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# *On the discrete Boltzmann equation with linear and nonlinear terms*

By Mitsuru Yamazaki

**Abstract.** For the discrete Boltzmann equation with linear and nonlinear terms, we show a boundedness of solutions with an explicit estimate and their asymptotic behavior when the momentum is conserved. Secondly when the Cauchy data are small, we show an exponential decay of solutions, for a model in which the nonlinear terms represent the 'binary collisions' and also 'multiple collisions'.

## 1. Formulation and results

In this paper, we study the discrete Boltzmann equation in one-dimensional space with linear and nonlinear terms. This system, which is different from the usual one by the intervention of linear terms, describes the gas motion of molecules which take only a finite number of velocities under the interactions between particles represented by the quadratic terms and also under the reflection of molecules at the inner wall of an infinite thin tube, represented by the linear terms. This linear terms are more general than the ones which are obtained by considering solutions around constant stationary solutions. Using the sign function and decomposing the solutions into two parts, which are explained later, under the conservation of momentum in the course of reflection, we prove the boundedness of solutions and asymptotic behavior of solutions which shows that all solutions tend to a free motion, hence we can define a 'wave operator' like in the sense of the scattering theory. This is the first work to define the wave operator for a large data even if in the case of the discrete models only with the quadratic terms. The wave operator was introduced by Bony [4] for a small Cauchy data. Finally,

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for the small Cauchy data, with binary and also multiple collisions, we have the estimates of solutions which show that the components of solutions with loss term in reflection decay exponentially.

We study the discrete model of Boltzmann equations in a thin infinite tube as follows :

$$
\text{(B)} \qquad \begin{cases} \dfrac{\partial u_i}{\partial t} + c_i \dfrac{\partial u_i}{\partial x} = Q_i(u) + L_i(u) \ , \\ \qquad u_i(x,0) = u_i^0(x) \ \ (\geqq 0) \quad \text{for} \quad x \in \mathbf{R}, t \in \mathbf{R}_+ \ . \end{cases}
$$

where

$$
Q_i(u) = \sum_{j,k,\ell \in I} (A_{ij}^{k\ell} u_k u_\ell - A_{k\ell}^{ij} u_i u_j) \ ,
$$

$$
L_i(u) = \sum_{k \in I} (\alpha_i^k u_k - \alpha_k^i u_i) \ .
$$

REMARK. $c_i$ is considered as the first component of $C_i \in \mathbf{R}^3$. Since $u_i$ represents the distribution function of molecules with velocity $C_i$, $i \neq j$ implies that $C_i \neq C_j$ but not that $c_i \neq c_j$. Nevertheless in this paper we assume, for simplicity, $i \neq j$ implies that $c_i \neq c_j$, which is not an essential hypothesis at all and without it we can recover the proof for all results obtained in this paper, using Bony's interesting induction argument [3], [15].

The natural physical conditions are the following :

CONDITION 1.

$$
\begin{gathered}
A_{ij}^{k\ell} \geqq 0, \quad A_{ij}^{k\ell} = A_{ji}^{k\ell} = A_{ij}^{\ell k} \ , \\
A_{k\ell}^{ij} \neq 0 \quad \Rightarrow \quad i \neq j \quad \textit{and} \quad c_i + c_j = c_k + c_\ell \ , \\
\forall i \ \exists (j,k,\ell) \quad \textit{such that} \quad A_{k\ell}^{ij} \neq 0 \ , \\
\alpha_i^k \geqq 0 \quad \textit{and} \quad \alpha_i^i = 0 \ .
\end{gathered}
$$

CONDITION 2.

$$
\forall i \in I \ , \ \sum_{k \in I} \alpha_k^i (c_k - c_i) = 0 \ .
$$

REMARK. In this paper, we never use the microreversibility condition $A_{ij}^{k\ell} = A_{k\ell}^{ij}$ nor $H$-theorem.

Here we consider the equations (B) which differs from the ordinary ones [5], [7], by the intervention of linear terms. Nevertheless the contribution of linear terms is important at least by two reasons : the linear terms must be considered when we study a solution near the constant stationary solutions [10] (in this case, the corresponding linear matrix is symmetric, which we won't assume later), and the equations with linear and nonlinear terms express a model of the motion of particles which are animated in a thin infinite tube under the binary collisions between particles and also under the linear reflections at the inner wall [12], [13], [14]. In this article, we improve the results obtained in [12] and extend an estimate of solution [3] to the case with linear and quadratic terms. Secondly, for small Cauchy data, we obtain an estimate of solutions in the Sobolev space under a similar assumption to the one in [4]. Furthermore we obtain an asymptotic result which expresses a closer look of the behavior of solutions even for the equations without linear terms.

We put $I_k$, $k = 0, 1, \cdots$, as follows :

$$
\begin{aligned}
I_0 &= \{i; \alpha_k^i = 0 \text{ for all } k \in I\} \\
&= \{i; \text{particles with velocity } c_i \text{ don't provoke any reflection}\}\ , \\
I_1 &= \{i \notin I_0; \text{there exists } j \in I_0 \text{ such that } \alpha_j^i > 0\} \\
&= \{i; \text{particles with velocity } c_i \text{ is transformed} \\
&\qquad \text{into a particle with velocity } \quad c_j, j \in I_0 \quad \text{by reflection}\}, \\
I_{k+1} &= \{i \notin \bigcup_{\ell=0}^{k} I_\ell; \quad \text{there exists} \quad j \in I_k \quad \text{such that} \quad \alpha_j^i > 0\}\ .
\end{aligned}
\tag{1.1}
$$

REMARK. As we see later, $I_0$ is not empty, if we assume Condition 2.

PROPOSITION 1.1. *Suppose Condition 1 is satisfied. Let* $u_i = u_i(x,t) \in C^1(\mathbf{R}_+, \mathcal{S}(\mathbf{R}))$ $(i \in I)$ *be a solution of (B). Then, for any* $t \in \mathbf{R}_+$, *we have*

$$
\int_{\mathbf{R}} \sum_i u_i(x,t)dx = \int_{\mathbf{R}} \sum_i u_i^0(x)dx \equiv \mu \tag{1.2}
$$

*(mass conservation law) .*

*Furthermore assuming Condition 2, we have*

$$\int_{\mathbf{R}} \sum_i c_i u_i(x,t)dx = \int_{\mathbf{R}} \sum_i c_i u_i^0(x)dx \tag{1.3}$$

*(momentum conservation law) .*

PROPOSITION 1.2. *Condition 2 implies that $I_0$ is not empty.*

PROOF. Let $M$ [resp. $m$ ] be an index such that $c_M > c_i$ for $i \neq M$ [resp. $c_m < c_i$ for $i \neq m$]. Then Condition 2 means

$$\sum_k \alpha_k^M (c_M - c_k) = 0 \ .$$

Then we have $\alpha_k^M = 0$ for all $k \in I$, because $\alpha_k^i \geqq 0$ .
Similarly we have $\alpha_k^m = 0$ for all $k \in I$. That is $M$ and $m \in I_0$. □

Our results are the following :
<u>under Condition 2</u>

THEOREM 1. *Suppose Conditions 1 and 2 are satisfied. For the Cauchy data $u_i^0$ positive, summable and bounded, there exists an unique global bounded solution $u_i(x,t) \in L^\infty(\mathbf{R} \times \mathbf{R}_+)$ and we obtain the estimate*

$$u_i(x,t) \leqq (1 + \sup_{i,x} u_i^0(x)) \exp(a\mu^2 + b\mu) \ , \tag{1.4}$$

*where $a$ and $b$ depend only on the equations, and $\mu$ is the total mass defined in Proposition 1.1.*

COROLLARY 2. *Suppose Conditions 1 and 2 are satisfied. For the Cauchy data $u_i^0$ positive and bounded, there exists an unique global solution $u_i(x,t) \in L^\infty_{\ell oc}(\mathbf{R} \times \mathbf{R}_+)$ and we obtain the estimate*

$$u_i(x,t) \leqq \exp(A\mu^2 t^2 + B) \ , \tag{1.5}$$

*where $A$ and $B$ don't depend on time.*

THEOREM 3. *Assume the same hypotheses as in Theorem 1. We have the asymptotic behavior of a solution : When $t$ tends to the infinity, $u_i(x+c_it, t)$ converge, in $L^p$ $(2 \leqq p \leqq \infty)$, to a function $\varphi_i(x)$ which is zero except for $i \in I_0$.*

without Condition 2

We put other hypotheses :

CONDITION 3.

$$\alpha_i^k > 0 \quad \text{for} \quad i \neq k \ .$$

Proposition 1.2 implies

PROPOSITION 1.3. *Condition 2 is not compatible with Condition 3.*

for the small Cauchy data :

i) Case with the binary collision terms.

In this case, we treat the general form of the binary collision terms :

$$Q_i(u) = \sum_{jk} B_i^{jk} u_j u_k, \tag{gQ}$$

which is introduced by Bony [4]. In this paper, he showed that the global existence of the solution for the small Cauchy data in the case $L_i = 0$ in $\mathbf{R}^N$ and defined the corresponding wave operators and scattering operators.

The equations are the following :

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + c_i \dfrac{\partial u_i}{\partial x} = Q_i(u) + L_i(u) \ , \\ \qquad u_i|_{t=0} = u_i^0(\cdot) \ , \end{cases} \tag{B}$$

with $L_i$ is of the form as before. On this system, we impose some assumptions :

CONDITION 4.

$$B_i^{jk} \neq 0 \quad \Rightarrow \quad j \neq k \ ,$$
$$B_i^{jk} \neq 0 \quad \Rightarrow \quad j \ \textit{and} \ k \notin I_0 \ ,$$
$$\alpha_i^k \geqq 0, \quad \forall j \exists i \alpha_j^i > 0 \ .$$

CONDITION 5.

$$\begin{cases} I_0 \neq \emptyset \ , \\ i \in I \backslash I_0 \Longrightarrow i \in I_1 \ . \end{cases}$$

REMARK. The first condition in Condition 4 is introduced in [4] and it is a reasonable condition for developing a general theory of global existence, because a blow-up example is known in the case without this condition. The second condition in Condition 4 means that the particles which don't provoke any reflection don't make any binary collision.

THEOREM 4. *Suppose Conditions 4 and 5 are satisfied. If the Cauchy data are sufficiently small in* $H^s (s = 1, 2, \cdots)$*, the solution has the decay estimate as follows :*

$$\begin{aligned} &\|u_i\|_{H^s} (\text{ so } \|u_i\|_{L^\infty}) \\ &\leqq \begin{cases} C_* \left\| u^0 \right\|_{H^s} & \text{for} \quad i \in I_0 \ , \\ C_* \left\| u^0 \right\|_{H^s} e^{-\lambda t} & \text{for} \quad i \in I_1 \ , \end{cases} \end{aligned} \tag{1.6}$$

*where* $C_*$ *and* $\lambda > 0$ *depend only on the equation.*

ii) Case with the multiple collision terms.

The case with the multiple collision terms is studied only in a few papers [1][2][6]. We consider the general multiple collision terms as follows :

$$R_i(u) = \sum_{p=2}^{\sigma} \sum_{j_1} \cdots \sum_{j_p} E_i^{j_1 \cdots j_p} u_{j_1} \cdots u_{j_p} \ , \tag{R}$$

where we permit the cases $j_k = j_\ell, k \neq \ell$.

Then the equations is following :

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + c_i \dfrac{\partial u_i}{\partial x} = R_i(u) + L_i(u) \ , \\ \qquad u_i|_{t=0} = u_i^0(\cdot) \ , \end{cases} \tag{M}$$

where $L_i$ is of the form as before. On this system, we impose the similar assumptions to Condition 4 :

CONDITION 6.

$$E_i^{j_1\cdots j_p} \neq 0 \quad \Rightarrow \quad \exists j_\alpha \neq j_\beta, j_\alpha, j_\beta \in \{j_1, \cdots j_p\} \ ,$$
$$E_i^{j_1\cdots j_p} \neq 0 \quad \Rightarrow \quad \begin{cases} \exists j_\alpha \notin I_0 & \text{if } i \in I_0 \\ \exists j_\alpha \neq j_\beta \notin I_0 & \text{if } i \notin I_0 \ , \end{cases}$$
$$\alpha_i^k \geqq 0, \quad \forall j \ \exists i \ \alpha_j^i > 0 \ .$$

*We obtain the result with the similar estimates as in Theorem 4 :*

THEOREM 5. *Suppose Conditions 5 and 6 are satisfied. If the Cauchy data are sufficiently small in $H^s(s = 1, 2, \cdots)$, the solution has the decay estimate as follows :*

$$\begin{aligned} &\|u_i\|_{H^s}( \text{ so } \|u_i\|_{L^\infty}) \\ &\leqq \begin{cases} C_* \|u^0\|_{H^s} & \text{for} \quad i \in I_0 \\ C_* \|u^0\|_{H^s} e^{-\lambda t} & \text{for} \quad i \in I_1 \ , \end{cases} \end{aligned} \tag{1.7}$$

*where $C_*$ and $\lambda > 0$ depend only on the equation.*

## 2. The proof

### 2.1 Estimates

In this section, assuming Conditions 1 and 2, we establish the estimates of solutions, improving the method due to Bony [3].

Let's define Bony's function [3] and its variation :

$$\varphi(t) = \sum_{i,j} (c_i - c_j) \iint \operatorname{sgn}(y - x) u_i(x,t) u_j(y,t) dx dy \ , \tag{2.1}$$

$$\psi(t; x_0, c_0) = \sum_i (c_i - c_0) \int \operatorname{sgn}\{x - (x_0 + c_0 t)\} u_i(x,t) dx \ . \tag{2.2}$$

Differentiating these functions, we have

LEMMA 2.1. *Suppose $T < T^*$. Under Conditions 1 and 2, we have*

$$\Delta(0, T) \leqq C\mu^2 \ , \tag{2.3}$$

$$\delta(0, T) \leqq C\mu \ , \tag{2.4}$$

*where $T^*$ is the existence time of solutions and*

$$\Delta(t_1, t_2) = \sup_{c_i \neq c_j} \int_{t_1}^{t_2} \int_{\mathbf{R}} u_i(x,t) u_j(x,t) dx dt \ , \tag{2.5}$$

$$\delta(t_1, t_2) = \sup_{c_i \neq c_j} \sup_{x \in \mathbf{R}} \int_{t_1}^{t_2} u_i(x + c_j t, t) dt \ . \tag{2.6}$$

PROOF. We differentiate $\varphi(t)$ with respect $t$ :

$$\varphi'(t) = -2 \sum_{i,j} (c_i - c_j)^2 \int u_i u_j dx \ .$$

Hence

$$2 \sum_{i,j} (c_i - c_j)^2 \int_0^T \int_{\mathbf{R}} u_i u_j dx = \varphi(0) - \varphi(T) \ . \tag{2.7}$$

On the other hand, we have

$$|\varphi(t)| \leqq C\mu^2 \text{ for all } t \ . \tag{2.8}$$

Therefore we obtain

$$\int_0^T \int_{\mathbf{R}} u_i(x,t) u_j(x,t) dx dt \leqq C\mu^2 \text{ for } c_i \neq c_j \ . \tag{2.9}$$

In the same way,

$$\psi'(t) = 2 \sum_i (c_i - c_0)^2 u_i(x_0 + c_0 t, t) \ . \tag{2.10}$$

This implies

$$2 \sum_i (c_i - c_0)^2 \int_0^T u_i(x_0 + c_0 t, t) \ dt = \psi(T) - \psi(0) \ . \tag{2.11}$$

On the other hand, we have

$$|\psi(t)| \leqq C\mu \quad \text{for all} \quad t \ . \tag{2.12}$$

Then we obtain

$$\int_0^T u_i(x_0 + c_0 t, t) dt \leqq C\mu \quad \text{for} \quad c_i \neq c_0 \ . \ \square \tag{2.13}$$

PROPOSITION 2.2. *Suppose Conditions 1 and 2 are satisfied. Then there exists* $p \geqq 0$ *such that*

$$I = I_0 \cup I_1 \cup \cdots \cup I_p \ . \tag{2.14}$$

PROOF. Suppose $i \notin I_0$. Then there exists $j$ such that $\alpha^i_j > 0$. Furthermore, there exists $j$ such that $\alpha^i_j > 0$ and $c_j > c_i$, because if not, every $j$ such that $\alpha^i_j > 0$ verifies $c_j < c_i$, then Condition 2 would fail. Therefore we have

$$i \in I \Rightarrow \begin{cases} \alpha^i_j = 0 \quad \text{for all} \quad j, \quad \text{that is} \quad i \in I_0 \\ \textit{or} \\ \text{there exists} \quad j \quad \text{such that} \quad \alpha^i_j > 0 \\ \quad \text{and} \quad c_j > c_i \ . \bigstar \end{cases} \tag{2.15}$$

If ★ holds, we repeat this procedure :

$$\begin{cases} j \in I_0 \quad \text{then} \quad i \in I_1 \\ \textit{or} \\ \text{there exists} \quad k \quad \text{such that} \quad \alpha^j_k > 0 \quad \text{and} \quad c_k > c_j \ . \bigstar\bigstar \end{cases} \tag{2.16}$$

If ★★ holds, we do as above :

$$\begin{cases} k \in I_0 \quad \text{then} \quad j \in I_1, i \in I_2 \\ \textit{or} \\ \text{there exists} \quad \ell \quad \text{such that} \quad \alpha^k_\ell > 0 \quad \text{and} \quad c_\ell > c_k \ . \bigstar\bigstar\bigstar \end{cases} \tag{2.17}$$

However this procedure must be finished because $\sharp I$ is finite. Then we prove the proposition. □

For analyzing closely our partial differential system, now we consider a simpler ordinary system ; this idea is motivated by the dissipation of the effect of the binary collision terms when the time is going to the infinity. This dissipation is suggested by the definability of the wave operator for the system without the linear term for the small Cauchy data, due to Bony [4] :

$$\begin{cases} \dfrac{df_i}{dt} = L_i(f) \ , \\ f_i|_{t=0} = f^0_i > 0 \ . \end{cases} \tag{O}$$

PROPOSITION 2.3. *Let $\mathcal{L}$ be a matrix corresponding to the linear operator $(L_i)_i$. Then each eigenvalue of $\mathcal{L}$ is* 0 *or of real part negative.*

To prove this proposition, we state a classical theorem :

LEMMA 2.4. (Geršgorin) *Let $A = [a_{ij}] \in M_n$, and let*

$$C'_j(A) \equiv \sum_{i \neq j} |a_{ij}| \tag{2.18}$$

*denote the deleted absolute column sums of $A$. Then all the eigenvalues of $A$ are located in the union of $n$ discs*

$$\bigcup_{i=1}^{n} \{z \in \mathbf{C} : |z - a_{ii}| \leqq C'_i(A)\} \equiv G(A) \ . \tag{2.19}$$

PROOF. See Chapter 6 in [8] for example. □

PROOF OF PROPOSITION 2.3. Simply apply Lemma 2.4 to the matrix $\mathcal{L}$. □

PROPOSITION 2.5. *Suppose Conditions 1 and 2 are satisfied. Then we have*

*1) for all $i \in I$, $f_i(t)$ is positive.*

*2) $\sum_{i \in I_0} f_i(t)$ is increasing and bounded, so tends to a limit $> 0$ as $t \to +\infty$.*

*3) for $i \notin I_0$, $f_i(t)$ tends to* 0 *exponentially as $t \to +\infty$.*

PROOF. 1) The assertion is clear when we write the system in the following form :

$$\frac{df_i}{dt} + (\sum_k \alpha_k^i) f_i = \sum_k \alpha_i^k f_k \ . \tag{2.20}$$

2) and 3) We show, for the first, for $i \in I_1$, $f_i(t)$ tends to 0 exponentially as $t \to +\infty$. Suppose that there exists $c > 0$ and $i \in I_1$ such that $f_i(t) > c$

for all $t \in \mathbf{R}_+$. Then

$$\begin{aligned} \frac{d}{dt}(\sum_{i\in I_0} f_i) &= \sum_j (\sum_{i\in I_0} \alpha_i^j) f_j - \sum_{i\in I_0}(\sum_j \alpha_j^i) f_i \\ &= \sum_{j\in I_1}(\sum_{i\in I_0} \alpha_i^j) f_j \\ &\geqq C_* c > 0 \quad \text{with} \quad C_* > 0\ . \end{aligned} \tag{2.21}$$

Then $\sum_{i\in I_0} f_i$ increases exponentially, which is a contradiction because of Proposition 2.3. Hence, for $i \in I_1$, $f_i(t)$ tends to 0 exponentially as $t \to +\infty$. Then we have

$$\frac{d}{dt}(\sum_{i\in I_0} f_i) = \sum_{j\in I_1}(\sum_{i\in I_0} \alpha_i^j) \exp(-\lambda_j t) \times P_j(t), \tag{2.22}$$

where $\Re e\ \lambda_j > 0$ and $P_j(t) > 0$ is a polynomial in $t$. Then the right-hand side is positive and integrable over $[0, \infty]$. Hence, $\sum_{i\in I_0} f_i(t)$ is increasing and bounded, so tends to a limit $> 0$ as $t \to +\infty$. Now we show, for $i \in I_2$, $f_i(t)$ tends to 0 exponentially as $t \to +\infty$, by reduction to absurdity. Suppose that there exists $c > 0$ and $i \in I_2$ such that $f_i(t) > c$ for all $t \in \mathbf{R}_+$. Then

$$\begin{aligned} \frac{d}{dt}(\sum_{i\in I_1} f_i) &= \sum_{j\in I_2}(\sum_{i\in I_1} \alpha_i^j) f_j - \sum_{i\in I_1}(\sum_j \alpha_j^i) f_i \\ &\geqq C_* c - \sum_{i\in I_1}(\sum_j \alpha_j^i) f_i \quad \text{with} \quad C_* > 0\ . \end{aligned} \tag{2.23}$$

Then, taking a sufficiently large $T$, for $t > T$, we have

$$\frac{d}{dt}(\sum_{i\in I_1} f_i) \geqq \frac{1}{2} C_* c > 0\ . \tag{2.24}$$

Hence $\sum_{i\in I_1} f_i$ increases exponentially, which is a contradiction. Hence, for $i \in I_2$, $f_i(t)$ tends to 0 exponentially as $t \to +\infty$. We continue this procedure until $I_p$, taking account of Proposition 2.2. Then we complete the proof. □

Now we fix $t_1$ and decompose $u_i$ into the sum of "(quasi-)linear part" $v_i$ and "(essential-)nonlinear part" $w_i$. Let $v_i$ be a solution for the equations :

$$\text{(V)} \qquad \begin{cases} \left(\dfrac{\partial}{\partial t} + c_i \dfrac{\partial}{\partial x}\right) v_i = L_i(v) - \displaystyle\sum_{j,k,\ell} A^{ij}_{k\ell} u_j \cdot v_i \ , \\ v_i|_{t=t_1} = u_i(\cdot, t_1) \ , \end{cases}$$

where $u_i$ is the solution of (B).

Then $w_i = u_i - v_i$ should satisfy

$$\text{(W)} \qquad \begin{cases} \left(\dfrac{\partial}{\partial t} + c_i \dfrac{\partial}{\partial x}\right) w_i = L_i(w) + Q_i(w) + r_i - s_i \ , \\ w_i|_{t=t_1} = 0 \ , \\ \text{with} \quad r_i = \displaystyle\sum_{j,k,\ell} A^{k\ell}_{ij} (v_k v_\ell + w_k v_\ell + v_k w_\ell) \ , \\ s_i = \displaystyle\sum_{j,k,\ell} A^{ij}_{k\ell} w_i v_j \ . \end{cases}$$

DEFINITION. The operator $\mathcal{P} = (\mathcal{P}_i)_i$ is said to be positively preserving if and only if the solution $u_i$ is nonnegative over $\mathbf{R} \times \mathbf{R}_+$, where $u_i(x, t)$ is a solution for the equations :

$$\text{(2.25)} \qquad \begin{cases} \left(\dfrac{\partial}{\partial t} + c_i \dfrac{\partial}{\partial x}\right) u_i = \mathcal{P}(u) \ , \\ u_i|_{t=0} = u_i^0(x) \geqq 0 \ . \end{cases}$$

COROLLARY 2.6. *The operators $(Q_i)_i$ and $(L_i)_i$ are positively preserving.*

PROOF. The assertions follow, for $L_i$, from Proposition 2.5 and, for $Q_i$, from a classical argument of the semi-linear equations, given in [11] for example. □

PROPOSITION 2.7.

$$v_i(x,t) \geqq 0 \text{ and } w_i(x,t) \geqq 0 \text{ for any } x \in \mathbf{R} \text{ and } t \in \mathbf{R}_+ \ . \tag{2.26}$$

PROOF. We take account of $u_i(x,t) \geqq 0$ for all $x \in \mathbf{R}$ and $t \in \mathbf{R}_+$. From the equations, we have

$$\left(\frac{\partial}{\partial t} + c_i \frac{\partial}{\partial x}\right) v_i + \sum_{j,k,\ell} A_{k\ell}^{ij} u_j \cdot v_i = L_i(v) \ . \tag{2.27}$$

Then we have $v_i(x,t) \geqq 0$, because the linear operator $(L_i)_i$ is positively preserving and $u_i(x,t) \geqq 0$. On the other hand,

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + c_i \dfrac{\partial}{\partial x}\right) w_i + \left(\displaystyle\sum_{j,k,\ell} A_{k\ell}^{ij} v_j\right) \cdot w_i = L_i(w) + Q_i(w) + r_i \ , \\ \qquad\qquad\qquad\qquad \text{with} \quad r_i \geqq 0 \ . \end{cases} \tag{2.28}$$

Then we have $w_i(x,t) \geqq 0$, because the linear operator $(L_i)_i$ and the binary operator $(Q_i)_i$ are positively preserving and $v_i(x,t) \geqq 0$. □

Now we have some remarks :

COROLLARY 2.8.

$$\int_{t_1}^{t_2} \int_{\mathbf{R}} r_i dx dt \leqq C_* \Delta(t_1, t_2), \quad \int_{t_1}^{t_2} \int_{\mathbf{R}} s_i dx dt \leqq C_* \Delta(t_1, t_2) \tag{2.29}$$

PROOF. Combine the previous propositions. □

We are now going to estimate the "linear part" solution $v_i$. Its estimate in "$L^\infty$" is as follows :

PROPOSITION 2.9. *The function* $V(t) \equiv \sup_{i,x} \dfrac{v_i(x,t)}{f_i(t)}$ *is strictly decreasing.*

PROOF. By a classical argument, it is sufficient to show $\dfrac{v_i(x,t)}{f_i(t)}$ is decreasing at the points $(i,x)$ where the *sup* is attained. At such point $(i,x)$, we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\frac{v_i(x,t)}{f_i(t)} &= \frac{\partial_t v_i}{f_i} - v_i\frac{\partial_t f_i}{{f_i}^2} \\
&= \frac{L_i(v) - \sum_{j,k,\ell} A^{ij}_{k\ell} u_j \cdot v_i}{f_i} - v_i\frac{L_i(f)}{{f_i}^2} \\
&< \sum_k \left(\alpha^k_i \frac{v_k}{f_i} - \alpha^i_k \frac{v_i}{f_i}\right) - \frac{v_i}{f_i}\sum_k \left(\alpha^k_i \frac{f_k}{f_i} - \alpha^i_k\right) \\
&= \sum_k \alpha^k_i \frac{f_k}{f_i}\left(\frac{v_k}{f_k} - \frac{v_i}{f_i}\right) \\
&\leqq 0,
\end{aligned}
\tag{2.30}
$$

where we used Condition 1 in the third inequality. □

The estimates for $v_i$ and $w_i$ along the characteristic are the following :

PROPOSITION 2.10. *For $c_i \neq c_j$ and $t_1 < t_2$, we have*

$$\sup_{x\in\mathbf{R}} \int_{t_1}^{t_2} v_i(x + c_j t, t)dt \leqq C_* \delta(t_1, t_2) \ , \tag{2.31}$$

$$\sup_{x\in\mathbf{R}} \int_{t_1}^{t_2} w_i(x + c_j t, t)dt \leqq C_* \Delta(t_1, t_2) \ , \tag{2.32}$$

*where these constants $C_*$ depend only on the equations.*

PROOF. The inequality for $v_i$ is evident, because $0 \leqq v_i \leqq u_i$ and the definition of $\delta(t_1, t_2)$. For proving the second inequality, we define, like as in §2.1 :

$$\psi_w(t; x_0, c_0) = \sum_i (c_i - c_0) \int \operatorname{sgn}\{x - (x_0 + c_0 t)\} w_i(x,t)dx. \tag{2.33}$$

Now we differentiate it. Then we have

$$\psi'_w(t) = 2\sum_i (c_i - c_0)^2 w_i(x_0 + c_0 t, t) + R - S, \tag{2.34}$$

where

$$R = \sum_i \int \operatorname{sgn}\{x - (x_0 + c_0 t)\} r_i(x,t)dx \ , \tag{2.35}$$

$$S = \sum_i \int \operatorname{sgn}\{x - (x_0 + c_0 t)\} s_i(x,t)dx \ . \tag{2.36}$$

On the other hand,

$$\int_{t_1}^{t_2} Rdt \quad \text{and} \quad \int_{t_1}^{t_2} Sdt \leqq C_* \Delta(t_1, t_2) \ ,$$

by virtue of Corollary 2.8. Furthermore we have

$$\begin{aligned} \frac{d}{dt} \left\| \sum_i w_i \right\|_{L^1} &= \left\| \sum_i (r_i - s_i) \right\|_{L^1} \\ &\leqq \left\| \sum_i r_i \right\|_{L^1} \ . \end{aligned} \tag{2.37}$$

Hence we have, for $t_1 < t < t_2$,

$$\left\| \sum_i w_i(\cdot, t) \right\|_{L^1} \leqq \int_{t_1}^{t_2} \left\| \sum_i r_i \right\|_{L^1} dt \leqq C_* \Delta(t_1, t_2) \ . \tag{2.38}$$

Then we have, for $t_1 < t < t_2$,

$$|\psi_w(t; x_0, c_0)| \leqq C_* \Delta(t_1, t_2) \ . \tag{2.39}$$

Therefore we have, by integration,

$$\begin{aligned} \int_0^T w_i(x_0 + c_0 t, t)dt &\leqq C_* \left|\psi_w(t_1; x_0, c_0)\right| + C_* \left|\psi_w(t_2; x_0, c_0)\right| \\ &\qquad + C_* \int_{t_1}^{t_2} Rdt + C_* \int_{t_1}^{t_2} Sdt \\ &\leqq C_* \Delta(t_1, t_2) \ . \ \square \end{aligned} \tag{2.40}$$

Now we estimate more closely $M(t_2)$ in terms of $M(t_1)$ for $t_1 < t_2 < T^*$, where

$$M(t) = \max_{i\in I} \sup_{s\le t} \sup_{x\in\mathbf{R}} u_i(x,s) \quad \text{for} \quad t < T^* \ . \tag{2.41}$$

First, we integrate the equations for $w_i$ along a characteristic curve. Then we have

$$\begin{aligned} w_i(t_2, x_*) \leqq C_* \int_{t_1}^{t_2} (\sum_{j\neq i} w_j + \sum_{k,j\neq i} w_k w_j + \sum_{k,j\neq i} v_k v_j \\ + \sum_{k,j\neq i} w_k v_j + \sum_{k,j\neq i} v_k w_j)(x + c_i t, t) dt \ , \end{aligned} \tag{2.42}$$

where $x_* = x + c_i(t_2 - t_1)$. Then we have

$$\int_{t_1}^{t_2} \sum_{j\neq i} w_j(x + c_i t, t) dt \leqq C_* \Delta(t_1, t_2) \ , \tag{2.43}$$

$$\int_{t_1}^{t_2} \sum_{k,j\neq i} w_k w_j(x + c_i t, t) dt \leqq C_* M(t_2) \Delta(t_1, t_2) \ , \tag{2.44}$$

$$\int_{t_1}^{t_2} \sum_{k,j\neq i} v_k v_j(x + c_i t, t) dt \leqq C_* M(t_2) \delta(t_1, t_2) \ . \tag{2.45}$$

Hence we have

$$\sup_x w_i(x, t_2) \leqq C_* \left(1 + M(t_2)\right) \cdot \Delta(t_1, t_2) + C_* M(t_2) \delta(t_1, t_2) \ . \tag{2.46}$$

Consequently for $u_i(x,t)$, we have

$$\begin{aligned} \sup_x u_i(x, t_2) &\leqq \sup_x v_i(x, t_2) + \sup_x w_i(x, t_2) \\ &\leqq C_* M(t_1) + C_* \left(1 + M(t_2)\right) \cdot \Delta(t_1, t_2) \\ &\quad + C_* M(t_2) \delta(t_1, t_2) \ . \end{aligned} \tag{2.47}$$

We take $T < T^*$ and thereafter a sequence $0 = t_0 < t_1 < \cdots < t_N = T$ such that $\Delta(t_j, t_{j+1}) < (4C_*)^{-1}$ and $\delta(t_j, t_{j+1}) < (4C_*)^{-1}$. Then, seeing that $\Delta(0,T) \leqq C_*\mu^2$ and $\delta(0,T) \leqq C_*\mu$ by virtue of Lemma 2.1, we have $N = O(\mu^2 + \mu)$ and

$$M(t_{j+1}) \leqq C_* M(t_j) + C_*\Delta(t_j, t_{j+1}) \leqq C_* M(t_j) + C_* \ . \tag{2.48}$$

Therefore we obtain

$$u_i(x,t) \leqq (1 + \sup_{i,x} u_i^0) \exp(a\mu^2 + b\mu) \ . \tag{2.49}$$

## 2.2 Proof of Theorem 3

We examine in a more detailed way the argument developed in the last section, that is, to decompose the solution $u_i$ into the sum of "(quasi-)linear part" $v_i$ and "(essential-) nonlinear part $w_i$. Later on, we specify $t_1$, which will be denoted $T$, and the dependence of $v_i$ and $w_i$ on a cut time $T$. Let's write down $u_i$ of the form $u_i = v_i^T + w_i^T$ :

$$(V^T) \qquad \begin{cases} \left(\dfrac{\partial}{\partial t} + c_i \dfrac{\partial}{\partial x}\right) v_i^T = L_i(v^T) - \displaystyle\sum_{j,k,\ell} A_{k\ell}^{ij} u_j \cdot v_i^T \ , \\ v_i^T|_{t=T} = u_i(\cdot, T) \ . \end{cases}$$

$$(W^T) \qquad \begin{cases} \left(\dfrac{\partial}{\partial t} + c_i \dfrac{\partial}{\partial x}\right) w_i^T = L_i(w^T) + Q_i(w^T) + r_i^T - s_i^T \ , \\ w_i^T|_{t=T} = 0 \ , \\ \text{with} \quad r_i^T = \displaystyle\sum_{j,k,\ell} A_{ij}^{k\ell}(v_k^T v_\ell^T + w_k^T v_\ell^T + v_k^T w_\ell^T) \ , \\ s_i^T = \displaystyle\sum_{j,k,\ell} A_{k\ell}^{ij} w_i^T v_j^T \ . \end{cases}$$

Proposition 2.11. *Assume the same hypotheses as in Theorem 1. Then $v_i^T(x,t)$ verifies that, for any $\varepsilon > 0$, there exists a large $T$ such that for $t > T$, we have*

$$\left\| u_i(\cdot,t) - v_i^T(\cdot,t) \right\|_{L^p} < \varepsilon \ (1 \leqq p \leqq \infty) \quad \textit{for all} \quad i \tag{2.50}$$

*and*

$$\left\| v_i^T(\cdot,t) \right\|_{L^p} \leqq \begin{cases} m(t) & i \in I_0 \\ m(t)\exp\{-\lambda t(1-\frac{1}{p})\} & i \notin I_0 \end{cases} \tag{2.51}$$

*for all* $i$, *where* $2 \leqq p \leqq \infty$, $\lambda > 0$ *and* $m(t)$ *is a strictly decreasing function.*

PROOF. Knowing that

$$u_i(x,t) \leqq M \equiv (1+\sup_{i,x} u_i^0)\exp(a\mu^2 + b\mu), \tag{2.52}$$

we have, for $t > T$,

$$\begin{aligned} \sup_x w_i^T(x,t) &\leqq C_*(1+M(t)) \cdot \Delta(T,t) + C_* M(t)\delta(T,t) \\ &\leqq C_*(1+M)(\Delta(T,t)+\delta(T,t)) \ , \end{aligned} \tag{2.53}$$

$$\left\| \sum_i w_i^T(\cdot,t) \right\|_{L^1} \leqq C_*\Delta(T,t) \ . \tag{2.54}$$

Using Lemma 2.1 which says $\Delta(0,\infty) \leqq C_*\mu^2$ and $\delta(0,\infty) \leqq C_*\mu$, we conclude that, for any $\varepsilon > 0$, there exists $T$ such that

$$\Delta(T,\infty) + \delta(T,\infty) < [C_*(1+M) \times (\sharp\{i \in I\})]^{-1} \cdot \varepsilon \ . \tag{2.55}$$

Then we have, for $t > T$,

$$\left\| \sum_i w_i^T(\cdot,t) \right\|_{L^1 \cap L^\infty} < \varepsilon \ . \tag{2.56}$$

Hence we have

$$\sup_i \left\| w_i^T(\cdot,t) \right\|_{L^p} < \varepsilon \ (1 \leqq p \leqq \infty) \ , \tag{2.57}$$

i.e.

$$\sup_i \left\| u_i(x,t) - v_i^T(x,t) \right\|_{L^p} < \varepsilon \ (1 \leqq p \leqq \infty), \tag{2.58}$$

save for trivial constants. Consequently we prove the first assertion of Proposition 2.11.

In order to prove the second assertion, we shall estimate $v_i$ :

PROPOSITION 2.12. *$\sum_i \|V_i(\cdot,t)\|_{L^2}^2$ is also strictly decreasing, where* $V_i(x,t) \equiv \dfrac{v_i^T(x,t)}{\sqrt{f_i(t)}}$.

PROOF. We have

$$\begin{aligned}\left(\frac{\partial}{\partial t} + c_i \frac{\partial}{\partial x}\right) V_i = \sum_k \left\{ \alpha_i^k \left(\frac{f_k(t)}{f_i(t)}\right)^{\frac{1}{2}} V_k - \alpha_k^i V_i \right\} \\ - \sum_{jk\ell} A_{k\ell}^{ij} u_j \cdot V_i - \frac{L_i(f(t))}{2f_i(t)} V_i \; .\end{aligned} \tag{2.59}$$

Hence we have

$$\begin{aligned}&\frac{d}{dt}\sum_i \|V_i(\cdot,t)\|_{L^2}^2 \\ &= -\sum_i \left(\sum_k \alpha_k^i\right) \|V_i\|^2 - \sum_{ik} \alpha_k^i \|V_i\|^2 - \sum_i \frac{L_i(f)}{f_i}\|V_i\|^2 \\ &\quad + 2\sum_{ik} \alpha_i^k \left(\frac{f_k}{f_i}\right)^{\frac{1}{2}} (V_k, V_i) - 2\sum_{ijk\ell} A_{k\ell}^{ij} (u_j V_i, V_i) \\ &= -\sum_{ik} \alpha_i^k f_k \left\| \frac{V_i}{\sqrt{f_i}} - \frac{V_k}{\sqrt{f_k}} \right\|^2 - 2\sum_{ijk\ell} A_{k\ell}^{ij} (u_j V_i, V_i) \\ &< 0,\end{aligned} \tag{2.60}$$

where we used the positivity of $u_i$ and $V_i$ and Condition 1. $\square$

We pursue the proof of Proposition 2.11. By Proposition 2.9 and 2.12, we have

$$\max_i \left\| \frac{v_i^T(\cdot,t)}{f_i(t)} \right\|_{L^\infty} \quad \text{and} \quad \max_i \left\| \frac{v_i^T(\cdot,t)}{f_i(t)} \right\|_{L^2} \tag{2.61}$$

are strictly decreasing, where a positive bounded functions $f_i(t)$ verifies the following condition :

– for $i \in I_0$, $f_i(t)$ is increasing and tends to a limit $> 0$ as $t \to +\infty$,
– for $i \notin I_0$, $f_i(t)$ tends to 0 exponentially as $t \to +\infty$.
The interpolation between $L^2$ and $L^\infty$ achieves then the proof. □

Now we show the theorem. We take a sequence $T_n$ which tends to the infinity. We show that $u_i(x+c_iT_n, T_n)$ is a Cauchy sequence for an adequate $T_n$. Owing to Proposition 2.11, $\sum_i w_i^{T_n}(x+c_iT_n, T_n)$ is a Cauchy sequence. On the other hand, we have

$$\begin{aligned}(2.62)\qquad &\int_{\mathbf{R}} \left| \sum_i v_i^{T_{n+1}}(x+c_iT_{n+1}, T_{n+1}) - \sum_i v_i^{T_n}(x+c_iT_n, T_n) \right|^2 dx \\ &\leqq C_* M^2 (T_{n+1} - T_n) \sum_{ij} \int_{\mathbf{R}} \int_{T_n}^{T_{n+1}} u_i u_j \, dt dx \\ &\leqq C_* M^2 (T_{n+1} - T_n) \Delta(T_n, T_{n+1}) \ .\end{aligned}$$

Hence $\sum_i v_i^{T_n}(x+c_iT_n, T_n)$ is a Cauchy sequence in $L^2$. Furthermore we have

$$\begin{aligned}(2.63)\qquad &\left| \sum_i v_i^{T_{n+1}}(x+c_iT_{n+1}, T_{n+1}) - \sum_i v_i^{T_n}(x+c_iT_n, T_n) \right| \\ &\leqq C_* M \delta_{x+c_iT_n}(T_n, T_{n+1}) \ ,\end{aligned}$$

where $\delta_x(t_1, t_2) = \sup_{c_i \neq c_j} \int_{t_1}^{t_2} u_i(x+c_jt, t)dt$. Hence $\sum_i v_i^{T_n}(x+c_iT_n, T_n)$ is a Cauchy sequence also in $L^\infty$. Consequently we prove the theorem. □

## 2.2 Proof of Theorem 4

Let's consider the following equations with parameter $\varepsilon$ :

$$(B_\varepsilon)\qquad \begin{cases} \dfrac{\partial u_i}{\partial t} + c_i \dfrac{\partial u_i}{\partial x} = \varepsilon Q_i(u) + L_i(u) \ , \\ u_i|_{t=0} = u_i^0(\cdot) \ , \end{cases}$$

where $\varepsilon$ is a positive constant.

For this Cauchy problem, we seek a solution $u_i(x,t)$ of type $u_i = \sum_{m=0}^{\infty} \varepsilon^m u_i^{(m)}$.

To prove Theorem 4, it is sufficient to show the following theorem :

THEOREM 2.12. *Suppose Conditions 4 and 5 are satisfied.*

*For* $\varepsilon \in \Big[0, C_{**}\big(\sum_{k=0}^{s} I_k^2\big)^{-\frac{1}{2}}\Big[$, *the series* $u_i = \sum_{m=0}^{\infty} \varepsilon^m u_i^{(m)}$ *converge in* $H^s (s = 1, 2, \cdots)$, *so* $L^\infty$, *uniformly with respect to* $t \in \mathbf{R}_+$ *and then we have*

$$\|u_i\|_{H^s}(\text{ so } \|u_i\|_{L^\infty}) \leqq \begin{cases} C_*\big(\sum_{k=0}^{s} I_k^2\big)^{\frac{1}{2}} & i \in I_0 \\ C_*\big(\sum_{k=0}^{s} I_k^2\big)^{\frac{1}{2}} e^{-\lambda t} & i \in I_1 \ , \end{cases} \tag{2.64}$$

*where* $I_k = \Big(\sum_i \left\|D^k u_i^0\right\|_{L^2}^2\Big)^{\frac{1}{2}}$, *the constants* $C_*$, $C_{**}$ *and* $\lambda > 0$ *depend only on the equations.*

As in the preceding section, we use

$$\begin{cases} \dfrac{df_i}{dt} = L_i(f) \ , \\ f_i|_{t=0} = f_i^0 > 0 \ . \end{cases} \tag{O}$$

Condition 5 implies a more precise estimate for $f_i$ than Proposition 2.3 :

PROPOSITION 2.13. *Suppose Conditions 4 and 5 are satisfied. There exists* $f_i^0 > 0$ *such that, for* $i \in I_1$, $f_i$ *tends to* $0$ *with the same order i.e. there is* $\lambda > 0$ *such that* $f_i(t) = e^{-2\lambda t} P_i(t)$ *with* $P_i$ *polynomial in* $t$ *and that, for* $i \in I_0$, $f_i(t)$ *is increasing and bounded, so tends to a limit* $> 0$ *as* $t \longrightarrow +\infty$.

PROOF. Let's put a matrix $\mathcal{L}' = (\alpha_i^j - \delta_{ij} \sum_k \alpha_k^i)_{ij\in I_1}$. Then let $n$ be a positive constant and $\mathcal{M} = \mathcal{L}' + n$. The matrix $\mathcal{M}$ is nonnegative for sufficiently large $n$. We have then

$$\exp t\mathcal{M} = e^{nt} \exp t\mathcal{L}' \quad \text{that is} \quad \exp t\mathcal{L}' = e^{-nt} \exp t\mathcal{M} \ . \tag{2.65}$$

Hence $\exp t\mathcal{L}'$ is a positive matrix. The Perron-Frobenius theorem for the matrix theory says that $\exp t\mathcal{L}'$ has a real and positive eigenvalue which is a simple root of the characteristic equation and exceeds the moduli of all the other eigenvalues and that, to this 'maximal' eigenvalue, there corresponds an eigenvector $z = (z_i)_i$ with positive coordinates $z_i > 0$ $(i \in I_1)$. Let $e^{-2\lambda}$

be its 'maximal' eigenvalue for $t = 1$. Let $f^0 = (f_i^0)_{i\in I_1}$ be a corresponding eigenvector with positive coordinates $f_i^0 > 0 (i \in I_1)$. It implies that

$$
\begin{aligned}
f_i(t) &= \exp(t\mathcal{L}')f_i^0 \\
&= e^{-2t\lambda} f_i^0 \quad \text{for} \quad i \in I_1 \ .
\end{aligned}
\tag{2.66}
$$

Moreover using Lemma 2.4 and Condition 5, we easily obtain that each eigenvalue of matrix $\mathcal{L}'$ is of real part negative and then we have $\lambda > 0$. We prove then the assertion for $i \in I_1$. On the other hand, the assertion for $i \in I_0$ is clear, because we have

$$
f_i(t) = f_i^0 + \int_0^t \sum_{k\in I_1} \alpha_i^k e^{-2\lambda\tau} f_k^0 d\tau \ .\Box
\tag{2.67}
$$

Let put $w_i(x,t) = \dfrac{u_i(x,t)}{\sqrt{f_i(t)}}$. Now we write down the equation for $w_i(x,t)$, and put $w_i(x,t) = \sum_{i=0}^{\infty} \varepsilon^m w_i^{(m)}(x,t)$. Then we have for $m = 0, 1, 2, \cdots$ ,

$$
\left\{
\begin{aligned}
&\left(\frac{\partial}{\partial t} + c_i \frac{\partial}{\partial x}\right) w_i^{(m)} = \sum_k \left\{ \alpha_i^k \left(\frac{f_k}{f_i}\right)^{\frac{1}{2}} w_k^{(m)} - \alpha_k^i w_i^{(m)} \right\} \\
&\qquad\qquad\qquad\qquad - \frac{L_i(f)}{2f_i} w_i^{(m)} + F_i^{(m)}(w) \ , \\
&w_i^{(m)}|_{t=0} = \begin{cases} u_i^0(x), & m = 0 \\ 0, & m = 1, 2, \cdots \end{cases} ,
\end{aligned}
\right.
\tag{2.68}
$$

where

$$
F_i^{(m)}(w) = \begin{cases}
\displaystyle\sum_{n=0}^{m-1} \sum_{jk} B_i^{jk} \left(\frac{f_j f_k}{f_i}\right)^{\frac{1}{2}} w_j^{(n)} w_k^{(m-n-1)} & \\
& \text{for } m = 1, 2, \cdots \ , \\
0 & \text{for } m = 0 \ .
\end{cases}
\tag{2.69}
$$

The energy estimate leads us :

Proposition 2.14. *Suppose Conditions 4 and 5 are satisfied. For* $s = 0, 1, 2, \cdots$ ,

$$\frac{d}{dt}\sum_i \left\| D^s w_i^{(m)} \right\|_{L^2}^2 = -\sum_{ik} \alpha_i^k f^k \left\| D^s \left( \frac{w_i^{(m)}}{f_i^{\frac{1}{2}}} - \frac{w_k^{(m)}}{f_k^{\frac{1}{2}}} \right) \right\|_{L^2}^2 + 2\sum_i \left\{ \left( D^s F_i^{(m)}(w), D^s w_i^{(m)} \right)_{L^2} \right\} , \tag{2.70}$$

*and especially for* $m = 0$,

$$\frac{d}{dt}\sum_i \left\| D^s w_i^{(0)} \right\|_{L^2}^2 = -\sum_{ik} \alpha_i^k f^k \left\| D^s \left( \frac{w_i^{(0)}}{f_i^{\frac{1}{2}}} - \frac{w_k^{(0)}}{f_k^{\frac{1}{2}}} \right) \right\|_{L^2}^2 \leqq 0 . \tag{2.71}$$

Corollary 2.15. *Suppose Conditions 4 and 5 are satisfied. Then we have, for* $s = 0, 1, 2, \cdots$ ,

$$\left\| D^s w_i^{(0)} \right\|_{L^2} \leqq C_* I_s \text{ for all } i \tag{2.72}$$

*where* $I_s = \left( \sum_i \left\| D^s u_i^0 \right\|_{L^2}^2 \right)^{\frac{1}{2}}$.

Proposition 2.16. *Suppose Conditions 4 and 5 are satisfied. For* $s = 1, 2, \cdots$ ,

$$\int_0^\infty \sum_i \left\| F_i^{(1)}(w) \right\|_{H^s} dt \leqq C_* \sum_{k=0}^{s} I_k^2 . \tag{2.73}$$

Proof. By the equation, we have

$$F_i^{(1)}(w) = \sum_{jk} B_i^{jk} \left( \frac{f_j f_k}{f_i} \right)^{\frac{1}{2}} w_j^{(0)} w_k^{(0)} . \tag{2.74}$$

On the other hand

$$\left\|w_j^{(0)}w_k^{(0)}\right\|_{H^s} \leqq C_*\left(\left\|w_j^{(0)}\right\|_{H^s}\left\|w_k^{(0)}\right\|_{L^\infty}+\left\|w_j^{(0)}\right\|_{L^\infty}\left\|w_k^{(0)}\right\|_{H^s}\right) \leqq C_*\left\|w_j^{(0)}\right\|_{H^s}\left\|w_k^{(0)}\right\|_{H^s} \quad \text{for} \quad s=1,2,\cdots . \tag{2.75}$$

Condition 4 and Proposition 2.13 give us

$$B_i^{jk}\left(\frac{f_j f_k}{f_i}\right)^{\frac{1}{2}} \leqq Ce^{-\lambda t} . \tag{2.76}$$

These estimates and Corollary 2.15 imply

$$\int_0^\infty \sum_i \left\|F_i^{(1)}(w)\right\|_{H^s} dt \leqq C_* \sum_{ijk} \left\|w_j^{(0)}\right\|_{H^s}\left\|w_k^{(0)}\right\|_{H^s} \int_0^\infty e^{-\lambda t} dt \leqq C_* \sum_{k=0}^{s} I_k^2 . \ \square \tag{2.77}$$

PROPOSITION 2.17. *Suppose Conditions 4 and 5 are satisfied. Then we have, for* $s=1,2,\cdots$ ,

$$\left\|w_i^{(1)}\right\|_{H^s} \leqq C_* \sum_{k=0}^{s} I_k^2 \quad \textit{for all} \quad i . \tag{2.78}$$

PROOF. Using Proposition 2.14, we have

$$\begin{aligned} &2\left(\sum_i \left\|w_i^{(1)}\right\|_{H^s}\right)\cdot\left(\frac{d}{dt}\sum_i \left\|w_i^{(1)}\right\|_{H^s}\right) \\ &= \frac{d}{dt}\sum_i \left\|w_i^{(1)}\right\|_{H^s}^2 \\ &\leqq C_* \sum_i \left\|w_i^{(1)}\right\|_{H^s}\cdot\left\|F_i^{(1)}(w)\right\|_{H^s} \\ &\leqq C_*\left(\sum_i \left\|w_i^{(1)}\right\|_{H^s}\right)\cdot\left(\sum_i \left\|F_i^{(1)}(w)\right\|_{H^s}\right) . \end{aligned} \tag{2.79}$$

This implies that

$$\frac{d}{dt}\sum_i \left\| w_i^{(1)} \right\|_{H^s} \leqq C_* \sum_i \left\| F_i^{(1)}(w) \right\|_{H^s} . \tag{2.80}$$

Hence we have,

$$\begin{aligned} 0 \leqq \sum_i \left\| w_i^{(1)} \right\|_{H^s} &\leqq \sum_i \left\| w_i^{(1)}|_{t=0} \right\|_{H^s} + C_* \sum_i \int_0^t \left\| F_i^{(1)}(w) \right\|_{H^s} dt \\ &\leqq 0 + C_* \sum_{k=0}^{s} I_k^2, \end{aligned} \tag{2.81}$$

by virtue of the previous proposition. □

Now let's put $\left\| w_i^{(m)} \right\|_{H^s} \leqq a_s^{(m)}$ for $s = 1, 2, \cdots$, then we have, by induction,

$$a_s^{(m+1)} \leqq C_* \sum_{n=0}^{m} a_s^{(n)} a_s^{(m-n)} . \tag{2.82}$$

Let's put $f(x) = f_s(x) = \sum_{n=0}^{\infty} a_s^{(n)} x^n$ and $F(x) = F_s(x) = \sum_{n=0}^{\infty} A_s^{(n)} x^n$, where

$$\begin{cases} A_s^{(m+1)} = C_* \sum_{n=0}^{m} A_s^{(n)} A_s^{(m-n)} , \\ A_s^{(0)} = C_* \left( \sum_{k=0}^{s} I_k^2 \right)^{\frac{1}{2}} . \end{cases} \tag{2.83}$$

Then the inequality (2.82) means that

$$a_s^{(m)} \leqq A_s(m) \text{ and } A_s^{(m)} \geqq 0 \text{ for } m = 0, 1, 2, \cdots \tag{2.84}$$

i.e. $F(x)$ is majorant series of $f(x)$. By the definition, $F(x)$ satisfies

$$\frac{F(x) - F(0)}{x} = C_* \{F(x)\}^2 . \tag{2.85}$$

Then we have

$$F(x)=\frac{1-\sqrt{1-4C_*xF(0)}}{2C_*x},\quad F(0)=C_*\left(\sum_{k=0}^{s}I_k^2\right)^{\frac{1}{2}}\ . \tag{2.86}$$

It is easy to see that the right-hand side can be written in infinite series with a positive convergence radius. Hence the series $F(x)$ and $f(x)$ have a positive convergence radius. Consequently we achieve the proof.

### 2.4 Proof of Theorem 5

As in the previous section, we consider the following equations with parameter $\varepsilon$ :

$$(M_\varepsilon)\qquad \begin{cases} \dfrac{\partial u_i}{\partial t}+c_i\dfrac{\partial u_i}{\partial x}=\varepsilon R_i(u)+L_i(u)\ , \\ \qquad u_i|_{t=0}=u_i^0(\cdot)\ , \end{cases}$$

where $\varepsilon$ is a positive constant.

The similar argument shows

$$\left\|w_i^{(m)}\right\|_{H^s}\leqq b_s^{(m)}\ \text{for}\ s=1,2,\cdots\ , \tag{2.85}$$

where

$$\begin{cases} b_s^{(m+1)}\leqq C\displaystyle\sum_{p=2}^{\sigma}\sum_{n_1+\cdots n_p=m}b_s^{(n_1)}\cdots b_s^{(n_p)}\ , \\ b_s^{(0)}=C\left(\displaystyle\sum_{k=0}^{s}I_k^2\right)^{\frac{1}{2}}\ . \end{cases} \tag{2.86}$$

In the same way, we complete the proof.

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Department of Mathematical Sciences
University of Tokyo
Komaba, Meguro–ku, Tokyo
Japan

Present address
Institute of Mathematics
University of Tsukuba
Tsukuba 305
Japan