J. Math. Sci. Univ. Tokyo
1 (1994), 617–629.

Some foliations on ruled surfaces

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Abstract. In this paper, (complex analytic) foliations on ruled surfaces leaving a curve invariant and having no singularities on it are observed. Classification of ruled surfaces and curves with such foliations is done and examples of each case are given.

§0. Introduction

Every ruled surface has a foliation—the ruling—, which characterizes ruled surfaces in all compact complex surfaces. Existence of another foliation characterizes some ruled surfaces in all ruled surfaces. In this paper, we classify ruled surfaces with a foliation on them leaving a curve invariant and having no singularities on it. §1 is a short review of ruled surfaces. The classification is executed in §2. (Main Theorem 2.1.) In §3, examples of each case are given. It follows that every case of the classification is worth listing. A part of these results is announced in [Sa4]. The author thanks Prof. T. Suwa for his helpful advices.

§1. A review of ruled surfaces

In this section, we review some properties of ruled surfaces, which may be found in eg. [Ha].

DEFINITION 1.0. A *ruled surface* $X \xrightarrow{\pi} C$ is a proper holomorphic map of a two-dimensional compact complex manifold X onto a closed Riemann surface C which makes X a $\mathbf{P}^1$-bundle over C.

1991 *Mathematics Subject Classification.* 58 Exx.

PROPOSITION 1.1.

0) *A ruled surface has a section, i.e. there exists a holomorphic map* $C \xrightarrow{\sigma} X$ *satisfying* $\pi \cdot \sigma = id_C$.

1) *For a ruled surface* $X \xrightarrow{\pi} C$, *there exists a section* C_0 *with the following properties:*

$${C_0}^2 = \text{the minimum of self-inersection numbers of sections of } X \xrightarrow{\pi} C.$$

We define a number e by

$$e = -{C_0}^2, \tag{1.2}$$

which satisfies the following inequality

$$e \geq -g, \tag{1.3}$$

where g is the genus of the Riemann surface C.

Since

$$\mathrm{H}^2(C, \mathcal{O}_C) = 0 \qquad \text{and} \qquad \mathrm{H}^2(X, \mathcal{O}_X) = 0$$

for a ruled surface $X \xrightarrow{\pi} C$, the exponential sequences

$$0 \to \mathbf{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

on X and C induce the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \uparrow & & & & & & \\
0 & \to & \mathrm{Pic}_0 X & \to & \mathrm{H}^1(X, \mathcal{O}_X^*) & \xrightarrow{c} & \mathrm{H}^2(X, \mathbf{Z}) & \to 0 & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathrm{Pic}_0 C & \to & \mathrm{H}^1(C, \mathcal{O}_C^*) & \xrightarrow{c} & \mathrm{H}^2(C, \mathbf{Z}) & \to 0, & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{1.4}$$

where

$$\begin{array}{rcl}
\mathrm{Pic}_0 C & = & ker[\mathrm{H}^1(C, \mathcal{O}_C^*) \xrightarrow{c} \mathrm{H}^2(C, \mathbf{Z})] \quad \text{and} \\
\mathrm{Pic}_0 X & = & ker[\mathrm{H}^1(X, \mathcal{O}_X^*) \xrightarrow{c} \mathrm{H}^2(X, \mathbf{Z})].
\end{array}$$

We denote by Num$X$ the group of numerically equivalent classes of divisors on X. This induces the following isomorphism.

$$\begin{array}{rcccc} \mathrm{Num}X & \simeq & \mathrm{H}^2(X,\mathbf{Z}) & & \\ & = & \mathbf{Z}c(C_0) & \oplus & \pi^*\mathrm{H}^2(C,\mathbf{Z}) \\ & = & \mathbf{Z}c(C_0) & \oplus & \mathbf{Z}c(f) \\ & \simeq & \mathbf{Z} & \oplus & \mathbf{Z} \end{array}$$

Here $c(C_0)$ and $c(f)$ are the images by the Chern map $\mathrm{H}^1(X,\mathcal{O}_X^*) \stackrel{c}{\longrightarrow} \mathrm{H}^2(X,\mathbf{Z})$ of the holomorphic line bundles defined by the divisors C_0 and f, the section of $X \stackrel{\pi}{\longrightarrow} C$ and f the fibre of $X \stackrel{\pi}{\longrightarrow} C$, respectively. We often denote them by C_0 and f. Thus we have the following intersection relations:

$$C_0{}^2 = -e, \qquad C_0 \cdot f = 1 \quad \text{and} \quad f^2 = 0. \tag{1.5}$$

The number e is very important. For ruled surfaces over a given Riemann surface C, the following holds.

Proposition 1.6.

0) *For a ruled surface $X \stackrel{\pi}{\longrightarrow} C$, there exists a locally free $\mathcal{O}_C$-module $\mathcal{E}$ of rank two satisfying $X = \mathbf{P}(\mathcal{E})$.*
1) *Two ruled surfaces $X_0 = \mathbf{P}(\mathcal{E}_0)$ and $X_1 = \mathbf{P}(\mathcal{E}_1)$ are isomorphic as ruled surfaces over C if and only if there exists an invertible sheaf $\mathcal{L}$ on C satisfying $\mathcal{E}_1 = \mathcal{L} \otimes \mathcal{E}_0$.*
2) *For a given ruled surface $X \stackrel{\pi}{\longrightarrow} C$, there exists a locally free $\mathcal{O}_C$-module $\mathcal{E}$ of rank two with $X = \mathbf{P}(\mathcal{E})$ satisfying*
   0) $\Gamma(C,\mathcal{E}) \neq 0$ *and*
   1) $\Gamma(C,\mathcal{L}\otimes\mathcal{E}) = 0$ *for any invertible sheaf $\mathcal{L}$ on C with deg$\mathcal{L} < 0$.*

For a ruled surface $X \stackrel{\pi}{\longrightarrow} C$, a locally free $\mathcal{O}_C$-module $\mathcal{E}$ of rank two is said to be *normalized* if $\mathcal{E}$ satisfies the conditions stated in 2) of Proposition 1.6. For such $\mathcal{E}$, the following equality holds.

$$e = -deg\mathcal{E} = -deg \bigwedge^2 \mathcal{E} \tag{1.7}$$

The number e is an invariant for X though a normalized invertible sheaf $\mathcal{E}$ is not always uniquely determined for $X \xrightarrow{\pi} C$.

Assume that $X = \mathbf{P}(\mathcal{E})$ with a normalized locally free $\mathcal{O}_C$-molule $\mathcal{E}$ of rank 2, as in Proposition 1.6. There are two cases with respect to the locally free sheaf $\mathcal{E}$:

i) $\mathcal{E}$ is decomposable or
ii) $\mathcal{E}$ is indecomposable.

Since $\mathcal{E}$ is normalized, these mean respectively,

i) $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$ where $\mathcal{L}$ is an invertible sheaf with $deg\mathcal{L} = deg\mathcal{E} = -e$ and
ii) $\mathcal{E}$ is a non-trivial extension

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{L} \to 0,$$

where $\mathcal{L}$ is an invertible sheaf with $deg\mathcal{L} = deg\mathcal{E} = -e$, which corresponds to a non-zero element of $\mathrm{H}^1(C, \mathcal{L}^*) \simeq$ the dual of $\mathrm{H}^0(C, \kappa\mathcal{L})$.

Here $\mathcal{L}^*$ is the dual $\mathcal{O}_C$-module of $\mathcal{L}$ and κ is the canonical sheaf on C. Note that it follows from the conditions 0) and 1) of 2) of Proposition 1.6 that $\mathcal{E}$ cannot be decomposable if $e < 0$.

Assume a ruled surface $X \xrightarrow{\pi} C$ is described as $X = \mathbf{P}(\mathcal{E})$ using a locally free $\mathcal{O}_C$-module $\mathcal{E}$ of rank two. Then there exists a one-to-one correspondence between sections $C \xrightarrow{\sigma} X$ of $X \xrightarrow{\pi} C$ and $\mathcal{O}_C$-morphisms $\mathcal{E} \to \mathcal{L} \to 0$ of $\mathcal{E}$ onto a certain invertible sheaf $\mathcal{L}$ on C. Thus 0) of Proposition 1.1 and 0) of Proposition 1.6 are equivalent to the fact that, on a Riemann surface, any locally free sheaf of rank two is an extension of an invertible sheaf by another.

Let $X \xrightarrow{\pi} C$ be a ruled surface and $\mathcal{E}$ a normalized locally free $\mathcal{O}_C$-module: $X = \mathbf{P}(\mathcal{E})$. There exists a section $C \xrightarrow{\sigma} X$ of $X \xrightarrow{\pi} C$ with the following property:

$$\mathcal{L}(C_0) \simeq \mathcal{O}_X(1),$$

where $\mathcal{L}(C_0)$ is the invertible sheaf on X defined by the divisor $\sigma(C) = C_0$ on X. A section satisfying this condition is, on account of 0) of the following Proposition 1.8, called a *normalized* section.

PROPOSITION 1.8. *Let $\mathcal{E}$ be a normalized locally free $\mathcal{O}_C$-module and C_0 a section of $X \xrightarrow{\pi} C$ with $\mathcal{L}(C_0) \simeq \mathcal{O}_X(1)$, which are fixed. Assume that*

ϵ is a divisor on C satisfying $\bigwedge^2 \mathcal{E} = \mathcal{L}(\epsilon)$. Then we have the following:

0) *Let C_1 be an arbitrary section of $X \xrightarrow{\pi} C$, $\mathcal{E} \to \mathcal{L} \to 0$ the corresponding surjection of $\mathcal{O}_C$-modules and δ a divisor on C with $\mathcal{L} = \mathcal{L}(\delta)$. Then*

$$C_1 \sim C_0 + \pi^*(\delta - \epsilon),$$

especially,

$$C_0^2 = deg\epsilon = deg \bigwedge^2 \mathcal{E} = deg\mathcal{E} = -e.$$

Here "$\sim$" represents the linear equivatence of divisors on X.

1) *Let f be a fibre of $X \xrightarrow{\pi} C$. Then we have*

$$\mathrm{Pic}X = \mathbf{Z}C_0 \oplus \pi^*\mathrm{Pic}C \simeq \mathbf{Z} \oplus \mathrm{Pic}C,$$

where $\mathrm{Pic}X$ $(\simeq \mathrm{H}^1(X, \mathcal{O}^*)$) *and* $\mathrm{Pic}C$ $(\simeq \mathrm{H}^1(X, \mathcal{O}^*)$) *are the Picard groups on X and C, respectively.*

We fix a normalized section C_0. The following holds.

Proposition 1.9. (cf. [Ha] p. 382) *Let $X = \mathbf{P}(\mathcal{E})$ and C_0 be as above.*

I) *The case $e \geq 0$. If an irreducible curve $C_1 \simeq_{num} aC_0 + bf$ on X is neither C_0 nor a fibre of $X \xrightarrow{\pi} C$ then*

$$a \geq 1 \qquad and \qquad b \geq ea.$$

II) *The case $e < 0$.*

II-0) *If an irreducible curve $C_1 \simeq_{num} aC_0 + bf$ on X is a section of $X \xrightarrow{\pi} C$ then*

$$a = 1 \qquad and \qquad b \geq 0.$$

II-1) *If an irreducible curve $C_1 \simeq_{num} aC_0 + bf$ on X is neither a section nor a fibre of $X \xrightarrow{\pi} C$ then*

$$a \geq 2 \qquad and \qquad b \geq \frac{1}{2}ea.$$

Here, "$\simeq_{num}$" represents the numerically equivalence of divisors on X.

§2. The main theorem

A foliation of dimension one can be defined in various ways. (cf. [GM1], [GM2], [Sa1], [Sa2] and [Sa3].) In this paper, we adopt the following one.

Let M be a complex manifold of dimension m, $\mathcal{O}_M$ the sheaf of germs of holomorphic functions on M and Θ_M the sheaf of germs of holomorphic vector fields on M.

DEFINITION 2.0.

0) A *foliation of dimension one* on M is an invertible subsheaf $\mathcal{F}$ of Θ_M with the following property. The analytic set

$$\{p \in M | (\Theta/\mathcal{F})_p \text{ is } \textit{not} \text{ a free } \mathcal{O}_p\text{-module of rank } m-1\},$$

which is called the *singular locus* of $\mathcal{F}$, is of codimension strictly greater than one.

1) A subvariety N of M defined by a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_M$ is said to be *invariant* with respect to a foliation $\mathcal{F} \subset \Theta_M$ on M if, at every $p \in M$,

$$\mathcal{F}_p \mathcal{I}_p \subset \mathcal{I}_p.$$

In what follows, we always assume that X is a ruled surface $X \xrightarrow{\pi} C$ with the invariant e, where C is a closed Riemann surface of genus g unless explicitly stated otherwise. We fix a normalized locally free $\mathcal{O}_C$-module $\mathcal{E}$ of rank two so that $X = \mathbf{P}(\mathcal{E})$. Let $C \xrightarrow{\sigma_0} X$ be a normalized section of $X \xrightarrow{\pi} C$, which is also fixed, and $C_0 = \sigma_0(C)$ (the divisor on X defined by) the image of C. Let f be a fibre of $X \xrightarrow{\pi} C$. We have

$$\mathrm{H}^2(X, \mathbf{Z}) = \mathbf{Z}C_0 \oplus \mathbf{Z}f \simeq \mathbf{Z}^2,$$

$$C_0^2 = deg\mathcal{E} = -e, \quad C_0 \cdot f = 1 \quad \text{and} \quad f^2 = 0.$$

MAIN THEOREM 2.1. *Let C be a closed Riemann surface of genus g, $X = \mathbf{P}(\mathcal{E}) \xrightarrow{\pi} C$ a ruled surface over C with the invariant e, a normalized locally free $\mathcal{O}_C$-module $\mathcal{E}$ and C_0 a normalized section of $X \xrightarrow{\pi} C$. Assume that a foliation $\mathcal{F} \subset \Theta_X$ on X leaves an irreducible curve $C_1 \simeq_{num} aC_0 + bf$*

with $a > 0$ on X invariant and has no singularities on C_1. Then one of the following is the case.

I-i) *$e = 0$ and $\mathcal{E}$ is decomposable and $b = 0$.*
I-ii) *$e = 0$ and $\mathcal{E}$ is indecomposable and $b = 0$.*
II) *$e < 0$, $a \geq 2$ and $b = \frac{1}{2}ea \in \mathbf{Z}$. (In this case, $\mathcal{E}$ is indecomposable.)*

To prove this theorem, Proposition 1.9 and the index formula of Camacho-Sad[Ca-S] are of essential importance.

Camacho-Sad's index formula is as follows: ([Ca-S] p. 592.)

THEOREM 2.2. (Camacho-Sad) *Let N be a closed Riemann surface embedded in a complex surface M and $\mathcal{F} \subset \Theta_M$ a foliation leaving N invariant. Then*

$$\sum_{q \in N} i_q(\mathcal{F}, N) = N^2,$$

where $i_q(\mathcal{F}, N)$ is the index of $\mathcal{F}$ at q with respect to N.

Note that $i_q(\mathcal{F}, N) = 0$ if $q \in N$ is not a singular point of $\mathcal{F}$.

PROOF OF THEOREM 2.1. All notations are as in Theorem 2.1. Since that the foliation $\mathcal{F} \subset \Theta$ leaves the curve $C_1 \simeq_{num} aC_0 + bf$ invariant and that $\mathcal{F}$ has no singularities on C_1, C_1 must be a non-singular curve and the index formula asserts that

$${C_1}^2 = a(2b - ea) = 0.$$

Thus $2b = ea$ holds. (Note that $a > 0$.) It follows from Proposition 1.9 that

I) if $e \geq 0$ then $b = 0$ and $e = 0$ and that
II) if $e < 0$ then $a \geq 2$ and $2b = ea$. □

§3. Examples

In this section, we give examples of all cases stated in Theorem 2.1 and prove that none of them cannot be omitted. It should be noted that a foliation $\mathcal{F} \subset \Theta_X$ on a complex manifold X defines, by taking local generators of $\mathcal{F}$, a morphism $L \xrightarrow{\varphi} TX$ of holomorphic vector bundles over X of a holomorphic line bundle L into the holomorphic tangent bundle TX, which also we call a foliation. (cf. [GM2].) The zero-locus $\{\varphi = 0\}$, which is the *singular locus* of the foliation $L \xrightarrow{\varphi} TX$, is of codimension strictly greater than one. Now that the complex manifold X is, in our case, a ruled surface, every holomorphic line bundle over X is meromorphically trivial. Thus the foliation $L \xrightarrow{\varphi} TX$ defines a global meromorphic vector field on X upto the multiplication of global meromorphic functions. We display examples by assigning meromorphic vector fields.

Let $X = \mathbf{P}(\mathcal{E})$ be a ruled surface over a closed Riemann surface C of genus g with a normalized locally free $\mathcal{O}_C$-module $\mathcal{E}$ and E the *dual* vector bundle of the vector bundle over C corresponding to the locally free sheaf $\mathcal{E}$, i.e. E is the vector bundle such that $\mathcal{E}$ is the *dual* $\mathcal{O}_C$-module of the sheaf of germs of holomorphic sections of E. If the vector bundle E is represented by a 1-cocycle $(E_{\alpha\beta})$ with respect to an open covering $\{U_\alpha\}$ of C, then X is obtained by patching $\{U_\alpha \times \mathbf{P}^1\}$ together, identfying $(z_\alpha, \zeta_\alpha) \in U_\alpha \times \mathbf{P}^1$ and $(z_\beta, \zeta_\beta) \in U_\beta \times \mathbf{P}^1$ if and only if $z_\alpha = z_\beta \in U_\alpha \cap U_\beta$ and $\zeta_\alpha = \frac{a(z)\zeta_\beta + b(z)}{c(z)\zeta_\beta + d(z)}$, where

$$E_{\alpha\beta} = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix}.$$

All notations are as in Theorem 2.1.

Case I-i) $X = \mathbf{P}(\mathcal{E})$ and $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$, where $\mathcal{L}$ is an invertible sheaf on C with $deg\mathcal{L} = 0$. Note that the direct product $X = C \times \mathbf{P}^1 = \mathbf{P}(\mathcal{O}_C \oplus \mathcal{O}_C)$ is obviously an example of this case and that there always exist such a foliation $\mathcal{F}$ and a curve C_1 that satisfy the conditions stated in Theorem 2.1.

0) The case $g = 0$. The Riemann surface C is none other than the complex projective line $\mathbf{P}^1$. Invertible sheaves over $\mathbf{P}^1$ are completely determined by the Chern class of them. Thus $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{O}_C$ and $X = \mathbf{P}^1 \times \mathbf{P}^1$.

1) The case $g = 1$. Take a coordinate covering $\{(U_\alpha, z_\alpha)\}$ of C such that $dz_\alpha = dz_\beta$ and that the vector bundle E is

represented by a 1-cocycle $(E_{\alpha\beta})$ of the following form:

$$E_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & d_{\alpha\beta} \end{bmatrix},$$

where $0 \neq d_{\alpha\beta} \in \mathbf{C}$. As a normalized section C_0 of $X \xrightarrow{\pi} C$, either $z_\alpha = 0$ or $z_\alpha = \infty$ will do. Take a constant $c \in \mathbf{C}$ arbitrary. On each U_α, we define a vector field

$$\frac{\partial}{\partial z_\alpha} + c\zeta_\alpha \frac{\partial}{\partial \zeta_\alpha} \in \Gamma(U_\alpha \times \mathbf{P}^1, \Theta).$$

These vector fields patch together to define a global vector field $\theta \in \Gamma(X, \Theta)$, which defines a foliation on X leaving C_0 invariant.

2) The case $g > 1$. As in the above case, we can take a coordinate covering $\{(U_\alpha, z_\alpha)\}$ of C such that the vector bundle E is represented by a 1-cocycle $(E_{\alpha\beta})$ of the form

$$E_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & d_{\alpha\beta} \end{bmatrix}$$

with $0 \neq d_{\alpha\beta} \in \mathbf{C}$. As a normalized section C_0 of $X \xrightarrow{\pi} C$, either $z_\alpha = 0$ or $z_\alpha = \infty$ does. Take a constant $c \in \mathbf{C}$ and a global meromorphic vector field $v \in \Gamma(C, \mathcal{M}(TC))$ with no zero arbitrary. Such a vector field v always exists. For a global holomorphic 1-form $(u_\alpha dz_\alpha) \in \Gamma(C, \mathcal{O}_C(T^*C))$, $(\frac{1}{u_\alpha}\frac{d}{dz_\alpha}) \in \Gamma(C, \mathcal{M}(TC))$ is the desired one. On each U_α, we define a meromorphic vector field

$$\frac{1}{u_\alpha}\frac{\partial}{\partial z_\alpha} + c\zeta_\alpha \frac{\partial}{\partial \zeta_\alpha} \in \Gamma(U_\alpha \times \mathbf{P}^1, \mathcal{M}(TX)).$$

These vector fields patch together to define a global meromorphic vector field $\theta \in \Gamma(X, \mathcal{M}(TX))$, which defines a foliation on X satisfying the conditions of Theorem 2.1.

Case I-ii) Let C be an elliptic curve with periods $(2\omega_1, 2\omega_2)$, i.e. a closed Riemann surface of genus $g = 1$ and $X = \mathbf{P}(\mathcal{E})$ with $e = 0$, where

$\mathcal{E}$ is an indecomposable locally free $\mathcal{O}_C$-module of rank two defined as follows. Note that $g = 1$, $e = 0$ and $\mathcal{E}$ is indecomposable implies that $\mathcal{E}$ is a non-trivial extension

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_C \to 0.$$

Let $\wp(z)$ be the Weierstrass $\wp$-function with periods $(2\omega_1, 2\omega_2)$ and put $\omega_3 = \omega_1 + \omega_2$ and $\alpha_i = \wp(\omega_i)$ for $i = 1$, 2 and 3. Considering the elliptic curve C as the quotient space of $\mathbf{C}$ defined by the lattice $\mathbf{Z}2\omega_1 \oplus \mathbf{Z}2\omega_2$, we denote by $[z] \in C$ the image of $z \in \mathbf{C}$. Let $a_i = [\omega_i]$ for $i = 1$, 2 and 3 and $U = C - a_3$. Taking a small enough open disk V with a coordinate x centred at 0, we identify V with an open set in C:

$$\begin{array}{ccc} V & \subset & C \\ x & \mapsto & [x + \omega_3]. \end{array}$$

We define a vector bundle E over C, with respect to this open covering $\{U, V\}$ of C, by

$$E_{VU} = \begin{bmatrix} 1 & \frac{1}{\wp'(z)} \\ 0 & 1 \end{bmatrix}.$$

Let X be a ruled surface defined by the vector bundle E, i.e. defined by patching $U \times \mathbf{P}^1$ and $V \times \mathbf{P}^1$ together, idenifying $(x, \xi) \in V \times \mathbf{P}^1$ and $([z], \zeta) \in U \times \mathbf{P}^1$ if and only if

$$\begin{cases} x &= [z - \omega_3] \\ \xi &= \zeta + \frac{1}{\wp'(z)} \end{cases}.$$

Taking fibre coordinates $\rho = \frac{1}{\xi}$ and $\eta = \frac{1}{\zeta}$, transition relation is written as follows:

$$\rho = \frac{\wp'(z)\eta}{\eta + \wp'(z)}.$$

Curves defined by $\rho = 0$ in $V \times \mathbf{P}^1$ and $\eta = 0$ in $U \times \mathbf{P}^1$ are patched together to define a normalized section C_0 of $X \xrightarrow{\pi} C$. On $U \times \mathbf{P}^1 \cap V \times \mathbf{P}^1$,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} - \frac{\wp''(z)}{\wp'(z)^2}\eta^2\frac{\partial}{\partial \eta} \qquad \text{and}$$
$$\rho^2\frac{\partial}{\partial \rho} = \eta^2\frac{\partial}{\partial \eta}.$$

Let θ be a holomorphic vector field

$$\frac{\partial}{\partial z} + \frac{\wp(z-\omega_3)}{\wp''(\omega_3)}\frac{\partial}{\partial \zeta} = \frac{\partial}{\partial z} - \frac{\wp(z-\omega_3)}{\wp''(\omega_3)}\eta^2\frac{\partial}{\partial \eta} \in \Gamma(U \times \mathbf{P}^1, \Theta_X).$$

θ extends to a vector field

$$\frac{\partial}{\partial x} + \left(\frac{\wp(x)}{\wp''(\omega_3)} - \frac{\wp''(\omega_3+x)}{\wp'(\omega_3+x)^2}\right)\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} - \left(\frac{\wp(x)}{\wp''(\omega_3)} - \frac{\wp''(\omega_3+x)}{\wp'(\omega_3+x)^2}\right)\rho^2\frac{\partial}{\partial \rho}$$

on $V \times \mathbf{P}^1$. Since

$$\wp'(\omega_3) = 0, \qquad \wp''(\omega_3) \neq 0 \qquad \text{and}$$
$$\wp(\omega_3+x) = \wp(\omega_3-x),$$

$\frac{\wp(x)}{\wp''(\omega_3)} - \frac{\wp''(\omega_3+x)}{\wp'(\omega_3+x)^2}$ is holomorphic at 0. Thus θ defines a global holomorphic vector field on X, which we denote by θ also. The foliation defined by this $\theta \in \Gamma(X, \Theta_X)$ satisfies the conditions.

Case II) We concider the case that the genus g of C is one. (cf. [Su1].) Since $e \geq -g$, we have $e = -1$. We construct an example of the case $a = 2$ and $b = -1$. Let C^- be an elliptic curve with periods $(2\omega_1, 2\omega_2)$ and $W = C^- \times \mathbf{P}^1$. We denote by $\wp$ the Weierstrass

$\wp$-function with periods $(2\omega_1, 2\omega_2)$ and define an elliptic function $\sigma(w)$ by

$$\sigma(w) = \frac{\wp'(w)}{2(\alpha_3 - \alpha_2)^{\frac{1}{2}}(\wp(w) - \alpha_1)},$$

where $\alpha_1 = \wp(\omega_1)$, $\alpha_2 = \wp(\omega_2)$ and $\alpha_3 = \wp(\omega_1 + \omega_2)$. function $\sigma(w)$ defines a section of $W \to C^-$.

Let G be a subgroup of the group of holomorphic automorphisms of W generated by

$$\begin{array}{ccc} W = C^- \times \mathbf{P}^1 & \longrightarrow & W = C^- \times \mathbf{P}^1 \\ ([w], \xi) & \mapsto & ([w + \omega_1], -\xi) \end{array}$$

and

$$\begin{array}{ccc} W = C^- \times \mathbf{P}^1 & \longrightarrow & W = C^- \times \mathbf{P}^1 \\ ([w], \xi) & \mapsto & ([w + \omega_2], \frac{1}{\xi}) \end{array}.$$

The quotient space $X = W/G$ is a ruled surface over an elliptic curve C with periods (ω_1, ω_2). The section of $W \to C^-$ defined by σ defines a section of $X \xrightarrow{\pi} C$, which we denote $C \xrightarrow{\sigma} X$. $C_0 = \sigma(C)$, satisfying $C_0^2 = 1$, is a normalized section. Let C_1 be a curve in X defined by curves in W with equations $\xi = 0$ or ∞. As a divisor on X, $C_1 \simeq_{num} 2C_0 - f$. $\frac{\partial}{\partial w} \in \Gamma(W, \Theta_W)$ defines a foliation on W, which induces a foliation on X leaving C_1 invariant. This is the desired one.

References

[B-B] Baum, P. and R. Bott, Singularities of holomorphic foliations, J. Differential Geometry **7** (1972), 279–342.

[Ca-S] Camacho, C. and P. Sad, Invariant varieties through singularities of holomorphic vector fields, Annals of Math. **115** (1982), 579–595.

[GM1] Gómez-Mont, X., Foliations by curves of complex analytic spaces, Contemporary Math. **58** (1987), 123–141.

[GM2] Gómez-Mont, X., Universal families of foliations by curves, Astérisque **150–151** (1987), 109–129.

[GM3] Gómez-Mont, X., Holomorphic foliations in ruled surfaces, Trans. of the American Math. Soc. **312** (1989), 179–201.

[Ha] Hartshorne, R., Algebraic Geometry, G.T.M. **52** (1977), Springer-Verlag.

[Sa1] Saeki, A., Foliations on complex spaces, to appear in Funkcialaj Ekvacioj.

[Sa2] Saeki, A., On foliations on complex spaces, Proc. Japan Acad. **68**A (1992), 261–265.
[Sa3] Saeki, A., On foliations on complex spaces II, Proc. Japan Acad. **69**A (1993), 5–9.
[Sa4] Saeki, A., On some foliations on ruled surfaces, Proc. Japan Acad. **70**A (1994), 17–21.
[Su1] Suwa, T., On ruled surfaces of genus 1, J. of the Math. Soc. of Japan **21** (1969), 291–311.
[Su2] Suwa, T., Unfoldings of complex analytic foliations with singularities, Japan. J. of Math. **9** (1983), 181–206.
[Su3] Suwa, T., Complex analytic singular foliations, (lecture notes).

(Received January 25, 1994)
(Revised April 11, 1994)

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