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Subtle statistical behavior in simple models for random advection-diffusion

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Abstract. Simple models for advection-diffusion with a statistical velocity field are studied here. These models involve advection by a time-independent random shear flow together with a constant mean flow. Several new and surprisingly subtle phenomena are developed here for the statistical behavior in these models. These new phenomena include: 1) mathematical criteria and examples with ill-posed evolution equations for the second order correlations and the mean statistics; 2) explicit sensitive dependence of the large scale, long time renormalization theory on parameters of the problem, such as the mean flow, the infrared cut-off, and the molecular diffusivity, for both the second order correlations and the mean statistics. This surprising sensitive dependence is explained in a self-consistent fashion both through mathematical theory and explicit examples.

1. Introduction

The advection-diffusion of a passive scalar by an incompressible velocity field is described by the equation

$$\frac{\partial T}{\partial t} + (v \cdot \nabla)\, T = \kappa \Delta T \tag{1.1}$$

where the incompressible velocity field, $v(x,t)$, satisfies div $v = 0$ and $\kappa \geq 0$ is the coefficient of molecular diffusion. The problem in (1.1) is especially important and difficult when the velocity field v involves a wide range of excited space and/or time scales and admits a statistical description. Practical applications where these are the circumstances include predicting temperature profiles in high Reynolds number turbulence ([1], [2]), the tracking of

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pollutants in the atmosphere ([3]), and the diffusion of tracers in heterogeneous porous media ([4]). Besides the practical interest in the equation from (1.1), the statistical behavior of solutions with random velocity fields is an important prototype problem for turbulence theories involving the Navier-Stokes equations ([1], [2]) since the equation in (1.1) is statistically nonlinear even though this equation is linear for a given realization. Statistical quantities that are of physical interest include the mean concentration, $\langle T \rangle$, and the second order correlations, $\langle T(x,t)\, T(x',t') \rangle$, which are related directly to the relative diffusion of pairs of particles, the pair distance function (see Section 2 and Chapter 8 of [1]). Here and below, the bracket $\langle Q \rangle$ is used to denote the statistical ensemble average of a quantity Q over suitable random velocity statistics. These problems are especially subtle when there are arbitrarily many spatial or temporal scales in the velocity field as typically occurs in the applications mentioned above. These issues have inspired a large theoretical effort in the physics and applied mathematics communities involving physical space and Fourier space renormalization theories which typically utilize partial summation of divergent perturbations series according to various recipes (see the references in [1], [2], [5]).

In response to all of the above issues, Avellaneda and one of the present authors have developed a mathematically rigorous theory for a class of models for (1.1) involving simple shear layers with many spatio-temporal scales ([6], [7], [8], [9]) which, despite their simplicity, capture a number of interesting phenomena from the more complex problems. This work also includes rigorous ([10]) and formal ([11]) extensions of this theory to more general settings for (1.1) beyond these special models as well as the use of these rigorous results in checking the behavior of various ad hoc physical renormalization theories ([12]). In their simplest form, these models are the special case of (1.1) given by

$$
\begin{aligned}
\frac{\partial T}{\partial t} + \overline{w}\frac{\partial T}{\partial x} + v(x)\frac{\partial T}{\partial y} &= \kappa \Delta T \\
T\big|_{t=0} &= T_0(x,y).
\end{aligned}
\tag{1.2}
$$

Here the random shearing velocity $v(x)$ is a stationary Gaussian random field with zero mean, i.e. $\langle v \rangle = 0$, and is completely characterized by the two point correlation function

$$R(x) = \langle v(x+x')v(x')\rangle = \int e^{2\pi ixk}E(k)\,dk \tag{1.3}$$

with $E(k)$ the real-valued energy density satisfying $E(-k) = E(k)$ (see [13]). The constant $\overline{w}$ in (1.2) represents the effect of the large scale mean flow.

Here the authors study several new phenomena in the simple models from (1.2). Next we give a brief summary of these results as well as an introduction to the remainder of this paper. In Section 2 exact closed evolution equations for the statistical quantities, $\langle T\rangle(x,y,t)$ and $\langle T(x+x',y+y',t)\,T(x',y',t)\rangle = P(x,y,t)$ are derived for the special case with $\kappa = 0$. Then simple mathematical criteria involving the correlation function, $R(x)$, and the mean flow, $\overline{w}$, are developed which yield necessary and sufficient conditions for these formal evolution equations for the statistics to yield a well-posed problem. At the end of Section 2, a family of examples involving velocity fields, $v(x)$, generated by the damped, stochastically forced, harmonic oscillator are utilized to present concrete examples where the equation for $P(x,y,t)$ is ill-posed and the equation for $\langle T\rangle(x,y,t)$ is either well-posed or ill-posed. These results have practical interest for the capability of Monte Carlo numerical methods for (1.1) to compute the statistical features of the special problem in (1.2); this is developed elsewhere by the authors ([14]).

In Section 3 the statistical behavior of solutions of (1.2) with $\kappa = 0$ is studied for Gaussian random velocity fields with a correlation function given by

$$R(x) = V^2\int_{-\infty}^{\infty} e^{2\pi ixk}|k|^{1-\epsilon}\psi_\infty(|k|)\,dk \tag{1.4}$$

where ϵ with $-\infty < \epsilon < 4$ is a parameter, motivated by renormalization theory in critical phenomena ([15]), that characterizes the statistical behavior of the velocity at large scales ([5], [6], [7]). Here $\psi_\infty(|k|)$ is a nonnegative rapidly decreasing cut-off with $\psi_\infty(0) = 1$; a prototypical example is $\psi_\infty(|k|) = e^{-|k|}$. First we demonstrate the phenomena developed in Section 2 for the problem in (1.2) with the velocity statistics in (1.4). Then, following references [6] and [7], we study the renormalization theory for (1.2), i.e. the universal large scale, long time behavior for the statistical quantities $\langle T \rangle$ and $P(x, y, t)$. We find remarkable behavior in the renormalization theory for the mean statistics when $\overline{w} \neq 0$ including trapping behavior for $\epsilon < 0$, sub-diffusive behavior for ϵ with $0 < \epsilon < 1$, and super-diffusive behavior for ϵ with $1 < \epsilon < 2$. We compute the renormalization theory for the second order correlations $P(x, y, t)$ at large scales in the super-diffusive regime, $1 < \epsilon < 4$, including the subtle singular behavior which occurs in the limit when the mean flow, $\overline{w}$, satisfies $\overline{w} \to 0$.

In Section 4 we assess the effects of molecular diffusion, i.e. $\kappa \neq 0$, for the equation in (1.2) with the velocity statistics in (1.4). For the case with $\overline{w} \neq 0$, we compute the mean square displacement, i.e. the second moment $\int_{\mathcal{R}^2} y^2 \langle T(x, y, t) \rangle \, dy$, and obtain diffusive scaling behavior for $\epsilon \leq 1$ and the same super-diffusive scaling behavior for ϵ with $1 < \epsilon < 2$ as occurred in Section 3 with $\kappa = 0$.

The renormalization theories from [6] and [7] also assume an infrared cut-off, i.e. the statistical velocity field is characterized by

$$R_\epsilon^\delta(x) = V^2 \int_{|k|>\delta} e^{2\pi i k x} |k|^{1-\epsilon} \psi_\infty(|k|) dk \tag{1.5}$$

for $-\infty < \epsilon < 4$ where δ is a small parameter, $\delta \to 0$. We remark here that the restricted integration in (1.5) is not essential for $\epsilon < 2$ since the integral converges in the limit $\delta \to 0$ and yields the correlation function in (1.4). However, for ϵ with $2 < \epsilon < 4$, there is an infrared divergence of energy so that

$$R_\epsilon^\delta(0) \to \infty \quad \text{as} \quad \delta \to 0 \quad \text{for} \quad 2 < \epsilon < 4 \tag{1.6}$$

and this cut-off is essential. The interesting physical value, $\epsilon = \frac{8}{3}$, corresponding to the "Kolmogoroff spectrum" is in this region ([5], [6], [7]). The large scale, long time renormalization theory for $\langle T \rangle$ and $P(x, y, t)$ developed in Sections 3 and 4 of this paper does not agree completely with the renormalization theory with an infrared cut-off from [7] for $\overline{w} \neq 0$, $\epsilon < 2$. The subtle mathematical differences are clarified in Section 5 in a consistent fashion.

The special case of the models in (1.2) that is treated in this paper and involves a time independent random velocity field is most appropriate as a simple model for statistical behavior for tracers in porous media ([4]). In fact, the statistical mean square displacements for (1.2) were computed by Matheron and de Marsily ([16]) in 1980 for some special cases to demonstrate features of super-diffusion. By providing a rich family of explicit but complex examples, all of the material developed in this paper has applications to the design of Monte Carlo methods (see [14], [17]) for computing turbulent diffusion statistics—an important practical topic. The version of the models in (1.2) which is most relevant for applications to turbulence involves velocity fields with time dependent correlations ([5], [6], [7]). Recent applications to the inertial range scaling theory and other non-Gaussian statistics with amusing links with N-body quantum mechanics can be found in references [18] and [19]. In fact, in this context with fixed correlations in time, one of the authors has recently generalized the behavior of the models in (1.2) to the important situation for (1.1) where the velocity fields are incompressible, homogeneous, and isotropic with a general spatial energy spectrum ([20]). We have intentionally written a lengthy introduction to this paper in order to attempt to attract more mathematicians to the interesting and subtle statistical questions regarding the mathematical theory for (1.1).

2. Exact Formulas for the Mean and Second Order Statistics for $T(x,y,t)$

We begin this section by deriving exact formulas for the evolution of the mean statistics, $\langle T\rangle(x,y,t)$, and the second order correlation statistics $P(x,y,t) = \langle T(x+x',y+y',t)\,T(x',y',t)\rangle$ for the equation in (1.2) under the assumption that the simple shear velocity $v(x)$ in (1.2) is a Gaussian random field and $\kappa = 0$. Throughout this paper we use the convention

$$f(x) = \int e^{2\pi i x\xi}\widehat{f}(\xi)d\xi$$

where $\frown$ denotes the Fourier transform.

2A) The Mean Statistics

To derive an evolution equation for the mean statistics for (1.2), we consider the transport equation

$$\begin{aligned} \frac{\partial T}{\partial t} + \overline{w}\frac{\partial T}{\partial x} + v(x)\frac{\partial T}{\partial y} &= 0 \\ T|_{t=0} &= T_0(x,y) \end{aligned} \tag{2.1}$$

where the initial value T_0 belongs to $C_0^\infty(\mathcal{R}^2)$. In order to solve equation (2.1), the Galilean change of variables, $\widetilde{x} = x - \overline{w}t$, $\widetilde{y} = y$, $\widetilde{t} = t$, is made yielding

$$\frac{\partial T}{\partial \widetilde{t}} + v(\widetilde{x} + \overline{w}\widetilde{t})\frac{\partial T}{\partial \widetilde{y}} = 0. \tag{2.2}$$

Dropping tildes in the notation and solving equation (2.2) using the method of characteristics results in the formula

$$T(x, y, t) = T_0\biggl(x, y - \int_0^t v(x + \overline{w}s)\, ds\biggr). \tag{2.3}$$

Now we look at $\langle T\rangle$ and write T_0 as a partial Fourier transform with respect to y. Thus,

$$\begin{aligned} \langle T\rangle &= \left\langle \int_{-\infty}^{\infty} e^{2\pi i\xi y} e^{-2\pi i\xi \int_0^t v(x+\overline{w}s)\, ds} \widehat{T}_0(x,\xi)\, d\xi \right\rangle \\ &= \int_{-\infty}^{\infty} e^{2\pi i\xi y} \left\langle e^{-2\pi i\xi \int_0^t v(x+\overline{w}s)\, ds} \right\rangle \widehat{T}_0(x,\xi)\, d\xi. \end{aligned} \tag{2.4}$$

Since the integral of v is a Gaussian field, the expectation in (2.4) is merely the characteristic function of a zero mean Gaussian random variable which is well known to be $e^{-\frac{4\pi^2\xi^2}{2}\sigma^2}$ where σ^2 is the variance of the Gaussian. Since the variance σ^2 in this case is given by $\sigma^2 = \int_0^t\int_0^t R\left(|\overline{w}(s-s')|\right) ds'ds$, we obtain the following closed formula for the mean:

$$\begin{aligned} \langle T\rangle &= \int_{-\infty}^{\infty} e^{2\pi i\xi y} e^{-\frac{4\pi^2\xi^2}{2}\left\langle \left(\int_0^t v(x+\overline{w}s)\, ds\right)^2\right\rangle} \widehat{T}_0(x,\xi)\, d\xi \\ &= \int_{-\infty}^{\infty} e^{2\pi i\xi y} e^{-\frac{4\pi^2\xi^2}{2}\int_0^t\int_0^t R(|\overline{w}(s-s')|) ds' ds} \widehat{T}_0(x,\xi)\, d\xi. \end{aligned} \tag{2.5}$$

To develop an evolution equation for the mean, we take the time derivative of (2.5) to yield

$$\begin{aligned} \frac{\partial\langle T\rangle}{\partial t} = \int_{-\infty}^{\infty} &-\frac{4\pi^2\xi^2}{2}\frac{\partial}{\partial t}\left[\int_0^t\int_0^t R\left(|\overline{w}(s-s')|\right) ds'ds\right] \\ &\times e^{2\pi i\xi y} e^{-\frac{4\pi^2\xi^2}{2}\int_0^t\int_0^t R(|\overline{w}(s-s')|) ds' ds} \widehat{T}_0(x,\xi)\, d\xi. \end{aligned} \tag{2.6}$$

We recognize the right hand side of (2.6) as nothing else but $D(t)\,\frac{\partial^2\langle T\rangle}{\partial y^2}$ where

$$D(t) = \frac{1}{2}\frac{\partial}{\partial t}\left[\int_0^t\int_0^t R\left(|\overline{w}(s-s')|\right)ds'ds\right] = \int_0^t R\left(|\overline{w}(s-t)|\right)ds. \tag{2.7}$$

Thus, we have obtained the *general evolution equation* for the *mean statistics* $\langle T\rangle\,(x,y,t)$ given by

$$\frac{\partial\langle T\rangle}{\partial t} = D(t)\,\frac{\partial^2\langle T\rangle}{\partial y^2} \tag{2.8}$$

with the "diffusion" coefficient $D(t)$ computed from the velocity statistics and the mean wind through the formula in (2.7).

2B) The Second Order Correlations

Under the assumption that the initial data, $T_0(x,y)$, for (2.1) is a stationary, zero mean Gaussian random field, it is not difficult to see that the random variable $T(x,y,t)$ is stationary for fixed t and thus, the second order correlation function $\langle T(x+x',y+y',t)\,T(x',y',t)\rangle = P(x,y,t)$ is well-defined. Here through a slight abuse of notation, we continue to use the expression, $\langle\cdot\rangle$, to denote ensemble average although in the derivation presented below, this is an iterated average over both the random initial data and the velocity statistics. For Gaussian random initial data with smooth realizations, we have the formula ([13])

$$T_0(x,y) = \int e^{2\pi i(x\xi_1+y\xi_2)}\widehat{T}_0(\vec{\xi}\,)\,dW(\xi_1)\otimes dW(\xi_2) \tag{2.9}$$

where $\widehat{T}_0(\vec{\xi}\,) = \overline{\widehat{T}_0(-\vec{\xi}\,)}$ is smooth and rapidly decreasing and $dW(\xi_j)$ for $j=1,2$ are formally two independent Gaussian white noises satisfying $\langle dW(\xi_j)\rangle = 0$ and

$$\langle dW(\xi_j)dW(\xi_j')\rangle = \delta(\xi_j+\xi_j')\,d\xi_j\,d\xi_j'. \tag{2.10}$$

With the same Galilean change of variables as used earlier, the equation in

(2.2) is solved by the method of characteristics to obtain

$$
\begin{aligned}
T(x,y,t) &= T_0\Big(x, y - \int_0^t v(x+\overline{w}s)\,ds\Big) \\
&= \int_{-\infty}^{\infty} e^{2\pi i(\xi_1 x + \xi_2 y)} e^{-2\pi i \xi_2 \int_0^t v(x+\overline{w}s)ds} \\
&\qquad \times \widehat{T}_0(\vec{\xi}\,)\, dW(\xi_1) \otimes dW(\xi_2)
\end{aligned}
\tag{2.11}
$$

Using the result from (2.11) in the expression for the product $T(x', y', t)$ $T(x+x', y+y', t)$ and averaging over the initial conditions utilizing (2.10) yields the formula

$$
\begin{aligned}
P(x,y,t) = \int_{-\infty}^{\infty} & e^{2\pi i(\xi_1 x + \xi_2 y)} \\
& \times \left\langle e^{2\pi i \xi_2 \int_0^t [v(x'+\overline{w}s) - v(x'+x+\overline{w}s)]ds} \right\rangle \left|\widehat{T}_0(\vec{\xi}\,)\right|^2 d\vec{\xi}.
\end{aligned}
\tag{2.12}
$$

The expression in brackets in (2.12) involves the characteristic function of the Gaussian random variable, $\int_0^t [v(x'+\overline{w}s) - v(x'+x+\overline{w}s)]ds$ associated with a stationary random field; thus, we have the formula

$$
\begin{aligned}
P(x,y,t) = \int_{-\infty}^{\infty} & e^{2\pi i(\xi_1 x + \xi_2 y)} e^{-\frac{4\pi^2\xi_2^2}{2}\left\langle \left[\int_0^t v(\overline{w}s) - v(x+\overline{w}s) ds\right]^2 \right\rangle} \\
& \times \left|\widehat{T}_0(\vec{\xi}\,)\right|^2 d\vec{\xi}.
\end{aligned}
\tag{2.13}
$$

In order to obtain an evolution equation for $P(x,y,t)$, we take the time derivative of (2.13) to yield

$$
\begin{aligned}
\frac{\partial}{\partial t} P(x,y,t) &= \int_{-\infty}^{\infty} - \frac{4\pi^2 \xi_2^2}{2} \frac{\partial}{\partial t} \left\langle \left[\int_0^t v(\overline{w}s) - v(x+\overline{w}s) ds\right]^2 \right\rangle \\
&\qquad \times e^{2\pi i(\xi_1 x + \xi_2 y)} e^{-\frac{4\pi^2\xi_2^2}{2}\left\langle \left[\int_0^t v(\overline{w}s) - v(x+\overline{w}s) ds\right]^2 \right\rangle} \\
&\qquad \times \left|\widehat{T}_0(\vec{\xi}\,)\right|^2 d\vec{\xi} \\
&= D(x,t) \frac{\partial^2 P}{\partial y^2}
\end{aligned}
\tag{2.14}
$$

where the "diffusion" coefficient $D(x,t)$ is given by the formula

$$
D(x,t) = \int_0^t 2R(|\overline{w}(\tilde{s}-t)|) - R(|x+\overline{w}(t-\tilde{s})|) - R(|x+\overline{w}(\tilde{s}-t)|)\, d\tilde{s}. \tag{2.15}
$$

The equation in (2.14) with $D(x,t)$ the function of the velocity statistics in (2.15) defines an explicit closed equation for the second order statistics for (2.1) in the Galilean shifted reference frame.

Why do we use the notation $P(x,y,t)$ for the second order correlation function? The reason is that the second order correlations are essentially the same quantity as Richardson's celebrated pair distance function from his pioneering work ([21]). Next we describe this quantity for the simple shear layer models in (2.1). Consider two particles denoted by subscripts 1 and 2. Let particle 1 be located at $(0,0)$ at time $t=0$ while particle 2 is at (x_0,y_0) for $t=0$. Denote the particle trajectories in the x-direction by $X_j(t)$ and in the y-direction by $Y_j(t)$ for $j=1,2$. For the simple shear layer model in (2.1), $X_j(t)=\overline{w}t+X_j(0)$ and $Y_j(t)=\int_0^t v(X_j(s))ds+Y_j(0)$. For the particle trajectories, the pair distance function is defined as

$$\begin{aligned} P(x,y,t) = \mathrm{Prob}\{\, & X_2(t)-X_1(t)=x, Y_2(t)-Y_1(t)=y \mid \\ & X_2(0)-X_1(0)=x_0, Y_2(0)-Y_1(0)=y_0 \,\} \end{aligned} \tag{2.16}$$

where Prob denotes the probability density with respect to the random velocity statistics and this probability is conditional on the initial separation (x_0,y_0). It is an amusing elementary exercise for the reader to verify that the quantity $P(x,y,t)$ defined in (2.16) satisfies the same equations in (2.14) and (2.15) with the initial condition

$$P(x,y,t)|_{t=0}=\delta(x-x_0)\otimes\delta(y-y_0) \tag{2.17}$$

depending on the initial separation distance.

2C) Well-Posed and Ill-Posed Evolution Equations for the Mean and the Second Order Statistics

Both the mean statistics, $\langle T\rangle(x,y,t)$, and the second order statistics, $P(x,y,t)$, satisfy simple second order equations with the form

$$\begin{aligned} \frac{\partial U}{\partial t} &= D(x,t)\frac{\partial^2 U}{\partial y^2} \\ U|_{t=0} &= U_0(x,y) \end{aligned} \tag{2.18}$$

where $D(x,t)$ is the function of the statistics and mean flow given in (2.7) and (2.15) respectively. Clearly the initial value problem is ill-posed if and only if there exists points (x_0,t_0) with $t_0>0$ so that

$$D(x_0,t_0)<0. \tag{2.19}$$

The equations in (2.8) and (2.14) always define a well-posed problem provided that the mean flow, $\overline{w}$, vanishes. However, more subtle behavior occurs in the case when the mean flow satisfies $\overline{w} \neq 0$. We have the following result which characterizes this behavior in the case with $\overline{w} \neq 0$ in terms of the statistical velocity correlation function.

Theorem: *For the situation with* $\overline{w} \neq 0$:

A) *The evolution equation in (2.8) for the mean statistics* $\langle T \rangle (x, y, t)$ *is ill-posed if and only if the velocity correlation function* $R(x)$ *from (1.3) satisfies*

$$\int_0^{t_0} R(s)\, ds < 0 \qquad \text{for some } t_0 > 0. \tag{2.20}$$

B) *The evolution equation in (2.14) for the second order statistics* $P(x, y, t) = \langle T(x + x', y + y', t)\, T(x', y', t)\rangle$ *or equivalently the pair distance function in (2.16) is ill-posed for some range of initial separation distances* x *if and only if the velocity correlation function* $R(x)$ *from (1.3) has an interval in* $(0, +\infty)$ *where* $R(x)$ *is monotone increasing.*

This theorem has the following immediate consequence:

Corollary: *If the velocity correlation function,* $R(x)$, *is a monotone decreasing function for* $0 < x < +\infty$, *then both of the evolution equations in (2.8) and (2.14) are always well-posed. Furthermore, consider any random velocity field with a smooth correlation function* $R(x)$ *satisfying the mild condition that* $\lim_{x\to\infty} R(x) = 0$. *Then for* $\overline{w} \neq 0$, *the evolution equation for the mean statistics,* $\langle T \rangle$, *cannot be ill-posed unless the evolution equation for the second order correlations,* $P(x, y, t)$, *is ill-posed for some range of initial separation distances.*

We leave the proof of the Corollary as an elementary exercise for the reader given the Theorem. Similarly, part A) of the Theorem is a direct consequence of the formula in (2.7). We prove part B) by utilizing the formula in (2.15) and the following elementary identities. First we change variables in (2.15) to obtain

$$D(x, t) = \int_0^t 2R(|\overline{w}s|) - R(|x + \overline{w}s|) - R(|x - \overline{w}s|)\, ds. \tag{2.21}$$

Looking at the formula in (2.21) as the sum of three integrals, it can be rewritten by setting $\frac{1}{\overline{w}}(x+\overline{w}s) = s'$ in the second integral and $\frac{1}{\overline{w}}(\overline{w}s-x) = \widetilde{s}$ in the third integral to get

$$\begin{aligned} D(x,t) &= \int_0^t 2R(|\overline{w}s|)\,ds - \int_{\frac{x}{\overline{w}}}^{\frac{x}{\overline{w}}+t} R(|\overline{w}s'|)\,ds' \\ &\quad - \int_{-\frac{x}{\overline{w}}}^{t-\frac{x}{\overline{w}}} R(|\overline{w}\,\widetilde{s}|)\,d\widetilde{s} \\ &= \int_0^{\frac{x}{\overline{w}}} R(|\overline{w}s|)\,ds - \int_t^{t+\frac{x}{\overline{w}}} R(|\overline{w}s|)\,ds \\ &\quad - \int_{-\frac{x}{\overline{w}}}^{0} R(|\overline{w}s|)\,ds + \int_{t-\frac{x}{\overline{w}}}^{t} R(|\overline{w}s|)\,ds. \end{aligned} \tag{2.22}$$

The second equality above is obtained by subtracting over common integration regions. Since R is even, cancellations in (2.22) yield the useful formula

$$D(x,t) = \int_{t-\frac{x}{\overline{w}}}^{t} R(|\overline{w}s|)\,ds - \int_t^{t+\frac{x}{\overline{w}}} R(|\overline{w}s|)\,ds. \tag{2.23}$$

The identity in (2.23) expresses $D(x,t)$ as the difference of two integrals of the velocity correlation function over equal and adjacent intervals of arbitrary length and location as x and t vary for $\overline{w} \neq 0$. Thus, from (2.23) the only way in which the coefficient $D(x,t)$ can remain non-negative for all x and t is for the correlation function $R(x)$ to be monotone decreasing on $[0,\infty)$. This completes the proof of the Theorem.

Remark: In the situation with ill-posed evolution equations for either the pair distance equation or the mean statistics, Monte Carlo simulation of the respective statistical quantity by any accurate procedure necessarily exhibits large fluctuations (see [14]).

2D) An Instructive Example—Velocity Statistics Generated by the Stochastic Damped Harmonic Oscillator

We consider stationary Gaussian velocity fields $v(x)$ which solve the damped linear harmonic oscillator forced by Gaussian white noise (see pages 74–76 of [13]). The random field $v(x)$ satisfies the stochastic O.D.E.

$$d\left(\frac{dv}{dx}\right) + 2\alpha\,dv + \omega^2 v\,dx = dW(x) \tag{2.24}$$

where α with $\alpha > 0$ is the damping constant, ω is the oscillation frequency, and $dW(x)$ is Gaussian white noise. There are three different regimes of behavior for the velocity correlation function, $R(x)$, depending on the ratio of the oscillation frequency, ω, to the damping coefficient, α. This correlation function is given explicitly in these three regions by the following ([13]):

$$\begin{aligned}
&\text{A)}\quad \text{For } \omega^2 > \alpha^2 \quad \text{with } \beta = \sqrt{\omega^2 - \alpha^2}\\
&\qquad R(x) = \frac{\pi}{2\alpha\,\omega^2}\, e^{-2\pi\alpha|x|}\left[\cos(2\pi\beta|x|) + \frac{\alpha}{\beta}\sin(2\pi\beta|x|)\right]\\
&\text{B)}\quad \text{For } \omega^2 = \alpha^2\\
&\qquad R(x) = \frac{\pi}{2\alpha^3}\, e^{-2\pi\alpha|x|}\,(1 + 2\pi\alpha|x|)\\
&\text{C)}\quad \text{For } \omega^2 < \alpha^2 \quad \text{with } \beta_1 = \sqrt{\alpha^2 - \omega^2}\\
&\qquad R(x) = \frac{\pi}{4\alpha\,\omega^2\beta_1}\\
&\qquad\qquad \times\left[(\alpha + \beta_1)\, e^{-2\pi(\alpha-\beta_1)|x|} - (\alpha - \beta_1)\, e^{-2\pi(\alpha+\beta_1)|x|}\right]
\end{aligned} \tag{2.25}$$

As a direct application of part B) of the Theorem, we claim the following result:

(2.26) For the stationary velocity statistics in (2.24), the evolution equation for the second order statistics $P(x, y, t)$ is ill-posed for some times and separation distances if and only if the oscillation frequency exceeds the damping coefficient, i.e. $\omega^2 > \alpha^2$.

The proof is an explicit calculation utilizing the Theorem and (2.25).

Next, we seek to determine the range of ω and α where the evolution equation for the mean statistics in (2.7) is an ill-posed equation. According to the Corollary and (2.26), this set is necessarily contained within the set where $\omega^2 > \alpha^2$. However, in general, we will see that the equation for the mean statistics, $\langle T \rangle$, is well-posed unless the parameters ω and α satisfy $\omega^2 \gg \alpha^2$. To quantify this statement, we calculate that for $\omega^2 > \alpha^2$, $D(t)$ in (2.7) is given explicitly by

$$\begin{aligned}
D(t) = \frac{1}{2\overline{w}\omega^4}\Big[&1 - e^{-2\pi\alpha\overline{w}t}\cos(2\pi\beta\overline{w}t)\\
&+\frac{(\omega^2 - 2\alpha^2)}{2\alpha\beta}\, e^{-2\pi\alpha\overline{w}t}\sin(2\pi\beta\overline{w}t)\Big]
\end{aligned} \tag{2.27}$$

where we recall that $\beta = \sqrt{\omega^2 - \alpha^2}$. This formula indicates that the only way in which we can get instability for some interval of time in the evolution equation for the mean statistics, $\langle T \rangle$, is provided that with $\frac{\omega^2}{\alpha^2} = c$ and $c \geq 2$, we have

$$e^{-\tau} \cos\left(\sqrt{c-1}\,\tau\right) - \frac{c-2}{2\sqrt{c-1}}\, e^{-\tau} \sin\left(\sqrt{c-1}\,\tau\right) > 1, \quad \text{for some } \tau > 0 \tag{2.28}$$

where $\tau = 2\pi\alpha\overline{w}t$. The equation in non-dimensional form in (2.28) can be solved numerically to determine c with the following results:

(2.29)
For $\frac{\omega^2}{\alpha^2} > 27.197$, the evolution equation for the mean statistics, $\langle T \rangle$, always has a region of instability in time.
For $\frac{\omega^2}{\alpha^2} < 27.196$, the evolution equation for the mean statistics, $\langle T \rangle$, is always well-posed in time.

Thus, the evolution equations for both of the statistical quantities, $\langle T \rangle$ and $P(x, y, t)$, are ill-posed in some intervals of time for $\frac{\omega^2}{\alpha^2} > 27.197$, but the parameter region for instability for the pair distance statistics, $\omega^2 > \alpha^2$, is significantly larger than that for the mean statistics.

3. Large Scale, Long Time Renormalization Theory for the Mean and Second Order Statistics with $\kappa = 0$

Here we consider the statistical behavior of the quantity T which satisfies the simple equation

$$\begin{aligned} \frac{\partial T}{\partial t} + \overline{w}\frac{\partial T}{\partial x} + v(x)\frac{\partial T}{\partial y} &= 0 \\ T\big|_{t=0} &= T_0(x, y) \end{aligned} \tag{3.1}$$

where the steady velocity $v(x)$ is a zero mean stationary Gaussian random field with correlation function given by

$$\begin{aligned} \langle v(x + x')\, v(x') \rangle &= R_\epsilon(x) \\ &= V^2 \int_{-\infty}^{\infty} e^{2\pi i k x} |k|^{1-\epsilon} \psi_\infty(|k|)\, dk \end{aligned} \tag{3.2}$$

and ϵ is a parameter characterizing the statistical behavior of the velocity at large scales. In the theory for the mean statistics, $\langle T \rangle$, for (3.1), the parameter ϵ ranges from $-\infty < \epsilon < 2$; on the other hand, for the second order statistics defined by the pair distance function, $P(x,y,t)$, despite the infinite behavior in (1.6) as $\delta \to 0$ for $2 < \epsilon < 4$, the behavior for $P(x,y,t)$ is finite for $-\infty < \epsilon < 4$. The parameter ϵ measures the strength of the infrared divergence; as ϵ increases, we expect to see "phase transitions" to regimes of anomalous enhanced diffusion for both statistical quantities, $\langle T \rangle$ and P, in the renormalized large scale, long time limit ([5], [6], [7]). First, we develop this renormalization theory as well as more explicit examples of the phenomena in Section 2 for the mean statistics with $-\infty < \epsilon < 2$; then we develop the theory for the second order statistics defined by $P(x,y,t)$ for the parameter range with $-\infty < \epsilon < 4$.

3A) Renormalization Theory for the Mean Statistics

In (2.7) and (2.8) of Section 2, we have derived an evolution equation for the mean statistics, $\langle T \rangle$, in a coordinate system moving with the constant mean flow given by

$$\begin{aligned} \frac{\partial \langle T \rangle}{\partial t} &= D(t)\, \frac{\partial^2 \langle T \rangle}{\partial y^2} \\ \langle T \rangle \big|_{t=0} &= T_0(x,y) \end{aligned} \tag{3.3}$$

where

$$D(t) = \int_0^t R(\overline{w}s)\, ds. \tag{3.4}$$

The most interesting functional of $\langle T \rangle$ which measures the spreading of the statistical cloud is the mean square displacement, $\langle Y^2(t) \rangle$, in the y-direction given by

$$\left\langle Y^2(t) \right\rangle = \int y^2 \langle T \rangle\, dy \tag{3.5}$$

where $\langle T(y,t) \rangle$ is the special solution of the equation in (3.3) with the point source initial data,

$$\langle T(y,t) \rangle \big|_{t=0} = \delta(0).$$

Thus, with (3.3), the mean square displacement, $\langle Y^2(t) \rangle$, is given by

$$\left\langle Y^2(t) \right\rangle = 2 \int_0^t D(s)\, ds. \tag{3.6}$$

3A) 1. Mean Statistics with $\overline{w} = 0$

In this special case the equations in (3.3) and (3.4) are extremely simple and reduce to

$$\begin{aligned} \frac{\partial \langle T \rangle}{\partial t} &= t\, R_\epsilon(0)\, \frac{\partial^2 \langle T \rangle}{\partial y^2} \\ \langle T(x,y,t) \rangle \,|_{t=0} &= T_0(x,y) \end{aligned} \tag{3.7}$$

with

$$R_\epsilon(0) = V^2 \int_{-\infty}^{\infty} |k|^{1-\epsilon} \psi_\infty(|k|)\, dk. \tag{3.8}$$

We remark that R_ϵ satisfies $R_\epsilon(0) < \infty$ for $-\infty < \epsilon < 2$. The universal large scale, long time behavior in this case is extremely easy to establish and serves as a prototype for the more difficult calculations which follow below. Consider the rescaling transformation

$$x' = \lambda x, \quad y' = \alpha y, \quad t' = \rho^2 t \tag{3.9}$$

where λ, α, ρ^2 are scaling constants perhaps functionally related with λ, α, $\rho \to 0$ so that the primed coordinates correspond to large scales and long times. Our use of the parameter ρ^2 for time rescaling is motivated both by an attempt to keep the notation somewhat consistent with references [6] and [7] for comparison and also from the fact that a standard diffusion equation is invariant under the special scaling group, $x' = \alpha x$, $y' = \alpha y$, $t' = \alpha^2 t$. To develop the large time rescaling theory for (3.7), we consider

$$\overline{T}^\Lambda(x,y,t) = (\lambda\alpha)^{-1} \langle T \rangle \left(\frac{x}{\lambda}, \frac{y}{\alpha}, \frac{t}{\rho^2} \right) \tag{3.10}$$

where $\Lambda = (\lambda,\, \alpha,\, \rho^2)$ under the assumption that

$$\int_{\mathcal{R}^2} T_0(x,y) = 1. \tag{3.11}$$

From (3.7)–(3.10) we calculate that $\overline{T}^\Lambda$ satisfies the equation

$$\frac{\partial \overline{T}^\Lambda}{\partial t} = \frac{\alpha^2}{\rho^4}\, t\, R_\epsilon(0)\, \frac{\partial^2 \overline{T}^\Lambda}{\partial y^2} \tag{3.12}$$

with

$$\overline{T}^{\Lambda}|_{t=0} = (\lambda\alpha)^{-1} T_0\left(\frac{x}{\lambda}, \frac{y}{\alpha}\right). \tag{3.13}$$

Thus, with (3.11) as $\Lambda \to 0$, $\overline{T}^{\Lambda}$ converges to $\overline{T}$, the universal scale invariant solution of the diffusion equation

$$\begin{aligned} \frac{\partial \overline{T}}{\partial t} &= t\, R_\epsilon(0)\, \frac{\partial^2 \overline{T}}{\partial y^2} \\ \overline{T}|_{t=0} &= \delta(x) \otimes \delta(y) \end{aligned} \tag{3.14}$$

provided that the scaling parameters are linked so that

$$\rho = \alpha^{\frac{1}{2}}. \tag{3.15}$$

Clearly, the solution $\overline{T}$ from (3.14) is invariant under the transformation group associated with the law in (3.15), i.e.

$$x' = \lambda x, \quad y' = \alpha y, \quad t' = \alpha t \tag{3.16}$$

and yields the universal large scale, long time behavior for initial data satisfying (3.11). There is no restriction on λ in (3.16) except that $\lambda \to 0$. We remark that other scaling behavior is possible. If instead we choose $\rho(\alpha) = \alpha^{\theta}$ with $\theta < \frac{1}{2}$, then the scale invariant solution satisfies the trivial equation

$$\begin{aligned} \frac{\partial \overline{T}}{\partial t} &= 0 \\ \overline{T}|_{t=0} &= \delta(x) \otimes \delta(y). \end{aligned} \tag{3.17}$$

Of course, the rescaling $\rho(\alpha) = \alpha^{\theta}$ with $\theta < \frac{1}{2}$ corresponds to a shorter renormalized time than the scaling $\rho(\alpha) = \alpha^{\frac{1}{2}}$ so that the trivial behavior in (3.17) is a short time limit of the universal behavior in (3.14).

3A) 2. Mean Statistics with $\overline{w} \neq 0$

Before computing the large scale, long time renormalization theory for the mean statistics, we examine the nature of the diffusion equation in (3.3) for the velocity statistics in (3.2) with the special ultraviolet cut-off

$$\psi_\infty(|k|) = e^{-|k|}. \tag{3.18}$$

For this specific cut-off, the expression for $D(t)$ in (3.4) utilizing (3.2) can be calculated in closed form yielding the formulas

$$D(t) = \frac{V^2}{|\overline{w}|\pi}\left(1 + 4\pi^2\overline{w}^2t^2\right)^{\frac{\epsilon}{2}-\frac{1}{2}}\Gamma(1-\epsilon)\sin\left[(1-\epsilon)\arctan(2\pi|\overline{w}t|)\right] \tag{3.19}$$

for $-\infty < \epsilon < 2$ and $\epsilon \neq 1$ and

$$D(t) = \frac{V^2}{|\overline{w}|\pi}\arctan(2\pi|\overline{w}t|) \tag{3.20}$$

for $\epsilon = 1$. With these explicit formulas, we leave it to the reader to easily verify that $D(t)$ is negative for some interval of time if and only if $\epsilon < -1$. Thus,

(3.21) for $\overline{w} \neq 0$ and the special cut-off $\psi_\infty = e^{-|k|}$, the evolution equation for the mean statistics is ill-posed for some interval of time if and only if ϵ satisfies $\epsilon < -1$.

It is also possible to calculate explicitly the mean square displacement for any time for this special cut-off with $\psi_\infty = e^{-|k|}$. Recalling that $\frac{\partial}{\partial t}\left\langle Y^2(t)\right\rangle = 2D(t)$, allows us to write the mean square displacement as follows

$$\begin{aligned}\left\langle Y^2(t)\right\rangle &= \frac{2V^2}{|\overline{w}|\pi}\int_0^t\int_0^\infty \sin(2\pi k\overline{w}s)|k|^{-\epsilon}\psi_\infty(|k|)\,dk\\ &= \frac{V^2}{\overline{w}^2\pi^2}\int_0^\infty [1-\cos(2\pi k\overline{w}t)]|k|^{-1-\epsilon}\psi_\infty(|k|)\,dk\end{aligned} \tag{3.22}$$

for $\epsilon < 2$. Doing the integration in (3.22) analytically with $\psi_\infty(|k|) = e^{-|k|}$ yields

$$\begin{aligned}\left\langle Y^2(t)\right\rangle = {}& \frac{V^2}{\overline{w}^2\pi^2}\\ &\times\left[\Gamma(-\epsilon) - \Gamma(-\epsilon)(1+4\pi^2\overline{w}^2t^2)^{\frac{\epsilon}{2}}\cos(\epsilon\arctan(2\pi|\overline{w}t|)\,)\right]\end{aligned} \tag{3.23}$$

for $\epsilon < 2$ and $\epsilon \neq 0, 1$. For $\epsilon = 0$, the calculation of (3.22) gives

$$\left\langle Y^2(t)\right\rangle = \frac{V^2}{2\overline{w}^2\pi^2}\log\left(1+4\pi^2\overline{w}^2t^2\right); \tag{3.24}$$

while for $\epsilon = 1$,

$$\left\langle Y^2(t)\right\rangle = \frac{V^2}{\overline{w}^2\pi^2}\left[2\pi|\overline{w}t|\arctan(2\pi|\overline{w}t|) - \frac{1}{2}\log\left(1+4\pi^2\overline{w}^2t^2\right)\right]. \tag{3.25}$$

Next we develop the general large scale long time renormalization theory for the mean statistics with $\overline{w} \neq 0$. For simplicity in exposition, we assume that the initial data $T_0(x,y)$ satisfies the requirement in (3.11). For $\overline{w} \neq 0$, the fact that the covariance function and the spectral density function are a Fourier transform pair is used to simplify (3.4). Since

$$R(x) = V^2 \int_{-\infty}^{\infty} \cos(2\pi kx)|k|^{1-\epsilon}\, \psi_\infty(|k|)\, dk, \tag{3.26}$$

the equation in (3.4) for the "diffusion" coefficient in (3.3) becomes

$$\begin{aligned} D(t) &= V^2 \int_0^t \int_{-\infty}^{\infty} \cos\left(2\pi k\overline{w}(s-t)\right)|k|^{1-\epsilon}\, \psi_\infty(|k|)\, dk\, ds \\ &= 2V^2 \int_0^{\infty} \int_0^t \cos\left(2\pi k\overline{w}(s-t)\right)\, ds\ |k|^{1-\epsilon}\, \psi_\infty(|k|)\, dk \\ &= \frac{V^2}{|\overline{w}|\pi} \int_0^{\infty} \sin(2\pi k\overline{w}t)|k|^{-\epsilon}\, \psi_\infty(k)\, dk. \end{aligned} \tag{3.27}$$

For large k, convergence of the formula in (3.27) is assured because of the ultraviolet cut-off, $\psi_\infty(|k|)$. For small k, the Taylor series of the sine term is of order k making the final formula in (3.27) absolutely convergent for $\epsilon < 2$. With the rescaling transformation from (3.9), $\overline{T}^\Lambda$ defined in (3.10) satisfies the diffusion equation

$$\begin{aligned} \frac{\partial \overline{T}^\Lambda}{\partial t} &= \frac{\alpha^2}{\rho^2} D\left(\frac{t}{\rho^2}\right) \frac{\partial^2 \overline{T}^\Lambda}{\partial y^2} \\ \overline{T}^\Lambda\big|_{t=0} &= (\lambda\alpha)^{-1} T_0\left(\frac{x}{\lambda}, \frac{y}{\alpha}\right). \end{aligned} \tag{3.28}$$

Setting $\widetilde{D} = \frac{\alpha^2}{\rho^2} D\left(\frac{t}{\rho^2}\right)$ and utilizing (3.27), we obtain that

$$\begin{aligned} \widetilde{D} &= \frac{V^2}{|\overline{w}|\pi} \frac{\alpha^2}{\rho^2} \int_0^{\infty} \sin\left(2\pi k\overline{w}\frac{t}{\rho^2}\right) |k|^{-\epsilon}\, \psi_\infty(k)\, dk \\ &= \frac{V^2}{|\overline{w}|\pi} \left(\frac{\alpha}{\rho^\epsilon}\right)^2 \int_0^{\infty} \sin(2\pi k\overline{w}t)\, |k|^{-\epsilon}\, \psi_\infty(\rho^2 k)\, dk. \end{aligned} \tag{3.29}$$

The limiting behavior of $\widetilde{D}$ as α, $\rho \to 0$ depends strongly on the value of ϵ with $-\infty < \epsilon < 2$.

First consider the region with $\epsilon < 0$; then for $|\overline{w}t| > 0$, the standard stationary phase integration by parts trick yields that the oscillatory integral

$$\int_0^{\infty} \sin(2\pi k\overline{w}t)\, |k|^{-\epsilon}\, \psi_\infty(\rho^2 k)\, dk = \mathrm{O}(1 + |\overline{w}t|^{-L}) \tag{3.30}$$

for $\epsilon < 0$, any $L > 0$, and any $\rho \to 0$. Also for $\epsilon < 0$, $\left(\frac{\alpha}{\rho^\epsilon}\right)^2 \to 0$ provided α and ρ tend to zero. Thus, the large scale diffusion coefficient satisfies $\widetilde{D} \to 0$ for any long time rescaling with $\rho \to 0$ and $\overline{T}^\Lambda$ converges to $\overline{T}$ which satisfies the trivial equation from (3.17).

For ϵ with $0 < \epsilon < 2$, we have the convergent formula for $t > 0$ given by

$$\begin{aligned} &\lim_{\rho\to 0}\int_0^\infty \sin(2\pi k\overline{w}t)\,|k|^{-\epsilon}\,\psi_\infty(\rho^2 k)\,dk \\ &\qquad = t^{\epsilon-1}(2\pi\overline{w})^{\epsilon-1}\sin\left(\frac{\pi}{2}(1-\epsilon)\right)\Gamma(1-\epsilon). \end{aligned} \tag{3.31}$$

With the equation in (3.28) satisfied by $\overline{T}^\Lambda$ with $\widetilde{D}$ given in (3.29), the fact in (3.31) implies that we need to pick the scaling law

$$\rho = \alpha^{\frac{1}{\epsilon}}, \qquad 0 < \epsilon < 2. \tag{3.32}$$

Then the scaled mean statistics $\overline{T}^\Lambda$ converges to $\overline{T}$ for any $t > 0$ where $\overline{T}$ satisfies

$$\begin{aligned} \frac{\partial\overline{T}}{\partial t} &= \widehat{C}_\epsilon\, t^{\epsilon-1}\,\frac{\partial^2\overline{T}}{\partial y^2} \\ \overline{T}|_{t=0} &= \delta(x)\otimes\delta(y) \end{aligned} \tag{3.33}$$

with

$$\begin{aligned} \widehat{C}_\epsilon(\overline{w}) &= 2V^2(2\pi\overline{w})^{\epsilon-2}\sin\left(\frac{\pi}{2}(1-\epsilon)\right)\Gamma(1-\epsilon) \\ &= \frac{1}{2}V^2\pi^{\epsilon-\frac{3}{2}}\,\overline{w}^{\epsilon-2}\,\frac{\Gamma\left(1-\frac{\epsilon}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\epsilon}{2}\right)} \end{aligned} \tag{3.34}$$

for $0 < \epsilon < 2$. Thus, with the hypothesis in (3.11) for the initial data, the universal scale invariant behavior of the mean statistics is determined by the scaling law in (3.32) and the Green's function for (3.33) which is readily calculated explicitly through time rescaled Gaussian kernels. Of course the function $\overline{T}$ remains invariant under the transformation group in (3.9) provided that ρ is determined from α through the renormalized scaling law in (3.32).

The large scale, long time behavior of the mean square displacement, $\left\langle Y^2(t)\right\rangle$, is readily calculated from (3.33) yielding the formula

$$\left\langle Y^2(t)\right\rangle = \begin{cases} 0, & \epsilon < 0 \\ \dfrac{2}{\epsilon}\widehat{C}_\epsilon(\overline{w})t^\epsilon, & 0 < \epsilon < 2. \end{cases} \tag{3.35}$$

For ordinary diffusion at large scales, we have $\langle Y^2(t)\rangle = D^* t$. From (3.35) we see that the effect on the mean statistics at large scales and long times of the mean flow, $\overline{w}$, together with the random velocity field, $v(x)$, is universal for a fixed ϵ but strongly depends on ϵ with

(3.36) trapping for $\epsilon < 0$ sub-diffusion for $0 < \epsilon < 1$ ordinary diffusion for $\epsilon = 1$ super-diffusion for $1 < \epsilon < 2$.

This behavior is summarized in the following table:

Table 3.1. Large Scale Renormalization Theory for Mean Statistics When $\overline{w} \neq 0$

Parameter	Scaling Law	Mean Square Displacement	Qualitative Behavior
$\epsilon < 0$	arbitrary	$\langle Y^2(t)\rangle \sim 0$	trapping
$0 < \epsilon < 1$	$\rho = \alpha^{\frac{1}{\epsilon}}$	$\langle Y^2(t)\rangle \sim t^{\epsilon}$	sub-diffusive
$\epsilon = 1$	$\rho = \alpha$	$\langle Y^2(t)\rangle \sim t$	diffusive
$1 < \epsilon < 2$	$\rho = \alpha^{\frac{1}{\epsilon}}$	$\langle Y^2(t)\rangle \sim t^{\epsilon}$	super-diffusive

The behavior of the large scale mean statistics is clearly a sensitive function of whether $\overline{w} = 0$ or $\overline{w} \neq 0$. For $\overline{w} = 0$, it follows from (3.14) that the large scale mean square displacement satisfies

$$\left\langle Y^2(t)\right\rangle = 2R_\epsilon(0)\, t^2, \qquad -\infty < \epsilon < 2 \tag{3.37}$$

i.e. "ballistic" scaling for any value of ϵ with $\epsilon < 2$. On the other hand, for any small value of $\overline{w} \neq 0$, we have the subtle dependence on ϵ described in Table 3.1. Thus, the large scale, long time behavior for the mean statistics exhibits singular dependence on $\overline{w}$ for $\overline{w}$ near zero.

The interested reader can confirm the renormalization theory for $\overline{w} \neq 0$ presented in (3.32) to (3.36) in the special case with $\psi_\infty(|k|) = e^{-|k|}$ by explicitly evaluating the formulas in (3.19) and (3.23)–(3.25) in large time asymptotic regimes. In particular, asymptotic evaluation of (3.24) at large times yields the transition behavior $\langle Y^2(t)\rangle \cong C\log(t)$ for $\epsilon = 0$. Such logarithmic corrections are typical in crossing phase transition boundaries in renormalization theory ([6], [15]).

3B) Renormalization Theory for the Second Order Statistics

Here we consider the renormalization theory for the correlation function $P(x,y,t) = \langle T(x+x', y+y', t)\, T(x', y', t)\rangle$ or equivalently the pair distance equation as described in (2.16). We begin with the simpler situation with no mean flow, i.e. $\overline{w} = 0$.

3B) 1. Second Order Statistics with $\overline{w} = 0$

The equation for the correlation function $P(x,y,t)$ derived in (2.14) for the special case with $\overline{w} = 0$ reduces to the evolution equation

$$\frac{\partial P}{\partial t} = 2t\,(R_\epsilon(0) - R_\epsilon(x))\,\frac{\partial^2 P}{\partial y^2} \tag{3.38}$$

$$P|_{t=0} = \langle T_0(x,y)\, T_0(0,0)\rangle \tag{3.39}$$

in the case of Gaussian random initial data $T_0(x,y)$ as given in (2.9). This is always a well-posed problem for any velocity statistics since $R_\epsilon(0) \geq R_\epsilon(x)$ for $0 \leq x < +\infty$. Next we follow ideas in references [5] and [18] to establish the equation in (3.38) for ϵ with $-\infty < \epsilon < 4$ despite the fact that $R_\epsilon(0)$ is formally infinite in the range $2 < \epsilon < 4$. The obvious strategy is first to use the pair distance equation with the stationary Gaussian random velocity field associated with the cut-off spectrum in (1.5) and then to pass to the limit (see [18] for details). Thus, the corresponding correlation function $P^\delta(x,y,t)$ satisfies the equation

$$\begin{aligned} \frac{\partial P^\delta}{\partial t} &= 2t\,(R^\delta_\epsilon(0) - R^\delta_\epsilon(x))\,\frac{\partial^2 P^\delta}{\partial y^2} \\ P^\delta|_{t=0} &= \langle T_0(x,y)\, T_0(0,0)\rangle\,. \end{aligned} \tag{3.40}$$

We calculate that

$$R^\delta_\epsilon(0) - R^\delta_\epsilon(x) = 2V^2 \int_\delta^\infty (1 - \cos(2\pi k x))|k|^{1-\epsilon}\,\psi_\infty(k)\,dk. \tag{3.41}$$

The key simple observation is that the integral in (3.41) converges absolutely as $\delta \to 0$ to

$$2V^2 \int_0^\infty (1 - \cos(2\pi k x))|k|^{1-\epsilon}\,\psi_\infty(k)\,dk < +\infty \tag{3.42}$$

for the entire range of spectral parameters, ϵ, with $-\infty < \epsilon < 4$; furthermore, Taylor expansion of $\cos(kx)$ for k near zero yields absolute convergence of the integral in (3.42) for ϵ with $-\infty < \epsilon < 4$. With a slight abuse of notation, we denote the expression in (3.42) as $R_\epsilon(0) - R_\epsilon(x)$ (even though $R_\epsilon(0) = +\infty$ for $2 \le \epsilon < 4$). Clearly it is not difficult to prove that the solutions P^δ of (3.40) with smooth initial data converge to P which satisfies the limiting equation in (3.38) for $-\infty < \epsilon < 4$ provided we make the identification between $R_\epsilon(0) - R_\epsilon(x)$ and (3.42) as mentioned above for $2 \le \epsilon < 4$.

To develop the large scale, long time renormalization theory for the correlation function, we consider the rescaling transformation in (3.9) and consider the rescaled correlation function P^Λ analogous to $\overline{T}^\Lambda$ from (3.10) for the mean statistics. Thus, P^Λ is given by

$$P^\Lambda(x, y, t) = (\lambda\alpha)^{-1} P\left(\frac{x}{\lambda}, \frac{y}{\alpha}, \frac{t}{\rho^2}\right). \tag{3.43}$$

From (3.38) P^Λ satisfies

$$\begin{gathered} \frac{\partial P^\Lambda}{\partial t} = \frac{\alpha^2}{\rho^4} 2t \left(R_\epsilon(0) - R_\epsilon\left(\frac{x}{\lambda}\right)\right) \frac{\partial^2 P^\Lambda}{\partial y^2} \\ P^\Lambda|_{t=0} = (\lambda\alpha)^{-1} \left\langle T_0\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right) T_0(0,0)\right\rangle. \end{gathered} \tag{3.44}$$

There is a completely different scaling behavior for $R_\epsilon(0) - R_\epsilon(\frac{x}{\lambda})$ as $\lambda \to 0$ in the regime $\epsilon < 2$ compared with the regime $2 < \epsilon < 4$.

First, in the regime with $-\infty < \epsilon < 2$, since $|k|^{1-\epsilon}\psi_\infty(|k|) \in L^1$, the Riemann-Lebesgue lemma implies that

$$R_\epsilon\left(\frac{x}{\lambda}\right) \to 0. \tag{3.45}$$

Therefore, P^Λ converges to $\overline{P}$ which satisfies the diffusion equation

$$\begin{gathered} \frac{\partial \overline{P}}{\partial t} = 2t\, R_\epsilon(0) \frac{\partial^2 \overline{P}}{\partial y^2} \\ \overline{P}|_{t=0} = C_0 \delta(x) \otimes \delta(y) \end{gathered} \tag{3.46}$$

for $-\infty < \epsilon < 2$ provided that we use the same scaling relation

$$\rho = \alpha^{\frac{1}{2}} \tag{3.47}$$

as in the case for the mean statistics with $\overline{w} = 0$ described in (3.14) to (3.16). In this regime pairs of particles are uncorrelated and the diffusion coefficient for the pair distance function is merely twice the coefficient of the mean statistics in (3.14). Incidentally, the constant C_0 in (3.46) is given by

$$\begin{aligned} C_0 &= \int_{\mathcal{R}^2} \langle T_0(x,y)\, T_0(0,0)\rangle\, dx\, dy \\ &= |\widehat{T}|^2(0) \end{aligned} \tag{3.48}$$

and for simplicity in exposition, we always assume the Gaussian random initial data in (2.9) satisfies $|\widehat{T}|^2(0) \neq 0$.

For the range with $2 < \epsilon < 4$, the function $R_\epsilon(0) - R_\epsilon(\frac{x}{\lambda})$ from (3.42) has completely different scaling behavior which corresponds, according to Barenblatt's classification from [22], to asymptotic behavior of the second kind in the parameter ϵ for $\epsilon > 2$ (see [18]). By rescaling (3.42), it is easy to establish the asymptotic behavior

$$R_\epsilon(0) - R_\epsilon\left(\frac{x}{\lambda}\right) = \left(\frac{|x|}{\lambda}\right)^{\epsilon-2} (C_\epsilon + o(1)) \tag{3.49}$$

for $2 < \epsilon < 4$ as $\lambda \to 0$ with

$$\begin{aligned} C_\epsilon &= 2V^2 \int_0^\infty (1 - \cos(2\pi k))|k|^{1-\epsilon}\, dk \\ &= \frac{-V^2 \pi^{\epsilon - \frac{3}{2}}\, \Gamma(1 - \frac{\epsilon}{2})}{\Gamma\left(-\frac{1}{2} + \frac{\epsilon}{2}\right)}. \end{aligned} \tag{3.50}$$

Thus, with (3.49) and (3.44), P^Λ satisfies the equation

$$\frac{\partial P^\Lambda}{\partial t} = \left(\frac{\alpha^2 \lambda^{2-\epsilon}}{\rho^4}\right) 2t\, |x|^{\epsilon-2} (C_\epsilon + o(1))\, \frac{\partial^2 P^\Lambda}{\partial y^2}. \tag{3.51}$$

Thus, we choose the scaling parameters $(\alpha,\, \lambda,\, \rho^2) = \Lambda$ to satisfy

$$\frac{\alpha^2 \lambda^{2-\epsilon}}{\rho^4} = 1, \qquad \Lambda \to 0 \tag{3.52}$$

for $2 < \epsilon < 4$. Then P^Λ converges to $\overline{P}$, the large scale, long time renormalized correlation function where $\overline{P}$ satisfies the equation

$$\begin{aligned} \frac{\partial \overline{P}}{\partial t} &= 2t\, |x|^{\epsilon-2} C_\epsilon\, \frac{\partial^2 \overline{P}}{\partial y^2} \\ \overline{P}|_{t=0} &= C_0 \delta(x) \otimes \delta(y) \end{aligned} \tag{3.53}$$

for $2 < \epsilon < 4$. The function $\overline{P}$ is clearly invariant under the scaling transformation from (3.9) provided that Λ satisfies (3.52). The function $\overline{P}$ in (3.53) is no longer Gaussian for fixed times for ϵ with $2 < \epsilon < 4$ unlike (3.46) and reflects the build up of large scale correlations. (See [18] for a complete discussion of the renormalization theory on a related problem and also including the effects of molecular diffusion.) The δ function initial data in (3.53) formally corresponds to the correlation function for scale invariant Gaussian white noise statistics.

The function $\overline{P}$ in (3.46) and (3.53) together with the scaling laws in (3.47) and (3.52) respectively describe the behavior of the renormalization theory for the correlation functions in the case with $\overline{w} = 0$. Clearly, there is a "phase transition" from standard scaling behavior with uncorrelated statistics for ϵ with $-\infty < \epsilon < 2$ to anomalous scaling behavior with strongly correlated statistics for ϵ with $2 < \epsilon < 4$.

3B) 2. Second Order Statistics with $\overline{w} \neq 0$

As in the case of the mean statistics, $\langle T \rangle$, a nontrivial mean flow with $\overline{w} \neq 0$ in (3.1) introduces subtle new phenomena in the renormalization theory for the correlation functions. A similar situation arises here for the case with $\overline{w} \neq 0$ as was discussed in (3.38)–(3.42) for the case with $\overline{w} = 0$; namely the pair distance evolution equation in (2.14) for $\overline{w} \neq 0$ with the velocity statistics in (3.2) extends to the regime $-\infty < \epsilon < 4$. Rather than repeat a similar argument as given in (3.38)–(3.42) in detail, we simply present the equation in (2.14) for $\overline{w} \neq 0$ with the formula for the diffusivity utilizing (2.15) and show that this formula converges for $-\infty < \epsilon < 4$.

With the velocity statistics described in (3.2), the equation for $P(x, y, t)$ in a shifted Galilean reference frame from (2.14) is given by

$$\begin{aligned} \frac{\partial P}{\partial t} &= D(x, t, \overline{w}) \frac{\partial^2 P}{\partial y^2} \\ P|_{t=0} &= \langle T_0(x, y)\, T_0(0, 0) \rangle \end{aligned} \tag{3.54}$$

where

$$
\begin{aligned}
&D(x,t,\overline{w}) \\
&\quad = 2\int_0^\infty\int_0^t [2\cos(2\pi k\overline{w}s) - \cos(2\pi k(x+\overline{w}s)) \\
&\qquad\qquad - \cos(2\pi k(x-\overline{w}s))]E(|k|)\,ds\,dk \\
&\quad = \frac{2}{\pi\overline{w}}\int_0^\infty \Big[\sin(2\pi k\overline{w}t) - \frac{1}{2}\sin(2\pi k(\overline{w}t+x)) \\
&\qquad\qquad -\frac{1}{2}\sin(2\pi k(\overline{w}t-x))\Big]\,\frac{E(k)}{k}\,dk
\end{aligned} \tag{3.55}
$$

with $E(k) = V^2\,k^{1-\epsilon}\,\psi_\infty(k)$. This form of $D(x,t,\overline{w})$ allows the rather straightforward determination of convergence properties of $D(x,t,\overline{w})$. For large k, the ultraviolet cut-off, $\psi_\infty(|k|)$, causes large wave number convergence. For small k, from the Taylor expansion of the sine terms in (3.55), it can readily be seen that the sum of the sine terms is of order k^3; thus, $D(x,t,\overline{w})$ converges for $\epsilon < 4$.

The first region of parameter space to be studied is the region in which $\epsilon < 1$. In this regime, the following inequality obtained from (3.55) holds:

$$
|D(x,t,\overline{w})| \le \frac{4V^2}{\pi|\overline{w}|}\int_0^\infty |k|^{-\epsilon}\psi_\infty(|k|)\,dk. \tag{3.56}
$$

In other words, $D(x,t,\overline{w})$ is bounded a priori by a constant that depends on the ultraviolet cut-off, ψ_∞. The function $\psi_\infty(|k|) = e^{-|k|}$ is chosen as an instructive example since the covariance R can be computed explicitly and is

$$
R\Big(\frac{x}{2\pi}\Big) = \frac{2V^2}{(1+x^2)^{1-\frac{\epsilon}{2}}}\cos\Big(2\Big(1-\frac{\epsilon}{2}\Big)\arctan(x)\Big)\,\Gamma(2-\epsilon) \tag{3.57}
$$

With the explicit formula in (3.57) we apply the Theorem in Section 2 to deduce the following fact:

(3.58) for $\overline{w} \neq 0$ and the special cut-off $\psi_\infty = e^{-|k|}$, the evolution equation for the pair correlation function is ill-posed for some separation distances in x and for some interval of time provided that ϵ satisfies $\epsilon < 1$.

It is amusing to compare the behavior in (3.58) for the pair correlation statistics with the behavior established earlier in (3.21) for the mean statistics.

Lack of space in this paper prevents a more detailed study of the regime with $\epsilon < 1$ and $\overline{w} \neq 0$ beyond these comments (see [14] for a numerical study).

Instead we concentrate on the regime $1 < \epsilon < 4$ for the correlation statistics with $\overline{w} \neq 0$ since the regime $1 < \epsilon < 2$ yields super-diffusive behavior for the mean statistics with $\overline{w} \neq 0$. For $1 < \epsilon < 4$, the formula in (3.55) is well-behaved even if $\psi_\infty(|k|) \equiv 1$. In this special case, the integral representation in (3.55) can be computed explicitly to give

$$\begin{aligned} D(x,t,\overline{w}) = {} & \overline{w}^{\,-1}\pi^{-\frac{3}{2}+\epsilon}\,\frac{\Gamma\left(1-\frac{\epsilon}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\epsilon}{2}\right)}\,V^2 \\ & \times\Big[|\overline{w}t|^{\epsilon-1}\mathrm{sgn}(\overline{w}t) - \frac{1}{2}|\overline{w}t+x|^{\epsilon-1}\mathrm{sgn}(\overline{w}t+x) \\ & -\frac{1}{2}|\overline{w}t-x|^{\epsilon-1}\mathrm{sgn}(\overline{w}t-x)\Big] \end{aligned} \tag{3.59}$$

for $1 < \epsilon < 4,\ \epsilon \neq 2$ and

$$D(x,t,\overline{w}) = 2V^2\left(t\,\log\left|\frac{x^2-\overline{w}^2t^2}{\overline{w}^2t^2}\right| + \frac{x}{\overline{w}}\,\log\left|\frac{x+\overline{w}t}{x-\overline{w}t}\right|\right) \tag{3.60}$$

for $\epsilon = 2$. In contrast to the behavior in (3.58) for $\epsilon < 1$, we claim the following fact:

(3.61) For $1 < \epsilon < 4$, the coefficient $D(x,t,\overline{w})$ from (3.59) and (3.60) satisfies $D(x,t,\overline{w}) > 0$ so that the evolution equation in (3.54) for the correlation function is well-posed.

The proof of this claim is given in the Appendix.

For the second order statistics, $P(x,y,t)$, an important physical quantity, that is analogous to the mean square displacement for the mean statistics, is the y-component of the mean square dispersion, $\left\langle \ell_y^2 \right\rangle$, related to the size of clouds ([21]) and defined by

$$\left\langle \ell_y^2 \right\rangle = \int y^2 P(x,y,t)\,dy \tag{3.62}$$

where $P(x,y,t)$ satisfies (2.14) with initial conditions given by a prescribed initial separation distance

$$P(x,y,t) = \delta(x-x_0)\otimes\delta(y-y_0). \tag{3.63}$$

Of course, the interpretation of $P(x,y,t)$ as the pair distance function, as described in (2.16), is most relevant for understanding the mean square dispersion from an intuitive point of view. With (2.14) and the initial condition in (3.63), it is easy to calculate the general formula

$$\left\langle \ell_y^2 \right\rangle = 2\int_0^t D(x_0, s, \overline{w})\, ds + y_0^2. \tag{3.64}$$

For the special case with $\psi_\infty \equiv 1$, $\overline{w} \neq 0$, and $1 < \epsilon < 4$, $\epsilon \neq 2$, the formula in (3.59) can be integrated explicitly to provide a useful example (see [14], [17]). This formula yields

$$\begin{aligned}\left\langle \ell_y^2 \right\rangle = &\frac{\pi^{\epsilon-\frac{3}{2}}}{\overline{w}^2}\frac{V^2\Gamma\left(-\frac{\epsilon}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\epsilon}{2}\right)} \\ &\times\left[-|x_0|^\epsilon - |\overline{w}t|^\epsilon + \frac{1}{2}|x_0 - \overline{w}t|^\epsilon + \frac{1}{2}|x_0 + \overline{w}t|^\epsilon\right] + y_0^2.\end{aligned} \tag{3.65}$$

Next we develop the large scale, long time renormalization theory for the correlation function, $P(x,y,t)$, in the regime $1 < \epsilon < 4$ ($\epsilon \neq 2$) with $\overline{w} \neq 0$ for the special case with $\psi_\infty \equiv 1$ and $D(x,t,\overline{w})$ given explicitly in (3.59). With the rescaling transformation in (3.9) and (3.54), P^Λ defined in (3.43) satisfies

$$\begin{aligned}\frac{\partial P^\Lambda}{\partial t} &= \widetilde{D}\left(\frac{x}{\lambda}, \frac{t}{\rho^2}, \overline{w}\right)\frac{\partial^2 P^\Lambda}{\partial y^2} \\ P^\Lambda|_{t=0} &= (\lambda\alpha)^{-1}\left\langle T_0\left(\frac{x}{\lambda}, \frac{y}{\alpha}\right) T_0(0,0)\right\rangle\end{aligned} \tag{3.66}$$

where

$$\widetilde{D}\left(\frac{x}{\lambda}, \frac{t}{\rho^2}, \overline{w}\right) = \frac{\alpha^2}{\rho^2}\, D\left(\frac{x}{\lambda}, \frac{t}{\rho^2}, \overline{w}\right) \tag{3.67}$$

with D given explicitly in (3.59). We denote by $\overline{C}_\epsilon$ the constant from (3.59) given by $\overline{C}_\epsilon = \pi^{-\frac{3}{2}+\epsilon}\frac{\Gamma\left(1-\frac{\epsilon}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\epsilon}{2}\right)} V^2$.

We are interested in all limits with universal scaling behavior for P^Λ provided that $\Lambda = (\lambda,\, \alpha,\, \rho) \to 0$. For $\overline{w} \neq 0$, another independent parameter emerges, namely the ratio $\frac{\lambda}{\rho^2}$, for the renormalization theory as compared with the case when $\overline{w} = 0$. The quantity $\widetilde{D}$ in (3.67) exhibits singular dependence for fixed ϵ and $1<\epsilon<4$ in both of the two scaling regimes $\frac{\lambda}{\rho^2}\to 0$ or $\frac{\lambda}{\rho^2}\to\infty$ in the sense of Barenblatt's intermediate asymptotics of the second kind ([22]). This results in a more complex renormalization theory for the correlations in the present case with $\overline{w}\neq 0$ when compared with the case with $\overline{w}=0$ which is already treated in (3.45)–(3.53).

First, we consider renormalization in the balanced regime where

$$\frac{\lambda}{\rho^2}=1, \quad \text{i.e.} \quad \rho=\lambda^{\frac{1}{2}}. \tag{3.68}$$

Then, from (3.59), we have the exact scaling behavior

$$\widetilde{D}\Big(\frac{x}{\lambda},\frac{t}{\rho^2},\overline{w}\Big)=\frac{\alpha^2\lambda^{1-\epsilon}}{\rho^2}\,D(x,t,\overline{w}). \tag{3.69}$$

Thus, with the nonlinear scaling laws

$$\alpha=\lambda^{\frac{\epsilon}{2}}, \quad \rho=\lambda^{\frac{1}{2}} \tag{3.70}$$

the correlation function P^{Λ} converges to $\overline{P}$ as $\Lambda\to 0$ where $\overline{P}$ is the solution of the equation

$$\begin{aligned}\frac{\partial\overline{P}}{\partial t}&=D(x,t,\overline{w})\,\frac{\partial^2\overline{P}}{\partial y^2}\\ \overline{P}|_{t=0}&=C_0\delta(x)\otimes\delta(y).\end{aligned} \tag{3.71}$$

The function $\overline{P}$ is clearly invariant under the scaling transformation in (3.9) with the nonlinear scaling laws from (3.70). The formulas in (3.70) and (3.71) together define one scale invariant universal renormalization theory which is distinct from the case with $\overline{w}=0$.

Next, we consider the regime where $\Lambda\to 0$ and

$$\frac{\lambda}{\rho^2(\lambda)}\to 0. \tag{3.72}$$

With the relation in (3.72), we scale and Taylor expand $\widetilde{D}$ from (3.67) to obtain

$$\begin{aligned}\widetilde{D}\Big(\frac{x}{\lambda},\frac{t}{\rho^2},\overline{w}\Big)=\frac{\overline{C}_\epsilon}{\overline{w}}\bigg[&\Big(\frac{\alpha}{\rho^\epsilon}\Big)^2|\overline{w}t|^{\epsilon-1}\\ &+\frac{\alpha^2\lambda^{2-\epsilon}}{\rho^4}\Big(|x|^{\epsilon-2}(1-\epsilon)\overline{w}t+o\Big(\Big(\frac{\lambda}{\rho^2}\Big)^\theta\Big)\Big)\bigg]\end{aligned} \tag{3.73}$$

for some $\theta > 0$. Thus, in the regime with $\frac{\lambda}{\rho^2(\lambda)} \to 0$, P^Λ satisfies the diffusion equation

$$\frac{\partial P^\Lambda}{\partial t} = \frac{\overline{C}_\epsilon}{\overline{w}} \Bigg[\left(\frac{\alpha}{\rho^\epsilon}\right)^2 |\overline{w}t|^{\epsilon-1} + \frac{\alpha^2 \lambda^{2-\epsilon}}{\rho^4} \left(|x|^{\epsilon-2}(1-\epsilon)\overline{w}t + o\left(\left(\frac{\lambda}{\rho^2}\right)^\theta\right)\right)\Bigg] \frac{\partial^2 P^\Lambda}{\partial y^2}. \tag{3.74}$$

In this regime where $\frac{\lambda}{\rho^2} \to 0$, there are different renormalization theories that occur for $1 < \epsilon < 2$ and $2 < \epsilon < 4$.

For $1 < \epsilon < 2$, we use the scaling law

$$\rho = \alpha^{\frac{1}{\epsilon}}. \tag{3.75}$$

Then with this scaling law, $\frac{\alpha^2 \lambda^{2-\epsilon}}{\rho^4} = \left(\frac{\lambda}{\rho^2}\right)^{2-\epsilon} \to 0$ for $1 < \epsilon < 2$. Therefore, P^Λ converges to $\overline{P}$ where $\overline{P}$ satisfies

$$\begin{aligned} \frac{\partial \overline{P}}{\partial t} &= 2\widehat{C}_\epsilon \, t^{\epsilon-1} \, \frac{\partial^2 \overline{P}}{\partial y^2} \\ \overline{P}|_{t=0} &= \delta(x) \otimes \delta(y) \end{aligned} \tag{3.76}$$

with $\widehat{C}_\epsilon$ the constant in (3.34). This is precisely the uncorrelated but super-diffusive behavior already documented for the mean statistics in (3.33) with the identical scaling law from (3.32).

For $2 < \epsilon < 4$ in the situation with $\frac{\lambda}{\rho^2} \to 0$, we look at (3.74) and select the nonlinear scaling law

$$\frac{\alpha^2 \lambda^{2-\epsilon}}{\rho^4} = 1. \tag{3.77}$$

With the choice in (3.77), $\left(\frac{\alpha}{\rho^\epsilon}\right)^2 = \left(\frac{\lambda}{\rho^2}\right)^{\epsilon-2} \to 0$ for $2 < \epsilon < 4$. Thus, from (3.74) we see that P^Λ converges to $\overline{P}$ where $\overline{P}$ satisfies the diffusion equation

$$\begin{aligned} \frac{\partial \overline{P}}{\partial t} &= (\overline{C}_\epsilon(1-\epsilon)) \, t \, |x|^{\epsilon-2} \, \frac{\partial^2 \overline{P}}{\partial y^2} \\ \overline{P}|_{t=0} &= C_0 \, \delta(x) \otimes \delta(y). \end{aligned} \tag{3.78}$$

Looking back at (3.52) and (3.53), since $\overline{C}_\epsilon(1-\epsilon) = 2C_\epsilon$ from (3.50), we see that the renormalization theory in (3.77) and (3.78) reduces to the one discussed earlier for $\overline{w} = 0$ and ϵ with $2 < \epsilon < 4$.

For completeness, without any details, we report the behavior for the third renormalization regime characterized by the requirements, $\Lambda \to 0$ but $\frac{\lambda}{\rho^2} \to \infty$. In this case, the nonlinear scaling law in given by

$$\alpha = \lambda \rho^{\epsilon-2} \tag{3.79}$$

for $1 < \epsilon < 4$ and P^Λ converges to $\overline{P}$ where $\overline{P}$ satisfies

$$\begin{aligned} \frac{\partial \overline{P}}{\partial t} &= \overline{C}_\epsilon(\epsilon-1)(\epsilon-2)|\overline{w}|^{\epsilon-4}|t|^{\epsilon-3}\, x^2 \, \frac{\partial^2 \overline{P}}{\partial y^2} \\ \overline{P}|_{t=0} &= \delta(x) \otimes \delta(y). \end{aligned} \tag{3.80}$$

It is worth remarking here that with the $\overline{C}_\epsilon$ given below (3.67), $\overline{C}_\epsilon(\epsilon - 1)(\epsilon-2)$ defines a positive diffusion coefficient for $1 < \epsilon < 4$. It is worth mentioning here that the nonlinear scaling laws in (3.77) and (3.79) both converge in the limit $\frac{\lambda}{\rho^2} \to 1$ to the scaling law in (3.70).

The following table summarizes the large scale, long time renormalization theory for the correlation functions which we have just developed. The expression "diffusion" coefficient in these tables always refers to the explicit diffusion coefficient for the evolution equation for P which we have calculated above. It is worth emphasizing here that for all of these different regimes of behavior with $-\infty < \epsilon < 4$, the renormalized second order correlation functions, P, are universal and scale invariant with the appropriate rescaling $x' = \lambda x,\ y' = \alpha y,\ t' = \rho^2 t$ with α, ρ determined by the corresponding nonlinear scaling law.

4. The Effect on the Mean Statistics of Molecular Diffusion with $\overline{w} \neq 0$

We consider the effects of molecular diffusion on the mean statistics for the model equation

$$\begin{aligned} \frac{\partial T}{\partial t} + \overline{w}\frac{\partial T}{\partial x} + v(x)\frac{\partial T}{\partial y} &= \kappa \Delta T \\ T|_{t=0} &= T_0(x,y). \end{aligned} \tag{4.1}$$

As in Section 3, we consider $v(x)$ to be a stationary zero mean Gaussian random field with correlation function given by

$$R(x) = V^2 \int_{-\infty}^{\infty} e^{2\pi i k x} |k|^{1-\epsilon} \psi_\infty(|k|)\, dk \tag{4.2}$$

Table 3.2. Large Scale Renormalization Theory for Second Order Statistics When **a)** $\overline{w} = 0$ and **b)** $\overline{w} \neq 0$

a)

$\overline{w} = 0$		
Parameter	Scaling Law	"Diffusion" Coefficient
$\epsilon < 2$	$\rho = \alpha^{\frac{1}{2}}$	$O(t)$
$2 < \epsilon < 4$	$\frac{\alpha^2\lambda^{2-\epsilon}}{\rho^4} = 1$	$O(t\lvert x\rvert^{\epsilon-2})$

b)

$\overline{w} \neq 0$			
Regime	Parameter	Scaling Law	"Diffusion" Coefficient
$\frac{\lambda}{\rho^2} = 1$	$1 < \epsilon < 4$	$\rho = \lambda^{\frac{1}{2}},\ \alpha = \lambda^{\frac{\epsilon}{2}}$	equation (3.59)†
$\frac{\lambda}{\rho^2} \to 0$	$1 < \epsilon < 2$	$\rho = \alpha^{\frac{1}{\epsilon}}$	$O(t^{\epsilon-1})^\ddagger$
	$2 < \epsilon < 4$	$\frac{\alpha^2\lambda^{2-\epsilon}}{\rho^4} = 1$	$O(t\lvert x\rvert^{\epsilon-2})^\S$
$\frac{\lambda}{\rho^2} \to \infty$	$1 < \epsilon < 4$	$\frac{\alpha\rho^{2-\epsilon}}{\lambda} = 1$	$O(t^{\epsilon-3}x^2)$

† Scale invariant universal theory.
‡ Uncorrelated and same as for mean statistics.
§ Same as for $\overline{w} = 0$; see Table a) above.

with the parameter ϵ varying for $-\infty < \epsilon < 2$. For the special case of $\overline{w} = 0$, the renormalized large scale, long time Green's function was calculated in reference [6] with super-diffusive and non-Gaussian behavior occurring for ϵ with $0 < \epsilon < 2$. Recently, interesting multidimensional generalizations of this model behavior have been developed in [10] and for the special value of $\epsilon = 1$ in [23].

Here we focus on the phenomena that occur with a nontrivial mean flow, $\overline{w} \neq 0$, and nonzero molecular diffusion, $\kappa \neq 0$. The main effect of molecular diffusion developed here for the mean statistics is to induce diffusive behavior for ϵ with $\epsilon < 1$ rather than the trapping or sub-diffusive behavior developed in Section 3A)2. for $\kappa = 0$ (see Table 3.1). For ϵ with $1 < \epsilon < 2$, the behavior remains super-diffusive at large scales and long times and is governed by the inviscid theory in Section 3A)2. Here we consider only the scaling theory for the y-component of the mean squared displacement, $\langle Y^2(t)\rangle$, rather than the complete behavior for the mean statistics, $\langle T\rangle$.

First we write the stochastic trajectory equations for (4.1)

$$\begin{aligned} dX &= \overline{w}\,dt + \sqrt{2\kappa}\,dW_1(t) \\ dY &= v(X)\,dt + \sqrt{2\kappa}\,dW_2(t) \end{aligned} \tag{4.3}$$

where W_1 and W_2 are independent Brownian motions. Integrating the equations in (4.3) yields

$$\begin{aligned} X(t) &= \overline{w}\,t + \sqrt{2\kappa}\,W_1(t) \\ Y(t) &= \int_0^t v\left(X(s)\right)ds + \sqrt{2\kappa}\,W_2(t). \end{aligned} \tag{4.4}$$

We now look at the mean square displacement where the expectation $\langle\cdot\rangle$ is with respect to both the velocity statistics and the Wiener process:

$$\begin{aligned} \langle Y^2(t)\rangle &= \Big\langle \Big(\int_0^t v(X(s))ds + \sqrt{2\kappa}\,W_2(t)\Big) \\ &\qquad \times \Big(\int_0^t v(X(s'))ds' + \sqrt{2\kappa}\,W_2(t)\Big)\Big\rangle_{v,W} \\ &= \Big\langle \int_0^t\int_0^t v(X(s))v(X(s'))\,ds'ds \\ &\qquad + 2\sqrt{2\kappa}\,W_2(t)\int_0^t v(X(s))\,ds + 2\kappa\,W_2^2(t)\Big\rangle_{v,W} \\ &= \int_0^t\int_0^t \langle R(X(s)-X(s'))\rangle_W\,ds'ds + 2\kappa\,t. \end{aligned} \tag{4.5}$$

Here in (4.5) we have moved expectations inside the double integral and used the fact that $\langle W_2(t)\rangle = 0$ and $\langle W_2^2(t)\rangle = t$. We substitute the integral representation for $R(x)$ from (4.2) and the representation of $X(s)$ from (4.4) into (4.5) and get

$$\begin{aligned} &\int_0^t\int_0^t \langle R(X(s)-X(s'))\rangle_W\,ds'ds \\ &\quad = V^2\int_0^t\int_0^t\Big\langle\int_{-\infty}^{\infty} e^{2\pi ik\left[\overline{w}(s-s')+\sqrt{2\kappa}\left(W_1(s)-W_1(s')\right)\right]} \\ &\qquad\qquad \times |k|^{1-\epsilon}\psi_\infty(|k|)\,dk\Big\rangle_W ds'ds \\ &\quad = V^2\int_0^t\int_0^t\int_{-\infty}^{\infty} e^{(2\pi i\overline{w}k-4\pi^2\kappa k^2)|s-s'|}|k|^{1-\epsilon}\psi_\infty(|k|)\,dk\,ds'ds. \end{aligned} \tag{4.6}$$

The second equality is true since $W_1(s) - W_1(s')$ is Gaussian and calculating the expectation merely involves calculating the characteristic function of $W_1(s) - W_1(s')$. At this point the order of integration is changed and the double integral is computed resulting in

$$
\begin{aligned}
& V^2 \int_{-\infty}^{\infty} \int_0^t \int_0^t e^{(2\pi i \overline{w} k - 4\pi^2 \kappa k^2)|s-s'|} ds' ds \, |k|^{1-\epsilon} \psi_\infty(|k|) \, dk \\
&\quad = 2V^2 \int_{-\infty}^{\infty} \left[\frac{e^{(2\pi i \overline{w} k - 4\pi^2 \kappa k^2)t} - 1}{(2\pi i \overline{w} k - 4\pi^2 \kappa k^2)^2} - \frac{t}{2\pi i \overline{w} k - 4\pi^2 \kappa k^2} \right] \\
&\qquad\qquad \times |k|^{1-\epsilon} \psi_\infty(|k|) \, dk \\
&\quad = 4V^2 \int_0^{\infty} \Bigg[\frac{4\pi^2 \kappa^2 k^2 - \overline{w}^2}{4\pi^2 k^2 (4\pi^2 \kappa^2 k^2 + \overline{w}^2)^2} \\
&\qquad\qquad \times \left[\cos(2\pi \overline{w} k t) \, e^{-4\pi^2 \kappa k^2 t} - 1 \right] \Bigg] |k|^{1-\epsilon} \psi_\infty(|k|) \, dk \\
&\qquad + 4V^2 t \int_0^{\infty} \frac{\kappa}{4\pi^2 \kappa^2 k^2 + \overline{w}^2} |k|^{1-\epsilon} \psi_\infty(|k|) \, dk.
\end{aligned}
\tag{4.7}
$$

Note that we have used the fact that the integral over a symmetric interval of an odd function is zero to eliminate all of the imaginary components in the above. Using the result from (4.7) in (4.5), we see that the mean square displacement is

$$
\begin{aligned}
\langle Y^2(t) \rangle = {} & 4\kappa^2 \int_0^{\infty} \frac{V^2}{(4\pi^2 \kappa^2 k^2 + \overline{w}^2)^2} \\
& \qquad \times \left[\cos(2\pi \overline{w} k t) \, e^{-4\pi^2 \kappa k^2 t} - 1 \right] |k|^{1-\epsilon} \psi_\infty(|k|) \, dk \\
& - \frac{\overline{w}^2}{\pi^2} \int_0^{\infty} \frac{V^2}{(4\pi^2 \kappa^2 k^2 + \overline{w}^2)^2} \\
& \qquad \times \left[\cos(2\pi \overline{w} k t) \, e^{-4\pi^2 \kappa k^2 t} - 1 \right] |k|^{-1-\epsilon} \psi_\infty(|k|) \, dk \\
& + 4\kappa \, t \int_0^{\infty} \frac{V^2}{4\pi^2 \kappa^2 k^2 + \overline{w}^2} |k|^{1-\epsilon} \psi_\infty(|k|) \, dk \\
& + 2\kappa \, t
\end{aligned}
\tag{4.8}
$$

Note that (4.8) implies that we must always observe at least diffusive type behavior.

The Large Scale, Long Time Behavior

In order to look at the large scale, long time behavior, the following change of scales is used: $y = \frac{\widetilde{y}}{\alpha}$, $t = \frac{\widetilde{t}}{\rho^2}$. Thus, (4.8) becomes

$$
\begin{aligned}
\left\langle Y^2(\widetilde{t})\right\rangle &= 4\kappa^2\overline{w}^{-4}\alpha^2\int_0^\infty \left[\cos\left(2\pi\overline{w}k\frac{\widetilde{t}}{\rho^2}\right)\, e^{-4\pi^2\kappa k^2\frac{\tilde{t}}{\rho^2}} - 1\right] \\
&\qquad\qquad\qquad \times |k|^{1-\epsilon}\widetilde{\widetilde{\psi}}_\infty(|k|)\, dk \\
&\quad -(\overline{w}\pi)^{-2}\alpha^2\int_0^\infty \left[\cos\left(2\pi\overline{w}k\frac{\widetilde{t}}{\rho^2}\right)\, e^{-4\pi^2\kappa k^2\frac{\tilde{t}}{\rho^2}} - 1\right] \\
&\qquad\qquad\qquad \times |k|^{-1-\epsilon}\widetilde{\widetilde{\psi}}_\infty(|k|)\, dk \\
&\quad +4\kappa\,\widetilde{t}\,\frac{\alpha^2}{\rho^2}\int_0^\infty \frac{V^2}{4\pi^2\kappa^2k^2+\overline{w}^2}\,\||k|^{1-\epsilon}\psi_\infty(|k|)\, dk \\
&\quad +2\kappa\,\widetilde{t}\,\frac{\alpha^2}{\rho^2} \\
&= \{1\} + \{2\} + \{3\} + \{4\}
\end{aligned}
\tag{4.9}
$$

with

$$\widetilde{\widetilde{\psi}}_\infty(k) = \frac{V^2\overline{w}^4}{(4\pi^2\kappa^2k^2+\overline{w}^2)^2}\,\psi_\infty(k).$$

We now drop tildes on the rescaled variables and assess each term in (4.9) in succession. We first note that term $\{4\}$ automatically guarantees that $\rho = \alpha^\theta$ for $\theta \leq 1$ since $\kappa \neq 0$ and predicts that we must have at least a diffusive time scale. The integral in $\{1\}$ can be bounded from above by

$$C\int_0^\infty |k|^{1-\epsilon}\psi_\infty(|k|)\, dk \tag{4.10}$$

where C is a constant since

$$\left|\cos\left(2\pi\overline{w}k\frac{t}{\rho^2}\right)\, e^{-4\pi^2\kappa k^2\frac{t}{\rho^2}} - 1\right| \leq 2$$

and

$$\left|\widetilde{\widetilde{\psi}}_\infty(|k|)\right| \leq V^2\psi_\infty(|k|).$$

Since this integral converges for $\epsilon < 2$, the magnitude of $\{1\}$ is dominated by $C\alpha^2$ for $\epsilon < 2$. Thus, $\{1\} \to 0$ as $\alpha \to 0$ for $\epsilon < 2$ regardless of the choice of scaling law ρ.

The integral in $\{2\}$ is bounded from above by

$$C\int_0^\infty |k|^{-1-\epsilon}\psi_\infty(|k|)\, dk \tag{4.11}$$

which converges for $\epsilon < 0$. Thus by the same argument used for term $\{1\}$, we see that $\{2\} \to 0$ as $\alpha \to 0$ for $\epsilon < 0$ without regard to the choice of time scaling law.

For the integral in $\{3\}$, an upper bound is given by

$$C \int_0^\infty |k|^{1-\epsilon} \psi_\infty(|k|)\, dk \tag{4.12}$$

which converges for $\epsilon < 2$. Thus, for $\rho = \alpha^\theta$ for $\theta < 1$, the term $\{3\} \to 0$ as $\alpha \to 0$ for $\epsilon < 2$; whereas if $\theta = 1$, $\{3\}$ does not approach 0, but instead converges to a constant. Likewise, when $\theta = 1$, $\{4\}$ does not approach 0 regardless of the value of ϵ. Thus, we conclude that for $\epsilon < 0$, the appropriate scaling law is the diffusive scaling with $\rho = \alpha$ and

$$\left\langle Y^2(t) \right\rangle = 4\kappa\, t \int_0^\infty \frac{V^2}{4\pi^2\kappa^2 k^2 + \overline{w}^2} |k|^{1-\epsilon} \psi_\infty(|k|)\, dk + 2\kappa\, t. \tag{4.13}$$

In order to determine the behavior for $0 < \epsilon < 2$, we need to analyze $\{2\}$ further. With the change of variables $k' = \frac{k}{\rho^2}$, the term $\{2\}$ can be rewritten as follows, where we drop primes for simplicity

$$\begin{aligned} \{2\} = |\pi \overline{w}|^{-2} \frac{\alpha^2}{\rho^{2\epsilon}} \int_0^\infty (1 - \cos(2\pi \overline{w} k t))\, |k|^{-1-\epsilon}\, \widetilde{\psi}_\infty(\rho^2 k)\, dk \\ + |\pi \overline{w}|^{-2} \frac{\alpha^2}{\rho^{2\epsilon}} \int_0^\infty \left(1 - e^{-4\pi^2 \kappa k^2 \rho^2 t}\right) |k|^{-1-\epsilon}\, \widetilde{\widetilde{\psi}}_\infty(\rho^2 k)\, dk \end{aligned} \tag{4.14}$$

with

$$\widetilde{\psi}_\infty(k) = e^{-4\pi\kappa k^2 t} \frac{V^2 \overline{w}^4}{(4\pi^2\kappa^2 k^2 + \overline{w}^2)^2}\, \psi_\infty(k)$$

and

$$\widetilde{\widetilde{\psi}}_\infty(k) = \frac{V^2 \overline{w}^4}{(4\pi^2\kappa^2 k^2 + \overline{w}^2)^2}\, \psi_\infty(k).$$

Both the first and second integrals in (4.14) are absolutely convergent integrals independent of ρ for $0 < \epsilon < 2$ since $|k|^{-1-\epsilon}$ is integrable for $|k| \geq 1$ and both $(1 - \cos(2\pi \overline{w} k t))$ and $(1 - e^{-4\pi^2 \kappa k^2 \rho^2 t})$ are bounded and vanish to second order at $k = 0$. Thus, by the dominated convergence theorem,

$$\lim_{\rho \to 0} \int_0^\infty \left(1 - e^{-4\pi^2 \kappa k^2 \rho^2 t}\right) |k|^{-1-\epsilon}\, \widetilde{\widetilde{\psi}}_\infty(\rho^2 k)\, dk = 0 \tag{4.15}$$

and

$$
\begin{aligned}
&\lim_{\rho\to 0}\int_0^\infty (1-\cos(2\pi\overline{w}kt))\,|k|^{-1-\epsilon}\,\widetilde{\psi}_\infty(\rho^2 k)\,dk \\
&\quad = V^2\int_0^\infty (1-\cos(2\pi\overline{w}kt))\,|k|^{-1-\epsilon}\,dk \\
&\quad = C_\epsilon\,|\overline{w}t|^\epsilon
\end{aligned} \tag{4.16}
$$

with

$$
C_\epsilon = \frac{-V^2\pi^{\epsilon+\frac{1}{2}}\,\Gamma\left(-\frac{\epsilon}{2}\right)}{2\Gamma\left(\frac{1}{2}+\frac{\epsilon}{2}\right)} \tag{4.17}
$$

Now for ϵ with $0<\epsilon<1$, term $\{4\}$ requires the diffusive scaling, $\rho=\alpha$, so that from (4.14)–(4.16), $\{2\}=O\left(\alpha^{2(1-\epsilon)}\right)$ which tends to zero. For $1<\epsilon<2$, term $\{2\}$ dominates so we choose the scaling law $\rho=\alpha^{\frac{1}{\epsilon}}$ and obtain through (4.14)–(4.16) that

$$
\left\langle Y^2(t)\right\rangle = |\pi\overline{w}|^{-2}C_\epsilon\,|\overline{w}t|^\epsilon \tag{4.18}
$$

for $1<\epsilon<2$. We note that the large time behavior of the mean square displacement from (4.18) has exactly the same coefficient as predicted in

Table 4.1. Large Scale Renormalization Theory for Mean Square Displacements When $\overline{w}\neq 0$

Parameter	Scaling Law	Mean Square Displacement	Qualitative Behavior
	$\kappa=0$		
$\epsilon<0$	arbitrary	$\left\langle Y^2(t)\right\rangle\sim 0$	trapping
$0<\epsilon<1$	$\rho=\alpha^{\frac{1}{\epsilon}}$	$\left\langle Y^2(t)\right\rangle\sim t^\epsilon$	sub-diffusive
$\epsilon=1$	$\rho=\alpha$	$\left\langle Y^2(t)\right\rangle\sim t$	diffusive
$1<\epsilon<2$	$\rho=\alpha^{\frac{1}{\epsilon}}$	$\left\langle Y^2(t)\right\rangle\sim t^\epsilon$	super-diffusive
	$\kappa\neq 0$		
$\epsilon\leq 1$	$\rho=\alpha$	$\left\langle Y^2(t)\right\rangle\sim t$	diffusive
$1<\epsilon<2$	$\rho=\alpha^{\frac{1}{\epsilon}}$	$\left\langle Y^2(t)\right\rangle\sim t^\epsilon$	super-diffusive

(3.35) by the scaling theory without molecular diffusivity in Section 3. The scaling behavior for $\overline{w} \neq 0$ for the cases with and without molecular diffusion is summarized now in the following table:
Observe that the addition of molecular diffusion overwhelms the sub-diffusive behavior observed when $\kappa = 0$. However, in the super-diffusive regime, the addition of molecular diffusion did not change the large scale, long time behavior.

5. The Renormalization Theory with an Infrared Cut-off

One of our goals here is to compare, briefly, the results obtained in Sections 3 and 4 of this paper to those in references [6] and [7] which utilize an infrared cut-off for the velocity statistics as given in (1.5). We consider the basic problem in (1.2) where necessarily the velocity field has a correlation function with an infrared cut-off, i.e. the velocity correlation function is given by

$$R_\epsilon^\delta(x) = 2V^2 \int_\delta^\infty \cos(2\pi kx)|k|^{1-\epsilon}\psi_\infty(k)\, dk. \tag{5.1}$$

First we consider the mean statistics, $\langle T \rangle$, in the simplest case with $\overline{w} = 0$ and $\kappa = 0$ for ϵ with $2 < \epsilon < 4$. In this special case, as in (3.7), the exact equation for the evolution of $\langle T \rangle$ is given by

$$\begin{aligned} \frac{\partial \langle T \rangle}{\partial t} &= t\, R_\epsilon^\delta(0)\, \frac{\partial^2 \langle T \rangle}{\partial y^2} \\ \langle T \rangle \left.\right|_{t=0} &= T_0(x, y). \end{aligned} \tag{5.2}$$

For a fixed value of ϵ with $2 < \epsilon < 4$, there is infrared divergence of energy because

$$R_\epsilon^\delta(0) \to \infty \qquad \text{for } 2 < \epsilon < 4. \tag{5.3}$$

Nevertheless, for a fixed value of δ, the mean square displacement, $\langle Y^2(t) \rangle$, from (3.6) is given by

$$\left\langle Y^2(t) \right\rangle = 2t^2 R_\epsilon^\delta(0). \tag{5.4}$$

Clearly the behavior in (5.4) is not uniform in δ and the constant in (5.4) explodes as $\delta \to 0$ due to (5.3); in fact,

$$\begin{aligned} R_\epsilon^\delta(0) &= 2V^2 \int_\delta^\infty |k|^{1-\epsilon}\, \psi_\infty(k)\, dk \\ &= \delta^{2-\epsilon}\, 2V^2 \int_1^\infty |k|^{1-\epsilon}\, \psi_\infty(\delta k)\, dk \end{aligned}$$

so that as $\delta \to 0$, for $2 < \epsilon < 4$,

$$R_\epsilon^\delta(0) = 2V^2(\epsilon - 2)^{-1}\delta^{2-\epsilon}(1 + o(\delta^\theta)) \tag{5.5}$$

with some value of θ with $\theta > 0$. The simplest remedy employed in references [6] and [7] and motivated by important practical problems involving isotropic models for eddy diffusivity is to consider the isotropic scaling transformation

$$x' = \delta x, \quad y' = \delta y, \quad t' = \rho^2(\delta) t \tag{5.6}$$

and

$$\overline{T}^\delta(x, y, t) = \delta^{-2} \langle T \rangle \left(\frac{x}{\delta}, \frac{y}{\delta}, \frac{t}{\rho^2(\delta)} \right). \tag{5.7}$$

With equation (5.2), $\overline{T}^\delta$ satisfies the equation

$$\frac{\partial \overline{T}^\delta}{\partial t} = \frac{\delta^2}{\rho^4}\, t\, R_\epsilon^\delta(0)\, \frac{\partial^2 \overline{T}^\delta}{\partial y^2} \tag{5.8}$$

when $2 < \epsilon < 4$. Thus, with the behavior for $R_\epsilon^\delta(0)$ in (5.5), we choose the nonlinear scaling law

$$\rho(\delta) = \delta^{1-\frac{\epsilon}{4}} \tag{5.9}$$

for $2 < \epsilon < 4$ and deduce that $\overline{T}^\delta$ converges to $\overline{T}$ which satisfies

$$\begin{aligned} \frac{\partial \overline{T}}{\partial t} &= t\, \overline{V}_\epsilon^2\, \frac{\partial^2 \overline{T}}{\partial y^2} \\ \overline{T}|_{t=0} &= \delta(x) \otimes \delta(y). \end{aligned} \tag{5.10}$$

Here $\overline{V}_\epsilon^2$ is the normalized energy from equation (5.1) with $\delta = 1$ and $\psi_\infty \equiv 1$, i.e. $\overline{V}_\epsilon^2 = 2V^2(\epsilon - 2)^{-1}$. In the renormalized super-diffusive time scale from (5.10), the mean squared displacement $\langle Y^2(t)\rangle$ is renormalized and given by

$$\left\langle Y^2(t)\right\rangle = 2\overline{V}_\epsilon^2 t^2. \tag{5.11}$$

In references [6] and [7], the same renormalized limiting behavior in (5.9)–(5.11) for $2 < \epsilon < 4$ is proved when the effects of a nonzero mean flow, $\overline{w} \neq 0$, and a nonzero molecular diffusivity, $\kappa \neq 0$, are included in (1.2).

It is worthwhile to demonstrate visually this convergence at large scales explicitly in an instructive example. We choose the Kolmogoroff value, $\epsilon = \frac{8}{3}, \overline{w} = 1,\ V = 1,\ \psi_\infty(|k|) \equiv 1$, and $\kappa = 0$ for a sequence of values of the infrared cutoff δ in (5.1) with $\delta = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$. With these assumptions, we have calculated a closed analytic expression for the mean square displacement, $\langle Y^2(t)\rangle$. In Figure 1 and Figure 2, we plot the normalized quantity

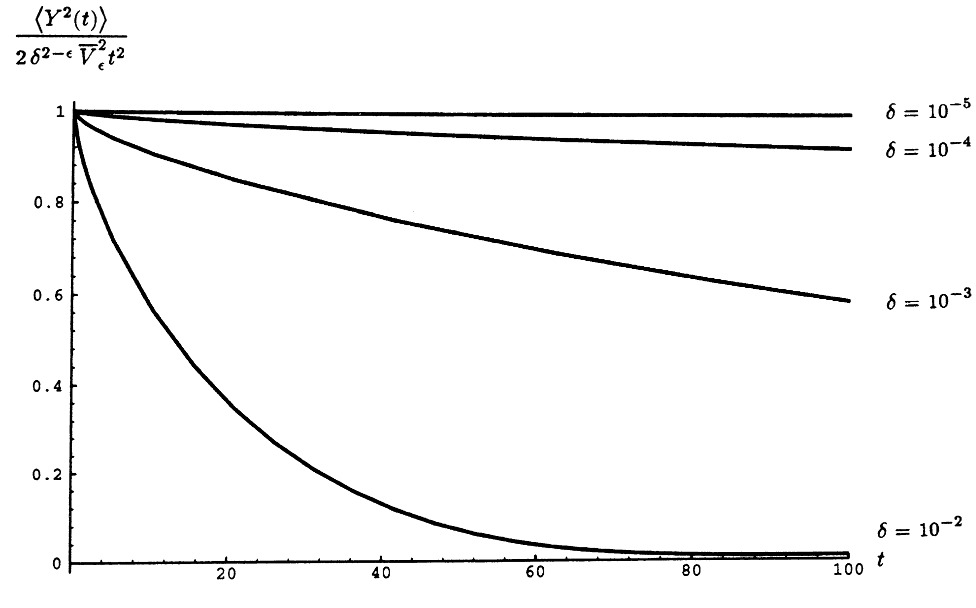

Fig. 1. Plot of $\frac{\langle Y^2(t)\rangle}{2\delta^{2-\epsilon}\overline{V}_\epsilon^2 t^2}$ versus time for various infrared cutoffs ($\delta = 10^{-2}$, 10^{-3}, 10^{-4}, 10^{-5}) when $\epsilon = \frac{8}{3}$, $\overline{w} = 1$, $V = 1$, and $\kappa = 0$ demonstrating the effect of decreasing δ.

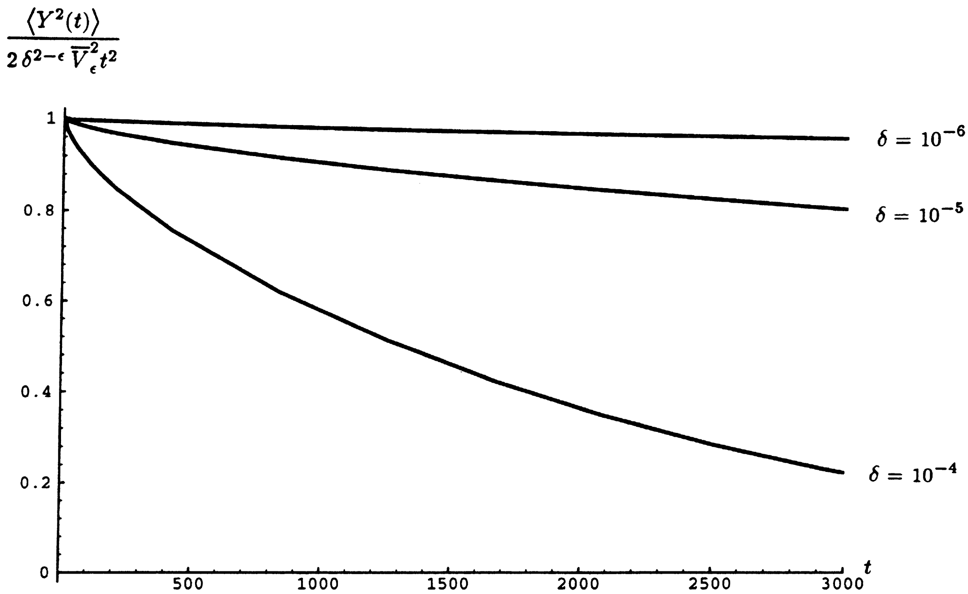

Fig. 2. Plot of $\frac{\langle Y^2(t)\rangle}{2\delta^{2-\epsilon}\overline{V}_\epsilon^2 t^2}$ versus time for various infrared cutoffs ($\delta = 10^{-4}$, 10^{-5}, 10^{-6}) when $\epsilon = \frac{8}{3}$, $\overline{w} = 1$, $V = 1$, and $\kappa = 0$ demonstrating the effect of decreasing δ.

$$\frac{\langle Y^2(t)\rangle}{2\,\delta^{2-\epsilon}\,\overline{V}_\epsilon^2 t^2} \tag{5.12}$$

as a function of time for these values of the cutoff parameter δ. According to the general theory in references [6] and [7] and demonstrated explicitly for $\overline{w} = 0$ in (5.6)–(5.11), the expression in (5.12) should tend to 1 as $\delta \to 0$. In Figure 1, we demonstrate this behavior for the values, $\delta = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$, on the shorter time interval, $0 \le t \le 100$, while Figure 2 illustrates this convergence dramatically on the longer time interval $0 \le t \le 3000$ for $\delta = 10^{-4}, 10^{-5}, 10^{-6}$. Of course, for any fixed value of δ, the *ultimate large time behavior as $t \to \infty$ is trapping* because, from (5.1), the Fourier transform of R_ϵ^δ has no mass at the origin. In Figure 3, we demonstrate this behavior for $\delta = 10^{-2}$ by graphing $\langle Y^2(t)\rangle$ on the interval $0 \le t \le 500$. Clearly the wiggles in this graph correspond to ill-posed behavior for the mean statistics at large enough times for fixed δ as discussed

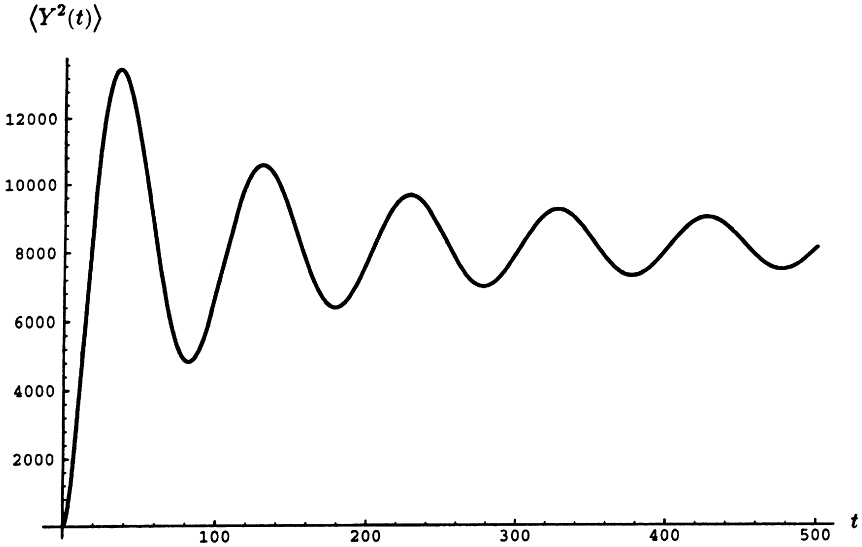

Fig. 3. Plot of mean square displacement for $\delta = 10^{-2}$, $\epsilon = \frac{8}{3}$, $\overline{w} = 1$, $V = 1$, and $\kappa = 0$ explicitly showing the trapping behavior at large times caused by the presence of the infrared cutoff.

in Section 2.

Next we show how to deduce the scaling in (5.9) as well as (5.10) through simple dimensional analysis. The quantity δ is a wavenumber so it has dimensions of $(\text{length})^{-1}$; with $\widetilde{t} = \rho^2 t$, consider the nondimensional mean squared displacement

$$\left\langle \widetilde{Y}^2(\widetilde{t}) \right\rangle = \delta^2 \left\langle Y^2 \left(\frac{\widetilde{t}}{\rho^2} \right) \right\rangle .$$

Then from (5.4) and (5.5), we calculate that as $\delta \to 0$

$$\left\langle \widetilde{Y}^2(\widetilde{t}) \right\rangle = 2\overline{V}_\epsilon^2 \, \frac{\delta^{4-\epsilon}}{\rho^4} \, \widetilde{t}^2 ,$$

from which we arrive at the nonlinear scaling law in (5.9) together with the behavior for the large scale, long time mean square displacement in (5.11).

As indicated above, as a consequence of the exact formulas in (5.2), (5.5), and (5.7), there is very little flexibility in calculating the large scale behavior of the mean statistics for $2 < \epsilon < 4$. With the more general anisotropic scaling laws from (3.9)

$$x' = \lambda x, \quad y' = \alpha y, \quad t' = \rho^2 t \tag{5.13}$$

we need to link α and ρ^2 explicitly with the cut-off parameter δ so that

$$\frac{\alpha^2 \delta^{2-\epsilon}}{\rho^4} = 1 \tag{5.14}$$

and the renormalized equation is given in (5.10). Furthermore, the equation in (5.10) is not invariant under the corresponding scaling group in (5.13). This behavior contrasts with the nonlinear scaling laws and universal renormalized equations developed throughout Section 3 of this paper for both the mean statistics and the correlation functions; all universal large scale, long time renormalized equations in Section 3 are scale invariant under the corresponding symmetry group of rescaling transformations from (3.9) associated with the appropriate nonlinear scaling laws. In fact, in Section 3 we have established this universal scale invariant behavior for the second order correlation statistics throughout the regime $2 < \epsilon < 4$ with $\overline{w} = 0$ and $\overline{w} \neq 0$ despite the infrared divergence of energy in (5.3).

Such differing behavior between the mean statistics and the second order statistics for the transported scalar for velocity spectra with infrared divergences of energy is not unexpected. It was suggested long ago by G.I. Taylor in reference [24] that the mean statistics of the scalar are dominated by the velocity scales with the most energy and thus, in our case, depend strongly on the infrared cut-off for $\delta \downarrow 0$ as established quantitatively in the simple model in (5.2)–(5.11). On the other hand, the second order statistics such as the pair distance function, $P(x, y, t)$, sense velocity differences on the scale of the separation and are expected to be universal, renormalizable, and scale invariant at large scales despite the infrared divergence; this is precisely what is established in Section 3B) for these second order statistics. In recent work by one of the authors ([18]), the complete dynamic statistical renormalization group involving all correlation functions for the scalar has been computed in another more complex exactly solvable model with strong infrared divergences with the same philosophical conclusions (see the discussion section in [18]). Nevertheless, in contradiction to the rigorous behavior established here and in reference [18] as well as the conventional wisdom of the turbulence community described earlier, some authors ([25], [26]) have repeated the calculations from reference [6] and claimed that the problem in (1.2) is not renormalizable for ϵ with $2 < \epsilon < 4$ simply because the mean statistics are not scale invariant; in that work ([25]) scale invariance is restored for the mean statistics by artificial, nonphysical devices involving a time dependent cut-off (to quote from reference [25], "time dependence in the infrared cut-off ... is utilized to achieve consistency between the asymptotic scaling exponents and the scaling behavior of the asymptotic equations"). The mathematical results presented in Section 3B) as well as those just discussed here and elsewhere ([18]) indicate a conceptual misunderstanding of those researchers ([25], [26]) as regards the inertial range renormalization theory for turbulent diffusion.

Next we discuss the role of the infrared cut-off in (5.1) in the regime of ϵ with $-\infty < \epsilon < 2$ where there is no infrared divergence of energy. In reference [7] the large scale, long time behavior of the model problem in (1.2) was determined for $\kappa \neq 0$, $\overline{w} \neq 0$, and *with an infrared cut-off.* With the usual diffusive scaling law, i.e. $\rho(\delta) = \delta$ in equation (5.6), it was established in reference [7] that there is a conventional homogenized diffusion equation satisfied by $\overline{T}$ for any ϵ with $-\infty < \epsilon < 2$. In particular at large times, the mean square displacement, $\langle Y^2(t) \rangle$, satisfies

$$\left\langle Y^2(t) \right\rangle = D^*_\epsilon \, t \tag{5.15}$$

for $-\infty < \epsilon < 2$. These results, involving the infrared cut-off explicitly for ϵ with $-\infty < \epsilon < 2$, do not agree with the rigorous results in Section 4 for the mean square displacement for $\overline{w} \neq 0$, $\kappa \neq 0$ without an infrared cut-off. Recall that without an infrared cut-off, super-diffusive behavior occurs for ϵ with $1 < \epsilon < 2$ (see for example the summary in Table 4.1). There is no contradiction in these different rigorous results; the use of the infrared cut-off simply removes most of the energy at the largest scales and this extra energy is responsible for the super-diffusive behavior for $\overline{w} \neq 0$ and $1 < \epsilon < 2$ as established in Section 3A)2. for $\kappa = 0$ and Section 4 for $\kappa \neq 0$.

To confirm this intuition rigorously, one can consider the renormalization theory for the mean statistics for $\overline{w} \neq 0$ and $\kappa = 0$ in the regime $-\infty < \epsilon < 2$ but with the velocity field having an infrared cut-off so that the corresponding correlation function is given in (5.1). Calculations following those in Section 3A)2. can be repeated with the following results:

(5.16) For the velocity field with the infrared cut-off in (5.1) and $\overline{w} \neq 0$, $\kappa = 0$, and any ϵ with $-\infty < \epsilon < 2$, consider the isotropic scaling law $x' = \delta x,\ y' = \delta y,\ t' = \rho^2 t$ where ρ^2 tends to zero in an arbitrary fashion. Then the large scale renormalization equation for the mean statistics is the trivial equation

$$\frac{\partial \overline{T}}{\partial t} = 0 \qquad\qquad \text{for } -\infty < \epsilon < 2.$$

The proof of this result follows similar calculations as were already presented in Section 3; we do not give the details here (see [27]). The result in (5.16) confirms the above intuition that the effect of the infrared cut-off is to remove energy from the largest scales and to prevent the super-diffusive behavior without the cut-off which we have established rigorously in Section 3 for ϵ with $1 < \epsilon < 2$ when $\overline{w} \neq 0$ for $\kappa = 0$. It should be clear to the reader that the result in (5.15) from [7] with nonzero molecular diffusivity but involving an infrared cut-off bears the same relationship to the results in Section 4 with $\kappa \neq 0$ as we have just described by comparing the results with $\kappa = 0$ from (5.16) with an infrared cut-off to those established in Section 3 for $\overline{w} \neq 0,\ \kappa = 0$ without such a cut-off of energy. This explains the important differences in the results reported here in Section 4 for $\kappa \neq 0$ and those from reference [7].

Appendix. Positivity of the Pair Distance Diffusion Coefficient for $1 < \epsilon < 4$ and $\overline{w} \neq 0$

Here we verify the positive character of the explicit diffusion coefficient in (3.59) and (3.60) for $1 < \epsilon < 4$. Recall the formula

$$\begin{aligned} D(x,t,\overline{w}) = \overline{w}^{\,-1}\pi^{-\frac{3}{2}+\epsilon}\,\frac{\Gamma\left(1-\frac{\epsilon}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\epsilon}{2}\right)}\,V^2 \\ \times\Big[|\overline{w}t|^{\epsilon-1}\mathrm{sgn}(\overline{w}t) - \frac{1}{2}|\overline{w}t+x|^{\epsilon-1}\mathrm{sgn}(\overline{w}t+x) \\ -\frac{1}{2}|\overline{w}t-x|^{\epsilon-1}\mathrm{sgn}(\overline{w}t-x)\Big] \end{aligned} \tag{A.1}$$

for $\epsilon \neq 2$. The formula for $\epsilon = 2$ is discussed at the end of this Appendix. In the following, the fact that $D(x, t, \overline{w}) > 0$ for $1 < \epsilon < 4$ is demonstrated.

For $1 < \epsilon < 2$, $\pi^{-\frac{3}{2}+\epsilon}\frac{\Gamma(1-\frac{\epsilon}{2})}{\Gamma(\frac{1}{2}+\frac{\epsilon}{2})}V^2 > 0$. For $|x| < |\overline{w}t|$, the remaining terms from (A.1) may be written as

$$\text{(A.2)} \qquad \frac{1}{|\overline{w}|}|\overline{w}t|^{\epsilon-1}\left[1 - \frac{1}{2}\left|1 + \frac{x}{\overline{w}t}\right|^{\epsilon-1} - \frac{1}{2}\left|1 - \frac{x}{\overline{w}t}\right|^{\epsilon-1}\right].$$

Since $f(y) = y^{\epsilon-1}$ is concave, $f(1 + \frac{x}{\overline{w}t}) + f(1 - \frac{x}{\overline{w}t}) < 2f(1) = 2$, making (A.2) positive. Therefore, $D(x, t, \overline{w}) > 0$ for $|x| < |\overline{w}t|$ and $1 < \epsilon < 2$. For $|x| > |\overline{w}t|$, the remaining terms from (A.1) may be written as

$$\text{(A.3a)} \quad \frac{1}{|\overline{w}|}|\overline{w}t|^{\epsilon-1}\left[1 - \frac{1}{2}\left|\frac{x}{\overline{w}t} + 1\right|^{\epsilon-1} + \frac{1}{2}\left|\frac{x}{\overline{w}t} - 1\right|^{\epsilon-1}\right] \qquad \text{for } \frac{x}{\overline{w}t} > 1$$

$$\text{(A.3b)} \quad \frac{1}{|\overline{w}|}|\overline{w}t|^{\epsilon-1}\left[1 + \frac{1}{2}\left|\frac{x}{\overline{w}t} + 1\right|^{\epsilon-1} - \frac{1}{2}\left|1 - \frac{x}{\overline{w}t}\right|^{\epsilon-1}\right] \qquad \text{for } \frac{x}{\overline{w}t} < -1.$$

It is clear that the terms in (A.3a) and (A.3b) are completely symmetric; thus, it suffices to consider the case in (A.3a). Again let $f(y) = y^{\epsilon-1}$ be a concave function. Introduce $g(\frac{x}{\overline{w}t}) = f(\frac{x}{\overline{w}t} + 1) - f(\frac{x}{\overline{w}t} - 1)$. Observe that $g(1) = f(2) - f(0) = f(2) < 2$ since f is concave. Also note that the derivative of g can be written as $g'(\frac{x}{\overline{w}t}) = f'(\frac{x}{\overline{w}t} + 1) - f'(\frac{x}{\overline{w}t} - 1) < 0$ since f is concave. Since g is decreasing and initially g is less than 2, $g(\frac{x}{\overline{w}t}) < 2$. In other words, $f(\frac{x}{\overline{w}t} + 1) - f(\frac{x}{\overline{w}t} - 1) < 2$ for $\frac{x}{\overline{w}t} > 1$, making (A.3a) positive. Hence, $D(x, t, \overline{w}) > 0$ for $1 < \epsilon < 2$.

For $2 < \epsilon < 4$, $\pi^{-\frac{3}{2}+\epsilon}\frac{\Gamma(1-\frac{\epsilon}{2})}{\Gamma(\frac{1}{2}+\frac{\epsilon}{2})}V^2 < 0$, and we argue similarly as above. For $|x| < |\overline{w}t|$, the remaining terms from (A.1) are as given in (A.2). Since $f(y) = y^{\epsilon-1}$ is convex, $f(1 + \frac{x}{\overline{w}t}) + f(1 - \frac{x}{\overline{w}t}) > 2f(1) = 2$, making (A.2) negative. Thus, $D(x, t, \overline{w}) > 0$ for $|x| < |\overline{w}t|$ and $2 < \epsilon < 4$. For $|x| > |\overline{w}t|$, (A.3a) and (A.3b) are still valid. By analogous arguments to those for the concave case, it can be seen that only (A.3a) need be considered and that $f(\frac{x}{\overline{w}t} + 1) - f(\frac{x}{\overline{w}t} - 1) > 2$ for $\frac{x}{\overline{w}t} > 1$, causing (A.3a) to be negative. This result could also have been seen by applying the previous argument to the concave function $-f$. Therefore, $D(x, t, \overline{w}) > 0$ for $2 < \epsilon < 4$ and $1 < \epsilon < 2$.

For $\epsilon = 2$, logarithmic terms appear in the formula for the covariance function. In this case, recall that $D(x,t,\overline{w})$ is as follows

$$D(x,t,\overline{w}) = 2V^2 t \log\left|\frac{x^2-\overline{w}^2t^2}{\overline{w}^2t^2}\right| + 2V^2\frac{x}{\overline{w}}\log\left|\frac{x+\overline{w}t}{x-\overline{w}t}\right|. \tag{A.4}$$

Since $D(x,t,\overline{w})$ does not change when either $-x$ or $-\overline{w}$ are substituted for x or $\overline{w}$ respectively, without loss of generality it is assumed that $x>0,\ \overline{w}>0$. For $x<\overline{w}t$,

$$\begin{aligned} D(x,t,\overline{w}) &> 2V^2\frac{x}{\overline{w}}\left[\log\left|\frac{x^2-\overline{w}^2t^2}{\overline{w}^2t^2}\right| + \log\left|\frac{x+\overline{w}t}{x-\overline{w}t}\right|\right] \\ &= 4V^2\frac{x}{\overline{w}}\log\left|\frac{\overline{w}t+x}{\overline{w}t}\right| > 0. \end{aligned} \tag{A.5}$$

For $x>\overline{w}t$,

$$\begin{aligned} D(x,t,\overline{w}) &> 2V^2 t\left[\log\left|\frac{x^2-\overline{w}^2t^2}{\overline{w}^2t^2}\right| + \log\left|\frac{x+\overline{w}t}{x-\overline{w}t}\right|\right] \\ &= 4V^2 t\log\left|\frac{x+\overline{w}t}{\overline{w}t}\right| > 0. \end{aligned} \tag{A.6}$$

Thus, $D(x,t,\overline{w})>0$ for $1<\epsilon<4$.

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