Weak Amenability for a Group Acting on a Finite Dimensional CAT(0) Cube Complex

(有限次元CAT(0)方体複体に作用する群の弱従順性)

水田　有一

WEAK AMENABILITY FOR A GROUP ACTING ON A FINITE DIMENSIONAL CAT(0) CUBE COMPLEX

NAOKAZU MIZUTA

ABSTRACT. We prove a Bożejko-Picardello type inequality for finite dimensional CAT(0) cube complex. As a consequence we obtain that a group acting properly on a finite dimensional CAT(0) cube complex is weakly amenable with the Cowling-Haagerup constant 1.

1. INTRODUCTION

In [BP], they considered the characteristic functions χ_n of $X_n = \{(x, y) \in X \times X : d(x, y) = n\}$ for a tree X and proved that the norms of their Schur multiplier grow at most linearly. In the present paper, we study the geometry of CAT(0) cube complex following [BP] and prove that in the case of finite dimensional CAT(0) cube complex, the corresponding norms grow at most polynomially. This generalizes of the result in [BP] since a tree is exactly a 1-dimensional CAT(0) cube complex. As a consequence, we obtain that a group acting properly on a finite dimensional CAT(0) cube complex is weakly amenable with the Cowling-Haagerup constant 1.

Over the years it has been important to consider finite dimensional approximation properties in operator algebra and among other things, amenability has been one of the central topic since the origin of operator algebra. Amenability of a group is characterized by nuclearity of its reduced group C^*-algebra, which is some kind of finite dimensional approximation property. The first example of a non-nuclear C^*-algebra which has MAP appeared in [Ha] and it was shown in [DH] that it has completely bounded approximation property, or CBAP in short. CBAP was thoroughly studied in [CH] and [HK]. Weak amenability of group is introduced there and turned out to be equivalent to CBAP of its reduced group C^*-algebra. Since nuclearity implies CBAP, weak amenability is actually a weak form of amenability. There is another important finite dimensional approximation property called exactness. In [CaN], it was shown that a group acting properly and

Date: December 29, 2009.

cocompactly on a finite dimensional CAT(0) cube complex is exact. It is known that weak amenability (or, more generally, AP) for a group implies exactness [HK] and hence our result extends the above result in [CaN]. The class of weakly amenable groups is fairly large including all amenable groups, free groups, lattices of simple Lie groups of $\mathbb{R}$-rank 1 [CH],[DH],[Ha2] and all Coxeter groups [F],[J]. On the other hand, Haagerup [Ha2] showed that lattices of simple Lie groups of higher rank are not weakly amenable. See also [Do]. It was proved in [NR2] that Coxeter groups act properly on (locally finite) finite dimensional CAT(0) cube complexes and hence our result reproves the result of [F],[J] (through [NR2]).

In [GH], they also showed weak amenability for a group acting properly on a finite dimensional CAT(0) cube complex using uniformly bounded representations.

The paper is organized as follows. In Section 2, we provide some background and notation used in the rest of the paper. In Section 3, we prove our main result, a Bożejko-Picardello type inequality for finite dimensional CAT(0) cube complex exploiting the fact that the 1-skeleton of a CAT(0) cube complex is a median graph.

2. Preliminaries

Notation. Throughout the paper, $\mathbb{Z}_+, \mathbb{N}$ denote the set of non-negative integers, positive integers respectively and $\#S$ denotes the cardinality of a set S.

2.1. **Schur multiplier.** Let A and B be C^*-algebras and $\varphi : A \to B$ be a linear map. We say that φ is completely bounded if $\|\varphi\|_{\mathrm{cb}} := \sup_n \|\varphi \otimes \mathrm{id}_n\| < \infty$ where id_n denotes the identity map on $n \times n$ matrix algebra and we refer to it as the completely bounded norm, or cb-norm of φ. Let X be a set and k be a function on $X \times X$. For any $T \in \mathbb{B}(\ell^2(X))$, we write it as a matrix $T = [T_{x,y}]_{x,y \in X}$ where $T_{x,y} = \langle T\delta_y, \delta_x \rangle$, δ_x denotes the characteristic function of $\{x\}$. Here we denote by $\mathbb{B}(H)$ the set of all bounded linear maps on a Hilbert space H. The Schur multiplier associated with k is a map m_k which sends $[T_{x,y}]$ to $[k(x,y)T_{x,y}]$. (This may not be an everywhere-defined map on $\mathbb{B}(\ell^2(X))$.) If this correspondence does define a map on all of $\mathbb{B}(H)$, we have that m_k is completely bounded with completely bounded norm $\leq C$ iff there exist families of vectors $(\xi_x)_{x \in X}$ and $(\eta_y)_{y \in X}$ in a Hilbert space H such that $k(x,y) = \langle \eta_y, \xi_x \rangle$ for every $x, y \in X$ and $\sup_{x,y \in X} \|\xi_x\| \|\eta_y\| \leq C$ [P, p.110]. The completely bounded norm of m_k is denoted by $\|k\|_{\mathrm{cb}}$. If k is the characteristic function χ_F of a subset $F \subseteq X \times X$, then we abbreviate $\|\chi_F\|_{\mathrm{cb}}$ to $\|F\|_{\mathrm{cb}}$.

Assume $X = \Gamma$ is a group. For a function φ on Γ, we associate it with the kernel $(s,t) \mapsto \varphi(st^{-1})$ and call the resulting Schur multiplier as the Herz-Schur multiplier. We still denote its cb-norm by $\|\varphi\|_{\mathrm{cb}}$. A function k on $X \times X$ is said to be positive definite if for any n in $\mathbb{N}$, any complex numbers $\alpha_1, \ldots, \alpha_n$ and any points $x_1, \ldots, x_n$ in X, we have $\sum_{i,j} \bar{\alpha}_i \alpha_j k(x_i, x_j) \geq 0$ where $\bar{\alpha}$ denotes the complex conjugate of the complex number α. Also, a function φ on a group Γ is said to be positive definite if the associated kernel is positive definite. The Schur multiplier of a positive definite function whose diagonals are 1 gives rise to a unital completely positive map on $\mathbb{B}(\ell^2(X))$, and the cb-norm of a unital completely positive map is equal to 1. For details of completely boundedness, complete positivity, and Schur multipliers, see [P].

2.2. **Weakly Amenable Group.** In this paper, all groups are assumed to be discrete and countable. A discrete group Γ is said to be *weakly amenable* if there exists a sequence (φ_n) of finitely supported functions on Γ and a constant C such that $\varphi_n \to 1$ pointwise and $\limsup_{n\to\infty} \|\varphi_n\|_{\mathrm{cb}} \leq C$. The Cowling-Haagerup constant $\Lambda_{\mathrm{cb}}(\Gamma)$ is the infimum of all such C for which such a sequence (φ_n) exists. For basics of weak amenability, see [BO],[CH],[DH], and [HK].

2.3. **Polytope.** Let $\mathbb{E}^n$ be the Euclidean space. A hyperplane is a codimension 1 linear subspace. A hyperplane splits $\mathbb{E}^n$ into two connected components. Each one of the component is called open halfspace, and the union of an open halfspace and the hyperplane is called closed halfspace. An affine subspace in $\mathbb{E}^n$ is a translate of a linear subspace. An affine open (closed) halfspace is a translate of open (closed) halfspace. A polyhedron is a finite intersection of closed affine halfspaces. A polytope $\mathcal{P}$ in $\mathbb{E}^n$ is the convex hull of a finite set in $\mathbb{E}^n$. A supporting affine hyperplane for $\mathcal{P}$ is a hyperplane one of whose associated closed halfspace contains $\mathcal{P}$ and their distance is 0. A face of $\mathcal{P}$ is an intersection of a supporting affine hyperplane and $\mathcal{P}$. It is a known fact that a polytope is a polyhedron, and a polyhedron is a polytope if and only if it is bounded. We also refer to a polytope as (convex) cell.

A polytopal complex is a family $\mathcal{F}$ of convex cells in $\mathbb{E}^n$ satisfying (i) if $\mathcal{P}$ is in $\mathcal{F}$ and $\mathcal{Q}$ is a face of $\mathcal{P}$, $\mathcal{Q}$ is also in $\mathcal{F}$, (ii) if $\mathcal{P}$, $\mathcal{Q} \in \mathcal{F}$, then their intersection is either empty or contained in $\mathcal{F}$. We glue cells along their faces and give it CW-topology.

We collect here some terminology from topology. Let X be a topological space. A subspace A is said to be a strong deformation retract of X if there exists a continuous map $H : X \times [0,1] \to X$ such that $H(x,0) = x$, $H(x,1) \in A$ for all $x \in X$, and $H(x,t) = x$ for all $x \in A$,

$t \in [0,1]$. If A is a strong deformation retract of X, then A and X are homotopy equivalent. For example, if P is a polytope and Q is one of its faces, then Q is a strong deformation retract of P ([BH, p.176]).

Let X be a CW complex. If $\mathcal{U} = \{X_\alpha\}$ is a cover of X by subcomplexes, the nerve of $\mathcal{U}$ is the abstract simplicial complex $N(\mathcal{U})$ having a vertex v_α for each X_α, and a simplex $\{v_{\alpha_0}, \dots, v_{\alpha_k}\}$ whenever $\bigcap_{i=0}^{k} X_{\alpha_i} \neq \emptyset$. The following property is useful when we decide homotopy type.

Lemma 1 (Ge, Proposition 9.3.20). *If the cover $\mathcal{U}$ is finite and if $\bigcap_{i=0}^{k} X_{\alpha_i}$ is contractible whenever it is non-empty, then $N(\mathcal{U})$ and X are homotopy equivalent.*

2.4. **CAT(0) Cube Complex.** A cube complex X is a metric polytopal complex in which each cell is isometric to the Euclidean cube $[-\frac{1}{2}, \frac{1}{2}]^n$ for some $n \in \mathbb{Z}_+$ (we follow the convention that $[-\frac{1}{2}, \frac{1}{2}]^0$ means a single point), and the gluing maps are isometries. We call $[-\frac{1}{2}, \frac{1}{2}]^n$ as an n-cube and the dimension of X, denoted by $\dim X$ is the supremum of such n. A cube complex X is equipped with the metric induced by the Euclidean metric on the cubes and it is called CAT(0) if the metric gives X a CAT(0) metric. If $\dim X < \infty$, then the complex carries a complete geodesic metric. (See [BH].) Let us notice that a 1-dimensional CAT(0) cube complex is exactly a tree.

We focus mainly on combinatorics of CAT(0) cube complex and we also give the combinatorial description of CAT(0) cube complex which is known to be equivalent to the above definition (at least in a finite dimensional case). A cube complex is a non-empty set X with a family $\mathcal{C}$ of non empty subsets of X called cubes satisfying that (1) $\mathcal{C}$ is a cover of X (2) for C_1 and C_2 in $\mathcal{C}$, $C_1 \cap C_2$ is also in $\mathcal{C}$ unless it is empty (3) for any C in $\mathcal{C}$, there is a bijection $\Phi : C \to \{0,1\}^n$ for some $n \in \mathbb{Z}_+$ (we refer to this cube C as an n-cube) preserving its faces *i.e.* for any $C' \subseteq C$, $C' \in \mathcal{C}$ iff $\Phi(C')$ is a face of $\{0,1\}^n$ where $A \subseteq \{0,1\}^n$ is called a face iff A is of the form $A_1 \times \cdots \times A_n$ for $\emptyset \neq A_i \subseteq \{0,1\}$. The dimension of X is still the supremum of n for which an n-cube exists. Notice that any 1 point set is a cube from (1) and (3) which we refer to as a vertex and also we refer to a 1-cube as an edge. For a vertex x, the vertex link $lk(x)$ of x is the set of vertices which are adjacent to x and it comes equipped with the simplicial structure given as follows: $S \subseteq lk(x)$ is a simplex iff there exists a cube including $\{x\}$ and S. A simplicial complex is said to be flag if any complete subgraph of n vertices is actually the 1-skeleton of an $(n-1)$-simplex. A cube complex is called locally CAT(0) (or non-positively curved) if every vertex link is a flag complex. A (combinatorial) cube complex has a geometric realization through the natural identification of $\{0,1\}^n$ with the Euclidean cube $[0,1]^n$ in $\mathbb{R}^n$. For a cube complex with its geometric realization simply-connected, it is known [Gr] that it is CAT(0) in the above sense, *i.e.* the metric induced by the Euclidean metric is CAT(0) iff it is locally CAT(0).

A combinatorial hyperplane is an equivalence class of unoriented edges where two edges e and f are called equivalent if there exists a finite sequence of edges $e = e_1, \ldots, e_n = f$ such that e_i and e_{i+1} are opposite sides of some 2-cube in X for all $i = 1, \ldots, n-1$. If an edge e belongs to a hyperplane H, we say e intersects with H. A combinatorial hyperplane admits a natural geometric realization obtained by the first barycentric subdivision which is called a geometric hyperplane [S]. A CAT(0) cube complex considered as a graph has another natural metric called graph metric. We often refer to the associated distance function on $X \times X$ as the combinatorial distance. In [S], it was shown that for any points $x, y \in X$ of combinatorial distance n, there are exactly n combinatorial hyperplanes which separates x and y and any geodesic path between x and y intersects with every hyperplanes separating x and y just once. In particular, the combinatorial distance of diagonal points of d-cubes is just d. The following is established in [S].

Theorem 1 (S, Thm 4.6). *If x and y are two vertices of X, and α and β are two geodesic paths from x to y, then there exists a finite sequence of geodesic paths α_i from x to y with $\alpha_1 = \alpha$ and $\alpha_n = \beta$, such that α_i and α_{i+1} differ by exchanging two consecutive edges for two edges that run on the opposite side of some 2-cube.*

Assume X is a connected graph. We consider it as a metric space via the graph metric and often identify X with its vertex set. For any $x, y \in X$, we define the geodesic interval $[x, y]$ as the set of vertices lying on a shortest path (geodesic path) from x to y. A graph is called median if, for each triple of vertices x, y, z, the geodesic intervals $[x, y], [y, z], [z, x]$ have a unique common point which is called the median of x, y, z. For example, if we divide the Euclidean plane into squares, it becomes trivially a CAT(0) cube complex with the vertex set $\mathbb{Z}^2$ and the combinatorial distance is just the ℓ^1-metric. In this case, for any vertices x, y, the geodesic interval $[x, y]$ is the set of vertices on a rectangle having x, y as one of its diagonal points and then it is easy to see that this graph is median. This holds more generally, that is, in [C], Chepoi showed that the 1-skeleton of a CAT(0) cube complex is a median graph [C, Thm 6.1]. This fact is a crucial ingredient for our paper. See also [ChN], [N], and [R].

Our aim is to show weak amenability for certain groups. To this end, we study the geometry of CAT(0) cube complex following the proof for trees in [BP].

Theorem 2. *Let X be a finite dimensional CAT(0) cube complex and let $X_n = \{(x,y) \in X \times X : d(x,y) = n\}$ for $n \in \mathbb{Z}_+$. Then the norms of Schur multipliers of the characteristic function of X_n grow polynomially i.e. there exists a polynomial p such that $\|X_n\|_{\text{cb}} \le p(n)$.*

Combining Theorem 2 with the previously established fact [NR1] that the combinatorial distance function on X is conditionally negative definite, we obtain

Theorem 3. *A group Γ which acts cellularly and properly on a finite dimensional CAT(0) cube complex X is weakly amenable with $\Lambda_{\text{cb}}(\Gamma) = 1$.*

Guentner-Higson has obtained the same result [GH] using uniformly bounded representations.

3. Main Results

Let X be a CAT(0) cube complex. We call ω an infinite geodesic in X if ω is an isometric map from $\mathbb{Z}_+$ into X where X is equipped with the combinatorial distance. If we fix a point $x \in X$ and add an infinite geodesic ω to X which starts at x, then the resulting cube complex $\tilde{X} = X \cup \{\omega\}$ is CAT(0) [BH, p.347] (or we can easily see this from the definition of CAT(0) cube complex). Note that the embedding of X into $\tilde{X}$ is isometric in the combinatorial distance.

Henceforth, we assume there exists an infinite geodesic ω_0 in X and fix it once and for all. We say that two infinite geodesics ω_1 and ω_2 eventually flow with if there exists $N \in \mathbb{Z}$ such that for any $n \in \mathbb{Z}_+$ with $n \ge |N|$, $\omega_1(n+N) = \omega_2(n)$. The following lemma is in [Wo, p.246].

Lemma 2. *Let X be a connected graph and ω be an infinite geodesic in X. Then for any $x \in X$ there exists an infinite geodesic ω_x which starts at x and eventually flows with ω.*

Recall that X is a median graph and for any three vertices x, y, z in a median graph, there exists a unique point $m(x,y,z)$ called the median of x, y, z which is on some geodesics connecting each pair of them.

Lemma 3. *For $x, y \in X$, there exists a unique point $m(x,y)$ in X with the following property: for all but finitely many z on ω_0, $m(x,y)$ is the median of x, y, z.*

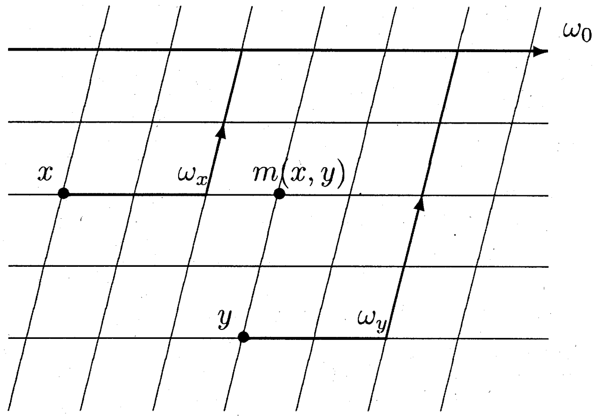

FIGURE 1

Proof. Uniqueness is clear from the uniqueness of the median for three points in X. For the existence, take any infinite geodesics ω_x, ω_y which start at x, y respectively and eventually flow with ω_0, which exist by Lemma 2. Since they eventually flow with, there is a point z which is on both ω_x and ω_y. It is easy to see that if ω is an infinite geodesic and if we take two points x', y' on ω and take another geodesic connecting x' and y' and substitute it for the corresponding geodesic on ω, then the resulting infinite path is a geodesic. With this observation in hand, the existence of $m(x, y)$ follows by considering geodesics between x and z, y and z which pass through $m(x, y, z)$ and substituting it for the corresponding geodesics on ω_x, ω_y respectively. □

For any $x \in X$, we denote by $A(x, k)$ the set of points of distance k from $x \in X$ on some infinite geodesic ω that starts at x and eventually flows with ω_0, *i.e.*

$$A(x,k) = \{y \in X : \text{there exists an infinite geodesic } \omega \text{ that starts at } x, \omega(k) = y, \text{ and eventually flows with } \omega\}.$$

By Lemma 2, these sets are non-empty for all $x \in X$ and $k \in \mathbb{Z}_+$.

Lemma 4. *For $x_1, x_2 \in X$, we write $y = m(x_1, x_2), \ell_1 = d(x_1, y)$, $\ell_2 = d(x_2, y)$. Then for $k_1, k_2 \in \mathbb{Z}_+$, $A(x_1, k_1) \cap A(x_2, k_2)$ is not empty iff $k_1 = \ell_1 + m$ and $k_2 = \ell_2 + m$ for some $m \in \mathbb{Z}_+$, and in this case, we have $A(x_1, k_1) \cap A(x_2, k_2) = A(y, m)$.*

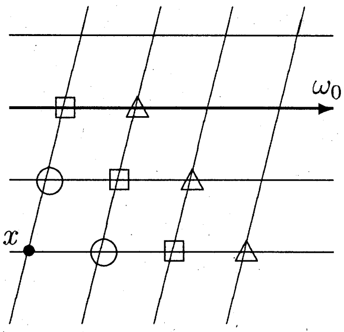

FIGURE 2

Proof. Assume $A(x_1,k_1)\cap A(x_2,k_2)$ is not empty and take z in $A(x_1,k_1)\cap A(x_2,k_2)$. Then there exist infinite geodesics $\omega_{x_1},\omega_{x_2}$ which start at x_1,x_2 respectively and eventually flow with ω_0 and pass through z. Consider the median $m(x_1,x_2,z)$ and take geodesics connecting x_1 to z and x_2 to z which pass through $m(x_1,x_2,z)$, and substitute them for the corresponding geodesics on $\omega_{x_1},\omega_{x_2}$, then we obtain two infinite geodesics $\tilde{\omega}_{x_1},\tilde{\omega}_{x_2}$ which start at x_1,x_2 respectively, eventually flow with ω_0 and pass through $m(x_1,x_2,z)$ and z. By the uniqueness, $m(x_1,x_2,z)$ coincides with y. If we define m to be $d(y,z)$, then $k_1=\ell_1+m$ and $k_2=\ell_2+m$ and z is in $A(y,m)$. Conversely, if $k_1=\ell_1+m, k_2=\ell_2+m$ for some $m\in\mathbb{Z}_+$, then we take two geodesics $\omega_{x_1},\omega_{x_2}$ which start at x_1,x_2 respectively, pass through y and eventually flow with ω_0. We can assume that ω_{x_1} and ω_{x_2} coincide after passing y by arguing as above. Then $\omega_{x_1}(k_1)=\omega_{x_2}(k_2)$ is in $A(x_1,k_1)\cap A(x_2,k_2)$ and it is not empty. Hence we have proved that the first statement and $A(x_1,k_1)\cap A(x_2,k_2)\subseteq A(y,m)$. Assume z is in $A(y,m)$ and take any geodesic ω which starts at y and eventually flows with ω_0 and $\omega(m)=z$. If we take any geodesic ω_{x_1} which starts at x_1 and passes through y and eventually flows with ω_0, there exists a point w which is on ω,ω_{x_1} with $d(y,w)\geq m$ since ω and ω_{x_1} eventually flow with ω_0. Then substituting the geodesic connecting y to w passing through z for the corresponding geodesic on ω_{x_1}, we obtain an infinite geodesic $\tilde{\omega}_{x_1}$ which starts at x_1 and passes through y,z and eventually flows with ω_0 and hence z is in $A(x_1,k_1)$. Similarly, we can show z is in $A(x_2,k_2)$ and we complete the proof. □

Next we consider the following polytopal structure. Its polytopes are obtained by sections of cubes, which are perpendicular to some diagonal lines of cubes: 0-polytopes are vertices of X and for $d\geq 1$, $P\subseteq X$ is a

d-polytope if there exists a $(d+1)$-cube C, $z \in C$ and l, $1 \leq l \leq d$ such that $P =$ the set of vertices of all l-th points on geodesics connecting z to $d_C(z)$ where we denote by $d_C(z)$ the point diagonal to z with respect to C. We make $A(x,k)$ into polytopal complexes by the above definition and denote this polytopal complex by $\mathcal{A}(x,k)$ (whose underlying vertex set is $A(x,k)$).

For example, a 1-polytope is a line segment corresponding to a diagonal line of some 2-cube, a 2-polytope is an equilateral triangle and for $d \geq 3$, there are several shapes of d-polytopes. For any d-polytope P, we write $d = \dim P$.

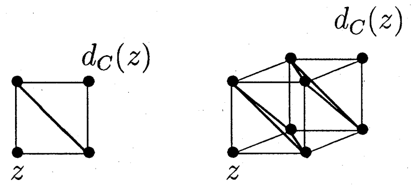

FIGURE 3. 1- and 2-dimensional polytopes

Lemma 5. *For any d-polytope* $P \in \mathcal{A}(x,k)$, *the* $(d+1)$*-cube* C *is unique in the above definition. Moreover, there exist infinite geodesics* $\omega_p, p \in P$ *which start at* x, *pass through* $z, p, d_C(z)$ *(not necessarily in this order) and eventually flow with* ω_0.

Proof. We first prove this for 1- and 2-polytopes. Assume $x_1, x_2 \in A(x,k)$ and a 2-cube C on which x_1, x_2 form a 1-polytope is given. Then the median of $\{x, x_1, x_2\}$, $m(x, x_1, x_2)$ is one vertex of C and another is $m(x_1, x_2)$. With the observation used in Lemma 3 in hand, the statement holds in this case. Assume $P = \{x_1, x_2, x_3\}$ is a 2-polytope. Since P is contained in a 3-cube, the 2-cubes formed by $\{x_1, x_2, m(x, x_1, x_2), m(x_1, x_2)\}$ and $\{x_1, x_3, m(x, x_1, x_3), m(x_1, x_3)\}$ intersects by an edge and hence either $m(x, x_1, x_2) = m(x, x_1, x_3)$ or $m(x_1, x_2) = m(x_1, x_3)$. Similarly, the 2-cube formed by $\{x_2, x_3, m(x, x_2, x_3), m(x_2, x_3)\}$ has common edges with these 2-cubes and hence there is a vertex which is in common for all these 2-cubes, and three points, either $\{m(x, x_1, x_2), m(x, x_1, x_3), m(x, x_2, x_3)\}$ or $\{m(x_1, x_2), m(x_1, x_3), m(x_2, x_3)\}$, each of which is in each 2-cubes form another 2-polytope in either $A(x, k-1)$ or $A(x, k+1)$. Continuing the above process with this new 2-polytope, we obtain the desired property. The general cases are essentially the same. We prove by induction in the dimension of cubes. Assume P is a d-polytope and a cube C on which P forms a d-polytope is given. Fix two points y, z in P and consider two points $m(x,y,z)$ and $m(y,z)$. We identify C with $\{0,1\}^{d+1}$ and we may assume $y = (1,\dots,1,0,\dots,0)$ (the first l coordinates are 1), $m(x,y,z) = (1,\dots,1,0,\dots,0)$ (the first $l-1$ coordinate are 1) and $m(y,z) = (1,\dots,1,0,\dots,0)$ (the first $l+1$ coordinates are 1) and P is the polytope formed by the points whose coordinates have 1 l-times with $l < d$. Considering $(l+1)$-cube formed by the first $l+1$ coordinates, we obtain an infinite geodesic ω which starts at x and passes through $(0,\dots,0), y$ and eventually flows with ω_0 by inductive hypothesis and hence we have that $(0,\dots,0)$ is in $A(x, k-l)$. By definition, there exist infinite geodesics ω_w for any $w \in P$ which start at x and pass through w and eventually flow with ω_0. Taking any geodesic between $(0,\dots,0)$ and w (which is necessarily in C) and tying it with a geodesic between x and $(0,\dots,0)$, we obtain a geodesic between x and w passing through $(0,\dots,0)$ and hence we may assume ω_w pass through $(0,\dots,0)$ by substituting the corresponding geodesics if necessary. Then, considering the medians $m(w,w')$ for $w, w' \in P$, it is easy to see that the set of $(l+1)$-th point of geodesics from $(0,\dots,0)$ to $(1,\dots,1)$ forms a d-polytope in $A(x, k+1)$ and continuing these process, we obtain an infinite geodesic which starts at x, passes through $(0,\dots,0), (1,\dots,1)$ and eventually flows with ω_0 and this implies the statement except for uniqueness.

Uniqueness follows from the construction. For example, the $(l+1)$-th points are obtained by the medians $m(w,w')$ and the $(l-1)$-th points are obtained by the medians $m(x,w,w')$, $w, w' \in P$ and the $(l+2)$-th points are obtained by the medians of two points in the $(l+1)$-th points and so on. □

Assume $A(x,1) = \{x_1,\dots,x_n\}$. We claim that for any i, j $(i \neq j)$ there are 2-cubes $C_{\{i,j\}}$ which is formed by $x, x_i, x_j, m(x_i, x_j)$. To see this, first we see that the distance between x_i and x_j is 2. Indeed, if we take a geodesic α between them and form a loop by tying α and edges $\{x, x_i\}$,$\{x, x_j\}$ together, we can conclude that the length of α is 2 since the length of any loop is even number [CR, Lem4.5]. Hence $m(x_i, x_j)$ is adjacent to x_i and x_j. Considering two geodesics between x_i and x_j formed by (x_i, x, x_j) and $(x_i, m(x_i,x_j), x_j)$, and using the cited Theorem [S, Thm 4.6], we conclude the existence of 2-cubes. Hence by the link condition in the definition of CAT(0) cube complex, there is an n-cube C which includes x and $A(x,1)$. Assume $k \geq 1$ and in this case, we have $A(x,k) = \bigcup_{i=1}^{n} A(x_i, k-1)$.

Proposition 1. *For non-empty $I \subseteq \{1, \ldots, n\}$, we have the following:*

$$\bigcap_{i \in I} A(x_i, k-1) = \begin{cases} \emptyset & (\text{if } k \leq \#I - 1), \\ A(d_I(x), k - \#I) & (\text{if } k \geq \#I). \end{cases}$$

where $d_I(x)$ is the point diagonal to x with respect to the $\#I$-cube associated with $\{x_i\}_{i \in I}$.

Proof. We prove by induction in $\#I$. The case for $\#I = 1$ is trivial. We assume $\#I \geq 2$ and the statement holds for $J \subseteq \{1, \ldots, n\}$ with $\#J < \#I$. First we handle the case for $k \leq \#I - 1$. Take $J \subseteq I$ with $\#J = \#I - 1$. If $k < \#I - 1$, then $\bigcap_{i \in I} A(x_i, k-1) \subseteq \bigcap_{i \in J} A(x_i, k-1) = \emptyset$ by the assumption. If $k = \#I - 1$ and we write $J \cup \{j\} = I$, then $\bigcap_{i \in J} A(x_i, k-1) = A(d_J(x), k - \#J)$ by the assumption. Since $d(x_j, d_J(x)) = \#I$, we obtain $\bigcap_{i \in I} A(x_i, k-1) = \bigcap_{i \in J} A(x_i, k-1) \cap A(x_j, k-1) = \emptyset$ by Lemma 4. Finally we handle the case for $k \geq \#I$. Again, if we write I as $J \cup \{j\}$ and by the assumption we have $\bigcap_{i \in J} A(x_i, k-1) = A(d_J(x), k - \#J)$. Noting that $m(x_j, d_J(x)) = d_I(x)$ by Lemma 5, we obtain $\bigcap_{i \in I} A(x_i, k-1) = \bigcap_{i \in J} A(x_i, k-1) \cap A(x_j, k-1) = A(d_I(x), k - \#I)$ by Lemma 4. □

Next we consider the topological property of the complex $\mathcal{A}(x, k)$ and show that $\mathcal{A}(x, k)$ is contractible. Although $A(x, k)$ is the union of $A(x_i, k-1)$, $\mathcal{A}(x, k)$ need not be the union of $\mathcal{A}(x_i, k-1)$. Assume $\mathcal{P}$ is a polytope in $\mathcal{A}(x, k)$. There exists a unique cube C which $\mathcal{P}$ is on, and vertices $z, d_C(z) \in C$ along with an infinite geodesic ω starting at x and passing through z and $d_C(z)$ as in Lemma 5. Here we assume z is closer to x than $d_C(z)$. If $z \neq x$, then $\mathcal{P}$ is in $\mathcal{A}(x_i, k-1)$ for some $i, x_i = \omega(1)$. In other words, if $\mathcal{P}$ is not in the complex $\bigcup_{i \in \{1, \ldots, n\}} \mathcal{A}(x_i, k-1)$, then $z = x$ and $\mathcal{P}$ should be on the cube spanned by the edges $\{x, x_i\}$.

Proposition 2. *$\mathcal{A}(x, k)$ is contractible for all $x \in X$ and $k \in \mathbb{Z}_+$.*

Proof. The proof proceeds by induction in k. It is obvious when $k = 0$.

If $1 \leq k < n$, it is shown as follows. We add the maximal polytope $\mathcal{P} \in \mathcal{A}(x, k)$ (and all of its faces) which is on the cube spanned by the edges $\{x, x_i\}$ to each subcomplex $\mathcal{A}(x_i, k-1)$ so that the resulting family of subcomplexes, denoted by $\tilde{\mathcal{A}}(x_i, k-1)$, consists a cover of $\mathcal{A}(x, k)$. Assume $I \subseteq \{x_1, \ldots, x_n\}$. If $\bigcap_{i \in I} \mathcal{A}(x_i, k-1) \neq \emptyset$, then the intersection of $\mathcal{P}$ and $\bigcap_{i \in I} \mathcal{A}(x_i, k-1)$ is a face of $\mathcal{P}$. Recall that any face of a polytope is a strong deformation retract of the polytope so that $\bigcap_{i \in I} \tilde{\mathcal{A}}(x_i, k-1) = \bigcap_{i \in I} \mathcal{A}(x_i, k-1) \bigcup \mathcal{P}$ and $\bigcap_{i \in I} \mathcal{A}(x_i, k-1)$ are homotopic. By Proposition 1 and induction, $\bigcap_{i \in I} \mathcal{A}(x_i, k-1)$ is contractible. If $\bigcap_{i \in I} \mathcal{A}(x_i, k-1) = \emptyset$, then $\bigcap_{i \in I} \tilde{\mathcal{A}}(x_i, k-1) = \mathcal{P}$ and it is contractible. By Lemma 1, $\mathcal{A}(x,k)$ is homotopic to the nerve of the cover which is a contractible simplex.

If $k \geq n$, the family of subcomplexes $\mathcal{A}(x_i, k-1)$ consists a cover of $\mathcal{A}(x,k)$ and any subfamily has a non-empty intersection by Proposition 1, whose contractibity follows from Proposition 1 and induction. Therefore, Lemma 1 applies and as in the above, we deduce that $\mathcal{A}(x,k)$ is contractible. □

Here we evaluate the number of vertices in $A(x,k)$. In [CR], they showed that it is bounded above by polynomial in k. We prove a similar result and calculate the exact bound. Intuitively, the maximal number is attained if the cubes of the maximal dimension are stuffed and in this case an easy calculation shows the number is equal to $\binom{k+N-1}{N-1}$ where N is the dimension of X. Indeed, this is the case.

Lemma 6. *There exists a polynomial q such that for any $x \in X$ and for $k \in \mathbb{Z}_+$, $\#A(x,k) \leq q(k)$. More precisely, we can take q to be $q_N(k) = \binom{k+N-1}{N-1}$ where N is the dimension of X.*

Proof. We prove by double induction in the dimension of the cubes which are involved and in k. The cases that $k = 0$ with arbitrary dimension, and $\dim X = 1$ with arbitrary k are trivial. Assume $A(x,1) = \{x_1, \ldots, x_n\}$ and write $A(x,k)$ as $A(x_1, k-1) \cup (A(x,k) \setminus A(x_1, k-1))$. We consider the hyperplane H separating x and x_1. Then H separates $A(x_1, k-1)$ from $A(x,k) \setminus A(x_1, k-1)$. To see this, take any $y \in A(x_1, k-1)$ and assume H does not separate x from y. Then there are k hyperplanes which separate x from y. Note that these hyperplanes are different from H since it does not separate x from y. Moreover, note that these hyperplanes separating x from y also separate x_1 from y since x and x_1 are separated only by H and so there exist $k+1$ hyperplanes separating x_1 and y, a contradiction. Conversely, take $y \in A(x,k)$ and assume H does not separate x_1 from y. Then there are just $k-1$ hyperplanes separating x_1 from y since H does not separate them *i.e.* $d(x_1, y) = k-1$. It is easy to see that there exists an infinite geodesic which starts at x and passes through x_1, y and eventually flows with ω_0, and hence in particular, y is in $A(x_1, k-1)$.

We claim by induction in k that for all y in $A(x,k) \setminus A(x_1, k-1)$ there exists z in $A(x_1, k)$ such that y and z are adjacent by an edge and this edge intersects with H. To prove this, take any y' in $A(x, k-1)$ which is adjacent to y. Then it is necessarily contained in $A(x, k-1) \setminus A(x_1, k-2)$ for if y' was in $A(x_1, k-2)$, then the hyperplane separating y from y' must be H, but H does not separate y from x, there must be $k+1$ hyperplanes which separates x from y', a contradiction. Hence, by induction, there exists z' in $A(x_1, k-1)$ such that y' and z' are adjacent by an edge that intersects with H. If we define z to be $m(y, z')$ which is in $A(x, k+1)$, it is easily seen that y', z', y, z form a 2-cube by arguing as in the preceding paragraph of Proposition 1. Note that the edges connecting y' to z' and y to z are hyperplane equivalent and so the edge connecting y to z intersects with H and hence we have proved the claim. If we consider $A(x, k) \setminus A(x_1, k-1)$ only, then the dimension of cubes that are involved is no more than $N-1$. Hence, by induction, we obtain

$$\begin{aligned}\#A(x, k) &= \#A(x_1, k-1) + \#(A(x, k) \setminus A(x_1, k-1)) \\ &\leq q_N(k-1) + q_{N-1}(k) \\ &= q_N(k)\end{aligned}$$

and we are done. □

Now, we can prove Theorem 2 using the above results and the construction used in [BP]. For the reader's convenience, we sketch the proof of Theorem 2 for the case of a tree seen in [BP]. For any $x \in X$, there exists a unique geodesic ω_x which starts at x and eventually flows with ω_0. We define maps f_k from X into $\ell^2(X)$ by $f_k(x) = \delta_{\omega_x(k)}$. Then it turns out that $\theta_n(x, y) := \sum_{k=0}^{n} \langle f_k(x), f_{n-k}(y)\rangle = \sum_{k=0}^{\lfloor n/2 \rfloor} \chi_{n-2k}(x, y)$ for any $x, y \in X$. Since $\|f_k(x)\| = 1$ for all $x \in X$, we have $\|\theta_n\|_{\mathrm{cb}} \leq n+1$ and hence $\|\chi_n\|_{\mathrm{cb}} \leq 2n$ since $\chi_n = \theta_n - \theta_{n-2}$. The variant for CAT(0) cube complex is more complicated, but we have prepared enough to prove it.

Proof of Theorem 2. Let χ_n be the characteristic function of X_n. To evaluate the cb-norm of m_{χ_n}, we consider the following Hilbert spaces. For any l, we define $\mathcal{X}^{(l)}$ to be the set of all l-polytopes in X and consider ℓ^2-spaces of them. Then we define maps $f_k^{(l)}$ from X into $\bigoplus_{d=0}^{N-1} \ell^2(\mathcal{X}^{(d)})$ as follows.

$$f_k^{(l)}(x) = \sum_{K \in \mathcal{A}(x,k)^{(l)}} \delta_K \in \ell^2(\mathcal{X}^{(l)}) \subseteq \bigoplus_{d=0}^{N-1} \ell^2(\mathcal{X}^{(d)})$$

where δ_K denotes the characteristic function of $\{K\}$, N is the dimension of X and $\ell^2(\mathcal{X}^{(l)})$ sits in $\bigoplus_{d=0}^{N-1} \ell^2(\mathcal{X}^{(d)})$ naturally. We define functions θ_n on $X \times X$ by $\theta_n(x, y) = \sum_{k=0}^{n} \sum_{l=0}^{N-1} (-1)^l \langle f_k^{(l)}(x), f_{n-k}^{(l)}(y)\rangle$ for any $x, y \in X$. Fix $x, y \in X$. If $d(x, y) \neq n - 2k$ for any k, $k \in \mathbb{Z}_+$, we have $\theta_n(x, y) = 0$ by Lemma 4 and if $d(x, y) = n - 2k$ for some k, $k \in \mathbb{Z}_+$, then $\theta_n(x, y)$=the Euler characteristic of $\mathcal{A}(m(x, y), \frac{n-d(x,y)}{2})$ by Lemma 4 and it equals to 1 by the contractibity of $\mathcal{A}(m(x, y), \frac{n-d(x,y)}{2})$. Hence we conclude that $\theta_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \chi_{n-2k}$. Since $\chi_n = \theta_n - \theta_{n-2}$, to prove that cb-norms of m_{χ_n} increase polynomially, it suffices to show that those of m_{θ_n} do. Note that the square of the norm of $f_k^{(l)}$ is equal to $\#\mathcal{A}(x,k)^{(l)}$. Recall that there are several shapes of l-polytopes and they consist of $\binom{l+1}{1}, \ldots, \binom{l+1}{l/2}$ vertices for l even, and $\binom{l+1}{1}, \ldots, \binom{l+1}{(l+1)/2}$ vertices for l odd respectively and hence $\#\mathcal{A}(x,k)^{(l)}$ is bounded above by $\sum_{m=1}^{\lfloor l/2 \rfloor+1} \binom{\#\mathcal{A}(x,k)}{\binom{l+1}{m}}$ (here we follow the convention that $\binom{n}{k} = 0$ if $n < k$). We define $r(n)$ to be $\sum_{l=0}^{N-1} \sum_{m=1}^{\lfloor l/2 \rfloor+1} \binom{n}{\binom{l+1}{m}}$. Since $\binom{n}{\binom{l+1}{m}}$ is an increasing function in n for fixed l, m, $\|f_k^{(l)}(x)\|^2 \leq r(q(k))$ holds for all $0 \leq l \leq N-1$, $k \in \mathbb{Z}_+$ and $x \in X$ where $q(k)$ is a polynomial in Lemma 6. Hence the completely bounded norm of θ_n is bounded above by $N\sum_{k=0}^{n} \sqrt{r(q(k))}\sqrt{r(q(n-k))}$ which is bounded by $N\sum_{k=0}^{n} r(q(k))r(q(n-k))$ since $\sqrt{t} \leq t$ for $t \geq 1$. □

We know from [NR1] that the combinatorial distance d on X gives rise to a conditionally negative definite function on $X \times X$ and also that, by Schönberg's theorem, $(x,y) \mapsto \exp(-\frac{1}{n}d(x,y))$ is a positive definite function on $X \times X$ for all $n \in \mathbb{N}$. If Γ acts on X cellularly (and hence isometrically), then we obtain unital positive definite functions on Γ defined by $s \mapsto \exp(-\frac{1}{n}d(sx_0, x_0))$ where $x_0 \in X$ is a fixed point in X. If, moreover, Γ-action is proper, then these functions are in $c_0(\Gamma)$, where $c_0(\Gamma)$ denotes the set of functions on Γ which vanish at infinity. Recall that a unital positive definite function gives rise to a unital completely positive Herz-Schur multiplier and hence its cb-norm is equal to 1. To obtain finitely supported functions with controlled cb-norms, we truncate the functions by using Theorem 2. This kind of idea has first appeared in [Ha] and been used by many authors.

Proof of Theorem 3. We fix $x_0 \in X$ as above and consider the functions $\psi_n(s) := \exp(-\frac{1}{n}d(sx_0, x_0))$ and $\chi_n(s) := \chi_n(sx_0, x_0)$. Then $\psi_n(s)\chi_k(s) = \exp(-\frac{k}{n})\chi_k(s)$ holds for all $s \in \Gamma$. We already know that there exists a polynomial p such that $\|X_n\|_{\mathrm{cb}} \leq p(n)$. For fixed $n \in \mathbb{N}$, $\sum_{k=0}^{\infty} \exp(-\frac{k}{n})p(k)$ converges and hence $\|\sum_{k=K}^{\infty} \psi_n\chi_k\|_{\mathrm{cb}}$ tends to 0, or equivalently, $\|\sum_{k=0}^{K} \psi_n\chi_k\|_{\mathrm{cb}}$ tends to $\|\psi_n\|_{\mathrm{cb}} = 1$ as K tends to ∞. Assume $s_1, \ldots, s_n$ and $\varepsilon > 0$ are given. Since ψ_n tends to 1 pointwise as n tends to ∞, we can take N so that $|1 - \psi_N(s_i)| < \varepsilon$ holds for all i, $i = 1, \ldots, n$. Also, we can take K so that $\|\sum_{k \geq K} \psi_N\chi_k\|_{\mathrm{cb}} < \varepsilon$ holds. Then $\varphi(s) := \sum_{k \leq K} \psi_N(s)\chi_k(s)$ satisfies that $|1 - \varphi(s_i)| < 2\varepsilon$ for all i, $i = 1, \ldots, n$ and $\|\varphi\|_{\mathrm{cb}} < 1 + \varepsilon$. Hence, if we choose N_n and K_n suitably for all $n \in \mathbb{N}$, $\varphi_n(s) = \sum_{k \leq K_n} \psi_{N_n}(s)\chi_k(s)$ defines a sequence of finitely supported functions on Γ which tends to 1 pointwise whose cb-norm tends to 1. □

Acknowledgment

The author is grateful to his adviser Narutaka Ozawa for constant encouragement and helpful comments and suggestions.

References

[BH] Bridson, Martin R.; Haefliger, André Metric spaces of non-positive curvature. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319. Springer-Verlag, Berlin, 1999. xxii+643 pp.

[BO] Brown, N.; Ozawa, N. C^*-Algebras and Finite-Dimensional Approximations, volume 88 of Graduate Studies in Mathematics. American Mathematical Society, 2008.

[BP] Bożejko, Marek; Picardello, Massimo A. Weakly amenable groups and amalgamated products. Proc. Amer. Math. Soc. 117 (1993), no. 4, 1039–1046.

[C] Chepoi, Victor Graphs of some CAT(0) complexes. Adv. in Appl. Math. 24 (2000), no. 2, 125–179.

[CH] Cowling, Michaël; Haagerup, Uffe Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. Invent. Math. 96 (1989), no. 3, 507–549.

[CaN] Campbell, Sarah; Niblo, Graham A. Hilbert space compression and exactness of discrete groups. J. Funct. Anal. 222 (2005), no. 2, 292–305.

[ChN] Chatterji, Indira; Niblo, Graham From wall spaces to CAT(0) cube complexes. Internat. J. Algebra Comput. 15 (2005), no. 5-6, 875–885.

[CR] Chatterji, Indira; Ruane, Kim Some geometric groups with rapid decay. Geom. Funct. Anal. 15 (2005), no. 2, 311–339.

[Do] Dorofaeff, B. Weak amenability and semidirect products in simple Lie groups. Math. Ann. 306 (1996), 737-742.

[DH] De Cannière, Jean; Haagerup, Uffe Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups. Amer. J. Math. 107 (1985), no. 2, 455–500.

[F] Fendler, Gero Weak amenability of Coxeter groups, ArXiv preprint math.GR/0203052.

[Ge] Geoghegan, R. Topological Methods in Group Theory, Graduate Texts in Mathematics, 243. Springer

[Gr] Gromov, M. Hyperbolic groups. Essays in group theory, 75–263, Math. Sci. Res. Inst. Publ., 8, Springer, New York, 1987.

[GH] Guentner, Erik; Higson, Nigel Weak amenability of CAT(0)-cubical groups, ArXiV preprint math.OA/0702568.

[Ha] Haagerup, Uffe An example of a nonnuclear C^*-algebra, which has the metric approximation property. Invent. Math. 50 (1978/79), no. 3, 279–293.

[Ha2] Haagerup, Uffe Group $C*$-algebras without the completely bounded approximation property, unpublished.

[HK] Haagerup, Uffe; Kraus, Jon Approximation properties for group C^*-algebras and group von Neumann algebras. Trans. Amer. Math. Soc. 344 (1994), no. 2, 667–699.

[J] Januszkiewicz, Tadeusz For Coxeter groups $z^{|g|}$ is a coefficient of a uniformly bounded representation. Fund. Math. 174 (2002), no. 1, 79–86.

[N] Nica, Bogdan Cubulating spaces with walls. (English summary) Algebr. Geom. Topol. 4 (2004), 297–309 (electronic).

[NR1] Niblo, Graham; Reeves, Lawrence Groups acting on CAT(0) cube complexes. Geom. Topol. 1 (1997), approx. 7 pp.

[NR2] Niblo, G. A.; Reeves, L. D. Coxeter groups act on CAT(0) cube complexes. J. Group Theory 6 (2003), no. 3, 399–413.

[P] Paulsen, Vern Completely bounded maps and operator algebras. Cambridge Studies in Advanced Mathematics, 78. Cambridge University Press, Cambridge, 2002. xii+300 pp.

[R] Roller, M Poc-sets, median algebras and group actions. An extended study of Dunwoody's construction and Sageev's theorem, preprint 1998

[S] Sageev, Michah Ends of group pairs and non-positively curved cube complexes. Proc. London Math. Soc. (3) 71 (1995), no. 3, 585–617.

[Wi] Wise, D. T. Cubulating small cancellation groups. Geom. Funct. Anal. 14 (2004), no. 1, 150–214.

[Wo] Woess, Wolfgang Random walks on infinite graphs and groups. Cambridge Tracts in Mathematics, 138. Cambridge University Press, Cambridge, 2000. xii+334 pp.

E-mail address: `mizuta@ms.u-tokyo.ac.jp`