits_{i=1}^{n+1} q_{n+1-i}^{<i>}(\theta)|x|^{-i}\}\Big|_{\partial\Omega} = 0. \end{cases}$$

Suppose that $U_{n+1}(x), P_{n+1}(x)$ are the solutions of the above problem such that

$$\int\limits_{\Omega} U_{n+1} u_0 dx = \int\limits_{\Omega} U_{n+1} p_0 dx = \int\limits_{\Omega} P_{n+1} u_0 dx = \int\limits_{\Omega} P_{n+1} p_0 dx = 0$$

A general solution must be found in a form :

$$u_{n+1} = U_{n+1} + c_{n+1} p_0, \;\; p_{n+1} = P_{n+1} + d_{n+1} u_0.$$

By analogy we should find $u_{n+2}(x), p_{n+2}(x), u_{n+3}(x), p_{n+3}(x)$ in form :

$$u_{n+2} = U_{n+2} + c_{n+2} p_0, \quad p_{n+2} = P_{n+2} + d_{n+2} u_0.$$

$$u_{n+3} = U_{n+3} + c_{n+3} p_0, \quad p_{n+3} = P_{n+3} + d_{n+3} u_0.$$

Then $v_{n+2}(\xi), q_{n+2}(\xi)$ satisfy :

$$\begin{cases} \Delta v_{n+2}(\xi) + \sum\limits_{i=0}^{n} \lambda_j^{<i>} \tilde{v}_{n-i}^{<2>}(\xi) = 0 \\ \{v_{n+2}(\theta) + u_0^{<n+2>}(\theta) + \cdots + u_{n+2}^{<0>}(\theta)\}\big|_{\partial B_1} = 0 \\ \lim\limits_{|\xi|\to\infty} v_{n+2}(\xi) = 0 \end{cases}$$

$$\begin{cases} \Delta q_{n+2}(\xi) + \sum\limits_{i=0}^{n} \lambda_{j+1}^{<i>} \tilde{q}_{n-i}^{<2>}(\xi) = 0 \\ \{q_{n+2}(\theta) + p_0^{<n+2>}(\theta) + \cdots + p_{n+2}^{<0>}(\theta)\}\big|_{\partial B_1} = 0 \\ \lim\limits_{|\xi|\to\infty} q_{n+2}(\xi) = 0. \end{cases}$$

Therefore $v_{n+2} = V_{n+2} - c_{n+1}A_2(\theta)|\xi|^{-2}, q_{n+2} = Q_{n+2} - d_{n+1}A_1(\theta)|\xi|^{-2}$. We denote by $V_{n+2}(\xi)$ and $Q_{n+2}(\xi)$ the solutions of the following problems :

$$\begin{cases} \Delta V_{n+2}(\xi) + \sum\limits_{i=0}^{n} \lambda_j^{<i>} \tilde{v}_{n-i}^{<2>}(\xi) = 0 \\ \{V_{n+2}(\theta) + E_{n+2}(\theta)\}|_{\partial B_1} = 0 \\ \lim\limits_{|\xi|\to\infty} V_{n+2}(\xi) = 0 \end{cases}$$

$$\begin{cases} \Delta Q_{n+2}(\xi) + \sum\limits_{i=0}^{n} \lambda_{j+1}^{<i>} \tilde{q}_{n-i}^{<2>}(\xi) = 0 \\ \{Q_{n+2}(\theta) + F_{n+2}(\theta)\}|_{\partial B_1} = 0 \\ \lim\limits_{|\xi|\to\infty} Q_{n+2}(\xi) = 0, \end{cases}$$

where the functions

$$E_{n+2}(\theta) = u_0^{<n+2>}(\theta) + \cdots + u_n^{<2>}(\theta) + U_{n+1}^{<1>}(\theta) + U_{n+2}^{<0>}(\theta)$$

$$F_{n+2}(\theta) = p_0^{<n+2>}(\theta) + \cdots + p_n^{<2>}(\theta) + P_{n+1}^{<1>}(\theta) + P_{n+2}^{<0>}(\theta)$$

are already defined from previous steps.

By analogy we should find $v_{n+3}(\xi), q_{n+3}(\xi)$ in a form

$$v_{n+3}(\xi) = V_{n+3}(\xi) - c_{n+2}A_2(\theta)|\xi|^{-2} - c_{n+1}B_2(\theta)|\xi|^{-3},$$

$$q_{n+3}(\xi) = Q_{n+3}(\xi) - d_{n+2}A_1(\theta)|\xi|^{-2} - d_{n+1}B_1(\theta)|\xi|^{-3}.$$

Finally we write equations for $u_{n+4}(x), p_{n+4}(x)$:

$$\begin{cases} \Delta u_{n+4} + \sum\limits_{i=0}^{n+4} \lambda_j^{<i>} u_{n+4-i} + \sum\limits_{i=0}^{n+3} \lambda_j^{<i>} v_{n-i+3}^{<1>}(\theta)|x|^{-1} + \\ \qquad\qquad\qquad\qquad \sum\limits_{i=0}^{n+2} \lambda_j^{<i>} v_{n-i+2}^{<2>}(\theta)|x|^{-2} = 0 \\ \{u_{n+4} + \sum\limits_{i=1}^{n+4} v_{n+4-i}^{<i>}(\theta)|x|^{-i}\}\Big|_{\partial\Omega} = 0 \end{cases}$$

$$\begin{cases} \Delta p_{n+4} + \sum\limits_{i=0}^{n+4} \lambda_{j+1}^{<i>} p_{n+4-i} + \sum\limits_{i=0}^{n+3} \lambda_{j+1}^{<i>} q_{n-i+3}^{<1>}(\theta)|x|^{-1} + \\ \qquad\qquad\qquad\qquad \sum\limits_{i=0}^{n+2} \lambda_{j+1}^{<i>} q_{n-i+2}^{<2>}(\theta)|x|^{-2} = 0 \\ \{p_{n+4} + \sum\limits_{i=1}^{n+4} q_{n+4-i}^{<i>}(\theta)|x|^{-i}\}\Big|_{\partial\Omega} = 0. \end{cases}$$

Note that $\lambda_j^{<0>} = \lambda_{j+1}^{<0>} = \lambda_j, \lambda_j^{<1>} = \lambda_{j+1}^{<1>} = \lambda_j^{<2>} = \lambda_{j+1}^{<2>} = 0$. So we have :

$$\begin{aligned} &\Delta\{u_{n+4}(x) - c_{n+1}A_2(\theta)|x|^{-2}\} + \lambda_j\{u_{n+4}(x) - c_{n+1}A_2(\theta)|x|^{-2}\} \\ &\quad + \lambda_j^{<3>}\{U_{n+1}(x) + c_{n+1}p_0(x)\} + \lambda_j^{<n+4>} u_0(x) = G_n(x) \end{aligned} \tag{24}$$

$$\{u_{n+4}(x) - c_{n+1}A_2(\theta)|x|^{-2}\}\big|_{\partial\Omega} = H_n(x) \tag{25}$$

where the functions $G_n(x), H_n(x)$ are defined from previous steps. Multiplying (24) by $u_0(x), p_0(x)$ and integrating over Ω_ε as $\varepsilon \to 0$ we obtain immediately c_{n+1} and $\lambda_j^{<n+4>}$. By analogy one can find d_{n+1} and $\lambda_{j+1}^{<n+4>}$. Our procedure is ended.

B. The case of a triple eigenvalue

We are interested only in the case of a bifurcation, i.e. $\lambda_j(\varepsilon) \le \lambda_{j+1}(\varepsilon) \le \lambda_{j+2}(\varepsilon)$ when ε is sufficiently small.

Suppose :

$$\lambda_{j+k}(\varepsilon) = \lambda_j + \lambda_{j+k}^{<1>}\varepsilon^1 + \lambda_{j+k}^{<2>}\varepsilon^2 + \cdots + \lambda_{j+k}^{<M>}\varepsilon^M + 0(\varepsilon^{M+1}) \quad (k = 0, 1, 2)$$

and

$$
\begin{aligned}
u_j(x,\varepsilon) &= [(u_0+v_0)+\varepsilon(u_1+v_1)+\varepsilon^2(u_2+v_2)+\dots] \\
u_{j+1}(x,\varepsilon) &= [(p_0+q_0)+\varepsilon(p_1+q_1)+\varepsilon^2(p_2+q_2)+\dots] \\
u_{j+2}(x,\varepsilon) &= [(r_0+s_0)+\varepsilon(r_1+s_1)+\varepsilon^2(r_2+s_2)+\dots].
\end{aligned}
$$

Putting $u_j(x,\varepsilon), u_{j+1}(x,\varepsilon), u_{j+2}(x,\varepsilon), \lambda_j(\varepsilon), \lambda_{j+1}(\varepsilon), \lambda_{j+2}(\varepsilon)$ into (1), (2) and comparing the coefficient in the identical order of ε we obtain the quations for $u_0(x), p_0(x), r_0(x)$ as the equations for $u_0(x), p_0(x)$ in the case of double eigenvalues.

Therefore :

$$
u_0(x) = a_0^1 u_j^*(x) + a_0^2 u_{j+1}^*(x) + a_0^3 u_{j+2}^*(x)
$$

$$
\begin{aligned}
p_0(x) &= b_0^1 u_j^*(x) + b_0^2 u_{j+1}^*(x) + b_0^3 u_{j+2}^*(x) \\
r_0(x) &= c_0^1 u_j^*(x) + c_0^2 u_{j+1}^*(x) + c_0^3 u_{j+2}^*(x)
\end{aligned}
$$

(see the definition of u_j, u_{j+1}, u_{j+2} in the introduction). Since we are only interested in the case of a bifurcation, it follows that the functions u_0, p_0, r_0 must be orthogonal. Then we have

$$
v_0(\xi) = -u_0(0)|\xi|^{-1}, q_0(\xi) = -p_0(0)|\xi|^{-1}, s_0(\xi) = -r_0(0)|\xi|^{-1}.
$$

Now we write the equations for $u_1(x), p_1(x), r_1(x)$

$$
\begin{cases}
\Delta u_1(x) + \lambda_j u_1(x) + \lambda_j^{<1>} u_0(x) - \lambda_j u_0(0)|x|^{-1} = 0 \quad \text{in } \Omega \\
u_1(x)|_{\partial\Omega} = u_0(0)|x|^{-1}\big|_{\partial\Omega}
\end{cases}
$$

....

From the conditions of their solvability and the conditions $\lambda_j(\varepsilon) < \lambda_{j+1}(\varepsilon) < \lambda_{j+2}(\varepsilon)$ when ε is sufficiently small. We have

$$
\lambda_j^{<1>} = \lambda_{j+1}^{<1>} = 0, \lambda_{j+2}^{<1>} = 4\pi\{u_{j+2}^*(0)\}^2, c_0^1 = c_0^2 = a_0^3 = b_0^3 = 0, c_0^3 = 1.
$$

So the function $r_0(x)$ is defined. Suppose provisionally the function $u_0(x)$, $p_0(x)$ are also defined. We show how to find $\lambda_{j+2}^{<2>}, s_1(\xi), r_1(x)$. Note the problems for $r_1(x)$ are sovable. However the solution is defined non-uniquely. Suppose that $R_1(x)$ is a solution such that $\int\limits_\Omega R_1 u_0 dx = \int\limits_\Omega R_1 p_0 dx = \int\limits_\Omega R_1 r_0 dx = 0$. A general solution $r_1(x)$ may be written as follows : $r_1(x) = R_1(x) + a_1 u_0(x) + b_1 p_0(x)$. Assume that a_1, b_1 are found. Then $s_1(\xi)$ satisfies :

$$\begin{cases} \Delta s_1(\xi) = 0, \quad \text{in } \mathbb{R}^3 \backslash B_1 \\ s_1(\xi)|_{\partial B_1} = -R_1(0) - r_0^{<1>}(\theta) \\ \lim\limits_{|\xi| \to \infty} s_1(\xi) = 0. \end{cases}$$

Therefore $s_1(\xi) = -R_1(0)|\xi|^{-1} - r_0^{<1>}(\theta)|\xi|^{-2}$. We obtain the equations for $r_2(x)$

$$\Delta r_2 - \lambda_j R_1(0)|x|^{-1} + \lambda_j r_2 + \lambda_{j+2}^{<1>}(R_1 + a_1 u_0 + b_1 p_0) + \lambda_{j+2}^{<2>} r_0 = 0 \tag{26}$$

$$\{r_2(x) - R_1(0)|x|^{-1}\}\big|_{\partial\Omega} = 0. \tag{27}$$

Multiplying (26) by $u_0(x), p_0(x), r_0(x)$ and intergrating over Ω_ε when $\varepsilon \to 0$ one deduce that $\lambda_{j+2}^{<2>} = 0, a_1 = b_1 = 0$. So we found $r_1(x), s_1(\xi)$, $\lambda_{j+2}^{<2>}$. By induction, as in the case of double eigenvalues, we can find all $r_n(x), s_n(\xi), \lambda_{j+2}^{<n+2>}$. Now, under some conditions, we show how to find $u_0(x)$ and $p_0(x)$. In the first step we had :

$$u_0(x) = a_0^1 u_j^*(x) + a_0^2 u_{j+1}^*(x), p_0(x) = b_0^1 u_j^*(x) + b_0^2 u_{j+1}^*(x), \\ \lambda_j^{<1>} = \lambda_{j+1}^{<1>} = 0.$$

Since : $u_0(0) = p_0(0) = 0$ it follows $v_0(\xi) = q_0(\xi) = 0$. From the equations for $u_1(x), p_1(x)$ we can find them in a form :

$$u_1(x) = c_1 p_0(x) + d_1 r_0(x), p_1(x) = e_1 u_0(x) + f_1 r_0(x).$$

Suppose that c_1, d_1, e_1, f_1 are known. Then, from the equations for $v_1(\xi)$, $q_1(\xi)$ we obtain immediately :

$$v_1(\xi) = -d_1|\xi|^{-1} - u_0^{<1>}(\theta)|\xi|^{-2}, q_1(\xi) = -f_1|\xi|^{-1} - p_0^{<1>}(\theta)|\xi|^{-2}.$$

Therefore the functions $p_2(x), u_2(x)$ satisfy :

$$\begin{cases} \Delta(u_2 - d_1|x|^{-1}) + \lambda_j(u_2 - d_1|x|^{-1}) + \lambda_j^{<2>} u_0(x) = 0 \\ (u_2 - d_1|x|^{-1})\big|_{\partial\Omega} = 0 \end{cases}$$

$$\begin{cases} \Delta(p_2 - f_1|x|^{-1}) + \lambda_j(p_2 - f_1|x|^{-1}) + \lambda_{j+1}^{<2>} p_0(x) = 0 \\ (p_2 - f_1|x|^{-1})\big|_{\partial\Omega} = 0. \end{cases}$$

From the conditions for solvability of this equation we deduce :

$$\lambda_j^{<2>} = \lambda_{j+1}^{<2>} = 0, \quad d_1 = f_1 = 0,$$

$$p_2(x) = P_2(x) + e_2 u_0(x) + f_2 r_0(x), \quad u_2(x) = U_2(x) + c_2 p_0(x) + d_2 r_0(x),$$

where $P_2(x), U_2(x)$ denote the solutions such that :

$$\int\limits_\Omega U_2 u_0 dx = \int\limits_\Omega U_2 p_0 dx = \int\limits_\Omega U_2 r_0 dx = \int\limits_\Omega P_2 u_0 dx = \int\limits_\Omega P_2 p_0 dx = \int\limits_\Omega P_2 r_0 dx = 0.$$

From the equations for $v_2(\xi), q_2(\xi)$ we have :

$$\begin{aligned} v_2(\xi) &= -U_2(0)|\xi|^{-1} - d_2 r_0(0)|\xi|^{-1} - c_1 p_0^{<1>}(\theta)|\xi|^{-2} - u_0^{<2>}(\theta)|\xi|^{-3} \\ q_2(\xi) &= -p_2(0)|\xi|^{-1} - f_2 r_0(0)|\xi|^{-1} - e_1 u_0^{<1>}(\theta)|\xi|^{-2} - p_0^{<2>}(\theta)|\xi|^{-3}. \end{aligned}$$

Finally we write the equations for $u_3(x), p_3(x)$

$$\begin{cases} \Delta u_3 + \lambda_j u_3 + \lambda_j^{<3>} u_0 - \lambda_j(U_2(0)|x|^{-1} + d_2 r_0(0)|x|^{-1} + \\ \qquad\qquad u_0^{<1>}(\theta)|x|^{-2} = 0 \\ (u_3 - U_2(0)|x|^{-1} - d_2 r_0(0)|x|^{-1} - u_0^{<1>}(\theta)|x|^{-2})\big|_{\partial\Omega} = 0 \end{cases}$$

$$\begin{cases} \Delta p_3 + \lambda_j p_3 + \lambda_{j+1}^{<3>} p_0 - \lambda_j(P_2(0)|x|^{-1} + f_2 r_0(0)|x|^{-1} + \\ \qquad\qquad p_0^{<1>}(\theta)|x|^{-2}) = 0 \\ (p_3 - P_2(0)|x|^{-1} - f_2 r_0(0)|x|^{-1} - p_0^{<1>}(\theta)|x|^{-2})\big|_{\partial\Omega} = 0. \end{cases}$$

From these conditions we have :

$$\lambda_j^{<3>} = 3 \int\limits_{\partial B_1} |u_0^{<1>}(\theta)|^2 d\theta, \quad \lambda_{j+1}^{<3>} = 3 \int\limits_{\partial B_1} |P_0^{<1>}(\theta)|^2 d\theta$$
$$d_2 = [r_0(0)]^{-1} U_2(0), \quad f_2 = [r_0(0)]^{-1} P_2(0).$$

As in the case of double eigenvalues we conclude that $3^{-1}\lambda_j^{<3>}$ and $3^{-1}\lambda_{j+1}^{<3>}$ are the eigenvalues of the matrix M^* (see the definition in the introduction) and the vector (a_0^1, a_0^2) is its eigenvector. So we found $\lambda_j^{<3>}, \lambda_{j+1}^{<3>}$, $u_0(x)$, $p_0(x), v_0(\xi), q_0(\xi), v_1(\xi), q_1(\xi)$.

A step of induction : Suppose that $\lambda_j^{<n+3>}, \lambda_{j+1}^{<n+3>}, u_n(x), p_n(x)$, $v_{n+1}(\xi)$, $q_{n+1}(\xi)$ are found. We shall find $\lambda_j^{<n+4>}, \lambda_{j+1}^{<n+4>}, u_{n+1}(x)$, $p_{n+1}(x), v_{n+2}(\xi), q_{n+2}(\xi)$ as follows. In previous steps we have known the equations for $u_{n+1}(x)$, $p_{n+1}(x)$ and found the conditions for their solvability. However, the solutions are defined non-uniquely. Assume that $U_{n+1}(x), P_{n+1}(x)$ are the solutions such that

$$\int\limits_{\Omega} U_{n+1} u_0 dx = \int\limits_{\Omega} U_{n+1} p_0 dx = \int\limits_{\Omega} U_{n+1} r_0 dx = 0$$

$$\int\limits_{\Omega} P_{n+1} u_0 dx = \int\limits_{\Omega} P_{n+1} p_0 dx = \int\limits_{\Omega} P_{n+1} r_0 dx = 0.$$

The functions $u_{n+1}(x), p_{n+1}(x)$ may be found in a form :

$$u_{n+1} = c_{n+1} p_0 + d_{n+1} r_0 + U_{n+1}, \; p_{n+1} = e_{n+1} u_0 + f_{n+1} r_0 + P_{n+1}.$$

By analogy we have :

$$u_{n+2} = c_{n+2} p_0 + d_{n+2} r_0 + U_{n+2}, \quad p_{n+2} = e_{n+2} u_0 + f_{n+2} r_0 + P_{n+2}$$

$$u_{n+3} = c_{n+3} p_0 + d_{n+3} r_0 + U_{n+3}, \quad p_{n+3} = e_{n+3} u_0 + f_{n+3} r_0 + P_{n+3}.$$

From the equations for $v_{n+2}(\xi), q_{n+2}(\xi)$ we claim that :

$$v_{n+2}(\xi) = V_{n+2}(\xi) - d_{n+1} A_3(\theta)|\xi|^{-2} - d_{n+2} r_0(0)|\xi|^{-1} - c_{n+1} A_2(\theta)|\xi|^{-2}$$

$$q_{n+2}(\xi) = Q_{n+2}(\xi) - f_{n+1}A_3(\theta)|\xi|^{-2} - f_{n+2}r_0(0)|\xi|^{-1} - e_{n+1}A_1(\theta)|\xi|^{-2},$$

where $A_1(\theta) = u_0^{<1>}(\theta), A_2(\theta) = p_0^{<1>}(\theta), A_3(\theta) = r_0^{<1>}(\theta)$ and $V_{n+2}(\xi)$, $Q_{n+2}(\xi)$ are defined by the equations as in the case of double eigenvalues.

By analogy we have

$$v_{n+3}(\xi) = V_{n+3}(\xi) - [d_{n+1}\{B_3(\theta) - 6^{-1}\lambda_j r_0(0)\} + c_{n+1}B_2(\theta)]|\xi|^{-3}$$

$$-\{c_{n+2}A_2(\theta) + d_{n+2}A_3(\theta)\}|\xi|^{-2} - \{d_{n+3}r_0(0) + 6^{-1}d_{n+1}\lambda_j r_0(0)\}|\xi|^{-1}$$

$$q_{n+3}(\xi) = Q_{n+3}(\xi) - [f_{n+1}\{B_3(\theta) - 6^{-1}\lambda_j r_0(0)\} + e_{n+1}B_1(\theta)]|\xi|^{-3}$$

$$-\{e_{n+2}A_1(\theta) + f_{n+2}A_3(\theta)\}|\xi|^{-2} - \{f_{n+3}r_0(0) + 6^{-1}f_{n+1}\lambda_j r_0(0)\}|\xi|^{-1},$$

where the functions $B_1(\theta) = u_0^{<2>}(\theta), B_2(\theta) = p_0^{<2>}(\theta), B_3(\theta) = r_0^{<2>}(\theta)$, $V_{n+3}(\xi), Q_{n+3}(\xi), d_{n+1}, d_{n+2}, f_{n+1}, f_{n+2}$ are defined.

Finally, we write the equations for $u_{n+4}(x), p_{n+4}(x)$:

$$\Delta\bar{u}_{n+4}(x) + \lambda_j\bar{u}_{n+4}(x) + \lambda_j^{<n+4>}u_0 + \lambda_j^{<3>}(U_{n+1} + c_{n+1}p_0 + d_{n+1}r_0) = G_{n+4}(x) \tag{28}$$

$$\bar{u}_{n+4}|_{\partial\Omega} = H_{n+4}(x) \tag{29}$$

$$\Delta\bar{p}_{n+4}(x) + \lambda_j\bar{p}_{n+4}(x) + \lambda_{j+1}^{<n+4>}p_0 + \lambda_{j+1}^{<3>}(P_{n+1} + e_{n+1}u_0 + f_{n+1}r_0) = I_{n+4}(x) \tag{30}$$

$$\bar{p}_{n+4}|_{\partial\Omega} = K_{n+4}(x) \tag{31}$$

where $\bar{u}_{n+4}(x) := (u_{n+4} - d_{n+3}r_0(0)|x|^{-1} - c_{n+1}A_2(\theta)|x|^{-2})$
and $\bar{p}_{n+4}(x) := (p_{n+4} - f_{n+3}r_0(0)|x|^{-1} - e_{n+1}A_1(\theta)|x|^{-2})$.

From the coditions for solvability of (28) - (31) we have :

$$c_{n+1} = (\lambda_j^{<3>} - \lambda_{j+1}^{<3>})^{-1}[\int\limits_{\Omega} G_{n+4} p_0 dx + \int\limits_{\partial\Omega} H_{n+4} \frac{\partial p_0}{\partial n} ds]$$

$$e_{n+1} = (\lambda_j^{<3>} - \lambda_{j+1}^{<3>})^{-1}[\int\limits_{\Omega} I_{n+4} u_0 dx + \int\limits_{\partial\Omega} K_{n+4} \frac{\partial u_0}{\partial n} ds]$$

$$\lambda_j^{<n+4>} - [\int\limits_{\Omega} G_{n+4} p_0 dx + \int\limits_{\partial\Omega} H_{n+4} \frac{\partial p_0}{\partial n} ds - \lambda_j^{<3>} c_{n+1}]$$

$$\lambda_{j+1}^{<n+4>} = -[\int\limits_{\Omega} I_{n+4} u_0 dx + \int\limits_{\partial\Omega} K_{n+4} \frac{\partial u_0}{\partial n} ds - \lambda_{j+1}^{<3>} e_{n+1}]$$

$$d_{n+3} = (4\pi r_0^2(0))^{-1}[\int\limits_{\Omega} G_{n+4} r_0 dx + \int\limits_{\partial\Omega} H_{n+4} \frac{\partial r_0}{\partial n} ds - \lambda_j^{<3>} d_{n+1}]$$

$$f_{n+3} = (4\pi r_0^2(0))^{-1}[\int\limits_{\Omega} I_{n+4} r_0 dx + \int\limits_{\partial\Omega} K_{n+4} \frac{\partial r_0}{\partial n} ds - \lambda_{j+1}^{<3>} f_{n+1}]$$

Our procedure is ended.

3. Proof

We shall prove our results only in the case of double eigenvalues. The case of triple eigenvalues may be proved similarly. Suppose

$$\alpha_N(x,\varepsilon) = \sum_{i=0}^{N} \varepsilon^i (u_i(x) + v_i(x\varepsilon^{-1})), \beta_N(x,\varepsilon) = \sum_{i=0}^{N} \varepsilon^i (p_i(x) + q_i(x\varepsilon^{-1}))$$

$$\lambda_j^{(N)}(\varepsilon) = \sum_{i=0}^{N} \lambda_j^{(i)} \varepsilon^i, \lambda_{j+1}^{(N)}(\varepsilon) = \sum_{i=0}^{N} \lambda_{i+1}^{(i)} \varepsilon^i.$$

We have

$$\Delta\alpha_N(x,\varepsilon)+\lambda_j^{<N>}(\varepsilon)\alpha_N(x,\varepsilon)=\sum_{i=0}^{N}\varepsilon^i[\Delta u_i+\sum_{p=0}^{i-1}\lambda_j^{<p>}u_{i-p-1}+\sum_{p=0}^{i-2}\lambda_j^{<p>}$$

$$|x|^{-1}v_{i-p-2}^{<1>}(\theta)$$

$$+\sum_{p=0}^{i-3}\lambda_j^{<p>}|x|^{-2}v_{i-p-3}^{<2>}(\theta)]+\sum_{i=0}^{N}\varepsilon^{i-2}[\Delta_\xi v_i(\xi)+\sum_{p=0}^{i-3}\lambda_j^{<p>}\tilde{v}_{i-p-3}^{<2>}(\xi)]$$

$$+\varepsilon\sum_{i=0}^{N-1}\varepsilon^i\lambda_j^{<i>}[\sum_{p=N-i}^{N}\varepsilon^p u_p+\sum_{p=N-i-2}^{N}\varepsilon^p\tilde{v}_p^{<2>}(x\varepsilon^{-1})$$

$$+\sum_{p=N-i-1}^{N}\varepsilon^{p+1}|x|^{-1}v_p^{<1>}(\theta)+\sum_{p=N-i-2}^{N}\varepsilon^{p+2}|x|^{-2}v_p^{<2>}(\theta)].$$

Obviously

$$|\Delta\alpha_N(x,\varepsilon)+\lambda_j^{<N>}(\varepsilon)\alpha_N(x,\varepsilon)|=0(\varepsilon^{N+1}|x|^{-2})=0(\varepsilon^{N-1})\quad(x\in\Omega_\varepsilon)$$
$$\alpha_N|_{\partial\Omega_\varepsilon}=0(\varepsilon^{N+1}).$$

By analogy we can see :

$$|\Delta\beta_N(x,\varepsilon)+\lambda_{j+1}^{<N>}(\varepsilon)\beta_N(x,\varepsilon)|=0(\varepsilon^{N+1}|x|^{-2})=0(\varepsilon^{N-1})\quad(x\in\Omega_\varepsilon)$$
$$\beta_N|_{\partial\Omega_\varepsilon}=0(\varepsilon^{N+1}).$$

Suppose that $\alpha_N^*(x,\varepsilon) = g_N(\varepsilon)[\alpha_N(x,\varepsilon) - \Gamma_1(x)\sum_{i=0}^{N}\varepsilon^i\tilde{v}_i^{(N-i)}(x\varepsilon^{-1}) - \Gamma_2(x\varepsilon^{-1})\sum_{i=0}^{N}\varepsilon^i\tilde{u}_i^{(N-i)}(x)]$,

$$\beta_N^*(x,\varepsilon)=k_N(\varepsilon)[\beta_N(x,\varepsilon)-\Gamma_1(x)\sum_{i=0}^{N}\varepsilon^i\tilde{v}_i^{(N-i)}(x\varepsilon^{-1})$$
$$-\Gamma_2(x\varepsilon^{-1})\sum_{i=0}^{N}\varepsilon^i\tilde{u}_i^{(N-i)}(x)]$$

where $\Gamma_1(x) \in C^\infty(\mathbb{R}^3), \Gamma_1(x) \equiv 1$ in a neighborhood of $\partial\Omega$ and $\Gamma_1(x) = 0$ in a neighborhood of $\{0\}$ and $\Gamma_2(x) \in C_0^\infty(\mathbb{R}^3), \Gamma_2(x) \equiv 1$ in a neighborhood of $\bar{B}_1$. The constants $g_N(\varepsilon), k_N(\varepsilon)$ are chosen such that

$$\|\alpha_N^*(x,\varepsilon)\|_{L^2(\Omega_\varepsilon)} = \|\beta_N^*(x,\varepsilon)\|_{L^2(\Omega_\varepsilon)} = 1.$$

It is easy to see

$$\begin{cases} \Delta\alpha_N^*(x,\varepsilon) + \lambda_j^{<N>}(\varepsilon)\alpha_N^*(x,\varepsilon) = L_N(x,\varepsilon) \ \text{ in } \ \Omega_\varepsilon \\ \alpha_N^*(x,\varepsilon)|_{\partial\Omega_\varepsilon} = 0 \end{cases}$$

$$\begin{cases} \Delta\beta_N^*(x,\varepsilon) + \lambda_{j+1}^{<N>}(\varepsilon)\beta_N^*(x,\varepsilon) = M_N(x,\varepsilon) \ \text{ in } \ \Omega_\varepsilon \\ \beta_N^*(x,\varepsilon)|_{\partial\Omega_\varepsilon} = 0. \end{cases}$$

Expand $\alpha_N^*(x,\varepsilon)$ and $\beta_N^*(x,\varepsilon)$ in the series of orthonormal eigenfunctions $u_1(x,\varepsilon), u_2(x,\varepsilon), \ldots$ in Ω_ε one have :

$$\alpha_N^*(x,\varepsilon) = \sum_{i=1}^{\infty} \alpha_i(\varepsilon)u_i(x,\varepsilon) \quad \text{where} \quad \sum_{i=1}^{\infty} \alpha_i^2(\varepsilon) = 1$$

$$\beta_N^*(x,\varepsilon) = \sum_{i=1}^{\infty} \beta_i(\varepsilon)u_i(x,\varepsilon) \quad \text{where} \quad \sum_{i=1}^{\infty} \beta_i^2(\varepsilon) = 1.$$

We claim that

$$\Delta\alpha_N^*(x,\varepsilon) = -\sum_{i=1}^{\infty} \lambda_i(\varepsilon)\alpha_i(\varepsilon)u_i(x,\varepsilon) = -\lambda_j^{<N>}(\varepsilon)\sum_{i=1}^{\infty} \alpha_i(\varepsilon)u_i(x,\varepsilon) + L_N(x,\varepsilon).$$

Obviously $|D^\alpha L_N(x,\varepsilon)|\big|_{\Omega_\varepsilon} = 0(\varepsilon^{N+1}|x|^{-|\alpha|})$.

Therefore $|\lambda_j^{<N>}(\varepsilon) - \lambda_j(\varepsilon)| \sim |\lambda_{j+1}^{<N>}(\varepsilon) - \lambda_{j+1}^{<N>}(\varepsilon)| = 0(\varepsilon^{N-1})$.

Since we have known $\lim\limits_{\varepsilon\to 0} \lambda_j(\varepsilon) = \lambda_j \ \ (j = 1, \ldots \infty)$ it follows that

$$\|\alpha_N^*(x,\varepsilon) - u_j(x,\varepsilon)\|_{L^2(\Omega_\varepsilon)} \sim \|\beta_N^*(x,\varepsilon) - u_{j+1}(x,\varepsilon)\|_{L^2(\Omega_\varepsilon)} = 0(\varepsilon^{N-1}).$$

We have also :

$$\Delta\{\alpha^*_N(x,\varepsilon)-u_j(x,\varepsilon)\}+\lambda_j(\varepsilon)\{\alpha^*_N(x,\varepsilon)-u_j(x,\varepsilon)\}=$$
$$L_N(x,\varepsilon)-\{\lambda_j^{<N>}(\varepsilon)-\lambda_j(\varepsilon)\}\alpha^*_N(x,\varepsilon)\ \text{ in }\ \Omega_\varepsilon,$$
$$|D^\alpha\{\alpha^*_N(x,\varepsilon)-u_j(x,\varepsilon)\}|\big|_{\partial\Omega_\varepsilon}=0(\varepsilon^{N-1-|\alpha|})\ \text{ for }\ |\alpha|\leqslant N-1,$$
$$\Delta\{\beta^*_N(x,\varepsilon)-u_{j+1}(x,\varepsilon)\}+\lambda_{j+1}(\varepsilon)\{\beta^*_N(x,\varepsilon)-u_{j+1}(x,\varepsilon)\}=$$
$$M_N(x,\varepsilon)-\{\lambda_{j+1}^{<N>}(\varepsilon)-\lambda_{j+1}(\varepsilon)\}\beta^*_N(x,\varepsilon)\ \text{ in }\ \Omega_\varepsilon,$$
$$|D^\alpha\{\beta^*_N(x,\varepsilon)-u_{j+1}(x,\varepsilon)\}|\big|_{\partial\Omega_\varepsilon}=0(\varepsilon^{N-1-|\alpha|})\ \text{ for }\ |\alpha|\leqslant N-1.$$

From a priori estimates for elliptic boundary value problems we conclude that :

$$\max_{X\in\Omega_\varepsilon}|D^\alpha\{\alpha^*_N(x,\varepsilon)-u_j(x,\varepsilon)\}|\leq C\varepsilon^{N-1}|x|^{-|\alpha|}$$
$$\max_{X\in\Omega_\varepsilon}|D^\alpha\{\beta^*_N(x,\varepsilon)-u_{j+1}(x,\varepsilon)\}|\leq C\varepsilon^{N-1}|x|^{-|\alpha|}$$

which completes the proof.

4. The final remark

The author of this note think we can study a bifurcations of any eigenvalues by our method under some conditions (for a bifurcation). These conditions are necessary because the bifurcation may be not occured when Ω is the ball (in general, when Ω is a domain with some symmetries).

References

[1] Vladimirov, V. S., *Equations of Mathematical Phisics*, USSR, Moscow (1988).

[2] Ozawa, S., Electrostatic capacity and eigenvalues of the Laplacian, J. Fac. Sci. Univ. Tokyo **30** (1983), 53–62.

[3] Ozawa, S., An asymptotic formula for the eigenvalues of the Laplacian in a three dimensional domain with a small hole, J. Fac. Sci. Univ. Tokyo **30** (1983), 243–257.

[4] Mazia, V. G., Nazarov, S. A. and B. A. Plamenevski, Asymptotic expansions of eigenvalues of boundary value problems for operator Laplace in a domain with small holes, Izvestia. Akad. Nayk. USSR **48** (1984), 347–371.

[5] Tri, M. N., An asymptotic formula of eigenvalues for boundary value problems in a domain with a small hole, Vestnik. Moscow. State Univ. **4** (1987), 17–20.
[6] Michlin, S. G., *Linear partial differential equations*, USSR, Moscow (1977).

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