論文題目

The structures of generalized principal series representations of $SL(3, \mathbf{R})$ and related Whittaker functions

($SL(3, \mathbf{R})$の一般主系列表現の構造と関連するWhittaker関数)

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THE STRUCTURES OF GENERALIZED PRINCIPAL SERIES REPRESENTATIONS OF $SL(3,\mathbf{R})$ AND RELATED WHITTAKER FUNCTIONS

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Introduction

When we consider Fourier expansions of automorphic forms on reductive groups, various kinds of spherical functions appear. Among others, one of the important functions is a Whittaker function. In the case of $GL(n)$, there exists a Fourier expansion of an automorphic form in terms of global Whittaker functions, which is found by Piatetski-Shapiro [35]. It is called *the Fourier-Whittaker expansion.* The precise information of local Whittaker functions plays important roles in the various aspects of automorphic forms on $GL(n)$, for example, analysis of local zeta integrals.

Our interest here is archimedean Whittaker functions on $GL(n)$. Jacquet introduced an integral expression of a (primary) Whittaker function for a principal series representation in [17], which is called *the Jacquet integral.* However, the Jacquet integrals are difficult to handle and accordingly archimedean zeta integrals defined by them are also difficult to understand. Hence many authors study the explicit formulas of Whittaker functions which are suitable for number theoretic applications ([2], [10], [16], [27], [40], [41], [42], [45]). They obtained the explicit formulas by evaluating the Jacquet integral or by solving the differential equations. The aim of this thesis is to give the explicit formulas of Whittaker functions on $SL(3,\mathbf{R})$ (or $GL(3,\mathbf{R})$).

Let us explain our problem in a more precise form. Let $G = NAK$ be a reductive Lie group with the Iwasawa decomposition. Here N is a maximal unipotent subgroup and K is a maximal compact subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{g}_{\mathbf{C}}$ the Lie algebra of G and its complexification, respectively. For a non-degenerate unitary character η of N, let $C^\infty_\eta(N\backslash G)$ be the subspace of $C^\infty(G)$ consisting of functions f such that

$$f(ng) = \eta(n)f(g), \qquad (n,g) \in N \times G,$$

on which G acts by the right translation. For an irreducible admissible representation (π, H_π) of G, we set $\mathcal{I}_{\eta,\pi} = \mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}},K)}(H_{\pi,K}, C^\infty_\eta(N\backslash G))$ and $\mathcal{I}^G_{\eta,\pi} = \mathrm{Hom}_G(H^\infty_\pi, C^\infty_\eta(N\backslash G))$. For an element Φ of $\mathcal{I}_{\eta,\pi}$ (resp. $\mathcal{I}^G_{\eta,\pi}$), the functions in the image of Φ are called *the secondary (resp. primary) Whittaker functions.* When $G = GL(n,\mathbf{R})$, if $\mathcal{I}_{\eta,\pi}$ is nonzero, it is known that π is isomorphic to an irreducible generalized principal series ([18, §2]). Moreover, Shalika [38] show that the space $\mathcal{I}^G_{\eta,\pi}$ is at most one dimensional for any irreducible admissible representation of $GL(n,\mathbf{R})$.

In this thesis, we concentrate our attention to the case of $G = GL(3,\mathbf{R})$ or $SL(3,\mathbf{R})$. The explicit formulas of Whittaker functions at the minimal K-type of a principal series representation of $SL(3,\mathbf{R})$ (induced from the minimal parabolic subgroup) have been obtained by Bump [2] and by Manabe, Ishii and Oda [27]. Now we settle two purposes of this thesis as follows:

1. Give the whole structure of the associated $(\mathfrak{g}_{\mathbf{C}},K)$-module of any generalized principal series representation of $SL(3,\mathbf{R})$.
2. Give the explicit formulas of Whittaker functions at the minimal K-type of a irreducible generalized principal series representation of $SL(3,\mathbf{R})$ induced from the maximal parabolic subgroup.

This thesis is divided into three self-complete parts. In Part 1, we accomplish the first purpose. We accomplish the second purpose in Part 2 and 3 by two different ways. Together with the results of principal series, we have the explicit formulas of Whittaker functions at the minimal K-type of any irreducible generalized principal series representation of $GL(3,\mathbf{R})$. In principle, by using the description of the $(\mathfrak{g}_{\mathbf{C}},K)$-module structure in Part 1, we can obtain the explicit formulas of the Whittaker functions at the whole K-types. Here we give a short introduction of each part as follows:

Part 1. The structures of standard $(\mathfrak{g},K)$-modules of $SL(3,\mathbf{R})$.

We give the whole structure of the associated $(\mathfrak{g}_{\mathbf{C}},K)$-module of any generalized principal series representation of $G=SL(3,\mathbf{R})$. The associated $(\mathfrak{g}_{\mathbf{C}},K)$-modules of generalized principal series (π,H_π) of G is realized as a subspace of $L^2(K)$ as a K-module. Peter-Weyl's theorem tells that $L^2(K)$ has a basis consisting of matrix coefficients of simple K-modules as a Hilbert space. Hence we can take the corresponding basis of H_π and the explicit description of the action of K on it. Because of the Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$, in order to describe the action of $\mathfrak{g}_{\mathbf{C}}$, it suffices to investigate the action of $\mathfrak{p}_{\mathbf{C}}$. Our main result of this part is the formulas of the action of $\mathfrak{p}_{\mathbf{C}}$ on the above basis. This part is the reproduction of the paper [30].

Part 2. Whittaker functions for generalized principal series representations of $SL(3,\mathbf{R})$.

We study Whittaker functions at the minimal K-type of a generalized principal series representation π of $G=SL(3,\mathbf{R})$ induced from the maximal parabolic subgroup. By evaluating the $(\mathfrak{g}_{\mathbf{C}},K)$-module structure of π, we give the system of partial differential equations characterizing Whittaker functions at the minimal K-type. We give 6 power series solutions of this system, that is, the power series expression of the secondary Whittaker functions for π. We also give the Mellin-Barnes type integral expressions of the unique solution having the moderate growth property. By Wallach's result [47], we note that this moderate growth solution is the primary Whittaker functions for π. This part is the reproduction of the paper [31].

Part 3. The Eisenstein series for $GL(3,\mathbf{Z})$ induced from cusp forms.

We study the Fourier-Whittaker coefficients of the Eisenstein series for $GL(3,\mathbf{Z})$ induced from cusp forms. First, we give the expression of the Fourier-Whittaker coefficients of the Eisenstein series in terms of the Jacquet integrals. Moreover, by evaluating the Jacquet integrals, we give the Mellin-Barnes type integral expressions of those at the minimal K-type. Of course, by the uniqueness of the primary Whittaker function, these formulas are coincides with the formulas in [2], [27] and Part 2 up to constant multiple. This part is the reproduction of the paper [29].

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Part 1. The structures of standard $(\mathfrak{g}, K)$-modules of $SL(3,\mathbf{R})$.

1. Introduction

For an admissible representation of a real reductive Lie group, the $(\mathfrak{g}, K)$-module structure is a fundamental data. As far as we know, for some 'small' reductive Lie groups G, the $(\mathfrak{g}, K)$-module structures of generalized principal series representations are completely described. For example, the description of them for $SL(2,\mathbf{R})$ is found in standard textbooks, and there are rather complete results for some groups of real rank 1, e.g. $SU(n,1)$ by Kraljević [22] and $Spin(1,2n)$ by Thieleker [44]. Moreover, in recent years, many authors give the explicit description of degenerate principal series representations of several groups, e.g. Fujimura [6], Howe and Tan [14], Lee [25], Lee and Loke [26]. However, for generalized principal series representations of Lie groups of higher rank, there are few references as far as the author knows. It seems to be difficult to describe the whole $(\mathfrak{g}, K)$-module structures of those representations, since their K-types are not multiplicity free. In the paper [34], the $(\mathfrak{g}, K)$-module structures of principal series representations of $Sp(2,\mathbf{R})$ are described by Oda. In a former paper [32], we extend the result for principal series representations of $Sp(3,\mathbf{R})$. The method in these papers is applicable to study of generalized principal series representations of other groups. In this part, we use this method to study the associated $(\mathfrak{g}, K)$-modules of generalized principal series of $SL(3,\mathbf{R})$.

Before describing the case of $SL(3,\mathbf{R})$, let us explain the problem in a more precise form for a general real semisimple Lie group G with its Lie algebra $\mathfrak{g}$. Fix a maximal compact subgroup K of G. Since the associated $(\mathfrak{g}, K)$-modules of generalized principal series are realized as subspaces of $L^2(K)$ as K-modules, we can investigate those K-module structures by Peter-Weyl's theorem. In order to describe the action of $\mathfrak{g}$ or $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$, it suffices to investigate the action of $\mathfrak{p}$ or $\mathfrak{p}_{\mathbf{C}}$, because of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. To study the action of $\mathfrak{p}_{\mathbf{C}}$, we compute the linear map $\Gamma_{\tau,i}$ defined as follows.

Let (π, H_π) be a generalized principal series representation of G with its subspace $H_{\pi,K}$ of K-finite vectors. For a K-type (τ, V_τ) of π and a K-homomorphism $\eta\colon V_\tau \to H_{\pi,K}$, we define a linear map $\tilde{\eta}\colon V_\tau \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}} \to H_{\pi,K}$ by $v \otimes X \mapsto \pi(X)\eta(v)$. Then $\tilde{\eta}$ is a K-homomorphism with $\mathfrak{p}_{\mathbf{C}}$ endowed with the adjoint action Ad of K. Let $V_\tau \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}} \simeq \bigoplus_{i\in I} V_{\tau_i}$ be the irreducible decomposition as a K-module and fix ι_i an injective K-homomorphism from V_{τ_i} to $V_\tau \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}$ for each i. We define a linear map $\Gamma_{\tau,i}\colon \mathrm{Hom}_K(V_\tau, H_{\pi,K}) \to \mathrm{Hom}_K(V_{\tau_i}, H_{\pi,K})$ by $\eta \mapsto \tilde{\eta} \circ \iota_i$. These linear maps $\Gamma_{\tau,i}$ $(i \in I)$ characterize the action of $\mathfrak{p}_{\mathbf{C}}$. The goal of this part is to give explicit expressions of ι_i and $\Gamma_{\tau,i}$ for any generalized principal series representation π of $G = SL(3,\mathbf{R})$. As a result, we obtain infinite number of 'contiguous relations', a kind of system of differential-difference relations among vectors in $H_\pi[\tau]$ and $H_\pi[\tau_i]$. Here $H_\pi[\tau]$ is τ-isotypic component of H_π. These are described in Proposition 3.2, Theorem 4.5 and 5.5. We remark that R. Howe give another description of $\Gamma_{\tau,i}$ in [12] when π is a principal series representation of $GL(3,\mathbf{R})$.

As an application, we can utilize the contiguous relations to obtain the explicit formulas of some spherical functions. In the paper [27], Manabe, Ishii and Oda give the explicit formulas of Whittaker functions for principal series representations of $SL(3,\mathbf{R})$ to solve the holonomic system of differential equations characterizing those functions, which is derived from the Capelli elements and the contiguous relations around minimal K-type. We can obtain the holonomic systems characterizing Whittaker functions for generalized principal series representations of $SL(3,\mathbf{R})$ induced from the maximal parabolic subgroup by using the result of this part. We give the explicit formulas of Whittaker functions by solving this system in Part 2. On the other hand, if we have the explicit formula of Whittaker function with a certain K-type, then we can give those with another K-type by using contiguous relations.

We give the contents of this part. In Section 2, we recall the structure of $SL(3,\mathbf{R})$ and define generalized principal series representations. In Section 3, we introduce the standard basis of a finite dimensional irreducible representation of K and give explicit expressions of $\iota_i\colon V_{\tau_i} \to V_\tau \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}$. In Section 4, we introduce the general setting of this part and give matrix representations of $\Gamma_{\tau,i}$ for principal series representations (Theorem 4.5). In Section 5, we give the matrix representations of $\Gamma_{\tau,i}$ for generalized principal series representations of $SL(3,\mathbf{R})$ induced from the maximal parabolic subgroup (Theorem 5.5). In Section 6, we give explicit expressions of the action of $\mathfrak{p}_{\mathbf{C}}$ (Proposition 6.2).

2. Preliminaries

2.1. **Groups and algebras.** We denote by $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ the ring of rational integers, the real number field and the complex number field, respectively. Let $\mathbf{Z}_{\geq 0}$ be the set of non-negative integers, 1_n the unit matrix of size n and $O_{m,n}$ the zero matrix of size $m\times n$ and E_{ij} the matrix of size 3 with 1 at (i,j)-th entry and 0 at other entries. We denote by δ_{ij} the Kronecker delta, i.e.

$$\delta_{ij} = \begin{cases} 1, & i=j, \\ 0, & \text{otherwise}. \end{cases}$$

For a Lie algebra $\mathfrak{l}$, we denote by $\mathfrak{l}_{\mathbf{C}} = \mathfrak{l} \otimes_{\mathbf{R}} \mathbf{C}$ the complexification of $\mathfrak{l}$.

Let G be the special linear group $SL(3,\mathbf{R})$ of degree three and $\mathfrak{g}$ its Lie algebra. We define a Cartan involution θ of G by $G \ni g \mapsto {}^tg^{-1} \in G$. Here ${}^tg$ and g^{-1} means the transpose and the inverse of g, respectively. Then the maximal compact subgroup of G is given by

$$K = \{g\in G \mid \theta(g) = g\} = SO(3).$$

If we denote the differential of θ again by θ, then we have $\theta(X) = -{}^tX$ for $X\in\mathfrak{g}$. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and the -1 eigenspaces of θ in $\mathfrak{g}$, respectively, that is,

$$\mathfrak{k} = \{X\in\mathfrak{g} \mid {}^tX = -X\} = \mathfrak{so}(3), \qquad \mathfrak{p} = \{X\in\mathfrak{g} \mid {}^tX = X\}.$$

Then $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{g}$ has the Cartan decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$.

Put $\mathfrak{a}_0 = \{\mathrm{diag}(t_1,t_2,t_3) \mid t_i\in\mathbf{R}\ (1\leq i\leq 3),\ t_1+t_2+t_3=0\}$. Then $\mathfrak{a}_0$ is a maximal abelian subalgebra of $\mathfrak{p}$. For each $1\leq i\leq 3$, we define a linear form e_i on $\mathfrak{a}_0$ by $\mathfrak{a}_0 \ni \mathrm{diag}(t_1,t_2,t_3) \mapsto t_i\in\mathbf{C}$. The set Σ of the roots for $(\mathfrak{a}_0,\mathfrak{g})$ is given by $\Sigma = \Sigma(\mathfrak{a}_0,\mathfrak{g}) = \{e_i - e_j \mid 1\leq i\neq j\leq 3\}$, and the subset $\Sigma^+ = \{e_i-e_j \mid 1\leq i<j\leq 3\}$ forms a positive root system. For each $\alpha\in\Sigma$, we denote the root space by $\mathfrak{g}_\alpha$ and choose a root vector E_α in $\mathfrak{g}_\alpha$ by $E_{e_i-e_j} = E_{ij}$ $(1\leq i\neq j\leq 3)$.

If we put $\mathfrak{n}_0 = \bigoplus_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha$, then $\mathfrak{g}$ has an Iwasawa decomposition $\mathfrak{g} = \mathfrak{n}_0\oplus\mathfrak{a}_0\oplus\mathfrak{k}$. Also we have $G = N_0A_0K$, where $N_0 = \exp(\mathfrak{n}_0)$ and $A_0 = \exp(\mathfrak{a}_0)$.

Let $\mathfrak{n}_1$, $\mathfrak{n}_2$ be subalgebras of $\mathfrak{n}_0$ defined by $\mathfrak{n}_1 = \mathfrak{g}_{e_1-e_2}\oplus\mathfrak{g}_{e_1-e_3}$, $\mathfrak{n}_2 = \mathfrak{g}_{e_1-e_3}\oplus\mathfrak{g}_{e_2-e_3}$. We take a basis $\{H_1,H_2\}$ of $\mathfrak{a}_0$ by

$$H_1 = \mathrm{diag}(1,0,-1), \qquad H_2 = \mathrm{diag}(0,1,-1),$$

and set $H^{(1)} = 2H_1 - H_2$, $H^{(2)} = H_1 + H_2$. we define subalgebras $\mathfrak{a}_1$, $\mathfrak{a}_2$ of $\mathfrak{a}_0$ by $\mathfrak{a}_1 = \mathbf{R}\cdot H^{(1)}$, $\mathfrak{a}_2 = \mathbf{R}\cdot H^{(2)}$. The group G has three non-trivial standard parabolic subgroups P_0, P_1, P_2 with Langland decompositions $P_i = N_iA_iM_i$ $(0\leq i\leq 2)$ where

$$\begin{aligned}
M_0 &= \{\mathrm{diag}(\varepsilon_1,\varepsilon_2,\varepsilon_1\varepsilon_2) \mid \varepsilon_i\in\{\pm1\}\ (1\leq i\leq 2)\},\\
M_1 &= \left\{\left.\begin{pmatrix} \det(h)^{-1} & O_{1,2} \\ O_{2,1} & h \end{pmatrix}\right| h\in SL^{\pm}(2,\mathbf{R})\right\},\\
M_2 &= \left\{\left.\begin{pmatrix} h & O_{2,1} \\ O_{1,2} & \det(h)^{-1} \end{pmatrix}\right| h\in SL^{\pm}(2,\mathbf{R})\right\},\\
A_i &= \exp(\mathfrak{a}_i) \quad N_i = \exp(\mathfrak{n}_i) \quad (i=1,2).
\end{aligned}$$

Here $SL^{\pm}(2,\mathbf{R}) = \{g\in GL(2,\mathbf{R}) \mid \det(g) = \pm1\}$. For $i=1,2$, let $\mathfrak{m}_i$ be a Lie algebra of M_i.

2.2. **Definition of the P_i-principal series representations of G.** For $0 \le i \le 2$, in order to define the P_i-principal series representation of G, we prepare the data (ν_i, σ_i) as follows.

For $\nu_0 \in \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}_0, \mathbf{C})$, we define a coordinate $(\nu_{0,1}, \nu_{0,2}) \in \mathbf{C}^2$ by $\nu_{0,i} = \nu_0(H_i)$ $(i = 1, 2)$. Then the half sum $\rho_0 = e_1 - e_3$ of the positive roots has coordinate $(\rho_{0,1}, \rho_{0,2}) = (2, 1)$. We define a quasicharacter $e^{\nu_0}\colon A_0 \to \mathbf{C}^\times$ by

$$e^{\nu_0}(a) = a_1^{\nu_{0,1}} a_2^{\nu_{0,2}}, \quad a = \mathrm{diag}(a_1, a_2, a_3) \in A_0.$$

We fix a character σ_0 of M_0. It is convenient to identify σ_0 with $(\sigma_{0,1}, \sigma_{0,2}) \in \{0,1\}^{\oplus 2}$ determined by

$$\sigma_0(\mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2)) = \varepsilon_1^{\sigma_{0,1}} \varepsilon_2^{\sigma_{0,2}}, \qquad \varepsilon_1, \varepsilon_2 \in \{\pm 1\}.$$

For each $i = 1, 2$, we identify $\nu_i \in \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}_i, \mathbf{C})$ with a complex number $\nu_i(H^{(i)}) \in \mathbf{C}$. Let ρ_i $(i = 1, 2)$ be the half sums of positive roots whose root spaces are contained in $\mathfrak{n}_i$, i.e. $\rho_1 = \frac{1}{2}(2e_1 - e_2 - e_3)$, $\rho_2 = \frac{1}{2}(e_1 + e_2 - 2e_3)$. Then both ρ_1 and ρ_2 are identified with 3. We fix a discrete series representation σ_i of $M_i \simeq SL^{\pm}(2, \mathbf{R})$ for $i = 1, 2$.

Definition 2.1. *For $0 \le i \le 2$, we define the P_i-principal series representation $\pi_{(\nu_i,\sigma_i)}$ of G by*

$$\pi_{(\nu_i,\sigma_i)} = \mathrm{Ind}_{P_i}^{G}(1_{N_i} \otimes e^{\nu_i + \rho_i} \otimes \sigma_i),$$

i.e. $\pi_{(\nu_i,\sigma_i)}$ is the right regular representation of G on the space $H_{(\nu_i,\sigma_i)}$ which is the completion of

$$H^{\infty}_{(\nu_i,\sigma_i)} = \left\{ f\colon G \to V^{\infty}_{\sigma_i} \text{ smooth} \;\middle|\; \begin{array}{l} f(namx) = e^{\nu_i+\rho_i}(a)\sigma_i(m) f(x) \\ \text{for } n \in N_i,\ a \in A_i,\ m \in M_i,\ x \in G \end{array} \right\}$$

with respect to the norm

$$\|f\|^2 = \int_K \|f(k)\|^2_{\sigma_i} dk.$$

Here $V^{\infty}_{\sigma_i}$ is the smooth part of the representation space V_{σ_i} of σ_i and $\|\cdot\|_{\sigma_i}$ is the norm on V_{σ_i}.

Remark 2.2. *The P_i-principal series representations are also called standard representations or generalized principal representations.*

3. Representations of $K = SO(3)$

3.1. **The spinor covering.** To describe the finite dimensional representations of $SO(3)$, the simplest way seems to be the one utilizing the double covering $\varphi\colon SU(2) = Spin(3) \to SO(3)$. We use the following realization introduced in [27].

We define $\varphi\colon SU(2) \to SO(3)$ by

$$\varphi(x) = \begin{pmatrix} p^2+q^2-r^2-s^2 & -2(ps-qr) & 2(pr+qs) \\ 2(ps+qr) & p^2-q^2+r^2-s^2 & -2(pq-rs) \\ -2(pr-qs) & 2(pq+rs) & p^2-q^2-r^2+s^2 \end{pmatrix}$$

for $x = \begin{pmatrix} p+\sqrt{-1}q & r+\sqrt{-1}s \\ -r+\sqrt{-1}s & p-\sqrt{-1}q \end{pmatrix} \in SU(2)$ $(p, q, r, s \in \mathbf{R})$. Then φ is a surjective homomorphism whose kernel is given by $\{\pm 1_2\}$.

The differential $d\varphi\colon \mathfrak{su}(2) \to \mathfrak{so}(3)$ of φ is an isomorphism and it maps the basis

$$u_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad u_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

to $-2K_{23}$, $2K_{13}$, $-2K_{12}$. Here $K_{ij} = E_{ij} - E_{ji}$ $(1 \le i < j \le 3)$.

3.2. **Representations of $SU(2)$.** The set of equivalence classes of the finite dimensional continuous representations of $SU(2)$ is exhausted by the symmetric tensor product τ_l ($l \in \mathbf{Z}_{\geq 0}$) of the representation $SU(2) \ni g \mapsto (v \mapsto g \cdot v) \in GL(\mathbf{C}^2)$. We use the following realizations of those which are introduced in [27].

Let V_l be the subspace consisting of degree l homogeneous polynomials of two variables x, y in the polynomial ring $\mathbf{C}[x, y]$. For $g \in SU(2)$ with $g^{-1} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ and $f(x, y) \in V_l$ we set

$$\tau_l(g)f(x, y) = f(ax + by, -\bar{b}x + \bar{a}y).$$

Passing to the Lie algebra $\mathfrak{su}(2)$, the derivation of τ_l, denoted by same symbol, is described as follows by using the standard basis $\{v_k = x^k y^{l-k} \mid 0 \leq k \leq l\}$ of V_l and the basis $\{u_1, u_2, u_3\}$ of $\mathfrak{su}(2)$. Namely we have

$$\tau_l(H)v_k = (l - 2k)v_k, \qquad \tau_l(E)v_k = -kv_{k-1}, \qquad \tau_l(F)v_k = (k - l)v_{k+1}.$$

Here $\{E,\ H,\ F\}$ is a $\mathfrak{sl}_2$-triple defined by

$$H = -\sqrt{-1}u_1, \qquad E = \frac{1}{2}(u_2 - \sqrt{-1}u_3), \qquad F = -\frac{1}{2}(u_2 + \sqrt{-1}u_3) \in \mathfrak{su}(2)_{\mathbf{C}}.$$

The condition that τ_l defines a representation of $SO(3)$ by passing to the quotient with respect to $\varphi \colon SU(2) \to SO(3)$ is that $\tau_l(-1_2) = (-1)^l = 1$, i.e. l is even. For $l \in \mathbf{Z}_{\geq 0}$, we denote the irreducible representation of $SO(3)$ induced from (τ_{2l}, V_{2l}) again by (τ_{2l}, V_{2l}).

3.3. **The adjoint representation of K on $\mathfrak{p}_{\mathbf{C}}$.** It is known that $\mathfrak{p}_{\mathbf{C}}$ becomes a K-module via the adjoint action of K. Concerning this, we have the following lemma.

Lemma 3.1. *Let $\{w_j \mid 0 \leq j \leq 4\}$ be the standard basis of (τ_4, V_4) and $\{X_j \mid 0 \leq j \leq 4\}$ be a basis of $\mathfrak{p}_{\mathbf{C}}$ defined as follows:*

$$X_0 = H_2 - \sqrt{-1}(E_{23} + E_{32}), \qquad X_1 = -\frac{1}{2}\{\sqrt{-1}(E_{12} + E_{21}) + (E_{13} + E_{31})\},$$
$$X_2 = -\frac{1}{3}(2H_1 - H_2), \qquad X_3 = -\frac{1}{2}\{\sqrt{-1}(E_{12} + E_{21}) - (E_{13} + E_{31})\},$$
$$X_4 = H_2 + \sqrt{-1}(E_{23} + E_{32}).$$

Then via the isomorphism between V_4 and $\mathfrak{p}_{\mathbf{C}}$ as K-modules we have the identification $w_j = X_j$ $(0 \leq j \leq 4)$.

Proof. By direct computation, we have Table. 1, which gives the adjoint actions of the basis $\{d\varphi(E),\ d\varphi(H),\ d\varphi(F)\}$ of $\mathfrak{k}_{\mathbf{C}}$ on the basis $\{X_j \mid 0 \leq j \leq 4\}$ of $\mathfrak{p}_{\mathbf{C}}$. Comparing the actions in the above with the actions in Subsection 3.2, we obtain the assertion. □

TABLE 1. The adjoint actions of $\mathfrak{k}_{\mathbf{C}}$ on $\mathfrak{p}_{\mathbf{C}}$.

	X_0	X_1	X_2	X_3	X_4
$d\varphi(H)$	$4X_0$	$2X_1$	0	$-2X_3$	$-4X_4$
$d\varphi(E)$	0	$-X_0$	$-2X_1$	$-3X_2$	$-4X_3$
$d\varphi(F)$	$-4X_1$	$-3X_2$	$-2X_3$	$-1X_4$	0

3.4. **Clebsch-Gordan coefficients for the representations of $\mathfrak{sl}(2, \mathbf{C})$ with respect to the standard basis.** For later use, we consider the irreducible decomposition of $V_l \otimes_{\mathbf{C}} V_4$ as $\mathfrak{sl}(2, \mathbf{C}) = \mathfrak{su}(2)_{\mathbf{C}}$-modules for arbitrary non-negative integer l.

Generically, the tensor product $V_l \otimes_{\mathbf{C}} V_4$ has five irreducible components $V_{l+4},\ V_{l+2},\ V_l,\ V_{l-2}$ and V_{l-4}. Here some components may vanish. We give an explicit expression of a nonzero $\mathfrak{sl}(2, \mathbf{C})$-homomorphism from each irreducible component to $V_l \otimes_{\mathbf{C}} V_4$ as follows.

Proposition 3.2. *Let* $\{v_k^{(l)} \mid 0 \le k \le l\}$ *be the standard basis of* V_l *for* $l \in \mathbf{Z}_{\ge 0}$. *We put* $v_k^{(l)} = 0$ *when* $k < 0$ *or* $k > l$.

If V_{l+2m}*-component of* $V_l \otimes_{\mathbf{C}} V_4$ *does not vanish, then we define linear maps* $I_{2m}^l \colon V_{l+2m} \to V_l \otimes_{\mathbf{C}} V_4$ $(-2 \le m \le 2)$ *by*

$$I_{2m}^l(v_k^{(l+2m)}) = \sum_{i=0}^{4} A_{[l,2m;k,i]} \cdot v_{k+2-m-i}^{(l)} \otimes w_i.$$

Here the coefficients $A_{[l,2m;k,i]} = a(l,2m;k,i)/d(l,2m)$ *are defined by following formulas.*

Formula 1: *The coefficients of* $I_4^l \colon V_{l+4} \to V_l \otimes_{\mathbf{C}} V_4$ *are given as follows:*

$$\begin{aligned}
a(l,4;k,0) =&(l+4-k)(l+3-k)(l+2-k)(l+1-k),\\
a(l,4;k,1) =&4(l+4-k)(l+3-k)(l+2-k)k,\\
a(l,4;k,2) =&6(l+4-k)(l+3-k)k(k-1),\\
a(l,4;k,3) =&4(l+4-k)k(k-1)(k-2),\\
a(l,4;k,4) =&k(k-1)(k-2)(k-3),\\
d(l,4) =&(l+4)(l+3)(l+2)(l+1).
\end{aligned}$$

Formula 2: *The coefficients of* $I_2^l \colon V_{l+2} \to V_l \otimes_{\mathbf{C}} V_4$ *are given as follows:*

$$\begin{aligned}
a(l,2;k,0) =&(l+2-k)(l+1-k)(l-k),\\
a(l,2;k,1) =&-(l+2-k)(l+1-k)(l-4k),\\
a(l,2;k,2) =&-3(l+2-k)(l-2k+2)k,\\
a(l,2;k,3) =&-(3l-4k+8)k(k-1),\\
a(l,2;k,4) =&-k(k-1)(k-2), \quad d(l,2) = (l+2)(l+1)l.
\end{aligned}$$

Formula 3: *The coefficients of* $I_0^l \colon V_l \to V_l \otimes_{\mathbf{C}} V_4$ *are given as follows:*

$$\begin{aligned}
a(l,0;k,0) =&(l-k)(l-1-k), & a(l,0;k,1) =&-2(l-k)(l-2k-1),\\
a(l,0;k,2) =&(l^2-6kl+6k^2-l), & a(l,0;k,3) =&2(l-2k+1)k,\\
a(l,0;k,4) =&k(k-1), & d(l,0) =&l(l-1).
\end{aligned}$$

Formula 4: *The coefficients of* $I_{-2}^l \colon V_{l-2} \to V_l \otimes_{\mathbf{C}} V_4$ *are given as follows:*

$$\begin{aligned}
a(l,-2;k,0) =&(l-k-2), & a(l,-2;k,1) =&-(3l-4k-6),\\
a(l,-2;k,2) =&3(l-2k-2), & a(l,-2;k,3) =&-(l-4k-2),\\
a(l,-2;k,4) =&-k, & d(l,-2) =&l-2.
\end{aligned}$$

Formula 5: *The coefficients of* $I_{-4}^l \colon V_{l-4} \to V_l \otimes_{\mathbf{C}} V_4$ *are given as follows:*

$$\begin{aligned}
a(l,-4;k,0) =&1, & a(l,-4;k,1) =&-4, & a(l,-4;k,2) =&6,\\
a(l,-4;k,3) =&-4, & a(l,-4;k,4) =&1, & d(l,-4) =&1.
\end{aligned}$$

Then I_{2m}^l *is a generator of* $\mathrm{Hom}_{\mathfrak{sl}(2,\mathbf{C})}(V_{l+2m}, V_l \otimes_{\mathbf{C}} V_4)$, *which is unique up to scalar multiple.*

Proof. We have

$$\begin{aligned}
&(\tau_l \otimes \tau_4)(E) \circ I_{2m}^l(v_0^{(l+2m)})\\
&= \sum_{i=0}^{4} A_{[l,2m;0,i]} \cdot (\tau_l(E) v_{2-m-i}^{(l)}) \otimes w_i + \sum_{i=0}^{4} A_{[l,2m;0,i]} \cdot v_{2-m-i}^{(l)} \otimes (\tau_4(E) w_i)\\
&= -\sum_{i=0}^{4} ((2-m-i) A_{[l,2m;0,i]} + (i+1) A_{[l,2m;0,i+1]}) \cdot v_{1-m-i}^{(l)} \otimes w_i.
\end{aligned}$$

Here we put $A_{[l,2m;0,5]}=0$. By direct computation, we confirm

$$(2-m-i)A_{[l,2m;0,i]}+(i+1)A_{[l,2m;0,i+1]}=0$$

for $-2\le m\le 2$ and $0\le i\le 4$. Hence

$$(\tau_l\otimes\tau_4)(E)\circ I^l_{2m}(v_0^{(l+2m)})=0.$$

Moreover, we have

$$(\tau_l\otimes\tau_4)(H)\circ I^l_{2m}(v_0^{(l+2m)})=(l+2m)I^l_{2m}(v_0^{(l+2m)}),$$

since

$$\begin{aligned}(\tau_l\otimes\tau_4)(H)(v_i^{(l)}\otimes w_j)&=(\tau_l(H)v_i^{(l)})\otimes w_j+v_i^{(l)}\otimes(\tau_4(H)w_j)\\&=(l+4-2i-2j)v_i^{(l)}\otimes w_j.\end{aligned}$$

This means $I^l_{2m}(v_0^{(l+2m)})$ is the highest weight vector of the V_{l+2m}-component of $V_l\otimes_{\mathbf{C}}V_4$ with respect to a Borel subalgebra $(\mathbf{C}\cdot H)\oplus(\mathbf{C}\cdot E)$ of $\mathfrak{sl}(2,\mathbf{C})$.

Therefore, in order to complete the proof, it suffices to confirm

$$(\tau_l\otimes\tau_4)(F)\circ I^l_{2m}(v_k^{(l+2m)})=I^l_{2m}\circ\tau_{l+2m}(F)(v_k^{(l+2m)})$$

for each $0\le k\le l+2m$.

We confirm these equations by direct computation. □

The coefficients $A_{[l,2m;k,i]}$ in the above proposition have the following relations.

Lemma 3.3. *The coefficients $A_{[l,2m;k,i]}$ in Proposition 3.2 satisfy following relations:*

$$A_{[l,2m;l+2m-k,0]}=(-1)^mA_{[l,2m;k,4]},\qquad A_{[l,2m;l+2m-k,2]}=(-1)^mA_{[l,2m;k,2]},$$
$$3\{(k-m+1)A_{[l,2m;k,1]}+(l-k+m+1)A_{[l,2m;k,3]}\}=(ml+m^2+m-6)A_{[l,2m;k,2]}.$$

for $-2\le m\le 2$ and $0\le k\le l+2m$.

Proof. These are obtained by direct computation. □

3.5. **The contragradient representation of** (τ_l,V_l). We denote by (τ^*,V^*) the contragradient representation of (τ,V). Here we note that V_l^* is equivalent to V_l as $SU(2)$-modules, since the irreducible $l+1$-dimensional representation of $SU(2)$ is unique up to isomorphism.

Lemma 3.4. *Let $\{v_k^{(l)*}\mid 0\le k\le l\}$ be the dual basis of the standard basis $\{v_k^{(l)}\mid 0\le k\le l\}$. Via the isomorphism between V_l and V_l^* as K-modules we have the identification*

$$v_k^{(l)}=(-1)^k\frac{(l-k)!k!}{l!}v_{l-k}^{(l)*}$$

for $0\le k\le l$.

Proof. We denote by $\langle,\rangle$ the canonical pairing on $V_l^*\otimes_{\mathbf{C}}V_l$.

Since

$$\langle\tau_l^*(H)v_k^{(l)*},v_m^{(l)}\rangle=-\langle v_k^{(l)*},\tau_l(H)v_m^{(l)}\rangle=(2m-l)\delta_{km}=(2k-l)\delta_{km},$$

we have $\tau_l^*(H)v_k^{(l)*}=(2k-l)v_k^{(l)*}$. Similarly, we obtain

$$\tau_l^*(E)v_k^{(l)*}=(k+1)v_{k+1}^{(l)*},\qquad \tau_l^*(F)v_k^{(l)*}=(l-k+1)v_{k-1}^{(l)*}.$$

From these equations, the identification $v_0^{(l)}=v_l^{(l)*}$ determines the isomorphism in the statement. □

4. THE $(\mathfrak{g}, K)$-MODULE STRUCTURES OF THE P_0-PRINCIPAL SERIES REPRESENTATIONS

4.1. **The irreducible decomposition of $\pi_{(\nu_0,\sigma_0)}|_K$ as a K-module.** We set

$$L^2_{(M_0,\sigma_0)}(K) = \{f \in L^2(K) \mid f(mx) = \sigma_0(m)f(x) \text{ for a.e. } m \in M,\ x \in K\}$$

and give a K-module structure by the right regular action of K. Then the restriction map $r_K\colon H_{(\nu_0,\sigma_0)} \ni f \mapsto f|_K \in L^2_{(M_0,\sigma_0)}(K)$ is an isomorphism of K-modules.

The space $L^2(K)$ has a $K \times K$-bimodule structure by the two sided regular action:

$$((k_1,k_2)f)(x) = f(k_1^{-1}xk_2), \quad x \in K,\ f \in L^2(K),\ (k_1,k_2) \in K \times K.$$

Then we define a homomorphism $\Phi_l\colon V_{2l}^* \otimes_{\mathbf{C}} V_{2l} \to L^2(K)$ of $K \times K$-bimodules by

$$w \otimes v \mapsto (x \mapsto \langle w, \tau_{2l}(x)v\rangle).$$

Then Peter-Weyl's theorem tells that

$$\widehat{\bigoplus_{l\in\mathbf{Z}_{\geq 0}}} \Phi_l\colon \widehat{\bigoplus_{l\in\mathbf{Z}_{\geq 0}}} V_{2l}^* \otimes_{\mathbf{C}} V_{2l} \to L^2(K)$$

is an isomorphism as $K \times K$-bimodules. Here $\widehat{\bigoplus}$ means a Hilbert space direct sum.

Since $L^2_{(M_0,\sigma_0)}(K) \subset L^2(K)$, we have an irreducible decomposition of $L^2_{(M_0,\sigma_0)}(K)$:

$$L^2_{(M_0,\sigma_0)}(K) \simeq \widehat{\bigoplus_{l\in\mathbf{Z}_{\geq 0}}} (V_{2l}^*[\sigma_0]) \otimes_{\mathbf{C}} V_{2l}.$$

Here $V[\sigma_0]$ means the σ_0-isotypic component in $(\tau|_{M_0}, V)$ for a K-module (τ, V). Therefore we obtain an isomorphism

$$r_K^{-1} \circ \widehat{\bigoplus_{l\in\mathbf{Z}_{\geq 0}}} \Phi_l\colon \widehat{\bigoplus_{l\in\mathbf{Z}_{\geq 0}}} (V_{2l}^*[\sigma_0]) \otimes_{\mathbf{C}} V_{2l} \to H_{(\nu_0,\sigma_0)}.$$

Since M_0 is generated by the two elements

$$m_{0,1} = \mathrm{diag}(-1,1,-1), \qquad m_{0,2} = \mathrm{diag}(1,-1,-1) \in M_0,$$

we note that $v \in V_{2l}[\sigma_0]$ if and only if

$$\tau_{2l}(m_{0,i})v = \sigma_0(m_{0,i})v = (-1)^{\sigma_{0,i}}v \qquad (i = 1,2)$$

for $v \in V_{2l}$. From the definition of (τ_{2l}, V_{2l}) and

$$\varphi^{-1}(m_{0,1}) = \left\{\pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}, \qquad \varphi^{-1}(m_{0,2}) = \left\{\pm\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}\right\},$$

we have $\tau_{2l}(m_{0,1})v_k^{(2l)} = (-1)^k v_{2l-k}^{(2l)}$ and $\tau_{2l}(m_{0,2})v_k^{(2l)} = (-1)^{l-k}v_k^{(2l)}$. Hence we have

$$V_{2l}[\sigma_0] = \bigoplus_{k\in Z(\sigma_0;l)} \mathbf{C}\cdot(v_{2l-k}^{(2l)} + (-1)^{\varepsilon(\sigma_0;l)}v_k^{(2l)}),$$

where $\varepsilon(\sigma_0;l) \in \{0,1\}$ such that $\varepsilon(\sigma_0;l) \equiv l - \sigma_{0,1} - \sigma_{0,2} \bmod 2$ and

$$Z(\sigma_0;l) = \begin{cases} \{k \in \mathbf{Z} \mid 0 \leq k \leq l,\ k \equiv l - \sigma_{0,2} \bmod 2\} & \text{if } \varepsilon(\sigma_0;l) = 0, \\ \{k \in \mathbf{Z} \mid 0 \leq k \leq l-1,\ k \equiv l - \sigma_{0,2} \bmod 2\} & \text{if } \varepsilon(\sigma_0;l) = 1. \end{cases}$$

We see that $\{v_{2l-k}^{(2l)*} + (-1)^{\varepsilon(\sigma_0;l)}v_k^{(2l)*} \mid k \in Z(\sigma_0;l)\}$ is the basis of $V_{2l}^*[\sigma_0]$, by using the identification $V_{2l}^* = V_{2l}$ in Lemma 3.3.

Now we define the elementary function $s(l;p,q) \in H_{(\nu_0,\sigma_0)}$ by

$$s(l;p,q) = r_K^{-1} \circ \Phi_l((v_{2l-p}^{(2l)*} + (-1)^{\varepsilon(\sigma_0;l)}v_p^{(2l)*}) \otimes v_q^{(2l)})$$

for $l \in \mathbf{Z}_{\geq 0}$, $p \in Z(\sigma_0;l)$ and $0 \leq q \leq 2l$.

For each $p \in Z(\sigma_0; l)$, we put $S(l;p)$ a column vector of degree $2l+1$ whose $q+1$-th component is $s(l;p,q)$, i.e. ${}^t(\ s(l;p,0),\ s(l;p,1),\ \cdots\ , s(l;p,2l)\ )$.

Moreover we denote by $\langle S(l;p)\rangle$ the subspace of $H_{(\nu_0,\sigma_0)}$ generated by the functions in the entries of the vector $S(l;p)$, i.e. $\langle S(l;p)\rangle = \bigoplus_{q=0}^{2l} \mathbf{C}\cdot s(l;p,q) \simeq V_{2l}$. Via the isomorphism between $\langle S(l;p)\rangle$ and V_{2l}, we identify $\{s(l;p,q) \mid 0 \leq q \leq 2l\}$ with the standard basis.

From above arguments, we obtain the following.

Proposition 4.1. *As an unitary representation of K, it has an irreducible decomposition:*

$$H_{(\nu_0,\sigma_0)} \simeq \widehat{\bigoplus_{l\in\mathbf{Z}_{\geq 0}}} (V_{2l}^*[\sigma_0]) \otimes_{\mathbf{C}} V_{2l}.$$

Then the τ_{2l}-isotypic component of $\pi_{(\nu_0,\sigma_0)}$ is given by

$$\bigoplus_{p\in Z(\sigma_0;l)} \langle S(l;p)\rangle.$$

Corollary 4.2. *The multiplicity $d(\sigma_0;l)$ of τ_{2l} in $\pi_{(\nu_0,\sigma_0),K}$ is given by*

$$d(\sigma_0;l) = \begin{cases} (l+2)/2 & \text{if } (\sigma_{0,1},\sigma_{0,2}) = (0,0) \text{ and } l \text{ is even,} \\ (l-1)/2 & \text{if } (\sigma_{0,1},\sigma_{0,2}) = (0,0) \text{ and } l \text{ is odd,} \\ l/2 & \text{if } (\sigma_{0,1},\sigma_{0,2}) \neq (0,0) \text{ and } l \text{ is even,} \\ (l+1)/2 & \text{if } (\sigma_{0,1},\sigma_{0,2}) \neq (0,0) \text{ and } l \text{ is odd.} \end{cases}$$

4.2. **General setting.** Let $H_{(\nu_i,\sigma_i),K}$ be the K-finite part of $H_{(\nu_i,\sigma_i)}$. In order to describe the action of $\mathfrak{g}$ or $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$, it suffices to investigate the action of $\mathfrak{p}$ or $\mathfrak{p}_{\mathbf{C}}$, because of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

For a K-type (τ_{2l}, V_{2l}) of $\pi_{(\nu_i,\sigma_i)}$ and a K-homomorphism $\eta\colon V_{2l} \to H_{(\nu_i,\sigma_i),K}$, we define a linear map

$$\tilde{\eta}\colon V_{2l} \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}} \to H_{(\nu_i,\sigma_i),K}$$

by $v \otimes X \mapsto \pi_{(\nu_i,\sigma_i)}(X)\eta(v)$. Here we denote differential of $\pi_{(\nu_i,\sigma_i)}$ again by $\pi_{(\nu_i,\sigma_i)}$. Then $\tilde{\eta}$ is K-homomorphism with $\mathfrak{p}_{\mathbf{C}}$ endowed with the adjoint action Ad of K.

Since

$$V_{2l} \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}} \simeq V_{2l} \otimes_{\mathbf{C}} V_4 \simeq \bigoplus_{-2\leq m\leq 2} V_{2(l+m)},$$

there are five injective K-homomorphisms

$$I_{2m}^{2l}\colon V_{2(l+m)} \to V_{2l} \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}, \qquad -2 \leq m \leq 2$$

for general $l \in \mathbf{Z}_{\geq 0}$. Then we define $\mathbf{C}$-linear maps

$$\Gamma_{l,m}^i\colon \mathrm{Hom}_K(V_{2l}, H_{(\nu_i,\sigma_i),K}) \to \mathrm{Hom}_K(V_{2(l+m)}, H_{(\nu_i,\sigma_i),K}), \quad -2 \leq m \leq 2$$

by $\eta \mapsto \tilde{\eta} \circ I_{2m}^{2l}$.

Now we settle the goal of this part:

- **(i):** Describe the injective K-homomorphism I_{2m}^{2l} in terms of the standard basis.
- **(ii):** Determine the matrix representations of the linear homomorphisms $\Gamma_{l,m}^i$ with respect to the induced basis defined in the next subsection.

We have already accomplished (i) in Proposition 3.2. We accomplish (ii) in Theorem 4.5 and 5.5. As a result, we obtain infinite number of 'contiguous relations', a kind of system of differential-difference relations among vectors in $H_{(\nu_i,\sigma_i)}[\tau_{2l}]$ and $H_{(\nu_i,\sigma_i)}[\tau_{2(l+m)}]$. Here $H_{(\nu_i,\sigma_i)}[\tau]$ is the τ-isotypic component of $H_{(\nu_i,\sigma_i)}$.

4.3. **The canonical blocks of elementary functions.** Let $\eta\colon V_{2l} \to H_{(\nu_i,\sigma_i),K}$ be a non-zero K-homomorphism. Then we identify η with the column vector of degree $2l+1$ whose $q+1$-th component is $\eta(v_q^{(2l)})$ for $0 \leq q \leq 2l$, i.e. ${}^t(\ \eta(v_0^{(2l)}),\ \eta(v_1^{(2l)}),\ \cdots,\eta(v_{2l}^{(2l)})\ )$.

By this identification, we identify $S(l;p)$ with the K-homomorphism

$$V_{2l} \ni v_q^{(2l)} \mapsto s(l;p,q) \in H_{(\nu_0,\sigma_0),K}, \quad 0 \leq q \leq 2l$$

for $p \in Z(\sigma_0;l)$. We note that $\{S(l;p) \mid p \in Z(\sigma_0;l)\}$ is a basis of the intertwining space $\mathrm{Hom}_K(V_{2l}, H_{(\nu_0,\sigma_0),K})$ and we call it *the induced basis from the standard basis.*

We define a certain matrix of elementary functions corresponding to the induced basis $\{S(l;p) \mid p \in Z(\sigma_0;l)\}$ of $\mathrm{Hom}_K(V_{2l}, H_{(\nu_0,\sigma_0),K})$ for each K-type τ_{2l} of our principal series representation $\pi_{(\nu_0,\sigma_0)}$.

Definition 4.3. *The following $(2l+1)\times d(\sigma_0;l)$ matrix $\mathbf{S}(\sigma_0;l)$ is called the canonical block of elementary functions for τ_{2l}-isotypic component of $\pi_{(\nu_0,\sigma_0)}$: When $(\sigma_{0,1},\sigma_{0,2}) = (0,0)$, we consider the matrix*

$$\mathbf{S}(\sigma_0;l) = \begin{cases} (\ S(l;0),\ S(l;2),\ S(l;4),\ \cdots,S(l;l)\ ) & \text{if } l \text{ is even,} \\ (\ S(l;1),\ S(l;3),\ S(l;5),\ \cdots,S(l;l-2)\ ) & \text{if } l \text{ is odd.} \end{cases}$$

When $(\sigma_{0,1},\sigma_{0,2}) = (1,0)$, we consider the matrix

$$\mathbf{S}(\sigma_0;l) = \begin{cases} (\ S(l;0),\ S(l;2),\ S(l;4),\ \cdots,S(l;l-2)\ ) & \text{if } l \text{ is even,} \\ (\ S(l;1),\ S(l;3),\ S(l;5),\ \cdots,S(l;l)\ ) & \text{if } l \text{ is odd.} \end{cases}$$

When $\sigma_{0,2} = 1$, we consider the matrix

$$\mathbf{S}(\sigma_0;l) = \begin{cases} (\ S(l;1),\ S(l;3),\ S(l;5),\ \cdots,S(l;l-1)\ ) & \text{if } l \text{ is even,} \\ (\ S(l;0),\ S(l;2),\ S(l;4),\ \cdots,S(l;l-1)\ ) & \text{if } l \text{ is odd.} \end{cases}$$

4.4. **The $\mathfrak{p}_{\mathbf{C}}$-matrix corresponding to I_{2m}^{2l}.** For two integers $c_0,\ c_1$ such that $c_0 \leq c_1$ and a rational function $f(x)$ in the variable x, we denote by

$$\mathop{\mathrm{Diag}}_{c_0 \leq n \leq c_1} (f(n))$$

the diagonal matrix of size $c_1 - c_0 + 1$ with an entry $f(n)$ at the $(n-c_0+1, n-c_0+1)$-th component. Let $\mathbf{e}_i^{(l)}$ $(0 \leq i \leq l)$ be the column unit vector of degree $l+1$ with its $i+1$-th component 1 and the remaining components 0. Moreover, let $\mathbf{e}_i^{(l)}$ be the column zero vector of degree $l+1$ when $i<0$ or $l<i$.

In this subsection, we define $\mathfrak{p}_{\mathbf{C}}$-matrix $\mathfrak{C}_{l,m}$ of size $(2(l+m)+1)\times(2l+1)$ corresponding to I_{2m}^{2l} with respect to the standard basis.

Let $\sum_{i=0}^{4} \iota_i^{(l,m)} \otimes X_i$ be the image of I_{2m}^{2l} under the composite of natural linear maps

$$\begin{aligned} \mathrm{Hom}_K(V_{2(l+m)}, V_{2l} \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}) &\to \mathrm{Hom}_{\mathbf{C}}(V_{2(l+m)}, V_{2l} \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}) \\ &\simeq \mathrm{Hom}_{\mathbf{C}}(V_{2(l+m)}, V_{2l}) \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}. \end{aligned}$$

Then we define $\mathfrak{p}_{\mathbf{C}}$-matrix $\mathfrak{C}_{l,m} = \sum_{i=0}^{4} R(\iota_i^{(l,m)}) \otimes X_i$ where $R(\iota_i^{(l,m)})$ is the matrix representation of $\iota_i^{(l,m)}$ with respect to the standard basis. The explicit expression of the matrix $R(\iota_i^{(l,m)})$ of size $(2(l+m)+1)\times(2l+1)$ is given by

$$\begin{aligned} &\left(O_{2(l+m)+1,m+2}, R(\iota_0^{(l,m)}), O_{2(l+m)+1,m+2}\right) \\ &= \left(O_{2(l+m)+1,4-i},\ \mathop{\mathrm{Diag}}_{0 \leq k \leq 2(l+m)} (A_{[2l,2m;k,i]}),\ O_{2(l+m)+1,i} \right) \end{aligned}$$

for $-2 \leq m \leq 2$ and $0 \leq i \leq 4$. Here we omit the symbol $O_{m,n}$ when $m=0$ or $n=0$.

For a column vector $\mathbf{v} = {}^t(v_0, v_1, \cdots, v_{2l}) \in (H_{(\nu_i,\sigma_i),K})^{\oplus 2l+1}$ which is identified with an element of $\mathrm{Hom}_K(V_{2l}, H_{(\nu_i,\sigma_i),K})$, we define $\mathfrak{C}_{l,m}\mathbf{v} \in (H_{(\nu_i,\sigma_i),K})^{\oplus 2(l+m)+1} \simeq \mathbf{C}^{2(l+m)+1} \otimes_{\mathbf{C}} H_{(\nu_0,\sigma_0),K}$ by

$$\mathfrak{C}_{l,m}\mathbf{v} = \sum_{\substack{0\le i\le 4\\ 0\le q\le 2l}} (R(\iota_i^{(l,m)}) \cdot \mathbf{e}_q^{(2l)}) \otimes (\pi_{(\nu_i,\sigma_i)}(X_i)v_q).$$

Here $R(\iota_i^{(l,m)}) \cdot \mathbf{e}_q^{(2l)}$ is the ordinal product of matrices $R(\iota_i^{(l,m)})$ and $\mathbf{e}_q^{(2l)}$.

From the definition of $\mathfrak{C}_{l,m}$, we note that the vector $\mathfrak{C}_{l,m}\mathbf{v}$ is identified with the image of $\mathbf{v}$ under $\Gamma^i_{l,m}$.

4.5. **The contiguous relations.**

Lemma 4.4. *The standard basis X_i $(0 \le i \le 4)$ in $\mathfrak{p}_{\mathbf{C}}$ have the following expressions according to the Iwasawa decomposition $\mathfrak{g}_{\mathbf{C}} = \mathfrak{n}_{\mathbf{C}} \oplus \mathfrak{a}_{\mathbf{C}} \oplus \mathfrak{k}_{\mathbf{C}}$:*

$$\begin{aligned}
X_0 =& -2\sqrt{-1}E_{e_2-e_3} + H_2 + \sqrt{-1}K_{23},\\
X_1 =& -(E_{e_1-e_3} + \sqrt{-1}E_{e_1-e_2}) + \frac{1}{2}(K_{13} + \sqrt{-1}K_{12}),\\
X_2 =& -\frac{1}{3}(2H_1 - H_2),\\
X_3 =&(E_{e_1-e_3} - \sqrt{-1}E_{e_1-e_2}) - \frac{1}{2}(K_{13} - \sqrt{-1}K_{12}),\\
X_4 =&2\sqrt{-1}E_{e_2-e_3} + H_2 - \sqrt{-1}K_{23}.
\end{aligned}$$

Proof. We obtain the assertion immediately from Lemma 6.6. □

We give the matrix representation of $\Gamma^0_{l,m}$ with respect to the induced basis as follows.

Theorem 4.5. *For $l \in \mathbf{Z}_{\ge 0}$, $-2 \le m \le 2$ such that $d(\sigma_0; l) > 0$ and $d(\sigma_0; l+m) > 0$, we have*

$$\mathfrak{C}_{l,m}\mathbf{S}(\sigma_0; l) = \mathbf{S}(\sigma_0; l+m) \cdot R(\Gamma^0_{l,m}) \tag{4.1}$$

with the matrix representation $R(\Gamma^0_{l,m}) \in M_{d(\sigma_0;l+m),d(\sigma_0;l)}(\mathbf{C})$ of $\Gamma^0_{l,m}$ with respect to the induced basis $\{S(l;p) \mid p \in Z(\sigma_0;l)\}$. We give the explicit expressions of the matrix

$$\begin{pmatrix} O_{n(\sigma_0;l,m),d(\sigma_0;l)} \\ R(\Gamma^0_{l,m}) \end{pmatrix}$$

as follows:

- *When $\sigma_{0,2} = 0$ and $(m, \sigma_{0,1} + l) \in \{0, \pm 2\} \times (2\mathbf{Z})$,*

$$\begin{pmatrix} \mathrm{Diag}_{0\le k\le d(\sigma_0;l)-1}\left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),-1]}\right) \\ O_{1,d(\sigma_0;l)} \end{pmatrix} + \begin{pmatrix} O_{1,d(\sigma_0;l)} \\ \mathrm{Diag}_{0\le k\le d(\sigma_0;l)-1}\left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),0]}\right) \end{pmatrix}$$
$$+ \begin{pmatrix} O_{2,d(\sigma_0;l)-1} & O_{2,1} \\ \mathrm{Diag}_{0\le k\le d(\sigma_0;l)-2}\left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),1]}\right) & \gamma^{(0)}_{[l,m;l,1]} \cdot e^{(d(\sigma_0;l)-2)}_{d(\sigma_0;l)-3} \end{pmatrix}.$$

- *When $\sigma_{0,2} = 0$ and $(m, \sigma_{0,1} + l) \in \{0, \pm 2\} \times (1 + 2\mathbf{Z})$,*

$$\begin{pmatrix} \mathrm{Diag}_{0\le k\le d(\sigma_0;l)-1}\left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),-1]}\right) \\ O_{1,d(\sigma_0;l)} \end{pmatrix} + \begin{pmatrix} O_{1,d(\sigma_0;l)} \\ \mathrm{Diag}_{0\le k\le d(\sigma_0;l)-1}\left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),0]}\right) \end{pmatrix}$$

$$+\begin{pmatrix} O_{2,d(\sigma_0;l)-1} & O_{2,1} \\ \operatorname*{Diag}_{0\le k\le d(\sigma_0;l)-2}\left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),1]}\right) & O_{d(\sigma_0;l)-1,1} \end{pmatrix}.$$

- *When* $\sigma_{0,2}=0$, $(m,\sigma_{0,1}+l)\in\{\pm1\}\times(2\mathbf{Z})$,

$$\operatorname*{Diag}_{0\le k\le d(\sigma_0;l)-1}\left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),-1]}\right)$$
$$+\begin{pmatrix} O_{1,d(\sigma_0;l)-1} & 0 \\ \operatorname*{Diag}_{0\le k\le d(\sigma_0;l)-2}\left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),0]}\right) & O_{d(\sigma_0;l)-1,1} \end{pmatrix}$$
$$+\begin{pmatrix} O_{2,d(\sigma_0;l)-2} & O_{2,1} & O_{2,1} \\ \operatorname*{Diag}_{0\le k\le d(\sigma_0;l)-3}\left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),1]}\right) & O_{d(\sigma_0;l)-2,1} & -\gamma^{(0)}_{[l,m;l,1]}\cdot e^{(d(\sigma_0;l)-3)}_{d(\sigma_0;l)-3} \end{pmatrix}.$$

- *When* $\sigma_{0,2}=0$ *and* $(m,\sigma_{0,1}+l)\in\{\pm1\}\times(1+2\mathbf{Z})$,

$$\begin{pmatrix} \operatorname*{Diag}_{0\le k\le d(\sigma_0;l)-1}\left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),-1]}\right) \\ O_{2,d(\sigma_0;l)} \end{pmatrix}+\begin{pmatrix} O_{1,d(\sigma_0;l)} \\ \operatorname*{Diag}_{0\le k\le d(\sigma_0;l)-1}\left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),0]}\right) \\ O_{1,d(\sigma_0;l)} \end{pmatrix}$$
$$+\begin{pmatrix} O_{2,d(\sigma_0;l)} \\ \operatorname*{Diag}_{0\le k\le d(\sigma_0;l)-1}\left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),1]}\right) \end{pmatrix}.$$

- *When* $\sigma_{0,2}=1$,

$$\begin{pmatrix} \operatorname*{Diag}_{0\le k\le d(\sigma_0;l)-1}\left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),-1]}\right) \\ O_{1,d(\sigma_0;l)} \end{pmatrix}+\begin{pmatrix} O_{1,d(\sigma_0;l)} \\ \operatorname*{Diag}_{0\le k\le d(\sigma_0;l)-1}\left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),0]}\right) \end{pmatrix}$$
$$+\begin{pmatrix} O_{2,d(\sigma_0;l)-1} & O_{2,1} \\ \operatorname*{Diag}_{0\le k\le d(\sigma_0;l)-2}\left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),1]}\right) & (-1)^{\varepsilon(\sigma_0;l+m)}\gamma^{(0)}_{[l,m;l-1,1]}\cdot e^{(d(\sigma_0;l)-2)}_{d(\sigma_0;l)-2} \end{pmatrix}.$$

Here

$$\begin{aligned}
\gamma^{(0)}_{[l,m;p,1]} &= (\nu_{0,2}+\rho_{0,2}-l+p)A_{[2l,2m;2l-p+m-2,0]},\\
\gamma^{(0)}_{[l,m;p,0]} &= -\frac{1}{3}\Big(2\nu_{0,1}-\nu_{0,2}+2\rho_{0,1}-\rho_{0,2}+lm-3+\frac{m(m+1)}{2}\Big)A_{[2l,2m;2l-p+m,2]},\\
\gamma^{(0)}_{[l,m;p,-1]} &= (\nu_{0,2}+\rho_{0,2}+l-p)A_{[2l,2m;2l-p+m+2,4]},\\
n(\sigma_0;l,m) &= \begin{cases} (2-m)/2 & \text{if } m\in\{0,\ \pm2\},\\ (3-m)/2 & \text{if } (m,l+\sigma_{0,2})\in\{\pm1\}\times(2\mathbf{Z}),\\ (1-m)/2 & \text{if } (m,l+\sigma_{0,2})\in\{\pm1\}\times(1+2\mathbf{Z}),\end{cases}
\end{aligned}$$

and $\delta(\sigma_0;l)\in\{0,1\}$ *such that* $\delta(\sigma_0;l)\equiv l-\sigma_{0,2} \bmod 2$.

In the above equations, we put $A_{[2l,2m;k,i]}=0$ *for* $k<0$ *or* $k>2(l+m)$, *and omit the symbols* $\operatorname*{Diag}_{c\le n\le c-1}(f(n))$, $O_{0,n}$, $O_{m,0}$ *and* $\mathbf{e}_i^{(-1)}$.

Proof. Since

$$s(l;p,q)(1_3)=\langle(v^{(2l)*}_{2l-p}+(-1)^{\varepsilon(\sigma_0;l)}v^{(2l)*}_p),v^{(2l)}_q\rangle=\delta_{2l-p\,q}+(-1)^{\varepsilon(\sigma_0;l)}\delta_{pq},$$

we have

$$S(l;p)(1_3) = \mathbf{e}^{(2l)}_{2l-p} + (-1)^{\varepsilon(\sigma_0;l)}\mathbf{e}^{(2l)}_p. \tag{4.2}$$

Hence $S(l;p)(1_3)$ ($p \in Z(\sigma_0;l)$) are linearly independent over $\mathbf{C}$. Thus we note that it suffices to evaluate the both side of the equation (4.1) at $1_3 \in G$.

First, we compute $\{\pi_{(\nu_0,\sigma_0)}(X_i)s(l;p,q)\}(1_3)$ for $0 \le i \le 4$, $p \in Z(\sigma_0;l)$ and $0 \le q \le 2l$. Since $\{s(l;p,q) \mid 0 \le q \le 2l\}$ is the standard basis of $\langle S(l;p)\rangle$, we have

$$\begin{aligned}
\{\pi_{(\nu_0,\sigma_0)}(\sqrt{-1}K_{23})s(l;p,q)\}(1_3) &= (l-q)(\delta_{2l-p\,q} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{pq}),\\
\{\pi_{(\nu_0,\sigma_0)}(K_{13}+\sqrt{-1}K_{12})s(l;p,q)\}(1_3) &= -q(\delta_{2l-p+1\,q} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{p+1\,q}),\\
\{\pi_{(\nu_0,\sigma_0)}(K_{13}-\sqrt{-1}K_{12})s(l;p,q)\}(1_3) &= (2l-q)(\delta_{2l-p-1\,q} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{p-1\,q}).
\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}
\{\pi_{(\nu_0,\sigma_0)}(E_\alpha)s(l;p,q)\}(1_3) &= 0 && (\alpha \in \Sigma^+),\\
\{\pi_{(\nu_0,\sigma_0)}(H_i)s(l;p,q)\}(1_3) &= (\nu_{0,i}+\rho_{0,i})s(l;p,q)(1_3) &&\\
&= (\nu_{0,i}+\rho_{0,i})(\delta_{2l-p\,q} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{pq}) && (i=1,2),
\end{aligned}$$

from the definition of principal series representation. From these computations and Iwasawa decomposition in Lemma 4.4, we obtain

$$\begin{aligned}
\{\pi_{(\nu_0,\sigma_0)}(X_0)s(l;p,q)\}(1_3) &= (\nu_{0,2}+\rho_{0,2}+l-q)(\delta_{2l-p\,q} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{pq}),\\
\{\pi_{(\nu_0,\sigma_0)}(X_1)s(l;p,q)\}(1_3) &= -\frac{q}{2}(\delta_{2l-p+1\,q} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{p+1\,q}),\\
\{\pi_{(\nu_0,\sigma_0)}(X_2)s(l;p,q)\}(1_3) &= -\frac{1}{3}(2\nu_{0,1}-\nu_{0,2}+2\rho_{0,1}-\rho_{0,2})(\delta_{2l-p\,q} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{pq}),\\
\{\pi_{(\nu_0,\sigma_0)}(X_3)s(l;p,q)\}(1_3) &= -\frac{2l-q}{2}(\delta_{2l-p-1\,q} - (-1)^{\varepsilon(\sigma_0;l)}\delta_{p-1\,q}),\\
\{\pi_{(\nu_0,\sigma_0)}(X_4)s(l;p,q)\}(1_3) &= (\nu_{0,2}+\rho_{0,2}-l+q)(\delta_{2l-p\,q} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{pq}).
\end{aligned}$$

We set

$$\pi_{(\nu_0,\sigma_0)}(X_i)S(l;p) = \sum_{0\le q\le 2l} \mathbf{e}^{(2l)}_q \otimes (\pi_{(\nu_0,\sigma_0)}(X_i)s(l;p,q)).$$

Then we obtain

$$\begin{aligned}
\{\pi_{(\nu_0,\sigma_0)}(X_0)S(l;p)\}(1_3) &= (\nu_{0,2}+\rho_{0,2}-l+p)\mathbf{e}^{(2l)}_{2l-p} + (-1)^{\varepsilon(\sigma_0;l)}(\nu_{0,2}+\rho_{0,2}+l-p)\mathbf{e}^{(2l)}_p,\\
\{\pi_{(\nu_0,\sigma_0)}(X_1)S(l;p)\}(1_3) &= -\frac{2l-p+1}{2}\mathbf{e}^{(2l)}_{2l-p+1} - (-1)^{\varepsilon(\sigma_0;l)}\frac{p+1}{2}\mathbf{e}^{(2l)}_{p+1},\\
\{\pi_{(\nu_0,\sigma_0)}(X_2)S(l;p)\}(1_3) &= -\frac{1}{3}(2\nu_{0,1}-\nu_{0,2}+2\rho_{0,1}-\rho_{0,2})(\mathbf{e}^{(2l)}_{2l-p} + (-1)^{\varepsilon(\sigma_0;l)}\mathbf{e}^{(2l)}_p),\\
\{\pi_{(\nu_0,\sigma_0)}(X_3)S(l;p)\}(1_3) &= -\frac{p+1}{2}\mathbf{e}^{(2l)}_{2l-p-1} - (-1)^{\varepsilon(\sigma_0;l)}\frac{2l-p+1}{2}\mathbf{e}^{(2l)}_{p-1},\\
\{\pi_{(\nu_0,\sigma_0)}(X_4)S(l;p)\}(1_3) &= (\nu_{0,2}+\rho_{0,2}+l-p)\mathbf{e}^{(2l)}_{2l-p} + (-1)^{\varepsilon(\sigma_0;l)}(\nu_{0,2}+\rho_{0,2}-l+p)\mathbf{e}^{(2l)}_p.
\end{aligned}$$

Let us compute $\{\mathfrak{C}_{l,m}S(l;p)\}(1_3)$. By the above equations, we have

$$\begin{aligned}
&\{\mathfrak{C}_{l,m}S(l;p)\}(1_3)\\
&= \sum_{\substack{0\le i\le 4\\ 0\le q\le 2l}} (R(\iota^{(l,m)}_i)\cdot\mathbf{e}^{(2l)}_q) \otimes \{(\pi_{(\nu_0,\sigma_0)}(X_i)s(l;p,q))\}(1_3)\\
&= \sum_{0\le i\le 4} R(\iota^{(l,m)}_i)\cdot\{(\pi_{(\nu_0,\sigma_0)}(X_i)S(l;p))\}(1_3)
\end{aligned}$$

$$
\begin{aligned}
&= R(\iota_0^{(l,m)})\cdot\{(\nu_{0,2}+\rho_{0,2}-l+p)\mathbf{e}^{(2l)}_{2l-p}+(-1)^{\varepsilon(\sigma_0;l)}(\nu_{0,2}+\rho_{0,2}+l-p)\mathbf{e}^{(2l)}_{p}\}\\
&+R(\iota_1^{(l,m)})\cdot\Big\{-\frac{2l-p+1}{2}\mathbf{e}^{(2l)}_{2l-p+1}-(-1)^{\varepsilon(\sigma_0;l)}\frac{p+1}{2}\mathbf{e}^{(2l)}_{p+1}\Big\}\\
&+R(\iota_2^{(l,m)})\cdot\Big\{-\frac{1}{3}(2\nu_{0,1}-\nu_{0,2}+2\rho_{0,1}-\rho_{0,2})(\mathbf{e}^{(2l)}_{2l-p}+(-1)^{\varepsilon(\sigma_0;l)}\mathbf{e}^{(2l)}_{p})\Big\}\\
&+R(\iota_3^{(l,m)})\cdot\Big\{-\frac{p+1}{2}\mathbf{e}^{(2l)}_{2l-p-1}-(-1)^{\varepsilon(\sigma_0;l)}\frac{2l-p+1}{2}\mathbf{e}^{(2l)}_{p-1}\Big\}\\
&+R(\iota_4^{(l,m)})\cdot\{(\nu_{0,2}+\rho_{0,2}+l-p)\mathbf{e}^{(2l)}_{2l-p}+(-1)^{\varepsilon(\sigma_0;l)}(\nu_{0,2}+\rho_{0,2}-l+p)\mathbf{e}^{(2l)}_{p}\}.
\end{aligned}
$$

Since

$$R(\iota_i^{(l,m)})\mathbf{e}^{(2l)}_q = A_{[2l,2m;i+q+m-2,i]}\mathbf{e}^{(2(l+m))}_{i+q+m-2},\quad -2\le m\le 2,$$

we obtain

$$\{\mathfrak{C}_{l,m}S(l;p)\}(1_3)=\sum_{-1\le i\le 1}\{\alpha_{[l,m;p,i]}\mathbf{e}^{(2(l+m))}_{2(l+m)-(p+m+2i)}+(-1)^{\varepsilon(\sigma_0;l)}\beta_{[l,m;p,i]}\mathbf{e}^{(2(l+m))}_{p+m+2i}\},\tag{4.3}$$

where

$$
\begin{aligned}
\alpha_{[l,m;p,1]} &=(\nu_{0,2}+\rho_{0,2}-l+p)A_{[2l,2m;2l-p+m-2,0]},\\
\alpha_{[l,m;p,0]} &=-\frac{1}{3}(2\nu_{0,1}-\nu_{0,2}+2\rho_{0,1}-\rho_{0,2})A_{[2l,2m;2l-p+m,2]}\\
&\quad -\frac{2l-p+1}{2}A_{[2l,2m;2l-p+m,1]}-\frac{p+1}{2}A_{[2l,2m;2l-p+m,3]},\\
\alpha_{[l,m;p,-1]} &=(\nu_{0,2}+\rho_{0,2}+l-p)A_{[2l,2m;2l-p+m+2,4]},\\
\beta_{[l,m;p,1]} &=(\nu_{0,2}+\rho_{0,2}-l+p)A_{[2l,2m;p+m+2,4]},\\
\beta_{[l,m;p,0]} &=-\frac{1}{3}(2\nu_{0,1}-\nu_{0,2}+2\rho_{0,1}-\rho_{0,2})A_{[2l,2m;p+m,2]}\\
&\quad -\frac{p+1}{2}A_{[2l,2m;p+m,1]}-\frac{2l-p+1}{2}A_{[2l,2m;p+m,3]},\\
\beta_{[l,m;p,-1]} &=(\nu_{0,2}+\rho_{0,2}+l-p)A_{[2l,2m;p+m-2,0]}.
\end{aligned}
$$

By the relations of the coefficients $A_{[2l,2m;k,i]}$ in Lemma 3.3, we see that

$$\alpha_{[l,m;p,i]}=(-1)^m\beta_{[l,m;p,i]}=\gamma^{(0)}_{[l,m;p,i]},\quad -1\le i\le 1.$$

Therefore, (4.3) become

$$\{\mathfrak{C}_{l,m}S(l;p)\}(1_3)=\sum_{-1\le i\le 1}\gamma^{(0)}_{[l,m;p,i]}\{\mathbf{e}^{(2(l+m))}_{2(l+m)-(p+m+2i)}+(-1)^{\varepsilon(\sigma_0;l)+m}\mathbf{e}^{(2(l+m))}_{p+m+2i}\}.\tag{4.4}$$

From the equations (4.2), (4.4) and $\varepsilon(\sigma_0;l)+m\equiv\varepsilon(\sigma_0;l+m)$ mod 2, we obtain the assertion. □

5. The $(\mathfrak{g},K)$-module structures of the P_i-principal series representations for $i=1,2$

In this section, we set $i=1$ or 2.

5.1. **The discrete series representations of $SL^{\pm}(2,\mathbf{R})$.** The set of equivalence classes of discrete series representations of $SL^{\pm}(2,\mathbf{R})$ is exhausted by the induced representation $D_k=\mathrm{Ind}_{SL(2,\mathbf{R})}^{SL^{\pm}(2,\mathbf{R})}(D_k^+)$. Here D_k^+ is the discrete series representation of $SL(2,\mathbf{R})$ with Blattner parameter k, i.e. the one whose minimal $SO(2)$-type is given by the character

$$SO(2)\ni\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\mapsto e^{\sqrt{-1}kt}\in\mathbf{C}^{\times}.$$

We denote by D_k^- the contragradient representation of D_k^+ and set $y_0 = \mathrm{diag}(1,-1) \in O(2)$. Then a discrete series representation D_k is uniquely determined by specifying the $SL(2,\mathbf{R})$-module structure together with the action of y_0. Since $D_k|_{SL(2,\mathbf{R})} = D_k^+ \oplus D_k^-$ and $D_k^+ \oplus D_k^-$ is infinitesimally equivalent with a subrepresentation of some principal series representation of $SL(2,\mathbf{R})$, we obtain the following realization of associated $(\mathfrak{sl}(2,\mathbf{C}), O(2))$-module of D_k:

$$V_{D_k,O(2)} = \bigoplus_{\alpha\in\mathbf{Z}_{\geq 0}} W_{k+2\alpha} \qquad (W_p = \mathbf{C}\cdot\chi_p + \mathbf{C}\cdot\chi_{-p})$$

and

$$D_k(\kappa_t)\chi_p = e^{\sqrt{-1}pt}\chi_p \qquad D_k(y_0)\chi_p = \chi_{-p}, \qquad D_k(w)\chi_p = \sqrt{-1}p\chi_p,$$
$$D_k(x_+)\chi_p = (k+p)\chi_{p+2}, \qquad D_k(x_-)\chi_p = (k-p)\chi_{p-2},$$

where

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; x_\pm = \begin{pmatrix} 1 & \pm\sqrt{-1} \\ \pm\sqrt{-1} & -1 \end{pmatrix} \in \mathfrak{sl}(2,\mathbf{C}),$$
$$\kappa_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \in SO(2) \quad (t\in\mathbf{R}).$$

Here we denote the differential of D_k again by D_k and the $O(2)$-finite part of V_{D_k} by $V_{D_k,O(2)}$. See [3, §2.5] for details.

5.2. **The irreducible decompositions of $\pi_{(\nu_1,\sigma_1)}|_K$ and $\pi_{(\nu_2,\sigma_2)}|_K$ as K-modules.** We identify M_i with $SL^\pm(2,\mathbf{R})$ by natural isomorphisms $m_i\colon SL^\pm(2,\mathbf{R}) \to M_i$ defined by

$$m_1(h) = \begin{pmatrix} \det(h)^{-1} & O_{1,2} \\ O_{2,1} & h \end{pmatrix}, \qquad m_2(h) = \begin{pmatrix} h & O_{2,1} \\ O_{1,2} & \det(h)^{-1} \end{pmatrix}.$$

for $h \in SL^\pm(2,\mathbf{R})$. Then we may put $\sigma_i = D_k \circ m_i^{-1}$ for some $k \geq 2$.

We analyzes the K-type of the representation space $H_{(\nu_i,\sigma_i)}$ of the P_i-principal series representation. the target V_{σ_i} of functions $\mathbf{f}$ in $H_{(\nu_i,\sigma_i)}$ has a decomposition:

$$V_{\sigma_i} = V_{D_k} = \widehat{\bigoplus_{\alpha\in\mathbf{Z}_{\geq 0}}} W_{k+2\alpha}.$$

Denote the corresponding decomposition of $\mathbf{f}$ by

$$\mathbf{f}(x) = \sum_{\alpha=0}^{\infty}(f_{k+2\alpha}(x)\otimes\chi_{k+2\alpha} + f_{-(k+2\alpha)}(x)\otimes\chi_{-(k+2\alpha)}).$$

From the definition of the space $H_{(\nu_i,\sigma_i)}$, we have

$$\mathbf{f}|_K(mx) = \sigma_i(m)\mathbf{f}|_K(x) \qquad (\text{a.e. } x\in K,\ m\in K_i = M_i\cap K \simeq O(2)).$$

For $m = m_i(\kappa_t)$, $m_i(y_0)$, comparing the coefficients of χ_p in the left hand side with those in the right hand side, we have the equations

$$f_p|_K(m_i(\kappa_t)x) = e^{\sqrt{-1}pt}f_p|_K(x), \qquad f_p|_K(m_i(y_0)x) = f_{-p}|_K(x).$$

Moreover, from the equality of inner products

$$\int_K \|\mathbf{f}|_K(x)\|_{\sigma_i}^2 dx = \sum_{\varepsilon\in\{\pm 1\},\ \alpha\in\mathbf{Z}_{\geq 0}} \left\{\int_K \left|f_{\varepsilon(k+2\alpha)}|_K(x)\right|^2 dx\right\} \|\chi_{\varepsilon(k+2\alpha)}\|_{\sigma_i}^2,$$

we have $f_p|_K \in L^2(K)$. Therefore $\mathbf{f}|_K$ belongs to

$$\widehat{\bigoplus_{\alpha\in\mathbf{Z}_{\geq 0}}} L_i^2(K; W_{k+2\alpha})$$

where

$$L_i^2(K;W_p)=\left\{\mathbf{f}\colon K\to W_p\ \middle|\ \begin{array}{l}\mathbf{f}(x)=f(x)\otimes\chi_p+f(m_i(y_0)x)\otimes\chi_{-p},\\ f\in L^2_{(K_i^\circ,\chi_p)}(K),\quad x\in K\end{array}\right\},$$

$$L^2_{(K_i^\circ,\chi_p)}(K)=\left\{f\in L^2(K)\ \middle|\ \begin{array}{l}f(m_i(\kappa_t)x)=e^{\sqrt{-1}pt}f(x),\\ m_i(\kappa_t)\in K_i^\circ,\ x\in K\end{array}\right\}.$$

Here K_i° means the connected component of K_i, which is isomorphic to $SO(2)$. We easily see that the restriction map

$$r_K^{(i)}\colon H_{(\nu_i,\sigma_i)}\ni\mathbf{f}\mapsto\mathbf{f}|_K\in\widehat{\bigoplus_{\alpha\in\mathbf{Z}_{\geq 0}}}L_i^2(K;W_{k+2\alpha})$$

is a K-isomorphism.

By Peter-Weyl's theorem, we have the following irreducible decomposition of $L^2_{(K_i^\circ,\chi_p)}(K)$:

$$L^2_{(K_i^\circ,\chi_p)}(K)\simeq\widehat{\bigoplus_{l\in\mathbf{Z}_{\geq 0}}}(V_{2l}^*[\xi_{(i;-p)}])\otimes_{\mathbf{C}}V_{2l}.$$

Here

$$\xi_{(i;p)}\colon K_i^\circ\ni m_i(\kappa_t)\mapsto e^{\sqrt{-1}pt}\in\mathbf{C}^\times$$

and $V[\xi_{(i;p)}]$ means the $\xi_{(i;p)}$-isotypic component in $(\tau|_{K_i^\circ},V)$ for a K-module (τ,V).

In this section, we denote by $\{v_{1,q}^{(2l)}\mid 0\leq q\leq 2l\}$ the standard basis of V_{2l}. We define an another basis $\{v_{2,q}^{(2l)}\mid 0\leq q\leq 2l\}$ of V_{2l} by

$$v_{2,q}^{(2l)}=\tau_{2l}(u_c)v_{1,q}^{(2l)}=\frac{1}{2^l}(x+y)^q(-x+y)^{2l-q}\quad(0\leq q\leq 2l)$$

where

$$u_c=\begin{pmatrix}0&0&-1\\0&1&0\\1&0&0\end{pmatrix}\in SO(3).$$

We note that $v\in V_{2l}[\xi_{(i;-p)}]$ if and only if

$$\tau_{2l}(m_i(\kappa_t))v=\xi_{(i;-p)}(m_i(\kappa_t))v=e^{-\sqrt{-1}pt}v\qquad(t\in\mathbf{R})$$

for $v\in V_{2l}$. From the definition of (τ_{2l},V_{2l}) and

$$\varphi^{-1}(m_1(\kappa_t))=\varphi^{-1}(u_c^{-1}m_2(\kappa_t)u_c)=\left\{\pm\operatorname{diag}(e^{-\sqrt{-1}t/2},e^{\sqrt{-1}t/2})\right\},$$

we have $\tau_{2l}(m_i(\kappa_t))v_{i,q}^{(2l)}=e^{\sqrt{-1}(q-l)t}v_{i,q}^{(2l)}$. Hence we have

$$V_{2l}[\xi_{(i;-p)}]=\begin{cases}\mathbf{C}\cdot v_{i,l-p}^{(2l)} & \text{if } -l\leq p\leq l,\\ 0 & \text{otherwise}.\end{cases}$$

By the identification $V_{2l}^*=V_{2l}$ in Lemma 3.3, we obtain

$$L^2_{(K_i^\circ,\chi_p)}(K)\simeq\widehat{\bigoplus_{l\in\mathbf{Z}_{\geq 0}}}(\mathbf{C}\cdot v_{i,l+p}^{(2l)*})\otimes_{\mathbf{C}}V_{2l}.$$

Here we put $v_{i,l+p}^{(2l)*}=0$ if $p<-l$ or $l<p$. Moreover, since

$$\varphi^{-1}(m_1(y_0))=\left\{\pm\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right\},\qquad\varphi^{-1}(u_c^{-1}m_2(y_0)u_c)=\left\{\pm\begin{pmatrix}0&\sqrt{-1}\\\sqrt{-1}&0\end{pmatrix}\right\},$$

we have

$$\tau_{2l}^*(m_1(y_0)^{-1})v_{1,l+p}^{(2l)*}=(-1)^{l+p}v_{1,l-p}^{(2l)*},\qquad\tau_{2l}^*(m_2(y_0)^{-1})v_{2,l+p}^{(2l)*}=(-1)^{l}v_{2,l-p}^{(2l)*}.$$

For $0 \le p \le l-k$ such that $p \equiv l-k \bmod 2$, we define the elementary function $t_i(l;p,q) \in H_{(\nu_i,\sigma_i)}$ by

$$t_i(l;p,q) = r_K^{(i)-1}(\tilde{t}_i(l;p,q))$$

where

$$\tilde{t}_1(l;p,q)(x) = \langle v_{1,2l-p}^{(2l)*}, \tau_{2l}(x) v_{1,q}^{(2l)} \rangle \otimes \chi_{l-p} + (-1)^p \langle v_{1,p}^{(2l)*}, \tau_{2l}(x) v_{1,q}^{(2l)} \rangle \otimes \chi_{p-l},$$
$$\tilde{t}_2(l;p,q)(x) = \langle v_{2,2l-p}^{(2l)*}, \tau_{2l}(x) v_{1,q}^{(2l)} \rangle \otimes \chi_{l-p} + (-1)^l \langle v_{2,p}^{(2l)*}, \tau_{2l}(x) v_{1,q}^{(2l)} \rangle \otimes \chi_{p-l}.$$

Let $T_i(l;p)$ be a column vector of degree $2l+1$ with its $q+1$-th component $t_i(l;p,q)$, i.e. ${}^t(\ t_i(l;p,0),\ t_i(l;p,1),\ \cdots\ ,t_i(l;p,2l)\ )$.

Moreover we denote by $\langle T_i(l;p) \rangle$ the subspace of $H_{(\nu_i,\sigma_i)}$ generated by the functions in the entries of the vector $T_i(l;p)$, i.e. $\langle T_i(l;p) \rangle = \bigoplus_{q=0}^{2l} \mathbf{C} \cdot t_i(l;p,q) \simeq V_{2l}$. Via the isomorphism between $\langle T_i(l;p) \rangle$ and V_{2l}, we identify $\{t_i(l;p,q) \mid 0 \le q \le 2l\}$ with the standard basis.

From above arguments, we obtain the following.

Proposition 5.1. *As an unitary representation of K, it has an irreducible decomposition:*

$$H_{(\nu_i,\sigma_i)} = \widehat{\bigoplus_{\substack{l \in \mathbf{Z}_{\ge 0},\ 0 \le p \le l-k \\ p \equiv l-k \bmod 2}}} \langle T_i(l;p) \rangle$$

for $i = 1, 2$. Then the τ_{2l}-isotypic component of $\pi_{(\nu_i,\sigma_i)}$ is given by

$$\bigoplus_{\substack{0 \le p \le l-k \\ p \equiv l-k \bmod 2}} \langle T_i(l;p) \rangle.$$

Corollary 5.2. *The multiplicity $d(\sigma_i;l)$ of τ_{2l} in $\pi_{(\nu_i,\sigma_i),K}$ is given by*

$$d(\sigma_i;l) = \begin{cases} (l-k+2)/2 & \text{if } k \le l \text{ and } l-k \text{ is even}, \\ (l-k+1)/2 & \text{if } k \le l \text{ and } l-k \text{ is odd}, \\ 0 & \text{if } k > l. \end{cases}$$

5.3. **The canonical blocks of elementary functions.** By the identification introduced in Subsection 4.3, we identify $T_i(l;p)$ with the K-homomorphism

$$V_{2l} \ni v_{1,q}^{(2l)} \mapsto t_i(l;p,q) \in H_{(\nu_i,\sigma_i),K}, \quad 0 \le q \le 2l$$

for $0 \le p \le l-k$ such that $p \equiv l-k \bmod 2$. We note that $\{T_i(l;p) \mid 0 \le p \le l-k,\ p \equiv l-k \bmod 2\}$ is a basis of $\mathrm{Hom}_K(V_{2l}, H_{(\nu_i,\sigma_i),K})$ and we call it *the induced basis from the standard basis.*

For each K-type τ_{2l} of our P_i-principal series representation $\pi_{(\nu_i,\sigma_i)}$, we define a certain matrix of elementary functions corresponding to the induced basis $\{T_i(l;p) \mid 0 \le p \le l-k,\ p \equiv l-k \bmod 2\}$ of $\mathrm{Hom}_K(V_{2l}, H_{(\nu_i,\sigma_i),K})$.

Definition 5.3. *For $l \in \mathbf{Z}_{\ge 0}$ such that $d(\sigma_i;l) > 0$, the following $(2l+1) \times d(\sigma_i;l)$ matrix $\mathbf{T}_i(\sigma_i;l)$ is called the canonical block of elementary functions for τ_{2l}-isotypic component of $\pi_{(\nu_i,\sigma_i)}$: When $l-k$ is even, we consider the matrix*

$$\mathbf{T}_i(\sigma_i;l) = (\ T_i(l;0),\ T_i(l;2),\ T_i(l;4),\ \cdots\ ,T_i(l;l-k)\ ).$$

When $l-k$ is odd, we consider the matrix

$$\mathbf{T}_i(\sigma_i;l) = (\ T_i(l;1),\ T_i(l;3),\ T_i(l;5),\ \cdots\ ,T_i(l;l-k)\ ).$$

5.4. The contiguous relations.

Lemma 5.4. *(i) The standard basis $\{X_j \mid 0 \le j \le 4\}$ of $\mathfrak{p}_{\mathbf{C}}$ have the following expressions according to the decomposition $\mathfrak{g}_{\mathbf{C}} = (\mathfrak{n}_{1,\mathbf{C}} \oplus \mathfrak{a}_{1,\mathbf{C}} \oplus \mathfrak{m}_{1,\mathbf{C}}) + \mathfrak{k}_{\mathbf{C}}$:*

$$\begin{aligned} X_0 &= m_1(x_-), & X_1 &= -(E_{e_1-e_3} + \sqrt{-1}E_{e_1-e_2}) + \frac{1}{2}(K_{13} + \sqrt{-1}K_{12}),\\ X_2 &= -\frac{1}{3}H^{(1)}, & X_3 &= (E_{e_1-e_3} - \sqrt{-1}E_{e_1-e_2}) - \frac{1}{2}(K_{13} - \sqrt{-1}K_{12}),\\ X_4 &= m_1(x_+). \end{aligned}$$

(ii) The standard basis $\{X_j \mid 0 \le j \le 4\}$ of $\mathfrak{p}_{\mathbf{C}}$ have the following expressions according to the decomposition $\mathfrak{g}_{\mathbf{C}} = \mathrm{Ad}(u_c^{-1})(\mathfrak{n}_{2,\mathbf{C}} \oplus \mathfrak{a}_{2,\mathbf{C}} \oplus \mathfrak{m}_{2,\mathbf{C}}) + \mathfrak{k}_{\mathbf{C}}$:

$$\begin{aligned} X_0 &= -Ad(u_c^{-1})m(x_-), & X_1 &= Ad(u_c^{-1})(E_{e_1-e_3} - \sqrt{-1}E_{e_2-e_3}) - \frac{1}{2}(K_{13} + \sqrt{-1}K_{12}),\\ X_2 &= \frac{1}{3}Ad(u_c^{-1})H^{(2)}, & X_3 &= -Ad(u_c^{-1})(E_{e_1-e_3} + \sqrt{-1}E_{e_2-e_3}) + \frac{1}{2}(K_{13} - \sqrt{-1}K_{12}),\\ X_4 &= -Ad(u_c^{-1})m(x_+), \end{aligned}$$

Proof. We obtain the assertion immediately from Lemma 6.6. □

We give the matrix representation of $\Gamma^i_{l,m}$ with respect to the induced basis as follows.

Theorem 5.5. *For $i = 1,2$ and $-2 \le m \le 2$, we have the following equation with the matrix representation $R(\Gamma^i_{l,m}) \in M_{d(\sigma_i;l+m),d(\sigma_i;l)}(\mathbf{C})$ of $\Gamma^i_{l,m}$ with respect to the induced basis $\{T_i(l;p) \mid 0 \le p \le l-k,\ p \equiv l-k \bmod 2\}$:*

$$\mathfrak{C}_{l,m}\mathbf{T}_i(\sigma_i;l) = \mathbf{T}_i(\sigma_i;l+m) \cdot R(\Gamma^i_{l,m}). \tag{5.1}$$

We give the explicit expressions of the matrix

$$\begin{pmatrix} O_{n(\sigma_i;l,m),d(\sigma_i;l)} \\ R(\Gamma^i_{l,m}) \end{pmatrix}$$

by

$$\begin{pmatrix} \underset{0\le j\le d(\sigma_i;l)-1}{\mathrm{Diag}}\left(\gamma^{(i)}_{[l,m;2j+\delta(\sigma_i;l),-1]}\right) \\ O_{1,d(\sigma_i;l)} \end{pmatrix} + \begin{pmatrix} O_{1,d(\sigma_i;l)} \\ \underset{0\le j\le d(\sigma_i;l)-1}{\mathrm{Diag}}\left(\gamma^{(i)}_{[l,m;2j+\delta(\sigma_i;l),0]}\right) \end{pmatrix}$$
$$+ \begin{pmatrix} O_{2,d(\sigma_i;l)-1} & O_{2,1} \\ \underset{0\le j\le d(\sigma_i;l)-2}{\mathrm{Diag}}\left(\gamma^{(i)}_{[l,m;2j+\delta(\sigma_i;l),1]}\right) & O_{d(\sigma_i;l)-1,1} \end{pmatrix}.$$

Here

$$\begin{aligned} \gamma^{(i)}_{[l,m;p,1]} &= (-1)^{i+1}(k-l+p)A_{[2l,2m;2l-p+m-2,0]},\\ \gamma^{(i)}_{[l,m;p,0]} &= \frac{(-1)^i}{3}\Big(\nu_i + \rho_i + lm - 3 + \frac{m(m+1)}{2}\Big)A_{[2l,2m;2l-p+m,2]},\\ \gamma^{(i)}_{[l,m;p,-1]} &= (-1)^{i+1}(k+l-p)A_{[2l,2m;2l-p+m+2,4]},\\ n(\sigma_i;l,m) &= \begin{cases} (2-m)/2 & \text{if } m \in \{0,\ \pm 2\},\\ (3-m)/2 & \text{if } (m,l-k) \in \{\pm 1\} \times (2\mathbf{Z}),\\ (1-m)/2 & \text{if } (m,l-k) \in \{\pm 1\} \times (1+2\mathbf{Z}), \end{cases} \end{aligned}$$

and $\delta(\sigma_i;l) \in \{0,1\}$ such that $\delta(\sigma_i;l) \equiv l-k \bmod 2$.

In the above equations, we put $A_{[2l,2m;p,j]} = 0$ for $p < 0$ or $p > 2(l+m)$, and omit the symbols $\underset{c_0\le n\le c_1}{\mathrm{Diag}}(f(n))$ $(c_0 > c_1)$, $O_{m,n}$ $(m \le 0$ or $n \le 0)$.

Proof. By the similar computation in the proof of Theorem 4.5 using Lemma 3.7 (i), we obtain the assertion in the case of $i=1$. In the case of $i=2$, the value of $T_2(l;p)$ at $u_c \in G$ is given by

$$T_2(l;p)(u_c) = \mathbf{e}^{(2l)}_{2l-p} \otimes \chi_{l-p} + (-1)^l \mathbf{e}^{(2l)}_p \otimes \chi_{p-l}.$$

Thus, by the similar computation using Lemma 3.7 (ii), we also obtain the assertion in the case of $i=2$ evaluating the both side of the equation (5.1) at $u_c \in G$. □

6. The action of $\mathfrak{p}_{\mathbf{C}}$

The linear map $\Gamma^i_{l,m}$ characterize the action of $\mathfrak{p}_{\mathbf{C}}$. In this section, we give the explicit description of the action of $\mathfrak{p}_{\mathbf{C}}$ on the elementary functions.

6.1. **The projectors for $V_l \otimes_{\mathbf{C}} V_4$.** For $-2 \le m \le 2$, we describe a surjective $\mathfrak{sl}(2,\mathbf{C})$-homomorphism P^l_{2m} from $V_l \otimes_{\mathbf{C}} V_4$ to V_{l+2m} in terms of the standard basis as follows.

Lemma 6.1. *Let $\{v^{(l)}_q \mid 0 \le q \le l\}$ be the standard basis of V_l for $l \in \mathbf{Z}_{\ge 0}$. We put $v^{(l)}_q = 0$ when $q<0$ or $q>l$.*

We define linear maps $P^l_{2m} \colon V_l \otimes_{\mathbf{C}} V_4 \to V_{l+2m}$ $(-2 \le m \le 2)$ by

$$P^l_{2m}(v^{(l)}_q \otimes w_r) = B_{[l,2m;q,r]} \cdot v^{(l+2m)}_{q+r+m-2},$$

when V_{l+2m}-component of $V_l \otimes_{\mathbf{C}} V_4$ does not vanish.

Here the coefficients $B_{[l,2m;q,r]} = b(l,2m;q,r)/d'(l,2m)$ are defined by following formulas.

Formula 1: *The coefficients of $P^l_4 \colon V_l \otimes_{\mathbf{C}} V_4 \to V_{l+4}$ are given as follows:*

$$b(l,4;q,r) = 1 \quad (0 \le r \le 4), \qquad d'(l,4) = 1.$$

Formula 2: *The coefficients of $P^l_2 \colon V_l \otimes_{\mathbf{C}} V_4 \to V_{l+2}$ are given as follows:*

$$\begin{aligned} &b(l,2;q,0) = 4q, && b(l,2;q,1) = -(l-4q), && b(l,2;q,2) = -2(l-2q),\\ &b(l,2;q,3) = -(3l-4q), && b(l,2;q,4) = -4(l-q), && d'(l,2) = l+4. \end{aligned}$$

Formula 3: *The coefficients of $P^l_0 \colon V_l \otimes_{\mathbf{C}} V_4 \to V_l$ are given as follows:*

$$\begin{aligned} &b(l,0;q,0) = 6q(q-1), && b(l,0;q,1) = -3q(l-2q+1),\\ &b(l,0;q,2) = l^2 - 6lq + 6q^2 - l, && b(l,0;q,3) = 3(l-2q-1)(l-q),\\ &b(l,0;q,4) = 6(l-q)(l-q-1), && d'(l,0) = (l+3)(l+2). \end{aligned}$$

Formula 4: *The coefficients of $I^l_{-2} \colon V_{l-2} \to V_l \otimes_{\mathbf{C}} V_4$ are given as follows:*

$$\begin{aligned} &b(l,-2;q,0) = 4q(q-1)(q-2), \quad b(l,-2;q,1) = -q(q-1)(3l-4q+2),\\ &b(l,-2;q,2) = 2q(l-2q)(l-q),\\ &b(l,-2;q,3) = -(l-4q-2)(l-q)(l-q-1),\\ &b(l,-2;q,4) = -4(l-q)(l-q-1)(l-q-2), \quad d'(l,-2) = (l+2)(l+1)l. \end{aligned}$$

Formula 5: *The coefficients of $I^l_{-4} \colon V_{l-4} \to V_l \otimes_{\mathbf{C}} V_4$ are given as follows:*

$$\begin{aligned} b(l,-4;q,0) &= q(q-1)(q-2)(q-3),\\ b(l,-4;q,1) &= -q(q-1)(q-2)(l-q),\\ b(l,-4;q,2) &= q(q-1)(l-q)(l-q-1),\\ b(l,-4;q,3) &= -q(l-q)(l-q-1)(l-q-2),\\ b(l,-4;q,4) &= (l-q)(l-q-1)(l-q-2)(l-q-3),\\ d'(l,-4) &= (l+1)l(l-1)(l-2). \end{aligned}$$

Then P^l_{2m} is the generator of $\mathrm{Hom}_{\mathfrak{sl}(2,\mathbf{C})}(V_l \otimes_{\mathbf{C}} V_4, V_{l+2m})$ such that $P^l_{2m} \circ I^l_{2m} = \mathrm{id}_{V_{l+2m}}$.

Proof. The composite

$$V_l \otimes_{\mathbf{C}} V_4 \simeq V_l^* \otimes_{\mathbf{C}} V_4^* \simeq (V_l \otimes_{\mathbf{C}} V_4)^* \ni f \mapsto f \circ I_{2m}^l \in V_{l+2m}^* \simeq V_{l+2m}$$

is a surjective $\mathfrak{sl}(2,\mathbf{C})$-homomorphism from $V_l \otimes_{\mathbf{C}} V_4$ to V_{l+2m}, which is unique up to scalar multiple. Therefore we obtain the assertion from Proposition 3.2 and Lemma 3.3. □

6.2. **The action of $\mathfrak{p}_{\mathbf{C}}$ on the elementary functions.**

Proposition 6.2. *(i) An explicit expression of the action of $\mathfrak{p}_{\mathbf{C}}$ on the basis $\{s(l;p,q) \mid l \geq 0,\ p \in Z(\sigma_0;l),\ 0 \leq q \leq 2l\}$ of $H_{(\nu_0,\sigma_0),K}$ is given by following equation:*

$$\pi_{(\nu_0,\sigma_0)}(X_r)s(l;p,q) = \sum_{\substack{-1\leq j\leq 1\\ -2\leq m\leq 2}} \gamma^{(0)}_{[l,m;p,j]} B_{[2l,2m;q,r]} s(l+m; p+m+2j, q+m+r-2).$$

Here we put

$$\gamma^{(0)}_{[0,m;0,j]} = B_{[0,2m;0,r]} = 0 \text{ for } m<2, \quad \gamma^{(0)}_{[1,m;p,j]} = B_{[2,2m;q,r]} = 0 \text{ for } m<0,$$

$$s(l;p,q) = 0 \text{ whenever } p \leq l \text{ such that } p \notin Z(\sigma_0;l) \text{ or } q<0 \text{ or } q>2l,$$

$$s(l;p,q) = (-1)^{\varepsilon(\sigma_0;l)} s(l;2l-p,q) \text{ for } p>l.$$

(ii) For $i=1,2$, the explicit expression of the action of $\mathfrak{p}_{\mathbf{C}}$ on the basis $\{t_i(l;p,q) \mid l \geq k,\ 0 \leq p \leq l-k,\ p \equiv l-k \bmod 2,\ 0 \leq q \leq 2l\}$ of $H_{(\nu_i,\sigma_i),K}$ is given by following equation:

$$\pi_{(\nu_i,\sigma_i)}(X_r)t_i(l;p,q) = \sum_{\substack{-1\leq j\leq 1\\ -2\leq m\leq 2}} \gamma^{(i)}_{[l,m;p,j]} B_{[2l,2m;q,r]} t_i(l+m; p+m+2j, q+m+r-2)$$

Here we put $t_i(l;p,q) = 0$ unless $0 \leq p \leq l-k$, $p \equiv l-k \bmod 2$ and $0 \leq q \leq 2l$.

Proof. Since

$$\pi_{(\nu_0,\sigma_0)}(X_r)s(l;p,q) = \sum_{-2\leq m\leq 2} \Gamma^0_{l,m}(S(l;p)) \circ P^l_{2m}(v_q^{(2l)} \otimes X_r),$$

$$\pi_{(\nu_i,\sigma_i)}(X_r)t_i(l;p,q) = \sum_{-2\leq m\leq 2} \Gamma^i_{l,m}(T_i(l;p)) \circ P^l_{2m}(v_q^{(2l)} \otimes X_r) \qquad (i=1,2),$$

we obtain the assertion from Theorem 4.5, 5.5 and Lemma 6.1. □

Part 2. Whittaker functions for generalized principal series representations of $SL(3, \mathbf{R})$.

1. Introduction

Whittaker functions on real reductive Lie groups play fundamental roles in the archimedean local theory of automorphic forms. Jacquet introduced an integral expression of a Whittaker function in [17], which is called the Jacquet integral. However, the Jacquet integrals are difficult to handle and accordingly archimedean zeta integrals defined by them are also difficult to understand. Hence many authors study the explicit formulas of Whittaker functions which are suitable for number theoretic applications.

For an archimedean local field k, an irreducible admissible representation π of $GL(n,k)$ ($n \geq 3$) has a non-degenerate (continuous) Whittaker model if and only if π is an irreducible principal series representation or an irreducible generalized principal series representation ([18, §2]). The explicit formulas of Whittaker functions for principal series representations of $GL(n,k)$ or $SL(n,k)$ already have some history. The explicit formulas of Whittaker functions for spherical principal series representations of $GL(n,\mathbf{R})$ have been developed by Bump [2], Vinogradov and Tahtajan [45], Stade [40], [41], [42]. Recently, Ishii and Stade have reached a simple recursive formula between those for $GL(n,\mathbf{R})$ and $GL(n-1,\mathbf{R})$ in [16]. Those for non-spherical principal series representations of $SL(n,\mathbf{R})$ are obtained, for $n=3$ by Manabe, Ishii and Oda [27], and for $n=4$ by Hina, Ishii and Oda [9]. Moreover, those for principal series representations of $GL(3,\mathbf{C})$ are obtained by Hirano and Oda [10]. They obtained the explicit formulas by evaluating the Jacquet integral or by solving the differential equations. As an extension of these studies, we discuss Whittaker functions for generalized principal series representations of $SL(3,\mathbf{R})$. Together with the results of principal series representations, we have the explicit formulas of Whittaker functions for any generic irreducible admissible representation of $GL(3,k)$. In the spherical case, the explicit formulas of Whittaker functions are utilized to evaluate the archimedean zeta integrals ([11], [42], [43]). We expect to utilize our result for the investigation of the archimedean zeta integrals attached to generic cuspidal representations of $GL(3)$.

Let us explain our problem in a more precise form. Before describing our situation for $SL(3,\mathbf{R})$, let us recall the general setting of the theory of Whittaker functions on a real reductive Lie group G. Fix a maximal compact subgroup K and a maximal unipotent subgroup N_0 of G. Take a non-degenerate unitary character ξ of N_0 and consider its C^∞-induction $C^\infty \operatorname{Ind}_{N_0}^G(\xi)$. For an irreducible admissible representation (π, H_π) of G, we consider the space $\operatorname{Hom}_{(\mathfrak{g}_{\mathbf{C}},K)}(\pi, C^\infty \operatorname{Ind}_{N_0}^G(\xi))$ of intertwining operators. Let (τ, V_τ) be a K-type of π and $\iota \colon V_\tau \to H_\pi$ a nonzero K-homomorphism. For $\Phi \in \operatorname{Hom}_{(\mathfrak{g}_{\mathbf{C}},K)}(\pi, C^\infty \operatorname{Ind}_{N_0}^G(\xi))$, we can define the function ϕ_{π,τ^*} contained in the space $C^\infty_{\xi,\tau^*}(N_0\backslash G/K)$ of V_{τ^*}-valued smooth functions on G satisfying $f(ngk) = \xi(n)\tau^*(k)^{-1}f(g)$ for all $(n,g,k) \in N_0 \times G \times K$ by $\Phi(\iota(v))(g) = \langle v, \phi_{\pi,\tau^*}(g)\rangle$ ($g \in G$, $v \in V_\tau$). Here (τ^*, V_{τ^*}) means the contragradient representation of τ and $\langle , \rangle$ is the canonical pairing of $V_\tau \times V_{\tau^*}$. We call ϕ_{π,τ^*} a Whittaker function with a K-type τ. Here we remark that any function $\varphi \in C^\infty_{\xi,\tau^*}(N_0\backslash G/K)$ is determined by its restriction $\varphi|_{A_0}$ to A_0 from the Iwasawa decomposition $G = N_0A_0K$ of G. We call $\varphi|_{A_0}$ the A_0-radial part of φ.

The purpose of this part is to give the explicit formulas of A_0-radial parts of Whittaker functions with the minimal K-type of π when π is an irreducible generalized principal series representation of $G = SL(3,\mathbf{R})$. Here a generalized principal series representation of G is an induced representation from a discrete series representation of the Levi part $GL(2,\mathbf{R})$ of a maximal parabolic subgroup. Firstly, we give the system of partial differential equations satisfied by Whittaker functions in Proposition 4.5. We obtain these equations from the investigation of the $(\mathfrak{g}_{\mathbf{C}},K)$-module structure around the minimal K-type of π and the Capelli elements which are generators of the center of the universal enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$. Secondly, we have 6 formal power series solutions in Theorem 5.8, which are considered as examples of confluent hypergeometric series of two variables. These solutions are called secondary Whittaker functions and form a basis of the space of Whittaker functions with the minimal K-type of π. Moreover, we also give the Mellin-Barnes type integral expressions of primary Whittaker functions, i.e. the Whittaker functions having the moderate growth property, and the relation between primary and secondary Whittaker functions in Theorem 5.9.

2. Preliminaries

2.1. **Groups and algebras.** We denote by $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. Let $\mathbf{Z}_{\geq 0}$ be the set of non-negative integers, 1_n the unit matrix in the space $M_n(\mathbf{C})$ of complex matrices of size n, $O_{m,n}$ the zero matrix of size $m \times n$ and E_{ij} the matrix unit in $M_3(\mathbf{C})$ with 1 at the (i,j)-th entry and 0 at other entries. We denote by δ_{ij} the Kronecker delta, i.e.

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise.} \end{cases}$$

For a Lie algebra $\mathfrak{l}$, we denote by $\mathfrak{l}_{\mathbf{C}} = \mathfrak{l} \otimes_{\mathbf{R}} \mathbf{C}$ the complexification of $\mathfrak{l}$.

Let G be the special linear group $SL(3,\mathbf{R})$ of degree three and $\mathfrak{g}$ its Lie algebra. We define a Cartan involution θ of G by $G \ni g \mapsto {}^tg^{-1} \in G$. Here ${}^tg$ and g^{-1} means the transpose and the inverse of g, respectively. Then $K = \{g \in G \mid \theta(g) = g\} = SO(3)$ is a maximal compact subgroup of G.

If we denote the differential of θ again by θ, then we have $\theta(X) = -{}^tX$ for $X \in \mathfrak{g}$. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and the -1 eigenspaces of θ in $\mathfrak{g}$, respectively, that is,

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid {}^tX = -X\} = \mathfrak{so}(3), \qquad \mathfrak{p} = \{X \in \mathfrak{g} \mid {}^tX = X\}.$$

Then $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Put $\mathfrak{a}_0 = \{\mathrm{diag}(t_1,t_2,t_3) \mid t_i \in \mathbf{R}\ (1 \leq i \leq 3),\ t_1+t_2+t_3 = 0\}$. Then $\mathfrak{a}_0$ is a maximal abelian subalgebra of $\mathfrak{p}$. For each $1 \leq i \leq 3$, we define a linear form e_i on $\mathfrak{a}_0$ by $\mathfrak{a}_0 \ni \mathrm{diag}(t_1,t_2,t_3) \mapsto t_i \in \mathbf{C}$. The set Σ of the restricted roots for $(\mathfrak{a}_0, \mathfrak{g})$ is given by $\Sigma = \Sigma(\mathfrak{a}_0,\mathfrak{g}) = \{e_i - e_j \mid 1 \leq i \neq j \leq 3\}$, and the subset $\Sigma^+ = \{e_i - e_j \mid 1 \leq i < j \leq 3\}$ forms a positive restricted root system. For each $\alpha \in \Sigma$, we denote its restricted root space by $\mathfrak{g}_\alpha$. Then E_{ij} is a restricted root vector in $\mathfrak{g}_{e_i - e_j}$ for $1 \leq i \neq j \leq 3$. If we put $\mathfrak{n}_0 = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$, then $\mathfrak{g}$ has an Iwasawa decomposition $\mathfrak{g} = \mathfrak{n}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{k}$. Also we have $G = N_0A_0K$, where $N_0 = \exp(\mathfrak{n}_0)$ and $A_0 = \exp(\mathfrak{a}_0)$. We take a basis $\{H_1, H_2\}$ of $\mathfrak{a}_0$ by $H_1 = \mathrm{diag}(1,0,-1)$, $H_2 = \mathrm{diag}(0,1,-1)$.

2.2. **Whittaker functions.** For a unitary character ξ of N_0, we denote the derivative of ξ by the same letter. Since

$$\mathfrak{n}_0/[\mathfrak{n}_0, \mathfrak{n}_0] \simeq \mathfrak{g}_{e_1-e_2} \oplus \mathfrak{g}_{e_2-e_3},$$

ξ is specified by two real numbers c_1 and c_2 such that

$$\xi(E_{12}) = 2\pi\sqrt{-1}c_1, \qquad \xi(E_{23}) = 2\pi\sqrt{-1}c_2.$$

When $c_1c_2 \neq 0$, the unitary character ξ of N_0 is called *non-degenerate*.

For a finite dimensional representation (τ, V) of K and a non-degenerate unitary character ξ of N_0, we consider the space $C^\infty_{\xi,\tau}(N_0\backslash G/K)$ of smooth functions $\varphi\colon G \to V$ with the property

$$\varphi(ngk) = \xi(n)\tau(k)^{-1}\varphi(g), \qquad (n,g,k) \in N_0 \times G \times K.$$

Here we remark that any function $\varphi \in C^\infty_{\xi,\tau}(N_0\backslash G/K)$ is determined by its restriction $\varphi|_{A_0}$ to A_0 from the Iwasawa decomposition $G = N_0A_0K$ of G. We call $\varphi|_{A_0}$ *the A_0-radial part of φ*. Also let $C^\infty \operatorname{Ind}_{N_0}^G(\xi)$ be the C^∞-induced representation from ξ with the representation space

$$C^\infty_\xi(N_0\backslash G) = \{\varphi \in C^\infty(G) \mid \varphi(ng) = \xi(n)\varphi(g),\ (n,g) \in N_0 \times G\},$$

on which G acts by right translation. Then we note that the space $C^{\infty}_{\xi,\tau}(N_0\backslash G/K)$ is isomorphic to $\mathrm{Hom}_K(V^*, C^{\infty}_{\xi}(N_0\backslash G))$ via the correspondence between $\iota \in \mathrm{Hom}_K(V^*, C^{\infty}_{\xi}(N_0\backslash G))$ and $F^{[\iota]} \in C^{\infty}_{\xi,\tau}(N_0\backslash G/K)$ given by the relation $\iota(v^*)(g) = \langle v^*, F^{[\iota]}(g)\rangle$ for $v^* \in V^*$ and $g \in G$ with the canonical pairing $\langle , \rangle$ on $V^* \times V$. Here (τ^*, V^*) means the contragradient representation of τ.

Let (π, H_π) be an admissible representation of G, and take a K-type (τ^*, V^*) of π with an injective K-homomorphism $\iota\colon V^* \to H_\pi$. Then, for each element I in the intertwining space $\mathcal{I}_{\xi,\pi} = \mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}},K)}(H_{\pi,K}, C^{\infty}_{\xi}(N_0\backslash G))$, the relation $I(\iota(v^*))(g) = \langle v^*, \Phi(I,\iota)(g)\rangle$ $(v^* \in V^*,\ g \in G)$ determines an element $\Phi(I,\iota) \in C^{\infty}_{\xi,\tau}(N_0\backslash G/K)$. Here $H_{\pi,K}$ is a subspace of H_π, consisting of all K-finite vectors. Now we put

$$\mathrm{Wh}(\pi,\xi,\tau) = \bigcup_{\iota \in \mathrm{Hom}_K(V^*, H_{\pi,K})} \{\Phi(I,\iota) \in C^{\infty}_{\xi,\tau}(N_0\backslash G/K) \mid I \in \mathcal{I}_{\xi,\pi}\}$$

and call $\mathrm{Wh}(\pi,\xi,\tau)$ *the space of Whittaker functions* for (π,ξ,τ). Moderate growth functions in $\mathrm{Wh}(\pi,\xi,\tau)$ are called *primary Whittaker functions* and we denote by $\mathrm{Wh}(\pi,\xi,\tau)^{\mathrm{mod}}$ the subspace of primary Whittaker functions in $\mathrm{Wh}(\pi,\xi,\tau)$. We consider Whittaker functions for irreducible generalized principal series representations of G.

2.3. **Generalized principal series representations of G.** We set $\mathfrak{n}$ and $\mathfrak{a}$ be subalgebras of $\mathfrak{g}$ defined by $\mathfrak{n} = \mathfrak{g}_{e_1-e_3} \oplus \mathfrak{g}_{e_2-e_3}$ and $\mathfrak{a} = \mathbf{R}\cdot(H_1+H_2)$, respectively. Let $P = NAM$ be a maximal parabolic subgroup of G with a Langlands decomposition, where

$$M = \left\{ \begin{pmatrix} h & O_{2,1} \\ O_{1,2} & \det(h)^{-1} \end{pmatrix} \middle| h \in SL^{\pm}(2,\mathbf{R}) \right\} \simeq SL^{\pm}(2,\mathbf{R}), \quad A = \exp(\mathfrak{a}), \quad N = \exp(\mathfrak{n}).$$

Here $SL^{\pm}(2,\mathbf{R}) = \{g \in GL(2,\mathbf{R}) \mid \det(g) = \pm 1\}$. Let $\mathfrak{m}$ be a Lie algebra of M.

We identify $\nu \in \mathfrak{a}^*_{\mathbf{C}} = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a},\mathbf{C})$ with a complex number $\nu = \nu\left(\frac{1}{3}(H_1+H_2)\right) \in \mathbf{C}$. Let ρ be the element of $\mathfrak{a}^*_{\mathbf{C}}$ defined by $2\rho = (e_1-e_3)+(e_2-e_3) = e_1+e_2-2e_3$. Then ρ is identified with 1. We define a character $e^{\nu}\colon A \to \mathbf{C}^{\times}$ by

$$e^{\nu}(a(r)) = r^{3\nu}, \qquad a(r) = \mathrm{diag}(r,r,r^{-2}) \in A.$$

We fix a discrete series representation $\sigma = D_k = \mathrm{Ind}_{SL(2,\mathbf{R})}^{SL^{\pm}(2,\mathbf{R})}(D_k^+)$ of $M \simeq SL^{\pm}(2,\mathbf{R})$ where D_k^+ is a discrete series representation of $SL(2,\mathbf{R})$ with the Blattner parameter $k > 0$.

Definition 2.1. *We define a generalized principal series representation $\pi_{(\nu,\sigma)}$ of G by*

$$\pi_{(\nu,\sigma)} = \mathrm{Ind}_P^G(1_N \otimes e^{\nu+\rho} \otimes \sigma),$$

i.e. $\pi_{(\nu,\sigma)}$ is the right regular representation of G on the space $H_{(\nu,\sigma)}$ which is the completion of

$$H^{\infty}_{(\nu,\sigma)} = \left\{ f\colon G \to V^{\infty}_{\sigma} \text{ smooth} \;\middle|\; \begin{array}{l} f(namx) = e^{\nu+\rho}(a)\sigma(m)f(x) \\ \text{for } n \in N,\ a \in A,\ m \in M,\ x \in G \end{array} \right\}$$

with respect to the norm

$$\|f\|^2 = \int_K \|f(k)\|^2_{\sigma} dk.$$

Here V^{∞}_{σ} is the smooth part of the representation space V_σ of σ and $\|\cdot\|_{\sigma_i}$ is the norm on V_σ.

For generic parameter ν, a generalized principal series representation $\pi_{(\nu,\sigma)}$ is irreducible. In this part, we always assume that $\pi_{(\nu,\sigma)}$ is irreducible.

3. The $(\mathfrak{g}_{\mathbf{C}},K)$-module structure of $\pi_{(\nu,\sigma)}$ around the minimal K-type

In this section, we explain some equations for weight vectors in the minimal K-type of $\pi_{(\nu,\sigma)}$, which are determined from the $(\mathfrak{g}_{\mathbf{C}},K)$-module structure of $\pi_{(\nu,\sigma)}$. Although we explain only a partial result here, we describe the whole $(\mathfrak{g}_{\mathbf{C}},K)$-module structure of $\pi_{(\nu,\sigma)}$ in Part 1.

3.1. **Irreducible K-modules.** Let $\mathcal{H}_l$ be the subspace consisting of degree l homogeneous polynomials of three variables x, y, z in the polynomial ring $\mathbf{C}[x, y, z]$. For $g = (g_{ij}) \in SO(3)$ and $f \in \mathcal{H}_l$ we set

$$\begin{aligned}\tilde{\tau}_l(g)f(x,y,z) &= f((x,y,z)\cdot g)\\ &= f(g_{11}x + g_{21}y + g_{31}z, g_{12}x + g_{22}y + g_{32}z, g_{13}x + g_{23}y + g_{33}z).\end{aligned}$$

We put $r^2 = x^2 + y^2 + z^2 \in \mathcal{H}_2$. Since r^2 is $SO(3)$-invariant, $r^2 \cdot \mathcal{H}_{l-2}$ is a $SO(3)$-invariant subspace of $\mathcal{H}_l$. Let τ_l be the quotient representation of $\tilde{\tau}_l$ on $V_l = \mathcal{H}_l/(r^2 \cdot \mathcal{H}_{l-2})$. Here we put $\mathcal{H}_{-1} = \mathcal{H}_{-2} = 0$. Then (τ_l, V_l) is an irreducible $2l+1$-dimensional representation and the set of equivalence classes of the irreducible finite dimensional continuous representations of $SO(3)$ is exhausted by τ_l $(l \in \mathbf{Z}_{\geq 0})$.

We put

$$v_{\varepsilon i}^{(l)} = (-\sqrt{-1}x)^{l-i}(y + \varepsilon\sqrt{-1}z)^i \mod r^2 \cdot \mathcal{H}_{l-2} \quad (\varepsilon \in \{\pm 1\},\ 0 \leq i \leq l).$$

Then $\{v_q^{(l)} \mid -l \leq q \leq l\}$ form a basis of V_l and each $v_q^{(l)}$ is a weight vector with respect to the Cartan subalgebra of $\mathfrak{k}_{\mathbf{C}}$ spanned by K_{23}. We have the following formulas of the $\mathfrak{k}_{\mathbf{C}}$-action:

$$\begin{aligned}\tau_l(K_{23})v_q^{(l)} &= \sqrt{-1}qv_q^{(l)},\\ \tau_l(K_{13} + \sqrt{-1}K_{12})v_q^{(l)} &= -(l+q)v_{q-1}^{(l)},\\ \tau_l(K_{13} - \sqrt{-1}K_{12})v_q^{(l)} &= (l-q)v_{q+1}^{(l)}.\end{aligned}$$

where $K_{ij} = E_{ij} - E_{ji} \quad (1 \leq i < j \leq 3)$.

Remark 3.1. The weight basis $\{v_q^{(l)} \mid -l \leq q \leq l\}$ is the same basis that Manabe, Ishii and Oda use in [27, §2] to treat the Whittaker functions for principal series representations of $SL(3,\mathbf{R})$.

3.2. **The adjoint representation of K on $\mathfrak{p}_{\mathbf{C}}$.** It is known that $\mathfrak{p}_{\mathbf{C}}$ becomes a K-module via the adjoint action Ad of K. Concerning this, we have the following lemma.

Lemma 3.2. *Let $\{X_j \mid -2 \leq j \leq 2\}$ be a basis of $\mathfrak{p}_{\mathbf{C}}$ defined as follows:*

$$\begin{aligned}X_{-2} &= H_2 - \sqrt{-1}(E_{23} + E_{32}), & X_{-1} &= -\frac{1}{2}\{\sqrt{-1}(E_{12} + E_{21}) + (E_{13} + E_{31})\},\\ X_0 &= -\frac{1}{3}(2H_1 - H_2), & X_1 &= -\frac{1}{2}\{\sqrt{-1}(E_{12} + E_{21}) - (E_{13} + E_{31})\},\\ X_2 &= H_2 + \sqrt{-1}(E_{23} + E_{32}).\end{aligned}$$

Then via the unique isomorphism between V_2 and $\mathfrak{p}_{\mathbf{C}}$ as K-modules we have the identification $v_j^{(2)} = X_j$ $(-2 \leq j \leq 2)$.

Proof. By direct computation, we have the following table of the adjoint actions of the basis $\{K_{23},\ K_{13} \pm \sqrt{-1}K_{12}\}$ of $\mathfrak{k}_{\mathbf{C}}$ on the basis $\{X_j \mid -2 \leq j \leq 2\}$ of $\mathfrak{p}_{\mathbf{C}}$.

	X_{-2}	X_{-1}	X_0	X_1	X_2
K_{23}	$-2\sqrt{-1}X_{-2}$	$-\sqrt{-1}X_{-1}$	0	$\sqrt{-1}X_1$	$2\sqrt{-1}X_2$
$K_{13} + \sqrt{-1}K_{12}$	0	$-X_{-2}$	$-2X_{-1}$	$-3X_0$	$-4X_1$
$K_{13} - \sqrt{-1}K_{12}$	$4X_{-1}$	$3X_0$	$2X_1$	X_2	0

TABLE. The adjoint actions of $\mathfrak{k}_{\mathbf{C}}$ on the basis $\{X_j \mid -2 \leq j \leq 2\}$ of $\mathfrak{p}_{\mathbf{C}}$.

Comparing the actions in the above table with the actions in Subsection 3.1, we have the assertion. □

3.3. **The contragradient representation of** (τ_l, V_l). We denote by (τ^*, V^*) the contragradient representation of (τ, V). Here we note that V_l^* is isomorphic to V_l as a $SO(3)$-module, since the irreducible $2l+1$-dimensional representation of $SO(3)$ is unique up to isomorphism.

Lemma 3.3. *Let* $\{v_q^{(l)*} \mid -l \le q \le l\}$ *be the dual basis of the weight basis* $\{v_q^{(l)} \mid -l \le q \le l\}$. *Via the unique isomorphism between* V_l *and* V_l^* *as a* K*-module we have the identification*

$$v_q^{(l)} = (-1)^{l+q}\frac{(l-q)!(l+q)!}{(2l)!}v_{-q}^{(l)*}$$

for $-l \le q \le l$.

Proof. We denote by $\langle , \rangle$ the canonical pairing on $V_l^* \times V_l$.

Since

$$\langle \tau_l^*(K_{23})v_q^{(l)*}, v_r^{(l)}\rangle = -\langle v_q^{(l)*}, \tau_l(K_{23})v_r^{(l)}\rangle = -\sqrt{-1}r\delta_{qr} = -\sqrt{-1}q\delta_{qr},$$

we have $\tau_l^*(K_{23})v_q^{(l)*} = -\sqrt{-1}qv_q^{(l)*}$. Similarly, we obtain

$$\tau_l^*(K_{13}+\sqrt{-1}K_{12})v_q^{(l)*} = (l+q+1)v_{q+1}^{(l)*}, \quad \tau_l^*(K_{13}-\sqrt{-1}K_{12})v_q^{(l)*} = -(l-q+1)v_{q-1}^{(l)*}.$$

From these equations, we obtain the assertion. □

3.4. **Some components of** $\mathfrak{p}_{\mathbf{C}} \otimes \tau_l$ **for** $l \ge 1$. For a K-module (π, H), we denote by $H[\tau_l]$ the τ_l-isotypic component of H. The tensor product $\mathfrak{p}_{\mathbf{C}} \otimes_{\mathbf{C}} V_l \simeq V_2 \otimes_{\mathbf{C}} V_l$ has at most five irreducible components V_{l+2}, V_{l+1}, V_l, V_{l-1} and V_{l-2}. Here some components may not appear. For our later use, we take a basis of $\bigoplus_{0\le m\le 2}(\mathfrak{p}_{\mathbf{C}} \otimes_{\mathbf{C}} V_l)[\tau_{l-m}]$ as follows.

Proposition 3.4. *We put*

$$v(i,q) = X_{i-1}\otimes v_{q+1}^{(l)} - 2X_i \otimes v_q^{(l)} + X_{i+1}\otimes v_{q-1}^{(l)}$$

for $l \ge 1$, $-1 \le i \le 1$ *and* $-l+1 \le q \le l-1$. *Then* $\{v(i,q) \mid -1 \le i \le 1,\ -l+1 \le q \le l-1\}$ *is a basis of* $\bigoplus_{0\le m\le 2}(\mathfrak{p}_{\mathbf{C}} \otimes_{\mathbf{C}} V_l)[\tau_{l-m}]$ *and satisfies the condition*

$$v(i,q) \equiv \tilde{v}_{i+q}^{(l)} \mod \bigoplus_{1\le m\le 2}(\mathfrak{p}_{\mathbf{C}} \otimes_{\mathbf{C}} V_l)[\tau_{l-m}]. \tag{3.1}$$

Here $\tilde{v}_q^{(l)}$ *is the image of* $v_q^{(l)}$ *under the unique isomorphism* $V_l \to (\mathfrak{p}_{\mathbf{C}} \otimes_{\mathbf{C}} V_l)[\tau_l]$ *which is determined by* $v_{-l}^{(l)} \mapsto \tilde{v}_{-l}^{(l)} = v(-1,-l+1)$.

Proof. Let W be the vector space spanned by the elements $v(i,q)$ $(-1 \le i \le 1,\ -l+1 \le q \le l-1)$.

By direct computation, we have

$$(\mathrm{ad}\otimes\tau_l)(K_{23})v(i,q) = \sqrt{-1}(i+q)v(i,q), \tag{3.2}$$

$$(\mathrm{ad}\otimes\tau_l)(K_{13}+\sqrt{-1}K_{12})v(i,q) = -(1+i)v(i-1,q) - (l+q-1)v(i,q-1), \tag{3.3}$$

$$(\mathrm{ad}\otimes\tau_l)(K_{13}-\sqrt{-1}K_{12})v(i,q) = (1-i)v(i+1,q) + (l-q-1)v(i,q+1) \tag{3.4}$$

for $-1 \le i \le 1$ and $-l+1 \le q \le l-1$. Here we denote the differential of the adjoint representation Ad by ad and put $v(-2,q) = v(2,q) = v(i,-l) = v(i,l) = 0$ for any $i, q \in \mathbf{Z}$. From these equations, we see that W is closed under the action of $\mathfrak{k}_{\mathbf{C}}$ and there are no elements in W whose eigenvalues of $(\mathrm{ad}\otimes\tau_l)(\sqrt{-1}K_{23})$ are larger than l. This implies W is a K-submodule of $\bigoplus_{0\le m\le 2}(\mathfrak{p}_{\mathbf{C}} \otimes_{\mathbf{C}} V_l)[\tau_{l-m}]$. Moreover, since the elements $v(i,q)$ $(-1 \le i \le 1,\ -l+1 \le q \le l-1)$ are linearly independent, we have

$$\dim_{\mathbf{C}} W = 6l-3 = \dim_{\mathbf{C}} \bigoplus_{0\le m\le 2}(\mathfrak{p}_{\mathbf{C}} \otimes_{\mathbf{C}} V_l)[\tau_{l-m}].$$

Hence we have $W = \bigoplus_{0\le m\le 2}(\mathfrak{p}_{\mathbf{C}} \otimes_{\mathbf{C}} V_l)[\tau_{l-m}]$.

By the equations (3.2) and (3.3), we see that $v(-1,-l+1)$ is the highest weight vector of $(\mathfrak{p}_{\mathbf{C}}\otimes_{\mathbf{C}} V_l)[\tau_l]$ with respect to the Borel subalgebra of $\mathfrak{k}_{\mathbf{C}}$ spanned by K_{23} and $K_{13}+\sqrt{-1}K_{12}$. Hence there is the unique isomorphism $V_l \to (\mathfrak{p}_{\mathbf{C}}\otimes_{\mathbf{C}} V_l)[\tau_l]$ which is determined by $v^{(l)}_{-l}\mapsto v(-1,-l+1)$.

We prove the equation (3.1) by induction with respect to $i+q$. When $i+q=-l$, the equation (3.1) holds since $v(-1,-l+1)=\tilde{v}^{(l)}_{-l}$. Assume that the equation (3.1) holds for $-1\le i\le 1$, $-l+1\le q\le l-1$ such that $i+q<n$, and take $-1\le i'\le 1$, $-l+1\le q'\le l-1$ such that $i'+q'=n$. From the equation (3.2), there is some constant $C_{(i',q')}$ such that $v(i',q')\equiv C_{(i',q')}\tilde{v}^{(l)}_{i'+q'}$ mod $\bigoplus_{1\le m\le 2}(\mathfrak{p}_{\mathbf{C}}\otimes_{\mathbf{C}} V_l)[\tau_{l-m}]$. Moreover, from the equation (3.3) and the inductive assumption, we have

$$\begin{aligned}
&(\mathrm{ad}\otimes\tau_l)(K_{13}+\sqrt{-1}K_{12})v(i',q')\\
&=-(1+i')v(i'-1,q')-(l+q'-1)v(i',q'-1)\\
&\equiv -(l+i'+q')\tilde{v}^{(l)}_{i'+q'-1} \quad \mathrm{mod} \bigoplus_{1\le m\le 2}(\mathfrak{p}_{\mathbf{C}}\otimes_{\mathbf{C}} V_l)[\tau_{2(l-m)}]\\
&\equiv (\mathrm{ad}\otimes\tau_l)(K_{13}+\sqrt{-1}K_{12})\tilde{v}^{(l)}_{i'+q'} \quad \mathrm{mod} \bigoplus_{1\le m\le 2}(\mathfrak{p}_{\mathbf{C}}\otimes_{\mathbf{C}} V_l)[\tau_{l-m}].
\end{aligned}$$

Hence we have $C_{(i',q')}=1$, and complete the induction. □

3.5. **Discrete series representations of $SL^{\pm}(2,\mathbf{R})$.** In this subsection, we consider a discrete series representation $D_k=\mathrm{Ind}_{SL(2,\mathbf{R})}^{SL^{\pm}(2,\mathbf{R})}(D_k^+)$ of $SL^{\pm}(2,\mathbf{R})$ where D_k^+ is a discrete series representation of $SL(2,\mathbf{R})$ with the Blattner parameter $k>0$. Let D_k^- be the contragradient representation of D_k^+ and set $y_0=\mathrm{diag}(1,-1)\in O(2)$. Then a discrete series representation D_k is uniquely determined by specifying the $SL(2,\mathbf{R})$-module structure together with the action of y_0. Since $D_k|_{SL(2,\mathbf{R})}=D_k^+\oplus D_k^-$ and $D_k^+\oplus D_k^-$ is infinitesimally equivalent with a subrepresentation of some principal series representation of $SL(2,\mathbf{R})$, we obtain the following realization of associated $(\mathfrak{sl}(2,\mathbf{C}),O(2))$-module of D_k:

$$V_{D_k,O(2)}=\bigoplus_{\alpha\in\mathbf{Z}_{\ge 0}} W_{k+2\alpha} \qquad (W_p=\mathbf{C}\cdot\chi_p+\mathbf{C}\cdot\chi_{-p})$$

and

$$\begin{aligned}
&D_k(w)\chi_p=\sqrt{-1}p\chi_p, && D_k(x_+)\chi_p=(k+p)\chi_{p+2}, && D_k(x_-)\chi_p=(k-p)\chi_{p-2},\\
&D_k(\kappa_t)\chi_p=e^{\sqrt{-1}pt}\chi_p && (t\in\mathbf{R}), && D_k(y_0)\chi_p=\chi_{-p},
\end{aligned}$$

where

$$w=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\ x_{\pm}=\begin{pmatrix}1&\pm\sqrt{-1}\\ \pm\sqrt{-1}&-1\end{pmatrix}\in\mathfrak{sl}(2,\mathbf{C}),\ \kappa_t=\begin{pmatrix}\cos t&\sin t\\-\sin t&\cos t\end{pmatrix}\in SO(2).$$

Here we denote differential of D_k again by D_k and the $O(2)$-finite part of V_{D_k} by $V_{D_k,O(2)}$. See [3, §2.5] for details.

3.6. **Irreducible decompositions of $(\pi_{(\nu,\sigma)}|_K, H_{(\nu,\sigma)})$ as a K-module.** We analyse the K-type of the representation space $H_{(\nu,\sigma)}$ of the generalized principal series representation. The target V_σ of functions $\mathbf{f}$ in $H_{(\nu,\sigma)}$ has a decomposition:

$$V_\sigma=V_{D_k}=\widehat{\bigoplus_{\alpha\in\mathbf{Z}_{\ge 0}}} W_{k+2\alpha}.$$

Denote the corresponding decomposition of $\mathbf{f}$ by

$$\mathbf{f}(x)=\sum_{\alpha=0}^{\infty}(f_{k+2\alpha}(x)\otimes\chi_{k+2\alpha}+f_{-(k+2\alpha)}(x)\otimes\chi_{-(k+2\alpha)}).$$

We define the natural isomorphism $\mathbf{m}\colon SL^{\pm}(2,\mathbf{R})\to M$ by

$$\mathbf{m}(h)=\begin{pmatrix} h & O_{2,1}\\ O_{1,2} & \det(h)^{-1}\end{pmatrix}\in M \qquad (h\in SL^{\pm}(2,\mathbf{R})),$$

and set $\sigma=D_k\circ\mathbf{m}^{-1}$. From the definition of the space $H_{(\nu,\sigma)}$, we have

$$\mathbf{f}(mx)=\sigma(m)\mathbf{f}(x) \qquad (\text{a.e. } x\in G,\ m\in M).$$

For $m=\mathbf{m}(\kappa_t)$, $\mathbf{m}(y_0)\in K_M=M\cap K\simeq O(2)$, comparing the coefficients of χ_p in the left hand side of above equation with those in the right hand side, we have

$$f_p(\mathbf{m}(\kappa_t)x)=e^{\sqrt{-1}pt}f_p(x), \qquad f_p(\mathbf{m}(y_0)x)=f_{-p}(x).$$

Moreover, from the equality

$$\int_K\|\mathbf{f}(x)\|_\sigma^2dx=\sum_{\varepsilon\in\{\pm1\},\ \alpha\in\mathbf{Z}_{\geq0}}\left\{\int_K|f_{\varepsilon(k+2\alpha)}(x)|^2dx\right\}\|\chi_{\varepsilon(k+2\alpha)}\|_\sigma^2,$$

we have $f_p|_K\in L^2(K)$. Therefore $\mathbf{f}|_K$ belongs to

$$\widehat{\bigoplus_{\alpha\in\mathbf{Z}_{\geq0}}}L^2_\sigma(K;W_{k+2\alpha})$$

where

$$L^2_\sigma(K;W_p)=\{\mathbf{f}\colon K\to W_p \mid \mathbf{f}(x)=f(x)\otimes\chi_p+f(\mathbf{m}(y_0)x)\otimes\chi_{-p},\ f\in L^2_{(K_M^\circ,\chi_p)}(K),x\in K\},$$
$$L^2_{(K_M^\circ,\chi_p)}(K)=\{f\in L^2(K)\mid f(\mathbf{m}(\kappa_t)x)=e^{\sqrt{-1}pt}f(x),\ \mathbf{m}(\kappa_t)\in K_M^\circ,\ x\in K\}.$$

Here K_M° means the identity component of K_M, which is isomorphic to $SO(2)$. It is easy to see that the restriction map

$$r_K\colon H_{(\nu,\sigma)}\ni\mathbf{f}\mapsto\mathbf{f}|_K\in\widehat{\bigoplus_{\alpha\in\mathbf{Z}_{\geq0}}}L^2_\sigma(K;W_{k+2\alpha})$$

is a K-isomorphism.

By the Peter-Weyl's theorem (see for example, [20]), we have a K-isomorphism

$$\widehat{\bigoplus_{l\in\mathbf{Z}_{\geq0}}}(V_l^*[\xi_{-p}])\otimes_{\mathbf{C}}V_l\to L^2_{(K_M^\circ,\chi_p)}(K)$$

which is determined by

$$(V_l^*[\xi_{-p}])\otimes_{\mathbf{C}}V_l\ni w\otimes v\mapsto(x\mapsto\langle w,\tau_l(x)v\rangle)\in L^2_{(K_M^\circ,\chi_p)}(K).$$

Here

$$\xi_p\colon K_M^\circ\ni\mathbf{m}(\kappa_t)\mapsto e^{\sqrt{-1}pt}\in\mathbf{C}^\times$$

and $V[\xi_p]$ means the ξ_p-isotypic component in $(\tau|_{K_M^\circ},V)$ for a K-module (τ,V).

We define a basis $\{w_q^{(l)}\mid -l\leq q\leq l\}$ of V_l by

$$w_q^{(l)}=\tau_l(u_c)v_q^{(l)} \quad (-l\leq q\leq l)$$

where

$$u_c=\begin{pmatrix}0&0&-1\\0&1&0\\1&0&0\end{pmatrix}\in SO(3).$$

We note that $v \in V_l[\xi_{-p}]$ if and only if $\tau_l(K_{12})v = -\sqrt{-1}pv$ for $v \in V_l$. Since $u_c^{-1}K_{12}u_c = K_{23}$, we have $\tau_l(K_{12})w_q^{(l)} = \sqrt{-1}qw_q^{(l)}$. Hence we have

$$V_l[\xi_{-p}] = \begin{cases} \mathbf{C}\cdot w_{-p}^{(l)} & \text{if } -l \leq p \leq l, \\ 0 & \text{otherwise .} \end{cases}$$

By the identification $V_l^* = V_l$ in Lemma 3.3, we obtain

$$L^2_{(K_M^\circ,\chi_p)}(K) \simeq \widehat{\bigoplus_{l\in\mathbf{Z}_{\geq 0}}} (\mathbf{C}\cdot w_p^{(l)*}) \otimes_{\mathbf{C}} V_l.$$

Here we put $w_p^{(l)*} = 0$ if $p < -l$ or $p > l$. Moreover, by direct computation, we have

$$\tau_l^*(\mathbf{m}(y_0)^{-1})w_p^{(l)*} = (-1)^l w_{-p}^{(l)*}.$$

For $k \leq p \leq l$ such that $p \equiv k \bmod 2$, we define the elementary function $t(l;p,q) \in H_{(\nu,\sigma)}$ by

$$t(l;p,q) = r_K^{-1}(\tilde{t}(l;p,q))$$

where

$$\tilde{t}(l;p,q)(x) = \langle w_p^{(l)*}, \tau_l(x)v_q^{(l)}\rangle \otimes \chi_p + (-1)^l\langle w_{-p}^{(l)*}, \tau_l(x)v_q^{(l)}\rangle \otimes \chi_{-p} \in L^2_\sigma(K;W_p).$$

Let $\langle T(l;p)\rangle$ be the subspace of $H_{(\nu,\sigma)}$ generated by the functions $t(l;p,q)$ $(-l \leq q \leq l)$. Via the unique isomorphism between $\langle T(l;p)\rangle$ and V_l, we identify $\{t(l;p,q) \mid -l \leq q \leq l\}$ with the weight basis $\{v_q^{(l)} \mid -l \leq q \leq l\}$.

From above arguments, we obtain the following.

Proposition 3.5. *As a K-module, $H_{(\nu,\sigma)}$ has an irreducible decomposition:*

$$H_{(\nu,\sigma)} = \widehat{\bigoplus_{l\in\mathbf{Z}_{\geq 0}}} \left\{ \bigoplus_{k\leq p\leq l,\ p-k\in 2\mathbf{Z}} \langle T(l;p)\rangle \right\}.$$

Corollary 3.6. *The multiplicity $[\pi_{(\nu,\sigma)}|_K : \tau_l]$ of τ_l in $\pi_{(\nu,\sigma)}|_K$ is given by*

$$[\pi_{(\nu,\sigma)}|_K : \tau_l] = \begin{cases} (l-k+2)/2 & \text{if } k \leq l \text{ and } l-k \text{ is even,} \\ (l-k+1)/2 & \text{if } k \leq l \text{ and } l-k \text{ is odd,} \\ 0 & \text{if } k > l. \end{cases}$$

3.7. **Dirac-Schmid equations.** We denote by $H_{(\nu,\sigma),K}$ the K-finite part of $H_{(\nu,\sigma)}$. From Corollary 3.6, we see that τ_k is the minimal K-type of $\pi_{(\nu,\sigma)}$ and its multiplicity is one. We put $\iota_k : V_k \ni v_q^{(k)} \mapsto t(k;k,q) \in H_{(\nu,\sigma),K}$.

Lemma 3.7. *For $0 \leq j \leq 2$, X_j has the following expressions according to the decomposition $\mathfrak{g}_{\mathbf{C}} = \mathrm{Ad}(u_c^{-1})(\mathfrak{n}_{\mathbf{C}} \oplus \mathfrak{a}_{\mathbf{C}} \oplus \mathfrak{m}_{\mathbf{C}}) + \mathfrak{k}_{\mathbf{C}}$:*

$$\begin{aligned} X_0 =& \frac{1}{3}\mathrm{Ad}(u_c^{-1})(H_1+H_2), \qquad X_1 = -\mathrm{Ad}(u_c^{-1})(E_{13}+\sqrt{-1}E_{23}) + \frac{1}{2}(K_{13}-\sqrt{-1}K_{12}), \\ X_2 =& -\mathrm{Ad}(u_c^{-1})m(x_+), \end{aligned}$$

Proof. We obtain the assertion immediately from Lemma 6.6. □

Lemma 3.8. *The weight basis $\{t(k;k,q) \mid -k \leq q \leq k\}$ of $H_{(\nu,\sigma),K}[\tau_k] = \langle T(k;k)\rangle$ satisfies the following relations:*

$$X_{i-1}\cdot t(k;k,q+1) - 2X_i \cdot t(k;k,q) + X_{i+1}\cdot t(k;k,q-1) = \nu t(k;k,q+i)$$

for $-1 \leq i \leq 1$ and $-k+1 \leq q \leq k-1$.

Proof. The image of the element $v(i,q)$ defined in Proposition 3.4 under the natural K-homomorphism $\tilde{\iota}_k\colon \mathfrak{p}_{\mathbf{C}} \otimes_{\mathbf{C}} V_k \ni X \otimes v \mapsto X \cdot \iota_k(v) \in H_{(\nu,\sigma),K}$ is given by

$$\tilde{\iota}_k(v(i,q)) = X_{i-1} \cdot t(k;k,q+1) - 2X_i \cdot t(k;k,q) + X_{i+1} \cdot t(k;k,q-1)$$

for $-1 \le i \le 1$ and $-k+1 \le q \le k-1$. On the other hand, from $\bigoplus_{1 \le m \le 2} H_{(\nu,\sigma),K}[\tau_{k-m}] = 0$ and Proposition 3.4, there is some constant $C \in \mathbf{C}$ such that $\tilde{\iota}_k(v(i,q)) = Ct(k;k,q+i)$ for any i,q. Hence we have

$$X_{i-1} \cdot t(k;k,q+1) - 2X_i \cdot t(k;k,q) + X_{i+1} \cdot t(k;k,q-1) = Ct(k;k,q+i) \tag{3.5}$$

for $-1 \le i \le 1$ and $-k+1 \le q \le k-1$.

We evaluate the both sides of the equation (3.5) at u_c when $i = 1$ and $q = k-1$. By the definition of $t(k;k,q)$, we see that $t(k;k,q)(u_c) = \delta_{k,q} \cdot \chi_k + (-1)^k \delta_{-k,q} \cdot \chi_{-k}$. Therefore, the right hand side of the equation (3.5) become

$$Ct(k;k,k)(u_c) = C \cdot \chi_k.$$

Since $t(k;k,q)$ can be identified with $v_q^{(k)}$, we have

$$((K_{13} - \sqrt{-1}K_{12}) \cdot t(k;k,q))(u_c) = (k-q)(\delta_{k,q+1} \cdot \chi_k + (-1)^k \delta_{-k,q+1} \cdot \chi_{-k}).$$

Moreover, we have

$$\begin{aligned}
&(\mathrm{Ad}(u_c^{-1})E \cdot t(k;k,q))(u_c) = 0 \qquad \text{for } E \in \mathfrak{n}_{\mathbf{C}},\\
&\left(\frac{1}{3}\mathrm{Ad}(u_c^{-1})(H_1 + H_2) \cdot t(k;k,q)\right)(u_c) = (\nu+1)(\delta_{k,q} \cdot \chi_k + (-1)^k \delta_{-k,q} \cdot \chi_{-k}),\\
&(\mathrm{Ad}(u_c^{-1})m(x_+) \cdot t(k;k,q))(u_c) = (2k)\delta_{k,q} \cdot \chi_{k+2},
\end{aligned}$$

from the definition of a generalized principal series representation. By above equations and expressions of X_i in Lemma 3.7, the left hand side of the equation (3.5) becomes

$$\begin{aligned}
&(X_0 \cdot t(k;k,k))(u_c) - 2(X_1 \cdot t(k;k,k-1))(u_c) + (X_2 \cdot t(k;k,k-2))(u_c)\\
&= (\nu+1) \cdot \chi_k - \chi_k + 0 = \nu \cdot \chi_k.
\end{aligned}$$

Hence we have $C = \nu$. □

4. Differential equations

4.1. **The Capelli elements.** The Capelli elements are known to give generators for the center $Z(\mathfrak{g}_{\mathbf{C}})$ of the universal enveloping algebra $U(\mathfrak{g}_{\mathbf{C}})$ of $\mathfrak{g}_{\mathbf{C}}$ (*cf.* [13, §11]). We put

$$E'_{ii} = E_{ii} - \frac{1}{3}\left(\sum_{j=1}^{3} E_{jj}\right), \quad (1 \le i \le 3).$$

We compute the Capelli elements by the vertical determinant of characteristic matrix $\mathcal{A} = (A_{ij})_{1 \le i,j \le 3} = x \cdot 1_3 - \mathcal{C} \in M_3(U(\mathfrak{g}_{\mathbf{C}})[x])$ of degree 3, where $\mathcal{C}$ is an element of $M_3(U(\mathfrak{g}_{\mathbf{C}}))$ defined by

$$\mathcal{C} = \begin{pmatrix} E'_{11} - 1 & E_{12} & E_{13} \\ E_{21} & E'_{22} & E_{23} \\ E_{31} & E_{32} & E'_{33} + 1 \end{pmatrix}.$$

Then we have

$$\sum_{\tau \in \mathfrak{S}_3} \mathrm{sgn}(\tau) A_{1\tau(1)} A_{2\tau(2)} A_{3\tau(3)} = x^3 + C_2 x - C_3$$

with the Capelli elements C_2 and C_3 of degree 2 and 3, respectively. Here $\mathfrak{S}_3$ is symmetric group of degree three. The explicit formulas of C_2 and C_3 are given by

$$\begin{aligned}
C_2 =& (E'_{11} - 1)E'_{22} + E'_{22}(E'_{33} + 1) + (E'_{11} - 1)(E'_{33} + 1) - E_{23}E_{32} - E_{13}E_{31} - E_{12}E_{21},\\
C_3 =& (E'_{11} - 1)E'_{22}(E'_{33} + 1) + E_{12}E_{23}E_{31} + E_{13}E_{21}E_{32}
\end{aligned}$$

$$- (E'_{11}-1)E_{23}E_{32} - E_{13}E'_{22}E_{31} - E_{12}E_{21}(E'_{33}+1).$$

We rewrite the Capelli elements $\{C_2,\ C_3\}$ as follows.

Lemma 4.1. *The Capelli elements $\{C_2,\ C_3\}$ have expressions of the form $\sum_i Z_iY_iX_i$ ($Z_i \in U(\mathfrak{n}_{0\mathbf{C}})$, $Y_i \in U(\mathfrak{a}_{0\mathbf{C}})$, $X_i \in U(\mathfrak{k}_{\mathbf{C}})$) as follows.*

$$\begin{aligned}
C_2 =&\Big\{\frac{1}{3}(2H_1-H_2)-1\Big\}\Big\{-\frac{1}{3}(H_1-2H_2)\Big\} + \Big\{-\frac{1}{3}(H_1-2H_2)\Big\}\Big\{-\frac{1}{3}(H_1+H_2)+1\Big\}\\
&+\Big\{\frac{1}{3}(2H_1-H_2)-1\Big\}\Big\{-\frac{1}{3}(H_1+H_2)+1\Big\}\\
&-E_{23}^2 - E_{13}^2 - E_{12}^2 + E_{23}K_{23} + E_{13}K_{13} + E_{12}K_{12},\\
C_3 =&\Big\{\frac{1}{3}(2H_1-H_2)-1\Big\}\Big\{-\frac{1}{3}(H_1-2H_2)\Big\}\Big\{-\frac{1}{3}(H_1+H_2)+1\Big\}\\
&-E_{23}^2\Big\{\frac{1}{3}(2H_1-H_2)-1\Big\} - E_{13}^2\Big\{-\frac{1}{3}(H_1-2H_2)\Big\} - E_{12}^2\Big\{-\frac{1}{3}(H_1+H_2)+1\Big\}\\
&+\Big\{E_{12}\Big\{-\frac{1}{3}(H_1+H_2)+1\Big\} - E_{13}E_{23}\Big\}K_{12} + \Big\{E_{13}\Big\{-\frac{1}{3}(H_1-2H_2)\Big\} - E_{12}E_{23}\Big\}K_{13}\\
&+\Big\{E_{23}\Big\{\frac{1}{3}(2H_1-H_2)-1\Big\} - E_{13}E_{12}\Big\}K_{23} + E_{12}E_{23}E_{13} + E_{13}E_{23}E_{12} + E_{13}K_{12}K_{23}.
\end{aligned}$$

Proof. We obtain the assertion by direct computation. □

4.2. **The eigenvalues of the Capelli elements.** The elements of $Z(\mathfrak{g}_{\mathbf{C}})$ acts on $H_{(\nu,\sigma),K}$ by scalar multiplication. We compute the eigenvalues of the Capelli elements by usual way.

We put $\mathcal{P} = \sum_{\alpha\in\Sigma^+} U(\mathfrak{g}_{\mathbf{C}})E_{-\alpha}$. Since $Z(\mathfrak{g}_{\mathbf{C}}) \subset U(\mathfrak{a}_{0\mathbf{C}}) \oplus \mathcal{P}$, we can define the projection γ_{Σ^+} of $Z(\mathfrak{g}_{\mathbf{C}})$ into the $U(\mathfrak{a}_{0\mathbf{C}})$ factor. We put $\sigma_{\Sigma^+}\colon \mathfrak{a}_{0\mathbf{C}} \ni H \mapsto H+\rho_0(H) \in U(\mathfrak{a}_{0\mathbf{C}})$ and extend σ_{Σ^+} to an algebra automorphism of $U(\mathfrak{a}_{0\mathbf{C}})$. Here $\rho_0 = e_1 - e_3$ is the half sum of positive roots. We define the Harish-Chandra homomorphism $\gamma\colon Z(\mathfrak{g}_{\mathbf{C}}) \to U(\mathfrak{a}_{0\mathbf{C}})$ by $\gamma = \sigma_{\Sigma^+}\circ\gamma_{\Sigma^+}$. By direct computation, the image of the Capelli element C_i under the Harish-Chandra homomorphism is given by

$$\gamma(C_i) = S_i\Big(E'_{11},\ E'_{22},\ E'_{33}\Big)$$

for $i=2,3$. Here $S_2(a,b,c) = ab+bc+ca$ and $S_3(a,b,c) = abc$ are the elementary symmetric functions of three variables of degree 2 and 3, respectively.

Let $\mathfrak{t}$ be the θ-stable Cartan subalgebra of $\mathfrak{m}$ generated by $H_1 - H_2$ and Λ_σ the infinitesimal character of σ relative to $\mathfrak{t}_{\mathbf{C}}$. It is easy to see that the value of Λ_σ at $H_1 - H_2$ is given by $k-1$. According to [20, Proposition 8.22], the infinitesimal character of $\pi_{(\nu,\sigma)}$ relative to $(\mathfrak{t}+\mathfrak{a})_{\mathbf{C}} = \mathfrak{a}_{0\mathbf{C}}$ is given by $\Lambda_\sigma + \nu$ and the eigenvalue of C_i is $\big((\Lambda_\sigma+\nu)\circ\gamma\big)(C_i)$ for $i=2,3$.

Since

$$\begin{aligned}
(\Lambda_\sigma+\nu)(E'_{11}) &= (\Lambda_\sigma+\nu)\left(\frac{1}{6}(H_1+H_2)+\frac{1}{2}(H_1-H_2)\right) = \frac{\nu+k-1}{2},\\
(\Lambda_\sigma+\nu)(E'_{22}) &= (\Lambda_\sigma+\nu)\left(\frac{1}{6}(H_1+H_2)-\frac{1}{2}(H_1-H_2)\right) = \frac{\nu-k+1}{2},\\
(\Lambda_\sigma+\nu)(E'_{33}) &= (\Lambda_\sigma+\nu)\left(-\frac{1}{3}(H_1+H_2)\right) = -\nu,
\end{aligned}$$

we obtain the following.

Lemma 4.2. *For $i=2,3$, the eigenvalue $\chi_{(\nu,\sigma)}(C_i)$ of C_i is given by*

$$\chi_{(\nu,\sigma)}(C_i) = S_i\Big(-\nu,\ \frac{\nu-k+1}{2},\ \frac{\nu+k-1}{2}\Big).$$

4.3. **The holonomic system of Whittaker functions.** Let $\Phi(I,\iota_k)\in \mathrm{Wh}(\pi_{(\nu,\sigma)},\xi,\tau_k^*)$ be a Whittaker function determined by an intertwining operator $I\in\mathcal{I}_{\xi,\pi_{(\nu,\sigma)}}$ and an injection $\iota_k\colon V_l\ni v_q^{(k)}\mapsto t(k;k,q)\in H_{(\nu,\sigma),K}$ $(-k\le q\le k)$. For $-k\le q\le k$, we define a scalar function F_q on A_0 by

$$F_q(a)=I(\iota_k(v_q^{(k)}))(a)=\langle v_q^{(k)},\Phi(I,\iota_k)(a)\rangle,\quad a\in A_0.$$

Then a Whittaker function $\Phi(I,\iota_k)$ is characterized by $\{F_q\mid -k\le q\le k\}$. In fact, A_0-radial part of a Whittaker function $\Phi(I,\iota_k)$ is written as

$$\Phi(I,\iota_k)|_{A_0}=\sum_{q=-k}^{k}F_q\otimes v_q^{(k)*}.$$

We write here the holonomic system for F_q with respect to the variables $y=(y_1,y_2)$ where $y_1=a_1/a_2$, $y_2=a_2/a_3=a_1a_2^2$ for $a=\mathrm{diag}(a_1,a_2,a_3)\in A_0$.

Lemma 4.3 ([13, Lemma 4.2 and 4.3]). *For $Z\in U(\mathfrak{n}_{0\mathbf{C}})$, $Y\in U(\mathfrak{a}_{0\mathbf{C}})$, $X\in U(\mathfrak{k}_{\mathbf{C}})$ and $v\in V_k$, we have*

$$(ZYX\cdot I(\iota_k(v)))(y)=\xi(\mathrm{Ad}(a)Z)(Y\cdot I(\iota_k(\tau_k(X)v)))(y).$$

In particular, we have

$$\begin{aligned}
(H_1\cdot I(\iota_k(v)))(y)&=(\partial_1+\partial_2)I(\iota_k(v))(y), & (H_2\cdot I(\iota_k(v)))(y)&=(-\partial_1+2\partial_2)I(\iota_k(v))(y),\\
(E_{12}\cdot I(\iota_k(v)))(y)&=2\pi\sqrt{-1}c_1y_1I(\iota_k(v))(y), & (E_{13}\cdot I(\iota_k(v)))(y)&=0,\\
(E_{23}\cdot I(\iota_k(v)))(y)&=2\pi\sqrt{-1}c_2y_2I(\iota_k(v))(y),
\end{aligned}$$

where $\partial_i=y_i\dfrac{\partial}{\partial y_i}$ $(i=1,2)$.

Proof. We obtain the assertion by easy computation. See [13] for details. □

Lemma 4.4. *The weight basis $\{X_i\mid -2\le i\le 2\}$ in $\mathfrak{p}_{\mathbf{C}}$ have the following expressions according to the Iwasawa decomposition $\mathfrak{g}_{\mathbf{C}}=\mathfrak{n}_{\mathbf{C}}\oplus\mathfrak{a}_{\mathbf{C}}\oplus\mathfrak{k}_{\mathbf{C}}$:*

$$\begin{aligned}
X_{-2}&=-2\sqrt{-1}E_{23}+H_2+\sqrt{-1}K_{23},\\
X_{-1}&=-(E_{13}+\sqrt{-1}E_{12})+\frac{1}{2}(K_{13}+\sqrt{-1}K_{12}),\\
X_0&=-\frac{1}{3}(2H_1-H_2),\\
X_1&=(E_{13}-\sqrt{-1}E_{12})-\frac{1}{2}(K_{13}-\sqrt{-1}K_{12}),\\
X_2&=2\sqrt{-1}E_{23}+H_2-\sqrt{-1}K_{23}.
\end{aligned}$$

Proof. We obtain the assertion immediately from Lemma 6.6. □

Proposition 4.5. *Put $F_q(y)=y_1y_2\tilde{F}_q(y)$ for $-k\le q\le k$. Then $\tilde{F}_q(y)$ satisfies the following system of partial differential equations:*
(i) For $-k+1\le q\le k-1$,

$$(-\partial_1+2\partial_2+4\pi c_2y_2-q)\tilde{F}_{q+1}-4\pi c_1y_1\tilde{F}_q+\big(-\partial_1+k+q-1-\nu\big)\tilde{F}_{q-1}=0,\tag{4.1}$$

$$\big(-\partial_1+k-q-1-\nu\big)\tilde{F}_{q+1}-4\pi c_1y_1\tilde{F}_q+(-\partial_1+2\partial_2-4\pi c_2y_2+q)\tilde{F}_{q-1}=0.\tag{4.2}$$

$$2\pi c_1y_1\tilde{F}_{q+1}+\big(2\partial_1-k+1-\nu\big)\tilde{F}_q+2\pi c_1y_1\tilde{F}_{q-1}=0,\tag{4.3}$$

(ii) For $-k\le q\le k$,

$$\{\Delta_2+2q\pi c_2y_2+\chi_{(\nu,\sigma)}(C_2)\}\tilde{F}_q+(k-q)\pi c_1y_1\tilde{F}_{q+1}+(k+q)\pi c_1y_1\tilde{F}_{q-1}=0,\tag{4.4}$$

$$\begin{aligned}(4.5)\qquad &\{\Delta_3 - 2q\pi c_2 y_2 \partial_1 - \chi_{(\nu,\sigma)}(C_3)\}\tilde{F}_q \\ &+ (k-q)\pi c_1 y_1 (\partial_2 + 2\pi c_2 y_2)\tilde{F}_{q+1} + (k+q)\pi c_1 y_1 (\partial_2 - 2\pi c_2 y_2)\tilde{F}_{q-1} = 0.\end{aligned}$$

Here

$$\Delta_2 = \partial_1^2 + \partial_2^2 - \partial_1\partial_2 - 4\pi^2(c_1^2 y_1^2 + c_2^2 y_2^2), \quad \Delta_3 = \partial_1(\partial_1 - \partial_2)\partial_2 + 4\pi^2 c_2^2 y_2^2 \partial_1 - 4\pi^2 c_1^2 y_1^2 \partial_2$$

and we put $\tilde{F}_q = 0$ *for* $q < -k$ *or* $q > k$.

Proof. We obtain the assertion from Lemma 3.8, 4.1, 4.2, 6.9 and 4.4. □

5. Whittaker functions for generalized principal series representations of $SL(3,\mathbf{R})$

5.1. **The dimension of the intertwining space $\mathcal{I}_{\xi,\pi_{(\nu,\sigma)}}$.** We recall two important invariants for finitely generated $U(\mathfrak{g}_{\mathbf{C}})$-modules, namely Gelfand-Kirillov dimension and multiplicity. See [46] for details.

Let H be a finitely generated $U(\mathfrak{g}_{\mathbf{C}})$-module and $v_1, v_2, \cdots, v_h$ its generators. For $n \in \mathbf{Z}_{\geq 0}$, $U_n(\mathfrak{g}_{\mathbf{C}})$ be the space of elements in $U(\mathfrak{g}_{\mathbf{C}})$ which may be written as a linear combination of products of at most n elements of $\mathfrak{g}_{\mathbf{C}}$ and put $H_n = \sum_{i=1}^{h} U_n(\mathfrak{g}_{\mathbf{C}})v_i$. Then, there exists some polynomial $p(x)$ in one variable over $\mathbf{Q}$ such that $\dim_{\mathbf{C}} H_n = p(n)$ for sufficiently large n. The Gelfand-Kirillov dimension $\operatorname{Dim} H$ is the degree of $p(x)$. Let d be any integer such that $d \geq \operatorname{Dim} H$. Then the multiplicity $c_d(H)$ of H is defined by

$$c_d(H) = \begin{cases} \text{the coefficient of } d!p(x) \text{ at } x^d & \text{if } d = \operatorname{Dim} H, \\ 0 & \text{if } d > \operatorname{Dim} H. \end{cases}$$

Multiplicities are always non-negative integers. The definitions of Gelfand-Kirillov dimensions and multiplicities do not depend on the choice of generators.

From the result of H. Matumoto (*cf.* [28, Corollary 2.2.2 and Theorem 6.2.1]), it follows that the dimension of the intertwining space $\mathcal{I}_{\xi,\pi}$ coincides with $c_3(H_{\pi,K})$ ($3 = \dim_{\mathbf{C}} \mathfrak{n}_{0\mathbf{C}}$) for an admissible representation (π, H_π) of G. We estimate $c_3(H_{(\nu,\sigma),K})$ as follows.

Let P_0 be a minimal parabolic subgroup of G with Langlands decomposition $P_0 = N_0 A_0 M_0$ where $M_0 = \{\operatorname{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2) \mid \varepsilon_1, \varepsilon_2 \in \{\pm 1\}\}$. By the subrepresentation theorem (see for example, [20, Theorem 8.37]), $\pi_{(\nu,\sigma)}$ is infinitesimally equivalent with a subrepresentation of a principal series representation $\pi_{(\nu_0,\sigma_0)} = \operatorname{Ind}_{P_0}^{G}(1_{N_0} \otimes e^{\nu_0+\rho_0} \otimes \sigma_0)$ for some character σ_0 of M_0 and $\nu_0 \in \mathfrak{a}_{0\mathbf{C}}^*$. We fix a such principal series representation $(\pi_{(\nu_0,\sigma_0)}, H_{(\nu_0,\sigma_0)})$ and define the $(\mathfrak{g}_{\mathbf{C}}, K)$-module $(\tilde{\pi}_{(\nu,\sigma)}, \tilde{H}_{(\nu,\sigma)})$ by $\tilde{H}_{(\nu,\sigma)} = H_{(\nu_0,\sigma_0),K}/H_{(\nu,\sigma),K}$. Then we have the following short exact sequence of $U(\mathfrak{g}_{\mathbf{C}})$-modules:

$$0 \to H_{(\nu,\sigma),K} \to H_{(\nu_0,\sigma_0),K} \to \tilde{H}_{(\nu,\sigma)} \to 0.$$

By [46, Lemma 2.4], we see that

$$c_3(H_{(\nu,\sigma),K}) = c_3(H_{(\nu_0,\sigma_0),K}) - c_3(\tilde{H}_{(\nu,\sigma)}).$$

From the result of Kostant [21, Theorem 5.5], it follows that $c_3(H_{(\nu_0,\sigma_0),K}) = \dim_{\mathbf{C}} \mathcal{I}_{\xi,\pi_{(\nu_0,\sigma_0)}}$ is the cardinality of the little Weyl group, i.e. $c_3(H_{(\nu_0,\sigma_0),K}) = 6$. Therefore, it suffices to estimate $c_3(\tilde{H}_{(\nu,\sigma)})$.

In order to estimate $c_3(\tilde{H}_{(\nu,\sigma)})$, we prepare the following lemma.

Lemma 5.1. *For a character* σ_0, *we take* $(\sigma_{0,1}, \sigma_{0,2}) \in \{0,1\} \times \{0,1\}$ *such that*

$$\sigma_0(\operatorname{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2)) = \varepsilon_1^{\sigma_{0,1}} \varepsilon_2^{\sigma_{0,2}} \quad (\varepsilon_1, \varepsilon_2 \in \{\pm 1\}).$$

Then the multiplicity $[\pi_{(\nu_0,\sigma_0)}|_K\colon \tau_l]$ *of* τ_l *in* $\pi_{(\nu_0,\sigma_0)}|_K$ *is given by*

$$[\pi_{(\nu_0,\sigma_0)}|_K\colon \tau_l] = \begin{cases} (l+2)/2 & \text{if } (\sigma_{0,1},\sigma_{0,2}) = (0,0) \text{ and } l \text{ is even}, \\ (l-1)/2 & \text{if } (\sigma_{0,1},\sigma_{0,2}) = (0,0) \text{ and } l \text{ is odd}, \\ l/2 & \text{if } (\sigma_{0,1},\sigma_{0,2}) \neq (0,0) \text{ and } l \text{ is even}, \\ (l+1)/2 & \text{if } (\sigma_{0,1},\sigma_{0,2}) \neq (0,0) \text{ and } l \text{ is odd}. \end{cases}$$

Proof. Let $V_l[\sigma_0]$ be the σ_0-isotypic component in $(\tau_l|_{M_0}, V_l)$. By Frobenius reciprocity (see for example, [20, Theorem 1.14]), we have

$$[\pi_{(\nu_0,\sigma_0)}|_K\colon \tau_l] = [\tau_l|_{M_0}\colon \sigma_0] = \dim_{\mathbf{C}} V_l[\sigma_0].$$

Since M_0 is generated by the two elements $m_{0,1} = \mathrm{diag}(-1,1,-1)$, $m_{0,2} = \mathrm{diag}(1,-1,-1)$, we note that $v \in V_l[\sigma_0]$ if and only if

$$\tau_l(m_{0,i})v = \sigma_0(m_{0,i})v = (-1)^{\sigma_{0,i}}v \qquad (i=1,2)$$

for $v \in V_l$. From the definition of (τ_l, V_l), we have $\tau_l(m_{0,1})v_q^{(l)} = (-1)^{l+q}v_{-q}^{(l)}$ and $\tau_l(m_{0,2})v_q^{(l)} = (-1)^q v_q^{(l)}$. From these equations, we obtain

$$V_l[\sigma_0] = \bigoplus_{q\in Z(\sigma_0;l)} \mathbf{C}\cdot (v_{-q}^{(l)} + (-1)^{\varepsilon(\sigma_0;l)}v_q^{(l)}),$$

where $\varepsilon(\sigma_0;l) \in \{0,1\}$ such that $\varepsilon(\sigma_0;l) \equiv l + \sigma_{0,1} + \sigma_{0,2} \bmod 2$ and

$$Z(\sigma_0;l) = \begin{cases} \{q\in\mathbf{Z} \mid 0\le q\le l,\ q\equiv \sigma_{0,2} \bmod 2\} & \text{if } \varepsilon(\sigma_0;l)=0, \\ \{q\in\mathbf{Z} \mid 1\le q\le l,\ q\equiv \sigma_{0,2} \bmod 2\} & \text{if } \varepsilon(\sigma_0;l)=1. \end{cases}$$

Since $[\pi_{(\nu_0,\sigma_0)}|_K\colon \tau_l]$ is the cardinality of the set $Z(\sigma_0;l)$, we obtain the assertion. □

From Corollary 3.6 and Lemma 5.1, there is some constant C_0 such that

$$[\tilde{\pi}_{(\nu,\sigma)}|_K\colon \tau_l] = [\pi_{(\nu_0,\sigma_0)}|_K\colon \tau_l] - [\pi_{(\nu,\sigma)}|_K\colon \tau_l] \le C_0 \qquad (l\in\mathbf{Z}_{\ge 0}). \tag{5.1}$$

Let $f_1, f_2, \cdots, f_h$ be generators of $\tilde{H}_{(\nu,\sigma)}$ as a $U(\mathfrak{g}_{\mathbf{C}})$-module. Then there is a non-negative integer L such that $f_i \in \bigoplus_{0\le l\le L}\tilde{H}_{(\nu,\sigma)}[\tau_l]$ $(1\le i\le h)$. Since $\mathfrak{p}_{\mathbf{C}}\otimes_{\mathbf{C}} V_l \simeq \bigoplus_{-2\le m\le 2} V_{l+m}$ and $\tilde{\iota}\colon \mathfrak{p}_{\mathbf{C}}\otimes_{\mathbf{C}} V_l \ni X\otimes v \mapsto X\cdot\iota(v) \in \tilde{H}_{(\nu,\sigma)}$ $(\iota\in \mathrm{Hom}_K(V_l, \tilde{H}_{(\nu,\sigma)}))$ is K-homomorphism, we see that

$$\mathfrak{p}_{\mathbf{C}}\cdot \tilde{H}_{(\nu,\sigma)}[\tau_l] \subset \bigoplus_{-2\le m\le 2} \tilde{H}_{(\nu,\sigma)}[\tau_{l+m}].$$

Here we denote by $\mathfrak{g}'\cdot\tilde{H}_{(\nu,\sigma)}[\tau_l]$ the space of elements in $\tilde{H}_{(\nu,\sigma)}$ which are written as a linear combination of $X\cdot f$ $(X\in\mathfrak{g}',\ f\in\tilde{H}_{(\nu,\sigma)}[\tau_l])$ for a subalgebra $\mathfrak{g}'$ of $\mathfrak{g}_{\mathbf{C}}$. Moreover, since $\mathfrak{g}_{\mathbf{C}} = \mathfrak{k}_{\mathbf{C}}\oplus\mathfrak{p}_{\mathbf{C}}$, we have

$$\mathfrak{g}_{\mathbf{C}}\cdot \tilde{H}_{(\nu,\sigma)}[\tau_l] \subset \bigoplus_{-2\le m\le 2} \tilde{H}_{(\nu,\sigma)}[\tau_{l+m}].$$

Therefore, we see that $\sum_{i=1}^h U_n(\mathfrak{g}_{\mathbf{C}})f_i \subset \bigoplus_{0\le l\le L+2n}\tilde{H}_{(\nu,\sigma)}[\tau_l]$. By the equation (5.1), we have

$$\dim_{\mathbf{C}}\sum_{i=1}^h U_n(\mathfrak{g}_{\mathbf{C}})f_i \le \dim_{\mathbf{C}}\bigoplus_{0\le l\le L+2n}\tilde{H}_{(\nu,\sigma)}[\tau_l] \le C_0\sum_{0\le l\le L+2n}(2l+1) = C_0(L+2n+1)^2.$$

This implies $\mathrm{Dim}\,\tilde{H}_{(\nu,\sigma)} \le 2$ and $c_3(\tilde{H}_{(\nu,\sigma)}) = 0$.

From the above arguments, we have the following.

Proposition 5.2. *The dimension of the intertwining space* $\mathcal{I}_{\xi,\pi_{(\nu,\sigma)}}$ *is 6.*

5.2. **Power series solutions at the origin.** In this section, we determine 6 linearly independent formal power series solutions at the origin $(y_1, y_2) = (0,0)$ of the system of partial differential equations in Proposition 4.5 for generic parameter ν. These solutions do not have exponential decay at infinity, different from a primary Whittaker function. We refer to these solutions as secondary Whittaker functions.

We use the same notation as in Subsection 4.3. After some computation, by inspection we find that it is convenient to introduce scalar functions $\Phi_{[\varepsilon_1,\varepsilon_2;p]}$ as follows:

We put

$$\Psi_q^0 = \frac{1}{2}(\tilde{F}_{-q} + \tilde{F}_q), \qquad \Psi_q^1 = \frac{1}{2}(\tilde{F}_{-q} - \tilde{F}_q)$$

for $0 \le q \le k$. Here we note that $\Psi_0^1 = 0$. Then, for $\varepsilon_1, \varepsilon_2 \in \{0,1\}$ and $0 \le p \le \left[\frac{k-\varepsilon_1-\varepsilon_2}{2}\right]$, we define the functions $\Phi_{[\varepsilon_1,\varepsilon_2;p]}$ by

$$\Psi^{\varepsilon_1}_{2q+\varepsilon_1+\varepsilon_2} = (-1)^{\varepsilon_1+\varepsilon_2} \sum_{p=0}^{q} h(\varepsilon_1, \varepsilon_2; p, q) \Phi_{[\varepsilon_1,\varepsilon_2;p]} \qquad \left(0 \le q \le \left[\frac{k-\varepsilon_1-\varepsilon_2}{2}\right]\right)$$

where

$$h(\varepsilon_1, \varepsilon_2; p, q) = \begin{cases} 1, & \text{if } (\varepsilon_1, p, q) = (0,0,0), \\ \left(\frac{(\varepsilon_2 p+q)(\varepsilon_1+1)+\varepsilon_1(\varepsilon_2+1)}{p+q+\varepsilon_1(p-q+1)}\right) \frac{2^{2p}(p+q)!}{(q-p)!(2p)!}, & \text{if } (\varepsilon_1, p, q) \ne (0,0,0),\ 0 \le p \le q, \\ 0, & \text{otherwise}, \end{cases}$$

and the symbol $[a]$ ($a \in \mathbf{R}$) means the maximal integer which is not larger than a. For $\varepsilon_1, \varepsilon_2 \in \{0,1\}$, $0 \le q \le \left[\frac{k-\varepsilon_1-\varepsilon_2}{2}\right]$, we call $(-1)^{\varepsilon_1-k}(\sqrt{-1})^{\varepsilon_2-k} y_1 y_2 \Phi_{[\varepsilon_1,\varepsilon_2;p]}$ the $[\varepsilon_1, \varepsilon_2; p]$-*component of a Whittaker function* $\Phi(I, \iota_k)$.

Remark 5.3. *We note that the* $[\varepsilon_1, \varepsilon_2; p]$*-component* $(-1)^{\varepsilon_1-k}(\sqrt{-1})^{\varepsilon_2-k} y_1 y_2 \Phi_{[\varepsilon_1,\varepsilon_2;p]}$ *is the image of the monomial* $f_{[\varepsilon_1,\varepsilon_2;p]} = x^{k-2p-\varepsilon_1-\varepsilon_2} y^{\varepsilon_2} z^{2p+\varepsilon_1} \bmod r^2 \cdot \mathcal{H}_{k-2} \in V_k$ *under the* K*-homomorphism* $I \circ \iota_k$, *i.e.*

$$(-1)^{\varepsilon_1-k}(\sqrt{-1})^{\varepsilon_2-k} y_1 y_2 \Phi_{[\varepsilon_1,\varepsilon_2;p]}(y) = I(\iota_k(f_{[\varepsilon_1,\varepsilon_2;p]}))(y) = \langle f_{[\varepsilon_1,\varepsilon_2;p]}, \Phi(I, \iota_k)(y) \rangle.$$

Lemma 5.4. *The coefficients* $h(\varepsilon_1, \varepsilon_2; p, q)$ *satisfy the following relations:*

$$h(\varepsilon_1, 0; p, q) + h(\varepsilon_1, 0; p, q-1) - 2h(\varepsilon_1, 1; p, q-1) - 2h(\varepsilon_1, 1; p-1, q-1) = 0,$$
$$h(\varepsilon_1, 1; p, q) + h(\varepsilon_1, 1; p, q-1) - 2h(\varepsilon_1, 0; p, q) = 0,$$

for $q \ge 1$, $0 \le p \le q$ *and* $\varepsilon_1 \in \{0,1\}$.

Proof. We obtain the assertion by direct computation. □

Proposition 5.5. *The functions* $\Phi_{[\varepsilon_1,\varepsilon_2;p]}$ *are determined from* $\Phi_{[0,0;0]}$ *recursively by the following equations:*

(i) $$(-2\partial_1 + 2\partial_2 + k - 1 - \nu)\Phi_{[0,1;0]} - 4\pi c_2 y_2 \Phi_{[1,0;0]} + 4\pi c_1 y_1 \Phi_{[0,0;0]} = 0.$$

(ii) For $\varepsilon_1 \in \{0,1\}$ *and* $1 \le p \le \left[\frac{k-\varepsilon_1}{2}\right]$,

$$4\pi c_1 y_1 \Phi_{[\varepsilon_1,0;p]} - (2\partial_1 - k + 1 - \nu)\Phi_{[\varepsilon_1,1;p-1]} + 4\pi c_1 y_1 \Phi_{[\varepsilon_1,0;p-1]} = 0.$$

(iii) For $\varepsilon_1 \in \{0,1\}$ *and* $0 \le p \le \left[\frac{k-\varepsilon_1-1}{2}\right]$,

$$4\pi c_1 y_1 \Phi_{[\varepsilon_1,1;p]} - (2\partial_1 - k + 1 - \nu)\Phi_{[\varepsilon_1,0;p]} = 0.$$

Proof. By rewriting the equations (4.1) and (4.2) in the case of $q = 0$ in terms of $\Phi_{[\varepsilon_1,\varepsilon_2;p]}$, we have

$$(-2\partial_1 + 2\partial_2 + 4\pi c_2 y_2 + k - 1 - \nu)\Phi_{[0,1;0]} + 4\pi c_1 y_1 \Phi_{[0,0;0]}$$
$$+ (-2\partial_1 - 4\pi c_2 y_2 + k - 1 - \nu)\Phi_{[1,0;0]} = 0,$$

$$(-2\partial_1 + 2\partial_2 - 4\pi c_2 y_2 + k - 1 - \nu)\Phi_{[0,1;0]} + 4\pi c_1 y_1 \Phi_{[0,0;0]} + (2\partial_1 - 4\pi c_2 y_2 - k + 1 + \nu)\Phi_{[1,0;0]} = 0.$$

Adding up these equations, we obtain (i).

We rewrite the equation (4.3) in terms of $\Psi_q^{\varepsilon_1}$ as follows:

$$4\pi c_1 y_1 \Psi_1^0 + (2\partial_1 - k + 1 - \nu)\Psi_0^0 = 0, \tag{5.2}$$

$$2\pi c_1 y_1 (\Psi_{q-1}^0 + \Psi_{q+1}^0 + \Psi_{q-1}^1 + \Psi_{q+1}^1) + (2\partial_1 - k + 1 - \nu)(\Psi_q^0 + \Psi_q^1) = 0, \tag{5.3}$$

$$2\pi c_1 y_1 (\Psi_{q-1}^0 + \Psi_{q+1}^0 - \Psi_{q-1}^1 - \Psi_{q+1}^1) + (2\partial_1 - k + 1 - \nu)(\Psi_q^0 - \Psi_q^1) = 0 \tag{5.4}$$

for $1 \le q \le k-1$. The equations (5.3) and (5.4) are equivalent to

$$2\pi c_1 y_1 \Psi_2^1 + (2\partial_1 - k + 1 - \nu)\Psi_1^1 = 0, \tag{5.5}$$

$$2\pi c_1 y_1 (\Psi_{q-2+\varepsilon_1}^{\varepsilon_1} + \Psi_{q+\varepsilon_1}^{\varepsilon_1}) + (2\partial_1 - k + 1 - \nu)\Psi_{q-1+\varepsilon_1}^{\varepsilon_1} = 0 \tag{5.6}$$

for $2 \le q \le k - \varepsilon_1$.

By rewriting the equations (5.2), (5.5) and (5.6) in terms of $\Phi_{[\varepsilon_1,\varepsilon_2;p]}$, we have

$$4\pi c_1 y_1 \Phi_{[\varepsilon_1,1;0]} - (2\partial_1 - k + 1 - \nu)\Phi_{[\varepsilon_1,0;0]} = 0, \tag{5.7}$$

$$(-1)^{\varepsilon_1}\left\{\sum_{p=0}^{q}(h(\varepsilon_1,0;p,q-1) + h(\varepsilon_1,0;p,q))2\pi c_1 y_1 \Phi_{[\varepsilon_1,0;p]} - \sum_{p=0}^{q-1}(2\partial_1 - k + 1 - \nu)h(\varepsilon_1,1;p,q-1)\Phi_{[\varepsilon_1,1;p]}\right\} = 0, \tag{5.8}$$

$$(-1)^{\varepsilon_1+1}\sum_{p=0}^{r}\Big\{(h(\varepsilon_1,1;p,r-1) + h(\varepsilon_1,1;p,r))2\pi c_1 y_1 \Phi_{[\varepsilon_1,0;p]} - (2\partial_1 - k + 1 - \nu)h(\varepsilon_1,0;p,r)\Phi_{[\varepsilon_1,1;p]}\Big\} = 0 \tag{5.9}$$

for $\varepsilon_1 \in \{0,1\}$, $1 \le q \le \left[\frac{k-\varepsilon_1}{2}\right]$ and $1 \le r \le \left[\frac{k-\varepsilon_1-1}{2}\right]$. By using the relations in Lemma 5.4, the equations (5.8) and (5.9) become

$$(-1)^{\varepsilon_1}\sum_{p=0}^{q-1} h(\varepsilon_1,1;p,q-1)\Big\{4\pi c_1 y_1(\Phi_{[\varepsilon_1,0;p]} + \Phi_{[\varepsilon_1,0;p+1]}) - (2\partial_1 - k + 1 - \nu)\Phi_{[\varepsilon_1,1;p]}\Big\} = 0,$$

$$(-1)^{\varepsilon_1+1}\sum_{p=0}^{r} h(\varepsilon_1,1;p,r)\Big\{4\pi c_1 y_1 \Phi_{[\varepsilon_1,0;p]} - (2\partial_1 - k + 1 - \nu)\Phi_{[\varepsilon_1,1;p]}\Big\} = 0,$$

respectively. From these equations and (5.7), we obtain (ii) and (iii). □

By above proposition, it suffices to consider only the function $\Phi_{[0,0;0]}$.

Proposition 5.6. *The function $\Phi_{[0,0;0]}$ satisfies the following system of partial differential equations.*

$$(4\Delta_2 - 4k\partial_1 - 3\nu^2 + 2k\nu + k^2 - 1)\Phi_{[0,0;0]} = 0, \tag{5.10}$$

$$\{4\Delta_3 - 4k\partial_1^2 + 4k^2\partial_1 + 16k\pi^2 c_1^2 y_1^2 + (\nu - k)(\nu + k + 1)(\nu + k - 1)\}\Phi_{[0,0;0]} = 0. \tag{5.11}$$

Proof. From Proposition 5.5 (i), (iii), we have

$$(-2\partial_1 + 2\partial_2 + k - 1 - \nu)\Phi_{[0,1;0]} - 4\pi c_2 y_2 \Phi_{[1,0;0]} + 4\pi c_1 y_1 \Phi_{[0,0;0]} = 0, \tag{5.12}$$

$$4\pi c_1 y_1 \Phi_{[0,1;0]} - (2\partial_1 - k + 1 - \nu)\Phi_{[0,0;0]} = 0. \tag{5.13}$$

We also have the following equations by rewriting the equations in Proposition 4.5 (ii) in terms of $\Phi_{[\varepsilon_1,\varepsilon_2;p]}$ in the case of $q=0$:

$$\{\Delta_2+\chi_{(\nu,\sigma)}(C_2)\}\Phi_{[0,0;0]}-2k\pi c_1y_1\Phi_{[0,1;0]}=0, \tag{5.14}$$

$$\{\Delta_3-\chi_{(\nu,\sigma)}(C_3)\}\Phi_{[0,0;0]}-2k\pi c_1y_1\partial_2\Phi_{[0,1;0]}+4k\pi^2c_1c_2y_1y_2\Phi_{[1,0;0]}=0. \tag{5.15}$$

Multiplying the both sides of the equations (5.14) and (5.13) by 4 and $2k$, respectively, we have

$$\begin{aligned}&\{4\Delta_2+4\chi_{(\nu,\sigma)}(C_2)\}\Phi_{[0,0;0]}-8k\pi c_1y_1\Phi_{[0,1;0]}=0,\\ &-2k(2\partial_1-k+1-\nu)\Phi_{[0,0;0]}+8k\pi c_1y_1\Phi_{[0,1;0]}=0.\end{aligned}$$

Adding up these equations, we obtain the equation (5.10).

Multiplying the both sides of the equations (5.13), (5.12) and (5.15) by $k(2\partial_1-k-1+\nu)$, $4k\pi c_1y_1$ and 4 from the left, respectively, we have

$$\begin{aligned}&-k(2\partial_1-k-1+\nu)(2\partial_1-k+1-\nu)\Phi_{[0,0;0]}+4k\pi c_1y_1(2\partial_1-k+1+\nu)\Phi_{[0,1;0]}=0,\\ &16k\pi^2c_1^2y_1^2\Phi_{[0,0;0]}+4k\pi c_1y_1(-2\partial_1+2\partial_2+k-1-\nu)\Phi_{[0,1;0]}-16k\pi^2c_1c_2y_1y_2\Phi_{[1,0;0]}=0,\\ &\{4\Delta_3-4\chi_{(\nu,\sigma)}(C_3)\}\Phi_{[0,0;0]}-8k\pi c_1y_1\partial_2\Phi_{[0,1;0]}+16k\pi^2c_1c_2y_1y_2\Phi_{[1,0;0]}=0.\end{aligned}$$

Adding up these equations, we obtain the equation (5.11). □

Proposition 5.7. *We put* $(l_1,l_2,l_3)=(-\nu+k,\ \frac{\nu+k-1}{2},\ \frac{\nu+k+1}{2})$ *and assume* $l_i-l_j\notin 2\mathbf{Z}$ $(1\le i\ne j\le 3)$. *Let* $\{\lambda_1,\lambda_2,\lambda_3\}$ *be a permutation of the three complex numbers* $\{l_1,l_2,l_3\}$. *Then the power series solutions around* $(y_1,y_2)=(0,0)$ *of the system in Proposition 5.6 are given by*

$$\Phi_{[0,0;0]}(y)=y_1^{\lambda_1}y_2^{-\lambda_2+k}\sum_{m,n\ge0}\frac{\left(\frac{\lambda_1-\lambda_2+2}{2}\right)_{m+n}(\pi c_1y_1)^{2m}(\pi c_2y_2)^{2n}}{m!n!\left(\frac{\lambda_1-\lambda_2+2}{2}\right)_m\left(\frac{\lambda_1-\lambda_3+2}{2}\right)_m\left(\frac{\lambda_1-\lambda_2+2}{2}\right)_n\left(\frac{\lambda_3-\lambda_2+2}{2}\right)_n}.$$

Here the symbol $(\alpha)_m$ *means* $\Gamma(\alpha+m)/\Gamma(\alpha)$.

Proof. Let

$$\Phi_{[0,0;0]}(y)=y_1^{\mu_1}y_2^{\mu_2}\sum_{m,n\ge0}A_{m,n}(\pi c_1y_1)^m(\pi c_2y_2)^n$$

be a formal power series solution around $(y_1,y_2)=(0,0)$ of the system in Proposition 5.6 with the normalization $A_{0,0}=1$. From the equations (5.10) and (5.11), we obtain the following recurrence relations:

$$\begin{aligned}&\{4(\mu_1+m)^2+4(\mu_2+n)^2-4(\mu_1+m)(\mu_2+n)-4k(\mu_1+m)-3\nu^2+2k\nu+k^2-1\}A_{m,n}\\ &-16A_{m-2,n}-16A_{m,n-2}=0,\end{aligned}$$

$$\begin{aligned}&\{4(\mu_1+m)(\mu_1-\mu_2+m-n)(\mu_2+n)-4k(\mu_1+m)^2+4k^2(\mu_1+m)\\ &+(\nu-k)(\nu+k+1)(\nu+k-1)\}A_{m,n}+16(\mu_1+m)A_{m,n-2}-16(\mu_2+n-k)A_{m-2,n}=0\end{aligned}$$

for $m,n\ge0$. Here we put $A_{m,n}=0$ when $m<0$ or $n<0$. From these equations, we may write

$$\Phi_{[0,0;0]}(y)=y_1^{\mu_1}y_2^{\mu_2}\sum_{m,n\ge0}C_{m,n}(\pi c_1y_1)^{2m}(\pi c_2y_2)^{2n}.$$

Then above recurrence relations become

(5.16)

$$\begin{aligned}&\{4(\mu_1+2m)^2+4(\mu_2+2n)^2-4(\mu_1+2m)(\mu_2+2n)-4k(\mu_1+2m)\\ &-3\nu^2+2k\nu+k^2-1\}C_{m,n}-16C_{m-1,n}-16C_{m,n-1}=0,\end{aligned}$$

(5.17)

$$\{4(\mu_1+2m)(\mu_1-\mu_2+2m-2n)(\mu_2+2n)-4k(\mu_1+2m)^2+4k^2(\mu_1+2m)$$
$$+(\nu-k)(\nu+k+1)(\nu+k-1)\}C_{m,n}+16(\mu_1+2m)C_{m,n-1}-16(\mu_2+2n-k)C_{m-1,n}=0$$

for $m,n\geq 0$. Here we put $C_{m,n}=0$ when $m<0$ or $n<0$.

Multiplying both sides of the equations (5.16) and (5.17) by $(\mu_1+2m)/4$ and $1/4$, respectively, we have

$$(\mu_1+2m)\{(\mu_1+2m)^2+(\mu_2+2n)^2-(\mu_1+2m)(\mu_2+2n)-k(\mu_1+2m)$$
$$+l_1l_2+l_2l_3+l_3l_1-k^2\}C_{m,n}-4(\mu_1+2m)C_{m-1,n}-4(\mu_1+2m)C_{m,n-1}=0,$$
$$\{(\mu_1+2m)(\mu_1-\mu_2+2m-2n)(\mu_2+2n)-k(\mu_1+2m)^2+k^2(\mu_1+2m)$$
$$-l_1l_2l_3\}C_{m,n}+4(\mu_1+2m)C_{m,n-1}-4(\mu_2+2n-k)C_{m-1,n}=0.$$

Adding up these equations, we have

$$(\mu_1-l_1+2m)(\mu_1-l_2+2m)(\mu_1-l_3+2m)C_{m,n}$$
$$-4(\mu_1+\mu_2-k+2m+2n)C_{m-1,n}=0. \tag{5.18}$$

Since

$$4\mu_1^2+4\mu_2^2-4\mu_1\mu_2-4k\mu_1-3\nu^2+2k\nu+k^2-1=0,$$
$$(\mu_1-l_1)(\mu_1-l_2)(\mu_1-l_3)=0$$

from the equations (5.16) and (5.18) in the case of $(m,n)=(0,0)$, we see that characteristic indices (μ_1,μ_2) take the following six values:

$$(\mu_1,\mu_2)=(l_i,-l_j+k) \qquad (1\leq i\neq j\leq 3).$$

Let $\{\lambda_1,\lambda_2,\lambda_3\}$ be a permutation of $\{l_1,l_2,l_3\}$ and we put $(\mu_1,\mu_2)=(\lambda_1,-\lambda_2+k)$. Then the equations (5.16) in the case of $m=0$ and (5.18) become

$$8n(\lambda_3-\lambda_2+2n)C_{0,n}-16C_{0,n-1}=0,$$
$$2m(\lambda_1-\lambda_2+2m)(\lambda_1-\lambda_3+2m)C_{m,n}-4(\lambda_1-\lambda_2+2m+2n)C_{m-1,n}=0.$$

From these equations, each coefficient $C_{m,n}$ is determined recursively, and we obtain the formal power series solutions in the statement. □

Theorem 5.8. *We put* $(l_1,l_2,l_3)=(-\nu+k,\ \frac{\nu+k-1}{2},\ \frac{\nu+k+1}{2})$ *and assume* $l_i-l_j\notin 2\mathbf{Z}$ $(1\leq i\neq j\leq 3)$. *We define the functions* $\Phi^{(i,j)}_{[\varepsilon_1,\varepsilon_2;p]}$ *by*

$$\Phi^{(1,2)}_{[\varepsilon_1,\varepsilon_2;p]}(y)=y_1^{l_1}y_2^{-l_2+k}$$
$$\times\sum_{m,n\geq 0}\frac{\left(\frac{l_1-l_2+2}{2}\right)_{m+n-\varepsilon_1-p}(\pi c_1y_1)^{-\varepsilon_1-\varepsilon_2-2p+2m}(\pi c_2y_2)^{-\varepsilon_1+2n}}{m!n!\left(\frac{l_1-l_2+2}{2}\right)_{m-\varepsilon_1-\varepsilon_2+\varepsilon_1\varepsilon_2-p}\left(\frac{l_1-l_3+2}{2}\right)_{m-\varepsilon_1\varepsilon_2-p}\left(\frac{l_1-l_2+2}{2}\right)_{n-\varepsilon_1-p}\left(\frac{l_3-l_2+2}{2}\right)_{n-\varepsilon_1}},$$

$$\Phi^{(1,3)}_{[\varepsilon_1,\varepsilon_2;p]}(y)=y_1^{l_1}y_2^{-l_3+k}$$
$$\times\sum_{m,n\geq 0}\frac{\left(\frac{l_1-l_3+2}{2}\right)_{m+n-p}(\pi c_1y_1)^{-\varepsilon_1-\varepsilon_2-2p+2m}(\pi c_2y_2)^{\varepsilon_1+2n}}{m!n!\left(\frac{l_1-l_2+2}{2}\right)_{m-\varepsilon_1-\varepsilon_2+\varepsilon_1\varepsilon_2-p}\left(\frac{l_1-l_3+2}{2}\right)_{m-\varepsilon_1\varepsilon_2-p}\left(\frac{l_1-l_3+2}{2}\right)_{n-p}\left(\frac{l_2-l_3+2}{2}\right)_{n+\varepsilon_1}},$$

$$\Phi^{(2,1)}_{[\varepsilon_1,\varepsilon_2;p]}(y)=y_1^{l_2}y_2^{-l_1+k}$$
$$\times\sum_{m,n\geq 0}\frac{\left(\frac{l_2-l_1+2}{2}\right)_{m+n+\varepsilon_1+\varepsilon_2-\varepsilon_1\varepsilon_2+p}(\pi c_1y_1)^{\varepsilon_1+\varepsilon_2-2\varepsilon_1\varepsilon_2+2m}(\pi c_2y_2)^{\varepsilon_1+2p+2n}}{m!n!\left(\frac{l_2-l_1+2}{2}\right)_{m+\varepsilon_1+\varepsilon_2-\varepsilon_1\varepsilon_2+p}\left(\frac{l_2-l_3+2}{2}\right)_{m+\varepsilon_1+\varepsilon_2-2\varepsilon_1\varepsilon_2}\left(\frac{l_2-l_1+2}{2}\right)_{n+\varepsilon_1+p}\left(\frac{l_3-l_1+2}{2}\right)_{n+p}},$$

$$\Phi^{(2,3)}_{[\varepsilon_1,\varepsilon_2;p]}(y)=y_1^{l_2}y_2^{-l_3+k}$$

$$\times \sum_{m,n\geq 0} \frac{\left(\frac{l_2-l_3+2}{2}\right)_{m+n+\varepsilon_1+\varepsilon_2-\varepsilon_1\varepsilon_2}(\pi c_1 y_1)^{\varepsilon_1+\varepsilon_2-2\varepsilon_1\varepsilon_2+2m}(\pi c_2 y_2)^{\varepsilon_1+2n}}{m!n!\left(\frac{l_2-l_1+2}{2}\right)_{m+\varepsilon_1+\varepsilon_2-\varepsilon_1\varepsilon_2+p}\left(\frac{l_2-l_3+2}{2}\right)_{m+\varepsilon_1+\varepsilon_2-2\varepsilon_1\varepsilon_2}\left(\frac{l_1-l_3+2}{2}\right)_{n-p}\left(\frac{l_2-l_3+2}{2}\right)_{n+\varepsilon_1}},$$

$$\Phi^{(3,1)}_{[\varepsilon_1,\varepsilon_2;p]}(y) = y_1^{l_3} y_2^{-l_1+k}$$

$$\times \sum_{m,n\geq 0} \frac{\left(\frac{l_3-l_1+2}{2}\right)_{m+n+\varepsilon_1\varepsilon_2+p}(\pi c_1 y_1)^{-\varepsilon_1-\varepsilon_2+2\varepsilon_1\varepsilon_2+2m}(\pi c_2 y_2)^{\varepsilon_1+2p+2n}}{m!n!\left(\frac{l_3-l_1+2}{2}\right)_{m+\varepsilon_1\varepsilon_2+p}\left(\frac{l_3-l_2+2}{2}\right)_{m-\varepsilon_1-\varepsilon_2+2\varepsilon_1\varepsilon_2}\left(\frac{l_2-l_1+2}{2}\right)_{n+\varepsilon_1+p}\left(\frac{l_3-l_1+2}{2}\right)_{n+p}},$$

$$\Phi^{(3,2)}_{[\varepsilon_1,\varepsilon_2;p]}(y) = y_1^{l_3} y_2^{-l_2+k}$$

$$\times \sum_{m,n\geq 0} \frac{\left(\frac{l_3-l_2+2}{2}\right)_{m+n-\varepsilon_1+\varepsilon_1\varepsilon_2}(\pi c_1 y_1)^{-\varepsilon_1-\varepsilon_2+2\varepsilon_1\varepsilon_2+2m}(\pi c_2 y_2)^{-\varepsilon_1+2n}}{m!n!\left(\frac{l_3-l_1+2}{2}\right)_{m+\varepsilon_1\varepsilon_2+p}\left(\frac{l_3-l_2+2}{2}\right)_{m-\varepsilon_1-\varepsilon_2+2\varepsilon_1\varepsilon_2}\left(\frac{l_1-l_2+2}{2}\right)_{n-\varepsilon_1-p}\left(\frac{l_3-l_2+2}{2}\right)_{n-\varepsilon_1}},$$

for $\varepsilon_1, \varepsilon_2 \in \{0,1\}$ *and* $0 \leq p \leq \left[\frac{k-\varepsilon_1-\varepsilon_2}{2}\right]$. *Then, for* $1 \leq i \neq j \leq 3$, *there is an element* $M^{(i,j)}_{[\nu,k]}$ *of* $\mathrm{Wh}(\pi_{(\nu,\sigma)}, \xi, \tau_k^*)$ *whose* $[\varepsilon_1,\varepsilon_2;p]$*-component is* $(-1)^{\varepsilon_1-k}(\sqrt{-1})^{\varepsilon_2-k} y_1 y_2 \Phi^{(i,j)}_{[\varepsilon_1,\varepsilon_2;p]}$. *Moreover, the set* $\left\{ M^{(i,j)}_{[\nu,k]} \,\middle|\, 1 \leq i \neq j \leq 3 \right\}$ *form a basis of* $\mathrm{Wh}(\pi_{(\nu,\sigma)}, \xi, \tau_k^*)$.

Proof. The function $\Phi^{(i,j)}_{[0;0;0]}$ is the same as the solution in Proposition 5.7 when $(\lambda_1, \lambda_2, \lambda_3) = (l_i, l_j, l_{6-i-j})$. Moreover, from Proposition 5.2 and the irreducibility of $\pi_{(\nu,\sigma)}$, it follows that the dimension of the space $\mathrm{Wh}(\pi_{(\nu,\sigma)}, \xi, \tau_k^*)$ is 6. Therefore, if we show that the functions $\Phi_{[\varepsilon_1,\varepsilon_2;p]} = \Phi^{(i,j)}_{[\varepsilon_1,\varepsilon_2;p]}$ are determined from $\Phi_{[0,0;0]} = \Phi^{(i,j)}_{[0,0;0]}$ by the equations in Proposition 5.5, the proof is completed.

We discuss only the case of $(i,j) = (1,2)$ here since other cases are similar. We put

$$C^{[\varepsilon_1,\varepsilon_2;p]}_{m,n} = \begin{cases} \dfrac{\left(\frac{l_1-l_2+2}{2}\right)_{m+n-\varepsilon_1-p}}{m!n!\left(\frac{l_1-l_2+2}{2}\right)_{m-\varepsilon_1-\varepsilon_2+\varepsilon_1\varepsilon_2-p}\left(\frac{l_1-l_3+2}{2}\right)_{m-\varepsilon_1\varepsilon_2-p}\left(\frac{l_1-l_2+2}{2}\right)_{n-\varepsilon_1-p}\left(\frac{l_3-l_2+2}{2}\right)_{n-\varepsilon_1}}, & \text{if } m,n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\Phi^{(1,2)}_{[\varepsilon_1,\varepsilon_2;p]}(y) = y_1^{l_1} y_2^{-l_2+k} \sum_{m,n\in\mathbf{Z}} C^{[\varepsilon_1,\varepsilon_2;p]}_{m,n}(\pi c_1 y_1)^{-\varepsilon_1-\varepsilon_2-2p+2m}(\pi c_2 y_2)^{-\varepsilon_1+2n}.$$

We have to confirm the following relations obtained from the equations in Proposition 5.5:

$$(-4m+4n+2)C^{[0,1;0]}_{m,n} - 4C^{[1,0;0]}_{m,n} + 4C^{[0,0;0]}_{m-1,n} = 0,$$

$$(-4m+4p+3\nu-k-1+2\varepsilon_1)C^{[\varepsilon_1,1;p-1]}_{m,n} + 4C^{[\varepsilon_1,0;p]}_{m,n} + 4C^{[\varepsilon_1,0;p-1]}_{m-1,n} = 0 \quad \left(1 \leq p \leq \left[\frac{k-\varepsilon_1}{2}\right]\right),$$

$$(-4m+4p+3\nu-k-1+2\varepsilon_1)C^{[\varepsilon_1,0;p]}_{m,n} + 4C^{[\varepsilon_1,1;p]}_{m,n} = 0 \quad \left(0 \leq p \leq \left[\frac{k-\varepsilon_1-1}{2}\right]\right)$$

for $m,n \in \mathbf{Z}$. We prove these equations by direct computation using

$$4C^{[1,0;0]}_{m,n} = \frac{(4n+2)(4n-3\nu+k+1)}{4m+4n-3\nu+k+1}C^{[0,1;0]}_{m,n}, \quad 4C^{[0,0;0]}_{m-1,n} = \frac{4m(4m-3\nu+k-1)}{4m+4n-3\nu+k+1}C^{[0,1;0]}_{m,n},$$

$$4C^{[\varepsilon_1,0;p]}_{m,n} = \frac{(4m-4p-3\nu+k+3-2\varepsilon_1)(4n-4p-3\nu+k+5-4\varepsilon_1)}{4m+4n-4p-3\nu+k+5-4\varepsilon_1}C^{[\varepsilon_1,1;p-1]}_{m,n},$$

$$4C^{[\varepsilon_1,0;p-1]}_{m-1,n} = \frac{4m(4m-4p-3\nu+k+3-2\varepsilon_1)}{4m+4n-4p-3\nu+k+5-4\varepsilon_1}C^{[\varepsilon_1,1;p-1]}_{m,n},$$

$$4C^{[\varepsilon_1,1;p]}_{m,n} = (4m-4p-3\nu+k+1-2\varepsilon_1)C^{[\varepsilon_1,0;p]}_{m,n}.$$

□

5.3. **Primary Whittaker functions.** From the multiplicity one theorem of Shalika [38] and the result of Wallach [47], it follows that a primary Whittaker function for $(\pi_{(\nu,\sigma)}, \xi, \tau_k^*)$ is unique up to scalar multiple. We give the explicit formula of the primary Whittaker function for $(\pi_{(\nu,\sigma)}, \xi, \tau_k^*)$ as follows.

Theorem 5.9. *We put* $(l_1, l_2, l_3) = (-\nu + k,\ \frac{\nu+k-1}{2},\ \frac{\nu+k+1}{2})$ *and assume* $l_i - l_j \notin 2\mathbf{Z}$ $(1 \le i \ne j \le 3)$. *For* $\varepsilon_1, \varepsilon_2 \in \{0,1\}$ *and* $0 \le p \le \left[\frac{k-\varepsilon_1-\varepsilon_2}{2}\right]$, *we put*

$$\Phi^{(W)}_{[\varepsilon_1,\varepsilon_2;p]}(y) = \frac{(-1)^{\varepsilon_2}\operatorname{sgn}(c_1^{\varepsilon_1+\varepsilon_2}c_2^{\varepsilon_1})}{4(2\pi\sqrt{-1})^2}\int_{\rho_1-\sqrt{-1}\infty}^{\rho_1+\sqrt{-1}\infty}\int_{\rho_2-\sqrt{-1}\infty}^{\rho_2+\sqrt{-1}\infty} V_{[\varepsilon_1,\varepsilon_2;p]}(s_1,s_2)(\pi|c_1 y_1|)^{-s_1}(\pi|c_2 y_2|)^{-s_2}ds_1 ds_2,$$

where

$$V_{[\varepsilon_1,\varepsilon_2;p]}(s_1,s_2) = \frac{\Gamma\left(\frac{s_1+l_1-\varepsilon_1-\varepsilon_2-2p}{2}\right)\Gamma\left(\frac{s_1+l_2}{2}\right)\Gamma\left(\frac{s_1+l_3}{2}\right)\Gamma\left(\frac{s_2-l_1+k+\varepsilon_1+2p}{2}\right)\Gamma\left(\frac{s_2-l_2+k}{2}\right)\Gamma\left(\frac{s_2-l_3+k}{2}\right)}{\Gamma\left(\frac{s_1+s_2+k-\varepsilon_2}{2}\right)}.$$

Here the symbol $\operatorname{sgn}(a)$ $(a \in \mathbf{R}^\times)$ *means* $\operatorname{sgn}(a) = a/|a|$ *and the lines of integration are taken as to the right of all poles of the integrand. Then there is an element* $W_{[\nu,k]}$ *of* $\mathrm{Wh}(\pi_{(\nu,\sigma)}, \xi, \tau_k^*)^{\mathrm{mod}}$ *whose* $[\varepsilon_1,\varepsilon_2;p]$*-component is* $(-1)^{\varepsilon_1-k}(\sqrt{-1})^{\varepsilon_2-k}y_1y_2\Phi^{(W)}_{[\varepsilon_1,\varepsilon_2;p]}$.

Moreover, the relation between $W_{[\nu,k]}$ *and* $M^{(i,j)}_{[\nu,k]}$ $(1 \le i \ne j \le 3)$ *is given by*

$$W_{[\nu,k]}(y) = \sum_{1\le i\ne j\le 3}\Gamma^{(i,j)}(\nu,k)\cdot M^{(i,j)}_{[\nu,k]}(y) \tag{5.19}$$

where

$$\Gamma^{(i,j)}(\nu,k) = (\pi|c_1|)^{l_i}(\pi|c_1|)^{-l_j+k}\Gamma\left(\frac{l_j-l_i}{2}\right)\Gamma\left(\frac{l_j-l_{6-i-j}}{2}\right)\Gamma\left(\frac{l_{6-i-j}-l_i}{2}\right).$$

Proof. Stirling's formula for the Gamma function ([48, §13.6]) shows that the integration of $\Phi^{(W)}_{[\varepsilon_1,\varepsilon_2;p]}$ is absolutely convergent and defines a moderate growth function. Hence, if we can show the expansion formula (5.19), we may conclude that the function $W_{[\nu,k]}$ is the A_0-radial part of the primary Whittaker function.

We may justify moving the line of integration to the left or right, avoiding the poles of the integrand of $\Phi^{(W)}_{[\varepsilon_1,\varepsilon_2;p]}$. Moving the line of integration to the left and summing the residues, we obtain

$$\Phi^{(W)}_{[\varepsilon_1,\varepsilon_2;p]}(y) = \frac{(-1)^{\varepsilon_2}\operatorname{sgn}(c_1^{\varepsilon_1+\varepsilon_2}c_2^{\varepsilon_1})}{4}$$
$$\times \sum_{(p_1,p_2):\ \text{pole}} \operatorname{Res}_{(s_1,s_2)=(p_1p_2)}\Big(V_{[\varepsilon_1,\varepsilon_2;p]}(s_1,s_2)(\pi|c_1y_1|)^{-s_1}(\pi|c_2y_2|)^{-s_2}\Big).$$

Now, we fix $\varepsilon_1,\ \varepsilon_2,\ p$ and put

$$\begin{aligned}
&r_{1,1} = -l_1+\varepsilon_1+\varepsilon_2+2p, && r_{1,2} = -l_2-\varepsilon_1-\varepsilon_2+2\varepsilon_1\varepsilon_2, && r_{1,3} = -l_3+\varepsilon_1+\varepsilon_2-2\varepsilon_1\varepsilon_2,\\
&r_{2,1} = l_1-k-\varepsilon_1-2p, && r_{2,2} = l_2-k+\varepsilon_1, && r_{2,3} = l_3-k-\varepsilon_1.
\end{aligned}$$

Then the set of the poles of the integrand of $\Phi^{(W)}_{[\varepsilon_1,\varepsilon_2;p]}(y)$ are given by

$$\{(s_1,s_2) = (r_{1,i}-2m, r_{2,j}-2n) \mid 1 \le i \ne j \le 3,\ m,n \in \mathbf{Z}_{\ge 0}\}.$$

The residue $\operatorname{Res}_{(s_1,s_2)=(r_{1,i}-2m,r_{2,j}-2n)} V_{[\varepsilon_1,\varepsilon_2;p]}(s_1,s_2)(\pi|c_1y_1|)^{-s_1}(\pi|c_2y_2|)^{-s_2}$ is given by

$$\frac{4(-1)^{m+n}}{m!n!}\frac{\Gamma\left(\frac{r_{1,i}-r_{1,j}}{2}-m\right)\Gamma\left(\frac{r_{1,i}-r_{1,6-i-j}}{2}-m\right)\Gamma\left(\frac{r_{2,i}-r_{2,j}}{2}-n\right)\Gamma\left(\frac{r_{2,i}-r_{2,6-i-j}}{2}-n\right)}{\Gamma\left(\frac{r_{1,i}+r_{2,j}+k-\varepsilon_2}{2}-m-n\right)}$$

$$\times (\pi|c_1|y_1)^{2m-r_{1,i}}(\pi|c_2|y_2)^{2n-r_{2,j}}.$$

By direct computation using

$$\Gamma(a-n) = \Gamma(a)\frac{(-1)^n}{(1-a)_n},$$

we have

$$\begin{aligned}
&\sum_{m,n\geq 0} \mathrm{Res}_{(s_1,s_2)=(r_{1,i}-2m,r_{2,j}-2n)}\left(V_{[\varepsilon_1,\varepsilon_2;p]}(s_1,s_2)(\pi|c_1|y_1)^{-s_1}(\pi|c_2|y_2)^{-s_2}\right)\\
&= 4(-1)^{\varepsilon_2}\,\mathrm{sgn}(c_1^{\varepsilon_1+\varepsilon_2}c_2^{\varepsilon_1})\Gamma^{(i,j)}(\nu,k)\cdot \Phi^{(i,j)}_{[\varepsilon_1,\varepsilon_2;p]}(y).
\end{aligned}$$

□

Remark 5.10. *By the duplication formula ([48, §12.15]) $2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\Gamma(2z)$, we can rewrite the integrand of $\Phi^{(W)}_{[\varepsilon_1,\varepsilon_2;p]}(y)$ as follows:*

$$\begin{aligned}
&V_{[\varepsilon_1,\varepsilon_2;p]}(s_1,s_2)(\pi|c_1|y_1)^{-s_1}(\pi|c_2|y_2)^{-s_2}\\
&= 2^{-k+3}\pi\frac{\Gamma\left(\frac{s_1-\nu+k-\varepsilon_1-\varepsilon_2-2p}{2}\right)\Gamma\left(s_1+\frac{\nu+k-1}{2}\right)\Gamma\left(\frac{s_2+\nu+\varepsilon_1+2p}{2}\right)\Gamma\left(s_2-\frac{\nu-k+1}{2}\right)}{\Gamma\left(\frac{s_1+s_2+k-\varepsilon_2}{2}\right)}\\
&\quad\times (2\pi|c_1|y_1)^{-s_1}(2\pi|c_2|y_2)^{-s_2}.
\end{aligned}$$

This simplification is compatible with the fact that the standard gamma factor defined from the Langlands parameter of $\pi_{(\nu,\sigma)}$ is also simplified.

Remark 5.11. Following the suggestion of the referee, we explain the reason why our explicit formulas of Whittaker functions for generalized principal series representations (Theorem 5.8 and 5.9) resemble to those for principal series representations ([2, §2], [27]). By the subrepresentation theorem ([20, Theorem 8.37]), our generalized principal series representation is embedding into some principal series representation. Therefore, 'in principle', our formulas should be obtained from the results in [2, §2] and [27], by considering the Whittaker realization of the explicit $(\mathfrak{g},K)$-module structure in Part 1. Actually, we can obtain our formulas in this way when Blattner parameter k is small. However, for general k, it seems to be difficult to compute in this way because of the combinatorial complexity.

Part 3. The Eisenstein series for $GL(3,\mathbf{Z})$ induced from cusp forms.

1. Introduction

In the theory of automorphic forms, the investigation of their Fourier expansions is a fundamental work and gives us many significant informations. Especially, Fourier coefficients of Eisenstein series have been studied by many mathematicians ([1], [7], [15], [23], [33], [37], [39], .etc).

In the case of $GL(n)$, Piatetski-Shapiro [35] introduced the Fourier expansion of an automorphic form in terms of Whittaker functions, which is called *the Fourier-Whittaker expansion.* This expansion plays an important role in the study of automorphic forms on $GL(n)$. However, there are few references of the explicit description of the Fourier-Whittaker expansions of the Eisenstein series, and they are not exhaustive even for the case of $n = 3$. The Fourier-Whittaker expansions of the minimal parabolic Eisenstein series for $GL(3,\mathbf{Z})$ are obtained by Vinogradov and Takhtazhyan [45] and by Bump [2]. Friedberg [5] also obtain those of the maximal parabolic Eisenstein series for $GL(3,\mathbf{Z})$ induced from the trivial character. The main purpose of this part is to investigate the Fourier-Whittaker expansions of the maximal parabolic Eisenstein series for $GL(3,\mathbf{Z})$ induced from cusp forms.

Let us explain our problem in a more precise form. Before describing our situation for $GL(3,\mathbf{R})$, we recall the general setting of the theory of the Eisenstein on a real reductive Lie group G. Fix an arithmetic subgroup Γ, a maximal unipotent subgroup N_0 and a maximal compact subgroup K of G. We take a parabolic subgroup $P = NAM$ of G with the Langlands decomposition. Here AM and A are the Levi component and the radial component of P, respectively. We put $\Gamma_M = \Gamma \cap M$ and let $L^2_\circ(\Gamma_M\backslash M)$ be the subspace consisting of all $\varphi \in L^2(\Gamma_M\backslash M)$ such that

$$\int_{(\Gamma_M\cap N')\backslash N'} \varphi(ng)dn = 0 \quad (g \in M)$$

for a unipotent radical N' of any parabolic subgroup of M. Via the right translation of M, $L^2_\circ(\Gamma_M\backslash M)$ becomes a M-module. If (π, H_π) is a irreducible subrepresentation of $L^2_\circ(\Gamma_M\backslash M)$, we call π a cuspidal representation of M. For $\nu \in \mathfrak{a}^*_{\mathbf{C}}$, let $I_\nu(\pi)$ be the associated $(\mathfrak{g}_{\mathbf{C}}, K)$-module of the parabolic induction $\mathrm{Ind}_P^G(1_N \otimes e^{\nu+\rho} \otimes \pi)$. Here $\mathfrak{g}$ and $\mathfrak{a}$ are the Lie algebras of G and A, respectively. We take a flat section $\mathfrak{a}^*_{\mathbf{C}} \times G \ni (\nu, g) \mapsto F_\nu(g) \in H_\pi$ for $I_\nu(\pi)$, i.e. for $\nu \in \mathfrak{a}^*_{\mathbf{C}}$, F_ν is contained in $I_\nu(\pi)$ and for $\kappa \in K$, the value of $F_\nu(\kappa)$ does not depend on ν. Then we define the Eisenstein series $E(F_\nu; g)$ by

$$E(F_\nu; g) = \sum_{\gamma\in(\Gamma\cap P)\backslash\Gamma} \lambda \circ F_\nu(\gamma g) \qquad (g \in G)$$

where $\lambda\colon H_\pi \ni \varphi \mapsto \varphi(1) \in \mathbf{C}$. For a unitary character ψ of N_0 such that $\psi(\Gamma \cap N_0) = 1$, we define the Fourier-Whittaker coefficient $E_\psi(F_\nu; g)$ by

$$E_\psi(F_\nu; g) = \int_{(\Gamma\cap N_0)\backslash N_0} E(F_\nu; ng)\psi(n)^{-1}dn.$$

In this part, we consider the case of $G = GL(3,\mathbf{R})$, $\Gamma = GL(3,\mathbf{Z})$ and P is a maximal parabolic subgroup of G. For a flat section F_ν contained in the minimal K-type of $I_\nu(\pi)$, we give the Mellin-Barnes type integral expression of $E_\psi(F_\nu; g)$ in Theorem 6.16, which is analogous to Bump's formula [2]. This explicit formula gives us another proof of the analytic continuation and the functional equation of $E(F_\nu; g)$, which were originally proved by Langlands [24] for general reductive groups. In principle, by using the description of the $(\mathfrak{g}_{\mathbf{C}}, K)$-module structure of $I_\nu(\pi)$ in Part 1, we can obtain the explicit formulas of the Eisenstein series at the whole K-types. We expect to utilize our formula for deeper study of the Eisenstein series, such as Rankin-Selberg convolution.

Let us explain the contents of this part. In §2, we introduce basic notions of the Fourier-Whittaker expansions and the Eisenstein series on $G = GL(3,\mathbf{R})$. In §3, we recall the basic facts about a cuspidal representation π of M. In §4, we give an explicit construction of a family of flat sections, which becomes a basis of $I_\nu(\pi)$. Moreover, we introduce the $(\mathfrak{g}_{\mathbf{C}}, K)$-embedding from $I_\nu(\pi)$ to some principal series of G. In §5, we give the expression of $E_\psi(F_\nu; g)$ in terms of the Jacquet integrals. Here the Jacquet integrals are Whittaker functions for principal series representations of G, introduced by Jacquet [17]. We may realize the Jacquet integrals as Whittaker functions for $I_\nu(\pi)$ via the embedding introduced in §3. In §6, we evaluate the Jacquet integrals and give the explicit formulas of $E_\psi(F_\nu; g)$ at the minimal K-type of $I_\nu(\pi)$ in Theorem 6.16.

2. Preliminaries

2.1. **The structure of $GL(3,\mathbf{R})$.** We denote by $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. Let $\mathbf{Z}_{\geq l}$ be the set of integers which are no less than $l \in \mathbf{Z}$ and $\mathbf{R}_+$ the set of positive real numbers. For $z \in \mathbf{C}$, we denote the real part and the imaginary part of z by $\mathbf{Re}(z)$ and $\mathbf{Im}(z)$, respectively. Let 1_3 be the unit matrix of degree 3 and $O_{m,n}$ the zero matrix of size $m \times n$. For a Lie algebra $\mathfrak{l}$, we denote by $\mathfrak{l}_{\mathbf{C}} = \mathfrak{l} \otimes_{\mathbf{R}} \mathbf{C}$ the complexification of $\mathfrak{l}$. Moreover, we denote the universal enveloping algebra of $\mathfrak{l}$ and its center by $U(\mathfrak{l})$ and $Z(\mathfrak{l})$, respectively.

Let $G = GL(3,\mathbf{R})$ and $\Gamma = GL(3,\mathbf{Z})$. For a Cartan involution $\theta\colon G \ni g \mapsto {}^tg^{-1} \in G$, its fixed part $K = \{g \in G \mid \theta(g) = g\} = O(3)$ is a maximal compact subgroup of G. Here ${}^tg$ and g^{-1} mean the transpose and the inverse of g, respectively.

Let $\mathfrak{g} = \mathfrak{gl}(3,\mathbf{R})$ be the Lie algebra of G. If we denote the differential of θ again by θ, then we have $\theta(X) = -{}^tX$ for $X \in \mathfrak{g}$. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and the -1 eigenspaces of θ in $\mathfrak{g}$, respectively, that is,

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid {}^tX = -X\} = \mathfrak{o}(3), \qquad \mathfrak{p} = \{X \in \mathfrak{g} \mid {}^tX = X\}.$$

Then $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{g}$ has the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Let E_{ij} be the matrix unit in $\mathfrak{g}$ with 1 at (i,j)-th entry and 0 at other entries. We put $\mathfrak{a}_0 = \bigoplus_{1\leq i\leq 3} \mathbf{R}E_{ii}$. Then $\mathfrak{a}_0$ is a maximal abelian subspace of $\mathfrak{p}$. If we put $\mathfrak{n}_0 = \bigoplus_{1\leq i<j\leq 3} \mathbf{R}E_{ij}$, then we have an Iwasawa decomposition $\mathfrak{g} = \mathfrak{n}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{k}$. Let A_0 and N_0 be the analytic subgroups corresponding to $\mathfrak{a}_0$ and $\mathfrak{n}_0$, respectively. Then also we have $G = N_0A_0K$.

We fix a complete system $\{w_i \in N_K(A_0) \mid 0 \leq i \leq 5\}$ of representatives of the Weyl group $W_G = N_K(A_0)/Z_K(A_0)$ as follows:

$$w_0 = 1_3, \qquad w_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad w_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$w_3 = w_1w_2, \qquad w_4 = w_2w_1, \qquad w_5 = w_1w_2w_1 = w_2w_1w_2.$$

Here $N_K(A_0)$ and $Z_K(A_0)$ mean the normalizer and the centralizer of A_0 in K, respectively.

It will be convenient to introduce the following notation:

$$n[x_1,x_2,x_3] = \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \in N_0,$$

$$a[y_1,y_2,y_3] = \mathrm{diag}(y_1y_2y_3, y_2y_3, y_3) \in A_0,$$

where $x_1, x_2, x_3 \in \mathbf{R}$, $y_1, y_2, y_3 \in \mathbf{R}_+$. Then we have

$$A_0 = \{a[y_1,y_2,y_3] \mid y_1,y_2,y_3 \in \mathbf{R}_+\}, \qquad N_0 = \{n[x_1,x_2,x_3] \mid x_1,x_2,x_3 \in \mathbf{R}\}.$$

2.2. **Automorphic forms for Γ on G.** For $x, g \in G$ and a function F on G, we put

$$(R(g)F)(x) = F(xg).$$

We denote the differential of R again by R.

An automorphic form for Γ on G is a smooth function ϕ on G satisfying the following properties:

(1) $\phi(\gamma g) = \phi(g)$ for $\gamma \in \Gamma$, $g \in G$.
(2) ϕ is K-finite, i.e. the $\mathbf{C}$-vector space spanned by $R(k)\phi$ $(k \in K)$ is finite dimensional.
(3) ϕ is $Z(\mathfrak{g}_{\mathbf{C}})$-finite, i.e. the $\mathbf{C}$-vector space spanned by $R(D)\phi$ $(D \in Z(\mathfrak{g}_{\mathbf{C}}))$ is finite dimensional.
(4) ϕ is moderate growth, i.e. there exists r, $C \in \mathbf{R}$ such that $|\phi(g)| < C\|g\|^r$ for $g \in G$. Here $\|\cdot\|$ is a norm on G defined by $\|g\|^2 = \sum_{ij}|g_{ij}|^2 + |\det(g)|^{-1}$ for $g = (g_{ij}) \in G$.

We denote by $\mathcal{A}(\Gamma\backslash G)$ the space of automorphic forms for Γ on G. Via the right translation R, $\mathcal{A}(\Gamma\backslash G)$ becomes a $(\mathfrak{g}_{\mathbf{C}}, K)$-module.

2.3. **Whittaker functions on G.** We put $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$. For $c_1, c_2 \in \mathbf{R}$, we define the unitary character $\psi = \psi_{c_1,c_2}$ of N_0 by

$$\psi(n[x_1, x_2, x_3]) = \mathbf{e}(c_1x_1 + c_2x_2), \qquad x_1, x_2, x_3 \in \mathbf{R}.$$

The unitary characters of N_0 is exhausted by characters of this type. Moreover, it holds that $\psi_{c_1,c_2}(\Gamma \cap N_0) = 1$ if and only if $c_1, c_2 \in \mathbf{Z}$.

We put

$$C^\infty_{\mathrm{mg}}(N_0\backslash G; \psi) = \left\{ W \in C^\infty(G) \;\middle|\; \begin{array}{l} W(ng) = \psi(n)W(g),\ (n, g) \in N_0 \times G, \\ W \text{ is moderate growth} \end{array} \right\}$$

and $\mathfrak{g}_{\mathbf{C}}$ and K act on this space by the right translation R.

Let (Π, H_Π) be an admissible $(\mathfrak{g}_{\mathbf{C}}, K)$-module. For a $(\mathfrak{g}_{\mathbf{C}}, K)$-homomorphism from H_Π to $C^\infty_{\mathrm{mg}}(N_0\backslash G; \psi)$, its image is called *Whittaker model of* Π and functions in its image are called *Whittaker functions for* Π.

2.4. **The Fourier-Whittaker expansions.** For $m_1, m_2 \in \mathbf{Z}$ and $\phi \in \mathcal{A}(\Gamma\backslash G)$, we define *the Fourier-Whittaker coefficient* ϕ_{m_1,m_2} *of* ϕ by

$$\phi_{m_1,m_2}(g) = \int_0^1\int_0^1\int_0^1 \phi(n[\xi_1, \xi_2, \xi_3]g)\mathbf{e}(-m_1\xi_1 - m_2\xi_2)d\xi_1 d\xi_2 d\xi_3.$$

We note that $\phi \mapsto \phi_{m_1,m_2}$ is a $(\mathfrak{g}_{\mathbf{C}}, K)$-homomorphism from $\mathcal{A}(\Gamma\backslash G)$ to $C^\infty_{\mathrm{mg}}(N_0\backslash G; \psi_{m_1,m_2})$. Hence ϕ_{m_1,m_2} is a Whittaker function for $\mathcal{A}(\Gamma\backslash G)$.

We put $\Gamma^2 = SL(2, \mathbf{Z})$ and

$$\Gamma^2_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;\middle|\; n \in \mathbf{Z} \right\}.$$

Proposition 2.1. *An automorphic form $\phi \in \mathcal{A}(\Gamma\backslash G)$ has the following expansion:*

$$\phi(g) = \sum_{m_1=-\infty}^{\infty} \phi_{m_1,0}(g) + \sum_{\gamma \in \Gamma^2_\infty\backslash\Gamma^2} \sum_{\substack{(m_1,m_2)\in\mathbf{Z}^2 \\ m_2>0}} \phi_{m_1,m_2}\left(\left(\begin{array}{c|c} \gamma & O_{2,1} \\ \hline O_{1,2} & 1 \end{array} \right) g \right).$$

This proposition was originally proved by Piatetski-Shapiro [35], in the adelic setting, on $GL(n)$. Also see Shalika [38, Theorem 5.8]. Bump [2, §4] have specialized the original proof by induction on n to the case $n = 3$, and translated from the adele group to $GL(3, \mathbf{R})$. The expansion in Proposition 2.1 is called *the Fourier-Whittaker expansion of* ϕ.

2.5. **The Eisenstein series induced from cusp forms.** In this section, we define the Eisenstein series on G which are induced from cusp forms of the standard maximal parabolic subgroup

$$P_1 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\}.$$

We specify the Langland decomposition $P_1 = N_1 A_1 M_1$ by

$$M_1 = \left\{ \left(\begin{array}{c|c} h & O_{2,1} \\ \hline O_{1,2} & \varepsilon \end{array} \right) \middle| \, h \in SL^{\pm}(2,\mathbf{R}),\ \varepsilon \in \{\pm 1\} \right\},$$
$$A_1 = \{a[1,y_2,y_3] \mid y_2, y_3 \in \mathbf{R}_+\}, \quad N_1 = \{n[0,x_2,x_3] \mid x_2,x_3 \in \mathbf{R}\}.$$

Here $SL^{\pm}(2,\mathbf{R}) = \{g \in GL(2,\mathbf{R}) \mid \det(g) = \pm 1\}$. Moreover, let $\mathfrak{m}_1$ be a Lie algebra of M_1 and put $\Gamma_{M_1} = \Gamma \cap M_1$.

For $\nu_1 = (\nu_{1,1}, \nu_{1,2}) \in \mathbf{C}^2$, we define the character $A_1 \ni a \mapsto a^{\nu_1} \in \mathbf{C}^{\times}$ by $a[1,y_2,y_3]^{\nu_1} = y_2^{2\nu_{1,1}} y_3^{2\nu_{1,1}+\nu_{1,2}}$. We put $\rho_1 = (1/2, -1)$.

For $\nu_1 \in \mathbf{C}^2$ and an admissible representation (π, H_π) of M_1, we define the induced $(\mathfrak{g}_{\mathbf{C}}, K)$-module $I_{\nu_1}(\pi)$ by

$$I_{\nu_1}(\pi) = \left\{ \begin{array}{c} F\colon G \to H_\pi^\infty \\ \text{smooth} \end{array} \,\middle|\, \begin{array}{l} F\colon K\text{-finite}, \\ F(namg) = a^{\nu_1+\rho_1}\pi(m)F(g), \\ (n,a,m,g) \in N_1 \times A_1 \times M_1 \times G. \end{array} \right\}$$

on which $\mathfrak{g}_{\mathbf{C}}$ and K act by the right translation R. Here H_π^∞ is the subspace consisting of all smooth vectors in H_π. We call $\mathbf{C}^2 \times G \ni (\nu_1, g) \mapsto F_{\nu_1}(g) \in H_\pi^\infty$ a flat section for $I_{\nu_1}(\pi)$ if F_{ν_1} is an element of $I_{\nu_1}(\pi)$ and for $\kappa \in K$, $F_{\nu_1}(\kappa)$ does not depend on $\nu_1 \in \mathbf{C}^2$.

We put

$$L^2_\circ(\Gamma_{M_1}\backslash M_1) = \left\{ \varphi \in L^2(\Gamma_{M_1}\backslash M_1) \,\middle|\, \int_0^1 \varphi(n[x,0,0]m)dx = 0,\ m \in M_1 \right\}$$

and M_1 acts on this space by the right translation R. If π is an irreducible subrepresentation of $L^2_\circ(\Gamma_{M_1}\backslash M_1)$, we call π a cuspidal representation of M_1.

Let (π, H_π) be a cuspidal representation of M_1 and define the evaluation map $\lambda\colon H_\pi \to \mathbf{C}$ by $\varphi \mapsto \varphi(1_3)$. If $\mathbf{Re}(\nu_{1,1} - \nu_{1,2}) > 3/2$, for $F \in I_{\nu_1}(\pi)$, the Eisenstein series

$$E(F;g) = \sum_{\gamma \in (\Gamma\cap P_1)\backslash\Gamma} \lambda \circ F(\gamma g) \tag{2.1}$$

is well-defined, and is absolutely convergent. Here we note that $F \mapsto E(F;g)$ is a $(\mathfrak{g}_{\mathbf{C}}, K)$-homomorphism from $I_{\nu_1}(\pi)$ to $\mathcal{A}(\Gamma\backslash G)$.

If we take a flat section F_{ν_1} for $I_{\nu_1}(\pi)$, the Eisenstein series $E(F_{\nu_1};g)$ can be realized as a function of ν_1. In §6, we show that $E(F_{\nu_1};g)$ has the analytic continuation to all $\nu_1 \in \mathbf{C}^2$ and satisfies the functional equation at the minimal K-types of $I_{\nu_1}(\pi)$ by using the Fourier-Whittaker expansions.

3. Cuspidal representations of M_1

In this section, we recall some facts for cuspidal representations of M_1. See [3] and its references for details. (Since $M_1 \simeq SL^{\pm}(2,\mathbf{R}) \times \{\pm 1\}$, cuspidal representations of M_1 are essentially same as those of $GL(2,\mathbf{R})$.)

3.1. The principal series of M_1. We put

$$N_{M_1}=\{n[x,0,0]\mid x\in\mathbf{R}\},\quad A_{M_1}=\{a[y,1/\sqrt{y},1]\mid y\in\mathbf{R}_+\},$$
$$K_{M_1}=K\cap M_1,\quad M_0=\{\mathrm{diag}(\varepsilon_1,\varepsilon_2,\varepsilon_3)\mid \varepsilon_i\in\{\pm1\},\ 1\le i\le 3\}.$$

Then $M_1=N_{M_1}A_{M_1}K_{M_1}$ is an Iwasawa decomposition of M_1. We take a set $\{m_{0,i}\mid 1\le i\le 3\}$ of generators of M_0 with

$$m_{0,1}=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix},\qquad m_{0,2}=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}=\tilde{\kappa}_\pi,\qquad m_{0,3}=-1_3.$$

It will be convenient to introduce the following notation:

$$\tilde{n}[x]=n[x,0,0]\in N_{M_1},\quad \tilde{a}[y]=a[y,1/\sqrt{y},1]\in A_{M_1},$$
$$\tilde{\kappa}_\theta=\begin{pmatrix}\cos\theta&\sin\theta&0\\-\sin\theta&\cos\theta&0\\0&0&1\end{pmatrix}\in K_{M_1},$$

where $x,\theta\in\mathbf{R}$, $y\in\mathbf{R}_+$. Moreover, we put $K^\circ_{M_1}=\{\tilde{\kappa}_\theta\mid 0\le\theta<2\pi\}$.

For $\tilde{\nu}\in\mathbf{C}$, we define the character $A_{M_1}\ni a\mapsto a^{\tilde{\nu}}\in\mathbf{C}^\times$ by $\tilde{a}[y]^{\tilde{\nu}}=y^{\tilde{\nu}}$. For $\delta_1,\delta_2,\delta_3\in\{0,1\}$, we define the character $\sigma=\sigma_{(\delta_1,\delta_2,\delta_3)}$ of M_0 by $\sigma(m_{0,i})=(-1)^{\delta_i}$ $(i=1,2,3)$. The characters of M_0 is exhausted by characters of this type.

For $\tilde{\nu}\in\mathbf{C}$ and a character σ of M_0, let $\pi_{(\tilde{\nu},\sigma)}$ be a principal series representation of M_1 with the representation space

$$H_{(\tilde{\nu},\sigma)}=\left\{\begin{array}{c}f\colon M_1\to\mathbf{C}\\ \textit{measurable}\end{array}\;\middle|\;\begin{array}{l}f(namg)=a^{\tilde{\nu}+\frac{1}{2}}\sigma(m)f(g),\\(n,a,m,g)\in N_{M_1}\times A_{M_1}\times M_0\times M_1,\\ f|_{K_{M_1}}\in L^2(K_{M_1})\end{array}\right\}.$$

on which M_1 acts by the right translation, that is, $\pi_{(\tilde{\nu},\sigma)}(g)=R(g)$ $(g\in M_1)$. We denote by $H_{(\tilde{\nu},\sigma),K_{M_1}}$ the K_{M_1}-finite part of $H_{(\tilde{\nu},\sigma)}$.

Since $M_1=N_{M_1}A_{M_1}K_{M_1}$ and $K_{M_1}=M_0K^\circ_{M_1}$, for $g\in M_1$, we have the decomposition $g=\tilde{n}(g)\tilde{a}(g)\tilde{m}(g)\tilde{\kappa}_{\theta_g}$ with $\tilde{n}(g)\in N_{M_1},\tilde{a}(g)\in A_{M_1},\tilde{m}(g)\in M_0$ and $0\le\theta_g<2\pi$. For $q\in\delta_2+2\mathbf{Z}$, we define $f_{(\tilde{\nu},\sigma;q)}\in H_{(\tilde{\nu},\sigma),K_{M_1}}$ by

$$f_{(\tilde{\nu},\sigma;q)}(g)=\tilde{a}(g)^{\tilde{\nu}+\frac{1}{2}}\sigma(\tilde{m}(g))\exp(\sqrt{-1}q\theta_g).$$

Then we have an irreducible decomposition

$$H_{(\tilde{\nu},\sigma),K_{M_1}}=\bigoplus_{q\in\delta_2+2\mathbf{Z}_{\ge0}}V_{(\tilde{\nu},\sigma;q)}$$

as a K_{M_1}-module, where $V_{(\tilde{\nu},\sigma;q)}=\mathbf{C}f_{(\tilde{\nu},\sigma;q)}+\mathbf{C}f_{(\tilde{\nu},\sigma;-q)}$. Here the action of K_{M_1} on $V_{(\tilde{\nu},\sigma;q)}$ is given as follows:

$$\pi_{(\tilde{\nu},\sigma)}(\tilde{\kappa}_\theta)f_{(\tilde{\nu},\sigma;r)}=\exp(\sqrt{-1}r\theta)f_{(\tilde{\nu},\sigma;r)},\tag{3.1}$$
$$\pi_{(\tilde{\nu},\sigma)}(m_{0,1})f_{(\tilde{\nu},\sigma;r)}=(-1)^{\delta_1}f_{(\tilde{\nu},\sigma;-r)},\tag{3.2}$$
$$\pi_{(\tilde{\nu},\sigma)}(m_{0,3})f_{(\tilde{\nu},\sigma;r)}=(-1)^{\delta_3}f_{(\tilde{\nu},\sigma;r)}\tag{3.3}$$

for $r\in\{\pm q\}$ and $0\le\theta<2\pi$.

3.2. **Some subrepresentations of** $\pi_{(\tilde{\nu},\sigma)}$. If $\delta_2 \equiv k \mod 2$, $\pi_{(\frac{k-1}{2},\sigma)}$ has a subrepresentation $D_{(k,\sigma)}$ whose representation space is

$$H_{D_{(k,\sigma)}} = \widehat{\bigoplus_{q\in k+2\mathbf{Z}_{\geq 0}}} V_{(\frac{k-1}{2},\sigma;q)}$$

for $k \in \mathbf{Z}_{\geq 2}$ and a character $\sigma = \sigma_{(\delta_1,\delta_2,\delta_3)}$ of M_0. Here $\widehat{\bigoplus}$ means the Hilbert space direct sum. Then $D_{(k,\sigma)}$ does not depend on δ_1, i.e. $D_{(k,\sigma_{(0,\delta_2,\delta_3)})} \simeq D_{(k,\sigma_{(1,\delta_2,\delta_3)})}$. Moreover, for each discrete series representation D of M_1, there exists some (k,σ) such that D is infinitesimally equivalent with $D_{(k,\sigma)}$.

Let (π, H_π) be a cuspidal representation of M_1 with its K_{M_1}-finite part $H_{\pi,K_{M_1}}$. Then it is known that π is isomorphic to unitary principal series or discrete series. Since $m_{0,2}$ and $m_{0,3}$ act on $L^2_\circ(\Gamma_{M_1}\backslash M_1)$ trivially, we see that π is infinitesimally equivalent with $\pi_{(\tilde{\nu},\sigma_{(\delta,0,0)})}$ for some $(\tilde{\nu},\delta) \in (\sqrt{-1}\mathbf{R}) \times \{0,1\}$ or $D_{(k,\sigma_{(0,0,0)})}$ for some $k \in 2\mathbf{Z}_{\geq 1}$.

It is convenient to introduce the following notations:

$$(\tilde{\nu}_\pi, \delta(\pi), S_+(\pi)) = \begin{cases} (\tilde{\nu}, \delta, 2\mathbf{Z}_{\geq 0}) & \text{if } H_{\pi,K_{M_1}} \simeq H_{(\tilde{\nu},\sigma_{(\delta,0,0)}),K_{M_1}}, \\ (\frac{k-1}{2}, 0, k+2\mathbf{Z}_{\geq 0}) & \text{if } H_{\pi,K_{M_1}} \simeq H_{D_{(k,\sigma_{(0,0,0)})},K_{M_1}}, \end{cases}$$

$$\sigma_\pi = \sigma_{(\delta(\pi),0,0)}, \quad S(\pi) = S_+(\pi) \cup (-S_+(\pi)).$$

We denote by $(\tilde{\pi}, H_{\tilde{\pi}})$ the subrepresentation of $\pi_{(\tilde{\nu}_\pi,\sigma_\pi)}$ whose representation space is

$$H_{\tilde{\pi}} = \widehat{\bigoplus_{q\in S_+(\pi)}} V_{(\tilde{\nu}_\pi,\sigma_\pi;q)} = \widehat{\bigoplus_{q\in S(\pi)}} \mathbf{C} f_{(\tilde{\nu}_\pi,\sigma_\pi;q)}.$$

Then $\tilde{\pi}$ is infinitesimally equivalent with π.

3.3. **The Jacquet integrals on** M_1. For $c \in \mathbf{R}$, we define the unitary character ψ_c of N_{M_1} by $\psi(\tilde{n}[x]) = \mathbf{e}(cx)$. We put

$$C^\infty_{\mathrm{mg}}(N_{M_1}\backslash M_1;\psi_c) = \left\{ W \in C^\infty(M_1) \;\middle|\; \begin{array}{l} W(ng) = \psi_c(n)W(g), \\ (n,g) \in N_{M_1} \times M_1, \\ W \text{ is moderate growth} \end{array} \right\}$$

and $\mathfrak{m}_{1\mathbf{C}}$ and K_{M_1} act on this space by the right translation R.

When $\mathbf{Re}(\tilde{\nu}) > 0$, we define the Jacquet integral $W_c(f;g)$ on M_1 by

$$W_c(f;g) = \int_{\mathbf{R}} f(w_1\tilde{n}[x]g)\mathbf{e}(-cx)dx \quad (g \in M_1)$$

for $c \in \mathbf{R}^\times$ and $f \in H_{(\tilde{\nu},\sigma),K_{M_1}}$. It is easy to see that $f \mapsto W_c(f;g)$ is an $(\mathfrak{m}_{1\mathbf{C}}, K_{M_1})$-homomorphism from $H_{(\tilde{\nu},\sigma),K_{M_1}}$ to $C^\infty_{\mathrm{mg}}(N_{M_1}\backslash M_1;\psi_c)$.

By using the flat section $f_{(\tilde{\nu},\sigma;q)}$ for $H_{(\tilde{\nu},\sigma),K_{M_1}}$, we consider the analytic continuation of the Jacquet integral to all $\tilde{\nu} \in \mathbf{C}$. From the definition, we have

$$W_c(f_{(\tilde{\nu},\sigma;q)};g) = \mathbf{e}(x)W_c(\pi_{(\tilde{\nu},\sigma)}(\tilde{\kappa})f_{(\tilde{\nu},\sigma;q)};\tilde{a}[y])$$

for $g \in M_1$ with the Iwasawa decomposition $g = \tilde{n}[x]\tilde{a}[y]\tilde{\kappa}$ ($x \in \mathbf{R}$, $y \in \mathbf{R}_+$, $\tilde{\kappa} \in K_{M_1}$). Therefore, from the formulas (3.1), (3.2) and (3.3), we see that $W_c(f_{(\tilde{\nu},\sigma;q)};g)$ is determined by $W_c(f_{(\tilde{\nu},\sigma;\pm q)};\tilde{a}[y])$. Since

$$f_{(\tilde{\nu},\sigma;q)}(w_1\tilde{n}[x]\tilde{a}[y]) = \left(\frac{y}{x^2+y^2}\right)^{\tilde{\nu}+\frac{1}{2}} \left(\frac{x-\sqrt{-1}y}{\sqrt{x^2+y^2}}\right)^q, \tag{3.4}$$

we have

$$(3.5)\qquad W_c(f_{(\tilde{\nu},\sigma;q)};\tilde{a}[y]) = \frac{(-\sqrt{-1})^q \pi^{\tilde{\nu}+\frac{1}{2}}|c|^{\tilde{\nu}-\frac{1}{2}}}{\Gamma(\tilde{\nu}+\frac{1}{2}+\mathrm{sgn}(c)\frac{q}{2})} W_{\mathrm{sgn}(c)\frac{q}{2},\tilde{\nu}}(4\pi|c|y)$$

by using the formula in [4, p.119 (12)]. Here $\mathrm{sgn}(c) = c/|c|$ ($c \in \mathbf{R}^\times$), $\Gamma(s)$ is the Gamma function and $W_{k,m}(z)$ is the confluent hypergeometric function introduced in [48, §16.12]. The formula (3.5) gives the analytic continuation of $W_c(f_{(\tilde{\nu},\sigma;q)};g)$ to all $\tilde{\nu} \in \mathbf{C}$.

3.4. **Explicit structures of cuspidal representations.** For $n \in \mathbf{Z}_{\geq 1}$, we define the Hecke operator $T(n)$ on $L^2_\circ(\Gamma_{M_1}\backslash M_1)$ by

$$(T(n)\varphi)(g) = \frac{1}{\sqrt{n}} \sum_{\substack{a,b,d\in\mathbf{Z}_{\geq 0},\\ ad=n,\ 0\leq b<d}} \varphi\left(\frac{1}{\sqrt{n}}\begin{pmatrix} a & b & 0\\ 0 & d & 0\\ 0 & 0 & \sqrt{n}\end{pmatrix} g\right)$$

where $\varphi \in L^2_\circ(\Gamma_{M_1}\backslash M_1)$. If a cuspidal representation π of M_1 is contained in a common eigenspace of the Hecke operators $T(n)$ ($n \in \mathbf{Z}_{\geq 1}$), we call π *a Hecke-eigen cuspidal representation.* We concentrate our attention to Hecke-eigen cuspidal representations since $L^2_\circ(\Gamma_{M_1}\backslash M_1)$ decompose into a Hilbert space direct sum of those.

Let (π, H_π) be a Hecke-eigen cuspidal representation of M_1. The element φ of H_π has the Fourier-Whittaker expansion $\varphi(g) = \sum_{m\neq 0}\varphi^{(m)}(g)$ where

$$\varphi^{(m)}(g) = \int_0^1 \varphi(\tilde{n}[\xi]g)\mathbf{e}(-m\xi)d\xi.$$

Here we note that $\varphi \mapsto \varphi^{(m)}$ is a homomorphism from $H_{\pi,K_{M_1}}$ to $C^\infty_{\mathrm{mg}}(N_{M_1}\backslash M_1;\psi_m)$ as $(\mathfrak{m}_{1\mathbf{C}}, K_{M_1})$-modules. We fix a $(\mathfrak{m}_{1\mathbf{C}}, K_{M_1})$-isomorphism $\iota_\pi\colon H_{\tilde{\pi},K_{M_1}} \to H_{\pi,K_{M_1}}$ and put $\varphi_{\pi,q} = \iota_\pi(f_{(\tilde{\nu}_\pi,\sigma_\pi;q)})$. By the multiplicity one theorem (See for example, [3, Theorem 2.8.1]), there exists $c_\pi(m)$ such that

$$\varphi^{(m)}_{\pi,q}(g) = c_\pi(m)|m|^{-\tilde{\nu}_\pi}W_m(f_{(\tilde{\nu}_\pi,\sigma_\pi;q)};g) \qquad (q \in S(\pi)).$$

Then we have

$$(3.6)\qquad \varphi_{\pi,q}(g) = \sum_{m\neq 0} c_\pi(m)|m|^{-\tilde{\nu}_\pi}W_m(f_{(\tilde{\nu}_\pi,\sigma_\pi;q)};g).$$

By the standard arguments (which is similar to [3, §1.4]), we obtain the following facts. Since $c_\pi(1) \neq 0$, we may assume that $c_\pi(1) = 1$. Then, for $n \in \mathbf{Z}_{\geq 1}$, $c_\pi(n)$ coincides with the eigenvalue of $T(n)$ on H_π and $c_\pi(-n) = (-1)^{\delta(\pi)}c_\pi(n)$. Moreover, it holds that

$$(3.7)\qquad c_\pi(m)c_\pi(n) = \sum_{0<d|\gcd(m,n)} c_\pi\left(\frac{mn}{d^2}\right).$$

Here $\gcd(m,n)$ means the greatest common divisor of m and n.

For each prime number p, we take two complex numbers $\alpha_\pi(p)$, $\beta_\pi(p)$ such that $\alpha_\pi(p) + \beta_\pi(p) = c_\pi(p)$ and $\alpha_\pi(p)\beta_\pi(p) = 1$. From (3.7), we see that any $c_\pi(n)$ is determined by $\{\alpha_\pi(p),\beta_\pi(p)\}_{p\colon \text{prime}}$, that is, if $\gcd(m,n) = 1$, $c_\pi(mn) = c_\pi(m)c_\pi(n)$ and for a prime number p and $e \in \mathbf{Z}_{\geq 0}$,

$$(3.8)\qquad c_\pi(p^e) = \frac{\alpha_\pi(p)^{e+1} - \beta_\pi(p)^{e+1}}{\alpha_\pi(p) - \beta_\pi(p)}.$$

For $s \in \mathbf{C}$ such that $\mathbf{Re}(s) > 3/2$, we define *the standard L-function for* π by $L(s,\pi) = \sum_{m>0} c_\pi(m)m^{-s}$. From (3.7), we see that $L(s,\pi)$ has the following Euler product expressions:

$$L(s,\pi) = \prod_{p\colon \text{prime}} (1 - c_\pi(p)p^{-s} + p^{-2s})^{-1}$$

$$= \prod_{p:\text{ prime}} (1-\alpha_\pi(p)p^{-s})^{-1}(1-\beta_\pi(p)p^{-s})^{-1}.$$

We put

$$\epsilon_\pi = \begin{cases} (-1)^\delta, & \text{if } \tilde{\pi} = \pi_{(\tilde{\nu},\sigma_{(\delta,0,0)})}, \\ (-1)^{\frac{k}{2}}, & \text{if } \tilde{\pi} = D_{(k,\sigma_{(0,0,0)})}, \end{cases}$$

$$L_\infty(s,\pi) = \begin{cases} \Gamma_{\mathbf{R}}(s+\tilde{\nu}+\delta)\Gamma_{\mathbf{R}}(s-\tilde{\nu}+\delta), & \text{if } \tilde{\pi} = \pi_{(\tilde{\nu},\sigma_{(\delta,0,0)})}, \\ \Gamma_{\mathbf{C}}\left(s+\frac{k-1}{2}\right), & \text{if } \tilde{\pi} = D_{(k,\sigma_{(0,0,0)})}, \end{cases}$$

where $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Then $\Lambda(s,\pi) = L_\infty(s,\pi)L(s,\pi)$ has the analytic continuation to all $s \in \mathbf{C}$ and satisfies the functional equation

$$\Lambda(s,\pi) = \epsilon_\pi \Lambda(1-s,\pi). \tag{3.9}$$

4. Constructions of flat sections for $I_{\nu_1}(\pi)$

4.1. **Irreducible representations of K.** Let $\tilde{V}_l$ be the subspace consisting of degree l homogeneous polynomials of three variables x_1, x_2, x_3 in the polynomial ring $\mathbf{C}[x_1,x_2,x_3]$. For $\kappa \in SO(3)$ and $f \in \tilde{V}_l$, we set

$$\tilde{\tau}_l(\kappa)f(x_1,x_2,x_3) = f((x_1,x_2,x_3)\cdot\kappa).$$

Here $(x_1,x_2,x_3)\cdot\kappa$ is the ordinal product of matrices. We put $r^2 = x_1^2+x_2^2+x_3^2 \in \tilde{V}_2$. Since r^2 is $SO(3)$-invariant, we can define the quotient representation τ_l of $\tilde{\tau}_l$ on $V_l = \tilde{V}_l/(r^2\cdot\tilde{V}_{l-2})$. Here we put $\tilde{V}_l = 0$ for $l < 0$.

For $\delta \in \{0,1\}$, we define the action τ_l^δ of $K = O(3)$ on $V_l^\delta = V_l$ by

$$\tau_l^\delta(\kappa) = (\det(\kappa))^\delta \tau_l(\det(\kappa)\kappa), \qquad \kappa \in K.$$

Then $(\tau_l^\delta, V_l^\delta)$ is an irreducible $(2l+1)$-dimensional representation and the set of equivalence classes of irreducible finite dimensional representations of K is exhausted by τ_l^δ $(\delta \in \{0,1\}, l \in \mathbf{Z}_{\geq 0})$.

We define the basis $\{v_q^{(l)}\}_{-l\leq q\leq l}$ of $V_l^\delta = V_l$ by

$$v_q^{(l)} = (\mathrm{sgn}(q)x_1 + \sqrt{-1}x_2)^{|q|}x_3^{l-|q|} \mod r^2\cdot\tilde{V}_{l-2}, \quad -l \leq q \leq l.$$

We denote by $(\tau_l^{\delta*}, V_l^{\delta*})$ the contragradient representation of $(\tau_l^\delta, V_l^\delta)$, and take the dual basis $\{v_q^{(l)*}\}_{-l\leq q\leq l}$ of $\{v_q^{(l)}\}_{-l\leq q\leq l}$. For later use, we compute the actions of $\tilde{\kappa}_\theta, m_{0,i} \in K$ on $\{v_q^{(l)*}\}_{-l\leq q\leq l}$ as follows:

$$\tau_l^{\delta*}(\tilde{\kappa}_\theta)v_q^{(l)*} = \exp(-\sqrt{-1}q\theta)v_q^{(l)*} \qquad (0 \leq \theta < 2\pi), \tag{4.1}$$

$$\tau_l^{\delta*}(m_{0,1})v_q^{(l)*} = (-1)^{l+\delta}v_{-q}^{(l)*}, \tag{4.2}$$

$$\tau_l^{\delta*}(m_{0,3})v_q^{(l)*} = (-1)^\delta v_q^{(l)*}. \tag{4.3}$$

4.2. **Flat sections.** We analyse the K-types of $I_{\nu_1}(\pi_{(\tilde{\nu},\sigma)})$ for $\tilde{\nu} \in \mathbf{C}$, $\nu_1 = (\nu_{1,1},\nu_{1,2}) \in \mathbf{C}^2$ and a character $\sigma = \sigma_{(\delta_1,\delta_2,\delta_3)}$ of M_0. The target $H_{(\tilde{\nu},\sigma)}$ of functions in $I_{\nu_1}(\pi_{(\tilde{\nu},\sigma)})$ has a decomposition:

$$H_{\pi_{(\tilde{\nu},\sigma)}} = \widehat{\bigoplus_{q\in\delta_2+2\mathbf{Z}}} \mathbf{C}f_{(\tilde{\nu},\sigma;q)}.$$

Denote the corresponding decomposition of $F \in I_{\nu_1}(\pi_{(\tilde{\nu},\sigma)})$ by

$$F(x) = \sum_{q\in\delta_2+2\mathbf{Z}} F_q(x)f_{(\tilde{\nu},\sigma;q)}.$$

From the definition of the space $I_{\nu_1}(\pi_{(\tilde{\nu},\sigma)})$, we have

$$F(mx) = \pi_{(\tilde{\nu},\sigma)}(m)F(x) \qquad x \in G,\ m \in M_1.$$

For $m = \tilde{\kappa}_\theta,\ m_{0,i} \in K_{M_1}$, comparing the coefficients of $f_{(\tilde{\nu},\sigma;q)}$ in the left hand side of the above equation with those in the right hand side, we have

$$F_q(\tilde{\kappa}_\theta x) = \exp(\sqrt{-1}q\theta)F_q(x) \quad (0 \le \theta < 2\pi),$$
$$F_q(m_{0,1}x) = (-1)^{\delta_1}F_{-q}(x), \qquad F_q(m_{0,3}x) = (-1)^{\delta_3}F_q(x).$$

Therefore $F|_K(x)$ belongs to

$$\bigoplus_{q\in\delta_2+2\mathbf{Z}_{\ge 0}} \{F_q(x)f_{(\tilde{\nu},\sigma;q)} + (-1)^{\delta_1}F_q(m_{0,1}x)f_{(\tilde{\nu},\sigma;-q)} \mid F_q \in C_{\mathrm{fin}}(K;q,\delta_3)\} \tag{4.4}$$

where

$$C_{\mathrm{fin}}(K;q,\delta) = \left\{ F_q \in C_{\mathrm{fin}}(K) \;\middle|\; \begin{array}{l} F_q(\tilde{\kappa}_\theta x) = \exp(\sqrt{-1}q\theta)F_q(x),\ \theta \in \mathbf{R}, \\ F_q(m_{0,3}x) = (-1)^{\delta}F_q(x),\ x \in K, \end{array} \right\}.$$

Here we denote by $C_{\mathrm{fin}}(K)$ the space of K-finite functions in $C^\infty(K)$. Since $P_1 \cap K = K_{M_1}$ and $G = P_1K$, we note that the restriction map $F \mapsto F|_K$ from $I_{\nu_1}(\pi_{(\tilde{\nu},\sigma)})$ to the space (4.4) is an isomorphism of K-modules.

Peter-Weyl's theorem (see for example, [20]) tells that $C_{\mathrm{fin}}(K)$ is generated by matrix coefficients of irreducible finite dimensional representations of K. From (4.1) and (4.3), we see that $C_{\mathrm{fin}}(K;q,\delta)$ is generated by $F_{(\delta,q,v)}$ ($v \in V_l^\delta,\ l \in \mathbf{Z}_{\ge|q|}$) with

$$F_{(\delta,q,v)}(x) = \langle v_q^{(l)*}, \tau_l^\delta(x)v\rangle, \qquad x \in K.$$

Here $\langle\cdot,\cdot\rangle$ is the canonical pairing on $V_l^{\delta*} \times V_l^\delta$. Moreover, from (4.2), we see that

$$F_{(\delta,q,v)}(m_{0,1}x) = (-1)^{l+\delta}F_{(\delta,-q,v)}(x).$$

Since $G = N_1A_1M_1K$, for $g \in G$, we have the decomposition $g = n_1(g)a_1(g)m_1(g)\kappa_1(g)$ with $n_1(g) \in N_1, a_1(g) \in A_1, m_1(g) \in M_1$ and $\kappa_1(g) \in K$. For $q \in \delta_2 + 2\mathbf{Z}$, we define the function $F^1_{(\nu_1,\tilde{\nu},\sigma;q,v)}$ in $I_{\nu_1}(\pi_{(\tilde{\nu},\sigma)})$ by

$$\begin{aligned} F^1_{(\nu_1,\tilde{\nu},\sigma;q,v)}(g) =& a_1(g)^{\nu_1+\rho_1}\pi_{(\tilde{\nu},\sigma)}(m_1(g))\{F_{(\delta_3,q,v)}(\kappa_1(g))f_{(\tilde{\nu},\sigma;q)} \\ &+ (-1)^{l+\delta_1+\delta_3}F_{(\delta_3,-q,v)}(\kappa_1(g))f_{(\tilde{\nu},\sigma;-q)}\}. \end{aligned}$$

Although the decomposition $g = n_1(g)a_1(g)m_1(g)\kappa_1(g)$ is not unique, this definition is well-defined. Then, by the above arguments, we have the decomposition

$$I_{\nu_1}(\pi_{(\tilde{\nu},\sigma)}) = \bigoplus_{q\in\delta_2+2\mathbf{Z}_{\ge0},\ l\in\mathbf{Z}_{\ge q}} \{F^1_{(\nu_1,\tilde{\nu},\sigma;q,v)} \mid v \in V_l^{\delta_3}\}.$$

Here, if $\delta_2 = q = 0$ and $l + \delta_1 + \delta_3 \equiv 1 \mod 2$, the subspace $\{F^1_{(\nu_1,\tilde{\nu},\sigma;q,v)} \mid v \in V_l^{\delta_3}\}$ vanish and if not, it is isomorphic to $\tau_l^{\delta_3}$ as K-modules. Moreover, by the definition of the induced $(\mathfrak{g}_{\mathbf{C}}, K)$-module, $I_{\nu_1}(D_{(k,\sigma)})$ is a $(\mathfrak{g}_{\mathbf{C}}, K)$-submodule of $I_{\nu_1}(\pi_{(\frac{k-1}{2},\sigma)})$ and we have

$$I_{\nu_1}(D_{(k,\sigma)}) = \bigoplus_{q\in k+2\mathbf{Z}_{\ge0},\ l\in\mathbf{Z}_{\ge q}} \{F^1_{(\nu_1,\frac{k-1}{2},\sigma;q,v)} \mid v \in V_l^{\delta_3}\},$$

Let (π, H_π) be a Hecke-eigen cuspidal representation of M_1. We extend the $(\mathfrak{m}_{1\mathbf{C}}, K_{M_1})$-isomorphism $\iota_\pi\colon H_{\tilde{\pi},K_{M_1}} \to H_{\pi,K_{M_1}}$ in §3.4 to a M_1-isomorphism from $H^\infty_{\tilde{\pi}}$ to H^∞_π. Then ι_π induces a $(\mathfrak{g}_{\mathbf{C}}, K)$-isomorphism $I_{\nu_1}(\tilde{\pi}) \ni F \mapsto \iota_\pi \circ F \in I_{\nu_1}(\pi)$. The image of $F^1_{(\nu_1,\tilde{\nu}_\pi,\sigma_\pi;q,v)}(g)$ under this isomorphism is given by

$$F^1_{(\nu_1,\pi;q,v)}(g) = a_1(g)^{\nu_1+\rho_1}\pi(m_1(g))\{F_{(0,q,v)}(\kappa_1(g))\varphi_{\pi,q}$$

$$+(-1)^{l+\delta(\pi)}F_{(0,-q,v)}(\kappa_1(g))\varphi_{\pi,-q}\},$$

and

$$I_{\nu_1}(\pi)=\bigoplus_{q\in S_+(\pi),\ l\in\mathbf{Z}_{\geq q}}\{F^1_{(\nu_1,\pi;q,v)}\mid v\in V_l^0\}.$$

We note that $\mathbf{C}^2\ni\nu_1\mapsto F^1_{(\nu_1,\pi;q,v)}\in C^\infty(G)\otimes H_\pi$ is a flat section for $I_{\nu_1}(\pi)$.

4.3. **The principal series of G.** For $\nu_0=(\nu_{0,1},\nu_{0,2},\nu_{0,3})\in\mathbf{C}^3$, we define the character $A_0\ni a\mapsto a^{\nu_0}\in\mathbf{C}^\times$ by

$$a[y_1,y_2,y_3]^{\nu_0}=y_1^{\nu_{0,1}}y_2^{\nu_{0,1}+\nu_{0,2}}y_3^{\nu_{0,1}+\nu_{0,2}+\nu_{0,3}}.$$

We put $\rho_0=(1,0,-1)$. For $\nu_0\in\mathbf{C}^3$ and a character $\sigma=\sigma_{(\delta_1,\delta_2,\delta_3)}$ of M_0, we put

$$I(\nu_0,\sigma)=\left\{F\in C^\infty(G)\;\middle|\;\begin{array}{l}F(namg)=a^{\nu_0+\rho_0}\sigma(m)F(g),\\(n,a,m,g)\in N_0\times A_0\times M_0\times G,\\F\colon K\text{-finite}\end{array}\right\}.$$

on which $\mathfrak{g}_\mathbf{C}$ and K act by the right translation R. Then $I(\nu_0,\sigma)$ is the associated $(\mathfrak{g}_\mathbf{C},K)$-module of a principal series representation of G.

We define the evaluation map $\lambda_{(\tilde\nu,\sigma)}\colon H_{(\tilde\nu,\sigma)}\to\mathbf{C}$ by $f\mapsto f(1_3)$. If $\nu_0=(\nu_{1,1}+\tilde\nu,\nu_{1,1}-\tilde\nu,\nu_{1,2})$, we see that $\lambda_{(\tilde\nu,\sigma)}\circ F\in I(\nu_0,\sigma)$ for $F\in I_{\nu_1}(\pi_{(\tilde\nu,\sigma)})$ and $\Phi_{(\tilde\nu,\sigma)}\colon I_{\nu_1}(\pi_{(\tilde\nu,\sigma)})\ni F\mapsto\lambda_{(\tilde\nu,\sigma)}\circ F\in I(\nu_0,\sigma)$ is a $(\mathfrak{g}_\mathbf{C},K)$-isomorphism. The image of $F^1_{(\nu_1,\tilde\nu,\sigma;q,v)}$ under this isomorphism is given by

$$F^0_{(\nu_0,\sigma;q,v)}(g)=a_0(g)^{\nu_0+\rho_0}\{F_{(\delta_3,q,v)}(\kappa_0(g))+(-1)^{l+\delta_1+\delta_3}F_{(\delta_3,-q,v)}(\kappa_0(g))\}.$$

Here we denote the Iwasawa decomposition of $g\in G$ by $g=n_0(g)a_0(g)\kappa_0(g)$ with $n_0(g)\in N_0,a_0(g)\in A_0$ and $\kappa_0(g)\in K$.

For a Hecke-eigen cuspidal representation π of M_1, we have

$$I_{\nu_1}(\pi)\simeq I_{\nu_1}(\tilde\pi)\subset I_{\nu_1}(\pi_{(\tilde\nu_\pi,\sigma_\pi)})\simeq I(\nu_0,\sigma_\pi)$$

where $\nu_0=(\nu_{1,1}+\tilde\nu_\pi,\nu_{1,1}-\tilde\nu_\pi,\nu_{1,2})$. Hence we may realize $I_{\nu_1}(\pi)$ as a $(\mathfrak{g}_\mathbf{C},K)$-submodule of $I(\nu_0,\sigma_\pi)$.

5. The Fourier-Whittaker expansions of the Eisenstein series

Throughout this section, let (π,H_π) be a Hecke-eigen cuspidal representation of M_1.

5.1. **The Jacquet integrals on G.** Jacquet [17] introduced Whittaker functions for principal series representations of arbitrary Chevalley groups. We specialize his general results to our situation for $GL(3,\mathbf{R})$. For $0\leq i\leq 5$, $c_1,c_2\in\mathbf{R}$ and $F\in I(\nu_0,\sigma)$, we can define the Jacquet integral $W_{c_1,c_2}(w_i,F;g)\in C^\infty_{\mathrm{mg}}(N_0\backslash G;\psi_{c_1,c_2})$.

In this part, we use

$$W_{c_1,0}(w_1,F;g)=\int_{\mathbf{R}}F(w_1n[x_1,0,0]g)\mathbf{e}(-c_1x_1)dx_1,$$
$$W_{c_1,c_2}(w_5,F;g)=\int_{\mathbf{R}^3}F(w_5n[x_1,x_2,x_3]g)\mathbf{e}(-c_1x_1-c_2x_2)dx_1dx_2dx_3.$$

Here $W_{c_1,c_2}(w_i,F;g)$ $(i=1,5)$ is well-defined and is absolutely convergent when $\nu_0\in D_{w_i}$ with

$$D_{w_1}=\{(\nu_{0,1},\nu_{0,2},\nu_{0,3})\in\mathbf{C}^3\mid\mathbf{Re}(\nu_{0,1}-\nu_{0,2})>0\},$$
$$D_{w_5}=\{(\nu_{0,1},\nu_{0,2},\nu_{0,3})\in\mathbf{C}^3\mid\mathbf{Re}(\nu_{0,i}-\nu_{0,i+1})>0\ (i\in\{1,2\})\}.$$

We note that $F\mapsto W_{c_1,c_2}(w_i,F;g)$ is a homomorphism from $I(\nu_0,\sigma)$ to $C^\infty_{\mathrm{mg}}(N_0\backslash G;\psi_{c_1,c_2})$ as a $(\mathfrak{g}_\mathbf{C},K)$-modules. Hence $W_{c_1,c_2}(w_i,F;g)$ is a Whittaker function for $I(\nu_0,\sigma)$.

Let $\mathbf{C}^3 \times G \ni (\nu_0, g) \mapsto F_{\nu_0}(g) \in \mathbf{C}$ be a flat section for $I(\nu_0, \sigma)$, i.e. F_{ν_0} is an element of $I(\nu_0, \sigma)$ and for $\kappa \in K$, $F_{\nu_0}(\kappa)$ does not depend on $\nu_0 \in \mathbf{C}^3$. It is known that $W_{c_1,c_2}(w_i, F_{\nu_0}; g)$ has the meromorphic continuation to all $\nu_0 \in \mathbf{C}^3$. We prove this fact by using explicit formulas in Lemma 5.3 and §6.

5.2. **The Fourier-Whittaker expansions of the Eisenstein series.** For $m_1, m_2 \in \mathbf{Z}$, let $E_{m_1,m_2}(F^1_{(\nu_1,\pi;q,v)}; g)$ be the Fourier-Whittaker coefficient of $E(F^1_{(\nu_1,\pi;q,v)}; g)$, that is,

$$E_{m_1,m_2}(F^1_{(\nu_1,\pi;q,v)}; g) = \int_0^1 \int_0^1 \int_0^1 E(F^1_{(\nu_1,\pi;q,v)}; n[\xi_1, \xi_2, \xi_3]g)\mathbf{e}(-m_1\xi_1 - m_2\xi_2)d\xi_1 d\xi_2 d\xi_3.$$

Because of the arguments in §4.3, the Jacquet integrals are realized as Whittaker functions for $I_{\nu_1}(\pi)$. In this subsection, we show that the Fourier-Whittaker coefficients $E_{m_1,m_2}(F^1_{(\nu_1,\pi;q,v)}; g)$ are expressed in terms of the Jacquet integrals.

Lemma 5.1. *We define the subsets S_i^Γ $(i = 0, 1, 2)$ in Γ by*

$$S_0^\Gamma = \{1_3\}, \qquad S_1^\Gamma = \{\gamma_2(a, b, c; l) \mid (a, b, c) \in \Sigma_\Gamma,\ l \in \mathbf{Z}\},$$
$$S_2^\Gamma = \{\gamma_2(a_2, b_2, c_2; l_2)\gamma_1(a_1, b_1, c_1; l_1) \mid (a_j, b_j, c_j) \in \Sigma_\Gamma,\ l_j \in \mathbf{Z}\ (j = 1, 2)\},$$

where

$$\Sigma_\Gamma = \{(a, b, c) \in \mathbf{Z}^3 \mid a > 0,\ 0 \le b, c < a,\ bc \equiv 1 \mod a\},$$
$$\gamma(a, b, c; l) = \begin{pmatrix} 1/a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & c/a + l \\ 0 & 1 \end{pmatrix},$$
$$\gamma_1(a, b, c; l) = \left(\begin{array}{c|c} \gamma(a, b, c; l) & O_{2,1} \\ \hline O_{1,2} & 1 \end{array}\right), \quad \gamma_2(a, b, c; l) = \left(\begin{array}{c|c} 1 & O_{1,2} \\ \hline O_{2,1} & \gamma(a, b, c; l) \end{array}\right).$$

The disjoint union $S_0^\Gamma \cup S_1^\Gamma \cup S_2^\Gamma$ forms a complete system of representatives of $(\Gamma \cap P_1)\backslash\Gamma$.

Proof. We define the surjective map $r \colon \Gamma \to \mathcal{P}_3$ by

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \mapsto (g_{31}, g_{32}, g_{33})$$

where $\mathcal{P}_3 = \{(m_1, m_2, m_3) \in \mathbf{Z}^3 \mid \gcd(m_1, m_2, m_3) = 1\}$. Then it is easy to see that r induces a bijection $(\Gamma \cap P_1')\backslash\Gamma \to \mathcal{P}_3$ with

$$P_1' = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix} \in P_1 \right\}.$$

By direct computation, we see that $S_i^\Gamma \subset \Gamma$, $r|_{S_i^\Gamma}$ is injective and

$$r(S_0^\Gamma) = \{(0, 0, 1)\}, \quad r(S_1^\Gamma) = \{(0, m_2, m_3) \in \mathcal{P}_3 \mid m_2 > 0\},$$
$$r(S_2^\Gamma) = \{(m_1, m_2, m_3) \in \mathcal{P}_3 \mid m_1 > 0\}.$$

Therefore, the disjoint union $\bigcup_{i\in\{0,1,2\}}\{S_i^\Gamma \cup (-S_i^\Gamma)\}$ forms a complete system of representatives of $(\Gamma \cap P_1')\backslash\Gamma$. Since $\Gamma \cap P_1 = (\Gamma \cap P_1') \cup (-\Gamma \cap P_1')$, we obtain the assertion. □

Remark 5.2. The choice of representatives of $(\Gamma \cap P_1)\backslash\Gamma$ in Lemma 5.1 is compatible with the Bruhat decomposition $G = \coprod_{0\le i\le 5} N_0A_0M_0w_iN_0$. Actually, we easily check that $S_i^\Gamma \subset N_0A_0M_0w_{2i}N_0$ for $i = 0, 1, 2$.

Lemma 5.3. *(i) Let $\tilde{\nu} \in \mathbf{C}$, $\nu_1 = (\nu_{1,1}, \nu_{1,2}) \in \mathbf{C}^2$ and a character $\sigma = \sigma_{(\delta_1,\delta_2,\delta_3)}$ of M_0. If $\mathbf{Re}(\tilde{\nu}) > 0$, the following equality holds:*

$$\begin{aligned} W_{c,0}(w_1, F^0_{(\nu_0,\sigma;q,v)}; g) =& a_1(g)^{\nu_1+\rho_1}\{F_{(\delta_3,q,v)}(\kappa_1(g))W_c(f_{(\tilde{\nu},\sigma;q)}; m_1(g)) \\ &+ (-1)^{l+\delta_1+\delta_3}F_{(\delta_3,-q,v)}(\kappa_1(g))W_c(f_{(\tilde{\nu},\sigma;-q)}; m_1(g))\}, \end{aligned}$$

where $\nu_0 = (\nu_{1,1}+\tilde{\nu}, \nu_{1,1}-\tilde{\nu}, \nu_{1,2})$. This equality gives the analytic continuation of the Jacquet integral $W_{c,0}(w_1, F^0_{(\nu_0,\sigma;q,v)}; g)$ to all $\nu_1 \in \mathbf{C}^2$, $\tilde{\nu} \in \mathbf{C}$.
(ii) The function $\lambda \circ F^1_{(\nu_1,\pi;q,v)}$ has the following expansion:

$$\lambda \circ F^1_{(\nu_1,\pi;q,v)}(g) = \sum_{m\neq 0} c_\pi(m)|m|^{-\tilde{\nu}_\pi} W_{m,0}(w_1, F^0_{(\nu_0,\sigma_\pi;q,v)}; g)$$

where $\nu_0 = (\nu_{1,1}+\tilde{\nu}_\pi, \nu_{1,1}-\tilde{\nu}_\pi, \nu_{1,2})$.

Proof. Since

$$\begin{aligned} F^0_{(\nu_0,\sigma;q,v)}(g) =& a_1(g)^{\nu_1+\rho_1}\{F_{(\delta_3,q,v)}(\kappa_1(g))f_{(\tilde{\nu},\sigma;q)}(m_1(g)) \\ &+ (-1)^{l+\delta_1+\delta_3}F_{(\delta_3,-q,v)}(\kappa_1(g))f_{(\tilde{\nu},\sigma;-q)}(m_1(g))\}, \end{aligned}$$

we have

$$\begin{aligned} W_{c,0}(w_1, F^0_{(\nu_0,\sigma;q,v)}; g) &= \int_{\mathbf{R}} F^0_{(\nu_0,\sigma;q,v)}(w_1 n[x,0,0]g)\mathbf{e}(-cx)dx \\ = a_1(g)^{\nu_1+\rho_1}&\left\{F_{(\delta_3,q,v)}(\kappa_1(g))\int_{\mathbf{R}} f_{(\tilde{\nu},\sigma;q)}(w_1 n[x,0,0]m_1(g))\mathbf{e}(-cx)dx\right. \\ + (-1)^{l+\delta_1+\delta_3}&\left.F_{(\delta_3,-q,v)}(\kappa_1(g))\int_{\mathbf{R}} f_{(\tilde{\nu},\sigma;-q)}(w_1 n[x,0,0]m_1(g))\mathbf{e}(-cx)dx\right\}. \end{aligned}$$

Hence we obtain the statement (i). The statement (ii) is obtained from the statement (i), (3.6) and the definition of $F^1_{(\nu_1,\pi;q,v)}$. □

We put

$$I_{m,(m_1,m_2)}(F_{\nu_0}, \gamma) = \int_0^1\int_0^1\int_0^1 W_{m,0}(w_1, F_{\nu_0}; \gamma n[\xi_1,\xi_2,\xi_3]g)\mathbf{e}(-m_1\xi_1 - m_2\xi_2)d\xi_1 d\xi_2 d\xi_3$$

where $m, m_1, m_2 \in \mathbf{Z}$, $\nu_0 \in \mathbf{C}^3$, $\gamma \in \Gamma$ and a flat section F_{ν_0} for $I(\nu_0,\sigma)$. Then, by the above two lemmas, we see that

$$E_{m_1,m_2}(F^1_{(\nu_1,\pi;q,v)}; g) = \sum_{m\neq 0} c_\pi(m)|m|^{-\tilde{\nu}_\pi} \sum_{\substack{\gamma\in S_i^\Gamma \\ i\in\{0,1,2\}}} I_{m,(m_1,m_2)}(F^0_{(\nu_0,\sigma_\pi;q,v)}, \gamma)$$

where $\nu_0 = (\nu_{1,1}+\tilde{\nu}_\pi, \nu_{1,1}-\tilde{\nu}_\pi, \nu_{1,2})$.

Lemma 5.4. *The following equalities hold.*
(i) For $m, m_1, m_2 \in \mathbf{Z}$ and $\nu_0 \in \mathbf{C}^3$,

$$I_{m,(m_1,m_2)}(F_{\nu_0}, 1_3) = \begin{cases} W_{m_1,0}(w_1, F_{\nu_0}; g), & \text{if } m_1 = m,\ m_2 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) For $(a,b,c) \in \Sigma_\Gamma$, $0 \neq m \in \mathbf{Z}$, $l, m_1, m_2 \in \mathbf{Z}$ and $\nu_0 \in \mathbf{C}^3$,

$$I_{m,(m_1,m_2)}(F_{\nu_0}, \gamma_2(a,b,c;l)) = 0.$$

(iii) For $(a_1,b_1,c_1), (a_2,b_2,c_2) \in \Sigma_\Gamma$, $m, m_1, m_2 \in \mathbf{Z}$ and $\nu_0 \in \mathbf{C}^3$,

$$\sum_{l_1,l_2\in\mathbf{Z}} I_{m,(m_1,m_2)}(F_{\nu_0}, \gamma_2(a_2,b_2,c_2;l_2)\gamma_1(a_1,b_1,c_1;l_1))$$

$$= \begin{cases} a_1^{-\nu_{0,2}+\nu_{0,3}+1} a_2^{-\nu_{0,1}+\nu_{0,3}+1} \mathbf{e}\left(\dfrac{m_1 c_1}{a_1} + \dfrac{m_2 b_1 c_2}{a_2}\right) W_{m_1,m_2}(w_5, F_{\nu_0}; g), & \\ \hfill \textit{if } m_2 a_1 = m a_2, & \\ 0, \hfill \textit{otherwise}. & \end{cases}$$

Proof. Since

$$W_{m,0}(w_1, F_{\nu_0}; n[\xi_1, \xi_2, \xi_3]g) = \mathbf{e}(m\xi_1) W_{m,0}(w_1, F_{\nu_0}; g), \tag{5.1}$$

we have

$$I_{m,(m_1,m_2)}(F_{\nu_0}, 1_3) = W_{m,0}(w_1, F_{\nu_0}; g) \int_0^1 \mathbf{e}((m - m_1)\xi_1) d\xi_1 \int_0^1 \mathbf{e}(-m_2\xi_2) d\xi_2.$$

Therefore, by the equality

$$\int_0^1 \mathbf{e}(n\xi) d\xi = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{otherwise}, \end{cases} \qquad (n \in \mathbf{Z}), \tag{5.2}$$

we have the statement (i).

Since

$$n[\xi_1, \xi_2, \xi_3] = n[0, 0, \xi_3] n[\xi_1, \xi_2, 0],$$
$$\gamma_2(a, b, c; l) n[0, 0, \xi_3] = n[-a\xi_3, 0, b\xi_3] \gamma_2(a, b, c; l),$$

and (5.1), we have

$$I_{m,(m_1,m_2)}(F_{\nu_0}, \gamma_2(a, b, c; l)) = \int_0^1 \mathbf{e}(-ma\xi_3) d\xi_3$$
$$\times \int_0^1 \int_0^1 W_{m,0}(w_1, F_{\nu_0}; \gamma_2(a, b, c; l) n[\xi_1, \xi_2, 0]g) \mathbf{e}(-m_1\xi_1 - m_2\xi_2) d\xi_1 d\xi_2.$$

Therefore, by the equality (5.2), we have the statement (ii).

Finally, we prove the statement (iii). Because of the analytic continuation, we may assume that $\mathbf{Re}(\nu_{0,1} - \nu_{0,2})$ and $\mathbf{Re}(\nu_{0,2} - \nu_{0,3})$ are sufficiently large. We put

$$\gamma[l_1, l_2] = \gamma_2(a_2, b_2, c_2; l_2) \gamma_1(a_1, b_1, c_1; l_1).$$

Then we have

$$\sum_{l_1, l_2 \in \mathbf{Z}} I_{m,(m_1,m_2)}(F_{\nu_0}, \gamma[l_1, l_2]) = \sum_{l_1, l_2 \in \mathbf{Z}} \int_0^1 \int_0^1 \int_0^1 \int_{\mathbf{R}} F_{\nu_0}(w_1 n[x, 0, 0] \gamma[l_1, l_2] n[\xi_1, \xi_2, \xi_3] g)$$
$$\times \mathbf{e}(-mx - m_1\xi_1 - m_2\xi_2) dx d\xi_1 d\xi_2 d\xi_3.$$

By direct computation, we have

$$w_1 n[x, 0, 0] \gamma[l_1, l_2] n[\xi_1, \xi_2, \xi_3] = \begin{pmatrix} 1/a_2 & 0 & -a_1 b_2 \\ 0 & 1/a_1 & a_1 b_2 x + b_1 \\ 0 & 0 & a_1 a_2 \end{pmatrix} w_5 n[x_1, x_2, x_3]$$

where

$$x_1 = \xi_1 + \frac{c_1}{a_1} + l_1, \qquad x_2 = \xi_2 + \frac{a_1}{a_2} x + \frac{b_1 c_2}{a_2} + b_1 l_2,$$
$$x_3 = \xi_3 + \left(\frac{c_1}{a_1} + l_1\right) \xi_2 + \frac{c_2}{a_1 a_2} + \frac{l_2}{a_1}.$$

Since $F_{\nu_0}(nag) = a^{\nu_0+\rho_0} F_{\nu_0}(g)$ $(n \in N_0,\ a \in A_0)$, by substituting

$$(\xi_1, \xi_2, \xi_3, x) \to (x_1, x_2, x_3, \xi_2),$$

we have

$$\begin{aligned}
&\sum_{l_1,l_2\in\mathbf{Z}} I_{m,(m_1,m_2)}(F_{\nu_0},\gamma[l_1,l_2]) \\
&= a_1^{-\nu_{0,2}+\nu_{0,3}-2} a_2^{-\nu_{0,1}+\nu_{0,3}-1} \sum_{l_1,l_2\in\mathbf{Z}} \mathbf{e}\left(\frac{m_1c_1}{a_1}+\frac{ma_2}{a_1}\left(\frac{b_1c_2}{a_2}+b_1l_2\right)\right) \\
&\times \int_{\frac{c_1}{a_1}+l_1}^{1+\frac{c_1}{a_1}+l_1} \int_{\mathbf{R}} \int_0^1 \int_{\left(\frac{c_1}{a_1}+l_1\right)\xi_2+\frac{c_2}{a_1a_2}+\frac{l_2}{a_1}}^{1+\left(\frac{c_1}{a_1}+l_1\right)\xi_2+\frac{c_2}{a_1a_2}+\frac{l_2}{a_1}} F_{\nu_0}(w_5n[x_1,x_2,x_3]g) \\
&\times \mathbf{e}\left(-m_1x_1-\frac{ma_2}{a_1}x_2+\left(\frac{ma_2}{a_1}-m_2\right)\xi_2\right) dx_3d\xi_2dx_2dx_1.
\end{aligned}$$

We decompose $l_2=a_1l_2'+r$ ($l_2'\in\mathbf{Z}$, $0\le r<a_1$) and sum up the terms for $l_1,l_2'\in\mathbf{Z}$. Then we have

$$\begin{aligned}
&\sum_{l_1,l_2\in\mathbf{Z}} I_{m,(m_1,m_2)}(F_{\nu_0},\gamma[l_1,l_2]) \\
&= a_1^{-\nu_{0,2}+\nu_{0,3}-2} a_2^{-\nu_{0,1}+\nu_{0,3}-1} \mathbf{e}\left(\frac{m_1c_1}{a_1}+\frac{ma_2}{a_1}\frac{b_1c_2}{a_2}\right) \\
&\times \left\{\sum_{0\le r<a_1} \mathbf{e}\left(\frac{ma_2b_1r}{a_1}\right)\right\} \int_0^1 \mathbf{e}\left(\left(\frac{ma_2}{a_1}-m_2\right)\xi_2\right) d\xi_2 \\
&\times \int_{\mathbf{R}^3} F_{\nu_0}(w_5n[x_1,x_2,x_3]g)\mathbf{e}\left(-m_1x_1-\frac{ma_2}{a_1}x_2\right) dx_1dx_2dx_3.
\end{aligned}\tag{5.3}$$

Since $\gcd(b_1,a_1)=1$, we have

$$\sum_{0\le r<a_1} \mathbf{e}\left(\frac{ma_2b_1r}{a_1}\right) = \sum_{r' \bmod a_1} \mathbf{e}\left(\frac{ma_2r'}{a_1}\right) = \begin{cases} a_1, & \text{if } a_1|ma_2, \\ 0, & \text{otherwise.}\end{cases}\tag{5.4}$$

Applying (5.2) and (5.4) to (5.3), we obtain the statement (iii). □

By this lemma, we see that $E_{m_1,m_2}(F^1_{(\nu_1,\pi;q,v)};g)$ is equal to

$$\begin{cases} 0 & \text{if } m_1=0, m_2=0, \\ c_\pi(m_1)|m_1|^{-\tilde{\nu}_\pi} W_{m_1,0}(w_1,F^0_{(\nu_0,\sigma_\pi;q,v)};g) & \text{if } m_1\neq 0, m_2=0, \\ \tilde{C}(m_1,m_2)|m_2|^{-\tilde{\nu}_\pi} W_{m_1,m_2}(w_5,F^0_{(\nu_0,\sigma_\pi;q,v)};g) & \text{if } m_2\neq 0, \end{cases}$$

where $\nu_0=(\nu_{1,1}+\tilde{\nu}_\pi,\nu_{1,1}-\tilde{\nu}_\pi,\nu_{1,2})$ and

$$\tilde{C}(m_1,m_2) = \sum_{\substack{(a_i,b_i,c_i)\in\Sigma_\Gamma,\\ i\in\{1,2\},\ a_2|m_2a_1}} c_\pi\left(\frac{m_2a_1}{a_2}\right)(a_1a_2)^{-\nu_{1,1}+\nu_{1,2}-1}\mathbf{e}\left(\frac{m_1c_1}{a_1}+\frac{m_2b_1c_2}{a_2}\right).$$

We evaluate $\tilde{C}(m_1,m_2)$ as follows.

Lemma 5.5. *For a nonzero integers m_1 and m_2, we define $C_{(\nu_1,\pi)}(m_1,m_2)$ by the following:*

(1) $C_{(\nu_1,\pi)}(m_1,m_2)=\operatorname{sgn}(m_2)^{\delta(\pi)}C_{(\nu_1,\pi)}(|m_1|,|m_2|)$.

(2) *If* $\gcd(m_1m_2,m_1'm_2')=1$,

$$C_{(\nu_1,\pi)}(m_1m_1',m_2m_2')=C_{(\nu_1,\pi)}(m_1,m_2)C_{(\nu_1,\pi)}(m_1',m_2').$$

(3) *For a prime number p and $n_1,n_2\in\mathbf{Z}_{\ge 0}$,*

$$C_{(\nu_1,\pi)}(p^{n_1},p^{n_2})=S_{n_1,n_2}(\alpha_\pi(p)p^{\nu_{1,1}},\beta_\pi(p)p^{\nu_{1,1}},p^{\nu_{1,2}}).$$

Here $S_{n_1,n_2}(\alpha,\beta,\gamma)$ is the Schur polynomial defined by

$$S_{n_1,n_2}(\alpha,\beta,\gamma)=\frac{\begin{vmatrix} 1 & \alpha^{n_1+1} & \alpha^{n_1+n_2+2} \\ 1 & \beta^{n_1+1} & \beta^{n_1+n_2+2} \\ 1 & \gamma^{n_1+1} & \gamma^{n_1+n_2+2} \end{vmatrix}}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}.$$

For $m_1\in\mathbf{Z}$, $0\neq m_2\in\mathbf{Z}$ and $\nu_1=(\nu_{1,1},\nu_{1,2})\in\mathbf{C}^2$ such that $\mathbf{Re}(\nu_{1,1}-\nu_{1,2})>3/2$, we have

$$\tilde{C}(m_1,m_2)=\begin{cases} \dfrac{L(\nu_{1,1}-\nu_{1,2},\pi)}{L(\nu_{1,1}-\nu_{1,2}+1,\pi)}c_\pi(m_2) & \text{if } m_1=0, \\ \dfrac{|m_1|^{-2\nu_{1,1}}|m_2|^{-\nu_{1,1}}}{L(\nu_{1,1}-\nu_{1,2}+1,\pi)}C_{(\nu_1,\pi)}(m_1,m_2) & \text{otherwise.} \end{cases}$$

Proof. In this proof, we use the following well-known formula (See for example, [8, Proposition 3.1.7]):

$$\sum_{\substack{0\le r<c \\ \gcd(r,c)=1}}\mathbf{e}\left(\frac{nr}{c}\right)=\sum_{0<l|\gcd(n,c)} l\mu\left(\frac{c}{l}\right) \tag{5.5}$$

for $c\in\mathbf{Z}_{\ge1}$ and $n\in\mathbf{Z}$. Here $\mu(n)$ $(n\in\mathbf{Z}_{\ge1})$ is *the Moebius function* defined by

$$\mu(n)=\begin{cases} \displaystyle\prod_{p:\text{ prime},\ p|n}(-1)^{\mathrm{ord}_p(n)} & \text{if } n\text{: square-free}, \\ 0 & \text{otherwise}, \end{cases}$$

where $\mathrm{ord}_p(n)$ is the largest integer e such that $p^e|n$.

We see that $\tilde{C}(m_1,m_2)$ is equal to

$$\sum_{a_1,a_2>0,\ a_2|m_2a_1} c_\pi\left(\frac{m_2a_1}{a_2}\right)(a_1a_2)^{-\nu_{1,1}+\nu_{1,2}-1}R(m_1,m_2,a_1,a_2) \tag{5.6}$$

where

$$R(m_1,m_2,a_1,a_2)=\sum_{\substack{0\le b_1,c_1<a_1 \\ b_1c_1\equiv1 \bmod a_1}}\mathbf{e}\left(\frac{m_1c_1}{a_1}\right)\sum_{\substack{0\le c_2<a_2 \\ \gcd(c_2,a_2)=1}}\mathbf{e}\left(\frac{m_2b_1c_2}{a_2}\right).$$

By the formula (5.5), we have

$$R(m_1,m_2,a_1,a_2)=\sum_{\substack{0\le b_1,c_1<a_1 \\ b_1c_1\equiv1 \bmod a_1}}\mathbf{e}\left(\frac{m_1c_1}{a_1}\right)\sum_{0<l_2|\gcd(m_2b_1,a_2)} l_2\mu\left(\frac{a_1}{l_2}\right).$$

If $a_2|m_2a_1$ and $\gcd(b_1,a_1)=1$, we see that $l_2|\gcd(m_2b_1,a_2)$ is equivalent to $l_2|\gcd(m_2,a_2)$. Hence, again by the formula (5.5), we have

$$R(m_1,m_2,a_1,a_2)=\sum_{0<l_1|\gcd(m_1,a_1)} l_1\mu\left(\frac{a_1}{l_1}\right)\sum_{0<l_2|\gcd(m_2,a_2)} l_2\mu\left(\frac{a_2}{l_2}\right)$$

if $a_2|m_2a_1$. Therefore, (5.6) becomes

$$\sum_{\substack{a_1,a_2>0 \\ a_2|m_2a_1}}\ \sum_{\substack{0<l_1|\gcd(m_1,a_1) \\ 0<l_2|\gcd(m_2,a_2)}} c_\pi\left(\frac{m_2a_1}{a_2}\right)(a_1a_2)^{-\nu_{1,1}+\nu_{1,2}-1}l_1l_2\mu\left(\frac{a_1}{l_1}\right)\mu\left(\frac{a_2}{l_2}\right). \tag{5.7}$$

From this expression, we have $\tilde{C}(m_1,m_2)=\mathrm{sgn}(m_2)^{\delta(\pi)}\tilde{C}(|m_1|,|m_2|)$ and may assume $m_1\ge 0, m_2>0$.

Decomposing $a_i = a_i' l_i$ $(i=1,2)$, (5.7) becomes

$$\sum_{\substack{0<l_1|m_1\\ 0<l_2|m_2}} \sum_{\substack{a_1',a_2'>0\\ a_2'l_2|m_2a_1'l_1}} c_\pi\left(\frac{m_2a_1'l_1}{a_2'l_2}\right)\mu(a_1')\mu(a_2')(a_1'a_2')^{-\nu_{1,1}+\nu_{1,2}-1}(l_1l_2)^{-\nu_{1,1}+\nu_{1,2}}.$$

By the definition of the Moebius function, it becomes

$$\begin{aligned}(5.8)\qquad &\sum_{\substack{0<l_1|m_1\\ 0<l_2|m_2}} \sum_{\substack{a_1',a_2'>0:\text{ square-free}\\ a_2'l_2|m_2a_1'l_1}} \prod_{p:\text{ prime}} c_\pi\left(p^{\mathrm{ord}_p(m_2a_1'l_1/a_2'l_2)}\right)\\ &\times\left(-p^{-\nu_{1,1}+\nu_{1,2}-1}\right)^{\mathrm{ord}_p(a_1'a_2')}\left(p^{-\nu_{1,1}+\nu_{1,2}}\right)^{\mathrm{ord}_p(l_1l_2)}\\ &=\prod_{p:\text{ prime}}\sum_{\substack{0\le\lambda_1\le\mathrm{ord}_p(m_1)\\ 0\le\lambda_2\le\mathrm{ord}_p(m_2)}}\sum_{\substack{\alpha_1,\alpha_2\in\{0,1\}\\ \alpha_2+\lambda_2\le\mathrm{ord}_p(m_2)+\alpha_1+\lambda_1}} c_\pi\left(p^{\mathrm{ord}_p(m_2)+\alpha_1+\lambda_1-\alpha_2-\lambda_2}\right)\\ &\times\left(-p^{-\nu_{1,1}+\nu_{1,2}-1}\right)^{\alpha_1+\alpha_2}\left(p^{-\nu_{1,1}+\nu_{1,2}}\right)^{\lambda_1+\lambda_2}.\end{aligned}$$

Here we put $\mathrm{ord}_p(0)=+\infty$.

From (3.7), we obtain the relation

$$(5.9)\qquad c_\pi(p^{n_1})c_\pi(p^{n_2}) = \sum_{e=0}^{\min\{n_1,n_2\}} c_\pi(p^{n_1+n_2-2e})$$

for $n_1,n_2\in\mathbf{Z}_{\ge1}$ and a prime number p. By using this relation, we have

$$\begin{aligned}&\sum_{\substack{\alpha_1,\alpha_2\in\{0,1\}\\ \alpha_2+\lambda_2\le\mathrm{ord}_p(m_2)+\alpha_1+\lambda_1}} c_\pi\left(p^{\mathrm{ord}_p(m_2)+\alpha_1+\lambda_1-\alpha_2-\lambda_2}\right)\left(-p^{-\nu_{1,1}+\nu_{1,2}-1}\right)^{\alpha_1+\alpha_2}\\ &=\left(1-c_\pi(p)p^{-(\nu_{1,1}-\nu_{1,2}+1)}+p^{-2(\nu_{1,1}-\nu_{1,2}+1)}\right)c_\pi\left(p^{\mathrm{ord}_p(m_2)+\lambda_1-\lambda_2}\right)\end{aligned}$$

for $0\le\lambda_i\le\mathrm{ord}_p(m_i)$ $(i=1,2)$. Therefore, (5.8) becomes

$$(5.10)\qquad \frac{1}{L(\nu_{1,1}-\nu_{1,2}+1,\pi)}\prod_{p:\text{ prime}}\sum_{\substack{0\le\lambda_1\le\mathrm{ord}_p(m_1)\\ 0\le\lambda_2\le\mathrm{ord}_p(m_2)}} c_\pi\left(p^{\mathrm{ord}_p(m_2)+\lambda_1-\lambda_2}\right)\left(p^{-\nu_{1,1}+\nu_{1,2}}\right)^{\lambda_1+\lambda_2}.$$

Now, we consider the case $m_1=0$. By using the relation (5.9), we have

$$\begin{aligned}\sum_{\lambda_1=0}^{\infty}\sum_{\lambda_2=0}^{n} c_\pi\left(p^{n+\lambda_1-\lambda_2}\right)\left(p^{-\nu_{1,1}+\nu_{1,2}}\right)^{\lambda_1+\lambda_2} &= \sum_{\lambda_1'=0}^{\infty}\sum_{\lambda_2=0}^{\min\{\lambda_1',n\}} c_\pi\left(p^{n+\lambda_1'-2\lambda_2}\right)\left(p^{-\nu_{1,1}+\nu_{1,2}}\right)^{\lambda_1'}\\ &=\sum_{\lambda_1'=0}^{\infty} c_\pi(p^n)c_\pi(p^{\lambda_1'})\left(p^{-\nu_{1,1}+\nu_{1,2}}\right)^{\lambda_1'}\end{aligned}$$

for $n\in\mathbf{Z}_{\ge0}$ and a prime number p. Therefore, we have

$$\tilde{C}(0,m_2)=\frac{\sum_{l=0}^{\infty}c_\pi(l)l^{-\nu_{1,1}+\nu_{1,2}}}{L(\nu_{1,1}-\nu_{1,2}+1,\pi)}c_\pi(m_2)=\frac{L(\nu_{1,1}-\nu_{1,2},\pi)}{L(\nu_{1,1}-\nu_{1,2}+1,\pi)}c_\pi(m_2).$$

Finally, we prove the case $m_1\neq0$. By (3.8), for $n_1,n_2\in\mathbf{Z}_{\ge0}$ and a prime number p, we have

$$(\alpha_\pi(p)p^{\nu_{1,1}}-\beta_\pi(p)p^{\nu_{1,1}})(\beta_\pi(p)p^{\nu_{1,1}}-p^{\nu_{1,2}})(p^{\nu_{1,2}}-\alpha_\pi(p)p^{\nu_{1,1}})$$

$$\times \sum_{\lambda_1=0}^{n_1} \sum_{\lambda_2=0}^{n_2} c_\pi \left(p^{n_2+\lambda_1-\lambda_2}\right) \left(p^{-\nu_{1,1}+\nu_{1,2}}\right)^{\lambda_1+\lambda_2}$$
$$= -\alpha_\pi(p)\beta_\pi(p)p^{3\nu_{1,1}}(\beta_\pi(p)^{-1}p^{-\nu_{1,1}+\nu_{1,2}}-1)(\alpha_\pi(p)^{-1}p^{-\nu_{1,1}+\nu_{1,2}}-1)$$
$$\times \sum_{\lambda_1=0}^{n_1} \sum_{\lambda_2=0}^{n_2} (\alpha_\pi(p)^{n_2+\lambda_1-\lambda_2+1} - \beta_\pi(p)^{n_2+\lambda_1-\lambda_2+1}) \left(p^{-\nu_{1,1}+\nu_{1,2}}\right)^{\lambda_1+\lambda_2}.$$

Since $\beta_\pi(p) = \alpha_\pi(p)^{-1}$, it becomes

$$(5.11) \quad -p^{3\nu_{1,1}}(\alpha_\pi(p)p^{-\nu_{1,1}+\nu_{1,2}}-1)(\alpha_\pi(p)^{-1}p^{-\nu_{1,1}+\nu_{1,2}}-1)$$
$$\times \left\{ \alpha_\pi(p)^{n_2+1} \sum_{\lambda_1=0}^{n_1} \left(\alpha_\pi(p)p^{-\nu_{1,1}+\nu_{1,2}}\right)^{\lambda_1} \sum_{\lambda_2=0}^{n_2} \left(\alpha_\pi(p)^{-1}p^{-\nu_{1,1}+\nu_{1,2}}\right)^{\lambda_2} \right.$$
$$\left. - \alpha_\pi(p)^{-n_2-1} \sum_{\lambda_1=0}^{n_1} \left(\alpha_\pi(p)^{-1}p^{-\nu_{1,1}+\nu_{1,2}}\right)^{\lambda_1} \sum_{\lambda_2=0}^{n_2} \left(\alpha_\pi(p)p^{-\nu_{1,1}+\nu_{1,2}}\right)^{\lambda_2} \right\}$$
$$= -p^{3\nu_{1,1}}\Big\{ \alpha_\pi(p)^{n_2+1}\big((\alpha_\pi(p)p^{-\nu_{1,1}+\nu_{1,2}})^{n_1+1}-1\big)\big((\alpha_\pi(p)^{-1}p^{-\nu_{1,1}+\nu_{1,2}})^{n_2+1}-1\big)$$
$$- \alpha_\pi(p)^{-n_2-1}\big((\alpha_\pi(p)^{-1}p^{-\nu_{1,1}+\nu_{1,2}})^{n_1+1}-1\big)\big((\alpha_\pi(p)p^{-\nu_{1,1}+\nu_{1,2}})^{n_2+1}-1\big)\Big\}.$$

Here we use the equality $(x-1)\sum_{\lambda=0}^{n} x^\lambda = x^{n+1}-1$. We can check that (5.11) is equal to

$$p^{-(2n_1+n_2)\nu_{1,1}} \begin{vmatrix} 1 & (\alpha_\pi(p)p^{\nu_{1,1}})^{n_1+1} & (\alpha_\pi(p)p^{\nu_{1,1}})^{n_1+n_2+2} \\ 1 & (\beta_\pi(p)p^{\nu_{1,1}})^{n_1+1} & (\beta_\pi(p)p^{\nu_{1,1}})^{n_1+n_2+2} \\ 1 & (p^{\nu_{1,2}})^{n_1+1} & (p^{\nu_{1,2}})^{n_1+n_2+2} \end{vmatrix}$$

by direct computation. Hence we have

$$\sum_{\lambda_1=0}^{n_1} \sum_{\lambda_2=0}^{n_2} c_\pi \left(p^{n_2+\lambda_1-\lambda_2}\right) \left(p^{-\nu_{1,1}+\nu_{1,2}}\right)^{\lambda_1+\lambda_2} = p^{-(2n_1+n_2)\nu_{1,1}} S_{n_1,n_2}(\alpha_\pi(p)p^{\nu_{1,1}}, \beta_\pi(p)p^{\nu_{1,1}}, p^{\nu_{1,2}}).$$

Applying this equality to (5.10), we obtain the assertion for $m_1 \neq 0$. □

From the above arguments, we have the following theorem.

Theorem 5.6. *The Fourier-Whittaker coefficients $E_{m_1,m_2}(F^1_{(\nu_1,\pi;q,v)};g)$ of the Eisenstein series $E(F^1_{(\nu_1,\pi;q,v)};g)$ are expressed in terms of the Jacquet integrals as follows:*

$$E_{m_1,m_2}(F^1_{(\nu_1,\pi;q,v)};g)$$
$$= \begin{cases} 0 & \text{if } m_1 = 0, m_2 = 0, \\ c_\pi(m_1)|m_1|^{-\tilde{\nu}_\pi} W_{m_1,0}(w_1, F^0_{(\nu_0,\sigma_\pi;q,v)};g) & \text{if } m_1 \neq 0, m_2 = 0, \\ \dfrac{L(\nu_{1,1}-\nu_{1,2},\pi)}{L(\nu_{1,1}-\nu_{1,2}+1,\pi)} c_\pi(m_2)|m_2|^{-\tilde{\nu}_\pi} W_{0,m_2}(w_5, F^0_{(\nu_0,\sigma_\pi;q,v)};g) & \text{if } m_1 = 0, m_2 \neq 0, \\ \dfrac{C_{(\nu_1,\pi)}(m_1,m_2)|m_1|^{-2\nu_{1,1}}|m_2|^{-\nu_{1,1}-\tilde{\nu}_\pi}}{L(\nu_{1,1}-\nu_{1,2}+1,\pi)} W_{m_1,m_2}(w_5, F^0_{(\nu_0,\sigma_\pi;q,v)};g) & \text{if } m_1 \neq 0, m_2 \neq 0, \end{cases}$$

for $m_1, m_2 \in \mathbf{Z}$. Here we take $C_{(\nu_1,\pi)}(m_1,m_2)$ as in Lemma 5.5 and $\nu_0 = (\nu_{1,1}+\tilde{\nu}_\pi, \nu_{1,1}-\tilde{\nu}_\pi, \nu_{1,2})$.

Remark 5.7. In adelic setting, this theorem is not essentially new except for the archimedean part. For the unramified non-archimedean places, there exist the explicit formulas of the non-degenerate Fourier-Whittaker coefficients and the constant terms of the Eisenstein series in more general situation. See [19] and its references for details. Here we remark that the sum

$$\sum_{m_1\in\mathbf{Z}} E_{m_1,0}(F^1_{(\nu_1,\pi;q,v)};g), \qquad \left(\text{resp. } \sum_{m_2\in\mathbf{Z}} E_{0,m_2}(F^1_{(\nu_1,\pi;q,v)};g)\right)$$

of the degenerate terms is the constant term of the Eisenstein series with respect to the unipotent radical N_1 (resp. $N_2 = w_5\theta(N_1)w_5^{-1}$) of the maximal parabolic subgroup P_1 (resp. $P_2 = w_5\theta(P_1)w_5^{-1}$).

6. Evaluations of the Jacquet integrals

6.1. **The radial parts of Whittaker functions.** After some computation, by inspection we find that it is convenient to introduce a system $\{v_{\mathbf{n}}\}_{\mathbf{n}\in S_l}$ of generators of $V_l^\delta = V_l$ with

$$v_{(n_1,n_2,n_3)} = x_1^{n_1}x_2^{n_2}x_3^{n_3} \mod r^2\cdot\tilde{V}_{l-2}, \qquad S_l = \{(n_1,n_2,n_3)\in\mathbf{Z}^3_{\geq 0} \mid n_1+n_2+n_3 = l\}.$$

Here $\{v_{\mathbf{n}}\}_{\mathbf{n}\in S_l}$ is closed under the action of the Weyl group up to sign, i.e.

$$\tau_l^\delta(w_1)v_{(n_1,n_2,n_3)} = (-1)^{n_2}v_{(n_2,n_1,n_3)}, \qquad \tau_l^\delta(w_2)v_{(n_1,n_2,n_3)} = (-1)^{n_3}v_{(n_1,n_3,n_2)}.$$

Of course, when $l\geq 2$, it is not linearly independent and is not a basis.

For a K-homomorphism $V_l^\delta \ni v \mapsto W(v;g)\in C^\infty_{\mathrm{mg}}(N_0\backslash G;\psi)$, we have

$$W(v;g) = \psi(n_0(g))W(\tau_l^\delta(\kappa_0(g))v;a_0(g)), \quad g\in G.$$

Here $g = n_0(g)a_0(g)\kappa_0(g)$ is the Iwasawa decomposition of $g\in G$ with $n_0(g)\in N_0, a_0(g)\in A_0$ and $\kappa_0(g)\in K$. Hence $v\mapsto W(v;g)$ is characterized by $\{W(v_{\mathbf{n}};a[y_1,y_2,y_3])\}_{\mathbf{n}\in S_l}$. We call $W(v_{\mathbf{n}};a[y_1,y_2,y_3])$ *the* **n**-*component of the* K-*homomorphism* $v\mapsto W(v;g)$.

The purpose of this section is to evaluate the **n**-component of

$$v\mapsto W_{c_1,c_2}(w_i,F^0_{(\nu_0,\sigma_\pi;q,v)};g)$$

corresponding to the minimal K-type of $I_{\nu_1}(\pi)$.

6.2. **The minimal K-type.** For a K-module H, we denote by $H[\tau_l^\delta]$ the τ_l^δ-isotypic component of H. From the arguments in §4.2 and §4.3, we have the following.

Lemma 6.1. *(i) The set of K-types of $I_{\nu_1}(\pi_{(\tilde{\nu},\sigma)})$ is given by $\{\tau_\delta^0\}\cup\{\tau_l^0 \mid l\geq 2\}$ for $\tilde{\nu}\in\mathbf{C}$ and $\sigma = \sigma_{(\delta,0,0)}$ ($\delta\in\{0,1\}$). Then the minimal K-type τ_δ^0 occurs with multiplicity one in $I_{\nu_1}(\pi_{(\tilde{\nu},\sigma)})$ and the image of $I_{\nu_1}(\pi_{(\tilde{\nu},\sigma)})[\tau_\delta^0]$ under $\Phi_{(\tilde{\nu},\sigma)}$ is given by*

$$\{F^0_{(\nu_0,\sigma;0,v)} \mid v\in V_\delta^0\}$$

where $\nu_0 = (\nu_{1,1}+\tilde{\nu},\nu_{1,1}-\tilde{\nu},\nu_{1,2})$. Here $\Phi_{(\tilde{\nu},\sigma)}\colon I_{\nu_1}(\pi_{(\tilde{\nu},\sigma)})\to I(\nu_0,\sigma)$ is the homomorphism defined in §4.3.
(ii) The set of K-types of $I_{\nu_1}(D_{(k,\sigma)})$ is given by $\{\tau_l^0 \mid l\geq k\}$ for $k\in\mathbf{Z}_{\geq 2}$ and $\sigma = \sigma_{(0,\delta,0)}$ such that $k\equiv\delta \mod 2$. Then the minimal K-type τ_k^0 occurs with multiplicity one in $I_{\nu_1}(D_{(k,\sigma)})$ and the image of $I_{\nu_1}(D_{(k,\sigma)})[\tau_k^0]$ under $\Phi_{(\frac{k-1}{2},\sigma)}$ is given by

$$\{F^0_{(\nu_0,\sigma;k,v)} \mid v\in V_k^0\}$$

where $\nu_0 = (\nu_{1,1}+\frac{k-1}{2},\nu_{1,1}-\frac{k-1}{2},\nu_{1,2})$.

6.3. **Some preparation.** In order to evaluate the Jacquet integrals, we prepare the following lemmas.

Lemma 6.2. *For $n = n[x_1, x_2, x_3]$ ($x_1, x_2, x_3 \in \mathbf{R}$), the Iwasawa decomposition $w_5 n = n_0(w_5 n) a_0(w_5 n) \kappa_0(w_5 n)$ is described as follows:*

$$n_0(w_5 n) = n\left[\frac{-(x_1x_2 - x_3)x_1 - x_2}{D_2}, \ \frac{-x_1 - x_2x_3}{D_1}, \ \frac{x_3}{D_1}\right] \in N_0,$$
$$a_0(w_5 n) = a\left[\sqrt{D_1}/D_2, \ \sqrt{D_2}/D_1, \ \sqrt{D_1}\right] \in A_0,$$
$$\kappa_0(w_5 n) = (\kappa_{ij}) = \begin{pmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{21} & \kappa_{22} & \kappa_{23} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} \end{pmatrix} \in SO(3) \subset K$$

where

$D_1 = 1 + x_1^2 + x_3^2, \quad D_2 = 1 + x_2^2 + (x_1x_2 - x_3)^2, \quad D_3 = 1 + x_1^2,$
$\kappa_{11} = (x_1x_2 - x_3)/\sqrt{D_2}, \quad \kappa_{12} = -x_2/\sqrt{D_2}, \quad \kappa_{13} = 1/\sqrt{D_2}, \quad \kappa_{21} = (x_1 + x_2x_3)/\sqrt{D_1D_2},$
$\kappa_{22} = \{x_3(x_1x_2 - x_3) - 1\}/\sqrt{D_1D_2}, \quad \kappa_{23} = \{-(x_1x_2 - x_3)x_1 - x_2\}/\sqrt{D_1D_2},$
$\kappa_{31} = 1/\sqrt{D_1}, \quad \kappa_{32} = x_1/\sqrt{D_1}, \quad \kappa_{33} = x_3/\sqrt{D_1}.$

Proof. By direct computation, we obtain the assertion. □

We denote by $\delta_{i,j}$ the Kronecker delta, i.e.

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 6.3. *(i) For $\delta \in \{0,1\}$ and $\mathbf{n} = (n_1, n_2, n_3) \in S_\delta$, the value of $F^0_{(\nu_0, \sigma_{(\delta,0,0)}; 0, v_\mathbf{n})}$ at $\kappa = (\kappa_{ij}) \in SO(3)$ is given by*

$$F^0_{(\nu_0, \sigma_{(\delta,0,0)}; 0, v_\mathbf{n})}(\kappa) = 2\kappa_{31}^{n_1}\kappa_{32}^{n_2}\kappa_{33}^{n_3}.$$

(ii) For $k \in \mathbf{Z}_{\geq 0}$, $\mathbf{n} = (n_1, n_2, n_3) \in S_k$ and $\delta \in \{0,1\}$ such that $\delta \equiv k \mod 2$, the value of $F^0_{(\nu_0, \sigma_{(0,\delta,0)}; k, v_\mathbf{n})}$ at $\kappa = (\kappa_{ij})_{ij} \in SO(3)$ is given by

$$F^0_{(\nu_0, \sigma_{(0,\delta,0)}; k, v_\mathbf{n})}(\kappa) = 2^{-k} \sum_{\varepsilon \in \{\pm 1\}} \left\{ \prod_{i=1}^{3} (\kappa_{1i} + \varepsilon\sqrt{-1}\kappa_{2i})^{n_i} \right\}.$$

Proof. Because of the definition of the flat section $F^0_{(\nu_0, \sigma; q, v_\mathbf{n})}$, it suffices to compute the values of the matrix coefficients $F_{(\delta_3, \pm q, v_\mathbf{n})}$ at $\kappa \in SO(3)$.

At first, we compute the values of the matrix coefficients at 1_3. Since

$$v_{(0,0,\delta)} = v_0^{(\delta)}, \qquad v_{(1,0,0)} = \frac{1}{2}(v_1^{(1)} - v_{-1}^{(1)}), \qquad v_{(0,1,0)} = \frac{-\sqrt{-1}}{2}(v_1^{(1)} + v_{-1}^{(1)}),$$

we have

$$F_{(0,0,v_\mathbf{n})}(1_3) = \langle v_0^{(\delta)*}, v_\mathbf{n} \rangle = \delta_{n_3,\delta} \tag{6.1}$$

for $\delta \in \{0,1\}$ and $\mathbf{n} = (n_1, n_2, n_3) \in S_\delta$.

By the binomial theorem

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i, \qquad \binom{n}{i} = \frac{n!}{(n-i)!i!}, \tag{6.2}$$

we have

$$x_1^{n_1}x_2^{n_2}x_3^{n_3} = \left\{\frac{1}{2}(z+\bar{z})\right\}^{n_1}\left\{\frac{-\sqrt{-1}}{2}(z-\bar{z})\right\}^{n_2} x_3^{n_3}$$
$$= \frac{(-\sqrt{-1})^{n_2}}{2^{n_1+n_2}}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}(-1)^j\binom{n_1}{i}\binom{n_2}{j} z^{n_1+n_2-i-j}\bar{z}^{i+j}x_3^{n_3}$$

with $z = x+\sqrt{-1}y$ and $\bar{z} = x-\sqrt{-1}y$. Since

$$(-1)^{m_2}v^{(k)}_{m_1-m_2} = z^{m_1}\bar{z}^{m_2}x_3^{m_3} \mod r^2\cdot\tilde{V}_{k-2}$$

for $(m_1,m_2,m_3)\in S_k$, there exists some $\{c_{\mathbf{n}}(q)\}_{-k<q<k}$ such that

$$v_{\mathbf{n}} = \delta_{n_3,0}\frac{(-\sqrt{-1})^{n_2}}{2^{n_1+n_2}}\{v^{(k)}_k + (-1)^{n_1}v^{(k)}_{-k}\} + \sum_{q=-k+1}^{k-1} c_{\mathbf{n}}(q)v^{(k)}_q$$

for $\mathbf{n} = (n_1,n_2,n_3)\in S_k$. Hence we have

$$F_{(0,\varepsilon k,v_{\mathbf{n}})}(1_3) = \langle v^{(k)*}_{\varepsilon k}, v_{\mathbf{n}}\rangle = \delta_{n_3,0}\varepsilon^{n_1}(-\sqrt{-1})^{n_2}2^{-k}. \tag{6.3}$$

By direct computation, for $l\in\mathbf{Z}_{\geq 0}$, $-l\leq q\leq l$ and $\mathbf{n} = (n_1,n_2,n_3)\in S_l$, we have

$$F_{(0,q,v_{\mathbf{n}})}(\kappa) = F_{(0,q,\tau_l^0(\kappa)v_{\mathbf{n}})}(1_3)$$
$$= \sum_{\substack{a_{ij}\in\mathbf{Z}_{\geq 0},\ 1\leq i,j\leq 3\\ a_{1j}+a_{2j}+a_{3j}=n_j}} \left\{\prod_{r=1}^{3}\frac{n_r!}{a_{1r}!a_{2r}!a_{3r}!}\kappa_{1r}^{a_{1r}}\kappa_{2r}^{a_{2r}}\kappa_{3r}^{a_{3r}}\right\} F_{(0,q,v_{\mathbf{n}'(a_{ij})})}(1_3)$$

where $\mathbf{n}'(a_{ij}) = (n'_1,n'_2,n'_3)$ with $n'_r = a_{r1}+a_{r2}+a_{r3}$. Applying (6.1) and (6.3) to this, we obtain the value of the matrix coefficients at κ which we need. □

6.4. **The case of** $I_{\nu_1}(\pi_{(\tilde{\nu},\sigma_{(\delta,0,0)})})$. For some (c_1,c_2,i), we give the explicit formulas of the $\mathbf{n}$-component of

$$V^0_\delta \ni v\mapsto W_{c_1,c_2}(w_i, F^0_{(\nu_0,\sigma;0,v)}; g)\in C^\infty_{\mathrm{mg}}(N_0\backslash G;\psi_{c_1,c_2})$$

as follows.

Proposition 6.4 ([2],[27]). *For $\tilde{\nu}\in\mathbf{C}$ and $\nu_1 = (\nu_{1,1},\nu_{1,2})\in\mathbf{C}^2$, we put $\nu_0 = (\nu_{1,1}+\tilde{\nu},\nu_{1,1}-\tilde{\nu},\nu_{1,2})$ and assume $\mathbf{Re}(\tilde{\nu})>0$ and $\mathbf{Re}(\nu_{1,1}-\nu_{1,2}-\tilde{\nu})>0$. We set*

$$W_{c_1,c_2}(w_i, F^0_{(\nu_0,\sigma;0,v_{\mathbf{n}})}; a[y_1,y_2,y_3]) = (-1)^\delta(-1)^{n_1}(\sqrt{-1})^{n_2}y_1y_2(y_2y_3)^{2\nu_{1,1}+\nu_{1,2}}\tilde{W}^{(\delta;i)}_{c_1,c_2,\mathbf{n}}(y_1,y_2)$$

where $\sigma = \sigma_{(\delta,0,0)}$ ($\delta\in\{0,1\}$). For $c_1,c_2\neq 0$ and $\mathbf{n} = (n_1,n_2,n_3)\in S_\delta$, we have

$$\tilde{W}^{(\delta;1)}_{c_1,0,\mathbf{n}}(y_1,y_2) = \delta_{n_1+n_2,0}\frac{(-1)^\delta 4|c_1|^{\tilde{\nu}}}{\Gamma_{\mathbf{R}}(2\tilde{\nu}+1)}y_1^{\nu_{1,1}}y_2^{-\nu_{1,2}}K_{\tilde{\nu}}(2\pi|c_1|y_1),$$

$$\tilde{W}^{(\delta;5)}_{0,c_2,\mathbf{n}}(y_1,y_2) = \delta_{n_2+n_3,0}\left\{\prod_{\varepsilon\in\{\pm1\}}\frac{\Gamma_{\mathbf{R}}(\nu_{1,1}-\nu_{1,2}+\varepsilon\tilde{\nu}+\delta)}{\Gamma_{\mathbf{R}}(\nu_{1,1}-\nu_{1,2}+1+\varepsilon\tilde{\nu}+\delta)}\right\}$$
$$\times\frac{4|c_2|^{\tilde{\nu}}}{\Gamma_{\mathbf{R}}(2\tilde{\nu}+1)}y_1^{\nu_{1,2}}y_2^{-\nu_{1,1}}K_{\tilde{\nu}}(2\pi|c_2|y_2),$$

$$\tilde{W}^{(\delta;5)}_{c_1,c_2,\mathbf{n}}(y_1,y_2) = \frac{\mathrm{sgn}(c_1)^{n_2+n_3}\,\mathrm{sgn}(c_2)^{\delta+n_1+n_2}2|c_1|^{-\nu_{1,2}}|c_2|^{\nu_{1,1}+\tilde{\nu}}}{\Gamma_{\mathbf{R}}(2\tilde{\nu}+1)\left\{\prod_{\varepsilon\in\{\pm1\}}\Gamma_{\mathbf{R}}(\nu_{1,1}-\nu_{1,2}+1+\varepsilon\tilde{\nu}+\delta)\right\}}$$
$$\times\frac{1}{(4\pi\sqrt{-1})^2}\int_{s_2}\int_{s_1}V^{(\delta)}_{\mathbf{n}}(s_1,s_2)(|c_1|y_1)^{-s_1}(|c_2|y_2)^{-s_2}ds_1ds_2,$$

where

$$V_{\mathbf{n}}^{(\delta)}(s_1,s_2) = \frac{\Gamma_{\mathbf{R}}(s_1+\nu_{1,2}+n_2+n_3)\Gamma_{\mathbf{R}}(s_2-\nu_{1,2}+n_1+n_2)}{\Gamma_{\mathbf{R}}(s_1+s_2+n_1+n_3)} \times \prod_{\varepsilon\in\{\pm1\}} \Gamma_{\mathbf{R}}(s_1+\nu_{1,1}+\varepsilon\tilde{\nu}+n_1)\Gamma_{\mathbf{R}}(s_2-\nu_{1,1}+\varepsilon\tilde{\nu}+n_3),$$

and the paths of integrations $\int_{s_i}$ are the vertical lines from $\mathbf{Re}(s_i)-\sqrt{-1}\infty$ to $\mathbf{Re}(s_i)+\sqrt{-1}\infty$ with sufficiently large real parts to keep the poles of the integrand on its left. Here $K_\nu(z)$ is the K-Bessel function (See for example, [3, p.66 (6.5)]*).*

Proof. The statement for $\delta=0$ is found in [2]. When $c_1c_2\neq 0$, the explicit formula for $W_{c_1,c_2}(w_5, F^0_{(\nu_0,\sigma_{(1,0,0)};0,v_{\mathbf{n}})}; a[y_1,y_2,y_3])$ is also found in [27, Proposition 7.2]. Here we slightly modify the normalization by Lemma 6.3 (i). Moreover, we obtain the explicit formula for $W_{c_1,0}(w_1, F^0_{(\nu_0,\sigma_{(1,0,0)};0,v_{\mathbf{n}})}; a[y_1,y_2,y_3])$ from (3.4), Lemma 5.3 (i), Lemma 6.3 (i) and the formula in [4, p.11 (7)]. As in the proof of Proposition 6.13, modifying the computation in [27, §6], we obtain the explicit formula of $W_{0,c_2}(w_5, F^0_{(\nu_0,\sigma_{(1,0,0)};0,v_{\mathbf{n}})}; a[y_1,y_2,y_3])$. □

Corollary 6.5. *We use the notation in Proposition 6.4. Then for $v\in V_\delta^0$, the Jacquet integrals $W_{c_1,0}(w_1, F^0_{(\nu_0,\sigma;0,v)}; g)$, $W_{0,c_2}(w_5, F^0_{(\nu_0,\sigma;0,v)}; g)$ and $W_{c_1,c_2}(w_5, F^0_{(\nu_0,\sigma;0,v)}; g)$ have the meromorphic continuations to all $(\tilde{\nu},\nu_1)\in\mathbf{C}\times\mathbf{C}^2$ and satisfy the functional equations*

$$W_{c_2,0}(w_1, F^0_{(\nu_0',\sigma;0,v)}; w_5{}^tg^{-1}) = \left\{\prod_{\varepsilon\in\{\pm1\}} \frac{\Gamma_{\mathbf{R}}(\nu_{1,1}-\nu_{1,2}+1+\varepsilon\tilde{\nu}+\delta)}{\Gamma_{\mathbf{R}}(\nu_{1,1}-\nu_{1,2}+\varepsilon\tilde{\nu}+\delta)}\right\} W_{0,c_2}(w_5, F^0_{(\nu_0,\sigma;0,v)}; g),$$

$$W_{c_2,c_1}(w_5, F^0_{(\nu_0',\sigma;0,v)}; w_5{}^tg^{-1}) = (-1)^\delta \operatorname{sgn}(c_1c_2)^\delta |c_1c_2|^{-\nu_{1,1}+\nu_{1,2}}|c_1/c_2|^{\tilde{\nu}} \times \left\{\prod_{\varepsilon\in\{\pm1\}} \frac{\Gamma_{\mathbf{R}}(\nu_{1,1}-\nu_{1,2}+1+\varepsilon\tilde{\nu}+\delta)}{\Gamma_{\mathbf{R}}(-\nu_{1,1}+\nu_{1,2}+1+\varepsilon\tilde{\nu}+\delta)}\right\} W_{c_1,c_2}(w_5, F^0_{(\nu_0,\sigma;0,v)}; g)$$

where $\nu_0' = (-\nu_{1,1}+\tilde{\nu}, -\nu_{1,1}-\tilde{\nu}, -\nu_{1,2})$.

6.5. **The Dirac-Schmid eigen-equations.** Since the values of $F^0_{(\nu_0,\sigma;k,v_{\mathbf{n}})}$ in Lemma 6.3 (ii) are complicated, it seems to be difficult to evaluate the corresponding Jacquet integrals, directly. To avoid this difficulty, we prepare some relation among their $\mathbf{n}$-components, which comes from $(\mathfrak{g}_{\mathbf{C}},K)$-module structure of $I_{\nu_1}(D_{(k,\sigma_{(0,\delta,0)})})$. The arguments in this subsection are essentially same as those in §4 of Part 2.

We put $I_3 = E_{11}+E_{22}+E_{33}\in\mathfrak{g}$ and decompose $\mathfrak{p} = Z_{\mathfrak{p}}\oplus\mathfrak{p}_0$ with

$$Z_{\mathfrak{p}} = \mathbf{R}I_3, \qquad \mathfrak{p}_0 = \{X\in\mathfrak{p} \mid \operatorname{tr}(X)=0\}.$$

Here $\operatorname{tr}(X)$ means the trace of X.

If we put $K_{ij} = E_{ij}-E_{ji}\in\mathfrak{k}$, then $\{K_{ij} \mid 1\le i<j\le 3\}$ is a basis of $\mathfrak{k}$. We denote the differential of $(\tau_l^\delta, V_l^\delta)$ by (τ_l, V_l) since the action of $\mathfrak{k}_{\mathbf{C}}$ does not depend on δ. By direct computation, we see that the action of K_{ij} on $v_{\mathbf{n}}$ is given by

$$\tau_l(K_{ij})v_{\mathbf{n}} = -n_iv_{\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j} + n_jv_{\mathbf{n}-\mathbf{e}_j+\mathbf{e}_i} \tag{6.4}$$

for $1\le i\neq j\le 3$ and $\mathbf{n}=(n_1,n_2,n_3)\in S_l$. Here $\mathbf{e}_1=(1,0,0)$, $\mathbf{e}_2=(0,1,0)$, $\mathbf{e}_3=(0,0,1)$ and $v_{\mathbf{n}}=0$ if $\mathbf{n}\notin S_l$.

It is known that $\mathfrak{p}_{0\mathbf{C}}$ becomes a $\mathfrak{k}_{\mathbf{C}}$-module via the adjoint action ad. Concerning this, we have the following lemma.

Lemma 6.6. *Let $\{X_{\mathbf{n}}\}_{\mathbf{n}\in S_2}$ be a system of generators of $\mathfrak{p}_{0\mathbf{C}}$ defined as follows:*

$$X_{2\mathbf{e}_i} = E_{ii}-\frac{1}{3}I_3, \qquad X_{\mathbf{e}_i+\mathbf{e}_j} = \frac{1}{2}(E_{ij}+E_{ji}) = E_{ij}-\frac{1}{2}K_{ij}$$

for $1 \le i \ne j \le 3$. *Then via the unique* $\mathfrak{k}_{\mathbf{C}}$*-isomorphism between* V_2 *and* $\mathfrak{p}_{0\mathbf{C}}$*, we have the identification* $v_{\mathbf{n}} = X_{\mathbf{n}}$ $(\mathbf{n} \in S_2)$.

Proof. We define the bijective linear map $\iota\colon V_2 \to \mathfrak{p}_{0\mathbf{C}}$ by $v_{\mathbf{n}} \mapsto X_{\mathbf{n}}$ $(\mathbf{n} \in S_2)$, which is well-defined because $\sum_{i=1}^{3} X_{2\mathbf{e}_i} = 0$. By direct computation, we can check $\iota(\tau_l(K_{ij})v_{\mathbf{n}}) = \mathrm{ad}(K_{ij})X_{\mathbf{n}}$ for $1 \le i < j \le 3$ and $\mathbf{n} \in S_2$. Hence ι is a $\mathfrak{k}_{\mathbf{C}}$-isomorphism and we obtain the assertion. □

For a $\mathfrak{k}_{\mathbf{C}}$-module (π, H), we denote by $H[\tau_l]$ the τ_l-isotypic component of H. The tensor product $\mathfrak{p}_{0\mathbf{C}} \otimes_{\mathbf{C}} V_l \simeq V_2 \otimes_{\mathbf{C}} V_l$ has the irreducible decomposition

$$V_2 \otimes_{\mathbf{C}} V_l = \begin{cases} V_{l+2} \oplus V_{l+1} \oplus V_l \oplus V_{l-1} \oplus V_{l-2}, & \text{if } l \ge 2, \\ V_3 \oplus V_2 \oplus V_1, & \text{if } l = 1, \\ V_2, & \text{if } l = 0. \end{cases} \tag{6.5}$$

Now we take a system of generators of $\bigoplus_{0 \le m \le 2} (\mathfrak{p}_{\mathbf{C}} \otimes_{\mathbf{C}} V_l)[\tau_{l-m}]$ as follows.

Lemma 6.7. *Let* $l \in \mathbf{Z}_{\ge 1}$. *We put* $v(\mathbf{n}, i) = \sum_{j=1}^{3} v_{\mathbf{e}_i+\mathbf{e}_j} \otimes v_{\mathbf{n}+\mathbf{e}_j}$ *for* $\mathbf{n} \in S_{l-1}$ *and* $1 \le i \le 3$. *Then* $\{v(\mathbf{n}, i) \mid \mathbf{n} \in S_{l-1},\ 1 \le i \le 3\}$ *is a system of generators of* $\bigoplus_{0 \le m \le 2} (V_2 \otimes_{\mathbf{C}} V_l)[\tau_{l-m}]$. *Here we put* $(V_2 \otimes_{\mathbf{C}} V_l)[\tau_{l'}] = 0$ *if* $l' < 0$. *Moreover, there exists a* $\mathfrak{k}_{\mathbf{C}}$*-homomorphism* $\bigoplus_{0 \le m \le 2} (V_2 \otimes_{\mathbf{C}} V_l)[\tau_{l-m}] \to V_l$ *such that* $v(\mathbf{n}, i) \mapsto v_{\mathbf{n}+\mathbf{e}_i}$.

Proof. We take $(\tilde{\tau}_l, \tilde{V}_l)$ as in §4.1 and denote the differential of $\tilde{\tau}_l$ again by $\tilde{\tau}_l$. Let $\Psi_l\colon \tilde{V}_l \to V_l$ be the natural surjective $\mathfrak{k}_{\mathbf{C}}$-homomorphism defined by $\tilde{v} \mapsto \tilde{v} \mod r^2 \cdot \tilde{V}_{l-2}$. If we put $\tilde{v}_{(n_1,n_2,n_3)} = x_1^{n_1} x_2^{n_2} x_3^{n_3}$ and $\tilde{v}(\mathbf{n}, i) = \sum_{j=1}^{3} \tilde{v}_{\mathbf{e}_i+\mathbf{e}_j} \otimes \tilde{v}_{\mathbf{n}+\mathbf{e}_j}$, then $\Psi_l(\tilde{v}_{\mathbf{n}}) = v_{\mathbf{n}}$ and $\Psi_2 \otimes \Psi_l(\tilde{v}(\mathbf{n}, i)) = v(\mathbf{n}, i)$.

Let $\tilde{V}(l, 2)$ be the the space spanned by $\{\tilde{v}(\mathbf{n}, i) \mid \mathbf{n} \in S_{l-1},\ 1 \le i \le 3\}$. By (6.4), we have

$$\begin{aligned} \tilde{\tau}_2 \otimes \tilde{\tau}_l(K_{ab})\tilde{v}(\mathbf{n}, i) = &- n_a \tilde{v}(\mathbf{n} - \mathbf{e}_a + \mathbf{e}_b, i) + n_b \tilde{v}(\mathbf{n} - \mathbf{e}_b + \mathbf{e}_a, i) \\ &- \delta_{a,i} \tilde{v}(\mathbf{n}, b) + \delta_{b,i} \tilde{v}(\mathbf{n}, a) \end{aligned} \tag{6.6}$$

for $1 \le a \ne b \le 3$, $1 \le i \le 3$ and $\mathbf{n} = (n_1, n_2, n_3) \in S_{l-1}$. Hence we see that $\tilde{V}(l, 2)$ is a $\mathfrak{k}_{\mathbf{C}}$-submodule of $\tilde{V}_2 \otimes_{\mathbf{C}} \tilde{V}_l$.

Since $\mathrm{Ker}(\Psi_2 \otimes \Psi_l) = \{(r^2 \cdot \tilde{V}_0) \otimes_{\mathbf{C}} V_l + V_2 \otimes_{\mathbf{C}} (r^2 \cdot \tilde{V}_{l-2})\}$, we see that

$$\mathrm{Ker}(\Psi_2 \otimes \Psi_l) \cap \tilde{V}(l, 2) = \begin{cases} 0, & \text{if } l \in \{1, 2\} \\ \displaystyle\bigoplus_{\substack{\mathbf{n} \in S_{l-3} \\ 1 \le i \le 3}} \mathbf{C} \left\{ \sum_{j=1}^{3} \tilde{v}(\mathbf{n} + 2\mathbf{e}_j, i) \right\}, & \text{if } l \ge 3. \end{cases}$$

Since $\Psi_2 \otimes \Psi_l(\tilde{V}(l, 2)) \simeq \tilde{V}(l, 2)/(\mathrm{Ker}(\Psi_2 \otimes \Psi_l) \cap \tilde{V}(l, 2))$, we have

$$\dim \Psi_2 \otimes \Psi_l(\tilde{V}(l, 2)) = \dim \tilde{V}(l, 2) - \dim(\mathrm{Ker}(\Psi_2 \otimes \Psi_l) \cap \tilde{V}(l, 2)) = 6l - 3.$$

From (6.5), we see that a $(6l-3)$-dimensional $\mathfrak{k}_{\mathbf{C}}$-submodule of $V_2 \otimes_{\mathbf{C}} V_l$ must be $\bigoplus_{0 \le m \le 2} (V_2 \otimes_{\mathbf{C}} V_l)[\tau_{l-m}]$. Hence $\bigoplus_{0 \le m \le 2} (V_2 \otimes_{\mathbf{C}} V_l)[\tau_{l-m}]$ coincides with $\Psi_2 \otimes \Psi_l(\tilde{V}(l, 2))$ and is spanned by $\{v(\mathbf{n}, i) \mid \mathbf{n} \in S_{l-1},\ 1 \le i \le 3\}$.

We define a surjective linear map $\tilde{\Phi}_{(l,2)}\colon \tilde{V}(l, 2) \to \tilde{V}_l$ by $\tilde{v}(\mathbf{n}, i) \mapsto \tilde{v}_{\mathbf{n}+\mathbf{e}_i}$. By (6.6), we note that $\tilde{\Phi}_{(l,2)}$ is a $\mathfrak{k}_{\mathbf{C}}$-homomorphism. Since $\tilde{\Phi}_{(l,2)}(\mathrm{Ker}(\Psi_2 \otimes \Psi_l) \cap \tilde{V}(l, 2)) \subset \mathrm{Ker}\, \Psi_l$, we can define a $\mathfrak{k}_{\mathbf{C}}$-homomorphism $\Phi_{(l,2)}\colon V(l, 2) \to V_l$ by $\Phi_{(l,2)} = \Psi_l \circ \tilde{\Phi}_{(l,2)} \circ (\Psi_2 \otimes \Psi_l)^{-1}$. From the construction of $\Phi_{(l,2)}$, we have $\Phi_{(l,2)}(v(\mathbf{n}, i)) = v_{\mathbf{n}+2\mathbf{e}_i}$. □

Lemma 6.8. *We set $\nu_1 = (\nu_{1,1}, \nu_{1,2}) \in \mathbf{C}^2$, $k \in \mathbf{Z}_{\geq 2}$ and $\sigma = \sigma_{(0,\delta,0)}$ such that $k \equiv \delta \mod 2$. Then for $1 \leq i \leq 3$ and $\mathbf{n} \in S_{k-1}$, it holds that*

$$\frac{\nu_{1,1}-\nu_{1,2}}{3} F^0_{(\nu_0,\sigma;k,v_{\mathbf{n}+\mathbf{e}_i})} = \sum_{j=1}^{3} R(X_{\mathbf{e}_i+\mathbf{e}_j}) F^0_{(\nu_0,\sigma;k,v_{\mathbf{n}+\mathbf{e}_j})}$$

where $\nu_0 = (\nu_{1,1} + \frac{k-1}{2}, \nu_{1,1} - \frac{k-1}{2}, \nu_{1,2})$.

Proof. We note that $\mathfrak{p}_{\mathbf{C}} \otimes_{\mathbf{C}} V_l \ni X \otimes v \mapsto R(X) F^0_{(\nu_0,\sigma;k,v)} \in I(\nu_0, \sigma)$ is a $\mathfrak{k}_{\mathbf{C}}$-homomorphism. Therefore, by Lemma 6.1 and 6.7, there exists some constant C such that

$$C F^0_{(\nu_0,\sigma;k,v_{\mathbf{n}+\mathbf{e}_i})} = \sum_{j=1}^{3} R(X_{\mathbf{e}_i+\mathbf{e}_j}) F^0_{(\nu_0,\sigma;k,v_{\mathbf{n}+\mathbf{e}_j})}$$

for any $1 \leq i \leq 3$ and $\mathbf{n} \in S_{k-1}$. In order to determine the constant C, we evaluate

$$C F^0_{(\nu_0,\sigma;k,v_{(k,0,0)})}(1_3) = \sum_{j=1}^{3} R(X_{\mathbf{e}_1+\mathbf{e}_j}) F^0_{(\nu_0,\sigma;k,v_{(k-1,0,0)+\mathbf{e}_j})}(1_3). \tag{6.7}$$

By Lemma 6.3 (ii), the left hand side of (6.7) is equal to $C 2^{-k+1}$. By the definition of the space $I(\nu_0, \sigma)$, we have

$$R(X_{2\mathbf{e}_1}) F(1_3) = \left(\frac{\nu_{1,1}-\nu_{1,2}}{3} + \frac{k+1}{2} \right) F(1_3),$$
$$R(X_{\mathbf{e}_i+\mathbf{e}_j}) F(1_3) = R(E_{ij}) F(1_3) - \frac{1}{2} R(K_{ij}) F(1_3) = -\frac{1}{2} R(K_{ij}) F(1_3),$$

for $F \in I(\nu_0, \sigma)$ and $1 \leq i < j \leq 3$. Moreover, by (6.4), we have

$$R(K_{ij}) F^0_{(\nu_0,\sigma;k,v_{\mathbf{n}})} = -n_i F^0_{(\nu_0,\sigma;k,v_{\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j})} + n_j F^0_{(\nu_0,\sigma;k,v_{\mathbf{n}-\mathbf{e}_j+\mathbf{e}_i})}$$

for $1 \leq i \neq j \leq 3$ and $\mathbf{n} = (n_1, n_2, n_3) \in S_k$. Therefore, by Lemma 6.3 (ii), we see that the right hand side of (6.7) is equal to $2^{-k+1}(\nu_{1,1} - \nu_{1,2})/3$. Comparing the both sides of (6.7), we have $C = (\nu_{1,1} - \nu_{1,2})/3$. $\square$

Lemma 6.9. *For $W \in C^\infty_{mg}(N_0 \backslash G; \psi_{c_1,c_2})$, we have*

$$R(X_{2\mathbf{e}_1}) W(a) = \Big(\partial_1 - \frac{1}{3} \partial_3 \Big) W(a),$$
$$R(X_{2\mathbf{e}_2}) W(a) = \Big(-\partial_1 + \partial_2 - \frac{1}{3} \partial_3 \Big) W(a),$$
$$R(X_{2\mathbf{e}_3}) W(a) = \Big(-\partial_2 + \frac{2}{3} \partial_3 \Big) W(v; a),$$
$$R(E_{i,i+1}) W(a) = 2\pi\sqrt{-1} c_i y_i W(a) \ (i \in \{1,2\}), \quad R(E_{13}) W(a) = 0,$$

where $a = a[y_1, y_2, y_3]$ and $\partial_i = y_i \dfrac{\partial}{\partial y_i}$.

Proof. By the definition of $C^\infty_{\mathrm{mg}}(N_0 \backslash G; \psi_{c_1,c_2})$ and easy computation, we obtain the assertion. $\square$

Proposition 6.10. *We set $\nu_1 = (\nu_{1,1}, \nu_{1,2}) \in \mathbf{C}^2$, $k \in \mathbf{Z}_{\geq 2}$ and $\sigma = \sigma_{(0,\delta,0)}$ such that $k \equiv \delta$ mod 2. We define the function $\tilde{W}^{(k;i)}_{c_1,c_2,\mathbf{n}}(y_1, y_2)$ by*

$$W_{c_1,c_2}(w_i, F^0_{(\nu_0,\sigma;k,v_{\mathbf{n}})}; a[y_1, y_2, 1]) = (-1)^{n_1} (\sqrt{-1})^{n_2} y_1 y_2^{2\nu_{1,1}+\nu_{1,2}+1} \tilde{W}^{(k;i)}_{c_1,c_2,\mathbf{n}}(y_1, y_2).$$

Then $\tilde{W}^{(k;i)}_{c_1,c_2,\mathbf{n}}(y_1,y_2)$ *satisfy the following system of partial differential equations:*
(i) For $(n_1,n_2,n_3)\in S_{k-1}$,

$$(2\partial_1-2\nu_{1,1}-k+1)\tilde{W}^{(k;i)}_{c_1,c_2,(n_1+1,n_2,n_3)}+4\pi c_1y_1\tilde{W}^{(k;i)}_{c_1,c_2,(n_1,n_2+1,n_3)}=0.$$

(ii) For $(n_1,n_2,n_3)\in S_{k-1}$,

$$\begin{aligned}&(-2\partial_1+2\partial_2+2\nu_{1,1}+2\nu_{1,2}+n_1-n_3)\tilde{W}^{(k;i)}_{c_1,c_2,(n_1,n_2+1,n_3)}-4\pi c_1y_1\tilde{W}^{(k;i)}_{c_1,c_2,(n_1+1,n_2,n_3)}\\&+4\pi c_2y_2\tilde{W}^{(k;i)}_{c_1,c_2,(n_1,n_2,n_3+1)}+n_2\tilde{W}^{(k;i)}_{c_1,c_2,(n_1+2,n_2-1,n_3)}-n_2\tilde{W}^{(k;i)}_{c_1,c_2,(n_1,n_2-1,n_3+2)}=0.\end{aligned}$$

(iii) For $(n_1,n_2,n_3)\in S_{k-1}$,

$$(-2\partial_2-2\nu_{1,1}+k-1)\tilde{W}^{(k;i)}_{c_1,c_2,(n_1,n_2,n_3+1)}-4\pi c_2y_2\tilde{W}^{(k;i)}_{c_1,c_2,(n_1,n_2+1,n_3)}=0.$$

(iv) For $(n_1,n_2,n_3)\in S_{k-2}$,

$$\tilde{W}^{(k;i)}_{c_1,c_2,(n_1+2,n_2,n_3)}-\tilde{W}^{(k;i)}_{c_1,c_2,(n_1,n_2+2,n_3)}+\tilde{W}^{(k;i)}_{c_1,c_2,(n_1,n_2,n_3+2)}=0.$$

Proof. From the relation $\sum_{1\le i\le 3}v_{\mathbf{n}+2\mathbf{e}_i}=0$, we obtain the statement (iv).

Since $I(\nu_0,\sigma)\ni F\mapsto W_{c_1,c_2}(w_i,F;g)\in C^\infty_{\mathrm{mg}}(N_0\backslash G;\psi_{c_1,c_2})$ is a homomorphism of $(\mathfrak{g}_{\mathbf{C}},K)$-modules, for $1\le i\le 3$ and $\mathbf{n}\in S_{k-1}$, we have

$$\frac{\nu_{1,1}-\nu_{1,2}}{3}W_{c_1,c_2}(w_i,F^0_{(\nu_0,\sigma;k,v_{\mathbf{n}+\mathbf{e}_i})};g)=\sum_{j=1}^{3}R(X_{\mathbf{e}_i+\mathbf{e}_j})W_{c_1,c_2}(w_i,F^0_{(\nu_0,\sigma;k,v_{\mathbf{n}+\mathbf{e}_j})};g)\tag{6.8}$$

from Lemma 6.8. By the statement (iv), (6.8), Lemma 6.9 and

$$\begin{aligned}&W_{c_1,c_2}(w_i,F;a[y_1,y_2,y_3])=y_3^{2\nu_{1,1}+\nu_{1,2}}W_{c_1,c_2}(w_i,F;a[y_1,y_2,1])\\&=(-1)^{n_1}(\sqrt{-1})^{n_2}y_1y_2(y_2y_3)^{2\nu_{1,1}+\nu_{1,2}}\tilde{W}^{(k;i)}_{c_1,c_2,\mathbf{n}}(y_1,y_2),\end{aligned}$$

we have the statements (i), (ii) and (iii). □

6.6. The case of $I_{\nu_1}(D_{(k,\sigma_{(0,\delta,0)})})$. In the evaluation of the Jacquet integral, prepare the following formulas play key role.

Lemma 6.11. *(i) For* $a,b,\mathbf{Re}(\nu)\in\mathbf{R}_+$,

$$a^{-\nu}=\frac{b^\nu}{\Gamma(\nu)}\int_{\mathbf{R}_+}\exp(-ab\xi)\xi^\nu\frac{d\xi}{\xi}.$$

(ii) For $a\in\mathbf{R}_+$ *and* $c\in\mathbf{R}$,

$$\int_{\mathbf{R}}\exp(-ax^2-2\pi\sqrt{-1}cx)dx=\left(\frac{\pi}{a}\right)^{\frac12}\exp\left(-\frac{\pi^2c^2}{a}\right).$$

(iii) For $y,A\in\mathbf{R}_+$ *and* $\nu\in\mathbf{C}$,

$$\int_{\mathbf{R}_+}\exp\left(-(\pi y)^2t-\frac{1}{t}\right)t^\nu\frac{dt}{t}=\frac{1}{4\pi\sqrt{-1}}\int_z\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z}{2}-\nu\right)(\pi y)^{-z}dz.$$

(iv) For $z\in\mathbf{C}$,

$$\Gamma(z)\Gamma\left(z+\frac12\right)=2^{1-2z}\sqrt{\pi}\Gamma(2z),$$

(v) For $z\in\mathbf{C}$ *and* $n\in\mathbf{Z}_{\ge0}$,

$$(z)_n=\frac{\Gamma(z+n)}{\Gamma(z)}=(-1)^n\frac{\Gamma(1-z)}{\Gamma(1-z-n)}.$$

Here $(z)_n = \prod_{i=0}^{n-1}(z+i)$.
(vi) For $a,b,c,d \in \mathbf{C}$,

$$\frac{1}{2\pi\sqrt{-1}}\int_z \Gamma(a+z)\Gamma(b+z)\Gamma(c-z)\Gamma(d-z)dz = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}.$$

Here the paths of integrations $\int_z$ *are the vertical lines from* $\mathbf{Re}(z)-\sqrt{-1}\infty$ *to* $\mathbf{Re}(z)+\sqrt{-1}\infty$ *with sufficiently large real parts to keep the poles of the integrand on its left.*

Proof. The statement (i) is obtained from the definition of the Gamma function. The statement (ii) is found in [4, p.15 (11)]. The statement (iii) is obtained from the two kinds of integral expressions of the K-Bessel function in [3, p.66 (6.5) and Lemma 1.9,1]:

$$K_\nu(y) = \frac{1}{2}\int_{\mathbf{R}} \exp\left(-\frac{y}{2}\left(t+\frac{1}{t}\right)\right)t^\nu \frac{dt}{t} = \frac{1}{4\cdot 2\pi\sqrt{-1}}\int_s \Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)\left(\frac{y}{2}\right)^{-s}ds$$

for $y>0$ and $\nu \in \mathbf{C}$. The statements (iv) and (v) are found in [36, Appendix II.3 and II.2]. The statement (vi) is Barnes' lemma, which is found in [48, §14.52]. $\square$

Moreover, we use the following formulas of the generalized Gauss hypergeometric function

$${}_{n+1}F_n\left(\begin{matrix} a_0, a_1, \cdots, a_n \\ b_1, b_2, \cdots, b_n \end{matrix}; z\right) = \sum_{m=0}^{\infty} \frac{(a_0)_m(a_1)_m\cdots(a_n)_m}{(b_1)_m(b_2)_m\cdots(b_n)_m}\frac{z^m}{m!}.$$

Lemma 6.12 ([36, §7.3.5.2 and §7.4.4.114]). *(i) For* $\mathbf{Re}(c-a-b)>0$,

$${}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

(ii) For $n \in \mathbf{Z}_{\geq 0}$ *and* $a,b,c \in \mathbf{C}$,

$${}_3F_2\left(\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2},\ a \\ b,\ \frac{3}{2}+a-b-n \end{matrix}; 1\right) = 4^{-n}\frac{(2b-2a-1)_n(2b+n-1)_n}{(b)_n\left(b-a-\frac{1}{2}\right)_n}.$$

Proposition 6.13. *For* $k \in \mathbf{Z}_{\geq 2}$ *and* $\nu_1 = (\nu_{1,1},\nu_{1,2}) \in \mathbf{C}^2$, *we put* $\nu_0 = (\nu_{1,1}+\frac{k-1}{2}, \nu_{1,1}-\frac{k-1}{2}, \nu_{1,2})$ *and assume* $\mathbf{Re}(\nu_{1,1}-\nu_{1,2}) > (k-1)/2$. *We set*

$$W_{c_1,c_2}(w_i, F^0_{(\nu_0,\sigma;k,v_{\mathbf{n}})}; a[y_1,y_2,y_3]) = (-1)^{n_1}(\sqrt{-1})^{n_2}y_1y_2(y_2y_3)^{2\nu_{1,1}+\nu_{1,2}}\tilde{W}^{(k;i)}_{c_1,c_2,\mathbf{n}}(y_1,y_2)$$

where $\sigma = \sigma_{(0,\delta,0)}$ *such that* $\delta \equiv k \mod 2$. *For* $c_1, c_2 \neq 0$ *and* $\mathbf{n} = (n_1,n_2,n_3) \in S_k$, *we have*

$$\tilde{W}^{(k;1)}_{c_1,0,\mathbf{n}}(y_1,y_2) = \delta_{n_3,0}\frac{(\sqrt{-1})^k\,\mathrm{sgn}(c_1)^{\delta+n_2}|c_1|^{k-1}}{\Gamma_{\mathbf{R}}(2k)} y_1^{\nu_{1,1}+\frac{k-1}{2}}y_2^{-\nu_{1,2}}\exp(-2\pi|c_1|y_1),$$

$$\tilde{W}^{(k;5)}_{0,c_2,\mathbf{n}}(y_1,y_2) = \delta_{n_1,0}\frac{\mathrm{sgn}(c_2)^{n_2}\Gamma_{\mathbf{C}}(\nu_{1,1}-\nu_{1,2}+\frac{k-1}{2})|c_2|^{k-1}}{\Gamma_{\mathbf{C}}(\nu_{1,1}-\nu_{1,2}+\frac{k+1}{2})\Gamma_{\mathbf{R}}(2k)} y_1^{\nu_{1,2}}y_2^{-\nu_{1,1}+\frac{k-1}{2}}\exp(-2\pi|c_2|y_2)$$

$$\begin{aligned}\tilde{W}^{(k;5)}_{c_1,c_2,\mathbf{n}}(y_1,y_2) =& \frac{\mathrm{sgn}(c_1)^{\delta+n_2+n_3}\,\mathrm{sgn}(c_2)^{n_1+n_2}|c_1|^{-\nu_{1,2}}|c_2|^{\nu_{1,1}+\frac{k-1}{2}}}{\Gamma_{\mathbf{C}}(\nu_{1,1}-\nu_{1,2}+\frac{k+1}{2})\Gamma_{\mathbf{R}}(2k)} \\ &\times \frac{1}{(4\pi\sqrt{-1})^2}\int_{s_2}\int_{s_1} V^{(k)}_{\mathbf{n}}(s_1,s_2)(|c_1|y_1)^{-s_1}(|c_2|y_2)^{-s_2}ds_1ds_2\end{aligned}$$

where

$$\begin{aligned}V^{(k)}_{\mathbf{n}}(s_1,s_2) =& \frac{\Gamma_{\mathbf{R}}(s_1+\nu_{1,2}+n_1)\Gamma_{\mathbf{R}}(s_2-\nu_{1,2}+n_3)}{\Gamma_{\mathbf{R}}(s_1+s_2+n_1+n_3)} \\ &\times \Gamma_{\mathbf{C}}\left(s_1+\nu_{1,1}+\frac{k-1}{2}\right)\Gamma_{\mathbf{C}}\left(s_2-\nu_{1,1}+\frac{k-1}{2}\right)\end{aligned}$$

and the paths of integrations $\int_{s_i}$ are the vertical lines from $\mathbf{Re}(s_i)-\sqrt{-1}\infty$ to $\mathbf{Re}(s_i)+\sqrt{-1}\infty$ with sufficiently large real parts to keep the poles of the integrand on its left.

Proof. We obtain the explicit formula for $\tilde{W}^{(k;1)}_{c_1,0,\mathbf{n}}$ from (3.4), Lemma 5.3 (i), (6.3) and the formula

$$\int_{\mathbf{R}}(u-\tau)^{-k}\mathbf{e}(uv)du=\begin{cases}\dfrac{(2\pi\sqrt{-1})^k}{(k-1)!}v^{k-1}\mathbf{e}(v\tau), & \text{if } v>0,\\ 0, & \text{if } v\le 0,\end{cases}$$

for $v\in\mathbf{R}$, $k\in\mathbf{Z}_{\ge 2}$ and $\tau\in\mathbf{C}$ such that $\mathbf{Re}(\tau)>0$. Here the above formula is obtained by the residue theorem.

Next, we evaluate

$$\tilde{W}^{(k;5)}_{c_1,c_2,\mathbf{n}}(y_1,y_2)=(-1)^{n_1}(-\sqrt{-1})^{n_2}(y_1y_2)^{-1}(y_2y_3)^{-2\nu_{1,1}-\nu_{1,2}}\\ \times\int_{\mathbf{R}^3}F^0_{(\nu_0,\sigma;k,v_\mathbf{n})}(w_5n[x_1,x_2,x_3]a[y_1,y_2,y_3])\mathbf{e}(-c_1x_1-c_2x_2)dx_1dx_2dx_3.$$

We change the variable by $(x_1,x_2,x_3)\to(y_1x_1,y_2x_2,y_1y_2x_3)$. It implies the replacement $n[x_1,x_2,x_3]a[y_1,y_2,y_3]\to a[y_1,y_2,y_3]n[x_1,x_2,x_3]$. By $w_5a[y_1,y_2,y_3]=a[y_2^{-1},y_1^{-1},y_1y_2y_3]w_5$ and the definition of the space $I(\nu_0,\sigma)$, we have

$$\tilde{W}^{(k;5)}_{c_1,c_2,\mathbf{n}}(y_1,y_2)=(-1)^{n_1}(-\sqrt{-1})^{n_2}y_1^{\nu_{1,2}}y_2^{-\nu_{1,1}-\frac{k-1}{2}}\\ \times\int_{\mathbf{R}^3}F^0_{(\nu_0,\sigma;k,v_\mathbf{n})}(w_5n[x_1,x_2,x_3])\mathbf{e}(-c_1y_1x_1-c_2y_2x_2)dx_1dx_2dx_3.$$

By using the equations in Proposition 6.10, for $c_1,c_2\neq 0$, $\{\tilde{W}^{(k;5)}_{0,c_2,\mathbf{n}}\}_{\mathbf{n}\in S_k}$ and $\{\tilde{W}^{(k;5)}_{c_1,c_2,\mathbf{n}}\}_{\mathbf{n}\in S_k}$ are determined from $\{\tilde{W}^{(k;5)}_{0,c_2,(1,0,k-1)},\tilde{W}^{(k;5)}_{0,c_2,(0,0,k)}\}$ and $\tilde{W}^{(k;5)}_{0,c_2,(0,0,k)}$, respectively. Moreover, by Lemma 6.2 and 6.3, we see that

$$\int_{\mathbf{R}}F^0_{(\nu_0,\sigma;k,v_{(1,0,k-1)})}(w_5n[x_1,x_2,x_3])dx_3$$

is an odd function of x_1. This implies $\tilde{W}^{(k;5)}_{0,c_2,(1,0,k-1)}(y_1,y_2)=0$. Hence it suffices to evaluate $\tilde{W}^{(k;5)}_{c_1,c_2,(0,0,k)}(y_1,y_2)$ for $c_1,c_2\in\mathbf{R}$ such that $c_2\neq 0$.

By Lemma 6.2 and 6.3, for $n=n[x_1,x_2,x_3]$, we have

$$F^0_{(\nu_0,\sigma;k,v_{(0,0,k)})}(w_5n)=a_0(w_5n)^{\nu_0+\rho_0}F^0_{(\nu_0,\sigma;k,v_{(0,0,k)})}(\kappa_0(w_5n))\\ =D_1^{\frac{\nu_{1,2}-\nu_{1,1}}{2}+\frac{k-3}{4}}D_2^{-\frac{k}{2}}2^{-k}(D_1D_2)^{-\frac{k}{2}}\sum_{\varepsilon\in\{\pm1\}}\left(\sqrt{D_1}-\varepsilon\sqrt{-1}((x_1x_2-x_3)x_1+x_2)\right)^k.$$

By the binomial theorem (6.2), it becomes

$$D_1^{\frac{\nu_{1,2}-\nu_{1,1}}{2}+\frac{k-3}{4}}D_2^{-\frac{k}{2}}2^{-k+1}(D_1D_2)^{-\frac{k}{2}}\sum_{0\le 2i\le k}\binom{k}{2i}\left(\sqrt{D_1}\right)^{k-2i}(-1)^i((x_1x_2-x_3)x_1+x_2))^{2i}.$$

Since $((x_1x_2-x_3)x_1+x_2)^2=-D_1+D_2D_3$, again by the binomial theorem, it becomes

$$D_1^{\frac{\nu_{1,2}-\nu_{1,1}}{2}+\frac{k-3}{4}}D_2^{-\frac{k}{2}}2^{-k+1}(D_1D_2)^{-\frac{k}{2}}\sum_{0\le 2i\le k}\binom{k}{2i}\left(\sqrt{D_1}\right)^{k-2i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}(-D_1)^{i-j}(D_2D_3)^j\\ =2^{-k+1}\sum_{0\le 2j\le k}\left\{\sum_{2j\le 2i\le k}\binom{k}{2i}\binom{i}{j}\right\}(-1)^jD_1^{\frac{\nu_{1,2}-\nu_{1,1}}{2}+\frac{k-3}{4}-j}D_2^{-k+j}D_3^j.$$

By Lemma 6.11 (iv), (v) and Lemma 6.12 (i), we have

$$\sum_{2j\le 2i\le k}\binom{k}{2i}\binom{i}{j}=\frac{\Gamma(k+1)}{\Gamma(k+1-2j)\Gamma(2j+1)}{}_2F_1\left(\begin{matrix}j-\frac{k}{2},\ j-\frac{k-1}{2}\\ j+\frac{1}{2}\end{matrix};1\right)$$
$$=\frac{\Gamma(k+1)\Gamma(j+\frac{1}{2})\Gamma(k-j)}{\Gamma(k+1-2j)\Gamma(2j+1)\Gamma(\frac{k+1}{2})\Gamma(\frac{k}{2})}=\frac{2^{k-1-2j}k\Gamma(k-j)}{\Gamma(k+1-2j)\Gamma(j+1)}.$$

Therefore, we have

$$F^0_{(\nu_0,\sigma;k,v_{(0,0,k)})}(w_5n[x_1,x_2,x_3])=\sum_{0\le 2j\le k}\frac{(-1)^jk\Gamma(k-j)}{4^j\Gamma(k+1-2j)\Gamma(j+1)}D_1^{\frac{\nu_{1,2}-\nu_{1,1}}{2}+\frac{k-3}{4}-j}D_2^{-k+j}D_3^j.$$

By the above computation, we see that

$$\tilde{W}^{(k;5)}_{c_1,c_2,(0,0,k)}(y_1,y_2)=\sum_{0\le 2j\le k}\frac{(-1)^jk\Gamma(k-j)y_1^{\nu_{1,2}}y_2^{-\nu_{1,1}-\frac{k-1}{2}}}{4^j\Gamma(k+1-2j)\Gamma(j+1)}$$
$$\times\int_{\mathbf{R}^3}D_1^{-\lambda-j}D_2^{-k+j}D_3^j\mathbf{e}(-c_1y_1x_1-c_2y_2x_2)dx_1dx_2dx_3 \tag{6.9}$$

where $\lambda=(2\nu_{1,1}-2\nu_{1,2}-k+3)/4$. We apply Lemma 6.11 (i) to (6.9) with $(a,\nu)=(D_1/D_3,\lambda+j)$, $(D_2,k-j)$ and $b=\pi|c_2|y_2$:

$$D_1^{-\lambda-j}=\frac{(\pi|c_2|y_2)^{\lambda+j}}{\Gamma(\lambda+j)}\int_{\mathbf{R}_+}\exp\left(-\pi|c_2|y_2\frac{D_1}{D_3}\xi_1\right)\left(\frac{\xi_1}{D_3}\right)^{\lambda+j}\frac{d\xi_1}{\xi_1},$$
$$D_2^{-k+j}=\frac{(\pi|c_2|y_2)^{k-j}}{\Gamma(k-j)}\int_{\mathbf{R}_+}\exp(-\pi|c_2|y_2D_2\xi_2)\xi_2^{k-j}\frac{d\xi_2}{\xi_2}.$$

Moreover, we change the variable by $x_2\to x_2+x_1x_3/D_3$. It implies

$$D_2\to 1+\frac{x_3^2}{D_3}+D_3x_2^2$$

and we have

$$\tilde{W}^{(k;5)}_{c_1,c_2,(0,0,k)}(y_1,y_2)=\sum_{0\le 2j\le k}\frac{(-1)^jky_1^{\nu_{1,2}}y_2^{-\nu_{1,1}-\frac{k-1}{2}}(\pi|c_2|y_2)^{\lambda+k}}{4^j\Gamma(k+1-2j)\Gamma(j+1)\Gamma(\lambda+j)}$$
$$\times\int_{\mathbf{R}^3}\int_{(\mathbf{R}_+)^2}\xi_1^{\lambda+j}\xi_2^{k-j}D_3^{-\lambda}\mathbf{e}\left(-c_1y_1x_1-c_2y_2\left(x_2+\frac{x_1}{D_3}x_3\right)\right)$$
$$\times\exp\left(-\pi|c_2|y_2\left(\xi_1+\xi_2+\frac{\xi_1+\xi_2}{D_3}x_3^2+D_3x_2^2\xi_2\right)\right)\frac{d\xi_1}{\xi_1}\frac{d\xi_2}{\xi_2}dx_1dx_2dx_3.$$

We integrate with respect to x_2 and x_3 by using Lemma 6.11 (ii) to find that

$$\tilde{W}^{(k;5)}_{c_1,c_2,(0,0,k)}(y_1,y_2)=\sum_{0\le 2j\le k}\frac{(-1)^jk\pi y_1^{\nu_{1,2}}y_2^{-\nu_{1,1}-\frac{k-1}{2}}(\pi|c_2|y_2)^{\lambda+k-1}}{4^j\Gamma(k+1-2j)\Gamma(j+1)\Gamma(\lambda+j)}$$
$$\times\int_{\mathbf{R}}\int_{(\mathbf{R}_+)^2}\xi_1^{\lambda+j}\xi_2^{k-j}D_3^{-\lambda}\{\xi_2(\xi_1+\xi_2)\}^{-\frac{1}{2}}\mathbf{e}(-c_1y_1x_1)$$
$$\times\exp\left(-\pi|c_2|y_2\left(\xi_1+\xi_2+\frac{1}{D_3\xi_2}+\frac{x_1^2}{D_3(\xi_1+\xi_2)}\right)\right)\frac{d\xi_1}{\xi_1}\frac{d\xi_2}{\xi_2}dx_1.$$

Changing variables from (ξ_1,ξ_2) to (u_1,u_2) by

$$\xi_1=\frac{u_2D_3}{u_1(1+u_2D_3)},\qquad \xi_2=\frac{1}{u_1(1+u_2D_3)},$$

we have

$$\tilde{W}^{(k;5)}_{c_1,c_2,(0,0,k)}(y_1,y_2) = \sum_{0\le 2j\le k} \frac{(-1)^j k\pi y_1^{\nu_{1,2}} y_2^{-\nu_{1,1}-\frac{k-1}{2}}(\pi|c_2|y_2)^{\lambda+k-1}}{4^j\Gamma(k+1-2j)\Gamma(j+1)\Gamma(\lambda+j)}$$
$$\times \int_{\mathbf{R}}\int_{(\mathbf{R}_+)^2} (1+u_2D_3)^{-\lambda-k+\frac{1}{2}} u_1^{-\lambda-k+1} u_2^{\lambda}(u_2D_3)^j$$
$$\times \exp\left(-\pi|c_2|y_2(u_1(1+u_2)+\frac{1}{u_1}) - 2\pi\sqrt{-1}c_1y_1x_1\right)\frac{du_1}{u_1}\frac{du_2}{u_2}dx_1.$$

Since

$$(u_2D_3)^j = (-1+1+u_2D_3)^j = \sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i}(1+u_2D_3)^i,$$

it becomes

$$\tilde{W}^{(k;5)}_{c_1,c_2,(0,0,k)}(y_1,y_2) = k\pi y_1^{\nu_{1,2}} y_2^{-\nu_{1,1}-\frac{k-1}{2}}(\pi|c_2|y_2)^{\lambda+k-1}$$
$$\times \sum_{0\le 2i\le k}\frac{(-1)^i}{i!}\left\{\sum_{2i\le 2j\le k}\frac{1}{4^j\Gamma(k+1-2j)\Gamma(\lambda+j)(j-i)!}\right\}$$
$$\times \int_{\mathbf{R}}\int_{(\mathbf{R}_+)^2}(1+u_2D_3)^{-\lambda-k+i+\frac{1}{2}}u_1^{-\lambda-k+1}u_2^{\lambda}$$
$$\times \exp\left(-\pi|c_2|y_2(u_1(1+u_2)+\frac{1}{u_1}) - 2\pi\sqrt{-1}c_1y_1x_1\right)\frac{du_1}{u_1}\frac{du_2}{u_2}dx_1.$$

By Lemma 6.11 (iv), (v) and Lemma 6.12 (i), we see that

$$\sum_{2i\le 2j\le k}\frac{1}{4^j\Gamma(k+1-2j)\Gamma(\lambda+j)(j-i)!} = \frac{4^{-i}}{\Gamma(\lambda+i)\Gamma(k+1-2i)}{}_2F_1\left(\begin{matrix} i-\frac{k}{2},\ i-\frac{k-1}{2} \\ \lambda+i\end{matrix};1\right)$$
$$= \frac{4^{-i}\Gamma(\lambda+k-i-\frac{1}{2})}{\Gamma(k+1-2i)\Gamma(\lambda+\frac{k}{2})\Gamma(\lambda+\frac{k-1}{2})} = \frac{2^{2\lambda+k-2-2i}\Gamma(\lambda+k-i-\frac{1}{2})}{\sqrt{\pi}\Gamma(k+1-2i)\Gamma(2\lambda+k-1)}.$$

Therefore, we have

$$\tilde{W}^{(k;5)}_{c_1,c_2,(0,0,k)}(y_1,y_2) = \sum_{0\le 2i\le k}\frac{(-1)^i k\sqrt{\pi}\Gamma(\lambda+k-i-\frac{1}{2})y_1^{\nu_{1,2}}y_2^{-\nu_{1,1}-\frac{k-1}{2}}(\pi|c_2|y_2)^{\lambda+k-1}}{2^{2-k-2\lambda+2i}i!\Gamma(k+1-2i)\Gamma(2\lambda+k-1)}$$
$$\times \int_{\mathbf{R}}\int_{(\mathbf{R}_+)^2}(1+u_2D_3)^{-\lambda-k+i+\frac{1}{2}}u_1^{-\lambda-k+1}u_2^{\lambda}$$
$$\times \exp\left(-\pi|c_2|y_2(u_1(1+u_2)+\frac{1}{u_1}) - 2\pi\sqrt{-1}c_1y_1x_1\right)\frac{du_1}{u_1}\frac{du_2}{u_2}dx_1.$$

We apply Lemma 6.11 (i) with $(a,b,\nu) = (1+u_2D_3, 1, \lambda+k-i-1/2)$:

$$(1+u_2D_3)^{-\lambda-k+i+\frac{1}{2}} = \frac{1}{\Gamma(\lambda+k-i-\frac{1}{2})}\int_{\mathbf{R}_+}\exp(-(1+u_2D_3)\xi_3)\xi_3^{\lambda+k-i-\frac{1}{2}}\frac{d\xi_3}{\xi_3}.$$

Then we have

$$\tilde{W}^{(k;5)}_{c_1,c_2,(0,0,k)}(y_1,y_2)$$
$$= \sum_{0\le 2i\le k}\frac{(-1)^i k\sqrt{\pi}y_1^{\nu_{1,2}}y_2^{-\nu_{1,1}-\frac{k-1}{2}}(\pi|c_2|y_2)^{\lambda+k-1}}{2^{2-k-2\lambda+2i}i!\Gamma(k+1-2i)\Gamma(2\lambda+k-1)}\int_{\mathbf{R}}\int_{(\mathbf{R}_+)^3}u_1^{-\lambda-k+1}u_2^{\lambda}\xi_3^{\lambda+k-i-\frac{1}{2}}$$

$$\times \exp\left(-u_2\xi_3 x_1^2 - 2\pi\sqrt{-1}c_1y_1x_1 - (1+u_2)\xi_3 - \pi|c_2|y_2(u_1(1+u_2)+\frac{1}{u_1})\right)\frac{du_1}{u_1}\frac{du_2}{u_2}\frac{d\xi_3}{\xi_3}dx_1.$$

We integrate with respect to x_1 by using Lemma 6.11 (ii) to find that

$$\begin{aligned}&\tilde{W}^{(k;5)}_{c_1,c_2,(0,0,k)}(y_1,y_2)\\ &= \sum_{0\le 2i\le k} \frac{(-1)^i k\pi y_1^{\nu_{1,2}} y_2^{-\nu_{1,1}-\frac{k-1}{2}}(\pi|c_2|y_2)^{\lambda+k-1}}{2^{2-k-2\lambda+2i}i!\Gamma(k+1-2i)\Gamma(2\lambda+k-1)}\int_{(\mathbf{R}_+)^3} u_1^{-\lambda-k+1}u_2^{\lambda-\frac{1}{2}}\xi_3^{\lambda+k-i-1}\\ &\quad\times \exp\left(-\frac{(\pi|c_1|y_1)^2}{u_2\xi_3} - (1+u_2)\xi_3 - \pi|c_2|y_2(u_1(1+u_2)+\frac{1}{u_1})\right)\frac{du_1}{u_1}\frac{du_2}{u_2}\frac{d\xi_3}{\xi_3}.\end{aligned}$$

We change the variables from (u_1,u_2,ξ_3) to (u,t_1,t_2) by

$$(u_1,u_2,\xi_3)\to(\pi|c_2|y_2t_2, u/t_1, u^{-1}).$$

Then it becomes

$$\begin{aligned}(6.10)\quad &\tilde{W}^{(k;5)}_{c_1,c_2,(0,0,k)}(y_1,y_2)\\ &= \sum_{0\le 2i\le k} \frac{(-1)^i k\pi y_1^{\nu_{1,2}} y_2^{-\nu_{1,1}-\frac{k-1}{2}}}{2^{2-k-2\lambda+2i}i!\Gamma(k+1-2i)\Gamma(2\lambda+k-1)}\int_{(\mathbf{R}_+)^3} t_1^{-\lambda+\frac{1}{2}}t_2^{-\lambda-k+1}u^{-k+i+\frac{1}{2}}\\ &\quad\times \exp\left(-(\pi|c_2|y_2)^2\frac{t_2}{t_1}u - \frac{1}{u} - (\pi|c_1|y_1)^2t_1 - \frac{1}{t_1} - (\pi|c_2|y_2)^2t_2 - \frac{1}{t_2}\right)\frac{du}{u}\frac{dt_1}{t_1}\frac{dt_2}{t_2}.\end{aligned}$$

When $c_1\ne 0$, we apply Lemma 6.11 (iii) for the integration with respect to u, t_1, t_2, successively:

$$\begin{aligned}\tilde{W}^{(k;5)}_{c_1,c_2,(0,0,k)}(y_1,y_2) &= \sum_{0\le 2i\le k} \frac{(-1)^i k\pi y_1^{\nu_{1,2}} y_2^{-\nu_{1,1}-\frac{k-1}{2}}}{2^{2-k-2\lambda+2i}i!\Gamma(k+1-2i)\Gamma(2\lambda+k-1)}\\ &\times \frac{1}{(4\pi\sqrt{-1})^3}\int_{s_2'}\int_{s_1}\int_{z'}\Gamma\left(\frac{z'}{2}\right)\Gamma\left(\frac{s_1}{2}\right)\Gamma\left(\frac{s_2'}{2}\right)\\ &\times\Gamma\left(\frac{z'}{2}+k-i-\frac{1}{2}\right)\Gamma\left(\frac{s_1-z'}{2}+\lambda-\frac{1}{2}\right)\Gamma\left(\frac{s_2'+z'}{2}+\lambda+k-1\right)\\ &\times(\pi|c_1|y_1)^{-s_1}(\pi|c_2|y_2)^{-s_2'-z'}dz'ds_1ds_2'.\end{aligned}$$

We change the variables from (s_2',z') to $(s_2,z)=(s_2'+z',z'/2)$, and apply Barne's lemma in Lemma 6.11 (vi). Then it becomes

$$\begin{aligned}(6.11)\quad \tilde{W}^{(k;5)}_{c_1,c_2,(0,0,k)}(y_1,y_2) &= \frac{k\pi 2^{2\lambda+k-2}}{\Gamma(2\lambda+k-1)}\frac{y_1^{\nu_{1,2}} y_2^{-\nu_{1,1}-\frac{k-1}{2}}}{(4\pi\sqrt{-1})^2}\\ &\times\int_{s_2}\int_{s_1}\Gamma\left(\frac{s_1}{2}\right)\Gamma\left(\frac{s_1}{2}+\lambda-\frac{1}{2}\right)\Gamma\left(\frac{s_2}{2}\right)\Gamma\left(\frac{s_2}{2}+\lambda+k-1\right)\\ &\times\left\{\sum_{0\le 2i\le k}\frac{(-1)^i\Gamma(\frac{s_1}{2}+\lambda+k-1-i)\Gamma(\frac{s_2}{2}+k-\frac{1}{2}-i)}{2^{2i}i!\Gamma(k+1-2i)\Gamma(\frac{s_1+s_2}{2}+\lambda+k-1-i)}\right\}\\ &\times(\pi|c_1|y_1)^{-s_1}(\pi|c_2|y_2)^{-s_2}ds_1ds_2.\end{aligned}$$

By Lemma 6.11 (iv), (v) and Lemma 6.12 (i), we see that

$$(6.12)\quad \sum_{0\le 2i\le k}\frac{(-1)^i\Gamma(\frac{s_1}{2}+\lambda+k-1-i)\Gamma(\frac{s_2}{2}+k-\frac{1}{2}-i)}{2^{2i}i!\Gamma(k+1-2i)\Gamma(\frac{s_1+s_2}{2}+\lambda+k-1-i)}$$

$$
\begin{aligned}
&= \frac{\Gamma\left(\frac{s_1}{2}+\lambda+k-1\right)\Gamma\left(\frac{s_2}{2}+k-\frac{1}{2}\right)}{\Gamma(k+1)\Gamma\left(\frac{s_1+s_2}{2}+\lambda+k-1\right)} {}_3F_2\left(\begin{matrix}-\frac{k}{2},\ \frac{1-k}{2},\ 2-\lambda-k-\frac{s_1+s_2}{2} \\ 2-\lambda-k-\frac{s_1}{2},\ \frac{3}{2}-k-\frac{s_2}{2}\end{matrix};1\right) \\
&= \frac{\Gamma\left(\frac{s_1}{2}+\lambda+k-1\right)\Gamma\left(\frac{s_2}{2}+k-\frac{1}{2}\right)4^{-k}(s_2-1)_k(3-2\lambda-k-s_1)_k}{\Gamma(k+1)\Gamma\left(\frac{s_1+s_2}{2}+\lambda+k-1\right)\left(2-\lambda-k-\frac{s_1}{2}\right)_k\left(\frac{s_2-1}{2}\right)_k} \\
&= \frac{2^{-s_1-s_2-2\lambda-2k+5}\pi\Gamma(s_2+k-1)\Gamma(s_1+2\lambda+k-2)}{\Gamma(k+1)\Gamma\left(\frac{s_2}{2}\right)\Gamma\left(\frac{s_1}{2}+\lambda-\frac{1}{2}\right)\Gamma\left(\frac{s_1+s_2}{2}+\lambda+k-1\right)}.
\end{aligned}
$$

Applying (6.12) to (6.11), we have

$$
\begin{aligned}
\tilde{W}^{(k;5)}_{c_1,c_2,(0,0,k)}(y_1,y_2) &= \frac{\pi^2 2^{-k+3}}{\Gamma(k)\Gamma(2\lambda+k-1)} \frac{y_1^{\nu_{1,2}}y_2^{-\nu_{1,1}-\frac{k-1}{2}}}{(4\pi\sqrt{-1})^2} \\
&\times \int_{s_2}\int_{s_1} \frac{\Gamma\left(\frac{s_1}{2}\right)\Gamma(s_1+2\lambda+k-2)\Gamma\left(\frac{s_2}{2}+\lambda+k-1\right)\Gamma(s_2+k-1)}{\Gamma\left(\frac{s_1+s_2}{2}+\lambda+k-1\right)} \\
&\times (2\pi|c_1|y_1)^{-s_1}(2\pi|c_2|y_2)^{-s_2}ds_1ds_2.
\end{aligned}
$$

By substituting $(s_1,s_2)\to(s_1+\nu_{1,2},s_2-\nu_{1,1}-(k-1)/2)$, we obtain the explicit formula of $\tilde{W}^{(k;5)}_{c_1,c_2,(0,0,k)}$.

Finally, we evaluate $\tilde{W}^{(k;5)}_{0,c_2,(0,0,k)}$. We apply Lemma 6.11 (iii) for the integration with respect to u, t_2 successively, and substitute $t_1\to t_1^{-1}$. Then we have

$$
\begin{aligned}
\tilde{W}^{(k;5)}_{0,c_2,(0,0,k)}(y_1,y_2) &= \sum_{0\le 2i\le k} \frac{(-1)^i k\pi y_1^{\nu_{1,2}}y_2^{-\nu_{1,1}-\frac{k-1}{2}}}{2^{2-k-2\lambda+2i}i!\Gamma(k+1-2i)\Gamma(2\lambda+k-1)} \\
&\times \frac{1}{(4\pi\sqrt{-1})^2}\int_{s_2'}\int_{z'}\Gamma\left(\frac{z'}{2}\right)\Gamma\left(\frac{s_2'}{2}\right)\Gamma\left(\frac{z'}{2}+k-i-\frac{1}{2}\right) \\
&\times\Gamma\left(-\frac{z'}{2}+\lambda-\frac{1}{2}\right)\Gamma\left(\frac{s_2'+z'}{2}+\lambda+k-1\right)(\pi|c_2|y_2)^{-s_2'-z'}dz'ds_2'.
\end{aligned}
$$

We change the variables from (s_2',z') to $(s_2,z)=(s_2'+z',z'/2)$, and apply Barne's lemma in Lemma 6.11 (vi). Then it becomes

$$
\begin{aligned}
\tilde{W}^{(k;5)}_{0,c_2,(0,0,k)}(y_1,y_2) =& \frac{k\pi 2^{2\lambda+k-2}\Gamma\left(\lambda-\frac{1}{2}\right)}{\Gamma(2\lambda+k-1)}\frac{y_1^{\nu_{1,2}}y_2^{-\nu_{1,1}-\frac{k-1}{2}}}{4\pi\sqrt{-1}}\int_{s_2}\Gamma\left(\frac{s_2}{2}\right)\Gamma\left(\frac{s_2}{2}+\lambda+k-1\right) \\
&\times\left\{\sum_{0\le 2i\le k}\frac{(-1)^i\Gamma(\lambda+k-1-i)\Gamma\left(\frac{s_2}{2}+k-\frac{1}{2}-i\right)}{2^{2i}i!\Gamma(k+1-2i)\Gamma\left(\frac{s_2}{2}+\lambda+k-1-i\right)}\right\}(\pi|c_2|y_2)^{-s_2}ds_2.
\end{aligned}
$$

By (6.12) for $s_1=0$, we have

$$
\tilde{W}^{(k;5)}_{0,c_2,(0,0,k)}(y_1,y_2) = \frac{\pi^2 2^{-k+3}\Gamma(2\lambda+k-2)}{\Gamma(k)\Gamma(2\lambda+k-1)}\frac{y_1^{\nu_{1,2}}y_2^{-\nu_{1,1}-\frac{k-1}{2}}}{4\pi\sqrt{-1}}\int_{s_2}\Gamma(s_2+k-1)(2\pi|c_2|y_2)^{-s_2}ds_2.
$$

By the substitution $s_2\to s_2-k+1$ and the Mellin-inversion formula, we obtain the explicit formula of $\tilde{W}^{(k;5)}_{0,c_2,(0,0,k)}$. □

Corollary 6.14. *We use the notation in Proposition 6.13. Then for $v\in V_k^0$, the Jacquet integrals $W_{c_1,0}(w_1,F^0_{(\nu_0,\sigma;k,v)};g)$, $W_{0,c_2}(w_5,F^0_{(\nu_0,\sigma;k,v)};g)$ and $W_{c_1,c_2}(w_5,F^0_{(\nu_0,\sigma;k,v)};g)$ have the meromorphic continuation to all $\nu_1\in\mathbf{C}^2$ and satisfy the functional equations*

$$
W_{c_2,0}(w_1,F^0_{(\nu_0',\sigma;k,v)};w_5{}^tg^{-1}) = \mathrm{sgn}(c_2)^\delta(-\sqrt{-1})^k\frac{\Gamma_{\mathbf{C}}\left(\nu_{1,1}-\nu_{1,2}+\frac{k+1}{2}\right)}{\Gamma_{\mathbf{C}}\left(\nu_{1,1}-\nu_{1,2}+\frac{k-1}{2}\right)}W_{0,c_2}(w_5,F^0_{(\nu_0,\sigma;k,v)};g),
$$

$$W_{c_2,c_1}(w_5, F^0_{(\nu_0',\sigma;k,v)}; w_5{}^tg^{-1}) = (-1)^k \operatorname{sgn}(c_1c_2)^\delta |c_1c_2|^{-\nu_{1,1}+\nu_{1,2}} |c_1/c_2|^{\frac{k-1}{2}} \times \frac{\Gamma_{\mathbf{C}}(\nu_{1,1}-\nu_{1,2}+\frac{k+1}{2})}{\Gamma_{\mathbf{C}}(-\nu_{1,1}+\nu_{1,2}+\frac{k+1}{2})} W_{c_1,c_2}(w_5, F^0_{(\nu_0,\sigma;0,v)}; g)$$

where $\nu_0' = (-\nu_{1,1}+(k-1)/2, -\nu_{1,1}-(k-1)/2, -\nu_{1,2})$.

Remark 6.15. In Part 2, by solving the system of partial differential equations, we obtain the explicit formula of the Whittaker function $W_{c_1,c_2}(w_5, F^0_{(\nu_0,\sigma;k,v)}; g)$ up to scalar multiple. Moreover, we also obtain the power series expressions of Whittaker functions for $I_{\nu_1}(D_{(k,\sigma_{(0,\delta,0)})})$ which do not satisfy the moderate growth property in it.

6.7. **The Fourier-Whittaker coefficients of the Eisenstein series.** Now, for a Hecke-eigen cuspidal representation (π, H_π) of M_1, we give the explicit formula of the Fourier-Whittaker coefficients of $E(F^1_{(\nu_1,\pi;q,v)}; g)$ at the minimal K-type. Put

$$(l_\pi, q_\pi, a_\pi) = \begin{cases} (\delta, 0, 1) & \text{if } \tilde\pi = \pi_{(\tilde\nu,\sigma_{(\delta,0,0)})}, \\ (k, k, 0) & \text{if } \tilde\pi = D_{(k,\sigma_{(0,0,0)})}. \end{cases}$$

Then $\tau^0_{l_\pi}$ is the minimal K-type of $I_{\nu_1}(\pi)$ and

$$I_{\nu_1}(\pi)[\tau^0_{l_\pi}] = \{F^1_{(\nu_1,\pi;q_\pi,v)} | v \in V^0_{l_\pi}\}.$$

Moreover, for $\mathbf{n} = (n_1, n_2, n_3) \in S_{l_\pi}$, we define $\zeta_{\pi,i}(\mathbf{n})$, $\hat\zeta_{\pi,i}(\mathbf{n})$ $(i \in \{1,2\})$ by

$$(\zeta_{\pi,1}(\mathbf{n}), \zeta_{\pi,2}(\mathbf{n})) = \begin{cases} (n_2+n_3, -n_2-n_3) & \text{if } \tilde\pi = \pi_{(\tilde\nu,\sigma_{(\delta,0,0)})}, \\ (n_1, 0) & \text{if } \tilde\pi = D_{(k,\sigma_{(0,0,0)})}, \end{cases}$$

$$\hat\zeta_{\pi,i}(\mathbf{n}) = \zeta_{\pi,i}(n_3, n_2, n_1) \quad (i \in \{1,2\}).$$

We define the normalized Eisenstein series $\tilde E(F^1_{(\nu_1,\pi;q_\pi,v)}; g)$ by

$$\tilde E(F^1_{(\nu_1,\pi;q_\pi,v)}; g) = 2^{-a_\pi}\Gamma_{\mathbf{R}}(2\tilde\nu_\pi + 1 + q_\pi)\Lambda(\nu_{1,1}-\nu_{1,2}+1, \pi) E(F^1_{(\nu_1,\pi;q_\pi,v)}; g).$$

From Theorem 5.6, Proposition 6.4, 6.13, we obtain the following.

Theorem 6.16. *For $v \in V^0_{l_\pi}$, we give the explicit formula of the Fourier-Whittaker coefficients $\tilde E_{m_1,m_2}(F^1_{(\nu_1,\pi;q_\pi,v)}; g)$ of the normalized Eisenstein series $\tilde E(F^1_{(\nu_1,\pi;q_\pi,v)}; g)$ as follows:*

$$\tilde E_{m_1,m_2}(F^1_{(\nu_1,\pi;q_\pi,v)}; g) = \begin{cases} 0 & \text{if } m_1 = 0, m_2 = 0, \\ \epsilon_\pi \Lambda(\nu_{1,1}-\nu_{1,2}+1, \pi) c_\pi(|m_1|)|m_1|^{-\nu_{1,1}} W^{(\nu_1,\pi)}_{m_1,0;v}(g) & \text{if } m_1 \neq 0, m_2 = 0, \\ \Lambda(\nu_{1,1}-\nu_{1,2}, \pi) c_\pi(|m_2|)|m_2|^{\nu_{1,1}} W^{(\nu_1,\pi)}_{0,m_2;v}(g) & \text{if } m_1 = 0, m_2 \neq 0, \\ C_{(\nu_1,\pi)}(|m_1|, |m_2|)|m_1|^{-2\nu_{1,1}-\nu_{1,2}} W^{(\nu_1,\pi)}_{m_1,m_2;v}(g) & \text{if } m_1 \neq 0, m_2 \neq 0, \end{cases}$$

where $V^0_{l_\pi} \ni v \mapsto W^{(\nu_1,\pi)}_{m_1,m_2;v} \in C^\infty_{mg}(N_0\backslash G; \psi_{m_1,m_2})$ is a K-homomorphism whose $\mathbf{n}$*-component is given by*

$$W^{(\nu_1,\pi)}_{m_1,m_2;v_{\mathbf{n}}}(a[y_1,y_2,y_3]) = (-1)^{\delta(\pi)}(-1)^{n_1}(\sqrt{-1})^{n_2} y_1y_2(y_2y_3)^{2\nu_{1,1}+\nu_{1,2}} \tilde W^{(\nu_1,\pi)}_{m_1,m_2,\mathbf{n}}(y_1,y_2).$$

Here the function $\tilde W^{(\nu_1,\pi)}_{m_1,m_2,\mathbf{n}}(y_1,y_2)$ is given as follows:

$$\tilde W^{(\nu_1,\pi)}_{m_1,0,\mathbf{n}}(y_1,y_2) = \delta_{\hat\zeta_{\pi,1}(\mathbf{n}),0} \operatorname{sgn}(m_1)^{n_2+n_3} \frac{y_2^{-\nu_{1,2}}}{4\pi\sqrt{-1}} \int_{s_1} L_\infty(s_1+\nu_{1,1}+\zeta_{\pi,2}(\mathbf{n}), \pi)(|m_1|y_1)^{-s_1} ds_1,$$

$$\tilde{W}^{(\nu_1,\pi)}_{0,m_2,\mathbf{n}}(y_1,y_2)=\delta_{\zeta_{\pi,1}(\mathbf{n}),0}\,\mathrm{sgn}(m_2)^{n_1+n_2}\frac{y_1^{\nu_{1,2}}}{4\pi\sqrt{-1}}\int_{s_2}L_\infty\big(s_2-\nu_{1,1}+\hat{\zeta}_{\pi,2}(\mathbf{n}),\pi\big)(|m_2|y_2)^{-s_2}ds_2,$$

$$\begin{aligned}&\tilde{W}^{(\nu_1,\pi)}_{m_1,m_2,\mathbf{n}}(y_1,y_2)\\&=\frac{\mathrm{sgn}(m_1)^{n_2+n_3}\,\mathrm{sgn}(m_2)^{n_1+n_2}}{(4\pi\sqrt{-1})^2}\int_{s_2}\int_{s_1}\frac{\Gamma_{\mathbf{R}}\big(s_1+\nu_{1,2}+\zeta_{\pi,1}(\mathbf{n})\big)\Gamma_{\mathbf{R}}\big(s_2-\nu_{1,2}+\hat{\zeta}_{\pi,1}(\mathbf{n})\big)}{\Gamma_{\mathbf{R}}(s_1+s_2+n_1+n_3)}\\&\quad\times L_\infty\big(s_1+\nu_{1,1}+\zeta_{\pi,2}(\mathbf{n}),\pi\big)L_\infty\big(s_2-\nu_{1,1}+\hat{\zeta}_{\pi,2}(\mathbf{n}),\pi\big)(|m_1|y_1)^{-s_1}(|m_2|y_2)^{-s_2}ds_1ds_2\end{aligned}$$

for $m_1,m_2\neq 0$. The paths of integrations $\int_{s_i}$ are the vertical lines from $\mathbf{Re}(s_i)-\sqrt{-1}\infty$ to $\mathbf{Re}(s_i)+\sqrt{-1}\infty$ with sufficiently large real parts to keep the poles of the integrand on its left.

Corollary 6.17. *For $v\in V^0_{l_\pi}$, $\tilde{E}(F^1_{(\nu_1,\pi;q_\pi,v)};g)$ has the analytic continuation to all $\nu_1\in\mathbf{C}^2$ and satisfies the functional equation $\tilde{E}(F^1_{(\nu_1,\pi;q_\pi,v)};g)=(-1)^{l_\pi}\tilde{E}(F^1_{(-\nu_1,\pi;q_\pi,v)};{}^tg^{-1})$.*

Proof. By Theorem 6.16, we see that $\tilde{E}_{m_1,m_2}(g)=\tilde{E}_{m_1,m_2}(F^1_{(\nu_1,\pi;q_\pi,v)};g)$ has the analytic continuation to all $\nu_1\in\mathbf{C}$, and the series

$$\sum_{m_1=-\infty}^{\infty}\tilde{E}_{m_1,0}(g)+\sum_{\gamma\in\Gamma^2_\infty\backslash\Gamma^2}\sum_{\substack{(m_1,m_2)\in\mathbf{Z}^2\\ m_2>0}}\tilde{E}_{m_1,m_2}\left(\left(\begin{array}{c|c}\gamma & O_{2,1}\\\hline O_{1,2} & 1\end{array}\right)g\right)$$

is absolutely convergent for any ν_1. This implies that $\tilde{E}(F^1_{(\nu_1,\pi;q_\pi,v)};g)$ has the analytic continuation to all $\nu_1\in\mathbf{C}^2$.

For $\phi\in\mathcal{A}(\Gamma\backslash G)$, the Fourier-Whittaker coefficient $\phi^*_{m_1,m_2}$ of $\phi^*(g)=\phi({}^tg^{-1})\in\mathcal{A}(\Gamma\backslash G)$ is given by

$$\phi^*_{m_1,m_2}(g)=\int_{\mathbf{R}^3}\phi({}^t(n[\xi_1,\xi_2,\xi_3]g)^{-1})\mathbf{e}(-m_1\xi_1-m_2\xi_2)d\xi_1d\xi_2d\xi_3.$$

Substituting $\xi_3\to-\xi_3+\xi_1\xi_2$, we have $\phi^*_{m_1,m_2}(g)=\phi_{m_2,m_1}(w_5{}^tg^{-1})$ because of the left Γ-invariance of ϕ and

$$w_5{}^tn[\xi_1,\xi_2,\xi_3]^{-1}=n[\xi_2,\xi_1,\xi_1\xi_2-\xi_3]w_5.$$

We can check $\tilde{E}_{m_2,m_1}(F^1_{(-\nu_1,\pi;q_\pi,v)};w_5{}^tg^{-1})=(-1)^{l_\pi}\tilde{E}_{m_1,m_2}(F^1_{(\nu_1,\pi;q_\pi,v)};g)$ by Theorem 6.16, (3.9) and

$$S_{n_1,n_2}(\alpha^{-1},\beta^{-1},\gamma^{-1})=S_{n_2,n_1}(\alpha,\beta,\gamma)(\alpha\beta\gamma)^{-n_1-n_2}.$$

Hence we obtain the functional equation of $\tilde{E}(F^1_{(\nu_1,\pi;q_\pi,v)};g)$. □

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