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On some generating functions for McKay numbers—prime power divisibilities of the hook products of Young diagrams

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Abstract. We discuss combinatorics related with p-adic valuations of hook products of Young diagrams and obtain some infinite product generating functions in two variables for McKay-Macdonald numbers of some classical finite groups.

1. Introduction

Let G be a finite group and p be a fixed prime number. The number of irreducible characters of G whose degrees are exactly k-times divisible by p is denoted by $m_p(k, G)$. These integers are called McKay numbers. If G runs over a series of groups $\{G_n\}$, the McKay numbers form a double sequence $\{m_p(k, G_n)\}$ indexed by pairs of natural numbers (k, n). We give generating functions for these numbers in the style of infinite product (or sum of two infinite products) in some special cases when $\{G_n\}$ = the symmetric groups $\{S_n\}$, alternating groups $\{A_n\}$, classical Weyl groups $\{W(B_n)\}, \{W(D_n)\}$, and finite general linear groups $\{GL(n, q)\}$.

We can also consider similar double sequence for the Macdonald number $\mu_p(k, G)$ which is the number of conjugacy classes of G whose sizes are exactly k-times divisible by p. We give generating functions for $\mu_p(k, G_n)$ in the style of infinite product when $\{G_n\} = \{S_n\}, \{GL(n, q)\}$ $(p \nmid q)$.

In [O1], J.B.Olsson gave a recursive formula for McKay numbers of the symmetric groups, in the context of the Alperin-McKay conjecture in the

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modular representation theory of finite groups (see also [F]). In this report, it is shown that this Olsson's formula can be put into a relatively simple generating function, which reflects the distribution of the p-adic valutations of the hook products of the Young diagrams (Theorem 3.5).

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Note. This paper is a revised and abridged version of my Master's thesis (Part 3) written in 1987 and submitted to the university of Tokyo in January 1989. I thank the referee who suggested a mistake to be corrected and several possible improvements which were very helpful in the latest revision process. Here, it would be appropriate to add a few remarks about recent related works. Firstly, generating functions for the number of p-defect 0 characters of some series of finite groups were given by J.B.Olsson "On the p-Blocks of Symmetric and Alternating Groups and Their Covering Groups" J. of Algebra 128, 188–213 (1990). More recently, a refinement of Theorem 6.8 (Theorem 8.9 of old version) was given in a more sophisticated context by J.B.Olsson and K.Uno "Dade's conjecture for general linear groups in the defining characteristic" preprint.

2. Some partitions

We prepare some notations in this paragraph. A partition $\lambda = (\lambda_1, \ldots, \lambda_d)$ is a finite sequence of nonnegative integers in non-increasing order. Each λ_i is called a part of λ. The sum of the parts is called the size of λ, denoted by $|\lambda|$. We define the multiplicity $m_i(\lambda)$ to be the number of parts of λ which equal i. Then we formally write as $\lambda = \oplus_{i=1}^{\infty} m_i(\lambda) \cdot [i]$.

Let us associate a Young diagram with λ by the ordinary method and identify it with λ. In particular, we often neglect the 0 parts of a partition. If two partitions $\lambda = (\lambda_1, \ldots, \lambda_d)$, $\mu = (\mu_1, \ldots, \mu_d)$ satisfy $\lambda_i \geq \mu_i$ for all i, then we write $\lambda \supset \mu$.

Example.

$$(3,2,2,0,0) = (3,2,2) = 2 \cdot [2] \oplus [3] = \begin{vmatrix} \square & \square & \square \\ \square & \square & \\ \square & \square & \end{vmatrix}.$$

Now let e be a fixed positive integer. It is well-known that any partition λ has a unique partition $\lambda^{(e)}$ called the e-core of λ, and has a unique e-tuple of partitions $(\lambda_0^{(e)}, \dots, \lambda_{e-1}^{(e)})$ called the e-quotients of λ (See [JK] or [O2]). The sum of the sizes of e-quotients of λ is called the e-weight of λ and denoted by $w_e(\lambda)$. We denote by $\mathrm{Hk}(\lambda)$ the multiset of the hook lengths of a Young diagram λ (see [Ma] for hook length, [St], [O2] for multiset). We also define $\mathrm{Hk}(\lambda)_e$ to be the submultiset of $\mathrm{Hk}(\lambda)$ consisting of the members divisible by e. On the other hand, $e \cdot \mathrm{Hk}(\lambda)$ denotes the multiset of the e-multiples of the members of $\mathrm{Hk}(\lambda)$.

PROPOSITION 2.1. *Let λ be a partition. Then,*

(1) *If $n \in \mathrm{Hk}(\lambda^{(e)})$, then $e \nmid n$.*
(2) $\mathrm{Hk}(\lambda)_e = \bigcup_{i=0}^{e-1} e \cdot \mathrm{Hk}(\lambda_i^{(e)})$.
(3) $|\lambda| = |\lambda^{(e)}| + w_e(\lambda) \cdot e$.
(4) *λ is uniquely determined by $\lambda^{(e)}$ and $(\lambda_0^{(e)}, \dots, \lambda_{e-1}^{(e)})$.*

PROOF. See [JK] or [O2]. □

DEFINITION 2.2. Let e, r be positive integers and n a nonnegative integer. The core number $C_e(r,n)$ is the number of re-cores of size rn whose r-cores are empty.

PROPOSITION 2.3. $\sum_{n=0}^{\infty} C_e(r,n)x^n = \prod_{n=1}^{\infty}(1-x^{en})^{er}(1-x^n)^{-r}$.

PROOF. After replacing the variable x by x^r, we may prove

$$\prod_{n=1}^{\infty}(1-x^{rn})^{-r} = (\sum_{n=0}^{\infty} C_e(r,n)x^{rn}) \prod_{n=1}^{\infty}(1-x^{ern})^{-er}.$$

But this follows from the observation through Proposition 2.1 that the both sides represent a generating function for the partitions with r-cores empty. □

Let ${}^t\lambda$ denote the conjugate partition of a partition λ (i.e., the partition whose Young diagram is the transpose of the diagram λ.) If ${}^t\lambda = \lambda$, we say λ to be self-conjugate.

LEMMA 2.4. *Let e be a positive integer. Then, a partition λ is self-conjugate if and only if $\lambda^{(e)}$ is self-conjugate and $\lambda_i^{(e)} = {}^t\lambda_{e-i-1}^{(e)}$ $(i = 0, \dots, e-1)$.*

PROOF. We use the "pictorial" description of the e-quotients in [JK, p.84–85]. It says that the i-th e-quotient is the partition which is formed by the corner nodes of all e-hooks of λ whose hand node's content number $\equiv i$ modulo e. Here an e-hook means a hook of length e, and the content number of the (i, j) node of the Young diagram λ is $j - i$. The above pictorial description justifies the formula

$$ {}^t(\lambda_i^{(e)}) = ({}^t\lambda)_{e-i-1}^{(e)} \quad (i = 0, \dots, e-1) $$

for any partition λ. The lemma follows from this easily. □

DEFINITION 2.5. Let e, r be positive integers, and n be a nonnegative integer. We define the self-conjugate core number $SC_e(r, n)$ to be the number of self-conjugate re-cores of size rn whose r-cores are empty.

By virtue of Lemma 2.4, the following proposition follows in a similar way to Proposition 2.3.

PROPOSITION 2.6.

$$ \sum_{n=0}^{\infty} SC_e(r,n)x^n = \begin{cases} \prod_{n=1}^{\infty} \frac{(1+x^{2n-1})(1-x^{2en})^{(re-1)/2}}{(1+x^{e(2n-1)})(1-x^{2n})^{(r-1)/2}}, & \text{if } r, e : \text{odd}, \\ \prod_{n=1}^{\infty} \frac{(1-x^{2en})^{re/2}(1+x^{2n-1})}{(1-x^{2n})^{(r-1)/2}}, & \text{if } r : \text{odd, } e : \text{even}, \\ \prod_{n=1}^{\infty} \frac{(1-x^{2en})^{re/2}}{(1-x^{2n})^{r/2}}, & \text{if } r : \text{even.} \ \square \end{cases} $$

Next, we study some classes of partitions. Let us fix a positive integer $q(> 1)$. For nonnegative integers k, we put $[k] = [k]_q = (q^k - 1)/(q - 1)$, and call these numbers q-projective numbers. We also call the integers of the form q^k $(k = 0, 1, ...)$ q-affine numbers. A q-projective (resp. q-affine) partition is a partition such that all (nontrivial) parts are q-projective (resp.

q-affine) numbers. A q-adic partition is a partition $\lambda = (\lambda_1, \dots)$ such that $\lambda_i \geq q\lambda_{i+1}$ $(i = 1, 2, \dots)$. We define an operator $^-$ which sends the q-affine number q^k to the q-projective number $[k]_q$, and extend it naturally to send q-affine partitions to q-projective partitions.

Example. Let $q = 3$. $\lambda = (27, 9, 9, 3, 1, 1) = 2[3^0] \oplus [3^1] \oplus 2[3^2] \oplus [3^3]$ is a 3-affine partition, and $\mu = (13, 4, 4, 1, 0, 0) = 2[0]_3 \oplus [1]_3 \oplus 2[2]_3 \oplus [3]_3$ is a 3-projective partition. Then $\bar{\lambda} = \mu$. □

PROPOSITION 2.7. *The number of q-projective partitions of size n is equal to the number of q-adic partitions of size n.*

PROOF. We construct a bijection between the two sets by using tableax. Let $\sigma = \oplus_{i \geq 0} b_i [i]_q$ be a q-projective partition. We draw the Young diagram of $\oplus_{i \geq 1} b_i [i]$ and write $1, q, q^2, \dots$ successively in the boxes of each rows from the right to the left. Let μ_i be the sum of the written numbers in the i-th column. Then $\mu = (\mu_1, \mu_2, \dots)$ is a q-adic partition clearly. We call this μ the *q-transposition* of σ and denote it by ${}^\dagger\sigma$. Apparently † gives a desired bijection. □

Example. Let $q = 3$ and $\sigma = (13, 4, 4, 1, 0, 0) = 2[0]_q \oplus [1]_q \oplus 2[2]_q \oplus [3]_q$ is a 3-projective partition. The corresponding tableau is the following:

$$\left|\begin{array}{lll} \boxed{9} & \boxed{3} & \boxed{1} \\ \boxed{3} & \boxed{1} & \\ \boxed{3} & \boxed{1} & \\ \boxed{1} & & \end{array}\right| .$$

Hence ${}^\dagger\sigma = (16, 5, 1)$. □

Let $\sigma = a_0[q^0] \oplus a_1[q^1] \oplus \cdots \oplus a_n[q^n]$ be a q-affine partition. By the reduction of σ at i $(1 \leq i \leq n)$, we mean the q-affine partition

$$\sigma_{(i)} = a_0[q^0] \oplus \cdots \oplus (a_{i-1} + q)[q^{i-1}] \oplus (a_i - 1)[q^i] \oplus \cdots \oplus a_n[q^n].$$

Note that we always have $|\sigma| = |\sigma_{(i)}|$. If a q-affine partition τ is obtained from σ by successive applications of reductions at various positions, then τ is just said to be a reduction of σ and denoted $\tau \prec_q \sigma$. Obviously $\prec_q$ gives an order structure on the set of the q-affine partitions of the same size.

Proposition 2.8. *Let σ, τ be q-affine partitions with $|\sigma| = |\tau|$, and suppose $\tau \prec_q \sigma$. Then $|\bar{\tau}| \le |\bar{\sigma}|$ and ${}^{\dagger}\bar{\tau} \subset {}^{\dagger}\bar{\sigma}$.*

Proof. The proof is reduced to the case $\tau = \sigma_{(i)}$. In this case, our claim can be verified directly. □

Definition 2.9. Let p be a prime number. We denote by $v_p(n)$ the exponential p-adic valuation of an integer n, i.e., $v_p(n) = k$ if and only if $p^k \mid n$, $p^{k+1} \nmid n$.

Corollary 2.10. *Let p be a prime number and n be a positive integer. We also assume that $n = \sum_{i=0}^{k} a_i p^i$ $(0 \le a_i < p)$.*

(1) *Each p-affine partition α of size n satisfies $\alpha \prec_p a_0[p^0] \oplus \cdots \oplus a_k[p^k]$; hence by Proposition 2.8, $|\bar{\alpha}| \le v_p(n!)$.*
(2) *The above equality holds only when $\alpha = a_0[p^0] \oplus \cdots \oplus a_k[p^k]$.*

3. Affine type and projective type

Let λ be a partition and p be a fixed prime number. We define a p-affine partition $\mathbb{A}_p(\lambda)$ and a p-projective partition $\mathbb{P}_p(\lambda)$ as follows.

$$\mathbb{A}_p(\lambda) = \oplus_{i \ge 0} a_i[p^i],$$
$$\mathbb{P}_p(\lambda) = \overline{\mathbb{A}_p(\lambda)},$$

where a_i is the p^i-weight of the p^{i+1}-core of λ. The former is called the p-affine type of λ and the latter is called the p-projective type of λ.

Definition 3.1. For a partition λ, we denote by h_λ the product of all the hook lengths of λ.

Proposition 3.2.

(1) $v_p(h_\lambda) = |\mathbb{P}_p(\lambda)|$.
(2) $|\lambda| = |\mathbb{A}_p(\lambda)|$.

Proof. As (2) is clear from the definition, we prove (1). If $\mathbb{A}_p(\lambda) = \oplus_{i \ge 0} a_i[p^i]$, then by the definition $a_i = (|\lambda^{(p^{i+1})}| - |\lambda^{(p^i)}|)/p^i$. Hence,

$$|\mathbb{P}_p(\lambda)| = \sum_i a_i[i]_p = \sum_{i \ge 0} (|\lambda| - |\lambda^{(p^{i+1})}|)/p^{i+1} = \sum_{i \ge 0} w_{p^{i+1}}(\lambda) = v_p(h_\lambda). \ \square$$

Definition 3.3. For a p-affine partition $\sigma = \oplus_{i\geq 0} b_i[p^i]$, we define $C(p,\sigma) := \prod_{i\geq 0} C_p(p^i, b_i)$ and $SC(p,\sigma) := \prod_{i\geq 0} SC_p(p^i, b_i)$.

Proposition 3.4. *For a p-affine partition σ, we have* $C(p,\sigma) = \#\{\lambda \mid \mathbb{A}_p(\lambda) = \sigma\}$.

Proof. We assume $\sigma = \oplus_{i=0}^{k} b_i[p^i]$ with $b_k \neq 0$, and prove the proposition by induction on k. For $k = 0$, the statement is clear from Definition 3.3. Put $S = \{\lambda \mid \mathbb{A}_p(\lambda) = \sigma\}$. Then Proposition 2.1 and the induction hypothesis imply

$$\#\{\lambda^{(p^k)} \mid \lambda \in S\} = C_p(1, b_0) \cdots C_p(p^{k-1}, b_{k-1}).$$

On the other hand, if $\lambda \in S$ then $\#\mathrm{Hk}(\lambda)_{p^k} = b_k$, $\mathrm{Hk}(\lambda)_{p^{k+1}} = \emptyset$. Hence by using Proposition 2.1 (2), we obtain

$$\begin{aligned}\#\{(\lambda_0^{(p^k)}, ..., \lambda_{p^k-1}^{(p^k)}) \mid \lambda \in S\} &= \#\{(\lambda_0, ..., \lambda_{p^k-1}) \mid \lambda_i : p\text{-core}, \sum |\lambda_i| = b_k\} \\ &= C_p(p^k, b_k). \ \square\end{aligned}$$

Now we are ready to prove the following

Theorem 3.5.

$$\sum_{\lambda} x^{v_p(h_\lambda)} y^{|\lambda|} = \prod_{k=0}^{\infty} \prod_{n=1}^{\infty} \frac{(1 - x^{[k]_p pn} y^{p^{k+1}n})^{p^{k+1}}}{(1 - x^{[k]_p n} y^{p^k n})^{p^k}}.$$

Here λ runs over all partitions including empty, and $[k]_p = (p^k - 1)/(p - 1)$ for $k \geq 0$.

Proof. By Proposition 3.4, we get

$$\sum_{\lambda} x^{v_p(h_\lambda)} y^{|\lambda|} = \sum_{\substack{\sigma \\ p\text{-affine}}} C(p,\sigma) x^{|\bar{\sigma}|} y^{|\sigma|} = \prod_{k=0}^{\infty} (\sum_{n=0}^{\infty} C_p(p^k, n) x^{[k]_p n} y^{p^k n}).$$

From this and Proposition 2.3, we conclude the theorem. $\square$

PROPOSITION 3.6. *For a p-affine partition σ, it holds that*

$$SC(p,\sigma) = \#\{\lambda \mid {}^t\lambda = \lambda,\ \mathbb{A}_p(\lambda) = \sigma\}. \ \square$$

Applying Proposition 2.6 and 3.6, we obtain the following

THEOREM 3.7.

$$\sum_{\substack{\lambda \\ \text{self-conjugate}}} x^{v_p(h_\lambda)} y^{|\lambda|} = \prod_{k=0}^{\infty} f_{p,k}(x^{[k]_p} y^{p^k}).$$

where $f_{p,k}(x)$ is the power series in variable x of the right hand side of Proposition 2.6 with $r = p^k$, $e = p$. $\square$

4. McKay numbers for classical Weyl groups, alternating groups

Let G be a finite group and p a prime number. The McKay number $m_p(k,G)$ is defined to be the number of complex irreducible characters χ of G with $v_p(\chi(1)) = k$. In this paragraph, we give generating functions for the double sequences $\{m_p(k,S_n)\}$, $\{m_p(k,A_n)\}$, $\{m_p(k,W(B_n))\}$ and $\{m_p(k,W(D_n))\}$ in the style involving infinite products, where S_n is the symmetric group of degree n ($S_0 = S_1 = \{1\}$), A_n is the alternating group of degree n, and $W(B_n)$ (resp. $W(D_n)$) is the classical Weyl group of type B_n (resp. D_n) with $W(B_0) = W(C_0) = W(C_1) = \{1\}$, $W(B_1) = \{\pm 1\}$. The irreducible characters of S_n (resp. $W(B_n)$) are well-known to be parametrized by the Young diagrams of size n (resp. the ordered pairs of Young diagrams of total size n), and they have simple degree formulae of the form $n!$ divided by the products of all the hook-lengths in the diagram(s). Their restriction laws from S_n to A_n (resp. $W(B_n)$ to $W(D_n)$) are described in simple manners in terms of Young diagrams, and all the irreducible characters of A_n (resp. $W(D_n)$) are obtained through such restrictions (cf. [JK], [May2-3]). Therefore, using the results of previous sections, we can estimate the p-adic valuations of the degrees of these irreducible characters and obtain the following

LEMMA 4.1.

(1) *(Olsson)* $m_p(k,S_n)=\sum_\tau C(p,\tau)$,

(2) $m_p(k,A_n)=\{\sum_\tau C(p,\tau)+3\sum_\tau SC(p,\tau)\}/2 \quad (p>2)$,

$\qquad = \{\sum_\tau C(2,\tau)-\sum_\tau SC(2,\tau)+4\sum_\sigma SC(2,\sigma)\}/2 \quad (p=2)$,

(3) $m_p(k,W(B_n))=\sum_{(\tau_1,\tau_2)} C(p,\tau_1)C(p,\tau_2)$,

(4) $m_p(k,W(D_n))=\{\sum_{(\tau_1,\tau_2)} C(p,\tau_1)C(p,\tau_2)+3\sum_\rho C(p,\rho)\}/2 \quad (p>2)$,

$\qquad = \{\sum_{(\tau_1,\tau_2)} C(2,\tau_1)C(2,\tau_2)-\sum_\rho C(2,\rho)+4\sum_\kappa C(2,\kappa)\}/2 \quad (p=2)$.

Here τ, σ, ρ, κ run over p-affine partitions with $|\tau|=n$, $|\bar\tau|=v_p(n!)-k$, $|\sigma|=n$, $|\bar\sigma|=v_p(n!)-k-1$, $2|\rho|=n$, $2|\bar\rho|=v_p(n!)-k$, $2|\kappa|=n$, $2|\bar\kappa|=v_p(n!)-k-1$, and (τ_1,τ_2) runs over pairs of p-affine partitions with $|\tau_1|+|\tau_2|=n$, $|\bar\tau_1|+|\bar\tau_2|=v_p(n!)-k$. □

If two power series $f(x,y)$ and $g(x,y)$ in two variables x, y have the same coefficients of x^my^n for $m\ge s$, $n\ge t$, we write

$$f(x,y)\sim g(x,y) \qquad \text{coeff}(x^sy^t).$$

By combining Theorems 3.5, 3.7 and Lemma 4.1, we obtain

THEOREM 4.2. *For a prime p, let $F_p(x,y)$ (resp. $G_p(x,y)$) denote the right hand side of Theorem 3.5 (resp. Theorem 3.7). Then,*

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} m_p(k,S_n)x^{v_p(n!)-k}y^n = F_p(x,y). \tag{1}$$

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} m_p(k,A_n)x^{v_p(n!)-k}y^n &\sim \frac{1}{2}\{F_p(x,y)^2+3G_p(x,y)\} \quad \textit{coeff}(y^2)\quad (p>2),\\ &\sim \frac{1}{2}\{F_2(x,y)^2+(4x-1)G_2(x,y)\} \quad \textit{coeff}(y^2)\quad (p=2).\end{aligned} \tag{2}$$

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} m_p(k,W(B_n))x^{v_p(n!)-k}y^n = F_p(x,y)^2. \tag{3}$$

$$(4)\qquad \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} m_p(k, W(D_n))x^{v_p(n!)-k}y^n$$
$$\sim \frac{1}{2}\{F_p(x,y)^2 + 3F_p(x^2,y^2)\} \quad \textit{coeff}(y) \quad (p>2),$$
$$\sim \frac{1}{2}\{F_2(x,y)^2 + (4x-1)F_2(x^2,y^2)\} \quad \textit{coeff}(y) \quad (p=2). \ \square$$

5. Formulae of $m_p(0, G_n)$

In [Ma1-2], Macdonald calculated $m_p(G_n) := m_p(0, G_n)$ for finite Coxeter groups. Here we deduce formulae for $G_n = S_n, W(B_n), W(D_n)$ and A_n. Let $k(r,s)$ be the numbers defined by $\sum_{s=0}^{\infty} k(r,s)T^s = \prod_{n=1}^{\infty}(1-T^n)^{-r}$, and $n = \sum_i a_i p^i$ $(0 \le a_i \le p-1)$ be the p-adic expansion of n. Then, from Lemma 4.1, we see $m_p(S_n) = \prod_i k(p^i, a_i)$ and $m_p(W(B_n)) = \prod_i k(2p^i, a_i)$ ([Ma1-2]). Observing Theorem 4.2, we further obtain the following for $W(D_n)$ $(n \ge 1)$. Suppose first that $p > 2$. If a_i =even for all i, then $m_p(W(D_n)) = \{m_p(W(B_n)) + 3m_p(S_{n/2})\}/2$, and if a_i =odd for some i, then $m_p(W(D_n)) = m_p(W(B_n))/2$. Next we suppose $p = 2$. If n is a positive power of 2, then $m_2(W(D_n)) = \{m_2(W(B_n)) + 4m_2(S_{n/2})\}/2$, otherwise, $m_2(W(D_n)) = m_2(W(B_n))/2$.

For the alternating groups, we introduce the numbers $\hat{k}(r,s)$ as follows. If r is even, $\sum_{s=0}^{\infty} \hat{k}(r,s)T^s = \prod_{n=1}^{\infty}(1-T^{2n})^{-r/2}$, and if r is odd, $\sum_{s=0}^{\infty} \hat{k}(r,s)T^s = \prod_{n=1}^{\infty}(1+T^{2n-1})(1-T^{2n})^{-(r-1)/2}$. Then for p odd, we can deduce from Lemma 4.1 that $m_p(A_n) = \{\prod_i k(p^i, a_i) + 3\prod_i \hat{k}(p^i, a_i)\}/2$. For $p = 2$, M.Sato's result is recorded in Note added in proof of [Mc]. It says that $m_2(A_1) = m_2(A_2) = 1$, $m_2(A_3) = 3$, and if n is of the form 2^t or $2^t + 1$ for some $t \ge 2$, then $m_2(A_n) = m_2(S_n)$, and otherwise $m_2(A_n) = m_2(S_n)/2$. We can obtain a proof for this result from Lemma 4.1 together with the following combinatorial lemma.

Lemma. *A 2-affine partition* $\sigma = \oplus_i b_i[2^i]$ *satisfies* $|\sigma| = n$, $|\bar{\sigma}| = v_2(n!) - 1$ *if and only if (1)* $0 \le b_i \le 3$ *for all* i, *(2)* $\#\{b_i \mid b_i = 2, 3\} = 1$ *and (3)* $b_i = 2, 3 \Rightarrow b_{i+1} = 0$.

PROOF. We use 2-transposition of Proposition 2.7. Let τ be the (unique multiplicity-free) 2-affine partition with $|\tau| = n$ and $|\bar{\tau}| = v_2(n!)$. Since $\sigma \prec_2 \tau$, we have ${}^\dagger\bar{\sigma} \subset {}^\dagger\bar{\tau}$. From this, it follows that the shape of ${}^\dagger\bar{\sigma}$ is obtained by removing one rim node of ${}^\dagger\bar{\tau}$. In other words, the tableax for $\bar{\sigma}$ is obtained by removing one box written 1 and splitting the row into two rows whose boxes have numbers half as much as before (see example below). Hence the multiplicities b_i of σ must satisfy (1)-(3). Conversely, if σ satisfies (1)-(3), then we can find a multiplicity free 2-affine partition τ with ${}^\dagger\bar{\sigma} \subset {}^\dagger\bar{\tau}$ and $|\bar{\sigma}| + 1 = |\bar{\tau}|$. Then our assertion follows. □

Example. Let $n = 23$. Then $\tau = [1] \oplus [2] \oplus [4] \oplus [16]$, $\bar{\tau} = [0] \oplus [1] \oplus [3] \oplus [15]$ and the corresponding tableau is

$$\left|\begin{array}{cccc} \boxed{8} & \boxed{4} & \boxed{2} & \boxed{1} \\ \boxed{2} & \boxed{1} & & \\ \boxed{1} & & & \end{array}\right|.$$

If we remove a box written 1 and split the row, we get one of the following tableax.

$$\left|\begin{array}{ccc} \boxed{4} & \boxed{2} & \boxed{1} \\ \boxed{4} & \boxed{2} & \boxed{1} \\ \boxed{2} & \boxed{1} & \\ \boxed{1} & & \end{array}\right|, \quad \left|\begin{array}{cccc} \boxed{8} & \boxed{4} & \boxed{2} & \boxed{1} \\ \boxed{1} & & & \\ \boxed{1} & & & \\ \boxed{1} & & & \end{array}\right|, \quad \left|\begin{array}{cccc} \boxed{8} & \boxed{4} & \boxed{2} & \boxed{1} \\ \boxed{2} & \boxed{1} & & \\ & & & \\ & & & \end{array}\right|,$$

which give 2-affine partitions $\sigma = [1] \oplus [2] \oplus [4] \oplus 2[8]$, $[1] \oplus 3[2] \oplus [16]$, $3[1] \oplus [4] \oplus [16]$ satisfying $|\sigma| = |\tau|$, $|\bar{\sigma}| + 1 = |\bar{\tau}|$ respectively. □

6. McKay numbers for finite general linear groups

Letting q be a fixed positive integer, we shall begin this paragraph by recalling prime factorization properties of the numbers $q^k - 1$ after e.g. [FS]. For a prime number l not dividing q, define $e_l(q)$ to be the multiplicative order of q mod l, and put $a_l(q) := v_l(q^{e_l(q)} - 1)$. Suppose that a positive integer k has the prime factorization $k = \prod_{l:prime} l^{b_l}$. Then, we have

$$q^k - 1 = \prod_{\substack{l:prime \\ e_l(q)|k}} l^{b_l + a_l(q)}.$$

The statement is equivalent to the following formula:

$$v_l(q^{e_l(q)m}-1)=v_l(m)+a_l(q).$$

The proof follows from a simple induction argument on $v_l(k)=v_l(k/e_l(q))$ after assuming $e_l(q)\mid k$ without loss of generality. For a partition λ, define $h_\lambda(q):=\prod_{h\in\mathrm{Hk}(\lambda)}(q^h-1)$.

THEOREM 6.1. *Let l be a prime number not dividing a positive integer q. Then*

$$\sum_\lambda x^{v_l(h_\lambda(q))}y^{|\lambda|}=\prod_{n=1}^\infty f_e(y^n)\prod_{k=0}^\infty f_l(x^{([k]_l+l^ka)n}y^{el^kn})^{el^k},$$

where $f_n(T)=(1-T^n)^n/(1-T)$, $e=e_l(q)$, $a=a_l(q)$.

PROOF. By the above properties of $v_l(q^k-1)$, we see that

$$v_l(h_\lambda(q))=\sum_{i=0}^{e-1}\sum_{h\in\mathrm{Hk}(\lambda_i^{(e)})}v_l(q^{eh}-1).$$

Since $|\lambda|=|\lambda^{(e)}|+e\sum_i|\lambda_i^{(e)}|$, the left hand side of the theorem can be written as

$$(\sum_{n=0}^\infty C_e(1,n)y^n)(\sum_\lambda x^{a|\lambda|+v_l(h_\lambda)}y^{e|\lambda|})^e.$$

Then we conclude the proof by Proposition 2.3 and Theorem 3.5. □

In the following, we let q be a power of a prime number p, and $\mathbf{F}_q$ denote the finite field with q elements. Write Φ for the set of monic irreducible polynomials $f(T)$ with $f(T)\neq T$, and let $\Phi_d\subset\Phi$ denote the subset of degree d. Then the cardinality $N(q,d)$ of the set Φ_d is equal to $d^{-1}\sum_{s|d}\mu(d/s)q^d$ for $d>1$ and $q-1$ for $d=1$. Let us introduce a new notation $(n)_q!$ to denote $(q^n-1)\cdots(q-1)$ for $n>0$ and 1 for $n=0$.

The irreducible characters of the general linear group $GL(n,q)$ were studied by J.A.Green [G], and they were parametrized by the partition-valued functions $\overrightarrow{\lambda}$ on Φ with $\sum_{f\in\Phi}\deg(f)|\overrightarrow{\lambda}(f)|=n$ (see [Ma] Chap.IV).

The degree of the character $\chi_{\overrightarrow{\lambda}}$ corresponding to $\overrightarrow{\lambda}$ is given by the formula:

$$\chi_{\overrightarrow{\lambda}}(1) = (n)_q! \prod_{f\in\Phi} \frac{q^{\deg(f)n({}^t\overrightarrow{\lambda}(f))}}{h_{\overrightarrow{\lambda}(f)}(q^{\deg(f)})}. \tag{6.2}$$

where $n(\lambda) = \sum_i (i-1)\lambda_i$ for a partition $\lambda = (\lambda_1, \dots)$. By combining this formula with Theorem 6.1, we can compute the following generating function.

THEOREM 6.3. *For a prime $l \nmid q$, we have*

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} m_l(k, GL(n,q)) x^{v_l((n)_q!)-k} y^n = \prod_{d=1}^{\infty} F_l(q^d; x, y^d)^{N(q,d)},$$

where $F_l(q;x,y)$ is the power series of the right hand side of Theorem 6.1. □

Next, we shall study the McKay numbers $m_p(k, GL(n,q))$.

THEOREM 6.4. *Let p be a prime number and $q = p^s$ for $s > 0$. Then,*

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} m_p(k, GL(n,q)) x^k y^n = \prod_{n=1}^{\infty} \frac{1 - x^{sn(n-1)/2} y^n}{1 - q x^{sn(n-1)/2} y^n}.$$

PROOF. Let $E(x,y)$ denote the left hand side. By the degree formula (6.2), we have

$$E(x,y) = \prod_{f\in\Phi} (\sum_{\lambda} x^{\deg(f) sn(\lambda)} y^{\deg(f)|\lambda|}).$$

If ${}^t\lambda = \bigoplus_i m_i[i]$, then $n(\lambda) = \sum_i m_i i(i-1)/2$. Therefore

$$\sum_{\lambda} x^{sn(\lambda)} y^{|\lambda|} = \prod_{k=1}^{\infty}\sum_{m=0}^{\infty} x^{msk(k-1)/2} y^{km} = \prod_{k=1}^{\infty} (1 - x^{sk(k-1)/2} y^k)^{-1}.$$

The proof is then reduced to the following elementary equation: $\prod_{d=1}^{\infty}(1 - T^d)^{-N(q,d)} = (1-T)/(1-qT)$. □

We obtain the following two corollaries by specializing $x = 0, 1$ respectively in Theorem 6.4.

COROLLARY 6.5 (Alperin [A]). $m_p(0, GL(n,q)) = q^n - q^{n-1}$. □

COROLLARY 6.6 (Feit and Fein [FF], [Ma3]). *If $c(n,q)$ denotes the number of conjugacy classes of $GL(n,q)$, then*

$$\sum_{n=0}^{\infty} c(n,q)y^n = \prod_{n=1}^{\infty} \frac{1-y^n}{1-qy^n}. \ \square$$

We remark that $q^n - q^{n-1}$ of 6.5 gives also the number of p-regular classes of $GL(n,q)$. From 6.6 follows that $c(n,q)$ is a polynomial in q. If we put $c(n,q) = \sum_k r(n,k)q^k$, then we can deduce the recurrence formula $r(n,k) = r(n-1,k-1) + r(n-k,k)$ by replacing q by qy in the power series in 6.6. The numbers $r(n,0)$ are well-known Euler's pentagonal numbers.

DEFINITION 6.7. For a partition $\lambda = (\lambda_1, \dots, 0)$, we define

$$\ell(\lambda) := \#\{i \mid \lambda_i \neq 0\},$$
$$\delta(\lambda) := \#\{i \mid \lambda_i > \lambda_{i+1}\}.$$

THEOREM 6.8. *If $q = p^s$ $(s \geq 1)$, then we have*

$$\sum_{k=0}^{\infty} m_p(k, GL(n,q))x^k = \sum_{|\lambda|=n} (q-1)^{\delta(\lambda)} q^{\ell(\lambda)-\delta(\lambda)} x^{sn({}^t\lambda)}.$$

PROOF. Let $\phi_n(x)$ denote the left hand side of the above, and consider the power series $\sum_{n=0}^{\infty} \phi_n(x)y^n$. Then by Theorem 6.4 it equals to

$$\begin{aligned}&\prod_{n=1}^{\infty}(1+\sum_{k=1}^{\infty}(q^k - q^{k-1})x^{skn(n-1)/2}y^{kn}) \\ &\quad = \sum_{\lambda}\{\prod_{\substack{i>0 \\ m_i(\lambda)>0}} (q^{m_i(\lambda)} - q^{m_i(\lambda)-1})\}x^{sn({}^t\lambda)}y^{|\lambda|}.\end{aligned}$$

From this Theorem 6.8 follows. □

REMARK. Let $G = GL(n,q)$. Then $c(n,q)$ is the number of G-conjugacy classes of pairs (s,u) such that s is a semisimple element of G, u is a unipotent element of G and $su = us$. Furthermore, $(q-1)^{\delta(\lambda)}q^{\ell(\lambda)-\delta(\lambda)}$ is the number of G-conjugacy classes of such (s,u) with u a unipotent element of type λ. A recent work of J.B.Olsson and K.Uno (see *Note* in Paragraph 1) gives an explanation of the meaning of $(q-1)^{\delta(\lambda)}q^{\ell(\lambda)-\delta(\lambda)}$ in terms of irreducible characters of $GL(n,q)$.

7. Macdonald numbers

Let G be a finite group, p a prime number and k a nonnegative integer. The Macdonald number $\mu_p(k,G)$ is defined to be the number of conjugacy classes C of G with $v_p(\#C) = k$.

We first consider the Macdonald number $\mu_p(k,S_n)$. Let $C(\lambda)$ denote the conjugacy class of S_n whose cycle type is a partition λ. The order z_λ of the centralizer of any element of $C(\lambda)$ is well-known to be equal to $\prod_i i^{m_i(\lambda)}m_i(\lambda)!$. Notice that $|\lambda| = n$ and $\#C(\lambda)\cdot z_\lambda = n!$. If k has the p-adic expansion $k = \sum_i a_i p^i$ $(0 \le a_i < p)$, then $v_p(k!) = \sum_i a_i [i]_p$. From this we have

$$\sum_{k=0}^{\infty} x^{v_p(k!)} y^k = \prod_{k=0}^{\infty} \frac{1 - x^{p[k]_p} y^{p^{k+1}}}{1 - x^{[k]_p} y^{p^k}}. \tag{7.1}$$

In this stage, we can deduce the following theorem easily.

THEOREM 7.2.

$$\begin{aligned}\sum_{\lambda} x^{v_p(z_\lambda)} y^{|\lambda|} &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \mu_p(k,S_n) x^{v_p(n!)-k} y^n \\ &= \prod_{k=0}^{\infty}\prod_{n=1}^{\infty} \frac{1 - x^{p[k]_p + v_p(n)p^{k+1}} y^{np^{k+1}}}{1 - x^{[k]_p + v_p(n)p^k} y^{np^k}}. \ \square\end{aligned}$$

Next we shall consider the case $G = GL(n,q)$, where q is a power of a prime p. For a partition $\lambda = \oplus_i m_i[i]$, we define $M_\lambda(q) = \prod_i (m_i)_q!$.

Theorem 7.3. *Let l be a prime number with $l \nmid q$, and put $e = e_l(q)$, $a = a_l(q)$. Then,*

$$\sum_{\lambda} x^{v_l(M_\lambda(q))} y^{|\lambda|} = \prod_{n=1}^{\infty} g_e(y^n) \prod_{k=0}^{\infty} g_l(x^{[k]_l + l^k a} y^{el^k n}),$$

where $g_n(T) = (1 - T^n)/(1 - T)$. If we denote the right hand side of the above by $G_l(q; x, y)$, then we have

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \mu_l(k, GL(n,q)) x^{v_l((n)_q!) - k} y^n = \prod_{d=1}^{\infty} G_l(q^d; x, y^d)^{N(q,d)}.$$

Proof. The first formula follows from (7.1) in a similar way to Theorem 6.1. Here notice that $v_l((se+i)_q!) = v_l(s!) + sa$ for $0 \le i \le e-1$. For the second, recall that the conjugacy classes of $GL(n,q)$ are parametrized by the partition-valued functions $\overrightarrow{\mu}$ on Φ with $\sum_{f \in \Phi} \deg(f) |\overrightarrow{\mu}(f)| = n$ and that the size of the class $C_{\overrightarrow{\mu}}$ corresponding to $\overrightarrow{\mu}$ satisfies multiplicatively the congruence:

$$\# C_{\overrightarrow{\mu}} \equiv \frac{(n)_q!}{\prod_{f \in \Phi} M_{\overrightarrow{\mu}(f)}(q^{\deg(f)})} \bmod q^{\mathbb{Z}}.$$

(See [Ma] Chap.IV for the precise formula.) Then the second formula follows at once. □

The author did not obtain a good expression of the generating function for the Macdonald numbers $\mu_p(k, GL(n,q))$.

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