On hyperkähler manifolds of type A_∞

(A_∞型超ケーラー多様体について)

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On hyperkähler manifolds of type A_∞

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Abstract

We study hyperkähler manifolds of type A_∞, which are noncompact real 4-dimensional manifolds of infinite topological type. Anderson-Kronheimer-LeBrun [1] and Goto [8] have constructed these hyperkähler manifolds whose metrics are complete and depend on the choice of parameters. We observe how the Riemannian metrics and the complex structures which are induced from hyperkähler structures depend on the choice of parameters. In particular, we focus on the asymptotic behavior of these hyperkähler metrics. By taking an appropriate parameter, we show that there exists a complete hyperkähler manifold of type A_∞ whose volume growth is r^α for each $3 < \alpha < 4$.

1 Introduction

A hyperkähler manifold is a Riemannian manifold (X, g) of real dimension $4n$ who has three integrable complex structures I_1, I_2, I_3 with relations $I_1^2 = I_2^2 = I_3^2 = I_1 I_2 I_3 = -1$ and each (g, I_i) satisfies Kähler condition. Then the holonomy group of g is a subgroup of $Sp(n)$ and g is Ricci-flat. If we take $y = (y_1, y_2, y_3) \in S^2 \subset \mathbb{R}^3$ and put $I_y := y_1 I_1 + y_2 I_2 + y_3 I_3$, then I_y is also a complex structure and (g, I_y) is Kähler manifold. Thus all hyperkähler manifolds have complex structures parametrized by the elements of S^2.

There are some systematic constructions of hyperkähler manifolds, and Gibbons-Hawking ansatz is one of them. In this construction, we can construct a 4-dimensional hyperkähler manifold from a finite subset of $\mathbb{R}^3$. It is known that all of ALE spaces of type A_k are constructed by Gibbons-Hawking ansatz. Moreover Anderson, Kronheimer and LeBrun constructed hyperkähler manifolds of type A_∞ from a countably infinite subset of $\mathbb{R}^3$ by Gibbons-Hawking ansatz [1]. The hyperkähler manifolds of type A_∞ are

4-dimensional noncompact complete hyperkähler manifolds of infinite topological type. Here infinite topological type means that the homology groups are infinitely generated. The same metrics were constructed due to Goto in [8] using hyperkähler quotinet method.

In this paper we study the differential geometric properties of hyperkähler manifold of type A_∞. First of all, we compute the period maps of hyperkähler manifold of type A_∞. The authors of [1] showed that the second homology group of each hyperkähler manifold of type A_∞ is infinitely generated. In this paper we study these generators more precisely. We show that each of generators is represented by the holomorphic curve with respect to a certain complex structure, and its volume with respect to the hyperkähler metric gives the minimum value in its homology class. Then we can also compute the period maps of the hyperkähler structure by integrating Kähler forms on the holomorphic curves. For the study of topology and the period maps, we apply the method for studying toric hyperkähler varieties argued in [2][18].

It is difficult to see the difference of local property between a hyperkähler manifolds of type A_∞ and an ALE space of type A_k which are constructed in the same way. As expected, we show that each hyperkähler manifold of type A_∞ can be approximated locally by a sequence of ALE space $\{(X_k, g_k)\}_k$, where (X_k, g_k) is an ALE space of type A_k.

On the other hand, the main purpose of this paper is to show that the asymptotic behavior of the hyperkähler manifolds of type A_∞ is different from that of type A_k. To observe the asymptotic behavior of hyperkähler metric, we study the volume growth. The notion of the volume growth is considered for a Riemannian manifold (X, g). We denote by $V_g(p_0, r)$ the volume of the ball $B_g(p_0, r) \subset X$ of radius r centered at $p_0 \in X$. Then we say that the volume growth of g is $f(r)$ if the condition

$$0 < \liminf_{r\to+\infty} \frac{V_g(p_0, r)}{f(r)} \le \limsup_{r\to+\infty} \frac{V_g(p_0, r)}{f(r)} < +\infty.$$

holds for some $p_0 \in X$. From Bishop-Gromov comparison theorem [4][10], the above condition is independent of $p_0 \in X$ if g has the nonnegative Ricci curvature.

In 4-dimension, for instance, the volume growth of Euclidean space $\mathbb{R}^4$ is r^4 and the volume growth of $\mathbb{R}^3 \times S^1$ with the flat metric is r^3. These are trivial examples and there are also nontrivial examples such as ALE hyperkähler metrics [5][6][19] whose volume growth is r^4, and multi-Taub-NUT metrics [24][20][15] whose volume growth is r^3. Thus there are several known examples of Ricci-flat metrics whose volume growth is r^n where n is integer.

In this paper we will show that the volume growth of the hyperkähler manifolds of type A_∞ is less than r^4 and more than r^3.

Each of the metrics in [1] is constructed from an element of

$$(Im\mathbb{H})_0^{\mathbb{N}} := \{\lambda = (\lambda_n)_{n\in\mathbb{N}} \in (Im\mathbb{H})^{\mathbb{N}};\ \sum_{n\in\mathbb{N}} \frac{1}{1+|\lambda_n|} < +\infty\},$$

where $\mathbb{H}$ is quaternion and $Im\mathbb{H}$ is its imaginary part. We denote by (X_λ, g_λ) the hyperkähler metric of type A_∞ constructed from $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in (Im\mathbb{H})_0^{\mathbb{N}}$. The purpose of this paper is studying the asymptotic behavior of $V_{g_\lambda}(p_0, r)$ for some $p_0 \in X_\lambda$, and observe how the volume growth of g_λ depends on the choice of λ. The main result is described as follows.

Theorem 1.1. *For each $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$ and $p_0 \in X_\lambda$, the function $V_{g_\lambda}(p_0, r)$ satisfies*

$$0 < \liminf_{r\to+\infty} \frac{V_{g_\lambda}(p_0, r)}{r^2\tau_\lambda^{-1}(r^2)} \leq \limsup_{r\to+\infty} \frac{V_{g_\lambda}(p_0, r)}{r^2\tau_\lambda^{-1}(r^2)} < +\infty,$$

where the function $\tau_\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is defined by

$$\tau_\lambda(R) := \sum_{n\in\mathbb{N}} \frac{R^2}{R + |\lambda_n|}$$

for $R \geq 0$. Moreover, we have

$$\lim_{r\to+\infty} \frac{V_{g_\lambda}(p_0, r)}{r^4} = 0, \quad \lim_{r\to+\infty} \frac{V_{g_\lambda}(p_0, r)}{r^3} = +\infty.$$

Applying Theorem 1.1 to some $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$, we can find hyperkähler manifolds whose volume growth is given as follows.

Theorem 1.2. (1) *Take $\alpha \in \mathbb{R}$ arbitrarily to be $3 < \alpha < 4$. Then there is a complete hyperkähler manifold (X_λ, g_λ) which satisfies*

$$0 < \liminf_{r\to+\infty} \frac{V_{g_\lambda}(p_0, r)}{r^\alpha} \leq \limsup_{r\to+\infty} \frac{V_{g_\lambda}(p_0, r)}{r^\alpha} < +\infty.$$

(2) *There is a complete hyperkähler manifold (X_λ, g_λ) which satisfies*

$$\lim_{r\to+\infty} \frac{V_{g_\lambda}(p_0, r)}{r^4} = 0, \quad \lim_{r\to+\infty} \frac{V_{g_\lambda}(p_0, r)}{r^\alpha} = +\infty$$

for any $\alpha < 4$.

Next we denote by $g_\lambda^{(s)}$ the Taub-NUT deformation of g_λ where $s > 0$ is the parameter of deformations. Then the volume growth of $g_\lambda^{(s)}$ is given by the following.

Theorem 1.3. *Let $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$ and $s > 0$. Then the volume growth of hyperkähler metric $g_\lambda^{(s)}$ satisfies*

$$\lim_{r \to +\infty} \frac{V_{g_\lambda^{(s)}}(p_0, r)}{r^3} = \frac{8\pi^2}{3\sqrt{s}}.$$

From Theorem 1.2, we can take a sequence $\{\lambda^{(k)} \in (Im\mathbb{H})_0^{\mathbb{N}}\}_{k \in \mathbb{Z}_{>0}}$ such that the volume growth of $g_{\lambda^{(k)}}$ is $r^{3+\frac{1}{k}}$. Then we consider what will happen if k goes to $+\infty$. Although the limit $\lambda^{(\infty)}$ may exists for some sequence $\{\lambda^{(k)}\}_{k \in \mathbb{Z}_{>0}}$, the sum $\sum_{n \in \mathbb{N}} \frac{1}{1+|\lambda_n^{(\infty)}|}$ always diverges to infinity and we cannot construct the complete hyperkähler metric from $\lambda^{(\infty)}$. Then we take $\rho^{(k)} \in (Im\mathbb{H})_0^{\mathbb{N}}$ such that $\rho_n^{(k)} \to \lambda_n^{(\infty)}$ $(k \to +\infty)$ for each $n \in \mathbb{N}$ with the volume growth of $g_{\rho^{(k)}}$ unchanged. If we consider the "limit" of $\{(X_{\rho^{(k)}}, g_{\rho^{(k)}})\}_{k \in \mathbb{Z}_{>0}}$, we obtain an incomplete hyperkähler metric. Moreover, if we take $\lambda^{(\infty)}$ to satisfy a certain condition, the above incomplete metric is isometric to the universal cover of Ooguri-Vafa metric constructed in [21].

This paper consists of 3 chapters. Chapter I consists of Sections 2-3, in which we construct hyperkähler manifolds of type A_∞ and study their basic geometric properties. In Section 2, we review the construction of hyperkähler manifolds of type A_∞. Although there are two constructions by Gibbons-Hawking ansatz and hyperkähler quotient method, we adopt the latter way along [8] since the argument in Section 4 is based on Goto's construction. Then we obtain the hyperkähler manifold (X_λ, g_λ) as a hyperkähler quotient, and there is a hyperkähler moment map $\mu_\lambda : X_\lambda \to Im\mathbb{H}$ with respect to an S^1-action on X_λ preserving the hyperkähler structure.

In Section 3, we study the homology group of X_λ and the period maps of hyperkähler structures. The homology group of X_λ are evaluated by constructing the deformation retracts of X_λ along [2][8]. The period maps of toric hyperkähler varieties are computed in [18], and we can also compute in the case of type A_∞ in the same way.

We study the volume growth of hyperkähler manifolds of type A_∞ in Chapter II which consists of Sections 4-8. For the proof we need the upper and lower estimate of the function $V_{g_\lambda}(p_0, r)$, which are discussed in Sections 4 and 5, respectively. In Section 4 the upper estimate of $V_{g_\lambda}(p_0, r)$ will be obtained as follows. Using the S^1-action and the hyperkähler moment map μ_λ, we can reduce the information of metric g_λ to a positive valued harmonic function with discrete poles on $Im\mathbb{H}$. Then the function $V_{g_\lambda}(p_0, r)$ can be described by using this harmonic function and we can compute explicitly the upper estimate of $V_{g_\lambda}(p_0, r)$.

Next we discuss the lower estimate $V_{g_\lambda}(p_0, r)$ in Section 5. If we try to estimate similarly to the way in Section 4, the estimate obtained in this way is weaker than what we expect. Accordingly we need to modify the way to estimate. In Section 5 we take an open subset of $B_{g_\lambda}(p_0, r)$ which is convenient to estimate its volume, and obtain the lower estimate of $V_{g_\lambda}(p_0, r)$.

From two types of estimates obtained in Sections 4 and 5, we prove Theorem 1.1 in Section 6.

We compute the volume growth of some examples concretely in Section 7, and obtain Theorem 1.2. Section 8 is devoted to studying the volume growth of the Taub-NUT deformations of (X_λ, g_λ).

In Chapter III, we will consider the sequences of hyperkähler quotients and their limit. In Section 9, we show that each hyperkähler manifold of type A_∞ is approximated by the sequence of ALE spaces. More precisely, for each bounded open subset U of (X_λ, g_λ) there exists a sequence of ALE spaces (X_k, g_k) of type A_k and bounded open subsets $U_k \subset X_k$ diffeomorphic to U, and $g_k|_{U_k}$ converge to $g_\lambda|_U$ for $k \to +\infty$ with respect to the C^0-norm on $\bigotimes^2 T^*U$. To consider the limit of $\{g_k|_{U_k}\}$, we construct an S^1-equivariant diffeomorphism from U_k to U.

On the other hand, in Section 10 we consider a sequence of hyperkähler manifold of type A_∞ using method in Section 9. If we take a sequence appropriately, an incomplete hyperkähler metric is constructed by considering the limit. In some special cases, these incomplete hyperkähler metrics are isometric to the universal cover of Ooguri-Vafa metric [21]. Moreover we show that there is a transitive $\mathbb{Z}$-action on this manifold preserving the metric and obtain hyperkähler metrics on the neighborhood of a singular fiber of the elliptic surface with singular fiber of type I_b $(b = 1, 2, \cdots)$ as a quotient space in Section 11.

The metrics constructed in Section 11 can be constructed also by Gibbons-Hawking ansatz. In Section 12 we construct them using Gibbons-Hawking ansatz along [21][11].

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Chapter I

Constructions and geometric properties of hyperkähler manifolds of type A_∞

2 Hyperkähler manifolds of type A_∞

In this section, we review the construction of hyperkähler manifolds of type A_∞ along [8]. Although they can be constructed by Gibbons-Hawking ansatz [1], we need hyperkähler quotient construction in [8] for arguments in Section 4. Before the construction, we start with some basic definitions.

Definition 2.1. Let (X, g) be a Riemannian manifold of dimension $4n$ and I_1, I_2, I_3 be complex structures on X. Then (X, g, I_1, I_2, I_3) is a hyperkähler manifold if (I_1, I_2, I_3) satisfies the relations $I_1^2 = I_2^2 = I_3^2 = I_1I_2I_3 = -1$ and each fundamental 2-form $\omega_i := g(I_i\cdot, \cdot)$ is closed for $i = 1, 2, 3$.

Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k = \mathbb{C} \oplus \mathbb{C}j$ be quaternion and $Im\mathbb{H} = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be its Imaginary part. Then an $Im\mathbb{H}$-valued 2-form $\omega := i\omega_1 + j\omega_2 + k\omega_3 \in \Omega^2(X) \otimes Im\mathbb{H}$ are constructed from the hyperkähler structure (g, I_1, I_2, I_3). Conversely, (g, I_1, I_2, I_3) is reconstructed from ω. Hence we call ω the hyperkähler structure on X instead of (g, I_1, I_2, I_3).

Next we consider the Lie group actions on hyperkähler manifolds. Let a Lie group G act on a manifold X. Then each element ξ of the Lie algebra $\mathfrak{g} = Lie(G)$ generates a vector field $\xi^* \in \mathcal{X}(X)$ defined by $\xi^*_x := \frac{d}{dt}|_{t=0} x \exp(t\xi) \in T_xX$ for $x \in X$.

Definition 2.2. Let n-dimensional torus $T^n = (S^1)^n$ act on a hyperkähler manifold (X, ω) preserving ω. Then the C^∞ map $\mu : X \to Im\mathbb{H} \otimes \mathbb{R}^n$ is a hyperkähler moment map of T^n-action if the following conditions are satisfied; (i) μ is T^n-invariant, (ii) $< d\mu_x(V), \xi >= \omega(\xi^*_x, V_x) \in Im\mathbb{H}$ for all $x \in X$ and $V \in T_xX$. Here $<,>$ is the standard inner product on $\mathbb{R}^n \cong Lie(T^n)$.

Next we review the construction of hyperkähler manifolds of type A_∞. Let $\mathbf{I}$ be an at most countable set. We will obtain the hyperkähler manifolds of type A_∞ if we assume $\mathbf{I}$ is infinite, and obtain the ALE spaces of type A_k if we assume $\sharp\mathbf{I} = k + 1$. For a set S, we denote by $S^{\mathbf{I}}$ the set of all maps from $\mathbf{I}$ to S. We denote an element of $x \in S^{\mathbf{I}}$ by $x = (x_n)_{n\in\mathbf{I}}$. Then we have a Hilbert space

$$M_{\mathbf{I}} := \{v \in \mathbb{H}^{\mathbf{I}};\ \|v\|_{\mathbf{I}}^2 < +\infty\},$$

where

$$< u, v >_{\mathbf{I}} := \sum_{n\in\mathbf{I}} u_n \bar{v}_n, \quad \|v\|_{\mathbf{I}}^2 :=< v, v >_{\mathbf{I}}$$

for $u, v \in \mathbb{H}^{\mathbf{I}}$.

For each

$$\Lambda \in \mathbb{H}_0^{\mathbf{I}} := \{\Lambda \in \mathbb{H}^{\mathbf{I}};\ \sum_{n\in\mathbf{I}} (1 + |\Lambda_n|^2)^{-1} < +\infty\},$$

we have the following Hilbert manifolds

$$\begin{aligned} M_\Lambda &:= \Lambda + M_{\mathbf{I}} = \{\Lambda + v;\ v \in M_{\mathbf{I}}\}, \\ U_\Lambda &:= \{g = (g_n)_{n\in\mathbf{I}} \in (S^1)^{\mathbf{I}};\ \|1_{\mathbf{I}} - g\|_\Lambda^2 < +\infty\}, \\ \mathfrak{u}_\Lambda &:= \{\xi = (\xi_n)_{n\in\mathbf{I}} \in \mathbb{R}^{\mathbf{I}};\ \|\xi\|_\Lambda^2 < +\infty\}, \\ G_\Lambda &:= \{g = (g_n)_{n\in\mathbf{I}} \in U_\Lambda;\ \prod_{n\in\mathbf{I}} g_n = 1\}, \\ \mathfrak{g}_\Lambda &:= \{\xi = (\xi_n)_{n\in\mathbf{I}} \in \mathfrak{u}_\Lambda;\ \sum_{n\in\mathbf{I}} \xi_n = 0\}, \end{aligned}$$

where

$$< \xi, \eta >_\Lambda := \sum_{n\in\mathbf{I}} (1 + |\Lambda_n|^2) \xi_n \bar{\eta}_n, \quad \|\xi\|_\Lambda^2 :=< \xi, \xi >_\Lambda$$

for $\xi, \eta \in \mathbb{C}^{\mathbf{I}}$. Here, $1_{\mathbf{I}} \in (S^1)^{\mathbf{I}}$ is the constant map $(1_{\mathbf{I}})_n := 1$. The convergence of $\prod_{n\in\mathbf{I}} g_n$ and $\sum_{n\in\mathbf{I}} \xi_n$ follows from the condition $\sum_{n\in\mathbf{I}}(1+\|\Lambda_n\|^2)^{-1} < +\infty$. Then G_Λ is a Hilbert Lie group whose Lie algebra is $\mathfrak{g}_\Lambda$. We can define a right action of G_Λ on M_Λ by $xg := (x_n g_n)_{n\in\mathbf{I}}$ for $x \in M_\Lambda$, $g \in G_\Lambda$. Here the product of x_n and g_n is given by regarding S^1 as the subset of $\mathbb{H}$ by the natural injections $S^1 \subset \mathbb{C} \subset \mathbb{H}$.

Since $M_{\mathbf{I}}$ is a left $\mathbb{H}$-module defined by $hv := (hv_n)_{n\in\mathbf{I}}$ for $v \in M_{\mathbf{I}}$ and $h \in \mathbb{H}$, $M_{\mathbf{I}}$ has a hyperkähler structure given by the left multiplication of i, j, k and inner product $<,>_{\mathbf{I}}$ on $M_{\mathbf{I}}$. We denote by $I_1, I_2, I_3 \in End(M_{\mathbf{I}})$ the complex structures induced by the left multiplication by i, j, k, respectively. Thus M_Λ is an infinite dimensional hyperkähler manifold and the action of G_Λ on M_Λ preserves the hyperkähler structure. Then define a map $\hat{\mu}_\Lambda : M_\Lambda \to Im\mathbb{H} \otimes \mathfrak{g}_\Lambda^*$ by

$$< \hat{\mu}_\Lambda(x), \xi >:= \sum_{n\in\mathbf{I}} (x_n i \bar{x}_n - \Lambda_n i \bar{\Lambda}_n)\xi_n \ \in Im\mathbb{H}$$

for $x \in M_\Lambda$, $\xi \in \mathfrak{g}_\Lambda$. If $\mathbf{I}$ is a finite set, then $\hat{\mu}_\Lambda$ is the hyperkähler moment map in the sense of Definition 2.2. If $\mathbf{I}$ is infinite, $\hat{\mu}_\Lambda$ is the hyperkähler moment map with respect to the action of infinite dimensional torus G_Λ on M_Λ.

Since $\hat{\mu}_\Lambda$ is G_Λ-invariant, then G_Λ acts on the inverse image $\hat{\mu}_\Lambda^{-1}(0)$. Hence we obtain the quotient space $\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ which is called hyperkähler quotient. In general, there is no guarantee that $\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ is a smooth manifold for every $\Lambda \in \mathbb{H}_0^{\mathbf{I}}$. For the smoothness of $\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ we need the following condition.

Definition 2.3. An element $\Lambda \in \mathbb{H}_0^{\mathbf{I}}$ is generic if $\Lambda_n i\bar{\Lambda}_n - \Lambda_m i \bar{\Lambda}_m \neq 0$ for all distinct $n, m \in \mathbf{I}$.

Proposition 2.4. *Take $\Lambda \in \mathbb{H}_0^{\mathbf{I}}$ to be generic. Then G_Λ acts on $\hat{\mu}_\Lambda^{-1}(0)$ freely.*

Proof. For each $l \in \mathbf{I}$, let $e_l \in \mathfrak{g}_\Lambda$ be defined by $e_l := (e_{l,n})_{n\in\mathbf{I}}$, where

$$e_{l,n} = \begin{cases} 1 & (n = l), \\ 0 & (n \neq l). \end{cases}$$

Since $\mathfrak{g}_\Lambda$ is generated by elements $e_l - e_m$, then $x \in M_\Lambda$ satisfies $\hat{\mu}_\Lambda(x) = 0$ if and only if $< \hat{\mu}_\Lambda(x), e_l - e_m >= 0$ for all $l, m \in \mathbf{I}$. Hence the value of $x_n i\bar{x}_n - \Lambda_n i \bar{\Lambda}_n$ is independent of $n \in \mathbf{I}$ for $x \in \hat{\mu}_\Lambda^{-1}(0)$.

Assume that there is a pair of $x \in \hat{\mu}_\Lambda^{-1}(0)$ and $g \in G_\Lambda$ satisfies $xg = x$. If $x_n \neq 0$ for any $n \in \mathbf{I}$, then $g = 1$. Therefore we may assume $x_s = 0$ for some $s \in \mathbf{I}$. Then we have $x_n i\bar{x}_n - \Lambda_n i\bar{\Lambda}_n = -\Lambda_s i \bar{\Lambda}_s$ for all $n \in \mathbf{I}$, which implies

$$x_n i \bar{x}_n = \Lambda_n i \bar{\Lambda}_n - \Lambda_s i \bar{\Lambda}_s \neq 0$$

for $n \neq s$. Since $x_n = 0$ if and only if $x_n i\bar{x}_n = 0$, we have $x_n \neq 0$ if $n \neq s$. Thus we have shown $g_n = 1$ if $n \neq s$, and also g_s should be 1 from the condition $\prod_{n\in\mathbf{I}} g_n = 1$. □

If $\mathbf{I}$ is a finite set, then $\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ is an ordinary hyperkähler quotient and the smoothness of $\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ for each generic $\Lambda \in \mathbb{H}_0^{\mathbf{I}}$ is ensured by [16]. If $\mathbf{I}$ is infinite, the smoothness of $\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ is ensured by the following.

Theorem 2.5 ([8]). *If $\Lambda \in \mathbb{H}_0^{\mathbf{I}}$ is generic, then $\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ is a smooth manifold of dimension 4, and the hyperkähler structure on M_Λ induces a hyperkähler structure $\hat{\omega}_\Lambda$ on $\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$.*

Remark 2.1. In [8], the above theorem is proved in the case of

$$\Lambda_n = \begin{cases} ni & (n \geq 0), \\ nk & (n < 0). \end{cases}$$

But it is easy to show the theorem for the case of any generic $\Lambda \in \mathbb{H}_0^{\mathbf{I}}$.

Take $\Lambda \in \mathbb{H}_0^{\mathbf{I}}$ and $(e^{i\theta_n})_{n\in\mathbf{I}} \in (S^1)^{\mathbf{I}}$ and put $\Lambda' = (\Lambda_n e^{i\theta_n})_{n\in\mathbf{I}}$. Then there is a canonical isomorphism of hyperkähler structure between $\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ and $\hat{\mu}_{\Lambda'}^{-1}(0)/G_{\Lambda'}$ as follows. Define a map $\hat{F} : \hat{\mu}_\Lambda^{-1}(0) \to \hat{\mu}_{\Lambda'}^{-1}(0)$ by

$$\hat{F}((x_n)_{n\in\mathbf{I}}) := (x_n e^{i\theta_n})_{n\in\mathbf{I}}.$$

Since $\hat{F}$ is equivariant with respect to G_Λ and $G_{\Lambda'}$-actions, we obtain $F : \hat{\mu}_\Lambda^{-1}(0)/G_\Lambda \to \hat{\mu}_{\Lambda'}^{-1}(0)/G_{\Lambda'}$, which satisfies $F^*\hat{\omega}_{\Lambda'} = \hat{\omega}_\Lambda$. In this case we have $\Lambda_n i\bar{\Lambda}_n = \Lambda'_n i\bar{\Lambda}'_n$ for all $n \in \mathbf{I}$. Conversely, if we take $\Lambda, \Lambda' \in \mathbb{H}_0^{\mathbf{I}}$ to be $(\Lambda_n i\bar{\Lambda}_n)_{n\in\mathbf{I}} = (\Lambda'_n i\bar{\Lambda}'_n)_{n\in\mathbf{I}}$, then there is $(e^{i\theta_n})_{n\in\mathbf{I}} \in (S^1)^{\mathbf{I}}$ such that $\Lambda'_n = \Lambda_n e^{i\theta_n}$. Thus $\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ and $\hat{\mu}_{\Lambda'}^{-1}(0)/G_{\Lambda'}$ are isomorphic as hyperkähler manifolds if $(\Lambda_n i\bar{\Lambda}_n)_{n\in\mathbf{I}} = (\Lambda'_n i\bar{\Lambda}'_n)_{n\in\mathbf{I}}$. Then for each

$$\begin{aligned} \lambda \in (Im\mathbb{H})_0^{\mathbf{I}} &:= \{(\Lambda_n i\bar{\Lambda}_n)_{n\in\mathbf{I}} \in (Im\mathbb{H})^{\mathbf{I}};\ \Lambda \in \mathbb{H}_0^{\mathbf{I}}\} \\ &= \{\lambda \in (Im\mathbb{H})^{\mathbf{I}};\ \sum_{n\in\mathbf{I}} \frac{1}{1+|\lambda_n|} < +\infty\}, \end{aligned}$$

we define the hyperkähler manifold $(X_\lambda, \omega_\lambda)$ by

$$\begin{aligned} X_\lambda &:= \hat{\mu}_\Lambda^{-1}(0)/G_\Lambda \\ &= \{x \in M_\Lambda; x_n i\bar{x}_n - \lambda_n \text{ is independent of } n \in \mathbf{I}\}/G_\Lambda, \\ \omega_\lambda &:= \hat{\omega}_\Lambda, \end{aligned}$$

where we take $\Lambda \in \mathbb{H}_0^{\mathbf{I}}$ to be $\Lambda_n i\bar{\Lambda}_n = \lambda_n$ for all $n \in \mathbf{I}$. Then this is well-defined from the above argument. The condition for X_λ to be smooth is written as follows.

Definition 2.6. An element $\lambda \in (Im\mathbb{H})_0^{\mathbf{I}}$ is generic if $\lambda_n - \lambda_m \neq 0$ for all distinct $n, m \in \mathbf{I}$.

Then Theorem 2.5 implies that $(X_\lambda, \omega_\lambda)$ is a smooth hyperkähler manifold for each generic $\lambda \in (Im\mathbb{H})_0^{\mathbf{I}}$. If $\mathbf{I}$ is infinite, then $(X_\lambda, \omega_\lambda)$ is called the hyperkähler manifold of type A_∞. If $\sharp\mathbf{I} = k+1 < +\infty$, then $(X_\lambda, \omega_\lambda)$ is an ALE hyperkähler manifold of type A_k [7].

We denote by g_λ the hyperkähler metric induced by ω_λ.

Theorem 2.7 ([8]). *Let (X_λ, g_λ) be as above. Then the Riemannian metric g_λ is complete.*

We denote by $[x] \in \hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ the equivalence class represented by $x \in \hat{\mu}_\Lambda^{-1}(0)$. Fix $m \in \mathbf{I}$ and put

$$[x]g := [x_m g, (x_n)_{n\in\mathbf{I}\backslash m}]$$

for $x = (x_m, (x_n)_{n\in\mathbf{I}\backslash m}) =\in \hat{\mu}_\Lambda^{-1}(0)$ and $g \in S^1$. This definition is independent of the choice of $m \in \mathbf{I}$, and we have the action of S^1 on X_λ. Then the hyperkähler moment map $\mu_\lambda : X_\lambda \to Im\mathbb{H} = \mathbb{R}^3$ is given by

$$\mu_\lambda([x]) := x_n i \bar{x}_n - \lambda_n \in Im\mathbb{H}.$$

Note that the above definition is independent of the choice of $n \in \mathbf{I}$ since x is an element of $\hat{\mu}_\Lambda^{-1}(0)$.

If we put $Stab([x]) := \{g \in S^1;\ [x]g = [x]\}$ for $[x] \in X_\lambda$, then it is obvious that $Stab([x]) = \{1\}$ if and only if $x_n \neq 0$ for any $n \in \mathbf{I}$, otherwise $Stab([x]) = S^1$. Hence we have a principal S^1-bundle $\mu_\lambda|_{X_\lambda^*} : X_\lambda^* \to Y_\lambda$ where

$$\begin{aligned} X_\lambda^* &:= \{[x] \in X_\lambda;\ x_n \neq 0 \text{ for all } n \in \mathbf{I}\}, \\ Y_\lambda &:= Im\mathbb{H}\backslash\{-\lambda_n;\ n \in \mathbf{I}\}. \end{aligned}$$

Definition 2.8. An S^1-action on a 4-dimensional hyperkähler manifold (X, ω) is tri-Hamiltonian if the action preserves ω and there exists a hyperkähler moment map $\mu : X \to Im\mathbb{H}$ with respect to this S^1-action.

Theorem 2.9 ([6]). *There exists a canonical one-to-one correspondence between the followings; (i) a hyperkähler manifold of real dimension 4 with free tri-Hamiltonian S^1-action, (ii) a principal S^1-bundle $\mu : X \to Y$ where Y is an open subset of $\mathbb{R}^3$, and an S^1-connection Γ on X and a positive valued harmonic function Φ on Y such that $\frac{d\Gamma}{2\sqrt{-1}} = \mu^*(*d\Phi)$. Here $*$ is the Hodge star operator with respect to the Euclidean metric on $\mathbb{R}^3$.*

Here we see a sketch of the proof, and the details can be seen in [8].

Let (X, ω) be a hyperkähler manifold of dimension 4 with a free S^1-action preserving ω, and $\mu : X \to Im\mathbb{H}$ be the hyperkähler moment map. Then (Y, Φ, Γ) is given by the followings. Let $Y = \mu(X)$. Then the function $\Phi : Y \to \mathbb{R}$ is defined by

$$\frac{1}{\Phi(\mu(x))} := 4g_x(\xi_x^*, \xi_x^*)$$

for $x \in X$, where g is the hyperkähler metric and $\xi := \sqrt{-1} \in Lie(S^1)$. For each $x \in X$, we denote by $V_x \subset T_x X$ the subspace spanned by ξ_x^*. Then the S^1-connection Γ on X is defined by the horizontal distribution $(H_x)_{x\in X}$ where $H_x \subset T_xX$ is the orthogonal complement of V_x.

Conversely, let $\mu = (\mu_1, \mu_2, \mu_3) : X \to Y \subset Im\mathbb{H}$ be a principal S^1-bundle and (Φ, Γ) with $\frac{d\Gamma}{2\sqrt{-1}} = \mu^*(*d\Phi)$ be given. Then the hyperkähler structure $\omega = (\omega_1, \omega_2, \omega_3) \in \Omega^2(X) \otimes Im\mathbb{H}$ is defined by

$$\omega_1 := d\mu_1 \wedge \frac{\Gamma}{2\sqrt{-1}} + \mu^*\Phi d\mu_2 \wedge d\mu_3,$$

$$\omega_2 := d\mu_2 \wedge \frac{\Gamma}{2\sqrt{-1}} + \mu^*\Phi d\mu_3 \wedge d\mu_1,$$

$$\omega_3 := d\mu_3 \wedge \frac{\Gamma}{2\sqrt{-1}} + \mu^*\Phi d\mu_1 \wedge d\mu_2.$$

The condition $\frac{d\Gamma}{2\sqrt{-1}} = \mu^*(*d\Phi)$ corresponds to the closedness of ω.

Theorem 2.9 gives the positive valued harmonic function Φ_λ on Y_λ and S^1-connection Γ_λ on X_λ^*. Then Φ_λ is given by

$$\Phi_\lambda(\zeta) = \frac{1}{4}\sum_{n\in\mathbf{I}} \frac{1}{|\zeta + \lambda_n|}$$

for $\zeta \in Y_\lambda$.

Let (X, ω) be a hyperkähler manifold satisfying the condition (i) of Theorem 2.9, and (Y, Φ, A) be what corresponds to (X, ω) satisfying the condition (ii). Denote by g the hyperkähler metric of ω and let $vol_g(B)$ be the volume of a subset $B \subset X$.

Lemma 2.10. *Let $U \subset Im\mathbb{H}$ be an open set. Then we have the following formula*

$$vol_g(\mu^{-1}(U)) = \pi \int_{\zeta\in U} \Phi(\zeta) d\zeta_1 d\zeta_2 d\zeta_3,$$

where $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in Im\mathbb{H}$ is the Cartesian coordinate.

Proof. It suffices to show the assertion for all open set $U \subset Y$.

First of all we suppose that the principal S^1-bundle $\mu : \mu^{-1}(U) \to U$ is trivial. Then we can take a C^∞ trivialization $\sigma : U \to \mu^{-1}(U)$ and define C^∞ map $t : \mu^{-1}(U) \to \mathbb{R}/2\pi\mathbf{I}$ by $t(\sigma(\zeta)e^{i\theta}) := \theta$ for $\zeta \in U$ and $\theta \in \mathbb{R}/2\pi\mathbf{I}$ and obtain a local coordinate (t, μ_1, μ_2, μ_3) on $\mu^{-1}(U)$. Since we may write $dt = -\sqrt{-1}\Gamma + \sum_{l=1}^3 a_l d\mu^l$ for some $a_1, a_2, a_3 \in C^\infty(\mu^{-1}(U))$, the volume form vol_g is given by

$$\begin{aligned} vol_g &= \frac{\mu^\circ\Phi}{2}(-\sqrt{-1}) d\mu_1 \wedge d\mu_2 \wedge d\mu_3 \wedge \Gamma \\ &= \frac{\mu^\circ\Phi}{2} d\mu_1 \wedge d\mu_2 \wedge d\mu_3 \wedge dt. \end{aligned}$$

Thus we have

$$\begin{aligned} vol_g(\mu^{-1}(U)) &= \int_{\mu^{-1}(U)} vol_g \\ &= \int_{\mu^{-1}(U)} \frac{\mu^\circ\Phi}{2} d\mu_1 \wedge d\mu_2 \wedge d\mu_3 \wedge dt \\ &= \pi \int_{\zeta\in U} \Phi(\zeta) d\zeta_1 d\zeta_2 d\zeta_3. \end{aligned}$$

For a general $U \subset Y$, we take open sets $U_1, U_2, \cdots, U_m \subset Y$ such that each principal S^1-bundle $\mu^{-1}(U_\alpha) \to U_\alpha$ is trivial and $\coprod_{\alpha=1}^m U_\alpha \subset U \subset \overline{\coprod_{\alpha=1}^m U_\alpha}$. Then we have

$$\begin{aligned} vol_g(\mu^{-1}(U)) &= \sum_{\alpha=1}^m vol_g(\mu^{-1}(U_\alpha)) \\ &= \sum_{\alpha=1}^m \pi \int_{\zeta\in U_\alpha} \Phi(\zeta) d\zeta_1 d\zeta_2 d\zeta_3 \\ &= \pi \int_{\zeta\in U} \Phi(\zeta) d\zeta_1 d\zeta_2 d\zeta_3. \end{aligned}$$

□

3 Geometry on $(X_\lambda, \omega_\lambda)$

3.1 Holomorphic curves

A complex manifold X of complex dimension $2n$ is a holomorphic symplectic manifold if there is a holomorphic $(2,0)$ form $\omega_{\mathbb{C}} \in \Omega^{(2,0)}(X)$ which satisfies a non-degenerate condition $\omega_{\mathbb{C}}^n \neq 0$.

Definition 3.1. Let $(X, \omega_{\mathbb{C}})$ be a holomorphic symplectic manifold of complex dimension $2n$. Then a complex submanifold $L \subset X$ of complex dimension n is holomorphic Lagrangian submanifold of X if $\omega_{\mathbb{C}}|_L = 0$.

Let (X, ω) be a hyperkähler manifold of real dimension $4n$. Take $y \in Im\mathbb{H}$ to be $|y| = 1$. Then $Im\mathbb{H}$ is decomposed into y-component and its orthogonal complement. Then we denote by $\omega_y \in \Omega^2(X)$ the y-component of $\omega \in \Omega^2(X) \otimes Im\mathbb{H}$. Let I_y be the complex structure corresponding to the Kähler form ω_y.

For each $\eta \in SO(3) = SO(Im\mathbb{H})$ we write $\eta_1 = \eta i$, $\eta_2 = \eta j$ and $\eta_3 = \eta k$. Then $\eta \in SO(3)$ gives the orthogonal decomposition $Im\mathbb{H} = \mathbb{R}\eta_1 \oplus \mathbb{R}\eta_2 \oplus \mathbb{R}\eta_3$.

Then the hyperkähler structure $\omega \in \Omega^2(X) \otimes Im\mathbb{H}$ can be written as $\omega = \eta_1\omega_{\eta_1} + \eta_2\omega_{\eta_2} + \eta_3\omega_{\eta_3}$ for every $\eta \in SO(3)$. Now we regard (X, I_{η_1}) as a complex manifold. Then $\omega_{\eta\mathbb{C}} := \omega_{\eta_2} + i\omega_{\eta_3}$ is a holomorphic symplectic structure on X.

Proposition 3.2. *Let (X, ω) be a 4-dimensional hyperkähler manifold and take $\eta \in SO(3)$. Then each holomorphic Lagrangian submanifold $L \subset X$ with respect to $\omega_{\eta\mathbb{C}}$ gives the minimum volume in their homology class.*

Proof. If we regard I_{η_3} as a complex structure on X, then the pair of a Kähler form ω_{η_3} and a holomorphic volume form $\omega_{\eta_1} + i\omega_{\eta_2}$ determines the Calabi-Yau structure on X. In this case the condition to be a holomorphic Lagrangian submanifold with respect to $\omega_{\eta\mathbb{C}}$ corresponds to the condition to be a special Lagrangian submanifold with respect to the Calabi-Yau structure $(\omega_{\eta_3}, \omega_{\eta_1} + i\omega_{\eta_2})$. The volume minimizing property of special Lagrangian submanifolds [12] gives the assertion. $\square$

In this section we put $\mathbf{I} = \mathbb{Z}$ for convenience. From now on we fix any generic $\lambda \in (Im\mathbb{H})_0^{\mathbb{Z}}$ and consider the hyperkähler manifold $(X_\lambda, \omega_\lambda)$ as constructed in Section 2. Next we are going to see that there are infinitely many minimal submanifold in X_λ by using the hyperkähler moment map μ_λ.

Put

$$\begin{aligned} [a,b] &:= \{ta + (1-t)b \in Im\mathbb{H};\ 0 \le t \le 1\}, \\ (a,b] &:= \{ta + (1-t)b \in Im\mathbb{H};\ 0 \le t < 1\}, \\ [a,b) &:= \{ta + (1-t)b \in Im\mathbb{H};\ 0 < t \le 1\}, \\ (a,b) &:= \{ta + (1-t)b \in Im\mathbb{H};\ 0 < t < 1\} \end{aligned}$$

for $a, b \in Im\mathbb{H}$.

Proposition 3.3. *Take $n, m \in \mathbb{Z}$ to be $n \ne m$ and $(-\lambda_n, -\lambda_m) \subset Y_\lambda$. If we put $y := \frac{\lambda_n - \lambda_m}{|\lambda_n - \lambda_m|}$ then $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m]) \cong \mathbb{C}P^1$ is a complex submanifold of X_λ with respect to I_y and gives the minimum volume in its homology class.*

Proof. Take $\eta \in SO(3)$ to be $\eta i = y$. If we write $\mu_\lambda = (\mu_{\lambda,1}, \mu_{\lambda,2}, \mu_{\lambda,3})$ according to the decomposition $Im\mathbb{H} = \mathbb{R}\eta_1 \oplus \mathbb{R}\eta_2 \oplus \mathbb{R}\eta_3$, then $\mu_{\lambda,2}$ and $\mu_{\lambda,3}$ are constant on $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m])$. Hence we have $d\mu_{\lambda,\alpha}|_{\mu_\lambda^{-1}([-\lambda_n,-\lambda_m])} = 0$ for $\alpha = 2, 3$, which gives $\omega_{\lambda,\eta\mathbb{C}}|_{\mu_\lambda^{-1}([-\lambda_n,-\lambda_m])} = 0$. $\square$

3.2 Topology

The author of [8] has constructed the deformation retract of X_λ for one $\lambda \in (Im\mathbb{H})_0^{\mathbb{Z}}$ and evaluated its homology group. In the case of finite topological type of toric hyperkähler varieties, the deformation retracts are constructed in [2]. In this subsection we review the construction of the deformation retracts of X_λ along [8][2].

If $n, m \in \mathbb{Z}$ are taken to be $(-\lambda_n, -\lambda_m) \subset Y_\lambda$, then the orientation of $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m])$ can be determined as follows. By taking a smooth section $(-\lambda_n, -\lambda_m) \to \mu_\lambda^{-1}((-\lambda_n, -\lambda_m))$ of μ_λ, we have a coordinate (s,t) on $\mu_\lambda^{-1}((-\lambda_n, -\lambda_m))$ where $t \in \mathbb{R}/2\pi\mathbb{Z}$ is the parameter of S^1-action and a function $s : \mu_\lambda^{-1}((-\lambda_n, -\lambda_m)) \to \mathbb{R}$ is defined by

$$s(p) := \frac{\lambda_n + \mu_\lambda(p)}{\lambda_n - \lambda_m}$$

for $p \in \mu_\lambda^{-1}((-\lambda_n, -\lambda_m))$. Thus the orientation of $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m])$ is by $ds \wedge dt$. Here we should remark that $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m])$ and $\mu_\lambda^{-1}([-\lambda_m, -\lambda_n])$ are same as manifolds but have opposite orientations.

For $n, m, l \in \mathbb{Z}$, $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m]) \cup \mu_\lambda^{-1}([-\lambda_m, -\lambda_l])$ and $\mu_\lambda^{-1}([-\lambda_n, -\lambda_l])$ determines the same homology class since the boundary of $\mu_\lambda^{-1}(\triangle_{n,m,l})$ is given by $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m]) \cup \mu_\lambda^{-1}([-\lambda_m, -\lambda_l]) \cup \mu_\lambda^{-1}([-\lambda_l, -\lambda_n])$, where

$$\triangle_{n,m,l} := \{-\alpha\lambda_n - \beta\lambda_m - \gamma\lambda_l \in Im\mathbb{H};\ \alpha + \beta + \gamma = 1,\ \alpha, \beta, \gamma \geq 0\}.$$

For any $n, m \in \mathbb{Z}$ we denote by $C_{n,m}$ the homology class determined by $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m])$. Then the above observation implies

$$C_{n,m} + C_{m,l} + C_{l,n} = C_{n,m} + C_{m,n} = 0.$$

for $n, m, l \in \mathbb{Z}$.

If $n, m, l, h \in \mathbb{Z}$ are taken to be $n \neq h$, $n \neq m$ and $l \neq h$ then the intersection number $C_{n,m} \cdot C_{l,h}$ is given by

$$C_{n,m} \cdot C_{l,h} = \begin{cases} 1 & (m = l) \\ 0 & (m \neq l) \end{cases}$$

and $C_{n,m} \cdot C_{n,m} = -2$.

Proposition 3.4. *By taking an appropriate bijection $\mathbb{Z} \to \mathbb{Z}$, there exists a deformation retract of $\bigcup_{n\in\mathbb{Z}}[-\lambda_n, -\lambda_{n+1}] \subset X_\lambda$.*

Proof. Fix a generic $\lambda \in (Im\mathbb{H})_0^{\mathbb{Z}}$. By taking an appropriate bijection $\mathbb{Z} \to \mathbb{Z}$, we may assume that λ satisfies the following conditions without loss of generality; (i) $(-\lambda_n, -\lambda_m) \subset Y_\lambda$ for each $n \in \mathbb{Z}$, (ii) there is a deformation retract

$$F : Im\mathbb{H} \times [0,1] \to Im\mathbb{H}$$

which satisfy $F(\cdot, 0) = id_{Im\mathbb{H}}$, $F(Im\mathbb{H}, 1) = \bigcup_{n\in\mathbb{Z}}[-\lambda_n, -\lambda_{n+1}]$ and $F(\zeta, 1) = \zeta$ for $\zeta \in \bigcup_{n\in\mathbb{Z}}[-\lambda_n, -\lambda_{n+1}]$. Then we have the horizontal lift $\tilde{F} : X_\lambda \times [0,1] \to X_\lambda$ of F by using the S^1-connection A_λ. The map $\tilde{F}$ is deformation retract what we expect. $\square$

Corollary 3.5. *For any generic $\lambda \in (Im\mathbb{H})_0^{\mathbb{Z}}$, the second homology group $H_2(X_\lambda, \mathbb{Z})$ is generated by $\{C_{n,m};\ n, m \in \mathbb{Z}\}$.*

Thus we obtain the followings.

Theorem 3.6. *Let $\lambda \in (Im\mathbb{H})_0^{\mathbb{Z}}$ be generic. Then $H_2(X_\lambda, \mathbb{Z})$ is a free $\mathbb{Z}$-module generated by $\{C_{n,m};\ n, m \in \mathbb{Z}\}$ with relations*

$$C_{n,m} + C_{m,l} + C_{l,n} = 0$$

for all $n, m, l \in \mathbb{Z}$. Moreover the intersection form on $H_2(X_\lambda, \mathbb{Z})$ is given by

$$C_{n,m} \cdot C_{l,h} = \begin{cases} 1 & (m = l) \\ 0 & (m \neq l) \end{cases}$$

and $C_{n,m} \cdot C_{n,m} = -2$ for $n, m, l, h \in \mathbb{Z}$ taken to be $n \neq h$, $n \neq m$ and $l \neq h$.

3.3 Period maps

We denote by $[\omega_\lambda] \in H^2(X_\lambda, \mathbb{R}) \otimes Im\mathbb{H}$ the cohomology class of ω_λ. In this subsection we compute $[\omega_\lambda]$, that is, compute the value of $< [\omega_\lambda], C_{n,m} >:= \int_{C_{n,m}} \omega_\lambda \in Im\mathbb{H}$ for all $n, m \in \mathbb{Z}$. In the case of finite topological type of toric hyperkähler varieties, the period maps are computed in [18].

Theorem 3.7. *Let $\lambda \in (Im\mathbb{H})_0^{\mathbb{Z}}$ be generic. Then the value of $< [\omega_\lambda], C_{n,m} >$ are given by*

$$< [\omega_\lambda], C_{n,m} >= \lambda_n - \lambda_m$$

for all $n, m \in \mathbb{Z}$.

Proof. Fix $n, m \in \mathbb{Z}$ and take a smooth curve $\gamma : [0,1] \to Im\mathbb{H}$ such that $\gamma(0) = -\lambda_n$, $\gamma(1) = -\lambda_m$ and $\gamma(s) \in Y_\lambda$ for $s \in (0,1)$. Since the homology class determined by $\mu_\lambda^{-1}(\gamma([0,1]))$ is $C_{n,m}$, we have

$$< [\omega_\lambda], C_{n,m} >= \int_{\mu_\lambda^{-1}(\gamma([0,1]))} \omega_\lambda.$$

Now we use the local coordinate $(t, \mu_{\lambda,1}, \mu_{\lambda,2}, \mu_{\lambda,3})$ as in the proof of Lemma 2.10. Then the local coordinate (s,t) on $\mu_\lambda^{-1}(\gamma([0,1]))$ is given by $(t, \mu_{\lambda,1} \circ \gamma(s), \mu_{\lambda,2} \circ \gamma(s), \mu_{\lambda,3} \circ \gamma(s))$. Using this coordinate, ω_λ is written as

$$\omega_{\lambda,\alpha} = \gamma'_\alpha(s)\frac{1}{2\pi}ds \wedge dt$$

for $\alpha = 1,2,3$, where $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s)) \in Im\mathbb{H} = \mathbb{R}^3$. Thus we obtain

$$\begin{aligned}\int_{\mu_\lambda^{-1}(\gamma([0,1]))} \omega_{\lambda,\alpha} &= \int_{\mu_\lambda^{-1}(\gamma([0,1]))} \gamma'_\alpha(s)\frac{1}{2\pi}ds \wedge dt \\ &= \int_0^{2\pi} \frac{1}{2\pi}dt \int_0^1 \gamma'_\alpha(s)ds \\ &= \gamma_\alpha(1) + \gamma_\alpha(0),\end{aligned}$$

which gives $< [\omega_\lambda], C_{n,m} >= \lambda_n - \lambda_m$. $\square$

Chapter II

The volume growth of hyperkähler manifolds of type A_∞

4 The upper estimate of the volume growth

For a Riemannian manifold (X, g) and a point $p_0 \in X$, we denote by $V_g(p_0, r)$ the volume of a ball $B_g(p_0, r) := \{p \in X;\ d_g(p_0, p) < r\}$ with respect to the Riemannian distance d_g. In this section, we will evaluate the upper estimate of $V_{g_\lambda}(p_0, r)$ for the hyperkähler manifold (X_λ, g_λ). From now on we are going to consider of type A_∞, so we put

$$\mathbf{I} = \mathbb{N} = \{n \in \mathbb{Z};\ n \geq 0\}$$

in Sections 4-8. In Section 2, we supposed $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$ to be generic for the smoothness of (X_λ, g_λ). But the function $V_{g_\lambda}(p_0, r)$ is determined only by the Riemannian measure vol_{g_λ} and the Riemannian distance d_{g_λ}. Even if $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$ is taken not to be generic, vol_{g_λ} and d_{g_λ} can be extended to X_λ naturally. Hence we take $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$ not necessary to be generic from now on.

Fix $p_0 \in X_\lambda$ to be $\mu_\lambda(p_0) = -\lambda_0$. We may assume $\lambda_0 = 0$ since the hyperkähler quotient constructed from $(\lambda_n - \lambda_0)_{n\in\mathbb{N}}$ is isometric to (X_λ, g_λ). For each $R > 0$, put

$$\varphi_\lambda(R) := \sum_{n\in\mathbb{N}} \frac{R}{R + |\lambda_n|},$$

and $\varphi_\lambda(0) := \lim_{R\to+0} \varphi_\lambda(R)$.

Proposition 4.1. *There exists a constant Q_- independent of λ and R, which satisfies*

$$d_{g_\lambda}(p_0, p)^2 \geq Q_- |\mu_\lambda(p)| \cdot \varphi_\lambda(|\mu_\lambda(p)|)$$

for any $p \in X_\lambda$.

Proof. Take $\Lambda = (\Lambda_n)_{n\in\mathbb{N}} \in \mathbb{H}_0^{\mathbb{N}}$ to be $\Lambda_n i \bar{\Lambda}_n = \lambda_n$ for all $n \in \mathbb{N}$. We fix $x \in \hat{\mu}_\Lambda^{-1}(0)$ such that $[x] = p$. If we regard $\hat{\mu}_\Lambda^{-1}(0)$ as the infinite dimensional Riemannian submanifold of M_Λ, then the quotient map $\hat{\mu}_\Lambda^{-1}(0) \to X_\lambda =$

$\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ is a Riemannian submersion and G_Λ acts on $\hat{\mu}_\Lambda^{-1}(0)$ as an isometric. Then the horizontal lift of the geodesic from p_0 to p has the same length as $d_{g_\lambda}(p_0, p)$. Since the Riemannian distance between Λ and $x' \in \hat{\mu}_\Lambda^{-1}(0)$ is larger than $\|\Lambda - x'\|_{\mathbb{N}}$, then we have

$$d_{g_\lambda}(p_0, p) \geq \inf_{\sigma \in G_\Lambda} \|\Lambda - x\sigma\|_{\mathbb{N}}.$$

If we put $\Lambda = (\alpha_n + \beta_n j)_{n\in\mathbb{N}}$, $x = (z_n + w_n j)_{n\in\mathbb{N}}$ and $\sigma = (e^{i\theta_n})_{n\in\mathbb{N}}$, then

$$\|\Lambda - x\sigma\|_{\mathbb{N}}^2 = \sum_{n\in\mathbb{N}} (|\alpha_n - z_n e^{i\theta_n}|^2 + |\beta_n - w_n e^{-i\theta_n}|^2).$$

If θ_n can be taken to give the minimum value of $|\alpha_n - z_n e^{i\theta_n}|^2 + |\beta_n - w_n e^{-i\theta_n}|^2$ for each $n \in \mathbb{N}$, then σ satisfies $\|\Lambda - x\sigma\|_{\mathbb{N}}^2 = \inf_{\tau\in G_\Lambda} \|\Lambda - x\tau\|_{\mathbb{N}}^2$.

Put $f_n(t) := |\alpha_n - z_n e^{it}|^2 + |\beta_n - w_n e^{-it}|^2$. Then we have

$$f_n(t) = |\lambda_n| + |\zeta + \lambda_n| - 2Re\{(\alpha_n \bar{z}_n + \bar{\beta}_n w_n)e^{-it}\},$$
$$|\alpha_n \bar{z}_n + \bar{\beta}_n w_n|^2 = \frac{1}{2}(|\lambda_n||\zeta + \lambda_n| + < \lambda_n, \zeta + \lambda_n >_{\mathbb{R}^3}),$$

where $<,>_{\mathbb{R}^3}$ is the standard inner products on $Im\mathbb{H} = \mathbb{R}^3$.

Suppose $n \neq 0$. If $\alpha_n \bar{z}_n + \bar{\beta}_n w_n \neq 0$, then we put $e^{i\theta_n} := \frac{\alpha_n \bar{z}_n + \bar{\beta}_n w_n}{|\alpha_n \bar{z}_n + \bar{\beta}_n w_n|}$. If $\alpha_n \bar{z}_n + \bar{\beta}_n w_n = 0$, we may put $\theta_n := 0$ since $f_n(t)$ is constant.

Let $S := \{n \in \mathbb{N};\ \alpha_n \bar{z}_n + \bar{\beta}_n w_n = 0\}$. For each $n \in S^c$, we have

$$\begin{aligned} |\alpha_n - z_n e^{i\theta_n}|^2 + |\beta_n - w_n e^{-i\theta_n}|^2 &= \frac{|\zeta|^2}{|\lambda_n| + |\zeta + \lambda_n| + 2\sqrt{|\alpha_n \bar{z}_n + \bar{\beta}_n w_n|}} \\ &\leq \frac{|\zeta|^2}{|\lambda_n|}. \end{aligned}$$

Hence we deduce

$$\sum_{n\in\mathbb{N}} |\Lambda_n - x_n e^{i\theta_n}|^2 \leq \sum_{n\in S} |\Lambda_n - x_n|^2 + \sum_{n\in S^c} \frac{|\zeta|^2}{|\lambda_n|} < +\infty.$$

Thus we obtain $\sum_{n\in\mathbb{N}\setminus\{0\}} |\Lambda_n|^2 |1 - e^{i\theta_n}|^2 < +\infty$, which ensures the convergence of $\Pi_{n\in\mathbb{N}\setminus\{0\}} e^{i\theta_n}$.

Since $f_0(t)$ is independent of t, we may put $e^{i\theta_0} := \Pi_{n\in\mathbb{N}\setminus\{0\}} e^{-i\theta_n}$. Thus we obtain $\sigma = (e^{i\theta_n})_{n\in\mathbb{N}}$ which gives the minimum value of $\|\Lambda - x\sigma\|_{\mathbb{N}}$. For this σ, we have

$$\|\Lambda - x\sigma\|_{\mathbb{N}}^2 = \sum_{n\in\mathbb{N}} (|\lambda_n| + |\zeta + \lambda_n| - \sqrt{2}\sqrt{|\lambda_n||\zeta + \lambda_n| + < \lambda_n, \zeta + \lambda_n >_{\mathbb{R}^3}})$$

$$= \sum_{n\in\mathbb{N}} \frac{|\zeta|^2}{|\lambda_n| + |\zeta+\lambda_n| + \sqrt{2}\sqrt{|\lambda_n||\zeta+\lambda_n| + <\lambda_n, \zeta+\lambda_n>_{\mathbb{R}^3}}}$$
$$\geq \sum_{n\in\mathbb{N}} \frac{|\zeta|^2}{8|\lambda_n| + 2|\zeta|} \geq \frac{1}{8}|\zeta| \cdot \varphi_\lambda(|\zeta|).$$

Then the assertion is obtained by putting $Q_- = \frac{1}{8}$. $\square$

Put $B_R := \{\zeta \in Im\mathbb{H};\ |\zeta| < R\}$. Next we discuss the upper estimate the volume of $\mu_\lambda^{-1}(B_R)$.

Lemma 4.2. *Put*

$$N_\lambda(R) := \{n \in \mathbb{N};\ |\lambda_n| \leq R\},$$
$$\psi_\lambda(R) := \sharp N_\lambda(R) + \sum_{n\in N_\lambda(R)^c} \frac{1}{|\lambda_n|} R$$

for $R \geq 0$. Then we have

$$\varphi_\lambda(R) \leq \psi_\lambda(R) \leq 2\varphi_\lambda(R).$$

Proof. Let

$$p(x) := \frac{x}{1+x}, \quad q(x) := \min\{1, x\}$$

for $x \geq 0$. Then the inequalities

$$p(x) \leq q(x) \leq 2p(x)$$

hold for each $x \geq 0$. Therefore the assertion follows from

$$\varphi_\lambda(R) = \sum_{n\in\mathbb{N}} p\Big(\frac{R}{|\lambda_n|}\Big), \quad \psi_\lambda(R) = \sum_{n\in\mathbb{N}} q\Big(\frac{R}{|\lambda_n|}\Big).$$

$\square$

Lemma 4.3. *For $\alpha \geq 1$ and $R \geq 0$, we have $\varphi_\lambda(\alpha R) \leq \alpha\varphi_\lambda(R)$ and $\psi_\lambda(\alpha R) \leq \alpha\psi_\lambda(R)$.*

Proof. Since $\frac{\varphi_\lambda(R)}{R}$ is strictly decreasing for R, we have

$$\varphi_\lambda(\alpha R) = \frac{\varphi_\lambda(\alpha R)}{\alpha R}\alpha R \leq \frac{\varphi_\lambda(R)}{R}\alpha R = \alpha\varphi_\lambda(R)$$

for $\alpha \geq 1$ and $R \geq 0$. It follows from the same argument that $\psi_\lambda(\alpha R) \leq \alpha\psi_\lambda(R)$. $\square$

Proposition 4.4. *There is a constant $P_+ > 0$ independent of λ and R such that*

$$vol_{g_\lambda}(\mu_\lambda^{-1}(B_R)) \le P_+ R^2 \varphi_\lambda(R)$$

for all $R \ge 0$.

Proof. From Lemma 2.10, the volume of $\mu_\lambda^{-1}(B_R)$ is given by

$$\begin{aligned} vol_{g_\lambda}(\mu_\lambda^{-1}(B_R)) &= \pi \int_{\zeta \in B_R} \Phi_\lambda(\zeta) d\zeta_1 d\zeta_2 d\zeta_3 \\ &= \frac{\pi}{4} \sum_{n \in \mathbb{N}} \int_{\zeta \in B_R} \frac{1}{|\zeta + \lambda_n|} d\zeta_1 d\zeta_2 d\zeta_3. \end{aligned}$$

Let $(r \ge 0, \Theta)$ be the polar coordinate over $Im\mathbb{H} = \mathbb{R}^3$, where Θ is a coordinate on $S^2 = \partial B_1$. If $n \in N_\lambda(R)$, then the change of variables $\zeta' = \zeta + \lambda_n$ gives

$$\begin{aligned} \int_{\zeta \in B_R} \frac{1}{|\zeta + \lambda_n|} d\zeta_1 d\zeta_2 d\zeta_3 &\le \int_{\zeta' \in B_{R+|\lambda_n|}} \frac{1}{|\zeta'|} d\zeta'_1 d\zeta'_2 d\zeta'_3 \\ &= 4\pi \int_0^{R+|\lambda_n|} r dr \\ &= 2\pi (R + |\lambda_n|)^2 \le 8\pi R^2. \end{aligned}$$

If $n \in N_\lambda(R)^c$, the mean value property of harmonic functions gives

$$\int_{\zeta \in B_R} \frac{1}{|\zeta + \lambda_n|} d\zeta_1 d\zeta_2 d\zeta_3 = \frac{4\pi R^3}{3} \frac{1}{|\lambda_n|},$$

since $\frac{1}{|\zeta + \lambda_n|}$ is harmonic on B_R. Hence the upper estimate of $vol_{g_\lambda}(\mu_\lambda^{-1}(B_R))$ is given by

$$\begin{aligned} vol_{g_\lambda}(\mu_\lambda^{-1}(B_R)) &\le 2\pi^2 \sharp N_\lambda(R) \cdot R^2 + \frac{\pi^2}{3} \sum_{n \in N_\lambda(R)^c} \frac{R}{|\lambda_n|} \cdot R^2 \\ &\le 2\pi^2 \psi_\lambda(R) R^2 \le 4\pi^2 \varphi_\lambda(R) R^2. \end{aligned}$$

Then we have the assertion by putting $P_+ := 4\pi^2$. □

Let $\theta_{\lambda,C}(R) := C\varphi_\lambda(R)R^2$, $\tau_{\lambda,C}(R) := C\varphi_\lambda(R)R$ for $C > 0$, and $\tau_{\lambda,C}^{-1} : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be the inverse function of $\tau_{\lambda,C}$.

Proposition 4.5. *Let P_+ and Q_- be as in Proposition 4.1 and 4.4. Then the inequality*

$$V_{g_\lambda}(p_0, r) \leq \theta_{\lambda,P_+} \circ \tau_{\lambda,Q_-}^{-1}(r^2)$$

holds for all $r \geq 0$.

Proof. It suffices to show that $B_{g_\lambda}(p_0, \sqrt{\tau_{\lambda,Q_-}(R)}) \subset \mu_\lambda^{-1}(B_R)$ for all $R \geq 0$.

Since each $p \in B_{g_\lambda}(p_0, \sqrt{\tau_{\lambda,Q_-}(R)})$ satisfies $d_{g_\lambda}(p_0, p)^2 \leq \tau_{\lambda,Q_-}(R)$, then we obtain $\tau_{\lambda,Q_-}(|\mu_\lambda(p)|) \leq \tau_{\lambda,Q_-}(R)$ from Proposition 4.1. Therefore $|\mu_\lambda(p)| \leq R$ follows from the strictly increasingness of τ_{λ,Q_-}. □

5 The lower estimate of the volume growth

In the previous section we obtained the upper estimate of $V_{g_\lambda}(p_0, r)$. The purpose of this section is to obtain the inequality

$$\theta_{\lambda,P_-} \circ \tau_{\lambda,Q_+}^{-1}(r^2) \leq V_{g_\lambda}(p_0, r),$$

where $P_-, Q_+ > 0$ are constants independent of λ and r. By considering similarly to Section 4, it seems to be enough to evaluate two types of inequalities

$$\begin{aligned} vol_{g_\lambda}(\mu_\lambda^{-1}(B_R)) &\geq P_- R^2 \varphi_\lambda(R), & (1) \\ d_{g_\lambda}(p_0, p)^2 &\leq Q_+ |\mu_\lambda(p)| \cdot \varphi_\lambda(|\mu_\lambda(p)|). & (2) \end{aligned}$$

In general we cannot obtain the latter type of inequality. Therefore we are going to evaluate two types of inequalities; one is a stronger inequality than (1), and the other is weaker than (2). First of all we consider the former estimate.

Let $U \subset S^2$ be a measurable set and put

$$B_{R,U} := \{t\zeta \in Im\mathbb{H};\ 0 \leq t \leq R,\ \zeta \in U\}.$$

We denote by m_{S^2} the measure induced from the Riemannian metric with constant curvature over S^2 whose total measure is given by $m_{S^2}(S^2) = 4\pi$. First of all we consider the lower estimate of $vol_{g_\lambda}(\mu_\lambda^{-1}(B_{R,U}))$.

Proposition 5.1. *There is a constant $C_- > 0$ independent of λ, R and $U \subset S^2$, which satisfies*

$$vol_{g_\lambda}(\mu_\lambda^{-1}(B_{R,U})) \geq C_- m_{S^2}(U) R^2 \cdot \varphi_\lambda(R).$$

Proof. From the triangle inequality, we have

$$
\begin{aligned}
vol_{g_\lambda}(\mu_\lambda^{-1}(B_{R,U})) &\geq \frac{\pi}{4}\sum_{n\in\mathbb{N}}\int_{\zeta\in B_{R,U}}\frac{1}{|\zeta|+|\lambda_n|}d\zeta_1 d\zeta_2 d\zeta_3 \\
&= \frac{\pi}{4}\sum_{n\in\mathbb{N}}\int_{r\in[0,R]}\int_{\Theta\in U}\frac{1}{r+|\lambda_n|}r^2 dr dm_{S^2} \\
&= \frac{\pi m_{S^2}(U)}{4}\sum_{n\in\mathbb{N}}\int_{r\in[0,R]}\frac{1}{r+|\lambda_n|}r^2 dr.
\end{aligned}
$$

If $n \in N_\lambda(R)$, then

$$
\int_{r\in[0,R]}\frac{1}{r+|\lambda_n|}r^2 dr \geq \int_{r\in[0,R]}\frac{1}{r+R}r^2 dr = (\log 2 - \frac{1}{2})R^2.
$$

If $n \in N_\lambda(R)^c$, then

$$
\begin{aligned}
\int_{r\in[0,R]}\frac{1}{r+|\lambda_n|}r^2 dr &= \int_{r\in[0,R]}\frac{1}{r+|\lambda_n|}r^2 dr \\
&= |\lambda_n|^2\Big\{-\frac{R}{|\lambda_n|}+\frac{1}{2}\Big(\frac{R}{|\lambda_n|}\Big)^2+\log\Big(1+\frac{R}{|\lambda_n|}\Big)\Big\}.
\end{aligned}
$$

Since the inequality

$$
\log(1+x) \geq x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4
$$

holds for $x \geq 0$, then we have

$$
\begin{aligned}
\int_{r\in[0,R]}\frac{1}{r+|\lambda_n|}r^2 dr &\geq |\lambda_n|^2\Big\{\frac{1}{3}\Big(\frac{R}{|\lambda_n|}\Big)^3-\frac{1}{4}\Big(\frac{R}{|\lambda_n|}\Big)^4\Big\} \\
&\geq \frac{1}{12}\frac{R^3}{|\lambda_n|}.
\end{aligned}
$$

By taking $\frac{4C_-}{\pi} := \min\{\log 2 - \frac{1}{2}, \frac{1}{12}\} > 0$, we have

$$
vol_{g_\lambda}(\mu_\lambda^{-1}(B_{R,U})) \geq C_- m_{S^2}(U)R^2 \cdot \psi_\lambda(R) \geq C_- m_{S^2}(U)R^2 \cdot \varphi_\lambda(R).
$$

□

Next we need the upper estimate of $d_{g_\lambda}(p_0, p)$. If we can calculate the length of a piecewise smooth path from p_0 to p, then the length is larger than $d_{g_\lambda}(p_0, p)$. Here we take the path as follows.

Put $\zeta = \mu_\lambda(p)$. If $|\zeta| \leq 1$, let γ_p be the geodesic from p_0 to p. If $|\zeta| > 1$, let $\delta : [1, |\zeta|] \to Im\mathbb{H}$ be defined by $\delta(t) := t\frac{\zeta}{|\zeta|}$ and take the horizontal lift $\tilde{\delta} : [1, |\zeta|] \to X_\lambda$ of δ with respect to the connection Γ_λ such that $\tilde{\delta}(|\zeta|) = p$. If there exist some $t \in [1, |\zeta|]$ such that $\delta(t) = -\lambda_n$ for some $n \in \mathbb{N}$, then $\tilde{\delta}$ is not always smooth but continuous and piecewise smooth. Then we obtain a path γ_p by connecting the geodesic from p_0 to $\tilde{\delta}(1)$ and $\tilde{\delta}$.

The length of γ_p is given by $d_{g_\lambda}(p_0, \tilde{\delta}(1)) + l_\lambda(\mu_\lambda(p))$ where $l_\lambda : Im\mathbb{H} \to \mathbb{R}$ is defined by

$$l_\lambda(\zeta) := \int_1^{|\zeta|} \sqrt{\Phi_\lambda\Big(t\frac{\zeta}{|\zeta|}\Big)}dt$$

if $|\zeta| \geq 1$, and $l_\lambda(\zeta) := 0$ if $|\zeta| \leq 1$. Thus we have $d_{g_\lambda}(p_0, p) \leq L_\lambda + l_\lambda(\mu_\lambda(p))$ where $L_\lambda := \sup_{p \in \mu_\lambda^{-1}(\bar{B}_1)} d_{g_\lambda}(p_0, p) < +\infty$.

Proposition 5.2. *There is a constant $C_+ > 0$ independent of λ and R which satisfies*

$$\int_{\Theta \in \partial B_1} l_\lambda(R\Theta)dm_{S^2} \leq 4\pi\sqrt{C_+ R\varphi_\lambda(R)}.$$

Proof. We may suppose $R \geq 1$, since the left hand side of the assertion is equal to 0 if $R < 1$. The definition of l_λ gives

$$\begin{aligned}\int_{\Theta \in \partial B_1} l_\lambda(R\Theta)dm_{S^2} &\leq \int_{\Theta \in \partial B_1}\int_1^R \sqrt{\Phi_\lambda(t\Theta)}dtdm_{S^2} \\ &= \int_{\zeta \in B_R \backslash B_1} \frac{\sqrt{\Phi_\lambda(\zeta)}}{|\zeta|^2}d\zeta, \qquad (3)\end{aligned}$$

where $d\zeta = d\zeta_1 d\zeta_2 d\zeta_3$.

Take $m \in \mathbb{Z}_{\geq 0}$ to be $2^m \leq R < 2^{m+1}$. Then the Cauchy-Schwarz inequality gives

$$\begin{aligned}(3) &= \sum_{l=0}^{m-1}\int_{\zeta \in B_{2^{l+1}} \backslash B_{2^l}} \frac{\sqrt{\Phi_\lambda(\zeta)}}{|\zeta|^2}d\zeta + \int_{\zeta \in B_R \backslash B_{2^m}} \frac{\sqrt{\Phi_\lambda(\zeta)}}{|\zeta|^2}d\zeta \\ &\leq \sum_{l=0}^{m-1}\sqrt{\int_{(r,\Theta) \in B_{2^{l+1}} \backslash B_{2^l}} \frac{r^2 dr d\Theta}{r^4}}\sqrt{\int_{\zeta \in B_{2^{l+1}} \backslash B_{2^l}} \Phi_\lambda(\zeta)d\zeta} \\ &\quad + \sqrt{\int_{(r,\Theta) \in B_R \backslash B_{2^m}} \frac{r^2 dr d\Theta}{r^4}}\sqrt{\int_{\zeta \in B_R \backslash B_{2^m}} \Phi_\lambda(\zeta)d\zeta}.\end{aligned}$$

From Proposition 4.4, the inequalities

$$\int_{\zeta \in B_t \backslash B_{t'}} \Phi_\lambda(\zeta)d\zeta \leq \int_{\zeta \in B_t} \Phi_\lambda(\zeta)d\zeta \leq \frac{P_+}{\pi}t^2\varphi_\lambda(t)$$

hold for any $0 \le t' \le t$. Therefore the assertion follows from

$$\begin{aligned}
(3) \quad &\le \sqrt{4\pi}\sum_{l=0}^{m-1}\sqrt{\frac{1}{2^l}-\frac{1}{2^{l+1}}}\sqrt{\frac{P_+}{\pi}2^{2(l+1)}\varphi_\lambda(2^{l+1})} \\
&\quad +\sqrt{4\pi}\sqrt{\frac{1}{2^m}-\frac{1}{R}}\sqrt{\frac{P_+}{\pi}R^2\varphi_\lambda(R)} \\
&\le 2\sum_{l=0}^{m-1}\sqrt{2}^{l+1}\sqrt{P_+\varphi_\lambda(R)}+2\sqrt{P_+R\varphi_\lambda(R)} \\
&\le 2(3+\sqrt{2})\sqrt{P_+R\varphi_\lambda(R)}
\end{aligned}$$

by putting $C_+ = (\frac{3+\sqrt{2}}{2\pi})^2 P_+$. □

Since $R\varphi_\lambda(R)$ diverges to $+\infty$ for $R \to +\infty$, there is a constant $R_0 > 0$ which satisfies

$$4\pi L_\lambda + \int_{\Theta\in S^2} l_\lambda(R\Theta)dm_{S^2} \le 4\pi\sqrt{2C_+R\varphi_\lambda(R)} \tag{4}$$

for all $R \ge R_0$. Now we put

$$U_{R,T} := \{\Theta \in S^2;\ L_\lambda + l_\lambda(R\Theta) \le \sqrt{TR\varphi_\lambda(R)}\}$$

for $R, T > 0$.

Lemma 5.3. *There exists a sufficiently large $R_0 > 0$ and we have*

$$m_{S^2}(U_{R,T}) \ge 4\pi\frac{\sqrt{T}-\sqrt{2C_+}}{\sqrt{T}-\sqrt{Q_-}}$$

for $R \ge R_0$ and $T > 2C_+$.

Proof. The definition of $U_{R,T}$ and Proposition 4.1 give

$$\begin{aligned}
\int_{\Theta\in S^2}(L_\lambda + l_\lambda(R\Theta))dm_{S^2} &= \int_{\Theta\in U_{R,T}}(L_\lambda + l_\lambda(R\Theta))dm_{S^2} \\
&\quad + \int_{\Theta\in S^2\setminus U_{R,T}}(L_\lambda + l_\lambda(R\Theta))dm_{S^2} \\
&\ge m_{S^2}(U_{R,T})\sqrt{Q_-R\varphi_\lambda(R)} \\
&\quad +(4\pi - m_{S^2}(U_{R,T}))\sqrt{TR\varphi_\lambda(R)}.
\end{aligned} \tag{5}$$

By combining (4) and (5), the inequality

$$4\pi(\sqrt{T}-\sqrt{2C_+})\sqrt{R\varphi_\lambda(R)} \le (\sqrt{T}-\sqrt{Q_-})m_{S^2}(U_{R,T})\sqrt{R\varphi_\lambda(R)}.$$

is obtained for $R \geq R_0$. Since $2C_+ > Q_-$, we have

$$m_{S^2}(U_{R,T}) \geq 4\pi \frac{\sqrt{T} - \sqrt{2C_+}}{\sqrt{T} - \sqrt{Q_-}}.$$

for $T > 2C_+$. $\square$

Lemma 5.4. *For each $R \geq 0$ and $T > 0$, $\mu_\lambda^{-1}(B_{R,U_{R,T}})$ is a subset of $B_{g_\lambda}(p_0, \sqrt{\tau_{\lambda,T}(R)})$.*

Proof. First of all we take $p \in \mu_\lambda^{-1}(B_{R,U_{R,T}})$ such that $|\mu_\lambda(p)| > 1$. Since $\frac{\mu_\lambda(p)}{|\mu_\lambda(p)|}$ is an element of $U_{R,T}$, we have

$$L_\lambda + l_\lambda(\mu_\lambda(p)) \leq L_\lambda + l_\lambda\Big(R\frac{\mu_\lambda(p)}{|\mu_\lambda(p)|}\Big) \leq \sqrt{TR\varphi_\lambda(R)}.$$

from $R \geq |\mu_\lambda(p)| > 1$. Then we obtain $d_{g_\lambda}(p_0, p) \leq \sqrt{TR\varphi_\lambda(R)}$ from $d_{g_\lambda}(p_0, p) \leq L_\lambda + l_\lambda(\mu_\lambda(p))$.

If $p \in \mu_\lambda^{-1}(B_{R,U_{R,T}})$ is taken to be $|\mu_\lambda(p)| \leq 1$, then we have the same conclusion as above since $l_\lambda(\mu_\lambda(p)) = 0$ in this case. $\square$

Now we fix a constant Q_+ to be $Q_+ > 2C_+$ and put $m_0 := 4\pi\frac{\sqrt{Q_+}-\sqrt{2C_+}}{\sqrt{Q_+}-\sqrt{Q_-}}$ and $P_- := m_0 C_-$.

Proposition 5.5. *Let $P_-, Q_+ > 0$ be as above. Then we have*

$$\liminf_{r\to\infty} \frac{V_{g_\lambda}(p_0, r)}{\theta_{\lambda,P_-} \circ \tau_{\lambda,Q_+}^{-1}(r^2)} > 0.$$

Proof. Let $R \geq 0$. From Lemma 5.4, we have

$$V_{g_\lambda}(p_0, \sqrt{\tau_{\lambda,Q_+}(R)}) \geq vol_{g_\lambda}(\mu_\lambda^{-1}(B_{R,U_{R,Q_+}})).$$

Then Proposition 5.1 gives

$$\begin{aligned} V_{g_\lambda}(p_0, \sqrt{\tau_{\lambda,Q_+}(R)}) &\geq m_{S^2}(U_{R,Q_+})C_- R^2\varphi_\lambda(R) \\ &\geq m_0 C_- R^2 \varphi_\lambda(R) = \theta_{\lambda,P_-}(R) \end{aligned}$$

for $R \geq R_0$. Thus we have the assertion by putting $R = \tau_{\lambda,Q_+}^{-1}(r^2)$. $\square$

6 The volume growth

In Sections 4 and 5, we estimate $V_{g_\lambda}(p_0, r)$ from the above by $\theta_{\lambda,P_+} \circ \tau^{-1}_{\lambda,Q_-}(r^2)$ and from the bottom by $\theta_{\lambda,P_-} \circ \tau^{-1}_{\lambda,Q_+}(r^2)$. In this section we show that the asymptotic behavior of the functions $\theta_{\lambda,P_+} \circ \tau^{-1}_{\lambda,Q_-}(r^2)$ and $\theta_{\lambda,P_-} \circ \tau^{-1}_{\lambda,Q_+}(r^2)$ are equal up to constant, and prove the main results.

The asymptotic behavior of $V_{g_\lambda}(p, r)$ is independent of the choice of $p \in X_\lambda$ from the next well-known fact.

Proposition 6.1. *Let (X, g) be a connected Riemannian manifold of dimension n, whose Ricci curvature is nonnegative. Then we have*

$$\lim_{r\to+\infty} \frac{V_g(p_1, r)}{V_g(p_0, r)} = 1$$

for any $p_0, p_1 \in X$.

Proof. From the Bishop-Gromov comparison inequality $\frac{V_g(p_1,r)}{r^n}$ is nonincreasing with respect to r. If we put $r_0 := d_g(p_0, p_1)$, then we have

$$\begin{aligned}
\frac{V_g(p_1, r)}{V_g(p_0, r)} &\le \frac{V_g(p_0, r + r_0)}{V_g(p_0, r)} \\
&= \frac{r^n}{V_g(p_0, r)} \frac{V_g(p_0, r + r_0)}{(r + r_0)^n} \frac{(r + r_0)^n}{r^n} \\
&\le \frac{r^n}{V_g(p_0, r)} \frac{V_g(p_0, r)}{r^n} \frac{(r + r_0)^n}{r^n} \\
&= \frac{(r + r_0)^n}{r^n} \to 1 \quad (r \to +\infty).
\end{aligned}$$

By considering the same argument after exchanging p_0 and p_1, we have the assertion. □

We denote by C^0_+ the set of the nondecreasing continuous functions from $\mathbb{R}_{\ge 0}$ to $\mathbb{R}_{\ge 0}$.

Definition 6.2. For $f_0, f_1 \in C^0_+$, we define $f_0(r) \preceq_r f_1(r)$ if

$$\limsup_{r\to+\infty} \frac{f_0(r)}{f_1(r)} < +\infty.$$

Definition 6.3. For $f_0, f_1 \in C^0_+$, we define $f_0(r) \simeq_r f_1(r)$ if $f_0(r) \preceq_r f_1(r)$ and $f_1(r) \preceq_r f_0(r)$.

For a Riemannian manifold (X, g) and a point $p_0 \in X$, the function $V_g(p_0, r)$ is an element of C^0_+. If (X, g) satisfies the assumption of Proposition 6.1, then the equivalence class of $V_g(p_0, r)$ with respect to $\simeq_r$ is independent of the choice of $p_0 \in X$. Therefore we denoted by $V_g(r)$ the equivalence class of $V_g(p_0, r)$. For example, if we write $V_g(r) \preceq_r r^n$, then it implies that

$$\limsup_{r \to +\infty} \frac{V_g(p_0, r)}{r^n} < +\infty$$

for some (hence all) $p_0 \in X$.

The main purpose of this section is to look for the function in C^0_+ which is equivalent to $V_{g_\lambda}(r)$.

Lemma 6.4. *Let $S_+, S_-, T_+, T_- > 0$. Then $\theta_{\lambda,S_+} \circ \tau^{-1}_{\lambda,T_-}$ and $\theta_{\lambda,S_-} \circ \tau^{-1}_{\lambda,T_+}$ are elements of C^0_+ and we have $\theta_{\lambda,S_+} \circ \tau^{-1}_{\lambda,T_-}(r^2) \simeq_r \theta_{\lambda,S_-} \circ \tau^{-1}_{\lambda,T_+}(r^2)$.*

Proof. We may suppose $T_- \le T_+$ without loss of generality. Since τ_{λ,T_+} and τ_{λ,T_-} are continuous, strictly increasing and satisfy $\tau_{\lambda,T_+}(0) = \tau_{\lambda,T_-}(0) = 0$, then τ^{-1}_{λ,T_+} and τ^{-1}_{λ,T_-} are also the elements of C^0_+. Hence the composite functions $\theta_{\lambda,S_+} \circ \tau^{-1}_{\lambda,T_-}$ and $\theta_{\lambda,S_-} \circ \tau^{-1}_{\lambda,T_+}$ are also the elements of C^0_+.

Next we show (i) $\theta_{\lambda,S_-} \circ \tau^{-1}_{\lambda,T_+}(r^2) \preceq_r \theta_{\lambda,S_+} \circ \tau^{-1}_{\lambda,T_-}(r^2)$ and (ii) $\theta_{\lambda,S_+} \circ \tau^{-1}_{\lambda,T_-}(r^2) \preceq_r \theta_{\lambda,S_-} \circ \tau^{-1}_{\lambda,T_+}(r^2)$.

(i) From $\tau_{\lambda,T_+}(R) \ge \tau_{\lambda,T_-}(R)$ and strictly increasingness of $\tau_{\lambda,T_\pm}$, then we have $\tau^{-1}_{\lambda,T_+}(r^2) \le \tau^{-1}_{\lambda,T_-}(r^2)$. Hence we obtain

$$\frac{\theta_{\lambda,S_-} \circ \tau^{-1}_{\lambda,T_+}(r^2)}{\theta_{\lambda,S_+} \circ \tau^{-1}_{\lambda,T_-}(r^2)} \le \frac{\theta_{\lambda,S_-} \circ \tau^{-1}_{\lambda,T_-}(r^2)}{\theta_{\lambda,S_+} \circ \tau^{-1}_{\lambda,T_-}(r^2)} \le \frac{S_-}{S_+}.$$

(ii) Put $R_\pm := \tau^{-1}_{\lambda,T_\pm}(r^2)$. Then we have

$$r^2 = T_+ R_+ \varphi_\lambda(R_+) = T_- R_- \varphi_\lambda(R_-).$$

Since φ_λ is nondecreasing, it holds

$$R_- \varphi_\lambda(R_-) = \frac{T_+}{T_-} R_+ \varphi_\lambda(R_+) \le \frac{T_+}{T_-} R_+ \varphi_\lambda\Big(\frac{T_+}{T_-} R_+\Big) \tag{6}$$

Since the function $R\varphi_\lambda(R)$ is strictly increasing with respect to R, the expression (6) gives $R_- \le \frac{T_+}{T_-} R_+$. Recall that φ_λ satisfies $\varphi_\lambda(\alpha R) \le \alpha \varphi_\lambda(R)$ for $\alpha \ge 1$, which implies $\theta_{\lambda,S_+}(\alpha R) \le \alpha^3 \theta_{\lambda,S_+}(R)$. Thus we have

$$\begin{aligned} \frac{\theta_{\lambda,S_+} \circ \tau^{-1}_{\lambda,T_-}(r^2)}{\theta_{\lambda,S_-} \circ \tau^{-1}_{\lambda,T_+}(r^2)} = \frac{\theta_{\lambda,S_+}(R_-)}{\theta_{\lambda,S_-}(R_+)} \le \frac{\theta_{\lambda,S_+}\Big(\frac{T_+}{T_-}R_+\Big)}{\theta_{\lambda,S_-}(R_+)} &\le \Big(\frac{T_+}{T_-}\Big)^3 \frac{\theta_{\lambda,S_+}(R_+)}{\theta_{\lambda,S_-}(R_+)} \\ &= \Big(\frac{T_+}{T_-}\Big)^3 \frac{S_+}{S_-}. \end{aligned}$$

□

Put $\theta_\lambda := \theta_{\lambda,1}$, $\tau_\lambda := \tau_{\lambda,1}$. Then the main result in this paper is described as follows.

Theorem 6.5. *For each $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$ and $p \in X_\lambda$, the function $V_{g_\lambda}(p,r)$ satisfies*

$$0 < \liminf_{r\to+\infty} \frac{V_{g_\lambda}(p,r)}{r^2\tau_\lambda^{-1}(r^2)} \leq \limsup_{r\to+\infty} \frac{V_{g_\lambda}(p,r)}{r^2\tau_\lambda^{-1}(r^2)} < +\infty.$$

Proof. We have shown that

$$\theta_{\lambda,P_-} \circ \tau_{\lambda,Q_+}^{-1}(r^2) \preceq_r V_{g_\lambda}(r) \preceq_r \theta_{\lambda,P_+} \circ \tau_{\lambda,Q_-}^{-1}(r^2)$$

in Sections 4 and 5, and

$$\theta_{\lambda,P_-} \circ \tau_{\lambda,Q_+}^{-1}(r^2) \simeq_r \theta_{\lambda,P_+} \circ \tau_{\lambda,Q_-}^{-1}(r^2) \simeq_r \theta_\lambda \circ \tau_\lambda^{-1}(r^2)$$

in Lemma 6.4. Thus we have the assertion from

$$\theta_\lambda \circ \tau_\lambda^{-1}(r^2) = \tau_\lambda^{-1}(r^2) \cdot \tau_\lambda(\tau_\lambda^{-1}(r^2)) = r^2\tau_\lambda^{-1}(r^2).$$

□

Corollary 6.6. *For $\lambda, \lambda' \in (Im\mathbb{H})_0^{\mathbb{N}}$, the condition $V_{g_\lambda}(r) \preceq_r V_{g_{\lambda'}}(r)$ is equivalent to $\varphi_\lambda(R) \preceq_R \varphi_{\lambda'}(R)$.*

Proof. The condition $V_{g_\lambda}(r) \preceq_r V_{g_{\lambda'}}(r)$ is equivalent to $\tau_\lambda^{-1}(r^2) \preceq_r \tau_{\lambda'}^{-1}(r^2)$ from Theorem 6.5.

If we assume $\tau_\lambda^{-1}(r^2) \preceq_r \tau_{\lambda'}^{-1}(r^2)$ then there are constants $r_0 \geq 0$ and $C \geq 1$ which satisfy $\frac{\tau_\lambda^{-1}(r^2)}{\tau_{\lambda'}^{-1}(r^2)} \leq C$ for all $r \geq r_0$. Now we put $r^2 := \tau_\lambda(R)$ and $(r')^2 := \tau_{\lambda'}(R)$ for $R \geq \tau_\lambda^{-1}(r_0^2)$. Since we have

$$R = \tau_{\lambda'}^{-1}((r')^2) = \tau_\lambda^{-1}(r^2) \leq C\tau_{\lambda'}^{-1}(r^2),$$

then

$$(r')^2 \leq \tau_{\lambda'}(C\tau_{\lambda'}^{-1}(r^2)) \leq C^2\tau_{\lambda'}(\tau_{\lambda'}^{-1}(r^2)) = C^2r^2$$

is given by the monotonicity of $\tau_{\lambda'}$. Thus we obtain $\frac{\tau_{\lambda'}(R)}{\tau_\lambda(R)} \leq C^2$ for all $R \geq \tau_{\lambda'}^{-1}(r_0^2)$, which implies $\tau_{\lambda'}(R) \preceq_R \tau_\lambda(R)$.

On the other hand, if we assume $\tau_{\lambda'}(R) \preceq_R \tau_\lambda(R)$ then $\tau_\lambda^{-1}(r^2) \preceq_r \tau_{\lambda'}^{-1}(r^2)$ is obtained in the same way. Thus we have the assertion since the condition $\tau_{\lambda'}(R) \preceq_R \tau_\lambda(R)$ is equivalent to $\varphi_{\lambda'}(R) \preceq_R \varphi_\lambda(R)$. □

Lemma 6.7. *For all* $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$, *we have*

$$\lim_{R\to+\infty} \varphi_\lambda(R) = +\infty.$$

Proof. From Lemma 4.2, it is enough to show $\lim_{R\to+\infty} \psi_\lambda(R) = +\infty$, which follows directly from $\lim_{R\to+\infty} \sharp N_\lambda(R) = +\infty$. □

Lemma 6.8. *For all* $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$, *we have*

$$\lim_{R\to+\infty} \frac{\varphi_\lambda(R)}{R} = 0.$$

Proof. For each $\varepsilon > 0$ there exists a sufficiently large $n_\varepsilon \in \mathbb{N}$ such that $\sum_{n>n_\varepsilon} \frac{1}{|\lambda_n|} < \frac{\varepsilon}{2}$. Hence we have

$$\begin{aligned}
\frac{\varphi_\lambda(R)}{R} &= \sum_{n\le n_\varepsilon} \frac{1}{R+|\lambda_n|} + \sum_{n>n_\varepsilon} \frac{1}{R+|\lambda_n|} \\
&\le \sum_{n\le n_\varepsilon} \frac{1}{R} + \sum_{n>n_\varepsilon} \frac{1}{|\lambda_n|} \\
&\le \frac{2n_\varepsilon+1}{R} + \frac{\varepsilon}{2}.
\end{aligned}$$

Then the inequality $\frac{\varphi_\lambda(R)}{R} \le \varepsilon$ holds for any $R \ge \frac{2(2n_\varepsilon+1)}{\varepsilon}$, which implies $\lim_{R\to+\infty} \frac{\varphi_\lambda(R)}{R} \le \varepsilon$. The assertion follows by taking $\varepsilon \to 0$. □

Corollary 6.9. *For all* $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$, *we have*

$$\lim_{r\to+\infty} \frac{V_{g_\lambda}(r)}{r^4} = 0, \quad \lim_{r\to+\infty} \frac{V_{g_\lambda}(r)}{r^3} = +\infty.$$

Proof. It suffices to show that

$$\lim_{r\to+\infty} \frac{\theta_\lambda \circ \tau_\lambda^{-1}(r^2)}{r^4} = 0, \quad \lim_{r\to+\infty} \frac{\theta_\lambda \circ \tau_\lambda^{-1}(r^2)}{r^3} = +\infty.$$

We put $R = \tau_\lambda^{-1}(r^2)$ and consider the limit of $R \to +\infty$. Then we have

$$\frac{\theta_\lambda \circ \tau_\lambda^{-1}(r^2)}{r^4} = \frac{\theta_\lambda(R)}{\tau_\lambda(R)^2} = \frac{1}{\varphi_\lambda(R)},$$

and

$$\frac{\theta_\lambda \circ \tau_\lambda^{-1}(r^2)}{r^3} = \frac{\theta_\lambda(R)}{\tau_\lambda(R)^{\frac{3}{2}}} = \sqrt{\frac{R}{\varphi_\lambda(R)}}.$$

Hence we obtain

$$\lim_{r\to+\infty}\frac{\theta_\lambda\circ\tau_\lambda^{-1}(r^2)}{r^4} = \lim_{R\to+\infty}\frac{1}{\varphi_\lambda(R)}=0,$$
$$\lim_{r\to+\infty}\frac{\theta_\lambda\circ\tau_\lambda^{-1}(r^2)}{r^3} = \lim_{R\to+\infty}\sqrt{\frac{R}{\varphi_\lambda(R)}}=+\infty.$$

from Lemmas 6.7 and 6.8. ☐

The condition $\varphi_\lambda(R)\preceq_R\varphi_{\lambda'}(R)$ is rather difficult to check. But we can describe the sufficient condition for $\varphi_\lambda(R)\preceq_R\varphi_{\lambda'}(R)$ easier as follows.

Proposition 6.10. *Let $\lambda,\lambda'\in(Im\mathbb{H})_0^{\mathbb{N}}$. Suppose that there are some $\alpha>0$ and $R_0>0$ such that $\sharp N_\lambda(R)\le\sharp N_{\alpha\lambda'}(R)$ for $R\ge R_0$, where $\alpha\lambda':=(\alpha\lambda'_n)_{n\in\mathbb{N}}$. Then we have $\varphi_\lambda(R)\preceq_R\varphi_{\lambda'}(R)$.*

Proof. We may assume that $|\lambda_n|\le|\lambda_{n+1}|$ and $|\lambda'_n|\le|\lambda'_{n+1}|$ without loss of generality since the hyperkähler metric g_λ depends only on the image of the map $n\mapsto\lambda_n$. Then the condition $\sharp N_\lambda(R)\le\sharp N_{\alpha\lambda'}(R)$ is equivalent to $N_\lambda(R)\subset N_{\alpha\lambda'}(R)$.

Take $n_0\in\mathbb{N}$ sufficiently large such that $|\lambda_{n_0}|\ge R_0$. Then we have $|\alpha\lambda'_n|\le|\lambda_n|$ for each $n\ge n_0$ from

$$N_\lambda(|\lambda_n|))\subset N_{\alpha\lambda'}(|\lambda_n|)).$$

If n is an element of $N_\lambda(R)^c\cap N_{\alpha\lambda'}(R)$ for $R\ge R_0$, then we have

$$\frac{1}{|\lambda_n|}\le\frac{1}{R},\quad \frac{1}{|\lambda_n|}\le\frac{1}{\alpha|\lambda'_n|}.$$

Thus we have

$$\begin{aligned}\psi_\lambda(R) &= \sharp N_\lambda(R)+\sum_{n\in N_\lambda(R)^c}\frac{R}{|\lambda_n|}\\ &= \sharp N_\lambda(R)+\sum_{n\in N_\lambda(R)^c\cap N_{\alpha\lambda'}(R)}\frac{R}{|\lambda_n|}+\sum_{n\in N_{\alpha\lambda'}(R)^c}\frac{R}{|\lambda_n|}\\ &\le \sharp N_\lambda(R)+(\sharp N_{\alpha\lambda'}(R)-\sharp N_\lambda(R))+\sum_{n\in N_{\alpha\lambda'}(R)^c}\frac{R}{\alpha|\lambda'_n|}\\ &= \psi_{\alpha\lambda'}(R)\end{aligned}$$

for $R\ge R_0$. From $\psi_{\lambda'}(R)\simeq_R\psi_{\alpha\lambda'}(R)$, we have $\psi_\lambda(R)\preceq_R\psi_{\lambda'}(R)$ which is equivalent to $\varphi_\lambda(R)\preceq_R\varphi_{\lambda'}(R)$. ☐

7 Examples

In this section we evaluate $V_{g_\lambda}(r)$ concretely for some $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$. Let $\lambda_{\mathbb{R}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous nondecreasing function which satisfies $\int_0^\infty \frac{dx}{1+\lambda_{\mathbb{R}}(x)} < +\infty$ and put $\hat{\varphi}_{\lambda_{\mathbb{R}}}(R) := \int_0^\infty \frac{Rdx}{R+\lambda_{\mathbb{R}}(x)}$. Then $\hat{\varphi}_{\lambda_{\mathbb{R}}}$ is strictly increasing and satisfies $\hat{\varphi}_{\lambda_{\mathbb{R}}}(R) \simeq_R \varphi_\lambda(R)$, where $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$ satisfies $|\lambda_n| = \lambda_{\mathbb{R}}(n)$. In this case it holds $V_{g_\lambda}(r) \simeq_r r^2\hat{\tau}_{\lambda_{\mathbb{R}}}^{-1}(r^2)$ where $\hat{\tau}_{\lambda_{\mathbb{R}}}$ is defined by $\hat{\tau}_{\lambda_{\mathbb{R}}}(R) := R\hat{\varphi}_{\lambda_{\mathbb{R}}}(R)$. Now we compute the volume growth of g_λ in the following two cases.

1. Fix $\alpha > 1$ and put $\lambda_{\mathbb{R}}(x) = x^\alpha$, $\lambda_n = \lambda_{\mathbb{R}}(n)i$. Then $\hat{\varphi}_{\lambda_{\mathbb{R}}}$ is given by

$$\hat{\varphi}_{\lambda_{\mathbb{R}}}(R) = \int_0^\infty \frac{Rdx}{R + x^\alpha} = R^{\frac{1}{\alpha}} \int_0^\infty \frac{dy}{1+y^\alpha},$$

where we put $y = \frac{x}{R^{\frac{1}{\alpha}}}$. Since $\int_0^\infty \frac{dy}{1+y^\alpha}$ is a constant, we have $\hat{\varphi}_{\lambda_{\mathbb{R}}}(R) \simeq_R R^{\frac{1}{\alpha}}$, which gives $\hat{\tau}_{\lambda_{\mathbb{R}}}(R) \simeq_R R^{1+\frac{1}{\alpha}}$ and $\hat{\tau}_{\lambda_{\mathbb{R}}}^{-1}(r^2) \simeq_r r^{\frac{2\alpha}{\alpha+1}}$. Hence the volume growth is given by

$$V_{g_\lambda}(r) \simeq_r r^{4-\frac{2}{\alpha+1}}.$$

Thus we obtain the following result.

Theorem 7.1. *Let $\alpha \in (3, 4)$. Then there is a hyperkähler manifold (X_λ, g_λ) whose volume growth is given by*

$$V_{g_\lambda}(r) \simeq_r r^\alpha.$$

2. Fix $\alpha > 0$ and put $\lambda_{\mathbb{R}}(x) = e^{\alpha x}$, $\lambda_n = \lambda_{\mathbb{R}}(n)i$. Then $\hat{\varphi}_{\lambda_{\mathbb{R}}}$ is given by

$$\hat{\varphi}_{\lambda_{\mathbb{R}}}(R) = \int_0^\infty \frac{Rdx}{R + e^{\alpha x}}.$$

By putting $y = e^{\alpha x}$, we have

$$\begin{aligned} \hat{\varphi}_{\lambda_{\mathbb{R}}}(R) &= \int_1^\infty \frac{Rdy}{\alpha y(y+R)} \\ &= \frac{1}{\alpha}\log(R+1). \end{aligned}$$

Hence we have $\hat{\varphi}_{\lambda_{\mathbb{R}}}(R) = \frac{1}{\alpha}\log(R+1)$ and $\hat{\tau}_{\lambda_{\mathbb{R}}}(R) = \frac{1}{\alpha}R\log(R+1)$.

Proposition 7.2. *Let $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$ be as above. Then the volume growth of g_λ satisfies $V_{g_\lambda}(r) \simeq_r \frac{r^4}{\log r}$.*

Proof. It is enough to see the behavior of $\frac{r^2\hat{\tau}_{\lambda_{\mathbb{R}}}^{-1}(r^2)\log r}{r^4}$ at $r \to +\infty$. Put $R = \hat{\tau}_{\lambda_{\mathbb{R}}}^{-1}(r^2)$. Then we have $r^2 = \frac{1}{\alpha}R\log(R+1)$ and $\log r = \frac{1}{2}(\log R + \log\log(R+1) - \log\alpha)$. Thus we have

$$\begin{aligned}\lim_{r\to+\infty}\frac{r^2\hat{\tau}_{\lambda_{\mathbb{R}}}^{-1}(r^2)\log r}{r^4} &= \lim_{R\to+\infty}\frac{\alpha}{2}\frac{R(\log R+\log\log(R+1)-\log\alpha)}{R\log(R+1)}\\ &= \frac{\alpha}{2}.\end{aligned}$$

□

Thus we have the following theorem.

Theorem 7.3. *There exists a hyperkähler manifold (X_λ, g_λ) whose volume growth satisfies*

$$\lim_{r\to+\infty}\frac{V_{g_\lambda}(r)}{r^4} = 0, \quad \lim_{r\to+\infty}\frac{V_{g_\lambda}(r)}{r^\alpha} = +\infty$$

for any $\alpha < 4$.

8 Taub-NUT deformations

We consider the volume growth of the Taub-NUT deformations of (X_λ, g_λ) in this section.

First of all, we define the Taub-NUT deformations of hyperkähler manifolds with tri-Hamiltonian S^1-actions. Let (X,ω) be a hyperkähler manifold of dimension 4 with tri-Hamiltonian S^1-action, and $\mu : X \to Im\mathbb{H}$ be the hyperkähler moment map. An action of $\mathbb{R}$ on $\mathbb{H}$ is defined by $x \mapsto x + \sqrt{s}^{-1}t$ for $x \in \mathbb{H}$ and $t \in \mathbb{R}$ by fixing a constant $s > 0$. This action preserves the standard hyperkähler structure on $\mathbb{H}$ and the hyperkähler moment map is given by $\sqrt{s}^{-1} \cdot Im : \mathbb{H} \to Im\mathbb{H}$. Then we obtain the hyperkähler quotient with respect to the action of $\mathbb{R}$ on $X \times \mathbb{H}$, that is, the quotient $(\mu^{(s)})^{-1}(\zeta)/\mathbb{R}$ where ζ is an element of $Im\mathbb{H}$ and $\mu^{(s)} : X \times \mathbb{H} \to Im\mathbb{H}$ is defined by $\mu^{(s)}(x,y) := \mu(x) + 2\sqrt{s}^{-1}Im(y)$. The hyperkähler structure on $(\mu^{(s)})^{-1}(\zeta)/\mathbb{R}$ is independent of ζ.

For each $\zeta \in Im\mathbb{H}$ we have an imbedding $\tilde{\iota}_{s,\zeta} : X \to (\mu^{(s)})^{-1}(\zeta)$ defined by $\tilde{\iota}_{s,\zeta}(x) := (x, \frac{\sqrt{s}}{2}(-\mu(x)+\zeta))$ which induces a diffeomorphism $\iota_{s,\zeta} : X \to (\mu^{(s)})^{-1}(\zeta)/\mathbb{R}$. Then we have another hyperkähler structure on X independent of ζ by the pull-back, which is called Taub-NUT deformation of ω denoted by $\omega^{(s)}$. If we denote by g the hyperkähler metric of (X,ω), then we denote by $g^{(s)}$ the hyperkähler metric of $(X,\omega^{(s)})$.

There is the pair of a harmonic function and an S^1-connection (Φ, Γ) corresponding to the hyperkähler structure ω by Theorem 2.9. Then the corresponding pair to $\omega^{(s)}$ is given by $(\Phi + \frac{s}{4}, \Gamma)$.

The S^1-action on X also preserves $\omega^{(s)}$ and $\mu : X \to Im\mathbb{H}$ is also the hyperkähler moment map with respect to $\omega^{(s)}$.

In this section we consider the Taub-NUT deformation of $(X_\lambda, \omega_\lambda)$, where $\mathbf{I} = \mathbb{N}$, and evaluate the volume growth by considering the same argument with Section 4, 5 and 6.

Lemma 8.1. *Let (X, g) be as above and take $p_0, p \in X$. We suppose that p_0 is a fixed point by the S^1-action. Then we have the inequality*

$$d_{g^{(s)}}(p_0, p)^2 \geq d_g(p_0, p)^2 + \frac{s}{4}|\mu(p) - \mu(p_0)|^2.$$

Proof. We apply the same argument as Proposition 4.1. Then the assertion follows from

$$\begin{aligned} d_{g^{(s)}}(p_0, p)^2 &\geq \inf_{t\in\mathbb{R}} d_{g\times g_{\mathbb{H}}}(\tilde{\iota}_{s,\zeta}(p_0)\cdot t, \tilde{\iota}_{s,\zeta}(p))^2 \\ &= \inf_{t\in\mathbb{R}} (d_g(p_0 e^{it}, p)^2 + |\frac{\sqrt{s}}{2}(\mu(p_0) - \mu(p)) + \frac{1}{\sqrt{s}} t|^2) \\ &= \inf_{t\in\mathbb{R}} \left(d_g(p_0, p)^2 + \frac{s}{4}|\mu(p_0) - \mu(p)|^2 + \frac{t^2}{s} \right) \\ &= d_g(p_0, p)^2 + \frac{s}{4}|\mu(p_0) - \mu(p)|^2, \end{aligned}$$

where $g_{\mathbb{H}}$ is the Euclidean metric on $\mathbb{H}$ and $g \times g_{\mathbb{H}}$ is the direct product metric. □

Lemma 8.2. *Let (X, g) be as above and $B \subset Im\mathbb{H}$ be a measurable set. Then we have*

$$vol_{g^{(s)}}(\mu^{-1}(B)) = vol_g(\mu^{-1}(B)) + \frac{\pi s}{4} m_{Im\mathbb{H}}(B),$$

where $m_{Im\mathbb{H}}$ is the Lebesgue measure of $Im\mathbb{H}$.

Proof. It follows directly from Lemma 2.10 and that $\omega^{(s)}$ corresponds to $(\Phi + \frac{s}{4}, \Gamma)$. □

For $s, C > 0$ and $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$, put

$$\theta_{\lambda,C}^{(s)}(R) := CR^2\varphi_\lambda(R) + \frac{\pi^2 s}{3}R^3, \quad \tau_{\lambda,C}^{(s)}(R) := CR\varphi_\lambda(R) + \frac{s}{4}R^2.$$

Proposition 8.3. *For $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$ and $s > 0$, we have*

$$\limsup_{r\to+\infty} \frac{V_{g_\lambda^{(s)}}(p_0, r)}{r^3} \le \frac{8\pi^2}{3\sqrt{s}}.$$

Proof. From Lemma 8.1 and 8.2, we have

$$V_{g_\lambda^{(s)}}(p_0, r) \le \theta_{\lambda,P_+}^{(s)} \circ (\tau_{\lambda,Q_-}^{(s)})^{-1}(r^2)$$

for $r > 0$. Then it suffices to show

$$\limsup_{r\to+\infty} \frac{\theta_{\lambda,P_+}^{(s)} \circ (\tau_{\lambda,Q_-}^{(s)})^{-1}(r^2)}{r^3} \le \frac{8\pi^2}{3\sqrt{s}}.$$

Put $R = (\tau_{\lambda,Q_-}^{(s)})^{-1}(r^2)$. Then we have

$$r^2 = Q_- R\varphi_\lambda(R) + \frac{s}{4}R^2 \ge \frac{s}{4}R^2.$$

Therefore Lemma 6.8 gives

$$\limsup_{r\to+\infty} \frac{\theta_{\lambda,P_+}^{(s)} \circ (\tau_{\lambda,Q_-}^{(s)})^{-1}(r^2)}{r^3} \le \limsup_{R\to+\infty} \frac{8(P_+R^2\varphi_\lambda(R) + \frac{\pi^2 s}{3}R^3)}{\sqrt{s}^3 R^3} = \frac{8\pi^2}{3\sqrt{s}}.$$

□

Next we consider the lower estimate of $V_{g_\lambda^{(s)}}(r)$. We apply the same way as Section 5. Put $l_\lambda^{(s)} : Im\mathbb{H} \to \mathbb{R}$ as

$$l_\lambda^{(s)}(\zeta) := \int_1^{|\zeta|} \sqrt{\Phi_\lambda\Big(t\frac{\zeta}{|\zeta|}\Big) + \frac{s}{4}}\, dt$$

on $|\zeta| > 1$, and $l_\lambda^{(s)}(\zeta) := 0$ on $|\zeta| \le 1$. Then the inequality $d_{g_\lambda^{(s)}}(p_0, p) \le L_\lambda^{(s)} + l_\lambda^{(s)}(\mu_\lambda(p))$ holds where $p_0 \in X$ is taken as in Sections 4 and 5, and we put $L_\lambda^{(s)} := \sup_{p\in\mu_\lambda^{-1}(\bar{B}_1)} d_{g_\lambda^{(s)}}(p_0, p)$.

Lemma 8.4. *Let $C_+ > 0$ be as in Proposition 5.2. Then we have*

$$\int_{\Theta\in S^2} l_\lambda^{(s)}(R\Theta)dm_{S^2} \le 4\pi\Big(\sqrt{\tau_{\lambda,C_+}(R)} + \sqrt{\frac{sR^2}{4}}\Big).$$

Proof. The assertion follows from

$$\begin{aligned}\int_{\Theta\in S^2}\Big(\int_1^R\sqrt{\Phi_\lambda(t\Theta)+\frac{s}{4}}\,dt\Big)dm_{S^2} &\le \int_{\Theta\in S^2}\Big(\int_1^R\sqrt{\frac{s}{4}}dt\Big)dm_{S^2}\\ &\quad+\int_{\Theta\in S^2} l_\lambda(R\Theta)dm_{S^2}\\ &\le 4\pi\sqrt{\frac{sR^2}{4}}+4\pi\sqrt{\tau_{\lambda,C_+}(R)}.\end{aligned}$$

□

Put $U^{(s)}_{R,T}:=\{\Theta\in S^2; L^{(s)}_\lambda+l^{(s)}_\lambda(R\Theta)\le\sqrt{\tau_{\lambda,T}(R)}+\sqrt{\frac{sR^2}{4}}\}$.

Lemma 8.5. *There is a constant $R_0>0$ such that*

$$m_{S^2}(U^{(s)}_{R,T})\ge\frac{4\pi(\sqrt{T}-\sqrt{2C_+})}{\sqrt{T}}$$

for any $R\ge R_0$ and $T>2C_+$.

Proof. We consider the same argument as in Lemma 5.3. First of all we remark that there exists sufficiently large $R_0>0$ such that

$$\int_{\Theta\in S^2}(L^{(s)}_\lambda+l^{(s)}_\lambda(R\Theta))dm_{S^2}\le 4\pi\Big(\sqrt{\tau_{\lambda,2C_+}(R)}+\sqrt{\frac{sR^2}{4}}\Big)$$

for any $R\ge R_0$. Then we have

$$\begin{aligned}\int_{\Theta\in S^2}(L^{(s)}_\lambda+l^{(s)}_\lambda(R\Theta))dm_{S^2} &= \int_{\Theta\in U^{(s)}_{R,T}}(L^{(s)}_\lambda+l^{(s)}_\lambda(R\Theta))dm_{S^2}\\ &\quad+\int_{\Theta\in S^2\setminus U^{(s)}_{R,T}}(L^{(s)}_\lambda+l^{(s)}_\lambda(R\Theta))dm_{S^2}\\ &\ge m_{S^2}(U^{(s)}_{R,T})\sqrt{\tau_{\lambda,Q_-}(R)+\frac{sR^2}{4}}\\ &\quad+(4\pi-m_{S^2}(U^{(s)}_{R,T}))\Big(\sqrt{\tau_{\lambda,T}(R)}+\sqrt{\frac{sR^2}{4}}\Big),\end{aligned}$$

where the constant $Q_->0$ is as in Proposition 4.1. Then an inequality $\sqrt{\tau_{\lambda,Q_-}(R)+\frac{sR^2}{4}}\ge\sqrt{\frac{sR^2}{4}}$ gives

$$\begin{aligned}\int_{\Theta\in S^2}(L^{(s)}_\lambda+l^{(s)}_\lambda(R\Theta))dm_{S^2} &\ge 4\pi\sqrt{\frac{sR^2}{4}}\\ &\quad+(4\pi-m_{S^2}(U^{(s)}_{R,T}))\sqrt{\tau_{\lambda,T}(R)}.\end{aligned}$$

Thus we have the conclusion. □

Lemma 8.6. *For each $R \geq 0$ and $T > 0$, $\mu_\lambda^{-1}(B_{R,U_{R,T}^{(s)}})$ is a subset of $B_{g_\lambda^{(s)}}\left(p_0, \sqrt{\tau_{\lambda,T}(R)} + \sqrt{\frac{sR^2}{4}}\right)$.*

Proof. It is shown by the same argument as Lemma 5.4. $\square$

Proposition 8.7. *For each $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$ and $s > 0$ we have*

$$\liminf_{r \to +\infty} \frac{V_{g_\lambda^{(s)}}(p_0, r)}{r^3} \geq \frac{8\pi^2}{3\sqrt{s}}.$$

Proof. For each sufficiently small $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that

$$m_{S^2}(U_{R,T_\varepsilon}^{(s)}) > 4\pi(1-\varepsilon)$$

from Lemma 8.5. Since $\frac{\varphi_\lambda(R)}{R}$ converges to 0, there exists $R_\varepsilon > 0$ such that $T_\varepsilon \varphi_\lambda(R) < \varepsilon^2 R$ for any $R > R_\varepsilon$. Then it holds

$$\begin{aligned} vol_{g_\lambda^{(s)}}\left(\mu_\lambda^{-1}\left(B_{R,U_{R,T_\varepsilon}^{(s)}}\right)\right) &\leq V_{g_\lambda^{(s)}}\left(p_0, \sqrt{\tau_{\lambda,T_\varepsilon}(R)} + \sqrt{\frac{sR^2}{4}}\right) \\ &\leq V_{g_\lambda^{(s)}}\left(p_0, (\varepsilon + \sqrt{\frac{s}{4}})R\right) \end{aligned} \tag{7}$$

for $R > R_\varepsilon$. On the other hand, we have the following inequality

$$vol_{g_\lambda^{(s)}}(\mu_\lambda^{-1}(B_{R,U})) \geq C_- m_{S^2}(U) R^2 \cdot \varphi_\lambda(R) + \frac{\pi m_{S^2}(U)}{12} sR^3 \tag{8}$$

for $U \dot{\subset} Im\mathbb{H}$ from Proposition 5.1. Thus we obtain

$$\begin{aligned} V_{g_\lambda^{(s)}}\left(p_0, \left(\varepsilon + \sqrt{\frac{s}{4}}\right)R\right) &\geq C_- m_{S^2}(U_{R,T_\varepsilon}^{(s)}) R^2 \cdot \varphi_\lambda(R) + \frac{\pi m_{S^2}(U_{R,T_\varepsilon}^{(s)})}{12} sR^3 \\ &\geq \frac{\pi^2}{3}(1-\varepsilon)sR^3 \end{aligned}$$

from (7) and (8). Hence substituting $r = (\varepsilon + \sqrt{\frac{s}{4}})R$ gives

$$\liminf_{r \to +\infty} \frac{V_{g_\lambda^{(s)}}(p_0, r)}{r^3} = \liminf_{R \to +\infty} \frac{V_{g_\lambda^{(s)}}(p_0, (\varepsilon + \sqrt{\frac{s}{4}})R)}{(\varepsilon + \sqrt{\frac{s}{4}})^3 R^3} \geq \frac{1}{3} \frac{\pi^2(1-\varepsilon)s}{(\varepsilon + \sqrt{\frac{s}{4}})^3}$$

for any sufficiently small $\varepsilon > 0$. Therefore the conclusion follows by taking the limit for $\varepsilon \to 0$. $\square$

From Proposition 8.3 and 8.7, we obtain the followings.

Theorem 8.8. *Let $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$ and $s > 0$. Then the volume growth of hyperkähler metric $g_\lambda^{(s)}$ is given by*

$$\lim_{r \to +\infty} \frac{V_{g_\lambda^{(s)}}(r)}{r^3} = \frac{8\pi^2}{3\sqrt{s}}.$$

Chapter III

The sequence of hyperkähler quotients

9 The approximations by ALE spaces

In this section we are going to show that each hyperkähler manifold of type A_∞ is approximated by the sequence of ALE spaces.

Let (X, g) be a Riemannian manifold. Then the norm $\| \cdot \|_{g_p}$ on $\Lambda^m T_p^* X$ is defined by giving the orthonormal basis $\{e^{i_1} \wedge \cdots \wedge e^{i_m}\}_{1 \le i_1 < \cdots < i_m \le dim X}$, where $\{e^1, \cdots e^{dim X}\}$ be the orthonormal basis of $T_p^* X$. Then the norm $\| \cdot \|_{(X,g)}$ on $\Omega^m(X)$ and $\Omega^m(X) \otimes Im\mathbb{H}$ is given by

$$\begin{aligned} \|\alpha\|_{(X,g)} &:= \sup_{p \in X} \|\alpha_p\|_{g_p}, \\ \|\beta\|_{(X,g)} &:= \sqrt{\|\beta_1\|^2_{(X,g)} + \|\beta_2\|^2_{(X,g)} + \|\beta_3\|^2_{(X,g)}} \end{aligned}$$

for $\alpha \in \Omega^m(X)$ and $\beta = i\beta_1 + j\beta_2 + k\beta_3 \in \Omega^m(X) \otimes Im\mathbb{H}$. In this section we give the proof of the following theorem.

Theorem 9.1. *Let $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$ be generic. Then there are ALE hyperkähler manifolds (X_n, ω_n) $(n = 1, 2, \cdots)$ of dimension 4, bounded open subsets $U_n \subset X_n$ and imbeddings $f_n : U_n \to X_\lambda$ such that $f_n(U_n) \subset f_{n+1}(U_{n+1})$ for all $n = 1, 2, \cdots$ and $\bigcup_n f_n(U_n) = X_\lambda$, which satisfy*

$$\|\omega_n - f_n^* \omega_\lambda\|_{(f_n^{-1}(B), f_n^* g_\lambda)} \to 0 \quad (n \to +\infty)$$

for any bounded open subset $B \subset X_\lambda$.

We fix a generic $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$. First of all we set ALE spaces (X_n, ω_n) and open subsets $U_n \subset X_n$ as follows.

We may take $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$ to be $|\lambda_n| \le |\lambda_{n+1}|$ for all $n \in \mathbb{N}$ without loss of generality. Then we set $\mathbf{I}(n) := \{0, 1, 2, \cdots, n\}$ and $(X_n, \omega_n) := (X_{\lambda^{<n>}}, \omega_{\lambda^{<n>}})$ where $\lambda^{<n>} \in (Im\mathbb{H})_0^{\mathbf{I}(n)}$ is defined by

$$\lambda^{<n>} = (\lambda_0, \lambda_1, \cdots, \lambda_n).$$

Put $U_n := \mu^{-1}_{\lambda^{<n>}}(B_{|\lambda_{n+1}|})$, where $B_R := \{\zeta \in Im\mathbb{H};\ |\zeta| < R\}$. Then it suffices to construct $f_n : U_n \to X_\lambda$ satisfying the assertion of Theorem 9.1.

For the construction of f_n, we prove some lemmas. Let $\nu : \mathbb{H} \to Im\mathbb{H}$ be the standard hyperkähler moment map given by $\nu(x) = xi\bar{x}$. For $\theta, \tau \in Im\mathbb{H}$, we put

$$[\theta, \tau] := \{s\theta + (1-s)\tau \in Im\mathbb{H};\ 0 \leq s \leq 1\}.$$

We often identify $Im\mathbb{H}$ with $\mathbb{R}i \oplus \mathbb{C}k$ and write $\theta = \theta_{\mathbb{R}}i - \theta_{\mathbb{C}}k$ for $\theta \in Im\mathbb{H}$, $\theta_{\mathbb{R}} \in \mathbb{R}$, $\theta_{\mathbb{C}} \in \mathbb{C}$.

Lemma 9.2. *Let $\theta, \tau \in Im\mathbb{H}\backslash\{0\}$. Then the following conditions are equivalent; (1) $0 \notin [\theta, \tau]$, (2) for each $x \in \nu^{-1}(\tau)$, there is a unique $y \in \nu^{-1}(\theta)$ which satisfies $|y - x| = \inf_{z \in \nu^{-1}(\theta)} |z - x|$.*

Proof. Fix $z, w, \alpha, \beta \in \mathbb{C}$ such that $\nu(z + wj) = \tau$, $\nu(\alpha + \beta j) = \theta$. If we put

$$f(t) := |(\alpha + \beta j)e^{it} - z + wj|^2 = |\alpha e^{it} - z|^2 + |\beta e^{-it} - w|^2$$

for $t \in \mathbb{R}$, then $f'(t) = 2iIm\{(\bar{\alpha}z + \beta\bar{w})e^{-it}\}$. Hence there is the unique point in $[0, 2\pi)$ which takes the minimum value of f if and only if $\bar{\alpha}z + \beta\bar{w} \neq 0$.

By putting $\theta = \theta_{\mathbb{R}}i - \theta_{\mathbb{C}}k$, $\tau = \tau_{\mathbb{R}}i - \tau_{\mathbb{C}}k$, we have

$$|\alpha|^2 = \frac{|\theta| + \theta_{\mathbb{R}}}{2},\ |\beta|^2 = \frac{|\theta| - \theta_{\mathbb{R}}}{2},\ \alpha\beta = \frac{\theta_{\mathbb{C}}}{2},$$
$$|z|^2 = \frac{|\tau| + \tau_{\mathbb{R}}}{2},\ |w|^2 = \frac{|\tau| - \tau_{\mathbb{R}}}{2},\ zw = \frac{\tau_{\mathbb{C}}}{2}.$$

Since we obtain $|\bar{\alpha}z + \beta\bar{w}|^2 = \frac{1}{2}(|\theta||\tau| + \theta_{\mathbb{R}}\tau_{\mathbb{R}} + Re(\bar{\theta_{\mathbb{C}}}\tau_{\mathbb{C}}))$, then the condition $\bar{\alpha}z + \beta\bar{w} \neq 0$ is equivalent to $0 \notin [\theta, \tau]$. ☐

If we take $\theta, \tau \in Im\mathbb{H}\backslash\{0\}$ to be $0 \notin [\theta, \tau]$, we can define $\psi_{\theta,\tau} : \nu^{-1}(\tau) \to \nu^{-1}(\theta)$ by

$$|\psi_{\theta,\tau}(x) - x| = \inf_{y \in \nu^{-1}(\theta)} |y - x|.$$

Lemma 9.3. *Let $\theta, \tau \in Im\mathbb{H}\backslash\{0\}$ and $0 \notin [\theta, \tau]$. Then $\psi_{\theta,\tau}$ is given by*

$$\psi_{\theta,\tau}(z + wj) = \frac{1}{\sqrt{2r}}\{(|\theta| + \theta_{\mathbb{R}})z + \theta_{\mathbb{C}}\bar{w} + (\theta_{\mathbb{C}}\bar{z} + (|\theta| - \theta_{\mathbb{R}})w)j\}$$

where $r := \sqrt{|\theta||\tau| + \theta_{\mathbb{R}}\tau_{\mathbb{R}} + Re(\bar{\theta_{\mathbb{C}}}\tau_{\mathbb{C}})}$.

Proof. From the proof of Lemma 9.2, $f(t)$ is minimum at

$$e^{it_0} = \frac{\bar{\alpha}z + \beta\bar{w}}{|\bar{\alpha}z + \beta\bar{w}|}.$$

Then we have

$$\begin{aligned}\psi_{\theta,\tau}(z+wj) &= (\alpha e^{it_0}+\beta e^{-it_0}j)\\ &= \frac{\bar{\alpha}z+\beta\bar{w}}{|\bar{\alpha}z+\beta\bar{w}|}\alpha+\frac{\alpha\bar{z}+\bar{\beta}w}{|\bar{\alpha}z+\beta\bar{w}|}\beta.\end{aligned}$$

From the proof of Lemma 9.2, $|\alpha|^2$, $|\beta|^2$, $\alpha\beta$ and $|\bar{\alpha}z+\beta\bar{w}|$ are dependent only on θ,τ. Hence we obtain $\psi_{\theta,\tau}(z+wj)=\frac{1}{\sqrt{2r}}\{(|\theta|+\theta_{\mathbb{R}})z+\theta_{\mathbb{C}}\bar{w}+(\theta_{\mathbb{C}}\bar{z}+(|\theta|-\theta_{\mathbb{R}})w)j\}$ □

In the case of $\theta=\tau=0$, we can also define $\psi_{\tau,\theta}$ since both $\nu^{-1}(\theta)$ and $\nu^{-1}(\tau)$ are consist of one element.

Next we define the map f_n. Take $\Lambda\in\mathbb{H}_0^{\mathbb{N}}$ and $\Lambda^{<n>}\in\mathbb{H}_0^{\mathbf{I}(n)}$ to be

$$\Lambda_n i\bar{\Lambda}_n=\lambda_n,\quad \Lambda^{<n>}=(\Lambda_0,\Lambda_1,\cdots,\Lambda_n)$$

for each $n\in\mathbb{N}$. Then we have identifications $X_\lambda=\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ and $X_n=X_{\lambda^{<n>}}=\hat{\mu}_{\Lambda^{<n>}}^{-1}(0)/G_{\Lambda^{<n>}}$. Now we take $x\in\hat{\mu}_{\Lambda^{<n>}}^{-1}(0)$ to satisfy

$$|\mu_{\lambda^{<n>}}([x])|=|x_l i\bar{x}_l-\lambda_l|<|\lambda_{n+1}|,$$

which is equivalent to taking $[x]\in U_n=\mu_{\lambda^{<n>}}^{-1}(B_{|\lambda_{n+1}|})$. Then we put $y_l(x)\in\mathbb{H}$ as

$$y_l(x):=\begin{cases} x_l & (0\le l\le n),\\ \psi_{\zeta+\lambda_l,\lambda_l}(\Lambda_l) & (n+1\le l),\end{cases}$$

where $\zeta=\mu_{\lambda^{<n>}}([x])$. Although the map $\psi_{\zeta+\lambda_l,\lambda_l}$ is defined if and only if $-\lambda_l\notin[0,\zeta]$ from Lemma 9.2, the inequality $|\mu_{\lambda^{<n>}}([x])|<|\lambda_{n+1}|$ ensures that $-\lambda_l\notin[0,\zeta]$ for all $l\ge n+1$. Thus $y_l(x)\in\mathbb{H}$ can be defined for all $x\in\hat{\mu}_{\Lambda^{<n>}}^{-1}(0)$ satisfying $|\mu_{\lambda^{<n>}}([x])|<|\lambda_{n+1}|$.

Lemma 9.4. *Let $x\in\hat{\mu}_{\Lambda^{<n>}}^{-1}(0)$ satisfy $|\mu_{\lambda^{<n>}}([x])|<|\lambda_{n+1}|$. Then $(y_l(x))_{l\in\mathbb{N}}$ is an element of $\hat{\mu}_\Lambda^{-1}(0)$, and we have $\mu_\lambda([(y_l(x))_{l\in\mathbb{N}}])=\mu_{\lambda^{<n>}}([x])$.*

Proof. It suffices to show that $(y_l(x))_{l\in\mathbb{N}}\in M_\Lambda$ and $y_l(x)i\overline{y_l(x)}-\lambda_l=\zeta=\mu_{\lambda^{<n>}}([x])$ for any $l\in\mathbb{N}$. Let $l\ge n+1$ and take $\alpha_l,\beta_l\in\mathbb{C}$ to be $\Lambda_l=\alpha_l+\beta_l j$. If we put

$$r_l^2:=\frac{1}{2}(|\zeta+\lambda_l||\lambda_l|+<\zeta+\lambda_l,\lambda_l>),$$

then $y_l(x)$ can be written as

$$\begin{aligned}y_l(x) &= \frac{1}{2r_l}\{(|\zeta+\lambda_l|+\zeta_{\mathbb{R}}+\lambda_{l,\mathbb{R}})\alpha_l+(\zeta_{\mathbb{C}}+\lambda_{l,\mathbb{C}})\bar{\beta}_l\\ &\quad +((\zeta_{\mathbb{C}}+\lambda_{l,\mathbb{C}})\bar{\alpha}_l+(|\zeta+\lambda_l|-\zeta_{\mathbb{R}}-\lambda_{l,\mathbb{R}})\beta_l)j\}.\end{aligned}$$

From $\Lambda_l i \overline{\Lambda_l} = \lambda_l$, we have

$$|\alpha_l|^2 = \frac{|\lambda_l| + \lambda_{l,\mathbb{R}}}{2}, \quad |\beta_l|^2 = \frac{|\lambda_l| - \lambda_{l,\mathbb{R}}}{2}, \quad \alpha_l \beta_l = \frac{\lambda_{l,\mathbb{C}}}{2}.$$

Then the value of $|y_l(x) - \Lambda_l|^2$ can be written as

$$|y_l(x) - \Lambda_l|^2 = |\zeta + \lambda_l| + |\lambda_l| - 2r_l = \frac{|\zeta|^2}{|\zeta + \lambda_l| + |\lambda_l| + 2r_l}.$$

Hence we obtain

$$\sum_{l=0}^{\infty} |y_l(x) - \Lambda_l|^2 \le \sum_{l=0}^{n} |y_l(x) - \Lambda_l|^2 + \sum_{l=n+1}^{\infty} \frac{|\zeta|^2}{|\lambda_l|} < +\infty,$$

which implies $(y_l(x))_{l\in\mathbb{N}} \in M_\Lambda$.

If $0 \le l \le n$, then $y_l(x) i \overline{y_l(x)} - \lambda_l = x_l i \overline{x}_l - \lambda_l = \zeta$. If $l \ge n+1$, then $y_l(x) i \overline{y_l(x)} = x_\zeta + \lambda_l$ from the definition of $\psi_{\zeta+\lambda_l,\lambda_l}$. Thus we have $y_l(x) i \overline{y_l(x)} - \lambda_l = \zeta$ for all $l \in \mathbb{N}$, which implies $\mu_\lambda([(y_l(x))_{l\in\mathbb{N}}]) = \mu_{\lambda^{<n>}}([x])$. $\square$

Now we can define open imbedding $f_n : U_n \to X_\lambda$ by $f_n([x]) := [(y_l(x))_{l\in\mathbb{N}}]$ due to Lemma 9.4. Then Lemma 9.4 says that $\mu_\lambda \circ f_n = \mu_{\lambda^{<n>}}$. Moreover the map f_n is S^1-equivariant which is easy to see from the definition. Since the image of f_n is given by $\mu_\lambda^{-1}(B_{|\lambda_{n+1}|})$, then we have

$$\bigcup_{n\in\mathbb{N}} f_n(U_n) = \bigcup_{n\in\mathbb{N}} \mu_\lambda^{-1}(B_{|\lambda_{n+1}|}) = X_\lambda.$$

Thus we have finished setting the datas in Theorem 9.1. What we have to do next is the estimate of the norm of $f_n^*\omega_\lambda - \omega_n$. From Theorem 2.9, the hyperkähler structures ω_λ and ω_n are determined perfectly by $(\Phi_\lambda, \Gamma_\lambda)$ and $(\Phi_n, \Gamma_n) := (\Phi_{\lambda^{<n>}}, \Gamma_{\lambda^{<n>}})$, respectively.

Next we are going to see that the norm of $f_n^*\omega_\lambda - \omega_n$ can be estimated by the norm of $\Phi_\lambda - \Phi_n$ and $f_n^*\Gamma_\lambda - \Gamma_n$ from the above. Since f_n is an S^1-equivariant map, $f_n^*\Gamma_\lambda$ is also an S^1-connection over $\mu_{\lambda^{<n>}} : U_n \cap X_\lambda^* \to B_{|\lambda_{n+1}|} \cap Y_\lambda$. Then there exists a unique 1-form $\gamma_n \in \Omega^1(B_{|\lambda_{n+1}|} \cap Y_\lambda)$ which satisfies $f_n^*\Gamma_\lambda - \Gamma_n = \mu_{\lambda^{<n>}}{}^*\gamma_n$ for each $n \in \mathbb{N}$. Let $U \subset Im\mathbb{H}$ be an open subset and $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in Im\mathbb{H} \cong \mathbb{R}^3$ be the Cartesian coordinate. Then we define the norm $\|\cdot\|_U$ on $C^\infty(U)$ and $\Omega^1(U)$ by

$$\begin{aligned} \|f\|_U &:= \sup_{x\in U} |f(x)| \\ \|\alpha\|_U &:= \sup_{x\in U} \sqrt{\alpha_1(x)^2 + \alpha_2(x)^2 + \alpha_3(x)^2} \end{aligned}$$

for $f \in C^\infty(U)$ and $\alpha = \alpha_1 d\zeta_1 + \alpha_2 d\zeta_2 + \alpha_3 d\zeta_3 \in \Omega^1(U)$.

Proposition 9.5. *There exists a constant $C > 0$ independent of $n \in \mathbb{N}$ and $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$ which satisfies*

$$\|f_n^*\omega_\lambda - \omega_n\|_{(f_n^{-1}(B_R), f_n^*g_\lambda)} \leq C\Big(\Big\|\frac{\gamma_n}{\Phi_\lambda}\Big\|_{B_R} + \Big\|\frac{\Phi_\lambda - \Phi_n}{\Phi_\lambda}\Big\|_{B_R}\Big)$$

for any $R > 0$.

Proof. From the description of ω_λ in the proof of Theorem 2.9, the orthonormal basis of $\Lambda^2 T^*X_\lambda$ with respect to g_λ is given by

$$\{\Gamma_\lambda \wedge d\mu_{\lambda,a},\ \Phi_\lambda d\mu_{\lambda,b} \wedge d\mu_{\lambda,c};\ 1 \leq a \leq 3,\ 1 \leq b < c \leq 3\}.$$

Since we have

$$\begin{aligned} f_n^*\omega_{\lambda,1} &= f_n^*\Big(d\mu_{\lambda,1} \wedge \frac{\Gamma_\lambda}{2\sqrt{-1}} + \Phi_\lambda d\mu_{\lambda,2} \wedge d\mu_{\lambda,3}\Big) \\ &= \frac{1}{2\sqrt{-1}} d\mu_{\lambda,1} \wedge (\Gamma_\lambda + \mu_{\lambda<n>}^*\gamma_n) + \Phi_\lambda d\mu_{\lambda<n>,2} \wedge d\mu_{\lambda<n>,3}, \\ \omega_{\lambda<n>,1} &= \frac{1}{2\sqrt{-1}} d\mu_{\lambda,1} \wedge \Gamma_\lambda + \Phi_n d\mu_{\lambda<n>,2} \wedge d\mu_{\lambda<n>,3}, \end{aligned}$$

then $f_n^*\omega_{\lambda,1} - \omega_{\lambda<n>,1}$ is given by

$$f_n^*\omega_{\lambda,1} - \omega_{\lambda<n>,1} = \mu_{\lambda<n>}{}^*\Big\{\frac{1}{2\sqrt{-1}} d\zeta_1 \wedge \gamma_n + (\Phi_\lambda - \Phi_n) d\zeta_2 \wedge d\zeta_3\Big\}.$$

Thus we have

$$\begin{aligned} &\|f_n^*\omega_{\lambda,1} - \omega_{\lambda<n>,1}\|_{(\mu_{\lambda<n>}{}^{-1}(B_R), f_n^*g_\lambda)} \\ \leq\ &\sqrt{\Big\|\frac{\gamma_{n,2}}{4\Phi_\lambda}\Big\|_{B_R}^2 + \Big\|\frac{\gamma_{n,3}}{4\Phi_\lambda}\Big\|_{B_R}^2 + \Big\|\frac{\Phi_\lambda - \Phi_n}{\Phi_\lambda}\Big\|_{B_R}^2}. \end{aligned}$$

By estimating $\|f_n^*\omega_{\lambda,a} - \omega_{\lambda<n>,a}\|_{(\mu_{\lambda<n>}{}^{-1}(B_R), f_n^*g_\lambda)}$ for $a = 2, 3$ similarly to the above, we obtain the assertion. □

From Proposition 9.5, we need to evaluate $\gamma_n \in \Omega^1(B_{|\lambda_{n+1}|} \cap Y_\lambda)$. First of all we write down the S^1-connection Γ_λ explicitly using coordinate $x_n = z_n + w_n$ for $(x_n)_{n\in\mathbf{I}} \in \hat{\mu}_\Lambda^{-1}(0)$. Let $\Gamma_{\mathbb{H}} \in \Omega^1(\mathbb{H}\backslash\{0\}, \sqrt{-1}\mathbb{R})$ be an S^1-connection on $\nu|_{\mathbb{H}\backslash\{0\}} : \mathbb{H}\backslash\{0\} \to Im\mathbb{H}\backslash\{0\}$ defined by

$$\begin{aligned} \nu(x) &:= xi\bar{x}, \\ (\Gamma_{\mathbb{H}})_{z+wj} &:= \frac{\sqrt{-1}Im(\bar{z}dz - \bar{w}dw)}{|z|^2 + |w|^2}. \end{aligned}$$

for $x = z + wj \in \mathbb{H}$. Then the pair $(\Gamma_{\mathbb{H}}, \Phi_{\mathbb{H}})$ induces the Euclidean metric on $\mathbb{H} = \mathbb{R}^4$ from the corresponding of Theorem 2.9, where the harmonic function $\Phi_{\mathbb{H}} : Im\mathbb{H}\backslash\{0\} \to \mathbb{R}$ is given by $\Phi_{\mathbb{H}}(h) := \frac{1}{4|h|}$.

By using $\Gamma_{\mathbb{H}}$, we can describe Γ_λ explicitly as follows. Let $d_x : \mathfrak{g}_\Lambda \to M$ be given by $d_x\xi := \xi_x^*$ for $x \in M_\Lambda$ and $\xi \in \mathfrak{g}_\Lambda$. Since M is a Hilbert space, we have the adjoint operator $d_x^* : M \to \mathfrak{g}_\Lambda$. Then for each $[x] \in X_\lambda$, $T_{[x]}X_\lambda$ is identified with $Kerd_x^* \cap Ker(d\hat{\mu}_\Lambda)_x$, which is a 4-dimensional subspace of M. Thus we can regard $(\Gamma_\lambda)_{[x]}$ as an element of the dual space of $Kerd_x^* \cap Ker(d\hat{\mu}_\Lambda)_x$. Then for all $P = (P_n)_{n\in\mathbb{Z}} \in Kerd_x^* \cap Ker(d\hat{\mu}_\Lambda)_x$, we have

$$(\Gamma_\lambda)_{[x]}(P) = \sum_{n\in\mathbf{I}} (\Gamma_{\mathbb{H}})_{x_n}(P_n).$$

Proposition 9.6. *The form γ_n is given by*

$$\gamma_n(\zeta) = \sum_{l\geq n+1} \frac{\sqrt{-1}Im\{(\zeta_{\mathbb{R}}\overline{\lambda_{l,\mathbb{C}}} - \overline{\zeta_{\mathbb{C}}}\lambda_{l,\mathbb{R}})d\zeta_{\mathbb{C}} + \overline{\zeta_{\mathbb{C}}}\lambda_{l,\mathbb{R}}d\zeta_{\mathbb{R}}\}}{2(|\zeta+\lambda_l||\lambda_l| + <\zeta+\lambda_l, \lambda_l>)|\zeta+\lambda_l|}$$

for $\zeta = \zeta_{\mathbb{R}}i - \zeta_{\mathbb{C}}k \in Im\mathbb{H}$.

Proof. Let $[x] \in U_n$ and take $z_l, w_l \in \mathbb{C}$ to be $x_l = z_l + w_l j$. Then Γ_n is given by

$$(\Gamma_n)_{[x]} = \sum_{l=0}^{n} \frac{\sqrt{-1}Im(\overline{z_l}dz_l - \overline{w_l}dw_l)}{|z_l|^2 + |w_l|^2}.$$

On the other hand, take $p_l, q_l \in \mathbb{C}$ to be

$$y_l(x) = p_l + q_l j.$$

Then $f_n^*\Gamma_\lambda$ is given by

$$(f_n^*\Gamma_\lambda)_{[x]} = \sum_{l=0}^{n} \frac{\sqrt{-1}Im(\overline{z_l}dz_l - \overline{w_l}dw_l)}{|z_l|^2 + |w_l|^2} + \sum_{l\geq n+1} \frac{\sqrt{-1}Im(\overline{p_l}dp_l - \overline{q_l}dq_l)}{|p_l|^2 + |q_l|^2}.$$

Hence $\mu_{\lambda<n>}{}^*\gamma_n$ is given by $\sum_{l\geq n+1} \frac{\sqrt{-1}Im(\overline{p_l}dp_l - \overline{q_l}dq_l)}{|p_l|^2+|q_l|^2}$. Recall that p_l and q_l are written as

$$\begin{aligned} p_l &= \frac{1}{2r_l}\{(|\zeta+\lambda_l| + \zeta_{\mathbb{R}} + \lambda_{l,\mathbb{R}})\alpha_l + (\zeta_{\mathbb{C}} + \lambda_{l,\mathbb{C}})\bar{\beta}_l\}, \\ q_l &= \frac{1}{2r_l}\{(\zeta_{\mathbb{C}} + \lambda_{l,\mathbb{C}})\bar{\alpha}_l + (|\zeta+\lambda_l| - \zeta_{\mathbb{R}} - \lambda_{l,\mathbb{R}})\beta_l\}. \end{aligned}$$

Therefore we have

$$\begin{aligned}
&\frac{\sqrt{-1}Im(\overline{p_l}dp_l - \overline{q_l}dq_l)}{|p_l|^2 + |q_l|^2} \\
= \quad & \mu_{\lambda < n >}{}^*\left\{\frac{\sqrt{-1}Im\{(\zeta_{\mathbb{R}}\overline{\lambda_{l,\mathbb{C}}} - \overline{\zeta_{\mathbb{C}}}\lambda_{l,\mathbb{R}})d\zeta_{\mathbb{C}} + \overline{\zeta_{\mathbb{C}}}\lambda_{l,\mathbb{R}}d\zeta_{\mathbb{R}}\}}{4r_l^2|\zeta + \lambda_l|}\right\}
\end{aligned}$$

for $l \geq n+1$ which gives the assertion. □

Now the Cartesian coordinate $d\zeta_1, d\zeta_2, d\zeta_3$ gives the identification

$$\Omega^1(B_{|\lambda_{n+1}|}) = C^\infty(B_{|\lambda_{n+1}|}) \otimes \mathbb{R}^3.$$

Then $\gamma_n \in C^\infty(B_{|\lambda_{n+1}|}) \otimes \mathbb{R}^3$ is also written as

$$\gamma_n(\zeta) = \sum_{l \geq n+1} \frac{\zeta \times \lambda_l}{2(|\zeta + \lambda_l||\lambda_l| + < \zeta + \lambda_l, \lambda_l >)|\zeta + \lambda_n|},$$

where $\times$ is the exterior product on $\mathbb{R}^3$.

The next proposition completes the proof of Theorem 9.1.

Proposition 9.7. *Let $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$ be generic. Then we have*

$$\lim_{n \to +\infty} \left\| \frac{\gamma_n}{\Phi_\lambda} \right\|_{B_R} = \lim_{n \to +\infty} \left\| \frac{\Phi_\lambda - \Phi_n}{\Phi_\lambda} \right\|_{B_R} = 0$$

for any $R > 0$.

Proof. Define $t_l \in [0, 2\pi)$ by

$$< \zeta + \lambda_l, \lambda_l > = |\zeta + \lambda_l||\lambda_l| \cos t_l$$

for $l \geq n+1$. Then we have

$$\begin{aligned}
|\gamma_n(\zeta)| &\leq \sum_{l \geq n+1} \frac{|(\zeta + \lambda_l) \times \lambda_l|}{2|\zeta + \lambda_l|^2 |\lambda_l| \cos t_l} \\
&= \sum_{l \geq n+1} \frac{|\zeta + \lambda_l||\lambda_l| \sin t_l}{2|\zeta + \lambda_l|^2 |\lambda_l| \cos t_l} \\
&= \frac{1}{2} \sum_{l \geq n+1} \frac{\tan \frac{t_l}{2}}{|\zeta + \lambda_l|},
\end{aligned}$$

which gives

$$\left\|\frac{\gamma_n}{\Phi_\lambda}\right\|_{B_R} \leq \frac{1}{2}\sup_{\zeta\in B_R}\frac{\sum_{l\geq n+1}\tan\frac{t_l}{2}/|\zeta+\lambda_l|}{\sum_{l=0}^{\infty}1/|\zeta+\lambda_l|}$$
$$\leq \frac{1}{2}\sup_{\zeta\in B_R}\frac{\sum_{l\geq n+1}1/|\zeta+\lambda_l|}{\sum_{l=0}^{\infty}1/|\zeta+\lambda_l|}$$

since $0\leq\tan\frac{t_l}{2}\leq 1$. On the other hand we have

$$\left\|\frac{\Phi_\lambda-\Phi_n}{\Phi_\lambda}\right\|_{B_R} = \sup_{\zeta\in B_R}\frac{\sum_{l\geq n+1}1/|\zeta+\lambda_l|}{\sum_{l=0}^{\infty}1/|\zeta+\lambda_l|}.$$

Thus it is enough to show that

$$\lim_{n\to+\infty}\sup_{\zeta\in B_R}\frac{\sum_{l\geq n+1}1/|\zeta+\lambda_l|}{\sum_{l=0}^{\infty}1/|\zeta+\lambda_l|}=0$$

for any $R>0$.

Now we put

$$S_n(\zeta) := \sum_{l=0}^{n}\frac{1}{|\zeta+\lambda_l|},$$
$$T_n(\zeta) := \sum_{l\geq n+1}\frac{1}{|\zeta+\lambda_l|},$$
$$S_{n,R} := \inf_{\zeta\in B_R}S_n(\zeta)>0.$$

Then we have

$$\frac{T_n(\zeta)}{S_n(\zeta)+T_n(\zeta)}\leq\frac{T_n(\zeta)}{S_{n,R}}\leq\frac{1}{S_{n,R}}\sum_{l\geq n+1}\frac{1}{|\lambda_l|-|\zeta|}.$$

If we take $n\in\mathbb{N}$ sufficiently large such that $|\lambda_{n+1}|>R$, then

$$\sup_{\zeta\in B_R}\frac{\sum_{l\geq n+1}1/|\zeta+\lambda_l|}{\sum_{l=0}^{\infty}1/|\zeta+\lambda_l|}\leq\frac{1}{S_{n,R}}\sum_{l\geq n+1}\frac{1}{|\lambda_l|-R}$$

Since $S_n(\zeta)\leq S_{n'}(\zeta)$ holds for $n\leq n'$, we have $S_n(R)\leq S_{n'}(R)$, which gives

$$\lim_{n'\to+\infty}\frac{1}{S_{n',R}}\sum_{l\geq n'+1}\frac{1}{|\lambda_l|-R}\leq\lim_{n'\to+\infty}\frac{1}{S_{n,R}}\sum_{l\geq n'+1}\frac{1}{|\lambda_l|-R}=0$$

by fixing $n\in\mathbb{N}$ to be $|\lambda_{n+1}|>R$. $\square$

The Taub-NUT deformations of ALE spaces of type A_k are called multi-Taub-NUT spaces. Then we can show the following theorem by the parallel argument to the proof of Theorem 9.1.

Theorem 9.8. *Let $\lambda \in (Im\mathbb{H})_0^{\mathbb{N}}$ be generic. Then there are multi-Taub-NUT spaces (X_n, ω_n) $(n = 1, 2, \cdots)$ of dimension 4, bounded open subsets $U_n \subset X_n$ and imbeddings $f_n : U_n \to X_\lambda$ such that $f_n(U_n) \subset f_{n+1}(U_{n+1})$ for all $n = 1, 2, \cdots$ and $\bigcup_n f_n(U_n) = X_\lambda$, which satisfy*

$$\|\omega_n - f_n^*\omega_\lambda^{(s)}\|_{(f_n^{-1}(B), f_n^*g_\lambda^{(s)})} \to 0 \quad (n \to +\infty)$$

for any bounded open subset $B \subset X_\lambda$.

10 The limit of the sequence of hyperkähler manifolds of type A_∞

In Sections 10 and 11, we will construct hyperkähler metrics on the elliptic fibration $f : X \to B$ for some open subset $B \subset \mathbb{C}$ contains 0, whose singular fiber $f^{-1}(0)$ is Kodaira type I_b. They will be constructed by considering the "limit" of the hyperkähler quotients of type A_∞. In particular, the metric of the case $b = 1$ is Ooguri-Vafa metric. We will construct a hyperkähler metric on a manifold which is diffeomorphic to the manifold of type A_∞ in Section 10, and construct $\mathbb{Z}$-action preserving the metrics in Section 11.

Let b be a positive integer and fix real numbers $0 < l_1 < l_2 < \cdots < l_b = l$. Then we define $\Lambda^{(\infty)} \in \mathbb{H}^{\mathbb{Z}}$ and $\Lambda^{(0)}, \Lambda^{(p)} \in \mathbb{H}_0^{\mathbb{Z}}$ by

$$\Lambda_{qb+r}^{(\infty)} := \begin{cases} \sqrt{ql + l_r}\, i & (a \geq 0), \\ -\sqrt{|ql + l_r|}\, k & (a < 0), \end{cases}$$

$$\Lambda_n^{(0)} := \begin{cases} max\{|\Lambda_n^{(\infty)}|^2,\ |\Lambda_n^{(\infty)}|\} i & (n \geq 0), \\ max\{|\Lambda_n^{(\infty)}|^2,\ |\Lambda_n^{(\infty)}|\} k & (n < 0), \end{cases}$$

$$\Lambda_n^{(p)} := \begin{cases} \Lambda_n^{(0)} & (|n| \geq p), \\ \Lambda_n^{(\infty)} & (|n| < p), \end{cases}$$

for $p \in \mathbb{Z}_{\geq 0}$. It is easy to see that each $\Lambda^{(p)} \in \mathbb{H}_0^{\mathbb{Z}}$ is generic, then we have the hyperkähler quotients $X_p := \hat{\mu}_{\Lambda^{(p)}}^{-1}(0)/G_{\Lambda^{(p)}}$ from Theorem 2.5. Then Theorem 2.9 gives

$$\begin{aligned} Y_p &= Im\mathbb{H}\backslash\{-\Lambda_n^{(p)} i \bar{\Lambda}_n^{(p)};\ n \in \mathbb{Z}\} = Im\mathbb{H}\backslash\{-\lambda_n^{(p)} i;\ n \in \mathbb{Z}\}, \\ \Phi_p(h) &= \frac{1}{4}\sum_{n\in\mathbb{Z}} \frac{1}{|h + \lambda_n^{(p)} i|}, \end{aligned}$$

where $\lambda_n^{(p)} \in \mathbb{R}$ is defined by $\lambda_n^{(p)} i = \Lambda_n^{(p)} i \bar{\Lambda}_n^{(p)}$. We denote by $\mu_p : X_p \to Im\mathbb{H}$ the hyperkähler moment map defined by $\mu_p([x]) := x_n i \bar{x}_n - \lambda_n^{(p)} i$ and by Γ_p the S^1-connection on principal S^1-bundle $\mu_p : X_p^* \to Y_p$ satisfying $\frac{d\Gamma_p}{2\sqrt{-1}} = \mu_p^*(*d\Phi_p)$.

Next we define the S^1-equivariant maps $\Psi_{q,p}$ from an open dense subset of X_p to X_q for $p < q$. We apply the method in Section 9. Put

$$L_p := \{ti;\ t \le \lambda_{-p}^{(\infty)},\ \lambda_p^{(\infty)} \le t\},$$
$$X_p^0 := X_p \backslash \mu_p^{-1}(L_p),$$

and define $\Psi_{q,p} : X_p^0 \to X_q$ by

$$\Psi_{q,p}([x]) := [(\psi_{\mu_q([x]) + \lambda_n^{(q)} i,\ \mu_p([x]) + \lambda_n^{(p)} i}(x_n))_{n\in\mathbb{Z}}]$$

for $x = (x_n)_{n\in\mathbb{Z}} \in \hat{\mu}_{\Lambda_p}^{-1}(0)$ and $p < q$. Then $\Psi_{q,p}$ is injective S^1-equivariant map and satisfies $\Psi_{q,p}(\mu_p^{-1}(h)) = \mu_q^{-1}(h)$ for all $h \in Im\mathbb{H}\backslash L_p$. Moreover the maps satisfies $\Psi_{r,q} \circ \Psi_{q,p} = \Psi_{r,p}$ for $p < q < r$ then we have a sequence of open embeddings

$$X_0^0 \subset X_1^0 \subset X_2^0 \subset \cdots \subset X_p^0 \subset X_{p+1}^0 \subset \cdots,$$

which defines a 4-dimensional manifold

$$X_\infty := \bigcup_{p=0}^{\infty} X_p^0$$

and the natural open embeddings $\Psi_p : X_p^0 \to X_\infty$.

From the S^1-equivariance of $\Psi_{q,p}$, X_∞ has an S^1-action and an S^1-invariant map $\mu_\infty : X_\infty \to Im\mathbb{H}$ induced from μ_p and the condition $\Psi_{q,p}(\mu_p^{-1}(h)) = \mu_q^{-1}(h)$.

Proposition 10.1. *The map $\mu_\infty : X_\infty \to Im\mathbb{H}$ is surjective. Moreover, S^1 acts on $\mu_\infty^{-1}(h)$ freely if h is an element of $Y_\infty := Im\mathbb{H}\backslash\{-\lambda_n^{(\infty)} i;\ n \in \mathbb{Z}\}$, trivially if h is an element of $\{-\lambda_n^{(\infty)} i;\ n \in \mathbb{Z}\}$.*

Proof. Since we have $\bigcup_{p=0}^{\infty} Im\mathbb{H}\backslash L_p = Im\mathbb{H}$, $\mu_\infty : X_\infty \to Im\mathbb{H}$ is surjective. For all $h \in Im\mathbb{H}$, there is a sufficiently large p such that $h \in Im\mathbb{H}\backslash L_p$. Then $\mu_\infty^{-1}(h)$ is included in X_p^0, so the S^1-action on $\mu_\infty^{-1}(h)$ is equivalent to the S^1-action on $\mu_p^{-1}(h)$. $\square$

To construct the hyperkähler metric on X_∞, next we consider the convergence of $\{(\Gamma_p, \Phi_p)\}_{p=0,1,\cdots}$.

First of all we deal with the convergence of Γ_p. Since $\Psi_{q,p}$ is S^1-equivariant, $\Psi_{q,p}^*\Gamma_q$ is an S^1-connection on X_p^0. Then there is a unique 1-form $\Gamma_{q,p} \in \Omega^1(Im\mathbb{H}\backslash L_p, \sqrt{-1}\mathbb{R})$ for $p < q$ which satisfies $\Psi_{q,p}^*\Gamma_q - \Gamma_p = \mu_p^*\Gamma_{q,p}$ on X_p^0. We are going to describe $\Gamma_{q,p}$ explicitly by using the Cartesian coordinate of $Im\mathbb{H} = \mathbb{R}^3$. For the description, we need some preparations.

For each $\rho \in Im\mathbb{H}\backslash\{0\}$, define $\psi_\rho : \mathbb{H}\backslash\nu^{-1}([0, -\rho]) \to \mathbb{H}$ by $\psi_\rho(x) := \psi_{\nu(x)+\rho,\nu(x)}(x)$. If $\rho = 0$ we define $\psi_0 := id_{\mathbb{H}}$. Since $\Psi_{q,p}$ is written as

$$\Psi_{q,p}([x]) = [(\psi_{(\lambda_n^{(q)} - \lambda_n^{(p)})i}(x_n))_{n\in\mathbb{Z}}],$$

hence we need to describe $\psi_\rho{}^*\Gamma_{\mathbb{H}} - \Gamma_{\mathbb{H}} \in \Omega^1(\mathbb{H}\backslash\nu^{-1}([0, -\rho]), \sqrt{-1}\mathbb{R})$ for the calculation of $\Psi_{q,p}{}^*\Gamma_q - \Gamma_p$. Now we put

$$(\gamma_{\rho,\lambda})_\zeta := \frac{\sqrt{-1}\rho(|\zeta + \lambda + i\rho| + |\zeta + \lambda|)Im\{(\overline{\zeta_{\mathbb{C}} + \lambda_{\mathbb{C}}})d\zeta_{\mathbb{C}}\}}{2\{|\zeta + \lambda + i\rho||\zeta + \lambda| + < \zeta + \lambda + i\rho, \zeta + \lambda >\}|\zeta + \lambda + i\rho||\zeta + \lambda|}$$

for $\zeta, \lambda \in Im\mathbb{H}$ and $\rho \in \mathbb{R}$.

Lemma 10.2. *Define $\nu_\lambda : \mathbb{H} \to Im\mathbb{H}$ by $\nu_\lambda(x) := \nu(x) - \lambda$ for $\lambda \in Im\mathbb{H}$. Then we have*

$$\psi_{i\rho}{}^*\Gamma_{\mathbb{H}} - \Gamma_{\mathbb{H}} = \nu_\lambda{}^*\gamma_{\rho,\lambda}$$

on $\mathbb{H}\backslash\nu^{-1}([0, -i\rho])$ for $\rho \in \mathbb{R}$.

Proof. For $z, w \in \mathbb{C}$, we have

$$\psi_\rho(z + wj) = \frac{1}{\sqrt{2}r}\{(\sigma + \rho)z + (\sigma - \rho)wj\}$$

where we put $r = \sqrt{|\theta||\tau| + < \theta, \tau >}$, $\sigma = |\theta| + |\tau|$, $\theta = \nu(z + wj) + i\rho$, $\tau = \nu(z + wj)$ from Lemma 9.3 and $\theta_{\mathbb{C}} = \tau_{\mathbb{C}}$. If we put $\alpha := \frac{\sigma+\rho}{\sqrt{2}r}z$ and $\beta := \frac{\sigma-\rho}{\sqrt{2}r}w$, then we have

$$\psi_{i\rho}{}^*\Gamma_{\mathbb{H}} = \frac{\sqrt{-1}Im(\bar{\alpha}d\alpha - \bar{\beta}d\beta)}{|\alpha|^2 + |\beta|^2}.$$

Since we have

$$\begin{aligned} \bar{\alpha}d\alpha &= |z|^2\frac{\sigma + \rho}{\sqrt{2}r}d\left(\frac{\sigma + \rho}{\sqrt{2}r}\right) + \left(\frac{\sigma + \rho}{\sqrt{2}r}\right)^2\bar{z}dz \\ \bar{\beta}d\beta &= |w|^2\frac{\sigma - \rho}{\sqrt{2}r}d\left(\frac{\sigma - \rho}{\sqrt{2}r}\right) + \left(\frac{\sigma - \rho}{\sqrt{2}r}\right)^2\bar{w}dw \end{aligned}$$

and $\frac{\sigma \pm \rho}{\sqrt{2}r}$ are real valued, we have

$$(\psi_{i\rho}{}^*\Gamma_{\mathbb{H}})_{z+wj} = \frac{\sqrt{-1}}{|\theta|}\left\{\left(\frac{\sigma+\rho}{\sqrt{2}r}\right)^2 Im(\bar{z}dz) - \left(\frac{\sigma-\rho}{\sqrt{2}r}\right)^2 Im(\bar{w}dw)\right\}.$$

Thus we obtain

$$\begin{aligned}(\psi_{i\rho}{}^*\Gamma_{\mathbb{H}} - \Gamma_{\mathbb{H}})_{z+wj} &= \frac{\sqrt{-1}}{|\theta||\tau|}\left\{\left(|\tau|\left(\frac{\sigma+\rho}{\sqrt{2}r}\right)^2 - |\theta|\right) Im(\bar{z}dz)\right. \\ &\qquad \left. -\left(|\tau|\left(\frac{\sigma-\rho}{\sqrt{2}r}\right)^2 - |\theta|\right) Im(\bar{w}dw)\right\}.\end{aligned}$$

From the definition of θ and τ, we have $\theta_{\mathbb{R}} = \tau_{\mathbb{R}} + \rho$ and $\theta_{\mathbb{C}} = \tau_{\mathbb{C}}$, which imply that we can put

$$< \theta, \tau >= |\tau|^2 + \rho\tau_{\mathbb{R}}, \ |\theta|^2 = |\tau|^2 + 2\rho\tau_{\mathbb{R}} + \rho^2.$$

Then it follows that

$$|\tau|\left(\frac{\sigma+\rho}{\sqrt{2}r}\right)^2 - |\theta| = \frac{\rho\sigma(-\tau_{\mathbb{R}} \pm |\tau|)}{r^2}.$$

Since we have $-\tau_{\mathbb{R}} + |\tau| = 2|w|^2$ and $-\tau_{\mathbb{R}} - |\tau| = 2|z|^2$, we obtain

$$\begin{aligned}(\psi_{i\rho}{}^*\Gamma_{\mathbb{H}} - \Gamma_{\mathbb{H}})_{z+wj} &= \frac{\sqrt{-1}\rho\sigma}{|\theta||\tau|r^2}\left\{2|w|^2 Im(\bar{z}dz) + 2|z|^2 Im(\bar{w}dw)\right\} \\ &= \frac{\sqrt{-1}\rho\sigma}{|\theta||\tau|r^2}\{\overline{zw}d(zw) - zwd(\overline{zw})\} \\ &= (\nu_\lambda{}^*\gamma)_{z+wj}\end{aligned}$$

□

Proposition 10.3. $\Gamma_{q,p} \in \Omega^1(Y_p, \sqrt{-1}\mathbb{R})$ *is given by*

$$\Gamma_{q,p} = \sum_{n=p}^{q-1} \gamma_{\lambda_n^{(q)} - \lambda_n^{(p)}, \lambda_n^{(p)}}.$$

Proof. We have the assertion from Lemma 10.4. □

From the explicit description of $\Gamma_{q,p}$, there is a limit

$$\Gamma_{\infty,p} := \lim_{q\to+\infty} \Gamma_{q,p} = \sum_{n=p}^{+\infty} \gamma_{\lambda_n^{(q)} - \lambda_n^{(p)}, \lambda_n^{(p)}}.$$

Proposition 10.4. *There is a unique S^1-connection $\Gamma_\infty \in \Omega^1(X^*_\infty, \sqrt{-1}\mathbb{R})$ whose restriction to $X^0_p \cap X^*_\infty$ is equal to $\Gamma_p + {\mu_p}^*\Gamma_{\infty,p}$ for all $p \geq 0$*

Proof. It suffices to show

$$\Psi_{q,p}{}^*(\Gamma_q + {\mu_q}^*\Gamma_{\infty,q}) = \Gamma_p + {\mu_p}^*\Gamma_{\infty,p}$$

on $X^0_p \cap X^*_\infty$ for $p < q$. By the definition of $\Gamma_{\infty,p}$, we have $\Gamma_{q,p} = \Gamma_{\infty,p} - \Gamma_{\infty,q}$. Then we have

$$\begin{aligned} \Psi_{q,p}{}^*(\Gamma_q + {\mu_q}^*\Gamma_{\infty,q}) &= \Gamma_p + {\mu_p}^*\Gamma_{q,p} + \Psi_{q,p}{}^*{\mu_q}^*\Gamma_{\infty,q} \\ &= \Gamma_p + {\mu_p}^*(\Gamma_{\infty,p} - \Gamma_{\infty,q}) + {\mu_p}^*\Gamma_{\infty,q} \\ &= \Gamma_p + {\mu_p}^*\Gamma_{\infty,p}. \end{aligned}$$

□

Thus we have obtained the limit Γ_∞ of $\{\Gamma_p\}_{p\geq 0}$. Next we consider the limit of $\{\Phi_p\}_{p\geq 0}$, but the sequence does not convergent. Then we consider the sequence of 1-forms $\{d\Phi_p\}_{p\geq 0}$ instead of $\{\Phi_p\}_{p\geq 0}$. Then it is easy to see that there is the limit

$$\alpha := \lim_{p\to+\infty} d\Phi_p = -\sum_{n\in\mathbb{Z}} \frac{(\zeta_{\mathbb{R}} + \lambda^\infty_n)d\zeta_{\mathbb{R}} + Re(\bar{\zeta}_{\mathbb{C}} d\zeta_{\mathbb{C}})}{|\zeta + \lambda^\infty_n i|^3}$$

which is closed 1-form over Y_∞. Then there is a primitive function Φ^λ_∞ of α given by

$$\Phi^\lambda_\infty(\zeta) := \int_c \alpha,$$

for $\lambda \in Y_\infty$, where we take a C^1 curve $c : [0,1] \to Y_\infty$ to be $c(0) = \lambda$ and $c(1) = \zeta$. Since Y_∞ is simply-connected, the definition of Φ^λ_∞ is not independent of c.

Proposition 10.5. *For all $\lambda \in Y_\infty$, Φ^λ_∞ is a harmonic function.*

Proof. By definition, we have $d\Phi^\lambda_\infty = \alpha = \lim_{p\to+\infty} d\Phi_p$. Then we have

$$\begin{aligned} *d\Phi^\lambda_\infty &= \lim_{p\to+\infty} *d\Phi_p \\ &= \lim_{p\to+\infty} \frac{F_{\Gamma_p}}{2\sqrt{-1}} = \frac{F_{\Gamma_\infty}}{2\sqrt{-1}} \end{aligned}$$

where F_Γ is the curvature of a connection Γ. Since the curvature form F_{Γ_∞} is closed, then Φ^λ_∞ is harmonic. □

Proposition 10.6. *Fix $\lambda \in Y_\infty$ and compact subset $K \subset \mathbb{C}$. Then there is a constant $a \in \mathbb{R}$ such that $\Phi_\infty^\lambda + a$ is positive valued on $p_{\mathbb{C}}^{-1}(K)$, where $p_{\mathbb{C}} : Im\mathbb{H} = i\mathbb{R} \oplus j\mathbb{C} \to j\mathbb{C}$ is the orthogonal projection.*

Proof. Define $\mathbb{Z}$-action on $Im\mathbb{H}$ by $\zeta \mapsto \zeta + il$. Since Φ_∞^λ is invariant under this $\mathbb{Z}$-action, we can regard Φ_∞^λ as a continuous function on $Y_\infty/\mathbb{Z}$.

If $0 \notin K$, then Φ_∞^λ is a continuous function on the compact set $p_{\mathbb{C}}^{-1}(K)/\mathbb{Z}$. In this case, there exists $a \in \mathbb{R}$ which satisfies the assertion.

Let $0 \in K$. Since $\lim_{\zeta \to -\lambda_n^{(infty)}{}_i} \Phi_\infty^\lambda(\zeta) = +\infty$, then Φ_∞^λ has a minimum value on $Y_\infty/\mathbb{Z}$. Hence the proof is finished by putting

$$a = 1 + \left| \min_{\zeta \in p_{\mathbb{C}}^{-1}(K)} \Phi_\infty^\lambda(\zeta) \right|.$$

□

Thus we obtain the pair $(\Gamma_\infty, \Phi_\infty^\lambda)$ on X_∞. Hence we have a hyperkähler structure on

$$X_\infty^\lambda := \{p \in X_\infty;\ \Phi_\infty^\lambda(\mu_\infty(p)) > 0 \text{ or } \mu_\infty(p) \in Y_\infty\}.$$

11 $\mathbb{Z}$-action on X_∞

In this section we are going to construct a $\mathbb{Z}$-action on X_∞ preserving the pair $(\Gamma_\infty, \Phi_\infty^\lambda)$, that is, the diffeomorphism $\hat{s} : X_\infty \to X_\infty$ which satisfies $\mu_\infty \circ \hat{s} = s$ and $\hat{s}^*\Gamma_\infty = \Gamma_\infty$ where $s : Im\mathbb{H} \to Im\mathbb{H}$ is defined by $s(\zeta) := \zeta + il$.

From now on, we fix $\lambda \in Y_\infty$ and put $\Phi_\infty := \Phi_\infty^\lambda$. If we put $X_\infty^\alpha := \{p \in X_\infty;\ \Phi_\infty(\mu_\infty(p)) + \alpha > 0\}$ for $\alpha \in \mathbb{R}$, then the pair $(\Gamma_\infty, \Phi_\infty + \alpha)$ induces a hyperkähler structure $\omega^\alpha = (\omega_1^\alpha, \omega_2^\alpha, \omega_3^\alpha)$ on X_∞^α. Let $I_1^\alpha, I_2^\alpha, I_3^\alpha$ be complex structures corresponding to the Kähler forms $\omega_1^\alpha, \omega_2^\alpha, \omega_3^\alpha$, respectively.

Proposition 11.1. *Let $\alpha, \beta \in \mathbb{R}$. Then for each $\xi \in \sqrt{-1}\mathbb{R}$ and $p \in X_\infty^\alpha \cap X_\infty^\beta$, we have*

$$\{\Phi_\infty(\mu_\infty(p)) + \alpha\} I_1^\alpha \xi_p^* = \{\Phi_\infty(\mu_\infty(p)) + \beta\} I_1^\beta \xi_p^*.$$

Proof. The holomorphic symplectic form $\omega_{\mathbb{C}}^\alpha \in \Omega^2(X_\infty^\alpha, \mathbb{C})$ with respect to I_1^α is given by $\omega_{\mathbb{C}}^\alpha := \omega_2^\alpha + \sqrt{-1}\omega_3^\alpha$. From the non-degeneracy of $\omega_{\mathbb{C}}^\alpha$, we have

$$T_p^{0,1} X_\infty^\alpha = \{v \in T_p X_\infty^\alpha \otimes \mathbb{C};\ \iota_v \omega_{\mathbb{C}}^\alpha = 0\}$$

for $p \in X_\infty^\alpha \cap X_\infty^*$, where ι is the interior product.

Fix $\xi \in \sqrt{-1}\mathbb{R}\backslash\{0\}$ and put

$$\partial_1^\alpha := \xi^* - \sqrt{-1} I_1^\alpha \xi^*,\ \partial_2^\alpha := I_2^\alpha \xi^* - \sqrt{-1} I_3^\alpha \xi^*$$

which is the basis of $T_p^{1,0}X_\infty^\alpha$. Since we can write

$$\omega_{\mathbb{C}}^\beta = \{\Gamma_\infty + \sqrt{-1}({\mu_\infty}^*\Phi_\infty + \beta)d\mu_{\mathbb{R}}\},$$

we can evaluate the basis δ_1, δ_2 of $T_p^{1,0}X_\infty^\beta$ as

$$\begin{aligned}
\delta_1 &= \frac{\beta-\alpha}{\Phi_\infty+\alpha}\overline{\partial_1^\alpha} + \Big(2 + \frac{\beta-\alpha}{\Phi_\infty+\alpha}\Big)\partial_1^\alpha \\
\delta_2 &= \partial_2^\alpha.
\end{aligned}$$

Then we have

$$\begin{aligned}
I_1^\beta\xi^* &= I_1^\beta\Big(\frac{\delta_1+\overline{\delta_1}}{4(1+(\beta-\alpha)(\Phi_\infty+\alpha)^{-1})}\Big) \\
&= \frac{\sqrt{-1}\delta_1 - \sqrt{-1}\overline{\delta_1}}{4(1+(\beta-\alpha)(\Phi_\infty+\alpha)^{-1})} \\
&= \frac{\Phi_\infty+\alpha}{\Phi_\infty+\beta}I_1^\alpha\xi^*.
\end{aligned}$$

□

From the above proposition, we obtain a vector field $V(\xi) \in \mathcal{X}(X_\infty)$ given by $V(\xi) := ({\mu_\infty}^*\Phi_\infty + \alpha)I_1^\alpha\xi^*$ on $X_\infty^\alpha \cap X_\infty^*$, since $X_\infty = \bigcup_{\alpha\in\mathbb{R}} X_\infty^\alpha$. Put $\xi = \sqrt{-1}$, $X_\infty^{**} := X_\infty\backslash\mu_\infty^{-1}(i\mathbb{R})$ and let $\{\exp tV(\xi)\}_{t\in\mathbb{R}} \subset \mathrm{Diff}(X_\infty^{**})$ be the 1-parameter group of transformations generated by $V(\xi)$. Then a curve $\tilde{c}_p : \mathbb{R} \to X_\infty$ given by $\tilde{c}_p(t) := \exp tV(\xi)(p)$ for each $p \in X_\infty^{**}$ is the horizontal lift of a curve $c_p(t) := \mu_\infty(p) + t\xi = \mu_\infty(p) + it \in Im\mathbb{H}$ with respect to $\mu_\infty : X_\infty^{**} \to Im\mathbb{H}\backslash i\mathbb{R}$. Hence $\exp lV(\xi) \in \mathrm{Diff}(X_\infty^{**})$ satisfies $\mu_\infty \circ \exp lV(\xi) = s \circ \mu_\infty$.

Proposition 11.2. *Let $\zeta = i\zeta_I + j\zeta_J + k\zeta_K \in Im\mathbb{H}$ be a Cartesian coordinate of $Im\mathbb{H} = i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$. Then we have*

$$\exp lV(\xi)^*\Gamma_\infty - \Gamma_\infty = 2\sqrt{-1}{\mu_\infty}^*\Big(-\frac{\partial\varphi_\infty}{\partial\zeta_J}d\zeta_K + \frac{\partial\varphi_\infty}{\partial\zeta_K}d\zeta_J\Big),$$

where $\varphi_\infty(\zeta_{\mathbb{C}}) := \int_0^l \Phi_\infty(\zeta_{\mathbb{C}}j + t\xi)dt$.

Proof. Put $f_t := \exp tV(\xi)$. Since $\Gamma_\infty(V(\xi)) = 0$, we have

$$\begin{aligned}
\frac{d}{dt}({f_t}^*\Gamma_\infty)_p &= ({f_t}^*\mathcal{L}_{V(\xi)}\Gamma_\infty)_p \\
&= ({f_t}^*\iota_{V(\xi)}d\Gamma_\infty)_p \\
&= ({f_t}^*\iota_{V(\xi)}(2\sqrt{-1}{\mu_\infty}^* * d\Phi_\infty))_p \\
&= 2\sqrt{-1}\Big(-\frac{\partial\Phi_\infty}{\partial\zeta_J}(\mu_\infty(p)+t\xi)d\zeta_K + \frac{\partial\Phi_\infty}{\partial\zeta_K}(\mu_\infty(p)+t\xi)d\zeta_J\Big)
\end{aligned}$$

for $p \in X_\infty^{**}$. Then we have the assertion by integrating from $t = 0$ to $t = 1$. $\square$

Since Φ_∞ is harmonic, φ_∞ is a harmonic function on $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$. Then there is a multi-valued holomorphic function F over $\mathbb{C}^\times$ whose real part is φ_∞. Let y be a holomorphic coordinate of $\mathbb{C}^\times$. Then note that $\frac{dF}{dy}$ is a single-valued holomorphic function.

We denote the imaginary part of F by ψ_∞. Let θ be one of the value of ψ_∞ at $y_0 \in \mathbb{C}^\times$. Then other values of ψ_∞ at y_0 are written as $\theta + \frac{n}{\sqrt{-1}}\mathrm{Res}(0, dF)$ for each $n \in \mathbb{Z}$, where

$$\mathrm{Res}(0, dF) := \int_{|y|=1} \frac{dF}{dy} dy$$

is the residue of dF at the origin $0 \in \mathbb{C}$. Now we define a gauge transformation $\sigma : X_\infty^{**} \to X_\infty^{**}$ by $\sigma(p) := p\exp(-2\sqrt{-1}\psi_\infty(\mu_{\infty,\mathbb{C}}(p)))$. Then we need to discuss the well-definedness of σ since ψ_∞ is multi-valued.

There is a $\mathbb{Z}$-action on Y_∞ generated by $s \in \mathrm{Diff}(Y_\infty)$ defined at the beginning of this section. We denote by $\pi_s : Y_\infty \to Y_\infty/\mathbb{Z}$ the quotient map. Recall that $*d\Phi_\infty$ is the first Chern form of principal S^1-bundle $\mu_\infty : X_\infty^* \to Y_\infty$ and invariant under the $\mathbb{Z}$-action. Then there is a cohomology class $\Theta \in H^2(Y_\infty/\mathbb{Z}, \mathbb{Z})$ such that $\pi_s{}^*\Theta = [\frac{*d\Phi_\infty}{\pi}]$, where $[\frac{*d\Phi_\infty}{\pi}]$ is the cohomology class determined by $\frac{*d\Phi_\infty}{\pi}$.

Now we put

$$S := \{it + jy \in i\mathbb{R} \oplus j\mathbb{C};\ t \in \mathbb{R},\ |y| = 1\}/\mathbb{Z} \subset Y_\infty/\mathbb{Z}$$

and denote its homology class by $[S] \in H_2(Y_\infty/\mathbb{Z}, \mathbb{Z})$. Then the next proposition ensures the well-definedness of σ.

Proposition 11.3. *Let* $<,>_{Y_\infty/\mathbb{Z}}$ *be the natural pairing*

$$H^2(Y_\infty/\mathbb{Z}, \mathbb{Z}) \times H_2(Y_\infty/\mathbb{Z}, \mathbb{Z}) \to \mathbb{Z}.$$

Then we have

$$\mathrm{Res}(0, dF) = \sqrt{-1}\pi < \Theta, [S] >_{Y_\infty/\mathbb{Z}} \in \sqrt{-1}\pi\mathbb{Z}$$

Proof. Put $y = y_1 + \sqrt{-1}y_2$. Then we have

$$\begin{aligned} \mathrm{Res}(0, dF) &= \int_{|y|=1} (d\varphi_\infty + \sqrt{-1}d\psi_\infty) \\ &= \sqrt{-1}\int_{|y|=1} d\psi_\infty \\ &= \sqrt{-1}\int_{|y|=1} \Big(-\frac{\partial\varphi_\infty}{dy_2}dy_1 + \frac{\partial\varphi_\infty}{dy_1}dy_2 \Big) dy \end{aligned}$$

On the other hand, we have

$$< \Theta, [S] >_{Y_\infty/\mathbb{Z}} = \int_{S'} \frac{*d\Phi_\infty}{\pi},$$

where $S' := \{it + jy \in i\mathbb{R} \oplus j\mathbb{C};\ 0 \le t \le l,\ |y| = 1\}$, since the push-forward $(\pi_s)_*([S'])[S]$ is equal to $[S]$. By using coordinate t, y_1, y_2 of Y_∞, we can show

$$\int_{S'} *d\Phi_\infty = \int_{|y|=1} \Big(- \frac{\partial \varphi_\infty}{dy_2} dy_1 + \frac{\partial \varphi_\infty}{dy_1} dy_2 \Big) dy.$$

$\square$

Thus we obtain two maps $\varphi_l, \sigma \in \mathrm{Diff}(X_\infty^{**})$. Then we define $\hat{s} \in \mathrm{Diff}(X_\infty^{**})$ by $\hat{s} := \varphi_l \circ \sigma = \sigma \circ \varphi_l$. By definition, we have $\mu_\infty \circ \hat{s} = s \circ \mu_\infty$ on X_∞^{**}.

Proposition 11.4. *$\hat{s}^*\Gamma_\infty = \Gamma_\infty$ on X_∞^{**}.*

Proof. For any S^1-connection Γ over $\mu_\infty : X_\infty^{**} \to Im\mathbb{H}\backslash i\mathbb{R}$, we have $\sigma^*\Gamma = \Gamma - 2\sqrt{-1}\mu_\infty{}^* d\psi_\infty$. By combining with Proposition 11.2, we have $\hat{s}^*\Gamma_\infty - \Gamma_\infty = 0$. $\square$

For each $\alpha \in \mathbb{R}$, $\hat{s}$ is an isomorphism of hyperkähler structure $(X_\infty^\alpha \cap X_\infty^{**}, \omega^\alpha = (\omega_1^\alpha, \omega_2^\alpha, \omega_3^\alpha))$. Since $\hat{s}$ preserves hyperkähler metric and the closure of $X_\infty^\alpha \cap X_\infty^{**}$ is $\overline{X_\infty^\alpha}$, we can extend $\hat{s}$ to the homeomorphism of X_∞^α. Then the extension of $\hat{s}$ is independent of $\alpha \in \mathbb{R}$ because there is some neighborhood $U_p \subset X_\infty^\alpha \cap X_\infty^\beta$ for each $p \in X_\infty \backslash X_\infty^{**}$ and $\alpha, \beta \in \mathbb{R}$ such that the Riemannian distance induced from hyperkähler metric ω^α is equivalent to one induced from ω^β. Thus we obtain a homeomorphism $\hat{s} : X_\infty \to X_\infty$ which preserves Riemannian distance induced from ω^α for all $\alpha \in \mathbb{R}$.

Proposition 11.5. *The map $\hat{s}$ is an isomorphism of the hyperkähler structure ω^α for all $\alpha \in \mathbb{R}$.*

Proof. Let $U \subset X_\infty$ be a sufficiently small neighborhood of $p \in X_\infty \backslash X_\infty^{**}$, and put $W := U \cap X_\infty^{**}$. We fix $\alpha \in \mathbb{R}$ to be $U \subset X_\infty^\alpha$. Since $\hat{s}$ preserves $\omega_{\mathbb{C}}^\alpha$ on W, $\hat{s}$ is holomorphic on W with respect to the complex structure I_1^α, and continuous on U. Since $U \backslash W$ is codimension 2, then $\hat{s}$ is holomorphic on U for any $p \in X_\infty \backslash X_\infty^{**}$, which implies $\hat{s} : X_\infty \to X_\infty$ is a diffeomorphism.

Since we have $\hat{s}^*\omega^\alpha - \omega^\alpha = 0$ on X_∞^{**}, the equality also holds on X_∞. $\square$

12 Gibbons-Hawking ansatz

In the previous sections, we obtain the hyperkähler metrics over the elliptic surface of type I_b by considering the limit of hyperkähler quotients. But these metrics also constructed by using Gibbons-Hawking ansatz as like the construction of Ooguri-Vafa metrics in [21][11].

Let $\lambda^\infty \in (Im\mathbb{H})_0^{\mathbb{Z}}$ and Y_∞ be as in Sections 10 and 11. Define a function $\hat{\Phi}_\infty^c : Y_\infty \to \mathbb{R}$ by

$$\hat{\Phi}_\infty^c(\zeta) := \frac{1}{4}\sum_{\mathbb{Z}}\Big(\frac{1}{|\zeta+\lambda_n^\infty|} + \frac{1}{1+|\lambda_n^\infty|}\Big) + c$$

for $\zeta \in Y_\infty$ and $c \in \mathbb{R}$. Then the series $\sum_{\mathbb{Z}}\Big(\frac{1}{|\zeta+\lambda_n^\infty|} + \frac{1}{1+|\lambda_n^\infty|}\Big)$ converges for any $\zeta \in Y_\infty$ and $\hat{\Phi}_\infty^c$ is a harmonic function from the same argument in [11]. Thus we have a closed 2-form $*d\hat{\Phi}_\infty^c \in \Omega^2(Y_\infty, \mathbb{R})$ independent of $c \in \mathbb{R}$.

Let a $\mathbb{Z}$-action on $Im\mathbb{H}$ be as in Section 11. Since $*d\hat{\Phi}_\infty^c$ is invariant under the $\mathbb{Z}$-action, it determines the cohomology class $[*d\hat{\Phi}_\infty^c] \in H^2(Y_\infty/\mathbb{Z}, \mathbb{R})$. Now put

$$S_n(r_n) := \{\zeta \in Y_\infty; |\zeta + \lambda_n^\infty| = r_n\}$$

and take $r_n > 0$ sufficiently small such that $r_n + r_{n+1} < |\lambda_n^\infty - \lambda_{n+1}^\infty|$. Then each $S_n(r_n)$ determines the homology class $\hat{C}_n \in H_2(Y_\infty, \mathbb{Z})$ and let $C_n \in H_2(Y_\infty/\mathbb{Z}, \mathbb{Z})$ be the push forward of $\hat{C}_n$ by the quotient map $Y_\infty \to Y_\infty/\mathbb{Z}$. Thus we have a basis $\{C_1, \cdots, C_b\}$ of $H_2(Y_\infty/\mathbb{Z}, \mathbb{Z})$ and let $\{e^1, \cdots, e^b\} \subset H^2(Y_\infty/\mathbb{Z}, \mathbb{Z})$ be its dual basis. Since the direct calculation gives

$$\frac{1}{\pi}\int_{\hat{C}_n} *d\hat{\Phi}_\infty^c = 1$$

by fixing an appropriate orientation on $\hat{C}_n$, we have

$$[\frac{1}{\pi} * d\hat{\Phi}_\infty^c] = e^1 + \cdots + e^b \in H^2(Y_\infty/\mathbb{Z}, \mathbb{Z}).$$

Next we denote by $\mu^\circ : X \to Y_\infty/\mathbb{Z}$ the principal S^1-bundle whose Euler class is given by $e^1 + \cdots + e^b$. From the same argument in [11], there is a C^∞-manifold $\bar{X}$ contains X as an open dense submanifold and μ° extends to S^1-fibration $\mu : \bar{X} \to Im\mathbb{H}/\mathbb{Z}$ such that $\mu|_X = \mu^\circ$, $\mu(\bar{X}\backslash X) = (Im\mathbb{H}\backslash Y_\infty)/\mathbb{Z}$ and $\mu|_{\bar{X}\backslash X}\bar{X}\backslash X \to (Im\mathbb{H}\backslash Y_\infty)/\mathbb{Z}$ is bijective. Let $\Gamma \in \Omega^1(X, \sqrt{-1}\mathbb{R})$ be an S^1-connection over $\mu^\circ : X \to Y_\infty/\mathbb{Z}$ and put

$$Y_\infty(c) \;:=\; \{\zeta \in Y_\infty;\ \hat{\Phi}_\infty^c(\zeta) > 0\},$$

$$
\begin{aligned}
\overline{Y_\infty(c)} &:= Y_\infty(c) \cup \{-\lambda_n^\infty;\ n \in \mathbb{Z}\}, \\
X(c) &:= \mu^{-1}(Y_\infty(c)/\mathbb{Z}), \\
\overline{X(c)} &:= \mu^{-1}(\overline{Y_\infty(c)}/\mathbb{Z}).
\end{aligned}
$$

Then we have a pair of a positive valued harmonic function $\hat{\Phi}_\infty^c$ on $X(c)$ and S^1-connection Γ on X such that $\frac{d\Gamma}{2\sqrt{-1}} = \mu^*(*d\hat{\Phi}_\infty^c)$. Thus we obtain the hyperkähler structure on $X(c)$ from Theorem 2.9.

For each closed 1-form $\gamma \in \Omega^1(Y_\infty, \mathbb{R})$, the pair $(\hat{\Phi}_\infty^c, \Gamma + \mu^*\gamma)$ also determines hyperkähler structure on $X(c)$. From the same discussion in [11], there exists a closed 1-form γ such that the hyperkähler structure extends to $\overline{X(c)}$.

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