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On the maximum value of the first coefficients of Kazhdan-Lusztig polynomials for symmetric groups

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Abstract. In this article, we show that $\max\{c^-(w); w \in \mathfrak{S}_n\} = [n^2/4]$, where $c^-(w)$ is the number of elements covered by $w \in \mathfrak{S}_n$ in the Bruhat order. Using this result, we can see that the maximum value of the first coefficients of Kazhdan-Lusztig polynomials for $\mathfrak{S}_n$ equals $[n^2/4] - n + 1$.

0. Introduction

Let (W, S) be a Coxeter system and $\leq_B$ denote the Bruhat order on W. We put

$$c^-(w) = \sharp\{y \in W; w \text{ covers } y \text{ in the Bruhat order}\},$$
$$g(w) = \sharp\{s \in S; s \leq_B w\}.$$

The purpose of this article is to show that, if W is the symmetric group $\mathfrak{S}_n$ of degree n, the maximum value of $c^-(w)$ (resp. $c^-(w) - g(w)$) over $w \in \mathfrak{S}_n$ is equal to $[n^2/4]$ (resp. $[n^2/4] - n + 1$), where $[x]$ denotes the Gaussian symbol, i.e. the greatest integer not exceeding x.

The maximum value of $c^-(w)$ plays a role in solving problems concerning with the Bruhat order with help of computers. Also, by results of Dyer [D] and Irving [I], the maximum value of $c^-(w) - g(w)$ gives the maximum value of the coefficient $p_1(x, y)$ of q in the Kazhdan-Lusztig polynomial $P_{x,y}(q) = \sum_{i \geq 0} p_i(x, y) q^i$.

This article is organized as follows: In Section 1, we associate a poset P_x to each permutation $x \in \mathfrak{S}_n$ and show that $c^-(x)$ (resp. $c^-(x) - g(x)$) is

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equal to the number of edges of the Hasse diagram of P_x (resp. $n-\mathrm{comp}(P_x)$, where $\mathrm{comp}(P_x)$ is the number of the connected components of the Hasse diagram of P_x). In Section 2, we use the Turán's theorem in the graph theory to evaluate the maximum values of $c^-(x)$ and $c^-(x)-g(x)$ (Theorem A and B). In Section 3, we combine Theorem A, B with results of Dyer [D] and Irving [I] and prove that the maximum value of the first coefficients of Kazhdan-Lusztig polynomials is given by $[n^2/4]-n+1$ (Theorem C).

1. Poset P_x associated to a permutation x

First, we define a poset P_x for $x \in \mathfrak{S}_n$.

DEFINITION 1.1. For each integer $n \geq 1$, we put $[n] := \{1, 2, \cdots, n\}$. For $x \in \mathfrak{S}_n$, we define a poset $(P_x, \leq_x)$ as follows:

$$P_x = \{\widetilde{i}; i \in [n]\} \text{ as a set, } \widetilde{j} \leq_x \widetilde{i} \Leftrightarrow i \leq j \text{ and } x(i) \geq x(j).$$

Example 1.2. Let $x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix} \in \mathfrak{S}_5$. Then the Hasse diagram of $(P_x, \leq_x)$ is the following.

$\widetilde{1}$ $\widetilde{3}$

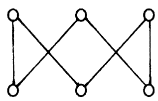
/ \ _/ \

$\widetilde{2}$ $\widetilde{4}$ $\widetilde{5}$

REMARKS 1.3.

(i) When $n \leq 5$, for any poset P with n elements, there exists $x \in \mathfrak{S}_n$ such that $P_x \simeq P$, where $P \simeq Q$ means that there exists a bijection f from P to Q satisfying $x \leq y$ in $P \Leftrightarrow f(x) \leq f(y)$ in Q.

(ii) When $n \geq 6$, the above statement is incorrect. For example, we cannot find $x \in \mathfrak{S}_6$ such that $P_x \simeq P$, where P is a poset with the following Hasse diagram.



(iii) It is easy to check that if $P_x = P_y$, then $x = y$.

Let us recall the definition of the Bruhat order on $\mathfrak{S}_n$ and we define some notations.

Definition 1.4. Let a, b be elements in $\mathfrak{S}_n$. We write $a <' b$ if there exist i, j such that $i < j$, $b(i) > b(j)$ and $a = b(i,j)$, where (i,j) is the permutation switching the number i and j and leaving the other numbers fixed. Then the Bruhat order denoted by $\leq_B$ is defined as follows:

$$x \leq_B y \Leftrightarrow \text{ there exist } z_0, z_1, \cdots, z_k \in \mathfrak{S}_n \text{ such that}$$
$$x = z_0 <' z_1 <' z_2 <' \cdots <' z_k = y.$$

For $x, y \in \mathfrak{S}_n$, we put $[x,y]:=\{z \in \mathfrak{S}_n; x \leq_B z \leq_B y\}$, $c^-(x):=\sharp\{z \in [e,x]; \ell(z) = \ell(x) - 1\}$, $G(x):=\{s \in [e,x]; \ell(s) = 1\}$, $g(x):=\sharp G(x)$, where e is the identity element and ℓ is the length function (cf. [Hu]). In other words, $c^-(x)$ (resp. $g(x)$) is the number of the coatoms (resp. atoms) of the interval $[e,x]$.

We define some more notations.

Definition 1.5. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be posets. We write $x \lessdot_P y$ if y covers x in P (i.e. $x <_P z \leq_P y \Rightarrow z = y$). If $P \cap Q = \emptyset$, then we define a new poset $(P + Q, \leq_{P+Q})$ as follows: $P + Q = P \cup Q$ as a set and $x \leq_{P+Q} y$ if and only if (i) $x, y \in P$ and $x \leq_P y$ or (ii) $x, y \in Q$ and $x \leq_Q y$. Also we define a new poset $(P \oplus Q, \leq_{P \oplus Q})$ as follows: $P \oplus Q = P \cup Q$ as a set and $x \leq_{P\oplus Q} y$ if and only if (i) $x, y \in P$ and $x \leq_P y$, (ii) $x, y \in Q$ and $x \leq_Q y$ or (iii) $x \in P$ and $y \in Q$. We put

$$h(P) := \sharp\{(x,y) \in P^2; y \lessdot_P x\},$$
$$\text{comp}(P) := \text{the number of the connected components of the Hasse diagram of } P.$$

In other words, $h(P)$ is the number of edges of the Hasse diagram of P. We say that P is connected if and only if $\text{comp}(P) = 1$.

Remark 1.6. For $x, y \in \mathfrak{S}_n$, it is well known that $y \lessdot_B x$ if and only if there exist i, j such that $y = x(i,j)$, $i < j$, $x(i) > x(j)$ and $x(k) \leq x(j)$ or $x(i) \leq x(k)$ for any $k \in [i,j]$, where $[i,j] := \{i, i+1, \cdots, j\}$.

Then we have the following.

Proposition 1.7. *For $x \in \mathfrak{S}_n$, we have*
(i) $c^-(x) = h(P_x)$,
(ii) $g(x) = n - \text{comp}(P_x)$.

Before the proof of Proposition 1.7, we prepare some more notations.

DEFINITION 1.8. For $x \in \mathfrak{S}_n$, we put

$$C(x) := \{(i,j); i < j, x(i,j) \lessdot_B x\},$$
$$H(x) := \{(\widetilde{i},\widetilde{j}) \in {P_x}^2; \widetilde{j} \lessdot_x \widetilde{i}\}.$$

REMARK 1.9. We can check that $\ell(x) = \sharp\{(\widetilde{i},\widetilde{j}) \in {P_x}^2; \widetilde{j} <_x \widetilde{i}\}$ for any $x \in \mathfrak{S}_n$.

PROOF OF PROPOSITION 1.7 (i). We define the map η from $C(x)$ to $H(x)$ by $\eta(i,j) := (\widetilde{i},\widetilde{j})$. Then, by Remark 1.6 and the definition of $\leq_x$, we have

$$\begin{aligned}
(i,j) \in C(x) &\Leftrightarrow i < j,\ x(i,j) \lessdot_B x \\
&\Leftrightarrow i < j,\ x(i) > x(j),\ x(k) \leq x(j) \text{ or } x(i) \leq x(k) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for any } k \in [i,j] \\
&\Leftrightarrow \widetilde{j} <_x \widetilde{i},\ x(k) \leq x(j) \text{ or } x(i) \leq x(k) \text{ for any } k \in [i,j] \\
&\Leftrightarrow \widetilde{j} \lessdot_x \widetilde{i} \\
&\Leftrightarrow (\widetilde{i},\widetilde{j}) \in H(x).
\end{aligned}$$

Hence, η is a bijection. It is easy to check that $\sharp C(x) = c^-(x)$ and $\sharp H(x) = h(P_x)$. So, we obtain $c^-(x) = h(P_x)$. □

Before the proof of Proposition 1.7 (ii), we will show a lemma.

LEMMA 1.10. *For $x \in \mathfrak{S}_n$, we have the following.*
(i) If P_x is connected, then $g(x) = n-1$.
(ii) Let P_1 be the connected component of P_x containing $\widetilde{1}$. Then $P_1 = \{\widetilde{1},\widetilde{2},\cdots,\widetilde{m}\}$ for some m and $x([m]) = [m]$.

PROOF. (i) Suppose that $g(x) \neq n-1$. Then there exists $k \in [n-1]$ such that $s_1, s_2, \cdots, s_{k-1} \in G(x)$ and $s_k \notin G(x)$, where $s_i := (i, i+1)$ for each $i \in [n-1]$. If there exist $\widetilde{r}, \widetilde{m}$ such that $r \in [k]$, $m \in [n]\backslash[k]$ and $\widetilde{r}$ and $\widetilde{m}$ are comparable, then we have $\widetilde{m} <_x \widetilde{r}$ (i.e. $r < m$ and $x(r) > x(m)$). On the other hand, since $r \leq k$, $k+1 \leq m$ and $s_k \notin G(x)$, we can see that $x(r) \leq k$ and $k+1 \leq x(m)$. This is a contradiction. So, we can get that every element in $\{\widetilde{1},\widetilde{2},\cdots,\widetilde{k}\}$ is incomparable to every element in $\{\widetilde{k+1},\widetilde{k+2},\cdots,\widetilde{n}\}$. This contradicts the assumption that P_x is connected. Hence, we have

$g(x) = n-1$. (ii) First, we will show that $P_1 = \{\widetilde{1}, \widetilde{2}, \cdots, \widetilde{m}\}$ as a set. Let $P_1 = \{\widetilde{i_1}, \widetilde{i_2}, \cdots, \widetilde{i_m}\}$, where $1 = i_1 < i_2 < \cdots < i_m$, as a set. Suppose that there exists $k \in [m]$ such that $i_p = p$ for any $p \in [k-1]$ and $i_k > k$. Then we can see that $\widetilde{k} \notin P_1$ and every element of P_1 is incomparable to $\widetilde{k}$. Hence, by the inequality $i_1 < i_2 < \cdots < i_{k-1} < k < i_k < \cdots < i_m$, we have $x(i_p) < x(k) < x(i_r)$ for any $p \in [k-1]$ and for any $r \in [m] \setminus [k-1]$. This means that every element in $\{\widetilde{i_1}, \widetilde{i_2}, \cdots, \widetilde{i_{k-1}}\}$ is incomparable to every element in $\{\widetilde{i_k}, \widetilde{i_{k+1}}, \cdots, \widetilde{i_m}\}$. This contradicts the assumption that P_1 is connected. Next, we will show that $x([m]) = [m]$. Suppose that there exists $k \in [m]$ such that $x(p) \leq m$ for any $p \in [k-1]$ and $x(k) > m$. Then it follows from $x(k) > m$ that

$$\sharp\{\widetilde{j}; \widetilde{j} \leq_x \widetilde{k}\} \geq \sharp\{j; j \geq k, x(j) \leq m\} + 1 = m - k + 2.$$

On the other hand, we have

$$\widetilde{1}, \widetilde{2}, \cdots, \widetilde{k-1} \notin \{\widetilde{j}; \widetilde{j} \leq_x \widetilde{k}\},$$

here we use the inequality that $x(p) \leq m < x(k)$ for any $p \in [k-1]$. Since P_1 is connected and $\widetilde{k} \in P_1$, we have

$$P_1 \supset \{\widetilde{1}, \widetilde{2}, \cdots, \widetilde{k-1}\} \sqcup \{\widetilde{j}; \widetilde{j} \leq_x \widetilde{k}\} \text{ (disjoint union)}.$$

It follows that we get $\sharp P_1 \geq m+1$. This is a contradiction. So, we obtain $x([m]) = [m]$. $\square$

PROOF OF PROPOSITION 1.7 (ii). Let $P_x = P_1 + P_2 + \cdots + P_k$ be the decomposition into connected components, and put $\sharp P_i = m_i \geq 1$ and $P_i = \{\widetilde{p_{i,1}}, \widetilde{p_{i,2}}, \cdots, \widetilde{p_{i,m_i}}\}$, where $p_{i,1} < p_{i,2} < \cdots < p_{i,m_i}$. We may assume that $p_{1,1} < p_{2,1} < \cdots < p_{k,1}$. Then, for each $i \in [k]$, it follows from Lemma 1.10 (ii) that there exists $x_i \in \mathfrak{S}_{m_i}$ such that P_i is isomorphic to P_{x_i}. Hence, by Lemma 1.10 (i), we have

$$\begin{aligned} g(x) &= g(x_1) + g(x_2) + \cdots + g(x_k) \\ &= (m_1 - 1) + (m_2 - 1) + \cdots + (m_k - 1) \\ &= m_1 + m_2 + \cdots + m_k - k \\ &= n - \mathrm{comp}(P_x). \ \square \end{aligned}$$

2. The maximum values of $c^-(w)$ and $c^-(w) - g(w)$

In this section, by the Turán's theorem, we evaluate the maximum values of $c^-(x)$ and $c^-(x) - g(x)$.

THEOREM (TURÁN). *The maximum number of the edges in n-vertex graphs which has no triangles is* $[n^2/4]$.

By the Turán's theorem, we can easily see the following.

COROLLARY 2.1. *If P is a poset with n elements, then we have* $h(P) \leq [n^2/4]$.

Hence, we have

THEOREM A.
$$\max\{c^-(x); x \in \mathfrak{S}_n\} = [n^2/4].$$

PROOF. By Proposition 1.7 (i) and Corollary 2.1, we have

$$\max\{c^-(x); x \in \mathfrak{S}_n\} = \max\{h(P_x); x \in \mathfrak{S}_n\} \leq [n^2/4].$$

We define $z_n \in \mathfrak{S}_n$ as follows:

$$(z_n(1), z_n(2), \cdots, z_n(n)) := \begin{cases} (m+1, m+2, \cdots, 2m, 1, 2, \cdots, m) & \text{if } n = 2m, \\ (m+1, m+2, \cdots, 2m+1, 1, 2, \cdots, m) & \text{if } n = 2m+1. \end{cases}$$

Then we can see that $c^-(z_n) = [n^2/4]$. Hence, we obtained this theorem. □

Also, we have the following.

PROPOSITION 2.2. *For a poset P with n elements, we have*

$$h(P) - (n - \mathrm{comp}(P)) \leq [n^2/4] - n + 1.$$

This proposition immediately follows from the next lemma.

Lemma 2.3. *Let P be a poset with n elements. If $P = P_1 + P_2 + \cdots + P_k$ is the decomposition into the connected components, then we have*

$$h(P) - (n - \mathrm{comp}(P)) \le h(P') - (n - \mathrm{comp}(P')),$$

where $P' = (P_1 \oplus P_2) + \cdots + P_k$.

Proof. Since $P_1, P_2 \ne \emptyset$, we have $h(P_1 + P_2) + 1 \le h(P_1 \oplus P_2)$. Hence, we can see $h(P) + 1 \le h(P')$. So, by the equality $n - \mathrm{comp}(P) = n - \mathrm{comp}(P') - 1$, we obtained this lemma. □

Proof of Proposition 2.2. Let $P = P_1 + P_2 + \cdots + P_k$ be the decomposition into the connected components. Then, by Corollary 2.1 and Lemma 2.3, we have

$$\begin{aligned} h(P) - (n - \mathrm{comp}(P)) &= h(P_1 + P_2 + \cdots + P_k) - (n - k) \\ &\le h((P_1 \oplus P_2) + \cdots + P_k) - (n - k + 1) \\ &\le h((P_1 \oplus P_2 \oplus P_3) + \cdots + P_k) - (n - k + 2) \\ &\le h(P_1 \oplus P_2 \oplus \cdots \oplus P_k) - (n - 1) \\ &\le [n^2/4] - n + 1. \ \square \end{aligned}$$

Hence, we have the following.

Theorem B.

$$\max\{c^-(x) - g(x); x \in \mathfrak{S}_n\} = [n^2/4] - n + 1.$$

Proof. By Proposition 1.7 and Proposition 2.2, we have

$$\begin{aligned} \max\{c^-(x) - g(x); x \in \mathfrak{S}_n\} &= \max\{h(P_x) - (n - \mathrm{comp}(P_x)); x \in \mathfrak{S}_n\} \\ &\le [n^2/4] - n + 1. \end{aligned}$$

On the other hand, for z_n defined in the proof of Theorem A, we can see that

$$c^-(z_n) - g(z_n) = [n^2/4] - n + 1.$$

Hence, we proved Theorem B. □

3. The maximum value of the first coefficient of Kazhdan-Lusztig polynomials

Here, we combine Theorem A, B with results of Dyer [D] and Irving [I] and prove that the maximum value of the first coefficients of Kazhdan-Lusztig polynomials is given by $[n^2/4]-n+1$.

First, we define Kazhdan-Lusztig polynomials.

DEFINITION 3.1. Let (W,S) be a Coxeter system. For $x,w\in W$, we define the Kazhdan-Lusztig polynomial for x,w denoted by $P_{x,w}(q)=\sum_{i\geq 0}p_i(x,w)q^i\in\mathbb{Z}[q]$ as follows:

$$P_{x,x}(q)=1 \text{ for all } x\in W,\quad P_{x,w}(q)=0 \text{ if } x\not\leq w.$$

If $x<w$, then choose $s\in S$ satisfying $\ell(sw)<\ell(w)$ and set

$$c:=\begin{cases}0 & \text{if } x<sx,\\ 1 & \text{if } sx<x.\end{cases}$$

Then $P_{x,w}(q)$ is defined inductively as follows:

$$P_{x,w}(q)=q^{1-c}P_{sx,sw}(q)+q^cP_{x,sw}(q)-\sum_{sz<z<sw}\mu(z,sw)q^{(\ell(w)-\ell(z))/2}P_{x,z}(q),$$

where $\mu(z,sw)$ is the coefficient of $q^{(\ell(sw)-\ell(z)-1)/2}$ of $P_{z,sw}(q)$.

REMARK 3.2. This definition is independent of the choice of s and is equivalent to the original definition in [KL]. See [Hu].

We can obtain the following.

THEOREM C.

$$\max\{p_1(x,w);x,w\in\mathfrak{S}_n\}=[n^2/4]-n+1.$$

PROOF. First, the following statements are valid. $p_1(e,w) = c^-(w) - g(w)$ for any $w \in \mathfrak{S}_n$([D]). $P_{x,z}(q) - P_{y,z}(q)$ has non-negative coefficients for any $x, y, z \in \mathfrak{S}_n$ with $x \leq_B y \leq_B z$([I]). Hence, by virtue of Theorem B, we have

$$\begin{aligned}\max\{p_1(x,w); x, w \in \mathfrak{S}_n\} &\leq \max\{p_1(e,w); w \in \mathfrak{S}_n\} \\ &= \max\{c^-(w) - g(w); w \in \mathfrak{S}_n\} \\ &= [n^2/4] - n + 1.\end{aligned}$$

In particular, for z_n defined in the proof of Theorem A, we have

$$p_1(e, z_n) = [n^2/4] - n + 1. \ \square$$

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