On the Runge theorem for instantons
(インスタントンに対するRungeの近似定理について)

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THE RUNGE THEOREM FOR INSTANTONS

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CONTENTS

1. Introduction

The last 30 years have seen the exciting development and application of "hard" techniques, in the sense of Gromov [Gro88], to the two fields of 4-dimensional topology and symplectic geometry. In the first case we are interested in smooth oriented 4-manifolds, and the problems of classification up to diffeomorphism, for example. In the other case we are interested in symplectic manifolds of any dimension, and the problems of existence or uniquness of symplectic structures, and so on. The hard techniques bring methods of nonlinear elliptic partial differential equations to attack these topological problems. First we introduce some appropriate geometric structure as an auxiliary tool, and then with this structure fixed we study the solutions of associated geometric partial differential equations. In the first case we consider Riemannian metrics on 4-manifolds and study the Yang-Mills instantons, that is, the solutions of the Yang-Mills anti-self-dual equations. In the other case we consider almost-Kähler metrics on symplectic manifolds and study the pseudo-holomorphic curves, that is, the solutions of the Cauchy-Riemann equations. These techniques are so powerful that many mathematicians have applied them to solve the problems of topology.

In the meanwhile, there seem to be striking parallels between these two hard techniques. In this paper we will pursue this meta-question, and the main theorem of this paper is the *Runge theorem for instantons*, which generalizes a theorem of Donaldson [Don93]. A classical theorem of Runge in complex analysis asserts that a meromorphic function defined on a domain in the Riemann sphere can be approximated, over compact subsets, by rational functions. Meromorphic functions are, by definition, holomorphic maps to the Riemann sphere, and rational functions are meromorphic functions from the Riemann sphere. Of course, holomorphic maps are solutions of the Cauchy-Riemann equations. Therefore, the classical Runge theorem can be paraphrased as saying that any local solution of the Cauchy-Riemann equations on a domain in the Riemann sphere can be approximated by global solutions. Our Runge theorem for instantons is an analogous result in which the Cauchy-Riemann equations on Riemann surfaces are replaced by the Yang-Mills anti-self-dual equations on oriented 4-manifolds.

We will now state our result precisely. Let X be an oriented closed Riemannian 4-manifold, U an open set in X, and K a compact subset of U. The Riemannian metric of X defines the Hodge star operator $*$. Let A_U be an anti-self-dual connection on a principal SU(2) bundle P_U over U, that is, a solution of the anti-self-dual equation over U:

$$F(A_U) + *F(A_U) = 0.$$

The Runge theorem for instantons. *There exists a sequence of principal* SU(2)*-bundles* P_n *over* X, $n = 1, 2, \ldots,$ *connections* A_n *on* P_n *which satisfy the Yang-Mills instanton equation over* X, *and bundle maps* $\rho_n \colon P_U \to P_n|_U$ *such that the sequence of connections* $\rho_n^*(A_n)$ *converge in* C^∞ *over a neighborhood of* K *to the connection* A_U.

Donaldson [Don93] proved this theorem for the case that X is the 4-sphere with the standard Riemannian metric. Our general case involves new analytic difficulties, and the main part of this paper will be devoted to develop further techniques to overcome these difficulties.

We will outline the contents of this paper. In Section 2 we will explain our strategy to prove the main theorem. In Section 3 we will review the notation and terminology on the Yang-Mills gauge theory and explain the basic approach to solve the anti-self-dual equations. The main theorem will be proved in Section 7. From Section 4 to Section 6 we will prepare analytic techniques to solve the anti-self-dual equations.

Last but not least, the author would like to his sincere gratitude to his adviser, Professor Mikio Furuta. He is not only a great mathematician but also an experienced and trusted adviser. He always inspires me. I learned from him that we have real freedom to do what we like when we do mathematics. I am very very happy to be his student.

2. Strategy

In this section we will explain the strategy to prove the main theorem. Our strategy generalizes that of [Don93].

Let X be an oriented closed Riemannian 4-manifold, U an open set in X, and K a compact subset of U. Let A_U be an anti-self-dual connection on a principal SU(2) bundle P_U over U. We will show that A_U can be approximated by a global solution of the anti-self-dual equations. Suppose a positive number ϵ is given. Then, we will construct

- a principal SU(2)-bundle $P_X \to X$,
- a bundle map $\rho\colon P_U \to P_X|_U$, and
- an anti-self-dual connection A_X on P_X

which satisfy the estimate

$$\|\rho^*(A_X) - A_U\|_{C^\infty(K)} < \epsilon.$$

Schematically, the strategy runs in three steps. First choose an arbitrary connection A_0' over X which extends the anti-self-dual connection A_U. This is possible because any principal SU(2)-bundle over an open set U is trivial. Then, we modify A_0, in a very explicit way by Taubes' cut and paste operation, to a new connection A_0, which is an approximate solution of the anti-self-dual equations in the sense that

$$\|F_{A_0} + *F_{A_0}\|_T \ll 1,$$

where $\|\cdot\|_T$ is the "Taubes norm", which we will introduce in Section 4.1. The final step is the main part. Since the space of the all connections are an affine space over the space of 1-forms, we can perturb by 1-form a the approximate solution A_0 to a nearby connection $A_0 + a$. Then, a new connection $A_0 + a$ has the anti-self-dual curvature if a satisfies some nonlinear partial differential equations. If the linearized equations are invertible, as in [Don93], then the implicit function theorem completes the proof. However, in our situation, this is not the case. The new difficulties are closely related to the small eigenvalue problem of the Laplacian, and will be explained in the next section.

3. The Anti-Self-Dual equations

In this section we will review our notation and terminology on the Yang-Mills gauge theory and explain the basic approach to solve the anti-self-dual equations. Our notation and terminology will basically follow that of [DK90].

Let X be an oriented closed Riemannian 4-manifold and P a principal SU(2)-bundle over X. Let $C(P)$ denote the space of all smooth connections on P. The space of all smooth automorphisms of P is denoted by $\mathcal{G}(P)$ and is called the gauge group of P. Fix any point x in X. The gauge group $\mathcal{G}(P)$ acts freely on $C(P) \times P|_x$, and the quotient is denoted by $\mathcal{B}'(P)$. The Riemannian metric of X defines the Hodge star operator $*$, and the bundle of self-dual 2-forms is denoted by $\wedge^+$. The bundle of Lie algebras associated to P via the adjoint representation is denoted by $\mathfrak{g}_P$. The space of $\mathfrak{g}_P$-valued 1-forms is denoted by $\Omega^1(\mathfrak{g}_P) = C^\infty(\mathfrak{g}_P \otimes T^*X)$, and the space of $\mathfrak{g}_P$-valued self-dual 2-forms is denoted by $\Omega^+(\mathfrak{g}_P) = C^\infty(\mathfrak{g}_P \otimes \wedge^+)$.

Fix a connection A on P. As the space of connections $C(P)$ is an affine space over the space of $\mathfrak{g}_P$-valued 1-forms $\Omega^1(\mathfrak{g}_P)$, any connection can be uniquely written as $A + a$ with $a \in \Omega^1(\mathfrak{g}_P)$. The connection $A + a$ has anti-self-dual curvature if and only if the 1-form a satisfies

$$d_A^+ a + (a \wedge a)^+ + F_A^+ = 0, \tag{1}$$

where d_A^+ is the first order operator

$$d_A^+\colon \Omega^1(\mathfrak{g}_P) \to \Omega^+(\mathfrak{g}_P)$$

and $(a \wedge a)^+$ is the self-dual part of the 2-form $a \wedge a$. We seek a solution of the above equation (1) in the form

$$a = d_A^* u$$

where $u \in \Omega^+(\mathfrak{g}_P)$ and

$$d_A^* \colon \Omega^+(\mathfrak{g}_P) \to \Omega^1(\mathfrak{g}_P)$$

is the formal adjoint of d_A^+. Then the equation (1) becomes the second order equation

$$d_A^+ d_A^* u + (d_A^* u \wedge d_A^* u)^+ + F_A^+ = 0. \tag{2}$$

We will call this the anti-self-dual equation.

The large part of this paper will be devoted to developing the analytic techniques to solve the anti-self-dual equations. If the Laplacian $d_A^+ d_A^*$ is invertible, then the anti-self-dual equation can be written as the fixed point equation

$$u = -(d_A^+ d_A^*)^{-1}[(d_A^* u \wedge d_A^* u)^+ + F_A^+],$$

and we can employ the contraction mapping principle to solve them. For example, the Weitzenböck formula implies that the Laplacian $d_A^+ d_A^*$ is invertible on the 4-sphere S^4 with the standard Riemannian metric. However, the Laplacian $d_A^+ d_A^*$ need not be invertible on arbitrary 4-manifolds, and this fact obstructs the direct application of the contraction mapping principle to solve the anti-self-dual equation.

Following Taubes [Tau84] we will use the Kuranishi map techniques to overcome these difficulties, and we will rewrite the anti-self-dual equation as the two equations as follows. The Laplacian $d_A^+ d_A^*$ is an essentially self-adjoint unbounded operator on $L^2(\mathfrak{g}_P \otimes \wedge^+)$, and has discrete spectrum with finite multiplicity. For any $E \geq 0$, define the projection operator

$$\Pi_{E,A} \colon L^2(\mathfrak{g}_P \otimes \wedge^+) \to L^2(\mathfrak{g}_P \otimes \wedge^+)$$

to be the finite rank spectral projections onto the subspace of L^2 spanned by the eigenvectors of $d_A^+ d_A^*$ with eigenvalues less than E. Define $\Pi_{E,A}^\perp$ to be the L^2-orthogonal complement of $\Pi_{E,A}$. We will rewrite the ASD equation (2) as two equations by using the operator identity $1 = \Pi_{E,A} + \Pi_{E,A}^\perp$ and consider only those $u \in \Omega^+(\mathfrak{g}_P)$ which satisfy $\Pi_{E,A}(u) = 0$. Thus, we are now considering the problem of finding a condition such that the two equations

$$d_A^+ d_A^* u + \Pi_{E,A}^\perp [(d_A^* u \wedge d_A^* u)^+ + F_A^+] = 0, \tag{3}$$

and

$$\Pi_{E,A} [(d_A^* u \wedge d_A^* u)^+ + F_A^+] = 0, \tag{4}$$

have simultaneously a solution $u \in \Omega^+(\mathfrak{g}_P)$ with $\Pi_{E,A}(u) = 0$. We will refer to the equation (3) as the anti-self-dual equation modulo obstructions, and the equation (4) the Kuranishi map equation. Since the Laplacian is invertible on $\mathrm{Im}[\Pi_{E,A}^\perp]$, we can again rewrite the equation (3) as the fixed point equation and employ the contraction mapping principle. This will be accomplished in Section 4. The equation (4) will be solved by the Kuranishi map techniques in Section 6.

4. The universal implicit function theorem

In this section we will prove "the universal implicit function theorem", which gives a general criteria under which an approximate solution can be deformed into a true solution of the anti-self-dual equation modulo obstructions (3). The main result in this section is Proposition 4.15, and it is *universal* in the sense that all estimates are independent of the Chern number of principal SU(2)-bundles.

Let X be an oriented closed Riemannian 4-manifold and $P \to X$ a principal SU(2)-bundle. For any connection A on P, the Laplacian $d_A^+ d_A^*$ is an essentially self-adjoint unbounded operator on $L^2(\mathfrak{g}_P \otimes \wedge^+)$, and has discrete spectrum with finite multiplicity. For any $E \geq 0$, define the projection operator

$$\Pi_{E,A} \colon L^2(\mathfrak{g}_P \otimes \wedge^+) \to L^2(\mathfrak{g}_P \otimes \wedge^+)$$

to be the finite rank spectral projections onto the subspace of L^2 spanned by the eigenvectors of $d_A^+ d_A^*$ with eigenvalues less than E. Define $\Pi_{E,A}^\perp$ to be the L^2-orthogonal complement of $\Pi_{E,A}$. It follows by definition that the Laplacian $d_A^+ d_A^*$ has bounded inverse $G_{A,E}$ on $\Pi_{E,A}^\perp\left(L^2(\mathfrak{g}_P \otimes \wedge^+)\right)$. Thus, if $u \in \Omega^+(\mathfrak{g}_P)$ with $\Pi_{E,A}(u) = 0$ is a solution of (3), then u solves the fixed point equation

$$u = -G_{A,E} \circ \Pi_{E,A}^\perp \left[(d_A^* u \wedge d_A^* u)^+ + F_A^+\right].$$

Proposition 4.15 will solve this fixed point equation.

4.1. **Taubes norms.** In this section we will introduce the *magical* norms à la Taubes. Let $\Delta = \nabla^* \nabla$ be the Laplacian on functions over X, and $g(x, y)$ the Green kernel of $\Delta + 1$. The kernel $g(x, y)$ is strictly positive, and smooth outside the diagonal with a singularity of order $1/d(x, y)^2$.

Definition 4.1 (Taubes norms)**.** For any section s of a Euclidean bundle over X with a Riemannian connection A, we define five norms by

- $||s||_{T^1} = \sup_{x \in X} \int_X g(x, y)|u(y)|\, dy$
- $||s||_{T^2}^2 = \sup_{x \in X} \int_X g(x, y)|u(y)|^2\, dy$

and

- $||s||_T = ||s||_2 + ||s||_{T^1}$
- $||s||_{TT}^2 = ||s||_\infty^2 + ||\nabla_A s||_{T^2}^2$
- $||s||_{TTT} = ||\nabla_A s||_\infty + ||s||_{TT}$

We will refer to these as the Taubes norms.

The rest of this section will be devoted to derive some inequalities related to those Taubes norms. The integrability of the Green kernel $g(x, y)$ leads to the following inequalities.

Proposition 4.2. *Let X be an oriented closed Riemannian manifold. There exists a positive constant z which depends only on the Riemannian metric such that, for any section s of a Euclidean bundle over X, we have*

$$||s||_{T^1} \le z||s||_\infty$$

and

$$||s||_{T^1} \le z||s||_4.$$

The compactness of X leads to the following inequalities.

Proposition 4.3. *Let X be an oriented closed Riemannian manifold. There exists a positive constant z which depends only on the Riemannian metric such that, for any section s of a Euclidean bundle over X, we have*

$$||s||_1 \le z||s||_{T^1}$$

and

$$||s||_2 \le z||s||_{T^2}.$$

We will frequently use the following Kato inequality.

Proposition 4.4. *Let X be an oriented closed Riemannian manifold. For any section s of a Euclidean bundle over X with a Riemannian connection A, we have*

$$|\nabla |s|| \le |\nabla_A s|. \tag{5}$$

Proof. First assume that s does not vanish anywhere on X. The metric compatibility of A yields that

$$\nabla(s,s) = 2(\nabla_A s, s).$$

On the other hand, we have

$$\nabla(s,s) = 2|s| \cdot \nabla|s|.$$

Therefore, Cauchy-Schwartz inequality proves the inequality (5). Next we lift the restriction that s be nowhere zero by a standard approximation argument. This completes the proof. □

We will refer to the next inequality as the Leray inequality.

Proposition 4.5. *Let X be an oriented closed Riemannian manifold. There exists a positive constant z which depends only on the Riemannian metric such that, for any section s of a Euclidean bundle over X with a Riemannian connection A, we have*

$$\|s\|_{T^2} \le z\|s\|_{L^2_1}. \tag{6}$$

Proof. First we prove that, for any compact support smooth function f on $\mathbb{R}^4$, we have

$$\sup_{x\in\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|f(y)|^2}{\operatorname{dist}(x,y)^2}\, dy \le \frac{1}{4}\int_{\mathbb{R}^4} |\nabla f|^2\, dy. \tag{7}$$

Fix a point $x \in \mathbb{R}^4$, and let (r,θ) denote polar coordinate centered at x. Then, via integration by parts, we have

$$\int_{\mathbb{R}^4} \frac{|f(y)|^2}{\operatorname{dist}(x,y)^2}\, dy = \int_{\mathbb{R}^4} \frac{|f(y)|^2}{r^2}\, dy = \int_{S^3}\int_{\mathbb{R}} \frac{|f(y)|^2}{r^2}\cdot r^3 dr d\theta$$
$$= -\frac{1}{2}\int_{S^3}\int_{\mathbb{R}} r^2 f\frac{\partial f}{\partial r}\, dr d\theta = -\frac{1}{2}\int_{\mathbb{R}^4} \frac{f}{r}\frac{\partial f}{\partial r}\, dy.$$

Therefore, by Cauchy-Schwartz inequality, we get

$$\int_{\mathbb{R}^4} \frac{|f(y)|^2}{\operatorname{dist}(x,y)^2}\, dy = -\frac{1}{2}\int_{\mathbb{R}^4} \frac{f}{r}\frac{\partial f}{\partial r}\, dy \le \frac{1}{2}\left(\int_{\mathbb{R}^4} \frac{|f|^2}{r^2}\, dy\right)^{1/2}\left(\int_{\mathbb{R}^4} |\nabla f|^2\, dy\right)^{1/2}.$$

Hence, we obtain the inequality (7).

Next we note that there exists a constant z which depends only on the Riemannian metric such that

$$z^{-1}\frac{1}{\operatorname{dist}(x,y)^2} \le g(x,y) \le z\frac{1}{\operatorname{dist}(x,y)^2}$$

for all $x \neq y$ in X. Therefore, for an oriented closed Riemannian manifold X, by using a partition of unity, we have the inequality (6) for any smooth function on X. For a section s of a Euclidean bundle with a Riemannian connection A, use the Kato inequality (5). Thus, we have completed the proof. □

4.2. **Linear analysis.** The purpose of this section is to derive linear estimates for the Green operator $G_{A,E}$ and for the projection operator $\Pi_{E,A}$.

First we will derive estimates for the Green operator $G_{A,E}$.

Lemma 4.6. *For any $u \in \Omega^+(\mathfrak{g}_P)$ with $\Pi_{E,A}(u) = 0$,*

$$\|u\|_2^2 \le E^{-1}\|u\|_\infty \|d_A^+ d_A^* u\|_1.$$

Proof. By definition, $\Pi_{E,A}(u) = 0$ implies

$$\|u\|_2^2 \le E^{-1}\langle u, d_A^+ d_A^* u\rangle_{L^2}.$$

Thus,

$$\|u\|_2^2 \le E^{-1}\langle u, d_A^+ d_A^* u\rangle_{L^2} \le E^{-1}\|u\|_\infty \|d_A^+ d_A^* u\|_1.$$

□

Lemma 4.7. *Suppose E satisfies $0 < E < 1$. There exist constants ϵ and z which depend only on the Riemannian metric such that, if $\|F_A^+\|_2 < \epsilon$, then for any $u \in \Omega^+(\mathfrak{g}_P)$ with $\Pi_{E,A}(u) = 0$,*

$$\|\nabla_A u\|_2^2 \leq zE^{-1}\|u\|_\infty \|d_A^+ d_A^* u\|_1.$$

Proof. The Weitzenböck formula implies that

$$\nabla_A^* \nabla_A u = 2d_A^+ d_A^* u + R \cdot u + F_A^+ \cdot u,$$

where R is the curvature term of the base manifold X. Here R and F_A^+ act on $\Omega^+(\mathfrak{g}_P)$ via natural linear actions, which we do not need to make explicit. Then, using the formula

$$|\nabla_A u| + \frac{1}{2}\Delta |u|^2 = (u, \nabla_A^* \nabla_A u),$$

we have the pointwise identity

$$|\nabla_A u|^2 + \frac{1}{2}\Delta |u|^2 = (2d_A^+ d_A^* u, u) + (R \cdot u, u) + (F_A^+ \cdot u, u).$$

Integrating both side of the above equation over X, we have

$$\begin{aligned}\|\nabla_A u\|_2^2 &= 2\int_X (d_A^+ d_A^* u, u) + \int_X (R \cdot u, u) + \int_X (F_A^+ \cdot u, u) \\ &\leq 2\|d_A^+ d_A^* u\|_1 \|u\|_\infty + \|R\|_\infty \|u\|_2^2 + \int_X (F_A^+ \cdot u, u).\end{aligned}$$

Note that the integration of $\Delta |u|^2$ over X vanishes. Cauchy-Schwarz inequality and Sobolev inequality imply

$$\int_X (F_A^+ \cdot u, u) \leq \|F_A^+\|_2 \|u\|_4^2 \leq z\|F_A^+\|_2 (\|\nabla_A u\|_2^2 + \|u\|_2^2).$$

Combining the above estimates and Lemma 4.6, we have

$$\begin{aligned}\|\nabla_A u\|_2^2 &\leq 2\|d_A^+ d_A^* u\|_1 \|u\|_\infty + \|R\|_\infty \|u\|_2^2 + z\|F_A^+\|_2 (\|\nabla_A u\|_2^2 + \|u\|_2^2) \\ &= 2\|d_A^+ d_A^* u\|_1 \|u\|_\infty + (\|R\|_\infty + z\|F_A^+\|_2)\|u\|_2^2 + z\|F_A^+\|_2 \|\nabla_A u\|_2^2 \\ &\leq (2 + \|R\|_\infty + z\|F_A^+\|_2) E^{-1} \|d_A^+ d_A^* u\|_1 \|u\|_\infty + z\|F_A^+\|_2 \|\nabla_A u\|_2^2.\end{aligned}$$

The last line uses the assumption that $E < 1$. Thus, by choosing ϵ so small that $z\|F_A^+\|_2 \leq 1/2$, we may use rearrangement to bring the right-hand term $\|\nabla_A u\|_2^2$ to the left hand side, and we get the desired result. □

Lemma 4.8. *Suppose E satisfies $0 < E < 1$. There exist constants ϵ and z which depend only on the Riemannian metric such that, if $\|F_A^+\|_T < \epsilon$, then for any $u \in \Omega^+(\mathfrak{g}_P)$ with $\Pi_{E,A}(u) = 0$,*

$$\|u\|_{L_1^2}^2 \leq zE^{-1}\|u\|_\infty \|d_A^+ d_A^* u\|_1.$$

Proof. This follows directly from Lemma 4.6 and Lemma 4.7. Note that, by definition, $\|F_A^+\|_2 \leq \|F_A^+\|_T$. □

The next lemma shows a miraculous power of the Taubes norms.

Lemma 4.9. *Suppose E satisfies $0 < E < 1$. There exist constants ϵ and z which depend only on the Riemannian metric such that, if $\|F_A^+\|_T < \epsilon$, then for any $u \in \Omega^+(\mathfrak{g}_P)$ with $\Pi_{E,A}(u) = 0$,*

$$\|u\|_{TT}^2 \leq zE^{-1}\|u\|_\infty \|d_A^+ d_A^* u\|_{T^1}.$$

Proof. We have the following formula

$$|\nabla_A u| + \frac{1}{2}\Delta |u|^2 = (u, \nabla_A^* \nabla_A u),$$

and thus

$$\frac{1}{2}\left(\Delta|u|^2 + |u|^2\right) + |\nabla_A u| = \frac{1}{2}|u|^2 + (u, \nabla_A^* \nabla_A u).$$

Recall that $g(x, y)$ is the Green kernel of the Laplacian on functions $\Delta + 1$. Multiplying the both sides by $g(x, y)$ and integrating over X, we obtain

$$\begin{aligned}&\frac{1}{2}|u(x)|^2 + \int_X g(x, y)|\nabla_A u(y)|^2\, dy \\ &= \frac{1}{2}\int_X g(x, y)|u(y)|^2\, dy + \int_X g(x, y)(u(y), \nabla_A^* \nabla_A u(y))\, dy.\end{aligned}$$

We will estimate the right hand side. The Leray inequality (6) implies that

$$\frac{1}{2}\int_X g(x, y)|u(y)|^2\, dy \leq \frac{1}{2}\sup_{x\in X}\int_X g(x, y)|u(y)|^2\, dy = \frac{1}{2}||u||_{T^2}^2 \leq z||u||_{L_1^2}^2,$$

The Weitzenböck formula imply that

$$\int_X g(x, y)(u(y), \nabla_A^* \nabla_A u(y))\, dy = \int_X g(x, y)(u(y), 2d_A^+ d_A^* u(y) + R \cdot u(y) + F_A^+ \cdot u(y))\, dy$$

By definition, we have

$$2\int_X g(x, y)(u(y), d_A^+ d_A^* u(y))\, dy \leq 2\int_X g(x, y)|u(y)||d_A^+ d_A^* u(y)|\, dy \leq 2||u||_\infty ||d_A^+ d_A^* u||_{T^1}$$

and

$$\int_X g(x, y)(u(y), F_A^+ \cdot u(y))\, dy \leq ||u||_\infty^2 ||F_A^+||_{T^1}.$$

By the Leray inequality, we have

$$\int_X g(x, y)(u(y), R \cdot u(y))\, dy \leq ||R||_\infty \int_X g(x, y)|u(y)|^2\, dy \leq z||u||_{L_1^2}^2.$$

Thus, we have

$$\begin{aligned}&\frac{1}{2}\int_X g(x, y)|u(y)|^2\, dy + \int_X g(x, y)(u(y), \nabla_A^* \nabla_A u(y))\, dy \\ &\leq z\left(||d_A^+ d_A^* u||_{T^1}||u||_\infty + ||u||_\infty^2 ||F_A^+||_{T^1} + ||u||_{L_1^2}^2\right).\end{aligned}$$

Finally, by choosing ϵ small enough to rearrange the term $||u||_\infty^2 ||F_A^+||_{T^1}$ and using Lemma 4.8, we have the desired result. Note that $||F_A^+||_{T^1} \leq ||F_A^+||_T$ by definition. □

We can now prove our main estimate for the Green operator.

Proposition 4.10. *Suppose E satisfies $0 < E < 1$. There exist constants ϵ and z which depend only on the Riemannian metric such that, for any connection A with $||F_A^+||_T < \epsilon$ and for any $u \in \Omega^+(\mathfrak{g}_P)$ with $\Pi_{E,A}(u) = 0$, we have*

$$||G_{A,E}(u)||_{TT} \leq zE^{-1}||u||_{T^1}.$$

Proof. We will prove

$$||u||_{TT} \leq zE^{-1}||d_A^+ d_A^* u||_{T^1},$$

which is equivalent to the above inequality. By Lemma 4.9,

$$||u||_{TT}^2 \leq zE^{-1}||u||_\infty ||d_A^+ d_A^* u||_{T^1}.$$

On the other hand, by definition, $||u||_\infty \leq ||u||_{TT}$. Therefore, we get the desired result. □

Next we will derive an estimate for the projection operator $\Pi_{E,A}^\perp$. We will use the following proposition due to Taubes [Tau89, Appendix].

Proposition 4.11 (Taubes). *Suppose E satisfies* $0 < E < 1$. *There exists a constant* L_0 *which depends only on the Riemannian metric such that, for any connection A with* $\|F_A^+\|_2 \leq 1$, *the number of linearly independent eigenvectors of* $d_A^+ d_A^*$ *with eigenvalues less than E is no greater than* L_0, *that is, we have*

$$\dim\left[\mathrm{Im}(\Pi_{E,A})\right] \leq L_0.$$

We will also use the following elliptic regularity estimates.

Proposition 4.12. *Suppose E satisfies* $0 < E < 1$. *There exists positive constants z and* ϵ *which depend only on the Riemannian metric such that, for any connection A with* $\|F_A^+\|_2 \leq \epsilon$ *and for any* $u \in \mathrm{Im}(\Pi_{E,A})$ *with* $\|u\|_2 = 1$, *we have*

$$\|u\|_\infty \leq z, \quad \|u\|_{T^1} \leq z.$$

Proof. Let $\omega_1, \dots, \omega_L$ be an L^2-orthonormal basis for $\mathrm{Im}(\Pi_{E,A})$. The eigenvalue corresponding to ω_j is denoted by μ_j. First we estimate $\|\nabla_A \omega_j\|_2$. The Weitzenböck formula implies that

$$\nabla_A^* \nabla_A \omega_j = 2 d_A^+ d_A^* \omega_j + R \cdot \omega_j + F_A^+ \cdot \omega_j = 2\mu_j \omega_j + R \cdot \omega_j + F_A^+ \cdot \omega_j.$$

Therefore, we have, by the Sobolev inequality,

$$\|\nabla_A \omega_j\|_2^2 \leq (2\mu_j + z)\|\omega_j\|_2^2 + \|F_A^+\|_{4/3}^2 (\|\nabla_A \omega_j\|_2^2 + \|\omega_j\|_2^2)$$

Note that we have $\|F_A^+\|_{4/3} \leq z\|F_A^+\|_2 \leq z\|F_A^+\|_T$ since X is compact. Thus, if $\|F_A^+\|_T$ is small enough, we have

$$\|\nabla_A \omega_j\|_2 \leq z. \tag{8}$$

Here, we use $E < 1$ and $\|\omega_j\|_2 = 1$. On the other hand, the pointwise inequality

$$\Delta|\omega_j|^2 + |\omega_j|^2 \leq 2(\omega_j, \nabla_A^* \nabla_A \omega_j) + |\omega_j|^2$$

and the Weitzenböck formula imply that

$$\|\omega_j\|_\infty \leq z\|\omega_j\|_{T^1}.$$

These two inequalities together with Proposition 4.3 and the Sobolev inequality imply that

$$\|\omega_j\|_\infty \leq z.$$

Then, for any $u \in \mathrm{Im}(\Pi_{E,A})$ with $\|u\|_2 = 1$, we have

$$u = \sum_1^L \langle u, \omega_j \rangle_{L^2} \omega_j,$$

and so,

$$\|u\|_\infty = \|\sum_{j=1}^L \langle u, \omega_j \rangle_{L^2} \omega_j\|_\infty \leq \sum_{j=1}^L \|u\|_2 \|\omega_j\|_2 \|\omega_j\|_\infty \leq z L_0,$$

where L_0 is a constant introduced in Proposition 4.11. The estimates for $\|u\|_{T^1}$ directly follow from the estimate for $\|u\|_\infty$ via Proposition 4.3. We have completed the proof. □

Finally we will prove our linear estimate for the projection operator $\Pi_{E,A}^\perp$.

Proposition 4.13. *Suppose E satisfies* $0 < E < 1$. *There exists positive constant z and* ϵ *which depend only on the Riemannian metric such that, for any connection A with* $\|F_A^+\|_T \leq \epsilon$ *and for any* $\omega \in \Omega^+(\mathfrak{g}_P)$, *we have*

$$\|\Pi_{E,A}^\perp(\omega)\|_{T^1} \leq z\|\omega\|_{T^1}.$$

Proof. Let $\omega_1, \dots, \omega_L$ be an L^2-orthonormal basis for $\mathrm{Im}(\Pi_{E,A})$. For any $\omega \in \Omega^+(\mathfrak{g}_P)$,

$$\begin{aligned}
\|\Pi_{E,A}(\omega)\|_{T^1} &= \|\sum_{j=1}^{L} \langle \omega, \omega_j \rangle_{L^2} \cdot \omega_j \|_{T^1} \\
&\le \sum_{j=1}^{L} |\langle \omega, \omega_j \rangle_{L^2}| \cdot \|\omega_j\|_{T^1} \\
&\le \sum_{j=1}^{L} \|\omega\|_1 \|\omega_j\|_\infty \|\omega_j\|_{T^1} = \left(\sum_{j=1}^{L} \|\omega_j\|_{T^1} \|\omega_j\|_\infty \right) \|\omega\|_1 \\
&\le z^2 L_0 \|\omega\|_{T^1}.
\end{aligned}$$

The last line uses Proposition 4.11 and Proposition 4.12 and $\|\omega\|_1 \le \|\omega\|_{T^1}$. Then,

$$\|\Pi_{E,A}^{\perp}(\omega)\|_{T^1} = \|\omega - \Pi_{E,A}(\omega)\|_{T^1} \le \|\omega\|_{T^1} + \|\Pi_{E,A}(\omega)\|_{T^1} \le (1 + z^2 L_0)\|\omega\|_{T^1}.$$

This completes the proof. □

4.3. **Quadratic estimates.** In this section we will show the following quadratic estimates.

Proposition 4.14. *Suppose E satisfies $0 < E < 1$. There exist positive constants ϵ and z which depend only on the Riemannian metric such that, if $\|F_A^+\|_T < \epsilon$, then for any $u \in \Omega^+(\mathfrak{g}_P)$ with $\Pi_{E,A}(u) = 0$,*

$$\|G_{A,E} \circ \Pi_{E,A}^{\perp}[(d_A^* u \wedge d_A^* u)^+]\|_{TT} \le z E^{-1} \|u\|_{TT}^2.$$

Proof. By Proposition 4.10,

$$\|G_{A,E} \circ \Pi_{E,A}^{\perp}[(d_A^* u \wedge d_A^* u)^+]\|_{TT} \le z E^{-1} \|\Pi_{E,A}^{\perp}[(d_A^* u \wedge d_A^* u)^+]\|_{T^1}.$$

By Proposition 4.13,

$$\|\Pi_{E,A}^{\perp}[(d_A^* u \wedge d_A^* u)^+]\|_{T^1} \le z \|(d_A^* u \wedge d_A^* u)^+\|_{T^1}.$$

Since we have the pointwise bound

$$|(d_A^* u \wedge d_A^* u)^+| \le 8\sqrt{2} |\nabla_A u|^2,$$

we have

$$\|(d_A^* u \wedge d_A^* u)^+\|_{T^1} \le 8\sqrt{2} \||\nabla_A u|^2\|_{T^1}.$$

By definition of Taubes norms,

$$\||\nabla_A u|^2\|_{T^1} = \|\nabla_A u\|_{T^2}^2 \le \|u\|_{TT}^2.$$

We have thus proved the proposition. □

4.4. **Contraction mapping principle.**

Proposition 4.15. *Let X be an oriented closed Riemannian 4-manifold and $P \to X$ a principal* SU(2)*-bundle. Let E be a positive number with $0 < E < 1$. There exist positive constants ϵ_1 and z which depend only on the Riemannian metric and are independent of P such that, for any connection A on P with $\|F_A^+\|_T < \epsilon_1 E^2$, there exists a unique solution $u = u(A) \in \Omega^+(\mathfrak{g}_P)$ to the equation*

$$d_A^+ d_A^* u + \Pi_{E,A}^{\perp}\left[(d_A^* u \wedge d_A^* u)^+ + F_A^+\right] = 0$$

with the property $\Pi_{E,A}(u) = 0$. The solution u satisfies

$$\|u\|_{TT} \le z E^{-1} \|F_A^+\|_T,$$

and

$$\|d_A^* u\|_2 \le z E^{-1} \|F_A^+\|_T.$$

Proof. For any $u \in \Omega^+(\mathfrak{g}_P)$ with $\Pi_{E,A}(u) = 0$, we will define

$$S(u) := -G_{A,E} \circ \Pi^{\perp}_{E,A}\left[(d^*_A u \wedge d^*_A u)^+ + F^+_A\right].$$

Note that it follows immediately as in the proof of Proposition 4.14 that

$$\|G_{A,E} \circ \Pi^{\perp}_{E,A}(F^+_A)\|_{TT} \leq z_1 E^{-1}\|F^+_A\|_T. \tag{9}$$

Then, by Proposition 4.14 and (9),

$$\|S(u)\|_{TT} \leq z_1 E^{-1}\|F^+_A\|_T + z_2 E^{-1}\|u\|^2_{TT}.$$

Therefore, if $\|u\|_{TT} \leq E/(4z_2)$ and $\|F^+_A\|_T \leq E^2/(16 z_1 z_2)$, then S satisfies

$$\|S(u)\|_{TT} \leq z_1 E^{-1} \cdot \frac{E^2}{16 z_1 z_2} + z_2 E^{-1} \cdot \left(\frac{E}{4z_2}\right)^2 < \frac{E}{4z_2}.$$

On the other hand, by Proposition 4.14,

$$\|S(u_1) - S(u_2)\|_{TT} \leq z_2 E^{-1}(\|u_1\|_{TT} + \|u_2\|_{TT}) \cdot \|u_1 - u_2\|_{TT}.$$

Therefore, if $\|u\|_{TT} \leq E/(4z_2)$, then S satisfies

$$\|S(u_1) - S(u_2)\|_{TT} \leq \frac{1}{2}\|u_1 - u_2\|_{TT}.$$

To sum up, the sequence u_j defined by

$$u_{j+1} = S(u_j)$$

starting with $u_0 = 0$ is Cauchy with respect to the norm $\|\cdot\|_{TT}$, and so converge to a limit u in the completion of $\Omega^+(\mathfrak{g}_P) \cap \mathrm{Im}(\Pi^{\perp}_{E,A})$ under $\|\cdot\|_{TT}$. The regularity of u is proved by observing that S is also a contraction with respect to the norm $\|\cdot\|_{TTT}$ and that the contraction mapping principle provides the unique solution.

The limit u satisfies $u = S(u)$, and we have

$$\|u\|_{TT} = \|S(u)\|_{TT} \leq z_1 E^{-1}\|F^+_A\|_T + z_2 E^{-1}\|u\|^2_{TT} \leq z_1 E^{-1}\|F^+_A\|_T + \frac{1}{4}\|u\|_{TT}.$$

Thus,

$$\|u\|_{TT} \leq 2 z_1 E^{-1}\|F^+_A\|_T.$$

The estimate for $\|d^*_A u\|_2$ is proved similarly. Thus, we have completed the proof. □

5. Cut and Paste operation

In this section we will describe a cut and paste operation for connections. Our description is largely motivated by Taubes in [Tau82, Tau84, Tau89].

5.1. Instantons on $\mathbb{R}^4$ and S^4.

In this section we will review the properties of instantons on $\mathbb{R}^4$ and S^4.

We begin by writing down explicitly a principal SU(2)-bundle $P_{\mathbb{R}^4}$ over $\mathbb{R}^4$ and an anti-self-dual connection $A_{\mathbb{R}^4}$ on it. We identify $\mathbb{R}^4$ with the quaternion $\mathbb{H}$ and SU(2) with the unit quaternions. Let U_1 and U_2 denote $\mathbb{R}^4$ and $\mathbb{R}^4 \setminus \{0\}$ respectively. The principal SU(2)-bundle $P_{\mathbb{R}^4}$ is specified by the transition function $g_{12}\colon U_1 \cap U_2 \to \mathrm{SU}(2)$ given by $g_{12}(x) := x/|x|$. The connection $A_{\mathbb{R}^4}$ is specified by $\mathfrak{su}(2)$-valued 1-forms W_1 on U_1 and W_2 on U_2 given by

$$W_1(x) := \frac{1}{1+|x|^2}(\sigma_1 \otimes \theta_1 + \sigma_2 \otimes \theta_2 + \sigma_3 \otimes \theta_3),$$
$$W_1(x) = g_{12}(x) W_2(x) g^{-1}_{12}(x) + g_{12}(x) d g^{-1}_{12}(x),$$

where $\{\sigma_i\}$ is a standard orthonormal basis for $\mathfrak{su}(2)$ and $\{\theta_i\}$ is a standard basis for $\wedge^+$ written down explicitly as

$$\theta_1 := x_1 dx_2 - x_2 dx_1 - x_3 dx_4 + x_4 dx_3,$$
$$\theta_2 := x_1 dx_3 - x_3 dx_1 - x_4 dx_2 + x_2 dx_4,$$
$$\theta_3 := x_1 dx_4 - x_4 dx_1 - x_2 dx_3 + x_3 dx_2.$$

Next we describe a principal SU(2)-bundle P_{S^4} on S^4 and an anti-self-dual connection A_{S^4} on it. Let s denote the south pole of S^4, and let $\pi\colon \mathbb{R}^4 \to S^4 \setminus \{s\}$ be the inverse to the stereographic projection from s. The map π is a conformal diffeomorphism. We identify U_1 and $S^4 \setminus \{s\}$ via π. We can extend $P_{\mathbb{R}^4}$ and $A_{\mathbb{R}^4}$ as a principal SU(2)-bundle P_{S^4} over S^4 and an anti-self-dual connection A_{S^4} on it. By definition, we have $\pi^* P_{S^4} = P_{\mathbb{R}^4}$ and $\pi^* A_{S^4} = A_{\mathbb{R}^4}$. We have $c_2(P_{S^4}) = 1$ by simple calculation. The connection A_{S^4} is centered in the sense of Taubes [Tau89, p.174] that

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4} y^j |\pi^* F_{A_{S^4}}(y)|^2\, dy = 0$$

for $j = 1, \dots 4$, and

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4} |y|^2 |\pi^* F_{A_{S^4}}(y)|^2\, dy = 1,$$

where $\{y^1, \dots, y^4\}$ is the standard coordinate of $\mathbb{R}^4$.

We collect simple properties of $A_{\mathbb{R}^4}$ and A_{S^4}. A positive number λ defines a conformal diffeomorphism of $\mathbb{R}^4$ by pulling back the coordinate functions y to y/λ. The pull-back $\lambda^* A_{\mathbb{R}^4}$ is an anti-self-dual-connection on $\lambda^* P_{\mathbb{R}^4}$ and satisfies

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4} |y|^2 |\pi^* F_{A_{S^4}}(y)|^2\, dy = \lambda^2.$$

There exists a positive constant z which depends only on the Riemannian metric such that

$$\|F_{A_{S^4}}\|_T \le z.$$

Finally we will recall a priori estimates for anti-self-dual-connections on $\mathbb{R}^4$ due to Uhlenbeck [Uhl82a, Uhl82b] and Taubes [Tau88, Lemma 9.1 and Lemma 9.2].

Proposition 5.1 (Uhlenbeck, Taubes). *There exist positive constants κ and z with the following significance: Let A be an anti-self-dual-connection on the trivial* SU(2)*-bundle $P = B_r \times \mathrm{SU}(2)$, where B_r is a ball in $\mathbb{R}^4$ centered at the origin with radius r. Suppose that*

$$\|F_A\|_{L^2(B_r)} < \kappa.$$

Then, for any point x in B_r with $|x| \le r/2$, we have

$$|F_A(x)| \le z r^{-2} \|F_A\|_{L^2(B_r)}.$$

Let Γ denote the product connection on P. Suppose that $a := A - \Gamma$ is in exponential gauge with respect to the origin. Then, for any point x in B_r with $|x| \le r/2$, we have

$$|a(x)| \le z r^{-1} \|F_A\|_{L^2(B_r)}.$$

Proposition 5.2 (Uhleneck, Taubes). *There exist positive constants κ and z with the following significance: Let A be an anti-self-dual-connection on the trivial* SU(2)*-bundle $P = \mathbb{R}^4 \times \mathrm{SU}(2)$ Suppose that*

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4} |y|^2 |F_A(y)|^2 = \lambda^2.$$

Then, for any point x in $\mathbb{R}^4$ with $|x| \ge 1/\kappa^2$, we have

$$|F_A(x)| \le z|x|^{-4}.$$

Let Γ denote the product connection on P. Suppose that $a := A - \Gamma$ is in exponential gauge with respect to the infinity. Then, for any point x in B_r with $|x| \geq 1/\kappa^2$, we have

$$|a(x)| \leq zr^{-3}.$$

5.2. **Cut and paste operation for connections.** Let (X, g) be an oriented closed Riemannian 4-manifold and P_0 a principal SU(2)-bundle over X. Let x_0 be any point in X. Fix a positive integer K, and choose K distinct points $\{x_1, \ldots, x_K\}$ in X. We also require that each point x_j is distinct from the base point x_0. Define $r > 0$ to be the one half of the smallest of

- the minimum geodesic distance between x_0 and x_j for $j = 1, \ldots, K$,
- the minimum geodesic distance between distinct points x_i and x_j, or
- the injectivity radius of the Riemannian manifold X.

Let $\boldsymbol{\lambda} = \{\lambda_1, \cdots, \lambda_K\}$ be a K-tuple of positive numbers satisfying

$$8\lambda_i + 8\lambda_j < \mathrm{dist}_g(x_i, x_j).$$

Let B_j denote the ball in X centered at x_j with radius r.

Let A_0 be a connection on P_0. Fix a point h_0 in $P_0|_{x_0}$ and a point h_j in $P_0|_{x_j}$ at each x_j. Parallel transport h_j along the radial geodesics through x_j to define a local section $\sigma(A, h_j)$ of $P_0|_{B_j}$. This section identifies $P_0|_{B_j}$ with $B_j \times \mathrm{SU}(2)$.

Let P_{S^4} and A_{S^4} be the principal SU(2)-bundle and the anti-self-dual connection written down explicitly in the previous section. Fix a point $h \in P_{S^4}|_s$, where s is the south pole. Recall that the inverse to a stereographic projection π defines a conformal diffeomorphism. Use this π to pull each A_{S^4} back to $\mathbb{R}^4$ as an anti-self-dual connection on $\pi^* P_{S^4}$. For each λ_j, $\lambda_j^* \pi^* A_{S^4}$ is an anti-self-dual connection on $\lambda_j^* \pi^* P_{S^4}$. By parallel transport of h by the connection $\lambda_j^* \pi^* A_{S^4}$ along the radial geodesics through the south pole on S^4, a section $\sigma_j(A_{S^4}, h, \lambda_j)$ of $\lambda_j^* \pi^* P_{S^4}|_{\mathbb{R}^4 \setminus \{0\}}$ is defined. Thus, we can identify $\lambda_j^* \pi^* P_{S^4}|_{\mathbb{R}^4 \setminus \{0\}}$ with $(\mathbb{R}^4 \setminus \{0\}) \times G$.

The set of all Gaussian coordinate system on X is parametrized by the oriented orthonormal frame bundle $\mathrm{Fr}(X) \to X$ via the exponential maps. Fix an oriented orthonormal frame $f_j \in \mathrm{Fr}(X)|_{x_j}$ at each point x_j. Each frame f_j defines a Gaussian coordinate system ϕ_j.

We will refer to the set

$$\{x_0; x_1, \ldots, x_K; f_1, \ldots, f_K; P_0, A_0, h_0; h_1, \ldots, h_K\}$$

as the gluing data.

For any $\rho > 0$, define a cut-off function $\beta_{j,\rho} \colon X \to [0, 1]$ by setting

$$\beta_{j,\rho}(x) := \beta\left(\mathrm{dist}_g(x_j, x)/\rho\right), \tag{10}$$

where $\beta \colon \mathbb{R} \to [0, 1]$ is a smooth function satisfying $\beta(t) = 1$ for $t \geq 1$ and $\beta(t) = 0$ for $t \leq 1/2$.

Definition 5.3. Define a new principal SU(2)-bundle $P(\boldsymbol{\lambda})$ over X and a new connection $A(\boldsymbol{\lambda})$ on $P(\boldsymbol{\lambda})$ as follows: If $\mathrm{dist}_g(x_j, \cdot) < \sqrt{\lambda_j}/4$, identify $P(\boldsymbol{\lambda})$ with $\lambda_j^* \pi^* P_{S^4}|_{\mathbb{R}^4 \setminus \{0\}}$ using the Gaussian coordinate system ϕ_j, and set

$$A(\boldsymbol{\lambda}) = \lambda_j^* \pi^* A_{S^4}.$$

If $4\sqrt{\lambda_j} < \mathrm{dist}_g(x_j, \cdot)$, identify $P(\boldsymbol{\lambda})$ with P_0 and set

$$A(\boldsymbol{\lambda}) = A_0.$$

If $\sqrt{\lambda_j}/4 \leq \mathrm{dist}_g(x_j, \cdot) \leq 4\sqrt{\lambda_j}$, identify $P(\boldsymbol{\lambda})$ with the trivial product bundle and set

$$A(\boldsymbol{\lambda}) = \Gamma + \beta_{j,4\sqrt{\lambda_j}} \cdot \sigma(A_0, h_j)^* A_0 + (1 - \beta_{j,\sqrt{\lambda_j}/2}) \cdot \sigma(A_{S^4}, h, \lambda_j)^* \lambda_j^* \pi^* A_{S^4},$$

where Γ is the product connection.

It is straightforward to check that $P(\boldsymbol{\lambda})$ and $A(\boldsymbol{\lambda})$ are well-defined. We have $c_2(P(\boldsymbol{\lambda})) = K + c_2(P_0)$ for any $\boldsymbol{\lambda}$, and therefore all the bundles $P(\boldsymbol{\lambda})$ are mutually isomorphic. It is very important in the subsequent sections to note that we can canonically identify P_0 with $P(\boldsymbol{\lambda})$ on $X \setminus \{x_1, \dots x_K\}$.

5.3. **Estimate of the self-dual curvature.** In this section we will derive the estimates for self-dual curvature of connections produced by the cut and paste operation in Section 5.2.

Let (X, g) be an oriented closed Riemannian 4-manifold and P_0 a principal SU(2)-bundle over X. Fix a gluing data

$$\{x_0; x_1, \dots, x_K; f_1, \dots, f_K; P_0, A_0, h_0; h_1, \dots, h_K\}.$$

Then, we can construct a principal SU(2)-bundle $P(\boldsymbol{\lambda})$ over X and a connection $A(\boldsymbol{\lambda})$ for any $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_K\}$ with $\lambda := \max \lambda_j \ll 1$.

Proposition 5.4. *There exist constant z and Λ, where z depends only on the Riemannian metric and Λ depends on the Riemannian metric and $||F_{A_0}||_2$, such that, if $\lambda < \Lambda$, then we have*

$$||F_A^+||_T \leq ||F_{A_0}^+||_T + z\lambda,$$

and

$$||F_A^+||_p \leq ||F_{A_0}^+||_p + z\lambda^{2/p}$$

for any $p \geq 1$.

Proof. The proof is now standard: Break X into five parts and compute the integrals over each part separately by using the estimates of Proposition 5.1 and Proposition 5.2.

□

5.4. **The small eigenvalues of the Laplacian.** In this section we will analyze the behavior of the small eigenvalues of the Laplacian for connections produced by the cut and paste operation in Section 5.2. The main result of this section is Proposition 5.11

Let (X, g) be an oriented closed Riemannian 4-manifold and P_0 a principal SU(2)-bundle over X. Fix a gluing data

$$\{x_0; x_1, \dots, x_K; f_1, \dots, f_K; P_0, A_0, h_0; h_1, \dots, h_K\}.$$

Then, we can construct a principal SU(2)-bundle $P(\boldsymbol{\lambda})$ over X and a connection $A(\boldsymbol{\lambda})$ on $P(\boldsymbol{\lambda})$ for any $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_K\}$ with $\lambda := \max \lambda_j \ll 1$.

Fix a positive number E with $0 < 4E < 1$.

Let $\{\omega_i(A_0)\}_{i=1}^{\infty}$ denote L^2-orthonormal basis of $L^2(\mathfrak{g}_{P_0} \otimes \wedge^+)$ consisting of the eigenvalues of the Laplacian $d_{A_0}^+ d_{A_0}^*$ and $\{\mu_i(A_0)\}_{i=1}^{\infty}$ the corresponding eigenvalues, repeated according to their multiplicity and in ascending order. Define the projection operator

$$\Pi_{4E, A_0} : L^2(\mathfrak{g}_{P_0} \otimes \wedge^+) \to L^2(\mathfrak{g}_{P_0} \otimes \wedge^+)$$

to be the finite rank spectral projections onto the subspace of L^2 spanned by the eigenvectors with eigenvalues less than $4E$. Let L denote the dimension of $\mathrm{Im}[\Pi_{4E, A_0}]$. Thus, $\mu_1(A_0) \leq \dots \leq \mu_L(A_0) < 4E \leq \mu_{L+1}(A_0)$.

We define a cut-off function $\gamma : X \to [0, 1]$ by

$$\gamma(x) := \beta\left(\frac{\mathrm{dist}_g(x, x_1)}{\lambda_1}\right) \times \dots \times \beta\left(\frac{\mathrm{dist}_g(x, x_K)}{\lambda_K}\right), \tag{11}$$

where $\beta : \mathbb{R} \to [0, 1]$ is a smooth function such that $\beta(t) = 1$ for $t \geq 1$ and $\beta(t) = 0$ for $t \leq 1/2$. The subset $\{x \in X \mid \mathrm{dist}_g(x, x_j) \leq \lambda_j \text{ for some } x_j\}$ is denoted by B. The support of $(1 - \gamma)$ is contained in B. By Proposition 4.11, there exists a positive constant z which depends only on the Riemannian metric such that

$$\mathrm{vol}(B) \leq zK\lambda^4. \tag{12}$$

Fix some $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_K\}$ with $\lambda := \max \lambda_j \ll 1$, and denote $P(\boldsymbol{\lambda})$ and $A(\boldsymbol{\lambda})$ by P and A. We construct a set of approximate eigenvectors $\omega_i(A_0)'$ of $d_A^+ d_A^*$ by setting

$$\omega_i(A_0)' := \gamma\omega_i(A_0).$$

Since P_0 and P are canonically identified on $X \setminus \{x_1, \dots, x_K\}$, we can regard $\omega_i(A_0)'$ as elements of $L^2(\mathfrak{g}_P \otimes \wedge^+)$. Let $\Pi_{4E,A}$ denote the the projection operator defined by A. We will show that $\{\Pi_{4E,A}[\omega_i(A_0)']\}$ are almost orthonormal in $L^2(\mathfrak{g}_P \otimes \wedge^+)$.

Lemma 5.5. *There exist positive constants z and Λ, where z only depends on the Riemannian metric and Λ depends on the Riemannian metric, A_0, such that, if $\lambda < \Lambda$, then*

$$\|d_A^* \omega_j(A_0)'\|_2^2 \le \mu_j(A_0) + zK\lambda$$

for $j = 1, \dots, L$.

Proof. We will divide X into three parts, and the integrals will be estimated over each part separately. Let $B_j(r)$ denote the geodesic ball in X centered at x_j with radius r.

First, on $X \setminus \bigcup B_j(4\sqrt{\lambda_j})$, we have $\omega_j(A_0)' = \omega_j(A_0)$, and we canonically identified $A(\boldsymbol{\lambda})$ with A_0. Thus,

$$\int_{X \setminus \bigcup B_j} |d_A^* \omega_j(A_0)'|^2 = \int_{X \setminus \bigcup B_j} |d_{A_0}^* \omega_j(A_0)|^2.$$

Next, on $B_j(4\sqrt{\lambda_j}) \setminus B_j(\sqrt{\lambda_j})$, we identified $P(\boldsymbol{\lambda})$ with P_0 via $\sigma(A_0, h_j)$, and

$$A(\boldsymbol{\lambda}) = \beta \cdot \sigma^* A_0 = \sigma^* A_0 + (\beta - 1) \cdot \sigma^* A_0,$$

where $\beta = \beta_{j, 4\sqrt{\lambda_j}}$ and $\sigma = \sigma(A_0, h_j)$. We also have $\omega_j(A_0)' = \omega_j(A_0)$ on this region. Thus,

$$d_A^* \omega_j(A_0)' = *d_A \omega_j(A_0)' = *(d_{\sigma^* A_0} \omega_j(A_0) + (\beta - 1)\sigma^* A_0 \wedge \omega_j(A_0)).$$

Choose Λ small enough so that $\|F_{A_0}\|_2$ is so small to apply Proposition 5.1. Since $\sigma^* A_0$ is in exponential gauge, we have

$$|\sigma^* A_0| \le z\lambda_j^{-1/2}.$$

Therefore, via Proposition 4.12,

$$\int_{B_j(4\sqrt{\lambda_j}) \setminus B_j(\sqrt{\lambda_j})} |(\beta - 1)\sigma^* A_0 \wedge \omega_j(A_0)|^2 \le z\lambda_j.$$

Thus,

$$\int_{B_j(4\sqrt{\lambda_j}) \setminus B_j(\sqrt{\lambda_j})} |d_A^* \omega_j(A_0)'|^2 \le \int_{B_j(4\sqrt{\lambda_j}) \setminus B_j(\sqrt{\lambda_j})} |d_{A_0}^* \omega_j(A_0)|^2 + z\lambda_j.$$

Finally, on $B_j(\sqrt{\lambda_j})$, in the same way as above, we have

$$\int_{B_j(\sqrt{\lambda_j})} |d_A^* \omega_j(A_0)'|^2 \le z\lambda_j.$$

Therefore, we have

$$\begin{aligned}
&\int_X |d_A^* \omega_j(A_0)'|^2 \\
&\le \int_{X \setminus \bigcup B_j(4\sqrt{\lambda_j})} |d_{A_0}^* \omega_j(A_0)|^2 + \sum_{j=1}^{K} \left(\int_{B_j(4\sqrt{\lambda_j}) \setminus B_j(\sqrt{\lambda_j})} |d_{A_0}^* \omega_j(A_0)|^2 + z\lambda + z\lambda \right) \\
&\le \mu_j(A_0) + zK\lambda
\end{aligned}$$

We have completed the proof. □

In the same way we can prove the following.

Lemma 5.6. *There exists a positive constant z which depends only on the Riemannian metric such that*

$$|\langle \omega_i', \omega_j' \rangle_{L^2} - \delta_{ij}| \le zK\lambda^4,$$

for $i = 1, \dots, L$.

Proof. By using the identity $\omega_j' = \omega_j + (\gamma - 1)\omega_j$, we have

$$\langle \omega_i', \omega_j' \rangle_{L^2} = \langle \omega_i, \omega_j \rangle_{L^2} + 2\langle (\gamma - 1)\omega_i, \omega_j \rangle_{L^2} + \langle (\gamma - 1)\omega_i, (\gamma - 1)\omega_j \rangle_{L^2}.$$

By definition, $\langle \omega_i, \omega_j \rangle_{L^2} = \delta_{ij}$, and, by (12) and Proposition 4.12, we have

$$\begin{aligned} &|2\langle (\gamma - 1)\omega_i, \omega_j \rangle_{L^2} + \langle (\gamma - 1)\omega_i, (\gamma - 1)\omega_j \rangle_{L^2}| \\ \le& 2\mathrm{vol}(B)\|\omega_i\|_\infty \|\omega_j\|_\infty + \mathrm{vol}(B)\|\omega_i\|_\infty \|\omega_j\|_\infty \le 3zK\lambda^4. \end{aligned}$$

This completes the proof. □

The following proposition shows that $\{\Pi_{4E,A}[\omega_i(A_0)']\}$ are almost orthonormal.

Proposition 5.7. *There exist positive constants z and* Λ, *where z only depends on the Riemannian metric and* Λ *depends on the Riemannian metric and* A_0, *such that, if* $\lambda < \Lambda$, *then*

$$|\langle \Pi_{4E,A}[\omega_i(A_0)'], \Pi_{4E,A}[\omega_j(A_0)'] \rangle_{L^2} - \delta_{ij}| \le zK\lambda,$$

for $i = 1, \dots, L$.

Proof. By Lemma 5.6 and Lemma **??**, we have

$$\begin{aligned} &|\langle \Pi_{4E,A}(\omega_i(A_0)'), \Pi_{4E,A}(\omega_j(A_0)') \rangle_{L^2} - \delta_{ij}| \\ \le& |\langle \Pi_{4E,A}(\omega_i(A_0)'), \Pi_{4E,A}(\omega_j(A_0)') \rangle_{L^2} - \langle \omega_i(A_0)', \omega_j(A_0)' \rangle_{L^2}| + |\langle \omega_i(A_0)', \omega_j(A_0)' \rangle_{L^2} - \delta_{ij}| \\ =& |\langle \Pi^\perp_{4E,A}(\omega_i(A_0)'), \Pi^\perp_{4E,A}(\omega_j(A_0)') \rangle_{L^2}| + |\langle \omega_i(A_0)', \omega_j(A_0)' \rangle_{L^2} - \delta_{ij}| \\ \le& \|\Pi^\perp_{4E,A}(\omega_i(A_0)'\|_2 \|\Pi^\perp_{4E,A}(\omega_i(A_0)'\|_2 + |\langle \omega_i(A_0)', \omega_j(A_0)' \rangle_{L^2} - \delta_{ij}| \\ \le& z\sqrt{\lambda} \cdot z\sqrt{\lambda} + zK\lambda^4 \le zK\lambda. \end{aligned}$$

Thus, we have completed the proof. □

Corollary 5.8. *There exist positive constants z and* Λ, *where z only depends on the Riemannian metric and* Λ *depends on the Riemannian metric and* A_0, *such that, if* $\lambda < \Lambda$, *then the linear map from* $\mathrm{Im}[\Pi_{4E,A_0}]$ *to* $\mathrm{Im}[\Pi_{4E,A}]$ *defined by*

$$\omega_i(A_0) \mapsto \Pi_{4E,A}(\omega_i(A_0)')$$

is injective.

Next we will analyze the small eigenvalue behavior of the Laplacian $d_A^+ d_A^*$. Recall we have denoted $P(\lambda)$ and $A(\lambda)$ by P and A. Let $\{\omega_i(A)\}_{i=1}^\infty$ denote L^2-orthonormal basis of $L^2(\mathfrak{g}_P \otimes \wedge^+)$ consisting of the eigenvalues of the Laplacian $d_A^+ d_A^*$ and $\{\mu_i(A)\}_{i=1}^\infty$ the corresponding eigenvalues, repeated according to their multiplicity and in ascending order. Let N denote the dimension of $\mathrm{Im}[\Pi_{E,A}]$. Thus, $\mu_1(A) \le \cdots \le \mu_N(A) < E \le \mu_{N+1}(A)$. Set

$$\mu_i(A)' := \gamma\mu_i(A),$$

where γ is the cut-off function defined in (11).

Lemma 5.9. *There exist positive constants z and* Λ, *where z only depends on the Riemannian metric and* Λ *depends on the Riemannian metric,* A_0 *and K, such that, if* $\lambda < \Lambda$, *then*

$$\|d_{A_0}^* \omega_i(A)'\|_2^2 \le \mu_i(A) + zK\lambda$$

for $i = 1, \dots, N$.

Proof. We will divide X into three parts, and the integrals will be estimated over each part separately. Let $B_j(r)$ denote the geodesic ball in X centered at x_j with radius r.

First, on $X \setminus \bigcup B_j(4\sqrt{\lambda_j})$, we have $\omega_j(A_0)' = \omega_j(A_0)$, and we canonically identified $A(\boldsymbol{\lambda})$ with A_0. Thus,

$$\int_{X \setminus \bigcup B_j} |d_A^* \omega_j(A_0)'|^2 = \int_{X \setminus \bigcup B_j} |d_{A_0}^* \omega_j(A_0)|^2.$$

Next, on $B_j(4\sqrt{\lambda_j}) \setminus B_j(\sqrt{\lambda_j})$, we identified $P(\boldsymbol{\lambda})$ with P_0 via $\sigma(A_0, h_j)$, and

$$A(\boldsymbol{\lambda}) = \beta \cdot \sigma^* A_0 = \sigma^* A_0 + (\beta - 1) \cdot \sigma^* A_0,$$

where $\beta = \beta_{j,4\sqrt{\lambda_j}}$ and $\sigma = \sigma(A_0, h_j)$. We also have $\omega_j(A_0)' = \omega_j(A_0)$ on this region. Thus,

$$d_A^* \omega_j(A_0)' = *d_A \omega_j(A_0)' = *(d_{\sigma^* A_0} \omega_j(A_0) + (\beta - 1)\sigma^* A_0 \wedge \omega_j(A_0)).$$

Choose Λ small enough so that $\|F_{A_0}\|_2$ is so small to apply Proposition 5.1. Since $\sigma^* A_0$ is in exponential gauge, we have

$$|\sigma^* A_0| \le z \lambda_j^{-1/2}.$$

Therefore, via Proposition 4.12,

$$\int_{B_j(4\sqrt{\lambda_j}) \setminus B_j(\sqrt{\lambda_j})} |(\beta - 1)\sigma^* A_0 \wedge \omega_j(A_0)|^2 \le z \lambda_j.$$

Thus,

$$\int_{B_j(4\sqrt{\lambda_j}) \setminus B_j(\sqrt{\lambda_j})} |d_A^* \omega_j(A_0)'|^2 = \int_{B_j(4\sqrt{\lambda_j}) \setminus B_j(\sqrt{\lambda_j})} |d_{A_0}^* \omega_j(A_0)|^2 + z \lambda_j.$$

Finally, on $B_j(\sqrt{\lambda_j})$, in the same way as above, we have

$$\int_{B_j(\sqrt{\lambda_j})} |d_A^* \omega_j(A_0)'|^2 \le z \lambda_j.$$

We have completed the proof. □

We will estimate upper bounds of $\mu_i(A)$. For every i, let $\langle \omega_1(A), \ldots, \omega_i(A) \rangle$ denote the linear span of $\omega_1(A), \ldots, \omega_i(A)$ in $L^2(\mathfrak{g}_P \otimes \wedge^+)$, and let $\langle \omega_1(A), \ldots, \omega_i(A) \rangle^\perp$ be its L^2-orthogonal complement in $L^2(\mathfrak{g}_P \otimes \wedge^+)$. By the Courant-Hilbert min-max principle, we have

$$\mu_i(A) = \inf \left\{ \frac{\|d_A^* \omega\|_2^2}{\|\omega\|_2^2} \mid \omega \in \langle \omega_1(A), \ldots, \omega_{i-1}(A) \rangle^\perp \right\}. \tag{13}$$

Proposition 5.10. *There exist positive constant z and Λ which depends on the Riemannian metric and A_0 such that, if $\lambda < \Lambda$, then*

$$\mu_i(A) \le 4(\mu_i(A_0) + zK\lambda)$$

for $j = 1, \ldots, L$.

Proof. By Lemma 5.6, if $z\lambda^2 \le 1/2$, we have $\|\omega_i(A_0)'\|_2 \ge 1/2$. Therefore, by Lemma 5.9,

$$\|d_A^* \omega_i(A_0)'\|_2^2 \le \mu_i(A_0) + zK\lambda \le (\mu_i(A_0) + zK\lambda) \cdot 4\|\omega_i(A_0)\|_2^2. \tag{14}$$

Let $\langle \omega_1(A_0)', \ldots, \omega_L(A_0)' \rangle$ denote the linear span of $\omega_1(A_0)', \ldots, \omega_L(A_0)'$ in $L^2(\mathfrak{g}_P \otimes \wedge^+)$ The above inequality (14) implies that, for any $\omega \in \langle \omega_1(A_0)', \ldots, \omega_L(A_0)' \rangle$, we have

$$\|d_A^* \omega\|_2^2 \le 4(\mu_i + z\lambda)\|\omega\|_2^2.$$

By the Courant-Hilbert min-max principle (13), we have

$$\mu_i(A) = \inf \left\{ \frac{\|d_A^* \omega\|_2^2}{\|\omega\|_2^2} \mid \omega \in \langle \omega_1(A), \ldots, \omega_{i-1}(A) \rangle^\perp \right\}.$$

Simple dimension counting yields

$$\dim[\langle \omega_1(A_0)', \ldots, \omega_L(A_0)' \rangle \cap \langle \omega_1(A), \ldots, \omega_{i-1}(A) \rangle^{\perp}] \geq 1.$$

Therefore, we have

$$\begin{aligned}\mu_i(A) &= \inf \left\{ \frac{\|d_A^* \omega\|_2^2}{\|\omega\|_2^2} \mid \omega \in \langle \omega_1(A), \ldots, \omega_{i-1}(A) \rangle^{\perp} \right\} \\ &\leq \inf \left\{ \frac{\|d_A^* \omega\|_2^2}{\|\omega\|_2^2} \mid \omega \in \langle \omega_1(A_0)', \ldots, \omega_L(A_0)' \rangle \cap \langle \omega_1(A), \ldots, \omega_{i-1}(A) \rangle^{\perp} \right\} \\ &\leq 4(\mu_i(A_0) + zK\lambda).\end{aligned}$$

This completes the proof. □

The next proposition is the main result of this section.

Proposition 5.11. *There exist positive constants z and Λ_2, where z only depends on the Riemannian metric and Λ_2 depends on the Riemannian metric and A_0, such that, if $\lambda < \Lambda_2$, then* $\mathrm{Im}[\Pi_{E,A}]$ *is contained in the linear span of* $\Pi_{4E,A}(\omega_1(A_0)'), \ldots, \Pi_{4E,A}(\omega_L(A_0)')$ *in* $L^2(\mathfrak{g}_P \otimes \wedge^+)$.

Proof. Suppose an element ω in $\mathrm{Im}[\Pi_{E,A}]$ satisfies

$$\langle \omega, \Pi_{4E,A}(\omega_i(A_0)') \rangle_{L^2} = 0$$

for every $i = 1, \ldots, L$. Since ω is contained in $\mathrm{Im}[\Pi_{E,A}]$, we have

$$\langle \omega, \Pi_{4E,A}(\omega_L(A_0)') \rangle_{L^2} = \langle \omega, \omega_i(A_0)' \rangle_{L^2}.$$

On the other hand, we have

$$\langle \omega, \omega_i(A_0)' \rangle_{L^2} = \langle \omega, \gamma\omega_i(A_0) \rangle_{L^2} = \langle \gamma\omega, \omega_i(A_0) \rangle_{L^2}.$$

Since $\{\omega_1(A_0), \ldots, \omega_L(A_0)\}$ is the basis of $\mathrm{Im}[\Pi_{4E,A_0}]$, the assumption implies that $\gamma\omega$ is perpendicular to $\mathrm{Im}[\Pi_{4E,A_0}]$. However, Proposition 5.10 shows that $\gamma\omega$ is contained in $\mathrm{Im}[\Pi_{4E,A_0}]$. Therefore, $\gamma\omega \equiv 0$. Since $\omega \mapsto \gamma\omega$ is injective, we have $\omega \equiv 0$. We have completed the proof. □

6. The Kuranishi map

6.1. **The Kuranishi map.** In this section we will define the Kuranishi map.

We begin by paraphrasing the results of the previous two sections. Given a gluing data

$$\{x_0; x_1, \ldots, x_K; f_1, \ldots, f_K; P_0, A_0, h_0; h_1, \ldots, h_K\},$$

we constructed a principal SU(2)-bundle $P(\boldsymbol{\lambda})$ over X and a connection $A(\boldsymbol{\lambda})$ on $P(\boldsymbol{\lambda})$ for any $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_K\}$ with $\lambda := \max \lambda_j \ll 1$. The bundles $P(\boldsymbol{\lambda})$ are mutually isomorphic with $c_2(P(\boldsymbol{\lambda})) = K + c_2(P_0)$. Choose and fix a principal SU(2)-bundle P with $c_2(P) = K + c_2(P_0)$. For each $\boldsymbol{\lambda}$, two isomorphisms from P to $P(\boldsymbol{\lambda})$ differ by an element in the gauge group $\mathcal{G}(P)$. For $\delta > 0$ we define C'_δ to be

$$C'_\delta(P) := \{(A, h) \in C(P) \times P|_{x_0} \mid \|F_A^+\|_T < \delta\}.$$

The gauge group $\mathcal{G}(P)$ acts on $C'_\delta(P)$, and we denote the quotient by $\mathcal{B}'_\delta(P)$. Since the gauge group also acts on the space of self-dual 2-forms $\Omega^+(\mathfrak{g}_P)$, we can form the associated infinite-dimensional vector bundle $\Omega(P) := C'_\delta(P) \times_{\mathcal{G}} \Omega^+(\mathfrak{g}_P)$ on $\mathcal{B}'_\delta(P)$. Fix a positive number E with $0 < 4E < 1$. Then, if $\delta \ll 1$, Proposition 4.15 yields a canonical section s_{4E} of $\Omega(P)$ by

$$[A, h] \mapsto [(A, h), \Pi_{E,A}([d_A^* u(A) \wedge d_A^* u(A)]^+ + F_A^+)],$$

where $u(A)$ is a unique solution of the anti-self-dual equation (3).

Proposition 5.4 implies that, if $\|F_{A_0}\|_T \ll 1$ and $\lambda \ll 1$, then $[A(\boldsymbol{\lambda}), h_0]$ is contained in $\mathcal{B}'_\delta(P)$.

In summary, there exist positive constants ϵ and Λ which depend on the Riemannian metric and E such that, given a gluing data with $\|F_{A_0}\|_T < \epsilon$, we can define a map σ_E from $(0,\Lambda)^K$ to $\Omega(P)$ by

$$\boldsymbol{\lambda} \mapsto [(A(\boldsymbol{\lambda}), h_0), \Pi_{E,A(\boldsymbol{\lambda})}([d_A^* u(A(\boldsymbol{\lambda})) \wedge d_A^* u(A(\boldsymbol{\lambda}))]^+ + F^+_{A(\boldsymbol{\lambda})}].$$

Note that $\sigma_E(\boldsymbol{\lambda}) = 0$ if and only if $A(\boldsymbol{\lambda}) + d^*_{A(\boldsymbol{\lambda})} u(A(\boldsymbol{\lambda}))$ has the anti-self-dual curvature.

Choose and fix an orthonormal basis $\{\omega_1(A_0), \ldots, \omega_L(A_0)\}$ for $\mathrm{Im}[\Pi_{4E,A_0}]$ consisting of the eigenvectors of the Laplacian $d^+_{A_0} d^*_{A_0}$. Then, Proposition 5.11 implies that, if $\lambda \ll 1$, then $\mathrm{Im}[\Pi_{E,A(\boldsymbol{\lambda})}]$ is contained in the linear span of $\{\Pi_{4E,A(\boldsymbol{\lambda})}(\omega_j(A_0)')\}$. Therefore, $\sigma_E(\boldsymbol{\lambda}) = 0$ if and only if

$$\langle \Pi_{4E,A(\boldsymbol{\lambda})}(\omega_j(A_0)'), (d_A^* u(A(\boldsymbol{\lambda})) \wedge d_A^* u(A(\boldsymbol{\lambda})))^+ + F^+_{A(\boldsymbol{\lambda})} \rangle_{L^2} = 0$$

for all $j = 1, \ldots, L$. Now we arrive:

Definition 6.1. There exist positive constant ϵ and Λ which depends on the Riemannian metric and E such that, given a gluing data with $\|F^+_{A_0}\|_T < \epsilon$, then we can define a map Ψ_E from $(0,\Lambda)^K$ to $\mathbb{R}^L$ by

$$\boldsymbol{\lambda} \mapsto \langle \Pi_{4E,A(\boldsymbol{\lambda})}(\omega_j(A_0)'), (d_A^* u(A(\boldsymbol{\lambda})) \wedge d_A^* u(A(\boldsymbol{\lambda})))^+ + F^+_{A(\boldsymbol{\lambda})} \rangle_{L^2}.$$

The map Ψ_E is called the Kuranishi map.

The rest of this paper is devoted to finding the zero's of the Kuranishi map Ψ_E.

6.2. **Approximation of the Kuranishi map.** In this section we will approximate the Kuranishi map Ψ_E.

Proposition 6.2. *There exists positive constant Λ and ϵ which depends only on the Riemannian metric such that, for any $\lambda < \Lambda$ and for any connection A with $\|F_A^+\|_T < \epsilon$, we have*

$$\begin{aligned}
&\langle \omega_l', \Pi_{E,A}[(d_A^* u \wedge d_A^* u)^+ + F_A^+] \rangle_{L^2} \\
&= - \sum_{k=1}^{K} \sqrt{2}\pi^2 \lambda_j^2 \left[\sum_{i=1}^{3} (\omega_l(x_k), (h_j \cdot \sigma_i \cdot h_j^{-1}) \otimes \theta_i) \right] + O(\lambda^{5/2}) + O(\epsilon).
\end{aligned}$$

Proof. It is a routine matter. We simply mimic the proof of [Tau84, Proposition 5.4.] by using the estimates in 4.15 and the explicit formula for A_{S^4}. □

6.3. **Approximate Kuranishi maps.** In this section we will define approximate Kuranishi maps.

We begin by introducing two maps I and Γ. First we will define a map I.

Definition 6.3. Let $P_0 \to X$ be a principal SU(2)-bundle and A_0 a connection on P. Fix a set of points $\mathbf{x} = \{x_1, \ldots, x_M\}$ in X. Let $V_{\mathbf{x}}$ denote $\bigoplus_{j=1}^M \Omega^+(\mathfrak{g}_{P_0})|_{x_j}$, and an inner product on $V_{\mathbf{x}}$ is defined by $(\{\omega(x_j)\}, \{\eta(x_j)\})_V := \frac{1}{M} \sum_{j=1}^M (\omega(x_j), \eta(x_j))$. Fix E with $0 < 4E < 1$, and we define a linear map

$$I_{\mathbf{x}} \colon \mathrm{Im}[\Pi_{4E,A_0}] \to V_{\mathbf{x}}$$

by $I_{\mathbf{x}}(\omega) := (\omega(x_j))$.

Lemma 6.4. *Let K be a compact subset of X. There exist a positive constant q and a set of points $\mathbf{x} = \{x_1, \ldots, x_M\}$ in $X \setminus K$ which depend on the Riemannian metric and A_0 such that, for any E with $0 < 4E < 1$, the map $I_{\mathbf{x}}$ is injective, and*

$$\|I_{\mathbf{x}}(\omega)\|_V \geq q \|\omega\|_2$$

for any $\omega \in \mathrm{Im}[\Pi_{4E,A_0}]$.

Proof. Suppose the contrary. Fix a countable dense subset $\{x_j\}_{j=1}^{\infty}$ of $X \setminus K$, and set $\mathbf{x}_n := \{x_1, \dots, x_n\}$. Suppose there were no such q. Then, for any n there exists a sequence $\{\omega_{n,i}\}_{i=1}^{\infty}$ in $\mathrm{Im}[\Pi_{4E,A_0}]$ satisfying both $\|\omega_{n,i}\|_2 = 1$ and $\|I_{\mathbf{x}_n}(\omega_{n,i})\|_V < 1/i$. Note that the set $\{\omega \in \mathrm{Im}[\Pi_{E,A_0}] \mid \|\omega\|_2 = 1\}$ is compact. Therefore, by diagonal sequence arguments, we have $\omega \in \mathrm{Im}[\Pi_{4E,A_0}]$ satisfying both $\|\omega\|_2 = 1$ and $\omega(x_i) = 0$ for any i. Since the set $\{x_j\}_{j=1}^{\infty}$ is dense in $X \setminus K$, we conclude that $\omega \equiv 0$ on $X \setminus K$. Thus, by the unique continuation theorem [Kaz88], we have $\omega \equiv 0$ on X. This is a contradiction. We have completed the proof. □

Next we will define a map Γ_x. The following proposition is due to Taubes [Tau89, Lemma 6.10.].

Proposition 6.5 (Taubes). *There exists a positive integer m with the following property. Let $P_0 \to X$ be a principal* SU(2)*-bundle, and x a point in X. Fix an isometric identification τ_x of $\Omega^+(\mathfrak{g}_{P_0})|_x$ with $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$. Let $\{\sigma_i\}_{i=1}^3$ be an orthonormal basis for $\mathfrak{su}(2)$. Then, there exist m points $\{g_a\}_{a=1}^m$ in* SU(2) *such that the map $\Gamma_x \colon (0,\infty)^m \to \Omega^+(\mathfrak{g}_{P_0})|_x$ defined by*

$$\Gamma_x(s_a) := \tau_x \left[-\sqrt{2}\pi^2 \sum_{a=1}^{m} s_a \cdot \left(\sum_{i=1}^{3} (g_a \cdot \sigma_i \cdot g_a^{-1}) \otimes \sigma_i \right) \right]$$

is surjective with the property that for any compact subset $K \subset \Omega^+(\mathfrak{g}_{P_0})|_x$ there exists a compact set $K' \subset (0,\infty)^m$ which is mapped surjectively onto K by Γ_x.

Finally we will define approximate Kuranishi maps.

Definition 6.6. Let $P_0 \to X$ be a principal SU(2)-bundle and A_0 a connection on P_0. Let E be a positive number with $0 < 4E < 1$. Fix a set of distinct points $\mathbf{x} = \{x_1, \dots, x_M\}$ in X, and choose an isometric identification τ_{x_j} of $\Omega^+(\mathfrak{g}_{P_0})|_{x_j}$ with $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$ at each x_j. Then, we can define a map $\Xi \colon (0,\infty)^{mM} \to \mathrm{Im}[\Pi_{4E,A_0}]$ as follows:

$$\begin{aligned}
&(\lambda_{1,1}, \dots, \lambda_{1,m}, \dots, \lambda_{M,1}, \dots, \lambda_{M,m}) \\
\mapsto &(\lambda_{1,1}^2, \dots, \lambda_{1,m}^2, \dots, \lambda_{M,1}^2, \dots, \lambda_{M,m}^2) \\
\mapsto &(\Gamma_{x_1}[(\lambda_{1,1}^2, \dots, \lambda_{1,m}^2)], \dots, \Gamma_{x_M}[(\lambda_{M,1}^2, \dots, \lambda_{M,m}^2)]) \\
\mapsto &I_{\mathbf{x}}^*[(\Gamma_{x_1}[(\lambda_{1,1}^2, \dots, \lambda_{1,m}^2)], \dots, \Gamma_{x_M}[(\lambda_{M,1}^2, \dots, \lambda_{M,m}^2)])],
\end{aligned}$$

where $I_{\mathbf{x}}^*$ is the adjoint operator of $I_{\mathbf{x}}$. The map Ξ is called the approximate Kuranishi map.

The crucial property of the approximate Kuranishi maps Ξ is the following.

Proposition 6.7. *There exist positive constants z and q such that, for any $\epsilon > 0$, the restriction of Ξ to $(z^{-1}\sqrt{\epsilon}, z\sqrt{\epsilon})^{mM}$ is a surjection to a neighborhood of $\{\omega \in \mathrm{Im}[\Pi_{4E,A_0}] \mid \|\omega\|_2 < \epsilon\}$ and the adjoint $(d\Xi)^*$ of the differential satisfies the uniform estimate*

$$\|(d\Xi)^*(\omega)\| \geq q\|\omega\|_2.$$

6.4. **Finding the zero's of the Kuranishi maps.** In this section we will find the zero's of the Kuranishi maps.

Let X be an oriented closed Riemannian 4-manifold, and K a compact subset of X. Let $P_0 \to X$ be a principal SU(2)-bundle on X, and A_0 a connection on P_0. First we will define the gluing data. Lemma 6.4 provides the points $\mathbf{x} = \{x_1, \dots, x_M\}$ in $X \setminus K$. Corollary **??** provides the constant Λ_2, and Proposition 6.5 provides the positive integer m. Elementary Riemannian geometry provides more points:

Lemma 6.8. *There exists positive constant Λ_3 and c which depend on the points $\mathbf{x}$ and the Riemannian metric such that Λ_3 is smaller than the injectivity radius of X and, for any $\lambda < \Lambda_3$, the geodesic balls $B(x_i, c\sqrt{\lambda})$ are contained in $X \setminus K$ and disjoint, and, for each point x_i, we can find a set of m-points $\{x_{i,1}, \dots, x_{i,m}\}$ in $B(x_i, c\sqrt{\lambda})$ with $16\lambda < \mathrm{dist}_g(x_{i,a}, x_{i,b})$ for $a, b = 1, \dots, m$ and $a \neq b$.*

The gluing points will be $\{x_{1,1}, \ldots, x_{M,m}\}$.

Next we will choose an orthonormal frame $f_{j,a}$ in $\mathrm{Fr}(X)|_{x_{j,a}}$ and a point $h_{j,a}$ in $P_0|_{x_{j,a}}$ at each point $x_{j,a}$. Fix an orthonormal frame f_j in $\mathrm{Fr}(X)|_{x_j}$ and a point h_j in $P_0|_{x_j}$. For each point $x_{j,a}$ there exists a unique geodesic from x_j to $x_{j,a}$, and this geodesic defines the orthonormal frame $f_{j,a}$ in $\mathrm{Fr}(X)|_{x_{j,a}}$ and a point $h'_{j,a}$ in $P_0|_{x_{j,a}}$ at each point $x_{j,a}$. Proposition 6.5 provides m-points $\{g_1, \ldots, g_m\}$ in SU(2), and set $h_{j,a} := g_a \cdot h_{j,a}$.

Finally we will choose some point x_0 in $X \setminus K$ distinct from $\{x_{1,1}, \ldots, x_{M,m}\}$ and a point h_0 in $P_0|_{x_0}$. Thus, we defined the gluing data.

Fix a positive constant $\delta \ll 1$, and define $E := \delta^{-1/8}$ and require that $\lambda < \delta^{2/5}$ and $\epsilon < \delta$. Then, Proposition 6.2 shows that

$$\sigma_E(\boldsymbol{\lambda}) = \Xi(\boldsymbol{\lambda}) + O(\delta),$$

where σ_E is the Kuranisi map and Ξ is the approximate Kuranishi map. Therefore, the implicit function theorem and Proposition 6.7 proves the following.

Proposition 6.9. *There exists a positive constant ϵ_2 which depends only on the Riemannian metric such that, for any connection A_0 with $\|F^+_{A_0}\|_T < \epsilon_2$, the Kuranishi map σ attains the zero.*

7. Proof of the main theorem

In this section we will prove the main theorem.

Let X be an oriented closed Riemannian 4-manifold, U an open set in X, and K a compact subset of U. Let A_U be an anti-self-dual connection on a principal SU(2) bundle $P_U \to U$. Suppose a positive number ϵ is given. Since any principal SU(2)-bundle on U is trivial, we can find a connection A'_0 on the trivial principal SU(2)-bundle on X which agrees with A_U on K. The same argument as in [Don93, pp. 196-198] provides a principal SU(2) bundle P_0 on X and a connection A_0 on P_0 which agree with P_U and A_U on K and the self-dual curvature $F^+_{A_0}$ has the decomposition $F^+_{A_0} = F_1 + F_2$ where $\sup|F_1| \ll 1$ and $\mathrm{vol}(\mathrm{supp}F_2) \ll 1$. Therefore, we obtain a principal SU(2) bundle P_0 on X and a connection A_0 on P_0 with $\|F_{A_0}\|_T \ll 1$. Then, Proposition 6.9 provides the zero of the Kuranishi map: $\sigma(\boldsymbol{\lambda}) = 0$. Thus, $A(\boldsymbol{\lambda}) + d^*_{A(\boldsymbol{\lambda})}u$ has the anti-self-dual curvature. Finally, standard blow-up techniques show that $\|d^*_{A(\boldsymbol{\lambda})}u\|_{C^\infty(K)} \ll 1$. This completes the proof of our main theorem.

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