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On the special values of abelian L-functions

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Abstract. Here we give a proof of the p-portion of a conjecture of Gross over the global function fields of characteristic p. In this case, the conjecture is in fact a refinement of the class number formula. Here the classic Dedekind Zeta function is generalized by a p-adic measure which interpolates the special values of abelian L-functions, and the regulator of the units group is generalized by a p-adic regulator.

The L-functions are associated to the characters of the maximal abelian extension of the given global field unramified outside a finite set $v_0, v_1, \ldots, v_r$, of places of the field. The case that $r = 1$ has been proved by Hayes. Gross also proved some congruence of the formula.

0. Introduction

In this note, we will prove the p-portion of a conjecture of Gross [G] over the global function fields.

The conjecture is about the special values of abelian L-functions. Let K be a global function field of characteristic p, and $S = \{v_0, v_1, \ldots, v_r\}$, $r \geq 1$, a finite set of primes of K. Suppose that L/K is an abelian extension unramified outside S, and G is the corresponding Galois group. There is an element θ_G in the ring $\mathbf{Z}[[G]]$ of integral measures of G, which interpolates special values of abelian L-function (see [G] or [T], V). Namely, for each continuous character χ of G, the value $\chi(\theta_G)$ equals the special value $L_T(\chi, 0)$ of the L-function relative to S, and modified at a non-empty finite set T of primes not contained in S (see [G],[T] or Section 1). The Gross conjecture is about the order of vanishing of θ_G as a measure and its leading term in a Taylor expansion. The leading term involves the (modified) class number and the G-valued regulator defined by Gross. Over function fields,

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this conjecture is a generalization of the classical class number formula. In [H], Hayes has proved the conjecture for $r = 1$, K a function field and any G. In [G], Gross prove the conjecture up to a constant, for G cyclic of prime order.

In this note, we consider the case that G equals $\mathbf{H}$, the Galois group of the maximal abelian pro-p-extension of K unramified outside S. We'll give a proof of the Gross conjecture for the case $G = \mathbf{H}$ and r any positive integer. An immediate consequence is that the conjecture is true for every quotient of $\mathbf{H}$.

Note that the conjecture can be stated universally for $G = G_S$ the Galois group of the maximal abelian extension of K unramified outside S. For each prime number l, the conjecture for $G = \mathbf{Z}_l$, the Galois group of the cyclic constant field extension, is just the class number formula and is true. Our result then implies that the conjecture is true for $G = \mathbf{H} \times \prod_{l \neq p} \mathbf{Z}_l$. This group is a quotient of G_S by a finite subgroup with order prime to p.

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1. The θ Element

We will follow the notations in [G]. The modified Zeta-function is defined as

$$\zeta_T(s) = \prod_{v \in T}(1 - N(v)^{1-s}) \prod_{v \notin S}(1 - N(v)^{-s})^{-1}.$$

For an abelian pro-finite Galois group G over K and a complex character χ of G, via Class Field Theory, the associated modified L-function is defined as

$$L_T(\chi, s) = \prod_{v \in T}(1 - \chi(v) \cdot N(v)^{1-s}) \prod_{v \notin S}(1 - \chi(v) \cdot N(v)^{-s})^{-1}.$$

For finite G, the theta element θ_G is the unique element of $\mathbf{Z}[G]$ such that

$$\chi(\theta_G) = L_T(\chi, 0).$$

The augmentation ideal $I_G \subset \mathbf{Z}[G]$ is defined as the kernel of the ring homomorphism

$$\mathbf{Z}[G] \longrightarrow \mathbf{Z},$$
$$g \mapsto 1.$$

Note that if H is a quotient of G and $\phi : G \longrightarrow H$ is the projection, then $\phi(\theta_G) = \theta_H$. Using this compatibility property, we define the theta element for pro-finite G as

$$\theta_G = \varprojlim_{G'} \theta_{G'},$$

where the projective limit is taken over all finite quotients of G. Thus θ_G can be viewed as an element of the ring $\mathbf{Z}[[G]]$ of integral measures of G. The augmentation ideal $I_G \subset \mathbf{Z}[[G]]$ is defined as the ideal of measures of volume zero. If χ_0 is the trivial character of G, then I_G is just the kernel of the induced ring homomorphism (by integration) $\chi_0 : \mathbf{Z}[[G]] \longrightarrow \mathbf{Z}$.

For a finite extension L/K, let $L(S)$ and $L(T)$ respectively be the sets of primes of L sitting over those in S and T. Also denote by $\zeta_{L(T)}(s)$ the modified Zeta-function of L relative to $L(S)$.

Suppose that χ is a continuous character of G of order p^n. Let $L^{(n)}$ be the cyclic extension determined by χ, and $L^{(n-1)}$ that determined by χ^p. Then we have

$$\prod_{\substack{a \pmod{p^n} \\ (a,p)=1}} \chi^a(\theta_G) = \lim_{s \to 0} \frac{\zeta_{L^{(n)}(T)}(s)}{\zeta_{L^{(n-1)}(T)}(s)}. \tag{1.1}$$

By the functional equation of Zeta-functions and the Dirichlet formula, the order of vanishing of $\zeta_{L^{(n)}(T)}(s)$ at zero equals the rank of the $L^{(n)}(S)$-units in $L^{(n)}$.

Lemma 1.2. *Let χ be a character of order exactly p^n, then $\chi(\theta_G) = 0$ if and only if the χ-eigen space of the $L^{(n)}(S)$-units is of positive rank.*

Proof. The complex numbers $\chi^a(\theta_G), (a, p) = 1$, are algebraic and conjugate to each other over $\mathbf{Q}$. They must be either all zero or all non-zero. The lemma is then a consequence of (1.1). □

In particular, we have $\chi_0(\theta_G) = 0$ for the trivial character χ_0, since we are assuming that S has at least two elements. Consequently,

$$\theta_G \in I_G. \tag{1.3}$$

Another direct consequence of Lemma 1.2 is the following:

Lemma 1.4. *If some place of S splits completely in L/K, then $\theta_G = 0$.*

2. The Gross Conjecture

The G-regulator is defined as follows (see [G], p. 179). Let Y be the free abelian group generated by the primes $v \in S$ and $X = \{\sum a_v \cdot v : \sum a_v = 0\}$ the subgroup of elements of degree zero in Y. Let U be the group of S-units, and U_T the rank r free abelian group of S-units which are $\equiv 1 \pmod{T}$. Let G be an abelian pro-finite Galois group over K. For each $v \in S$, let f_v be the homomorphism

$$f_v : U_T \longrightarrow K_v^* \longrightarrow \mathbf{A}_K^*/K^* \longrightarrow G.$$

Also let λ_G be the morphism

$$\begin{aligned} \lambda_G : U_T &\longrightarrow G \otimes X \\ \epsilon &\mapsto \sum_S f_v(\epsilon) \cdot v. \end{aligned} \tag{2.1}$$

Choosing bases $< \epsilon_1, \ldots, \epsilon_r >$ and $< x_1, \ldots, x_r >$, we obtain an $r \times r$ matrix (g_{ij}) for λ_G with entries in G. Recall that G is an abelian group and hence a $\mathbf{Z}$-module. The determinant $det(g_{ij}) =: \sum_{\pi \in S_r} sign(\pi) g_{1\pi(1)} \cdot g_{2\pi(2)} \cdot \ldots \cdot g_{r\pi(r)}$ is an element of the symmetric product $Sym_r(G)$ of G over $\mathbf{Z}$.

It is well-known that for finite G, the map $g \mapsto g - 1 \pmod{I_G^2}$ gives an isomorphism $G \xrightarrow{\sim} I_G/I_G^2 \simeq H_1(G, \mathbf{Z})$ (see [Hi]). If G is pro-finite, we have $G = \lim_{\substack{\leftarrow \\ \mu}} G_\mu$ over the finite quotients. If I_μ is the augmentation ideal of G_μ, then $I_G = \lim_{\substack{\leftarrow \\ \mu}} I_\mu$ and $I_G^2 = \lim_{\substack{\leftarrow \\ \mu}} I_\mu^2$. Since for $\mu > \mu'$, the map $I_\mu^2 \longrightarrow I_{\mu'}^2$ is surjective, the map $G \xrightarrow{\sim} I_G/I_G^2$ is also an isomorphism (Mittag-Leffler Condition, see [A], p.104). We have the induced map

$$Sym_r(G) \longrightarrow I_G^r/I_G^{r+1}.$$

Let $det_G\lambda$ be the image of $det(g_{ij})$ in I_G^r/I_G^{r+1}. The Gross Conjecture says that

$$\theta_G \equiv m \cdot det_G\lambda \pmod{I_G^{r+1}}. \tag{2.2}$$

Here $m = \pm h_T$ where h_T is the modified class number of the S-integers of K and the sign depends on the choice of the orientation of the bases $< \epsilon_i >$ and $< x_i >$ (see [G], p. 182).

Let $\mathbf{H}$ be the por-p completion of the group $\mathbf{A}_K^*/K^* \cdot \prod_{v \notin S} \mathcal{O}_v^*$. It is the Galois group of the maximal abelian pro-p-extension of K unramified outside S. In this note, we show the following :

THEOREM 2.3. *The Gross conjecture* (2.2) *is true for* $G = \mathbf{H}$.

Note that for fixed G, the conjecture does not depend on S, as we have the following (see [G], p. 182):

(2.4) COMPATIBILITY OF THE CONJECTURE. *Suppose that* $S \subset S'$ *and* G' *is a quotient of* G. *If the conjecture is true for* (G, S), *then it is also true for* (G', S').

As a consequence of Theorem 2.3 and (2.4), the conjecture is true for every pro-p group G. When G is a pro-p group, it is advantageous to work over $\mathbf{Z}_p$. Let $\mathbf{Z}_p[[G]]$ be the ring of $\mathbf{Z}_p$-measures over G and $I_{G,p}$ the augmentation ideal of it.

LEMMA 2.5. *If G is a pro-p group, then for each non-negative integer r,*

(1) $$I_{G,p}^r \cap \mathbf{Z}[[G]] = I_G^r$$

(2) *the natural map* $I_G^r/I_G^{r+1} \longrightarrow I_{G,p}^r/I_{G,p}^{r+1}$ *is an injection.*

PROOF. The second statement is a direct consequence of the first one. If G is finite, then the first statement can be deduced from the fact that $\mathbf{Z}_p[[G]] = \mathbf{Z}[[G]] \otimes_{\mathbf{Z}} \mathbf{Z}_p$ and $I_{G,p}^r = I_G^r \otimes_{\mathbf{Z}} \mathbf{Z}_p$. For infinite G, we just take the projective limit over the finite quotients of G. □

As a corollary, we have the following.

LEMMA 2.6. *If G is a pro-p group, then θ_G and $det_G\lambda$ can be viewed respectively as elements of $\mathbf{Z}_p[[G]]$ and $I_{G,p}^r/I_{G,p}^{r+1}$. The Gross conjecture is then equivalent to*

$$\theta_G \equiv m \cdot det_G\lambda \pmod{I_{G,p}^{r+1}}.$$

Remark. As pointed out to the author by J. Tate, if we are working over $\mathbf{Z_p}$, then the auxiliary set T can be discarded. This is explained as follows. For each v not in S, let $[v] \in G$ be the Frobenius element. Define

$$\theta' = \theta \cdot \prod_{v \in T} (1 - [v]N(v))^{-1},$$

and

$$L(\chi, s) = \prod_{v \notin S} (1 - \chi(v) \cdot N(v)^{-s})^{-1}.$$

Then for each character χ of G, we have $\chi(\theta') = L(\chi, 0)$. Since $(1-[v]N(v))$ is invertible in $\mathbf{Z}_p[[G]]$, $\theta' \in \mathbf{Z}_p[[G]]$, too. The S-unit group U is in general not free. As in the beginning of this section, by choosing a subgroup (e.g. U_T) of U of rank r we can define a regulator $det\lambda$ in $I^r_{G,p}/I^{r+1}_{G,p}$. Since the order of the group of roots of unit is relatively prime to p and hence invertible in $\mathbf{Z}_p$, the regulator $det\lambda$ is independent of the choice of the free subgroup of U. In particular, the regulator $det_G\lambda$ in the conjecture (2.2) is independent of the set T. In (2.2), if we replace θ by θ' and m by a suitable $m' \in \mathbf{Z}_p$ then we get a new conjecture $(2.2)'$ which does not depend on T. By Lemma 2.6, $(2.2)'$ is equivalent to (2.2).

3. Reduction of the Conjecture for $G = \mathbf{H}$ and a Sketch of the Proof

In this section, we fix the sets S and T described before. A projective system of quotient groups of $\mathbf{H}$ will be called a fine system, if every finite quotient group of $\mathbf{H}$ is a quotient of some group in this system. By (2.4), the proof of Theorem 2.3 can be reduced to proving (2.2) for every G in a fine system. Let $\mathcal{H}$ be the sub-system of quotients of $\mathbf{H}$ consisting of those which are isomorphic to $\mathbf{Z}_p^d$, for some positive integer d.

Lemma 3.1 (see also [K]).

(1) (*Local Leopoldt*) *For each v, the homomorphism $U \otimes \mathbf{Z}_p \longrightarrow K_v^*$ is injective.*

(2) *$\mathbf{H}$ is isomorphic, as a topological group, to a countable infinite product of $\mathbf{Z}_p$.*

(3) *The system $\mathcal{H}$ is a fine system.*

PROOF. Given an element $a \in K^*$, which is not a pth power in K^*, we need to show that a is not a pth power in K_v^*. Then the first statement follows. By the structure theorem of K_v^* ([W]), the second statement also follows. If a is a pth power in K_v^*, then the non-trivial pure inseparable extension $K(u^{\frac{1}{p}})/K$ would be locally trivial at v. This is impossible.

The third statement is a consequence of the second statement. □

Let $gr(\mathbf{Z}_p[[G]]) = \mathbf{Z}_p \oplus I_{G,p}/I_{G,p}^2 \oplus I_{G,p}^2/I_{G,p}^3 \oplus \ldots.$ be the graded algebra associated to the group ring. Suppose that $G \in \mathcal{H}$. Then the natural morphism $Sym_k(G) \longrightarrow I_{G,p}^k/I_{G,p}^{k+1}$ induces an isomorphism

$$\delta = \delta_G : \bigoplus_{k \geq 0} Sym_k(G) \otimes \mathbf{Z}_p \xrightarrow{\sim} gr(\mathbf{Z}_p[[G]]).$$

Suppose that $G \simeq \mathbf{Z}_p^d$ and $< \sigma_1, \ldots, \sigma_d >$ is a basis of G over $\mathbf{Z}_p$. Let $t_i = \sigma_i - 1 \in \mathbf{Z_P}[[G]]$. Then it is well known that $\mathbf{Z}_p[[G]] = \mathbf{Z}_p[[\underline{t}]], \underline{t} = (t_i)_{i=1,\ldots,d}$. In this case, $I_{G,p}^k = (\underline{t}^k)$, and we have a non-canonical isomorphism of algebras

$$gr(\mathbf{Z}_p[[G]]) \xrightarrow{\sim} \mathbf{Z}_p[\underline{t}] \subset \mathbf{Z}_p[[G]].$$

This and the isomorphism δ induce the following non-canonical isomorphism

$$\delta(\underline{t}) = \delta_G(\underline{t}) : \bigoplus_{k \geq 0} Sym_k(G) \otimes \mathbf{Z}_p \xrightarrow{\sim} \mathbf{Z}_p[\underline{t}] \subset \mathbf{Z}_p[[G]].$$
$$\sigma_i \mapsto t_i.$$

Let $\theta_1(\underline{t}) = \theta_{1,G}(\underline{t}) = \theta_G$. Recall that (g_{ij}) is the matrix associated with the morphism λ_G defined in Section 2. Let $t_{ij} = \delta(\underline{t})(g_{ij})$. Then $\theta_2 = \theta_{2,G} = \theta_{2,(\underline{t})} := det(t_{ij}) = \delta(\underline{t})(\delta^{-1}(det_G \lambda))$ is the unique homogeneous polynomial of degree r such that

$$\theta_2 \equiv det_G \lambda \pmod{I_{G,p}^{r+1}}.$$

Then by Lemma 2.6, for $G \in \mathcal{H}$, the Gross Conjecture is equivalent to

$$\theta_1 \equiv m \cdot \theta_2. \pmod{(\underline{t})^{r+1}} \tag{3.2}$$

The polynomial $\theta_{2,(\underline{t})}$ depends on the choice of the basis of G. If $\theta_{2,(\underline{t})} = F(\underline{t})$ for a form F of degree r and $< \sigma_1', \ldots, \sigma_d' >$ is another basis of G such that $(\sigma'_1, \ldots, \sigma'_d) = (\sigma_1, \ldots, \sigma_d) \cdot A$ for some $A \in GL(d, \mathbf{Z}_p)$, then we have $\theta_{2,(\underline{t}')} = F(\underline{t}' \cdot A^{-1})$.

Suppose $G \in \mathcal{H}$ and $G \simeq \mathbf{Z}_p^d$. Let $V = V_G$ be the d-dimensional affine space over $\mathbf{Z}_p$ such that $V(\mathbf{Z}_p) = Hom_{\mathbf{Z}_p}(G, \mathbf{Z}_p)$. Then the coordinate ring of V equals $\bigoplus_{k\geq 0} Sym_k(G) \otimes \mathbf{Z}_p$. By the non-canonical map $\delta(\underline{t})$, we identify this coordinate ring with $\mathbf{Z}_p[\underline{t}]$. Then the homogeneous polynomial θ_2 defines a Zariski closed set $D = D_G$ in V. It is known that $\theta_2 = det(t_{ij})$ is absolutely irreducible provided the t_{ij} are independent variables (see [V] Vol.1, p.94). Note that the absolute irreducibility of θ_2 does not depend on the choice of the basis of G. This motivates the following definition. Let $\mathcal{H}'$ be the set consisting of all those $G \in \mathcal{H}$ with $g_{ij},\ i, j = 1, \ldots, r$, linearly independent over $\mathbf{Z}_p$. In particular, if G is in $\mathcal{H}'$, then $d =: rk_{\mathbf{Z}_p}(G) \geq r^2$.

Lemma 3.3.

(1) *The system $\mathcal{H}'$ is a fine system.*
(2) *If G is in $\mathcal{H}'$, then θ_2 is absolutely irreducible and $D(\mathbf{Z}_p)$ is Zariski dense in D.*

Proof. For (1), since every $G \in \mathcal{H}$ is a quotient of $\mathbf{H}$, it is sufficient to show that for $G = \mathbf{H}$ the associated g_{ij} are linearly independent over $\mathbf{Z}_p$. We can then take $< x_i = v_i - v_0, i = 1, \ldots, r >$ as a basis of X. For a chosen basis $< \epsilon_i >$ of U_T, we have $g_{ij} = f_{v_i}(\epsilon_j)$. By Lemma 3.1 and the definiton of $\mathbf{H}$, we see that the g_{ij} are linearly independent over $\mathbf{Z}_p$.

For (2), we may assume that $\{t_{ij},\ i, j = 1, \ldots, r\}$ is a subset of $\{t_i, i = 1, \ldots, d\}$, and $t_{11} = t_1$. Denote $t^- = (t_2, \ldots, t_d)$, and denote by V^- the locus of $t_1 = 0$. The coordinate ring of V^- can be identified with $\mathbf{Z}_p[t^-]$. This identification corresponds to a projection ϕ from V to V^-. Let $g(t^-)$ be the (1,1)-minor of (t_{ij}). Then D is defined by an equation of the from

$$t_{11} \cdot g(t^-) = F(t^-).$$

Let $W = V - \{zeros\ of\ g\}$ and $W^- = V^- - \{zeros\ of\ g\}$. Then ϕ induces an isomorphism from $D \cap W$ to W^-. Since $W^-(\mathbf{Z}_p)$ is dense in W^- and $D \cap W$ is dense in D (which is irreducible), $D(\mathbf{Z}_p)$ is dense in D. $\square$

We now sketch the proof of Theorem 2.3. Two method will be applied frequently. The first is to *go up* to a larger group of which G is a quotient.

If Theorem 2.3 is true for this group, then by (2.4), it is also true for G. This will allow us to work on a group which is more "generic" than G. The second method is to *go down* to the $\mathbf{Z}_p$-quotients of G, i.e. those which are isomorphic to $\mathbf{Z}_p$. In Section 4, by this going down method, we reduce the proof to the case that G is *degenerate*, namely, $G \simeq \mathbf{Z}_p$ and $\theta_{2,G} = 0$. For degenerate G, Theorem 2.3 is equivalent to that at the origin, the theta element $\theta_{1,G}$ has order of vanishing greater than r. In Section 5, we study the order of vanishing of $\theta_{1,G}$ for degenerate G. There are in general two possibilities :

(1) We can show directly that $\theta_{1,G}$ is 0.

(2) From G, we can go up to a canonical Γ in $\mathcal{H}$. The corresponding $\mathbf{Z}_p$-affine space V_Γ will contain the $\mathbf{Z}_p$-line V_G. We show that V_Γ also contains (maybe not distinct) $r+1$ affine hyperplanes to which the restrictions of $\theta_{1,\Gamma}$ are all zero. The theta element $\theta_{1,\Gamma}$ has then $r+1$ homogeneous linear factors. If the $r+1$ hyperplanes are distinct (in this case, we say that G is good, this is true when $r = 1$), then $\theta_{1,\Gamma}$ is divisible by $r+1$ distinct homogeneous linear factors hence has order of vanishing larger than r. Then the order of vanishing of $\theta_{1,G}$ is also greater than r. In particular, the theorem is proved for $r = 1$. In Section 6, our last step of the proof is to go up again from G to an $H \in \mathcal{H}$ such that a generic $\mathbf{Z}_p$-quotient of H is degenerate and good. By the going down method, we show that at the origin, $\theta_{1,H}$ has order of vanishing greater than r, and so does $\theta_{1,G}$.

4. The Degeneration

For each $H \simeq \mathbf{Z}_p$ contained in $\mathcal{H}$, recall the homomorphisms $f_v, v \in S$, defined in Section 2 (taking $G = H$). We say that H is degenerate if there is a non-trivial $u \in U_T \otimes \mathbf{Z}_p$ such that $f_v \otimes 1(u) = 0, \forall v \in S$. It is easy to see that $H \simeq \mathbf{Z}_p$ contained in $\mathcal{H}$ is degenerate if and only if $\theta_{2,H} = 0$. For each $G \in \mathcal{H}$, we use $ord(\theta_{1,G})$ to denote the order of vanishing of $\theta_{1,G}$ at the origin.

Proposition 4.1. *The Gross Conjecture is true for* $\mathbf{H}$ *if and only if, for all* $H \simeq \mathbf{Z}_p$ *which is degenerate,* $ord(\theta_{1,H}) \geq r+1$.

Proof. "$\Rightarrow$" is a direct consequence of (3.2). To show "$\Leftarrow$", we need the following geometric lemma. For each $G \simeq \mathbf{Z}_p^d$ contained in $\mathcal{H}$, we have the following identifications.

$$\begin{aligned}&\{(H \simeq \mathbf{Z}_p \text{ a quotient of } G, \sigma \text{ a topological generator of } H)\}\\ \simeq&\{\mathbf{Z}_p\text{-linear projection } G \longrightarrow \mathbf{Z}_p\}\\ \simeq&\{\text{surjective, zero-degree, } \mathbf{Z}_p\text{-graded-algebra homomorphism}\\ &\mu : \mathbf{Z}_p[\underline{t}] \longrightarrow \mathbf{Z}_p[x]\}\\ \simeq&\{\text{embedding of } \mathbf{Z}_p \text{ into } V_G(\mathbf{Z}_p)\}.\end{aligned}$$

For a $\mu : \mathbf{Z}_p[\underline{t}] \longrightarrow \mathbf{Z}_p[x]$ corresponding to an embedding of $\mathbf{Z}_p$ into $V_G(\mathbf{Z}_p)$, we also denote by μ the induced morphism from $\mathbf{Z}_p[[\underline{t}]]$ to the one variable power series ring $\mathbf{Z}_p[[x]]$. Let D be the affine scheme defined by θ_2. We have $\mu(\theta_2) = 0$ if and only if H is degenerate. In this case, we say that μ belongs to D.

Lemma 4.2. *Let $\theta_2(t_1, \dots, t_d)$ be an rth degree absolutely irreducible homogeneous polynomial over $\mathbf{Z}_p$ and D the affine scheme defined by θ_2. Suppose that $D(\mathbf{Z}_p)$ is Zariski dense in D. Let $\theta_1 \in \mathbf{Z}_p[[t_1, \dots, t_d]]$ be such that for every μ belonging to D, we have $ord(\mu(\theta_1)) \geq r+1$. Then we have*

$$\theta_1 \in \mathbf{Q}_p \cdot \theta_2 + (t_1, \dots, t_d)^{r+1}.$$

Proof. Let $\theta_1 = \sum_i \theta_1^{(i)}$ be such that each $\theta_1^{(i)}$ is the homogeneous polynomial of degree i. Then for each μ belonging to $D, \mu(\theta_1^{(i)}) = 0$ for $i \leq r$. This shows that for $i \leq r$, $\theta_1^{(i)} = 0$ on each $\mathbf{Z}_p$-line inside $D(\mathbf{Z}_p)$. Since D is a $\mathbf{Z}_p$-cone, we have, for $i \leq r$, $\theta_1^{(i)} = 0$ on $D(\mathbf{Z}_p)$ hence on D. As θ_2 is absolutely irreducible of degree r, we must have $\theta_1^{(i)} = 0$ for $i < r$ and $\theta_1^{(r)} \in \mathbf{Q}_p \cdot \theta_2$. $\square$

We now consider the proof of "$\Leftarrow$". Suppose the condition of the proposition holds. We need to show that (3.2) is true for each $G \in \mathcal{H}'$. For such G, we consider all its degenerate $\mathbf{Z}_p$ quotients. By the compatibility of θ_1 under quotients, Lemma 3.3 and Lemma 4.2, we deduce that

$$\theta_1 \equiv m' \cdot \theta_2 \pmod{(\underline{t})^{r+1}},$$

for some $m' \in \mathbf{Q}_p$. We can assume that G is large enough that some quotient of it is G_p, the Galois group of the $\mathbf{Z}_p$-extension obtained by constant field extensions. On G_p, the formula (2.2) is nothing but the (modified) classical class number formula and so is true (see [G], Section 1). By projecting to G_p, we must have $m' = m$. $\square$

5. Consequences of the Degeneration

By Proposition 4.1, to complete the proof of Theorem 2.3 for each r, it is enough to show that $\theta_1 = \theta_{1,G}$ is divisible by t^{r+1} for each degenerate $G \simeq \mathbf{Z}_p$ contained in $\mathcal{H}$. In this section, we study some useful consequence of the degeneration. To illustrate the main idea for the proof of Theorem 2.3, at the end of this section we complete the proof for the special case $r = 1$.

Suppose that $G \simeq \mathbf{Z}_p$ is degenerate. Then there is a non-zero u in $U_T \otimes \mathbf{Z}_p$ such that $f_v \otimes 1(u) = 0$ for all $v \in S$. For each v, let $N_v \subset K_v^*$ be the kernel of

$$K_v^* \longrightarrow \mathbf{A}_K^*/K^* \longrightarrow G.$$

For each $v \in S$, there are three possibilities : $K_v^*/N_v \simeq \{1\}, \mathbf{Z}$ or $\mathbf{Z}_p$. Suppose that for some $v \in S$,

$$K_v^*/N_v \simeq \{1\}. \tag{5.1}$$

Then v splits completely under the corresponding $\mathbf{Z}_p$-extension. By Lemma 1.4, $\theta_1 = \theta_G = 0$.

Suppose that

$$K_v^*/N_v \simeq \mathbf{Z} \text{ } or \text{ } \mathbf{Z}_p, \forall v \in S. \tag{5.2}$$

Then G_v, the local Galois group, is isomorphic to $\mathbf{Z}_p$ for all $v \in S$. Note that if $K_v^*/N_v \simeq \mathbf{Z}$, then $\mathcal{O}_v^* = N_v$. Since for a non-trivial u, we can not have $u_v \in \mathcal{O}_v^*$ for all v, we must have $\mathcal{O}_v^* \neq N_v$ for some v. Let Γ be the pro-p completion of the group

$$(A_K^*/K^* \prod_{v \in S} N_v \prod_{v \notin S} \mathcal{O}_v^*)/\{torsions\}.$$

Consider the natural homomorphism

$$\rho : \bigoplus_{v \in S} G_v \longrightarrow \Gamma.$$

The cokernel of ρ is finite, since it is the Galois group over K of an unramified abelian extension which splits over S. The kernel of ρ contains, with finite index, the image of the natural map

$$U_T \otimes \mathbf{Z}_p \longrightarrow \bigoplus_{v \in S} G_v.$$

Since G is degenerate, this kernel has rank at most $r-1$. By computing the rank, we see that $\Gamma \simeq \mathbf{Z}_p^d$, for some $d \geq 2$. Note that Γ is in $\mathcal{H}$ and G is a quotient of Γ.

For each $v \in S$, let Γ_v be the image in Γ of G_v and γ_v a topological generator of Γ_v. Let L/K, L_G/K and $L/L^{(v)}$ be the field extensions with Galois groups equal to Γ, G and Γ_v respectively. We have the following picture:

$$\begin{tikzcd}
& L \arrow[dl, dash, "\Gamma_v"'] \arrow[dr, dash] \arrow[dd, dash, "\Gamma"'] & \\
L^{(v)} \arrow[dr, dash] & & L_G \arrow[dl, dash, "G"] \\
& K &
\end{tikzcd}$$

Since v splits under the abelian extension $L^{(v)}/K$, Lemma 1.4 again implies that θ_{Γ/Γ_v} is zero. This shows that as an element of $\mathbf{Z}_p[[\Gamma]], \theta_\Gamma$ is in the kernel of the homomorphism of algebras

$$\mathbf{Z}_p[[\Gamma]] \longrightarrow \mathbf{Z}_p[[\Gamma/\Gamma_v]],$$

i.e., θ_Γ is divisible by $\gamma_v - 1$. Thus we have proved the following.

Lemma 5.3. *If* (5.2) *holds, then θ_Γ is divisible by $\gamma_v - 1$ for each v.*

Proof (of Theorem 2.3 for the case $r=1$). Suppose that $G \in \mathcal{H}$ is degenerated. If (5.1) holds, then $\theta_G = 0$. Suppose that (5.2) holds. Since we are in the case that $r=1$, the morphism ρ is injective and γ_{v_0} and γ_{v_1} are linearly independent. As a consequence of Lemma 5.3, θ_Γ is divisible by $(\gamma_{v_0}-1)(\gamma_{v_1}-1)$. By the projection $\Gamma \longrightarrow G$, we see that $\theta_1 = \theta_G$ is divisible by t^2. By Lemma 4.1, the proof for $r=1$ is completed. $\square$

6. The General Case

In this section, we complete the proof of Theorem 2.3 for the general r. As explained in Section 5, for each degenerate $G \simeq \mathbf{Z}_p$, there are two possibilities, i.e., one of (5.1) and (5.2) will hold. The main difficulty is that when (5.2) holds, for distinct v and v' in S, the associated γ_v and $\gamma_{v'}$ may be proportional to each other over $\mathbf{Z}_p$. To overcome this, we need further consideration of linear algebra.

Recall that for each $v \in S$ and each $H' \in \mathcal{H}$, $f_v = f_{v,H'} : U_T \longrightarrow H'$ is the map defined in Section 2. For each u in $U_T \otimes \mathbf{Z}_p$ not divisible by p, let $p_u(H')$ be the maximal $\mathbf{Z}_p$-free quotient of

$$H'/ < f_v \otimes 1(\mathbf{Z}_p \cdot u), v \in S > .$$

Let $\mathcal{H}'_u$ be the set of all the $p_u(H'), H' \in \mathcal{H}'$.

Suppose that H is in $\mathcal{H}'_u$ for some u. Then H is a Galois group over K of certain abelian extension unramified outside S. Suppose that $H = p_u(H')$ and $H' \xrightarrow{\phi} H$ is the projection. Let $< \epsilon_1, \ldots, \epsilon_r >$ be a basis of $U_T \otimes \mathbf{Z}_p$. Without loss of generality, we may assume that $u = \epsilon_r$. Since $H' \in \mathcal{H}'$, by definition, the entries of the matrix $(f_{v_i,H'}(\epsilon_j))_{\substack{i=1,\ldots,r \\ j=1,\ldots,r}}$ are linearly independent over $\mathbf{Z}_p$ (see the proof of Lemma 3.3). Consequently, the matrix $(f_{v_i,H}(\epsilon_j)(= \phi f_{v_i,H'}(\epsilon_j)))_{\substack{i=0,1,\ldots,r \\ j=1,\ldots,r-1}}$ has the property that each $(r-1)$-minor of it has rank $r-1$ over $\mathbf{Z}_p$. For each subset R of S with cardinality equal to $r-1$, the symmetric tensor $f_{R,H} := det((f_{v,H}(\epsilon_j))_{\substack{v \in R \\ j=1,\ldots,r-1}}) \in Sym_{r-1}(H) \otimes \mathbf{Z}_p$ is non-zero. Let $\underline{t} = (t_1, \ldots, t_d)$, where $< \sigma_i := t_i + 1, i = 1, \ldots, d >$ is a basis of H over $\mathbf{Z}_p$. The non-canonical isomorphism $\delta_H(\underline{t})$ (defined in Section 3) then identifies $\bigoplus_{k \geq 0} Sym_k(H) \otimes \mathbf{Z}_p$ with $\mathbf{Z}_p[\underline{t}]$. Under this, $f_{R,H}$ is identified with a homogeneous polynomial, which will be denoted by $f_{R,H}(\underline{t})$.

Let $V = V_H$ be the d-dimensional $\mathbf{Z}_p$ affine space defined in Section 3 (taking $G = H$). Recall that each projection $\pi : H \longrightarrow \mathbf{Z}_p$ can be viewed as an embedding of $\mathbf{Z}_p$ into $V(\mathbf{Z}_p)$. Let

$$V_0 = (V - \cup_{all\ R}\{zeros\ of\ f_{R,H}(\underline{t})\}) \cup \{0\}.$$

LEMMA 6.1. *Suppose that* $H \in \mathcal{H}'_u$ *and the projection* $\pi : H \longrightarrow G \xrightarrow{\sim} \mathbf{Z}_p$ *belongs to* $V_0(\mathbf{Z}_p)$. *Then*

(1) $f_{v,G} \otimes 1(u) = 0$ *in* G, *for all* $v \in S$. *Hence* G *is degenerate.*
(2) *If for some subset* R *of* S *with cardinality* $r-1$, $u' \in U_T \otimes \mathbf{Z}_p$ *satisfies* $f_{v,G} \otimes 1(u') = 0, \forall v \in R$, *then* $u' \in \mathbf{Z}_p \cdot u$.
(3) *If* (5.2) *holds for* G, *then all the* γ_v *defined in Section 5 are not proportional to each other.*

PROOF. Since $f_{v,G}$ factors through $f_{v,H}$, (1) follows from the fact that $H \in \mathcal{H}'_u$. Since π is in $V_0(\mathbf{Z}_p)$, $det((f_{v,G}(\epsilon_i))_{\substack{v \in R \\ i=1,\dots,r-1}}) = \pi_*(f_{R,H}) \neq 0$ and (2) follows.

Suppose that (5.2) holds for G. Let $\alpha \in G_{v_0}$ and $\beta \in G_{v'_0}$ be such that $\rho(\alpha) = \gamma_{v_0}$ and $\rho(\beta) = \gamma_{v'_0}$. If γ_{v_0} and $\gamma_{v'_0}$ are linearly dependent over $\mathbf{Z}_p$, then there are non-zero $a, b \in \mathbf{Z}_p$ such that $a \cdot \alpha + b \cdot \beta$ is in the kernel of ρ. For each $v \in S$, denote by i_v the natural map from $U_T \otimes \mathbf{Z}_p$ to G_v. Since the kernel of ρ contains, with finite index, the image of $U_T \otimes \mathbf{Z}_p$ in $\bigoplus_{v \in S} G_v$, there is a $u' \in U_T \otimes \mathbf{Z}_p$ and a non-zero $c \in \mathbf{Z}_p$ such that

$$i_v(u') = 0, for\ v \in R =: S - \{v_0, v'_0\},$$

and

$$i_{v_0}(u') = ac \cdot \alpha, i_{v'_0}(u') = bc \cdot \beta.$$

Since f_v factors through i_v, by (2) we then have $u' \in \mathbf{Z}_p \cdot u$. But then we must have $ac \cdot \gamma_{v_0} = f_{v_0}(u') = 0$, a contradiction. □

PROOF (of Theorem 2.3). By Proposition 4.1, we need to show that for any S of cardinality r and any degenerate $G \simeq \mathbf{Z}_p$, $\theta_{1,G}$ is of order of vanishing at least $r+1$. Being degenerate, G is a quotient of certain H in $\mathcal{H}'_u$ for some u. Let $\pi : H \longrightarrow G$ be a projection. Since $\pi_*(\theta_{1,H}) = \theta_{1,G}$, it is sufficient to show that $\theta_{1,H} \in (t_1, \dots t_d)^{r+1}$.

Suppose that $H \longrightarrow G_0 \xrightarrow{\sim} \mathbf{Z}_p$ is a projection belonging to $V_0(\mathbf{Z}_p)$. Then by Lemma 6.1, G_0 is degenerate. As showed in Section 5, if (5.1) holds for G_0, then $\theta_{1,G_0} = 0$. If (5.2) holds for G_0, then by Lemma 6.1 again, all the γ_v are not proportional to each other. As a consequence of Lemma 5.3, $ord(\theta_{1,G_0}) > r$.

Let $\theta_{1,H} = \sum_i \theta_{1,H}^{(i)}$ be the decomposition of $\theta_{1,H}$ into homogeneous polynomials. Then we just showed that for $i \leq r$, $\theta_{1,H}^{(i)} = 0$ on each $\mathbf{Z}_p$-line inside $V_0(\mathbf{Z}_p)$ hence on $V_0(\mathbf{Z}_p)$. Since $V_0(\mathbf{Z}_p)$ is Zariski dense in V, we have $\theta_{1,H}^{(i)} = 0$ for $i \leq r$. This shows that $\theta_{1,H} \in (t_1, \ldots, t_d)^{r+1}$. $\square$

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