Singularities for Solutions to Schrödinger Equations
(シュレーディンガー方程式の解の特異性)

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Contents

1 Introduction and summary

We consider the Cauchy problem for the following linear Schrödinger equation

$$\begin{cases} D_t u + Hu = 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $u = u(t,x) \in L^2(\mathbb{R}^n)$ denotes the time evolution of wave function at time $t \in \mathbb{R}$; $D_t = \frac{1}{i}\partial_t$; and

$$H = -\frac{1}{2}\sum_{ij=1}^{n} \partial_{x_i} a_{ij}(x) \partial_{x_j} + W(x) \tag{1.2}$$

denotes the Hamiltonian of system, in which $W(x)$ is the potential energy.

In the vast literature, many aspects of Schrödinger equations have been extensively studied, such as spectral theory, scattering theory, and fundamental solutions, and so on, since Schrödinger equation is the fundamental equation of quantum mechanics. Also, such results have been extended to various operators in mathematical physics. (See, for example, [27].)

In this thesis we study the microlocal singularities for solutions of the above Schrödinger equations under certain assumptions on $a_{ij}(x)$ and $W(x)$. Studies of microlocal singularities of Schrödinger equations goes back (at least) to a work by Boutet de Monvel [1], which shows that the singularities of the solutions to Schrödinger equations propagate at infinite speed, different from the wave equation, which propagate at finite speed as explained by the celebrated Hörmander's singularities propagation theorem [10]. However, it does not indicate how the singularities of the solution $u(t)$ at time t is related to the initial state u_0. Craig, Kappeler and Strauss studied the propagation of singularities for the variable coefficient Schrödinger equation in 1996 ([2]). They showed that the microlocal regularity of the solution along a nontrapped geodesic follows from rapid decay of the initial state in a conic neighborhood of the asymptotic velocity of the nontrapping geodesic. This property is called the microlocal smoothing property. Since then the microlocal smoothing property has been studied by several authors. (See [3–5, 7, 11, 17, 23, 26, 30] and the references therein.)

In 2004, Hassell and Wunsch [9] studied a characterization of singularities of solutions to Schrödinger equations on scattering manifolds, which was defined by Melrose [22]. Nakamura [24] studied the same problem (on Euclidean spaces) using a completely different method and different formulation for Schrödinger equation of variable coefficients. The method of Nakamura is relatively simple and is later adopted by other works on propagation of singularities as well as other aspects. The results of Nakamura are generalized and extended to Schrödinger equations on scattering manifolds, and also to the study of analytic singularities of solutions. (See [12, 18–20, 25].)

On the other hand, the singularities for perturbed harmonic oscillators have also been studied by several authors. Smoothing property, fundamental solutions, and propagation of singularities for solutions of the harmonic oscillator equations have been studied in the following works: [6, 8, 13–16, 21, 28, 29, 31–35]. In most of these works, the authors considered the constant coefficients cases, i.e., $a_{ij}(x) = \delta_{ij}$. In this thesis we study the variable coefficients cases for the harmonic oscillators and also a related model, i.e., magnetic fields Schrödinger equations. Specifically, we study the following problems.

- Short-range perturbation of harmonic oscillators. More precisely, we will consider the case $W(x) = \frac{|x|^2}{2} + V(x)$ in (1.2), and $a_{ij}(x)$ and $V(x)$ are assumed to satisfy the following conditions:

  Assumption A. $a_{jk}(x), V(x) \in C^\infty(\mathbb{R}^n;\mathbb{R})$ for $j,k = 1,\dots,n$, and $(a_{jk}(x))_{j,k}$ is positive symmetric for each $x \in \mathbb{R}^n$. Moreover, there exists $\mu > 1$ such that for any $\alpha \in \mathbb{Z}_+^n$,

  $$\left|\partial_x^\alpha (a_{jk}(x) - \delta_{jk})\right| \le C_\alpha \langle x\rangle^{-\mu-|\alpha|},$$
  $$\left|\partial_x^\alpha V(x)\right| \le C_\alpha \langle x\rangle^{2-\mu-|\alpha|}$$

  for $x \in \mathbb{R}^n$ with some $C_\alpha > 0$, where $\langle x\rangle = \sqrt{1+|x|^2}$.

  This model is studied in Chapter 2 using an argument similar to [24]. We characterize the singularities of solutions in terms of the classical scattering data and the propagator for unperturbed harmonic oscillator $H_0 = -\frac{1}{2}\triangle + \frac{1}{2}|x|^2$.

- Short-range perturbed constant magnetic fields: We consider the Hamiltonian of the following form

$$H=\frac{1}{2}\sum_{j,k=1}^{2}\big(D_{x_j}-(Mx)_j-A_j(x)\big)a_{jk}(x)\big(D_{x_k}-(Mx)_k-A_k(x)\big)$$
$$+V(x)$$

with

$$M=\begin{pmatrix}0 & \frac{1}{2}\\ -\frac{1}{2} & 0\end{pmatrix}.$$

The conditions on the perturbations are similar to the short-range perturbed harmonic oscillators, i.e., we impose the following conditions on the coefficients:

Assumption B. $a_{jk}(x)$, $A_j(x)$, $V(x)\in C^\infty(\mathbb{R}^2;\mathbb{R})$ for $j,k\in\{1,2\}$, and $\big(a_{jk}(x)\big)_{j,k}$ is positive symmetric for each $x\in\mathbb{R}^2$. Moreover, there exists $\mu_1>1$, $\mu_2>1$, $\mu_3>0$ such that for any $\alpha\in\mathbb{Z}_+^2$,

$$\begin{aligned}&\big|\partial_x^\alpha(a_{jk}(x)-\delta_{jk})\big|\le C_\alpha\langle x\rangle^{-\mu_1-|\alpha|},\\ &\big|\partial_x^\alpha V(x)\big|\le C_\alpha\langle x\rangle^{2-\mu_2-|\alpha|},\\ &\big|\partial_x^\alpha A_j(x)\big|\le C_\alpha\langle x\rangle^{-\mu_3-|\alpha|}\end{aligned}$$

for $x\in\mathbb{R}^2$ with some $C_\alpha>0$, where $\langle x\rangle=\sqrt{1+|x|^2}$.

This model is studied in Chapter 3, using an argument similar to perturbed harmonic oscillators. The singularities of solutions are characterized by using the classical scattering data and the propagator for the constant magnetic operator $H_0=\frac{1}{2}(D_x-Mx)^2$.

- Propagation of singularities for long-range perturbations: In Chapter 4, we study the long-range perturbation of harmonic oscillators, i.e., we assume the same assumption as in Chapter 2 except for the condition $\mu>1$, which is replaced by the condition $\mu>0$ instead. We use similar argument as in [25] to construct a modified free propagator and then use this modified free propagator multiplied by the propagator of the unperturbed harmonic oscillator to characterize the singularities of solutions, combined with the modified classical scattering data.

For more detailed information we refer the introduction section of each chapter.

For the reader's convenience, here we list some notations, which are used throughout this thesis:

The Hilbert space is denoted by $\mathcal{H} = L^2(\mathbb{R}^n)$, and the space of the bounded operators on $\mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$. The Fourier transform is denoted by

$$\hat{u}(\xi) = \mathcal{F}u(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) dx,$$

and the inverse Fourier transform is denoted by $\check{u}(x) = \mathcal{F}^* u(x)$. For a smooth symbol $a(x, \xi)$ on $\mathbb{R}^{2n}$, we denote the Weyl quantization by $a^w(x, D_x)$, i.e., for $u \in \mathcal{S}(\mathbb{R}^n)$, a Schwartz function on $\mathbb{R}^n$,

$$a^w(x, D_x)u(x) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} a(\tfrac{x+y}{2}, \xi) u(y) dy\, d\xi.$$

We use the $S(m, g)$ symbol notation and we denote the wave front set of $u \in \mathcal{S}'(\mathbb{R}^n)$ by WFu (in the sense of Hörmander [10]).

2 Wave front set for perturbed harmonic oscillators

2.1 Introduction

In this chapter we consider a Schrödinger operator with variable coefficients and the harmonic potential:

$$H = -\frac{1}{2}\sum_{j,k=1}^{n} \partial_{x_j} a_{jk}(x)\partial_{x_k} + \frac{1}{2}|x|^2 + V(x)$$

on $\mathcal{H} = L^2(\mathbb{R}^n)$, $n \geq 1$. We denote the unperturbed harmonic oscillator by H_0:

$$H_0 = -\frac{1}{2}\triangle + \frac{1}{2}|x|^2 \qquad \text{on } \mathcal{H},$$

and we suppose H is a short-range perturbation of H_0 in the following sense:

Assumption A. $a_{jk}(x)$, $V(x) \in C^\infty(\mathbb{R}^n;\mathbb{R})$ for $j,k = 1,\dots,n$, and $(a_{jk}(x))_{j,k}$ is positive symmetric for each $x \in \mathbb{R}^n$. Moreover, there exists $\mu > 1$ such that for any $\alpha \in \mathbb{Z}_+^n$,

$$\begin{aligned} &\left|\partial_x^\alpha (a_{jk}(x) - \delta_{jk})\right| \leq C_\alpha \langle x\rangle^{-\mu-|\alpha|}, \\ &\left|\partial_x^\alpha V(x)\right| \leq C_\alpha \langle x\rangle^{2-\mu-|\alpha|} \end{aligned}$$

for $x \in \mathbb{R}^n$ with some $C_\alpha > 0$, where $\langle x\rangle = \sqrt{1+|x|^2}$.

Then it is well-known that H is essentially self adjoint on $C_0^\infty(\mathbb{R}^n)$, and we denote the unique self-adjoint extension by the same symbol H. We denote the symbols of H, H_0, the kinetic energy and the free Schrödinger operator by p, p_0, k and k_0, respectively. Namely, we denote

$$\begin{aligned} p(x,\xi) &= \frac{1}{2}\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k + \frac{1}{2}|x|^2 + V(x), \\ p_0(x,\xi) &= \frac{1}{2}|\xi|^2 + \frac{1}{2}|x|^2, \\ k(x,\xi) &= \frac{1}{2}\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k, \quad k_0(x,\xi) = \frac{1}{2}|\xi|^2. \end{aligned}$$

We denote the Hamilton flow generated by a symbol $a(x,\xi)$ on $\mathbb{R}^{2n}$ by $\exp(tH_a) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. We also denote

$$\pi_1(X) = x, \quad \pi_2(X) = \xi \quad \text{for } X = (x,\xi) \in \mathbb{R}^{2n}.$$

Let $(x_0,\xi_0) \in \mathbb{R}^{2n}$. (x_0,ξ_0) is called *forward (backward, resp.) nontrapping* (with respect to k) if

$$|\pi_1(\exp(tH_k)(x_0,\xi_0)| \to \infty$$

as $t \to +\infty$ ($t \to -\infty$, resp.). If (x_0,ξ_0) is forward/backward nontrapping, then it is well-known

$$(x_\pm,\xi_\pm) = \lim_{t\to\pm\infty} \exp(-tH_{k_0}) \circ \exp(tH_k)(x_0,\xi_0)$$

exists, and $S_\pm$: $(x_0,\xi_0) \mapsto (x_\pm,\xi_\pm)$ are locally diffeomorphic (see, e.g., Nakamura [24], Section 2.).

Now we present our main results of this paper. Let us recall our harmonic oscillator H_0 has a period 2π, i.e., $e^{-i2\pi H_0}\varphi = \varphi$ for $\varphi \in \mathcal{H}$. Moreover, we have

$$e^{\mp i\pi H_0}\varphi(x) = \varphi(-x), \quad \varphi \in \mathcal{H}.$$

Our first result concern the evolution by H up to time π. We denote

$$u(t) = e^{-itH}u_0, \quad u_0 \in \mathcal{H}.$$

We denote the wave front set of a distribution f by $WF(f)$.

Theorem 2.1. *(i) Suppose (x_0,ξ_0) is backward nontrapping, and let $0 < t_0 < \pi$, $u_0 \in \mathcal{H}$. Then*

$$(x_0,\xi_0) \in WF(u(t_0)) \iff (x_-,\xi_-) \in WF(e^{-it_0H_0}u_0).$$

(ii) Suppose (x_0,ξ_0) is forward nontrapping, and let $-\pi < t_0 < 0$, $u_0 \in \mathcal{H}$. Then

$$(x_0,\xi_0) \in WF(u(t_0)) \iff (x_+,\xi_+) \in WF(e^{-it_0H_0}u_0).$$

Remark 2.2. We note that microlocally e^{-itH_0} is a rotation in the phase space. More precisely, for any reasonable symbol $a = a(x,\xi)$,

$$e^{-itH_0}a^w(x,D_x)e^{itH_0} = a^w\big(\cos(t)x + \sin(t)D_x, -\sin(t)x + \cos(t)D_x\big),$$

where $a^w(x, D_x)$ denotes the Weyl-quantization of a. Hence, in particular, $(x_0, \xi_0) \in WF(e^{-itH_0}u_0)$ if and only if there exists a symbol: $a \in C_0^\infty(\mathbb{R}^{2n})$ such that $a(x_0, \xi_0) \neq 0$ and

$$\left\| a^w(\cos(t)x - \sin(t)D_x, h(\sin(t)x + \cos(t)D_x))u_0 \right\| = O(h^\infty)$$

as $h \to 0$.

At the time $t = \pm\pi$, $u(t)$ behaves differently. We denote the set of forward (backward, resp.) nontrapping points by $\mathcal{T}_+$ ($\mathcal{T}_-$, resp.) $\subset \mathbb{R}^{2n} \setminus 0 := \{(x, \xi) \in \mathbb{R}^{2n}, \xi \neq 0\}$. $S_\pm$ are diffeomorphism from $(\mathbb{R}^{2n} \setminus 0) \setminus \mathcal{T}_\pm$ to $\mathbb{R}^{2n} \setminus 0$, and hence $S_\pm^{-1}$ are well-defined from $\mathbb{R}^{2n} \setminus 0$ to $(\mathbb{R}^{2n} \setminus 0) \setminus \mathcal{T}_\pm$. We also denote the antipodal map in $\mathbb{R}^{2n}$ by Γ, i.e., $\Gamma(x, \xi) = (-x, -\xi)$.

Theorem 2.3. *(i) Suppose (x_0, ξ_0) is backward nontrapping, and let $u_0 \in \mathcal{H}$. Then*

$$(x_0, \xi_0) \in WF(u(\pi)) \iff S_+^{-1} \circ \Gamma \circ S_-(x_0, \xi_0) \in WF(u_0).$$

(ii) Suppose (x_0, ξ_0) is forward nontrapping, and let $u_0 \in \mathcal{H}$. Then

$$(x_0, \xi_0) \in WF(u(-\pi)) \iff S_-^{-1} \circ \Gamma \circ S_+(x_0, \xi_0) \in WF(u_0).$$

The microlocal singularities of solutions to Schrödinger equations have been attracted attention during the past years, especially after the publication of the break-through paper by Craig-Kappeler-Strauss [2] in 1996 (for more literature, see references of [2] and [24]). On the other hand, the singularities of solutions to the harmonic oscillator type Schrödinger equations have been studied by several authors, including Zelditch [35], Yajima [32], Kapitanski-Rodnianski-Yajima [13], Doi [8] and Wunsch [31]. Most of these works concern the case with constant coefficients with potential perturbations. In particular, if the metric is flat, i.e., if $a_{jk}(x) = \delta_{jk}$, then $S_\pm$ is the identity map, and Theorem 2.3 recovers results in [8, 13, 32, 35]. In fact, if the metric is flat, Theorem 2.1 is also obtained by Doi [8], and this paper is partially motivated by this beautiful work. Moreover, he also considered a class of long range type perturbations, i.e, when $V(x) = O(|x|)$ as $|x| \to \infty$, and demonstrated that a shift of singularities occurs. Microlocal smoothing effect for the Schrödinger equations on scattering manifolds with harmonic potential is studied by Wunsch [31].

In a sense, this paper is an analogue of a work by Nakamura [24] where the microlocal singularities of asymptotically flat Schrödinger equations is studied (see also a closely related work by Hassel-Wuncsh [9] and Ito-Nakamura [12]). The main difference (and the novelty) is the analysis of the classical trajectories with high energies. In [24], the standard classical scattering theory is sufficient to prove the propagation of singularities (with a scaling argument). However, in the presence of the harmonic potential, the high energy asymptotics of the classical trajectories is completely different from the long-time asymptotics. Thus we need to obtain precise high energy asymptotics of the trajectories using the time evolution of the harmonic oscillator, and it is carried out in Section 2.2. This situation is somewhat similar to the case of such analysis for Schrödinger equations with long-range perturbations [25], but the asymptotics itself is naturally completely different. We also note that our results have much in common with a paper by Zelditch [35], at least in spirit, and our results may be considered as generalizations of his results to variable coefficients cases.

The main results are proved in Section 2.3, and the argument is similar to [24]. However, the scaling argument is slightly more complicated, and we try to give a more transparent formulation (see the last part of Section 2.2 and the beginning of Section 2.3). In the last section, we discuss generalizations of our main theorems to the case when the harmonic potential is inhomogeneous.

2.2 High energy asymptotics of the classical flow

In this section, we study the high energy behavior of the classical flow generated by $p(x,\xi)$. More precisely, we consider the properties of $\exp(tH_p)(x_0,\lambda\xi_0)$ as $\lambda\to+\infty$. Thoughout this section, we suppose (x_0,ξ_0) is forward nontrapping, and consider the case $t>0$. The case $t\leq 0$ can be considered similarly.

For $\lambda>0$, we write

$$p^\lambda(x,\xi)=\frac{1}{2}\sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k+\frac{|x|^2}{2\lambda^2}+\frac{1}{\lambda^2}V(x),$$

$$p_0^\lambda(x,\xi)=\frac{1}{2}|\xi|^2+\frac{1}{2\lambda^2}|x|^2.$$

Then, by direct computations, we learn

$$\pi_1(\exp(tH_p)(x,\lambda\xi)) = \pi_1(\exp(\lambda t H_{p^\lambda})(x,\xi)), \tag{2.1}$$

$$\pi_2(\exp(tH_p)(x,\lambda\xi)) = \lambda\cdot\pi_2(\exp(\lambda t H_{p^\lambda})(x,\xi)). \tag{2.2}$$

Hence, it suffices to consider $\exp(tH_{p^\lambda})(x,\xi)$ for $0\le t\le \lambda t_0$, instead of $\exp(tH_p)(x,\lambda\xi)$ for $0\le t\le t_0$.

We note, for each fixed $t\in\mathbb{R}$,

$$\lim_{\lambda\to\infty}\exp(tH_{p^\lambda})(x,\xi) = \exp(tH_k)(x,\xi) \tag{2.3}$$

by the continuity of the solutions to ODEs with respect to the coefficients. Hence, if $t>0$ is large and then $\lambda>0$ is taken sufficiently large (after fixing t), $\pi_1(\exp(tH_{p^\lambda})(x_0,\xi_0))$ is far away from the origin by virtue of the nontrapping condition. The next lemma claims that this statement holds for $0\le t\le\lambda\delta$ with sufficiently small $\delta>0$.

Lemma 2.4. *There exists $\delta>0$ and a small neighborhood Ω of (x_0,ξ_0) such that*

$$|\pi_1(\exp(tH_{p^\lambda})(x,\xi))|\ge c_1t-c_2 \quad \text{for } 0\le t\le\lambda\delta,\ (x,\xi)\in\Omega$$

with some $c_1,c_2>0$.

Proof. In the following, we denote

$$\exp(tH_{p^\lambda})(x,\xi) = (y^\lambda(t;x,\xi),\eta^\lambda(t;x,\xi)).$$

By the conservation of the energy: $p^\lambda(y^\lambda(t),\eta^\lambda(t)) = \text{const.}$, and the ellipticity of the principal symbol, we easily see

$$\frac{1}{\lambda^2}|y^\lambda(t;x,\xi)|^2 + |\eta^\lambda(t;x,\xi)|^2\le C,\quad (x,\xi)\in\Omega, t\in\mathbb{R},$$

where Ω is a small neighborhood of (x_0,ξ_0). Hence, in particular, we have

$$|y^\lambda(t;x,\xi)|\le C\langle t\rangle,\quad |\eta^\lambda(t;x,\xi)|\le C$$

for $t>0$ with some $C>0$ by the Hamilton equations. On the other hand, by direct computations, we have

$$\begin{aligned}\frac{d^2}{dt^2}|y^\lambda(t)|^2 &= 2\frac{d^2}{dt^2}\left(y^\lambda\cdot\frac{dy^\lambda}{dt}\right) = 2\frac{d}{dt}\left(\sum_{j,k}a_{jk}(y^\lambda)y_j^\lambda\eta_k^\lambda\right)\\ &= 4p^\lambda(y^\lambda,\eta^\lambda)+2W(y^\lambda,\eta^\lambda),\end{aligned}$$

where

$$
\begin{aligned}
W(y^\lambda,\eta^\lambda) &= \sum_{j,k,\ell} a_{jk}(y^\lambda)(a_{k\ell}(y^\lambda)-\delta_{k\ell})\eta_j^\lambda\eta_k^\lambda \\
&\quad + \sum_{j,k,\ell,m} \frac{\partial a_{jk}}{\partial x_\ell}(y^\lambda)\, a_{\ell m}(y^\lambda)\eta_m^\lambda y_j^\lambda \eta_k^\lambda \\
&\quad - \sum_{j,k,\ell,m} a_{jk}(y^\lambda)\frac{\partial a_{\ell m}}{\partial x_k}(y^\lambda) y_j^\lambda \eta_\ell^\lambda \eta_m^\lambda - \frac{1}{\lambda^2}\sum_{j,k} a_{jk}(y^\lambda)y_j^\lambda y_k^\lambda \\
&\quad - \frac{1}{\lambda^2}\sum_{j,k} a_{jk}(y^\lambda)\frac{\partial V}{\partial x_k}(y^\lambda)y_j^\lambda - \frac{1}{\lambda^2}|y^\lambda|^2 - \frac{2}{\lambda^2}V(y^\lambda).
\end{aligned}
$$

Combining these, we learn

$$
\frac{d^2}{dt^2}|y^\lambda(t)|^2 \geq 4p^\lambda(y^\lambda,\eta^\lambda) - c_4(\langle y^\lambda\rangle^{-\mu} + \lambda^{-2}\langle y^\lambda\rangle^2).
$$

We note $p^\lambda(x_0,\xi_0) = k(x_0,\xi_0) + O(\lambda^{-2})$ and $k(x_0,\xi_0) > 0$, and hence $p^\lambda(x_0,\xi_0) > 0$ for large λ. Since $\lambda^{-2}\langle y^\lambda\rangle^2 = O(\langle t\rangle^2/\lambda^2)$, if $0 \leq t \leq \delta\lambda$ with sufficiently small $\delta > 0$, the last term is small and

$$
\frac{d^2}{dt^2}|y^\lambda(t)|^2 \geq 3p^\lambda(y^\lambda,\eta^\lambda) - c_4\langle y^\lambda\rangle^{-\mu}
$$

for the initial conditon $(x,\xi) \in \Omega$. By the nontrapping condition and (2.3), if $T_0 > 0$ is sufficiently large and λ is large (depending on T_0), then

$$
c_4\langle y^\lambda(T_0)\rangle^{-\mu} \leq p^\lambda(x,\xi) \quad \text{for } (x,\xi)\in\Omega, \text{ and } \quad \frac{d}{dt}|y^\lambda(T_0)| > 0.
$$

Then by the standard convexity argument, we learn

$$
|y^\lambda(t)|^2 \geq |y^\lambda(T_0)|^2 + p^\lambda(x,\xi)(t-T_0)^2 \quad \text{for } t \in [T_0,\delta\lambda],
$$

and this implies the assertion. □

Lemma 2.5. *Let $\delta > 0$ and Ω as in the previous lemma, and let $\sigma \in (0,\delta)$. Then*

$$
\lim_{\lambda\to\infty} \exp(-\sigma\lambda H_{p_0^\lambda}) \circ \exp(\sigma\lambda H_{p^\lambda})(x,\xi) = S_+(x,\xi),
$$

for $(x,\xi)\in\Omega$.

Proof. We denote

$$(z^\lambda(t;x,\xi),\zeta^\lambda(t;x,\xi)) = \exp(-tH_{p_0^\lambda})\circ\exp(tH_{p^\lambda})(x,\xi),$$

and we show the convergence of $(z^\lambda(\sigma\lambda),\zeta^\lambda(\sigma\lambda))$ to $S_+(x,\xi)$ for $(x,\xi)\in\Omega$. We recall

$$\exp(-tH_{p_0^\lambda})(x,\xi) = \big(\cos(\tfrac{t}{\lambda})x-\lambda\sin(\tfrac{t}{\lambda})\xi, \tfrac{1}{\lambda}\sin(\tfrac{t}{\lambda})x+\cos(\tfrac{t}{\lambda})\xi\big)$$

since p_0^λ is the scaled harmonic oscillator. Thus

$$z^\lambda(t) = \cos(\tfrac{t}{\lambda})y^\lambda(t)-\lambda\sin(\tfrac{t}{\lambda})\eta^\lambda(t),\ \zeta^\lambda = \tfrac{1}{\lambda}\sin(\tfrac{t}{\lambda})y^\lambda(t)+\cos(\tfrac{t}{\lambda})\eta^\lambda(t).$$

By direct computations, we have

(2.4)

$$\begin{aligned}
\frac{d}{dt}z_k^\lambda &= -\frac{1}{\lambda}\sin(\tfrac{t}{\lambda})y_k^\lambda+\cos(\tfrac{t}{\lambda})\frac{dy_k^\lambda}{dt}-\cos(\tfrac{t}{\lambda})\eta_k^\lambda-\lambda\sin(\tfrac{t}{\lambda})\frac{d\eta_k^\lambda}{dt}\\
&= -\frac{1}{\lambda}\sin(\tfrac{t}{\lambda})y_k^\lambda+\cos(\tfrac{t}{\lambda})\sum_j a_{jk}(y^\lambda)\eta_j^\lambda-\cos(\tfrac{t}{\lambda})\eta_k^\lambda\\
&\quad+\frac{1}{\lambda}\sin(\tfrac{t}{\lambda})y_k^\lambda+\sin(\tfrac{t}{\lambda})\Big(\frac{\lambda}{2}\sum_{i,j}\frac{\partial a_{ij}}{\partial x_k}(y^\lambda)\eta_i^\lambda\eta_j^\lambda+\frac{1}{\lambda}\frac{\partial V}{\partial x_k}(y^\lambda)\Big)\\
&= \cos(\tfrac{t}{\lambda})\sum_j(a_{jk}(y^\lambda)-\delta_{jk})\eta_j^\lambda\\
&\quad+\sin(\tfrac{t}{\lambda})\Big(\frac{\lambda}{2}\sum_{i,j}\frac{\partial a_{ij}}{\partial x_k}(y^\lambda)\eta_i^\lambda\eta_j^\lambda+\frac{1}{\lambda}\frac{\partial V}{\partial x_k}(y^\lambda)\Big)\\
&= O(\langle y^\lambda\rangle^{-\mu})+O(\lambda\langle y^\lambda\rangle^{-\mu-1}+\lambda^{-1}\langle y^\lambda\rangle^{1-\mu})\\
&= O(\langle t\rangle^{-\mu})
\end{aligned}$$

for $0\le t\le\delta\lambda$. Similarly, we have

(2.5)

$$\begin{aligned}
\frac{d}{dt}\zeta_k^\lambda &= \frac{1}{\lambda^2}\cos(\tfrac{t}{\lambda})y_k^\lambda+\frac{1}{\lambda}\sin(\tfrac{t}{\lambda})\frac{dy_k^\lambda}{dt}-\frac{1}{\lambda}\sin(\tfrac{t}{\lambda})\eta_k^\lambda+\cos(\tfrac{t}{\lambda})\frac{d\eta_k^\lambda}{dt}\\
&= \frac{1}{\lambda^2}\cos(\tfrac{t}{\lambda})y_k^\lambda+\frac{1}{\lambda}\sin(\tfrac{t}{\lambda})\sum_j a_{jk}(y^\lambda)\eta_j^\lambda-\frac{1}{\lambda}\sin(\tfrac{t}{\lambda})\eta_k^\lambda\\
&\quad-\frac{1}{\lambda^2}\cos(\tfrac{t}{\lambda})y_k^\lambda-\cos(\tfrac{t}{\lambda})\Big(\frac{1}{2}\sum_{i,j}\frac{\partial a_{ij}}{\partial x_k}(y^\lambda)\eta_i^\lambda\eta_j^\lambda+\frac{1}{\lambda^2}\frac{\partial V}{\partial x_k}(y^\lambda)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\lambda}\sin(\tfrac{t}{\lambda})\sum_j (a_{jk}(y^\lambda)-\delta_{jk})\eta_j^\lambda \\
&\quad - \cos(\tfrac{t}{\lambda})\Big(\frac{1}{2}\sum_{i,j}\frac{\partial a_{ij}}{\partial x_k}(y^\lambda)\eta_i^\lambda\eta_j^\lambda + \frac{1}{\lambda^2}\frac{\partial V}{\partial x_k}(y^\lambda)\Big) \\
&= O(\lambda^{-1}\langle y^\lambda\rangle^{-\mu}) + O(\langle y^\lambda\rangle^{-\mu-1} + \lambda^{-2}\langle y^\lambda\rangle^{-(\mu-1)}) \\
&= O(\langle t\rangle^{-\mu-1})
\end{aligned}
$$

for $0 \leq t \leq \delta\lambda$. Moreover, for each $t \in \mathbb{R}$, we have

$$
\begin{aligned}
\lim_{\lambda\to\infty}\frac{d}{dt}z_k^\lambda(t) &= \sum_j a_{jk}(\tilde{y})\tilde{\eta}_j - \tilde{\eta}_k + \frac{t}{2}\sum_{i,j}\frac{\partial a_{ij}}{\partial x_k}(\tilde{y})\tilde{y}_j\tilde{\eta}_j \\
&= \frac{d}{dt}(\tilde{y}_k - t\tilde{\eta}_k), \\
\lim_{\lambda\to\infty}\frac{d}{dt}\zeta_k^\lambda(t) &= -\frac{1}{2}\sum_{i,j}\frac{\partial a_{ij}}{\partial x_k}(\tilde{y})\tilde{\eta}_i\tilde{\eta}_j = \frac{d}{dt}\tilde{\eta}_k,
\end{aligned}
$$

where $(\tilde{y}(t), \tilde{\eta}(t)) = \exp(tH_k)(x,\xi)$. By using the dominated convergence theorem, we conclude

$$
\begin{aligned}
\lim_{\lambda\to\infty} z^\lambda(\sigma\lambda) &= x + \lim_{\lambda\to\infty}\int_0^{\sigma\lambda}\frac{dz^\lambda}{dt}dt = x + \int_0^\infty \frac{d}{dt}(\tilde{y} - t\tilde{\eta})dt \\
&= \lim_{t\to+\infty}(\tilde{y}(t) - t\tilde{\eta}(t)) = \pi_1(S_+(x,\xi)), \\
\lim_{\lambda\to\infty}\zeta^\lambda(\sigma\lambda) &= \xi + \lim_{\lambda\to\infty}\int_0^{\sigma\lambda}\frac{d\zeta^\lambda}{dt}dt = \xi + \int_0^\infty\frac{d}{dt}\tilde{\eta}(t)dt \\
&= \lim_{\tau\to+\infty}\tilde{\eta}(t) = \pi_2(S_+(x,\xi)).
\end{aligned}
$$

This completes the proof of the lemma. □

Lemma 2.6. *Let $0 < \sigma < \pi$, and let Ω be a small neighborhood of (x_0, ξ_0) as in the previous lemmas. Then*

$$
\lim_{\lambda\to\infty}\exp(-\sigma\lambda H_{p_0^\lambda})\circ\exp(\sigma\lambda H_{p^\lambda})(x,\xi) = S_+(x,\xi)
$$

for $(x,\xi)\in\Omega$.

Proof. It suffices to consider the case $\delta < \sigma < \pi$, and we fix such σ. Let $\varepsilon > 0$, and we show that if

$$
\max(|z^\lambda(\sigma\lambda) - x_+|, |\zeta^\lambda(\sigma\lambda) - \xi_+|) > \varepsilon
$$

then λ is bounded from above, where $(x_+, \xi_+) = S_+(x, \xi)$, and $(z^\lambda, \zeta^\lambda)$ is as in the proof of the previous lemma. Our claim then follows from this assertion.

We first note

$$y^\lambda(t) = \cos(\tfrac{t}{\lambda}) z^\lambda(t) + \lambda \sin(\tfrac{t}{\lambda}) \zeta^\lambda(t).$$

For the moment, we suppose $|z^\lambda(t) - x_+|$, $|\zeta^\lambda(t) - \xi_+| < \varepsilon$. Then we have

$$|y^\lambda(t)| \geq \lambda \sin(\tfrac{t}{\lambda})(|\xi_+| - \varepsilon) - (|x_+| + \varepsilon) \geq \delta_1 \lambda$$

with some $\delta_1 > 0$ provided $\delta\lambda \leq t \leq \sigma\lambda$, $\varepsilon < |\xi_+|/2$ and $\lambda \geq \lambda_0$, where δ_1 and λ_0 depend only on $|x_+|$ and $|\xi_+|$. Then, by using formulas (2.4) and (2.5), we learn

$$\left|\frac{d}{dt} z^\lambda(t)\right| \leq C\lambda^{-\mu}, \quad \left|\frac{d}{dt} \zeta^\lambda(t)\right| \leq C\lambda^{-\mu-1}.$$

Now we choose λ sufficiently large that

$$\max(|z^\lambda(\delta\lambda) - x_+|, |\zeta^\lambda(\delta\lambda) - \xi_+|) \leq \frac{\varepsilon}{2},$$

and suppose

$$\max(|z^\lambda(\sigma\lambda) - x_+|, |\zeta^\lambda(\sigma\lambda) - \xi_+|) \geq \varepsilon.$$

Then there exists $t_0 \in (\delta\lambda, \sigma\lambda)$ such that

$$\max(|z^\lambda(t_0) - x_+|, |\zeta^\lambda(t_0) - \xi_+|) = \varepsilon$$

and

$$\max(|z^\lambda(t) - x_+|, |\zeta^\lambda(t) - \xi_+|) \leq \varepsilon \quad \text{for } \delta\lambda \leq t \leq t_0..$$

By the above observation, we learn

$$\begin{aligned}
|z^\lambda(t_0) - x_+| &= \left| z^\lambda(\delta\lambda) - x_+ + \int_{\delta\lambda}^{t_0} \frac{dz^\lambda}{dt}(t)dt \right| \\
&\leq |z^\lambda(\delta\lambda) - x_+| + C(t_0 - \delta\lambda)\lambda^{-\mu} \\
&\leq |z^\lambda(\delta\lambda) - x_+| + C(\sigma - \delta)\lambda^{-(\mu-1)}, \\
|\zeta^\lambda(t_0) - \xi_+| &= \left| \zeta^\lambda(\delta\lambda) - \xi_+ + \int_{\delta\lambda}^{t_0} \frac{d\zeta^\lambda}{dt}(t)dt \right| \\
&\leq |\zeta^\lambda(\delta\lambda) - \xi_+| + C(\sigma - \delta)\lambda^{-\mu}.
\end{aligned}$$

Thus we have

$$\varepsilon = \max(|z^\lambda(t_0) - x_+|, |\zeta^\lambda(t_0) - \xi_+|) \le \frac{\varepsilon}{2} + C(\sigma - \delta)\lambda^{-(\mu-1)},$$

and hence $\lambda \le (2C(\sigma-\delta)/\varepsilon)^{1/(\mu-1)}$, and this completes the proof of the assertion. □

The next theorem follows immediately from Lemma 2.3.

Theorem 2.7. *(i) Suppose (x_0, ξ_0) is forward nontrapping, and let $0 < \sigma < \pi$. Then there exists a neighborhood Ω of (x_0, ξ_0) such that*

$$\lim_{\lambda\to\infty} \pi_1\big(\exp(-\sigma H_{p_0}) \circ \exp(\sigma H_p)(x, \lambda\xi)\big) = \pi_1(S_+(x,\xi)),$$
$$\lim_{\lambda\to\infty} \lambda^{-1}\pi_2\big(\exp(-\sigma H_{p_0}) \circ \exp(\sigma H_p)(x, \lambda\xi)\big) = \pi_2(S_+(x,\xi))$$

for $(x,\xi) \in \Omega$, and the convergence is uniform in Ω.
(ii) Suppose (x_0, ξ_0) be backward nontrapping, and let $-\pi < \sigma < 0$. Then there exists a neighborhood Ω of (x_0, ξ_0) such that

$$\lim_{\lambda\to\infty} \pi_1\big(\exp(-\sigma H_{p_0}) \circ \exp(\sigma H_p)(x, \lambda\xi)\big) = \pi_1(S_-(x,\xi)),$$
$$\lim_{\lambda\to\infty} \lambda^{-1}\pi_2\big(\exp(-\sigma H_{p_0}) \circ \exp(\sigma H_p)(x, \lambda\xi)\big) = \pi_2(S_-(x,\xi))$$

for $(x,\xi) \in \Omega$, and the convergence is uniform in Ω.

We introduce several notations as a preparation for the proof of our main results. We set

$$\begin{aligned}\ell(t; x, \xi) &= (p \circ \exp(tH_{p_0}))(x,\xi) - p_0(x,\xi)\\ &= \sum_{j,k=1}^{n} \big(a_{jk}(\cos(t)x + \sin(t)\xi) - \delta_{jk}\big)(-\sin(t)x_j + \cos(t)\xi_j)\\ &\qquad \times (-\sin(t)x_k + \cos(t)\xi_k) + V(\cos(t)x + \sin(t)\xi).\end{aligned}$$

Then it is easy to show that $\ell(t; x, \xi)$ generates the *scattering time evolution*:

$$S_t = \exp(-tH_{p_0}) \circ \exp(tH_p).$$

Similarly,

$$\ell^\lambda(t; x, \xi) = (p^\lambda \circ \exp(tH_{p_0^\lambda}))(x,\xi) - p_0^\lambda(x,\xi)$$

generates the time evolution:

$$S_t^\lambda = \exp(-tH_{p_0^\lambda}) \circ \exp(tH_{p^\lambda}).$$

We denote the scaling with respect to ξ by $\mathfrak{d}_\lambda$, i.e.,

$$\mathfrak{d}_\lambda(x,\xi) = (x, \lambda\xi) \quad \text{for } (x,\xi) \in \mathbb{R}^{2n}.$$

Then by (2.1) and (2.2), we have

$$\exp(tH_p)\circ\mathfrak{d}_\lambda = \mathfrak{d}_\lambda\circ\exp(\lambda tH_{p^\lambda}), \quad \exp(tH_{p_0})\circ\mathfrak{d}_\lambda = \mathfrak{d}_\lambda\circ\exp(\lambda tH_{p_0^\lambda}),$$

and hence we also have

$$S_t \circ \mathfrak{d}_\lambda = \mathfrak{d}_\lambda \circ S_{\lambda t}^\lambda. \tag{2.6}$$

2.3 Proof of main results

In this section, we mainly concern the case (x_0,ξ_0) is forward non-trapping, and prove the part (ii) of Theorem 2.1.

We first consider the property of $e^{itH_0}e^{-itH}$ for $t \leq 0$. Let $v_0 \in C_0^\infty(\mathbb{R}^n)$, and we consider

$$v(t) = e^{itH_0}e^{-itH}v_0.$$

Then it is easy to observe

$$\begin{aligned}\frac{d}{dt}v(t) &= -ie^{itH_0}(H-H_0)e^{-itH}v_0\\ &= -i\big(e^{itH_0}He^{-itH_0} - H_0\big)v(t) = -iL(t)v(t),\end{aligned}$$

where $L(t) = e^{itH_0}He^{-itH_0} - H_0$. We recall, for any reasonable symbol $a(x,\xi)$, we have

$$e^{itH_0}a^w(x,D_x)e^{-itH_0} = (a\circ\exp(tH_{p_0}))^w(x,D_x),$$

without the remainder terms, since $p_0(x,\xi)$ is a quadratic form in (x,ξ), and we employ the Weyl quantization. Thus we have

$$L(t) = (p\circ\exp(tH_{p_0}))^w(x,D_x) - p_0(x,D_x) = \ell^w(t;x,D_x),$$

i.e., $\ell(t;x,\xi)$ is the Weyl-symbol of $L(t)$. This is in fact expected, since $e^{itH_0}e^{-itH}$ is the quantization of S_t.

Let Ω be a small neighborhood of (x_0, ξ_0) as in the last section, and let $f \in C_0^\infty(\Omega)$ be such that $f(x_0, \xi_0) > 0$, and $f(x, \xi) \geq 0$ on $\mathbb{R}^{2n}$. We then set

$$f_h(x, \xi) = f(x, h\xi), \quad h = \lambda^{-1},$$

where $h > 0$ is our semiclassical parameter. We consider the behavior of

$$G(t) = e^{itH_0} e^{-itH} f^w(x, hD_x) e^{itH} e^{-itH_0}$$

as $h \to 0$. The operator valued function $G(t)$ satisfies the Heisenberg equation:

$$\frac{d}{dt} G(t) = -i[L(t), G(t)], \quad G(0) = f^w(x, hD_x). \tag{2.7}$$

The corresponding canonical equation of the classical mechanics is

$$\frac{\partial \psi_0}{\partial t}(t; x, \xi) = -\{\ell, \psi_0\}(t; x, \xi), \quad \psi(0; x, \xi) = f(x, h\xi)$$

and the solution is given by

$$\psi_0(t; x, \xi) = (f_h \circ S_t^{-1})(x, \xi)$$

since S_t is the Hamilton flow generated by $\ell(t; x, \xi)$. Now we note

$$f_h(x, \xi) = f(x, \xi/\lambda) = (f \circ \mathfrak{J}_\lambda^{-1})(x, \xi).$$

Hence, recalling (2.6), we learn

$$\begin{aligned} f_h \circ S_t^{-1} &= f \circ \mathfrak{J}_\lambda^{-1} \circ S_t^{-1} = f \circ (S_t \circ \mathfrak{J}_\lambda)^{-1} \\ &= f \circ (\mathfrak{J}_\lambda \circ S_{\lambda t}^\lambda)^{-1} = f \circ (S_{\lambda t}^\lambda)^{-1} \circ \mathfrak{J}_\lambda^{-1}. \end{aligned}$$

In other words, we have

$$\psi_0(t; x, \xi) = (f_h \circ S_t^{-1})(x, \xi) = (f \circ (S_{\lambda t}^\lambda)^{-1})(x, h\xi).$$

We expect

$$G(t) \sim \psi_0(t; x, D_x) = (f \circ (S_{\lambda t}^\lambda)^{-1})(x, hD_x)$$

for small $h > 0$, and we construct the asymptotic solution to the Heisenberg equation (2.7) with the principal symbol $\psi_0(t; x, \xi)$.

Lemma 2.8. *Let $-\pi < t_0 < 0$, and set $I = [t_0, 0]$. There exists $\psi(t; x, \xi) \in C_0^\infty(\mathbb{R}^{2n})$ for $t \in I$ such that*

(i) $\psi(0; x, \xi) = f(x, h\xi)$.

(ii) $\psi(t; x, \xi)$ *is supported in* $S_t \circ \mathfrak{J}_\lambda(\Omega) = \mathfrak{J}_\lambda \circ S^\lambda_{\lambda t}(\Omega)$.

(iii) For any $\alpha, \beta \in \mathbb{Z}^n_+$, *there is* $C_{\alpha\beta} > 0$ *such that*

$$\left|\partial_x^\alpha \partial_\xi^\beta \psi(t; x, \xi)\right| \le C_{\alpha\beta} h^{|\beta|}, \quad t \in I, x, \xi \in \mathbb{R}^n.$$

(iv) The principal symbol of ψ *is given by* ψ_0, *i.e., for any* $\alpha, \beta \in \mathbb{Z}^n_+$, *there is* $C_{\alpha\beta} > 0$ *such that*

$$\left|\partial_x^\alpha \partial_\xi^\beta (\psi(t; x, \xi) - \psi_0(t; x, \xi))\right| \le C_{\alpha\beta} h^{1+|\beta|}, \quad t \in I, x, \xi \in \mathbb{R}^n.$$

(v) If we set $G(t) = \psi^w(t; x, D_x)$, *then*

$$\left\| \frac{d}{dt} G(t) + i[L(t), G(t)] \right\|_{\mathcal{L}(\mathcal{H})} = O(h^\infty)$$

as $h \to 0$, *uniformly in* $t \in I$.

Proof. Given the classical mechanical construction above, the construction of the asymptotic solution is quite similar to (or slightly simpler than) the proof of Lemma 4 of [24]. We note $\ell(t; x, \xi) \in S(\langle\xi\rangle^2, dx^2 + d\xi^2/\langle\xi\rangle^2)$ locally in x, and $\pi_1(\mathrm{supp}\ (\psi_0(t; \cdot, \cdot)))$ is contained in a compact set by virtue of the asymptotic property: $S^\lambda_{\lambda t} \sim S_+$ as $\lambda \to \infty$. We omit the detail. □

Now the proof of Theorem 2.1 is almost the same as the proof of Theorem 1 of [24], and we simply refer the reader to the paper.

Finally, we show Theorem 2.3 follows from Theorem 2.1.

Proof of Theorem 2.3. We prove the part (i) only. We note

$$WF(e^{-i(\pi/2)H_0} u) = WF(\hat{u}), \tag{2.8}$$

$$WF(e^{i(\pi/2)H_0} u) = WF(\check{u}) = \Gamma(WF(\hat{u})). \tag{2.9}$$

In fact, $e^{-i(\pi/2)H_0}$ is the Fourier transform. (See also Remark 2.2.)

We set $(x', \xi') = S_+^{-1} \circ \Gamma \circ S_-(x_0, \xi_0)$ so that

$$(x_-, \xi_-) = \Gamma(x'_+, \xi'_+), \quad \text{where } (x'_+, \xi'_+) = S_+(x', \xi'). \tag{2.10}$$

By Theorem 2.1 (i) with $t_0 = \pi/2$, and $u(\pi/2)$ as the initial condition, we have

$$\begin{aligned}(x_0,\xi_0) \in WF(u(\pi)) &\iff (x_-,\xi_-) \in WF(e^{-i(\pi/2)H_0}u(\pi/2)) \\ &\iff (x_-,\xi_-) \in WF(\hat{u}(\pi/2)).\end{aligned}$$

We have used (2.8) in the second step. On the other hand, by Theorem 2.1 (ii) with $t_0 = -\pi/2$, and $u(\pi/2)$ as the initial condition, and using (2.9), we also have

$$\begin{aligned}(x',\xi') \in WF(u_0) &\iff (x'_+,\xi'_+) \in WF(e^{i(\pi/2)H_0}u(\pi/2)) \\ &\iff (x'_+,\xi'_+) \in WF(\check{u}(\pi/2)) = \Gamma(WF(\hat{u}(\pi/2))).\end{aligned}$$

By (2.10), this implies the claim of Theorem 2.3 (i). The part (ii) is proved similarly. □

2.4 Inhomogeneous harmonic oscillators

Here we consider the case when the harmonic potential is inhomogeneous, i.e.,

$$H_0 = -\frac{1}{2}\triangle + \frac{1}{2}\sum_{i,j=1}^{n} b_{ij}x_ix_j,$$

with a positive symmetric matrix (b_{ij}), and

$$H = -\frac{1}{2}\sum_{i,j=1}^{n} \partial_{x_i}a_{ij}(x)\partial_{x_j} + \frac{1}{2}\sum_{i,j=1}^{n} b_{ij}x_ix_j + V(x).$$

We assume $(a_{jk}(x))$ and $V(x)$ satisfy Assumption A. By an orthogonal transform, we can diagonalize the harmonic potential, and hence we may assume $\sum b_{ij}x_ix_j = \sum_{j=1}^{n} \nu_j^2 x_j^2$, where $\nu_j^2 > 0$, $j = 1, \dots, n$, are eigenvalues of (b_{ij}). The behavior of the inhomogeneous harmonic oscillator depends on the number theoretical properties of $(\nu_j)_{j=1}^{n}$. If there exist no $t_0 > 0$ such that

$$(2.11) \qquad t_0\nu_j \in \pi\mathbb{Z}, \quad j = 1, \dots, n,$$

then it is well-known that the recurrence of the evolution operator does not occur, i.e., there are no $t_0 \neq 0$ such that $e^{-itH_0} = I$. In this case we have the following result:

Theorem 2.9. *Suppose* (x_0, ξ_0) *is backward nontrapping, and suppose that there are no* $t_0 > 0$ *such that (2.11) hold. Then for any* $t > 0$,

$$(x_0, \xi_0) \in WF(e^{-itH}u_0) \iff (x_-, \xi_-) \in WF(e^{-itH_0}u_0).$$

Obviously, an analogous result holds for $t < 0$, but we omit it here.

If there exists $t_0 > 0$ such that (2.11) holds, then we have the following result:

Theorem 2.10. *Let* $t_0 > 0$ *be the smallest positive number satisfying (2.11), and let* $m_j = t_0\nu_j/\pi \in \mathbb{Z}$. *We set*

$$\tilde{\Gamma}(x_1, \dots, x_n, \xi_1, \dots, \xi_n) = (\sigma_1 x_1, \dots, \sigma_n x_n, \sigma_1 \xi_1, \dots, \sigma_n \xi_n)$$

for $(x, \xi) \in \mathbb{R}^{2n}$, *where* $\sigma_j = 1$ *if* m_j *is even, and* $\sigma_j = -1$ *if* m_j *is odd. Suppose* (x_0, ξ_0) *is backward nontrapping. Then for* $0 < t < t_0$,

$$(x_0, \xi_0) \in WF(e^{-itH}u_0) \iff (x_-, \xi_-) \in WF(e^{-itH_0}u_0),$$

and

$$(x_0, \xi_0) \in WF(e^{-it_0H}u_0) \iff S_+^{-1} \circ \tilde{\Gamma} \circ S_-(x_0, \xi_0) \in WF(u_0).$$

The proofs of these theorems are similar to Theorems 2.1 and 2.3, and we omit the detail. We only note the fact that

$$\exp\left[-it_0\left(-\frac{1}{2}\frac{d^2}{dx^2} + \nu_j^2\frac{x^2}{2}\right)\right]u(x) = (\mathcal{F}^{2m_j}u)(x) = u(\sigma_j x)$$

for $u \in L^2(\mathbb{R})$, $j = 1, \dots, n$.

3 Singularities for solutions to Schrödinger equations with asymptotically constant magnetic fields

3.1 Introduction

In this chapter we consider a Schrödinger operator with variable coefficients and magnetic potentials:

$$H = \frac{1}{2}\sum_{j,k=1}^{2}\Big(D_{x_j} - (Mx)_j - A_j(x)\Big)a_{jk}(x)\Big(D_{x_k} - (Mx)_k - A_k(x)\Big) + V(x)$$

on $\mathcal{H} = L^2(\mathbb{R}^2)$, where

$$M = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}.$$

We denote the unperturbed constant magnetic operator by H_0:

$$H_0 = \frac{1}{2}(D_x - Mx)^2 \qquad \text{on } \mathcal{H},$$

and we suppose H is a short-range perturbation of H_0 in the following sense:

Assumption B. $a_{jk}(x)$, $A_j(x)$, $V(x) \in C^\infty(\mathbb{R}^2;\mathbb{R})$ for $j,k \in \{1,2\}$, and $\big(a_{jk}(x)\big)_{j,k}$ is positive symmetric for each $x \in \mathbb{R}^2$. Moreover, there exists $\mu_1 > 1$, $\mu_2 > 1$, $\mu_3 > 0$ such that for any $\alpha \in \mathbb{Z}_+^2$,

$$\big|\partial_x^\alpha(a_{jk}(x) - \delta_{jk})\big| \le C_\alpha \langle x\rangle^{-\mu_1-|\alpha|},$$
$$\big|\partial_x^\alpha V(x)\big| \le C_\alpha \langle x\rangle^{2-\mu_2-|\alpha|}$$
$$\big|\partial_x^\alpha A_j(x)\big| \le C_\alpha \langle x\rangle^{-\mu_3-|\alpha|}$$

for $x \in \mathbb{R}^2$ with some $C_\alpha > 0$, where $\langle x\rangle = \sqrt{1+|x|^2}$.

Then it is well-known that H is essentially self adjoint on $C_0^\infty(\mathbb{R}^2)$, and we denote the unique self-adjoint extension by the same symbol H. This problem is closely related to the perturbed harmonic oscillators in the previous chapter. Thus the assumption and formulation and the argument are almost the same. For our reader's convenience, we formulate it in a parallel way. So in the following we denote the symbols of H, H_0, the kinetic energy and the free Schrödinger operator by p, p_0, k and k_0, respectively. Namely, we denote

$$
\begin{aligned}
p(x,\xi) &= \frac{1}{2}\sum_{j,k=1}^{2} a_{jk}(x)\Big(\xi_j-(Mx)_j-A_j(x)\Big)\Big(\xi_k-(Mx)_k-A_k(x)\Big)\\
&\quad +V(x),\\
p_0(x,\xi) &= \frac{1}{2}(\xi-Mx)^2,\\
k(x,\xi) &= \frac{1}{2}\sum_{j,k=1}^{2} a_{jk}(x)\xi_j\xi_k, \quad k_0(x,\xi)=\frac{1}{2}|\xi|^2.
\end{aligned}
$$

Like the previous chapter, we denote the Hamilton flow generated by a symbol $a(x,\xi)$ on $\mathbb{R}^{2n}$ by $\exp(tH_a) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. We also denote

$$\pi_1(X)=x, \quad \pi_2(X)=\xi \quad \text{for } X=(x,\xi)\in\mathbb{R}^{2n}.$$

Let $(x_0,\xi_0)\in\mathbb{R}^{2n}$. (x_0,ξ_0) is called *forward* (*backward, resp.*) *nontrapping* (with respect to k) if

$$\Big|\pi_1\big(\exp(tH_k)(x_0,\xi_0)\big)\Big| \to \infty$$

as $t\to+\infty$ ($t\to-\infty$, resp.). If (x_0,ξ_0) is forward/backward nontrapping, then

$$(x_\pm,\xi_\pm)=\lim_{t\to\pm\infty}\exp(-tH_{k_0})\circ\exp(tH_k)(x_0,\xi_0)$$

exists, and $S_\pm$: $(x_0,\xi_0)\mapsto(x_\pm,\xi_\pm)$ are locally diffeomorphic (see, the previous chapter).

Now we present our main results of this paper. Let us recall the constant magnetic operator H_0 has a period 2π, i.e., $e^{-i2\pi H_0}\varphi=\varphi$ for $\varphi\in\mathcal{H}$. Our first result concerns the evolution by H up to time 2π. We denote

$$u(t)=e^{-itH}u_0, \quad u_0\in\mathcal{H}.$$

We denote the wave front set of a distribution f by $WF(f)$.

Theorem 3.1. (i) *Suppose* (x_0,ξ_0) *is backward nontrapping, and let* $0<t_0<2\pi$, $u_0\in\mathcal{H}$. *Then*

$$(x_0,\xi_0)\in WF(u(t_0)) \iff (x_-,\xi_-)\in WF(e^{-it_0H_0}u_0).$$

(ii) *Suppose* (x_0,ξ_0) *is forward nontrapping, and let* $-2\pi<t_0<0$, $u_0\in\mathcal{H}$. *Then*

$$(x_0,\xi_0)\in WF(u(t_0)) \iff (x_+,\xi_+)\in WF(e^{-it_0H_0}u_0).$$

Remark 3.2. We note that microlocally e^{-itH_0} is a rotation in the phase space. More precisely, for any reasonable symbol $a=a(x,\xi)$,

$$e^{itH_0}a^w(x,D_x)e^{-itH_0}=a^w\big(A(t)x+B(t)D_x,-B(t)x+A(t)D_x\big),$$

where $a^w(x,D_x)$ denotes the Weyl-quantization of a and

$$A(t)=\begin{pmatrix}\frac{1+\cos t}{2} & -\frac{\sin t}{2}\\ \frac{\sin t}{2} & \frac{1+\cos t}{2}\end{pmatrix}=\cos\tfrac{t}{2}\begin{pmatrix}\cos\frac{t}{2} & -\sin\frac{t}{2}\\ \sin\frac{t}{2} & \cos\frac{t}{2}\end{pmatrix},$$

$$B(t)=\begin{pmatrix}-\frac{\sin t}{2} & -\frac{1-\cos t}{2}\\ \frac{1-\cos t}{2} & \frac{\sin t}{2}\end{pmatrix}=\sin\tfrac{t}{2}\begin{pmatrix}\cos\frac{t}{2} & -\sin\frac{t}{2}\\ \sin\frac{t}{2} & \cos\frac{t}{2}\end{pmatrix}.$$

Hence, in particular, $(x_0,\xi_0)\in WF(e^{-itH_0}u_0)$ if and only if there exists a symbol: $a\in C_0^\infty(\mathbb{R}^{2n})$ such that $a(x_0,\xi_0)\neq 0$ and

$$\big\|a^w\big(A(t)x+B(t)D_x,h(-B(t)x+A(t)D_x)\big)u_0\big\|=O(h^\infty)$$

as $h\to 0$.

At the time $t=\pm 2\pi$, $u(t)$ behaves differently. We denote the set of forward (backward, resp.) trapping points by $\mathcal{T}_+$ ($\mathcal{T}_-$, resp.) $\subset\mathbb{R}^{2n}\setminus 0:=\{(x,\xi)\in\mathbb{R}^{2n};\xi\neq 0\}$. $S_\pm$ are diffeomorphism from $(\mathbb{R}^{2n}\setminus 0)\setminus\mathcal{T}_\pm$ to $\mathbb{R}^{2n}\setminus 0$, and hence $S_\pm^{-1}$ are well-defined from $\mathbb{R}^{2n}\setminus 0$ to $(\mathbb{R}^{2n}\setminus 0)\setminus\mathcal{T}_\pm$.

Theorem 3.3. (i) *Suppose* (x_0,ξ_0) *is backward nontrapping, and let* $u_0\in\mathcal{H}$. *Then*

$$(x_0,\xi_0)\in WF\big(u(2\pi)\big) \iff S_+^{-1}\circ S_-(x_0,\xi_0)\in WF(u_0).$$

(ii) *Suppose* (x_0,ξ_0) *is forward nontrapping, and let* $u_0\in\mathcal{H}$. *Then*

$$(x_0,\xi_0)\in WF\big(u(-2\pi)\big) \iff S_-^{-1}\circ S_+(x_0,\xi_0)\in WF(u_0).$$

If the metric is flat, i.e., $a_{jk}(x) = \delta_{jk}$, then we have in particular the following

Corollary 3.4. *If* $a_{jk}(x) = \delta_{jk}$, *then*

$$WF\big(u(2\pi)\big) = WF\big(u(-2\pi)\big) = WF(u_0).$$

There are many results about the microlocal singularities for Schrödinger equations since the pioneer work of Craig-Kappeler-Strauss [2] in 1996. Here we just list some of them (For more information, see [2, 16, 24]). For perturbed free Schrödinger equations, see [9, 12, 24, 25]. For perturbed harmonic oscillators see [8, 13, 16, 31, 32, 35]. In [16], we studied the perturbed harmonic oscillators. Since in many aspects magnetic fields operators appear to have similar properties to harmonic oscillators, such as spectrum, periodicity of the corresponding classical trajectories, and so on, we naturally expect the analogue in microlocal singularities too.

In this paper we study the magnetic and metric perturbation effect of the constant magnetic fields. By the similar argument to that in the paper of perturbed harmonic oscillators[16], we obtain analogue of microlocal singularities between harmonic oscillator and magnetic fields. Moreover, from the results and the argument, we see that the perturbation of magnetic fields does not affect the propagation of singularities, while the metric perturbation does.

The rest of this paper is arranged as follows. Section 3.2 is devoted to the analysis for the asymptotic behavior of the classical Hamilton flow, and the main results are proved in Section 3.3. In the last section we generalize the results to the case of inhomogeneous magnetic fields perturbation in higher dimension.

3.2 High energy asymptotics of the classical flow

In this section, we study the high energy behavior of the classical flow generated by $p(x, \xi)$. More precisely, we consider the properties of $\exp(tH_p)(x_0, \lambda\xi_0)$ as $\lambda \to +\infty$. Throughout this section, we suppose (x_0, ξ_0) is forward nontrapping, and consider the case $t > 0$. The case $t \leq 0$ can be considered similarly.

For $\lambda > 0$, we write

$$p_0^\lambda(x, \xi) = \frac{1}{2}(\xi - \frac{1}{\lambda}Mx)^2$$

and

$$p^\lambda(x,\xi)=\frac{1}{2}\sum_{j,k=1}^{2}a_{jk}(x)\Big(\xi_j-\frac{1}{\lambda}(Mx)_j-\frac{1}{\lambda}A_j(x)\Big)$$
$$\times\Big(\xi_k-\frac{1}{\lambda}(Mx)_k-\frac{1}{\lambda}A_k(x)\Big)$$
$$+\frac{1}{\lambda^2}V(x).$$

Then, by direct computations, we learn

$$\pi_1\big(\exp(tH_p)(x,\lambda\xi)\big)=\pi_1\big(\exp(\lambda tH_{p^\lambda})(x,\xi)\big), \tag{3.1}$$

$$\pi_2\big(\exp(tH_p)(x,\lambda\xi)\big)=\lambda\cdot\pi_2\big(\exp(\lambda tH_{p^\lambda})(x,\xi)\big). \tag{3.2}$$

Hence, it suffices to consider $\exp(tH_{p^\lambda})(x,\xi)$ for $0\le t\le\lambda t_0$, instead of $\exp(tH_p)(x,\lambda\xi)$ for $0\le t\le t_0$.

We note, for each fixed $t\in\mathbb{R}$,

$$\lim_{\lambda\to\infty}\exp(tH_{p^\lambda})(x,\xi)=\exp(tH_k)(x,\xi) \tag{3.3}$$

by the continuity of the solutions to ODEs with respect to the coefficients. Hence, if $t>0$ is large and then $\lambda>0$ is taken sufficiently large (after fixing t), $\pi_1\big(\exp(tH_{p^\lambda})(x_0,\xi_0)\big)$ is far away from the origin by virtue of the nontrapping condition. The next lemma claims that this statement holds for $0\le t\le\lambda\delta$ with sufficiently small $\delta>0$.

Lemma 3.5. *There exists $\delta>0$ and a small neighborhood Ω of (x_0,ξ_0) such that*

$$\big|\pi_1\big(\exp(tH_{p^\lambda})(x,\xi)\big)\big|\ge c_1t-c_2\quad\text{for } 0\le t\le\lambda\delta,\ (x,\xi)\in\Omega$$

with some $c_1,c_2>0$.

Proof. In the following, we denote

$$\exp(tH_{p^\lambda})(x,\xi)=\big(y^\lambda(t;x,\xi),\eta^\lambda(t;x,\xi)\big).$$

And we let $\mu=\min\{\mu_1,\mu_2,\mu_3+1\}$. By the conservation of the energy: $p^\lambda(y^\lambda(t),\eta^\lambda(t))=\text{const.}$, and the ellipticity of the principal symbol, we easily see

$$\frac{1}{\lambda}\Big|\frac{d}{dt}y^\lambda(t;x,\xi)\Big|\le\frac{C}{\lambda}\Big\langle\frac{y^\lambda(t)}{\lambda}\Big\rangle,\quad(x,\xi)\in\Omega,t\in\mathbb{R},$$

where Ω is a small neighborhood of (x_0, ξ_0). Then by Gronwall inequality, we have for any $T > 0$ fixed,

$$\frac{1}{\lambda}\left|y^\lambda(t;x,\xi)\right| \le C_T.$$

By the conservation of the energy and the ellipticity again, $|\eta^\lambda(t)| \le C_T$ follows.

Hence, in particular, we have

$$\left|y^\lambda(t;x,\xi)\right| \le C\langle t\rangle, \quad \left|\eta^\lambda(t;x,\xi)\right| \le C$$

for $t > 0$ with some $C > 0$. On the other hand, by direct computations, we have

$$\begin{aligned}\frac{d^2}{dt^2}\left|y^\lambda(t)\right|^2 &= 2\frac{d^2}{dt^2}\left(y^\lambda \cdot \frac{dy^\lambda}{dt}\right) = 2\frac{d}{dt}\left(\sum_{j,k} a_{jk}(y^\lambda) y_j^\lambda \eta_k^\lambda\right) \\ &= 4p^\lambda(y^\lambda, \eta^\lambda) + 2W(y^\lambda, \eta^\lambda),\end{aligned}$$

where

$$\begin{aligned} W(y^\lambda, \eta^\lambda) = &\sum_{j,k,\ell} a_{jk}(y^\lambda)\Big(a_{k\ell}(y^\lambda) - \delta_{k\ell}\Big)\Big(\eta_j^\lambda - \frac{1}{\lambda}(My^\lambda)_j - \frac{A_j(y^\lambda)}{\lambda}\Big) \\ &\quad\times \Big(\eta_\ell^\lambda - \frac{1}{\lambda}(My^\lambda)_\ell - \frac{A_\ell(y^\lambda)}{\lambda}\Big) \\ &+ \sum_{j,k,\ell,m} y_k^\lambda \Big(\frac{\partial a_{jk}}{\partial x_\ell}(y^\lambda)\, a_{\ell m}(y^\lambda) - \frac{1}{2}\frac{\partial a_{jm}}{\partial x_\ell}(y^\lambda)\, a_{\ell k}(y^\lambda)\Big) \\ &\quad\times \Big(\eta_j^\lambda - \frac{1}{\lambda}(My^\lambda)_j - \frac{A_j(y^\lambda)}{\lambda}\Big) \\ &\quad\times \Big(\eta_m^\lambda - \frac{1}{\lambda}(My^\lambda)_m - \frac{A_m(y^\lambda)}{\lambda}\Big) \\ &+ \sum_{j,k,\ell,m} \frac{y_k^\lambda}{\lambda} a_{\ell k}(y^\lambda) a_{\ell m}(y^\lambda)\Big(M_{\ell j} - M_{j\ell}\Big) \times \\ &\quad\times \Big(\eta_m^\lambda - \frac{1}{\lambda}(My^\lambda)_m - \frac{A_m(y^\lambda)}{\lambda}\Big) \\ &+ \sum_{j,k,\ell,m} \frac{y_k^\lambda}{\lambda} a_{\ell k}(y^\lambda) a_{\ell m}(y^\lambda)\Big(\frac{\partial A_\ell}{\partial x_j}(y^\lambda) - \frac{\partial A_j}{\partial x_j}(y^\lambda)\Big) \times \\ &\quad\times \Big(\eta_m^\lambda - \frac{1}{\lambda}(My^\lambda)_m - \frac{A_m(y^\lambda)}{\lambda}\Big)\end{aligned}$$

$$-\sum_{j,k}\frac{y_k^\lambda}{\lambda^2}a_{jk}(y^\lambda)\frac{\partial V}{\partial x_j}(y^\lambda)-\frac{2}{\lambda^2}V(y^\lambda).$$

Combining these, we learn

$$\frac{d^2}{dt^2}\left|y^\lambda(t)\right|^2 \geq 4p^\lambda(y^\lambda,\eta^\lambda)-c_4\big(\langle y^\lambda\rangle^{-\mu}+\lambda^{-1}\langle y^\lambda\rangle\big).$$

We note $p^\lambda(x_0,\xi_0)=k(x_0,\xi_0)+O(\lambda^{-2})$ and $k(x_0,\xi_0)>0$, and hence $p^\lambda(x_0,\xi_0)>0$ for large λ. Since $\lambda^{-1}\langle y^\lambda\rangle = O\big(\langle t\rangle/\lambda\big)$, if $0\leq t\leq \delta\lambda$ with sufficiently small $\delta>0$, the last term is small and

$$\frac{d^2}{dt^2}\left|y^\lambda(t)\right|^2 \geq 3p^\lambda(y^\lambda,\eta^\lambda)-c_4\langle y^\lambda\rangle^{-\mu}$$

for the initial condition $(x,\xi)\in\Omega$. By the nontrapping condition and (3.3), if $T_0>0$ is sufficiently large and λ is large (depending on T_0), then

$$c_4\langle y^\lambda(T_0)\rangle^{-\mu}\leq p^\lambda(x,\xi)\quad\text{for } (x,\xi)\in\Omega,\text{ and}\quad \frac{d}{dt}\left|y^\lambda(T_0)\right|>0.$$

Then by the standard convexity argument, we learn

$$|y^\lambda(t)|^2\geq |y^\lambda(T_0)|^2+p^\lambda(x,\xi)(t-T_0)^2\quad\text{for } t\in[T_0,\delta\lambda],$$

and this implies the assertion. □

Lemma 3.6. *Let $\delta>0$ and Ω as in the previous lemma, and let $\sigma\in(0,\delta)$. Then*

$$\lim_{\lambda\to\infty}\exp(-\sigma\lambda H_{p_0^\lambda})\circ\exp(\sigma\lambda H_{p^\lambda})(x,\xi)=S_+(x,\xi),$$

for $(x,\xi)\in\Omega$.

Proof. We denote

$$\big(z^\lambda(t;x,\xi),\zeta^\lambda(t;x,\xi)\big)=\exp(-tH_{p_0^\lambda})\circ\exp(tH_{p^\lambda})(x,\xi),$$

and we show the convergence of $\big(z^\lambda(\sigma\lambda),\zeta^\lambda(\sigma\lambda)\big)$ to $S_+(x,\xi)$ for $(x,\xi)\in\Omega$. We recall

$$\exp(-tH_{p_0^\lambda})(x,\xi)=\Big(A^\tau(\tfrac{t}{\lambda})x-\lambda B^\tau(\tfrac{t}{\lambda})\xi,\tfrac{1}{\lambda}B^\tau(\tfrac{t}{\lambda})x+A^\tau(\tfrac{t}{\lambda})\xi\Big),$$

since p_0^λ is the principal symbol of the scaled constant magnetic operator. Thus we have

$$z^\lambda(t) = A^\tau(\tfrac{t}{\lambda})y^\lambda(t) - \lambda B^\tau(\tfrac{t}{\lambda})\eta^\lambda(t)$$

and

$$\zeta^\lambda(t) = \tfrac{1}{\lambda}B^\tau(\tfrac{t}{\lambda})y^\lambda(t) + A^\tau(\tfrac{t}{\lambda})\eta^\lambda(t),$$

where M^τ denotes the transpose of the matrix M. And by direct computations, we have

(3.4)

$$\begin{aligned}
\frac{d}{dt}z_k^\lambda &= \sum_{j=1}^{2}\Big\{\frac{dA_{jk}^\tau(\frac{t}{\lambda})}{dt}y_j^\lambda - \lambda\frac{dB_{jk}^\tau(\frac{t}{\lambda})}{dt}\eta_j^\lambda + A_{jk}(\tfrac{t}{\lambda})\frac{dy_j^\lambda}{dt} - \lambda B_{jk}(\tfrac{t}{\lambda})\frac{d\eta_j^\lambda}{dt}\Big\} \\
&= \sum_{j=1}^{2}\Big\{-\frac{1}{\lambda}\Big(\frac{1}{4}B_{jk}(\tfrac{t}{\lambda}) - \frac{1}{2}C_{jk}(\tfrac{t}{\lambda})\Big)y_j^\lambda - \Big(A_{jk}(\tfrac{t}{\lambda}) + D_{jk}(\tfrac{t}{\lambda})\Big)\eta_j^\lambda \\
&\qquad + A_{jk}(\tfrac{t}{\lambda})\frac{dy_j^\lambda}{dt} - \lambda B_{jk}(\tfrac{t}{\lambda})\frac{d\eta_j^\lambda}{dt}\Big\} \\
&= \sum_{j=1}^{2}\Big\{-\frac{1}{\lambda}\Big(\frac{1}{4}B_{jk}(\tfrac{t}{\lambda}) - \frac{1}{2}C_{jk}(\tfrac{t}{\lambda})\Big)y_j^\lambda - \Big(A_{jk}(\tfrac{t}{\lambda}) + D_{jk}(\tfrac{t}{\lambda})\Big)\eta_j^\lambda\Big\} \\
&\quad + \sum_{j=1}^{2}A_{jk}(\tfrac{t}{\lambda})\sum_{\ell=1}^{2}a_{j\ell}(y^\lambda)\Big(\eta_\ell^\lambda - \frac{1}{\lambda}(My^\lambda)_\ell - \frac{A_\ell(y^\lambda)}{\lambda}\Big) \\
&\quad + \sum_{j=1}^{2}\lambda B_{jk}(\tfrac{t}{\lambda})\Big\{\frac{1}{2}\sum_{\ell,m}\frac{\partial a_{\ell m}(y^\lambda)}{\partial x_j}\Big(\eta_m^\lambda - \frac{1}{\lambda}(My^\lambda)_m - \frac{A_m(y^\lambda)}{\lambda}\Big) \\
&\qquad\qquad \times\Big(\eta_\ell^\lambda - \frac{1}{\lambda}(My^\lambda)_\ell - \frac{A_\ell(y^\lambda)}{\lambda}\Big) \\
&\qquad - \frac{1}{\lambda}\sum_{m\ell}a_{m\ell}(y^\lambda)\Big(M_{mj} + \frac{\partial A_m}{\partial x_j}(y^\lambda)\Big) \\
&\qquad\qquad \times\Big(\eta_\ell^\lambda - \frac{1}{\lambda}(My^\lambda)_\ell - \frac{A_\ell(y^\lambda)}{\lambda}\Big) \\
&\qquad + \frac{1}{\lambda^2}\frac{\partial V}{\partial x_j}(y^\lambda)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\lambda}\Big\{\sum_{j=1}^{2}\Big(A_{jk}(\tfrac{t}{\lambda}) - B_{jk}(\tfrac{t}{\lambda})\Big)A_j(y^\lambda) \\
&\qquad + \sum_{j\ell}\Big(B_{jk}(\tfrac{t}{\lambda})\frac{\partial A_j}{\partial x_\ell}(y^\lambda)\eta_\ell^\lambda - B_{jk}(\tfrac{t}{\lambda})M_{\ell j}A_\ell(y^\lambda)\Big) \\
&\qquad - \sum_{j\ell m} B_{jk}(\tfrac{t}{\lambda})\frac{\partial A_j}{\partial x_\ell}(y^\lambda)M_{\ell m}y_m^\lambda\Big\} \\
&\quad + \sum_{j=1}^{2}\lambda B_{jk}(\tfrac{t}{\lambda})\Big\{\frac{1}{2}\sum_{\ell,m}\frac{\partial a_{\ell m}(y^\lambda)}{\partial x_j}\Big(\eta_m^\lambda - \frac{1}{\lambda}(My^\lambda)_m - \frac{A_m(y^\lambda)}{\lambda}\Big) \\
&\qquad\qquad \times\Big(\eta_\ell^\lambda - \frac{1}{\lambda}(My^\lambda)_\ell - \frac{A_\ell(y^\lambda)}{\lambda}\Big) \\
&\qquad\quad - \frac{1}{\lambda}\sum_{m\ell}(a_{m\ell}(y^\lambda) - \delta_{m\ell})\Big(M_{mj} + \frac{\partial A_m}{\partial x_j}(y^\lambda)\Big) \\
&\qquad\qquad \times\Big(\eta_\ell^\lambda - \frac{1}{\lambda}(My^\lambda)_\ell - \frac{A_\ell(y^\lambda)}{\lambda}\Big) \\
&\qquad\quad + \frac{1}{\lambda^2}\frac{\partial V}{\partial x_j}(y^\lambda)\Big\} \\
&\quad + \sum_{j=1}^{2}A_{jk}(\tfrac{t}{\lambda})\sum_{\ell=1}^{2}\Big(a_{j\ell}(y^\lambda) - \delta_{j\ell}\Big)\Big(\eta_\ell^\lambda - \frac{1}{\lambda}(My^\lambda)_\ell - \frac{A_\ell(y^\lambda)}{\lambda}\Big) \\
&= O\big(\langle t\rangle^{-\mu}\big)
\end{aligned}
$$

for $0 \le t \le \delta\lambda$, where

$$
C(t) = \cos\tfrac{t}{2}\begin{pmatrix} -\sin\frac{t}{2} & -\cos\frac{t}{2} \\ \cos\frac{t}{2} & -\sin\frac{t}{2}\end{pmatrix},
$$

$$
D(t) = \sin\tfrac{t}{2}\begin{pmatrix} -\sin\frac{t}{2} & -\cos\frac{t}{2} \\ \cos\frac{t}{2} & -\sin\frac{t}{2}\end{pmatrix},
$$

come from the derivative of $A(t)$ and $B(t)$ write w.r.t. t.

Similarly, we have

(3.5)

$$
\frac{d}{dt}\zeta_k^\lambda = \sum_{j=1}^{2}\Big\{\frac{1}{\lambda}\frac{dB_{jk}^\tau(\frac{t}{\lambda})}{dt}y_j^\lambda + \frac{dA_{jk}^\tau(\frac{t}{\lambda})}{dt}\eta_j^\lambda + \frac{1}{\lambda}B_{jk}(\tfrac{t}{\lambda})\frac{dy_j^\lambda}{dt} + A_{jk}(\tfrac{t}{\lambda})\frac{d\eta_j^\lambda}{dt}\Big\}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{2}\Big\{\frac{1}{\lambda^2}\Big(A_{jk}(\tfrac{t}{\lambda}) + D_{jk}(\tfrac{t}{\lambda})\Big)y_j^\lambda + \frac{1}{\lambda}\Big(-\frac{1}{4}B_{jk}(\tfrac{t}{\lambda}) + \frac{1}{2}C_{jk}(\tfrac{t}{\lambda})\Big)\eta_j^\lambda\Big\} \\
&\quad + \sum_{j=1}^{2}\Big(\frac{1}{\lambda}B_{jk}(\tfrac{t}{\lambda})\frac{dy_j^\lambda}{dt} + A_{jk}(\tfrac{t}{\lambda})\frac{d\eta_j^\lambda}{dt}\Big) \\
&= \sum_{j=1}^{2}\Big\{\frac{1}{\lambda^2}\Big(A_{jk}(\tfrac{t}{\lambda}) + D_{jk}(\tfrac{t}{\lambda})\Big)y_j^\lambda + \frac{1}{\lambda}\Big(-\frac{1}{4}B_{jk}(\tfrac{t}{\lambda}) + \frac{1}{2}C_{jk}(\tfrac{t}{\lambda})\Big)\eta_j^\lambda\Big\} \\
&\quad + \sum_{j=1}^{2}\frac{1}{\lambda}B_{jk}(\tfrac{t}{\lambda})\sum_{\ell=1}^{2}a_{j\ell}(y^\lambda)\Big(\eta_\ell^\lambda - \frac{1}{\lambda}(My^\lambda)_\ell - \frac{A_\ell(y^\lambda)}{\lambda}\Big) \\
&\quad + \sum_{j=1}^{2}A_{jk}(\tfrac{t}{\lambda})\Big\{-\frac{1}{2}\sum_{\ell,m}\frac{\partial a_{\ell m}(y^\lambda)}{\partial x_j}\Big(\eta_m^\lambda - \frac{1}{\lambda}(My^\lambda)_m - \frac{A_m(y^\lambda)}{\lambda}\Big) \\
&\qquad\qquad \times \Big(\eta_\ell^\lambda - \frac{1}{\lambda}(My^\lambda)_\ell - \frac{A_\ell(y^\lambda)}{\lambda}\Big) \\
&\qquad + \frac{1}{\lambda}\sum_{m\ell}a_{m\ell}(y^\lambda)\Big(M_{mj} + \frac{\partial A_m}{\partial x_j}(y^\lambda)\Big) \\
&\qquad\qquad \times \Big(\eta_\ell^\lambda - \frac{1}{\lambda}(My^\lambda)_\ell - \frac{A_\ell(y^\lambda)}{\lambda}\Big) \\
&\qquad - \frac{1}{\lambda^2}\frac{\partial V}{\partial x_j}(y^\lambda)\Big\} \\
&= \sum_{j=1}^{2}\frac{1}{\lambda}B_{jk}(\tfrac{t}{\lambda})\sum_{\ell=1}^{2}\Big(a_{j\ell}(y^\lambda) - \delta_{j\ell}\Big)\Big(\eta_\ell^\lambda - \frac{1}{\lambda}(My^\lambda)_\ell - \frac{A_\ell(y^\lambda)}{\lambda}\Big) \\
&\quad - \sum_{j=1}^{2}A_{jk}(\tfrac{t}{\lambda})\frac{1}{2}\sum_{\ell,m}\frac{\partial a_{\ell m}(y^\lambda)}{\partial x_j}\Big(\eta_m^\lambda - \frac{1}{\lambda}(My^\lambda)_m - \frac{A_m(y^\lambda)}{\lambda}\Big) \\
&\qquad\qquad \times \Big(\eta_\ell^\lambda - \frac{1}{\lambda}(My^\lambda)_\ell - \frac{A_\ell(y^\lambda)}{\lambda}\Big) \\
&\quad + \sum_{j\ell m}\frac{1}{\lambda}A_{jk}(\tfrac{t}{\lambda})\Big(a_{m\ell}(y^\lambda) - \delta_{m\ell}\Big)\Big(M_{mj} + \frac{\partial A_m}{\partial x_j}(y^\lambda)\Big) \\
&\quad \times \Big(\eta_\ell^\lambda - \frac{1}{\lambda}(My^\lambda)_\ell - \frac{A_\ell(y^\lambda)}{\lambda}\Big) \\
&\quad - \frac{1}{\lambda^2}\sum_{j=1}^{2}A_{jk}(\tfrac{t}{\lambda})\Big(\frac{\partial V(y^\lambda)}{\partial x_j} + \sum_{\ell=1}^{2}M_{\ell j}A_\ell(y^\lambda) + A_j(y^\lambda)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\lambda}\sum_{j\ell} A_{jk}(\tfrac{t}{\lambda})\frac{\partial A_j(y^\lambda)}{\partial x_\ell}\eta_\ell^\lambda - \frac{1}{\lambda^2}\sum_{j\ell m} A_{jk}(\tfrac{t}{\lambda})\frac{\partial A_j(y^\lambda)}{\partial x_\ell} M_{\ell m} y_m^\lambda \\
&-\frac{1}{\lambda^2}\sum_{j=1}^{2} B_{jk}(\tfrac{t}{\lambda})A_j(y^\lambda) \\
&= O\big(\langle t\rangle^{-(\mu+1)}\big)
\end{aligned}
$$

for $0 \le t \le \delta\lambda$. Moreover, for each $t \in \mathbb{R}$, we have

$$
\begin{aligned}
\lim_{\lambda\to\infty} \frac{d}{dt} z_k^\lambda(t) &= \sum_j a_{jk}(\tilde{y})\tilde{\eta}_j - \tilde{\eta}_k + \frac{t}{2}\sum_{i,j}\frac{\partial a_{ij}}{\partial x_k}(\tilde{y})\tilde{y}_j\tilde{\eta}_j \\
&= \frac{d}{dt}(\tilde{y}_k - t\tilde{\eta}_k), \\
\lim_{\lambda\to\infty} \frac{d}{dt} \zeta_k^\lambda(t) &= -\frac{1}{2}\sum_{i,j}\frac{\partial a_{ij}}{\partial x_k}(\tilde{y})\tilde{\eta}_i\tilde{\eta}_j = \frac{d}{dt}\tilde{\eta}_k,
\end{aligned}
$$

where $(\tilde{y}(t), \tilde{\eta}(t)) = \exp(tH_k)(x,\xi)$. By using the dominated convergence theorem, we conclude

$$
\begin{aligned}
\lim_{\lambda\to\infty} z^\lambda(\sigma\lambda) &= x + \lim_{\lambda\to\infty}\int_0^{\sigma\lambda}\frac{dz^\lambda}{dt}dt = x + \int_0^\infty \frac{d}{dt}(\tilde{y} - t\tilde{\eta})dt \\
&= \lim_{t\to+\infty}\big(\tilde{y}(t) - t\tilde{\eta}(t)\big) = \pi_1\big(S_+(x,\xi)\big), \\
\lim_{\lambda\to\infty} \zeta^\lambda(\sigma\lambda) &= \xi + \lim_{\lambda\to\infty}\int_0^{\sigma\lambda}\frac{d\zeta^\lambda}{dt}dt = \xi + \int_0^\infty \frac{d}{dt}\tilde{\eta}(t)dt \\
&= \lim_{t\to+\infty}\tilde{\eta}(t) = \pi_2\big(S_+(x,\xi)\big).
\end{aligned}
$$

This completes the proof of the lemma. □

Lemma 3.7. *Let $0 < \sigma < 2\pi$, and let Ω be a small neighborhood of (x_0, ξ_0) as in the previous lemmas. Then*

$$
\lim_{\lambda\to\infty} \exp(-\sigma\lambda H_{p_0^\lambda}) \circ \exp(\sigma\lambda H_{p^\lambda})(x,\xi) = S_+(x,\xi)
$$

for $(x,\xi) \in \Omega$.

Proof. It suffices to consider the case $\delta < \sigma < 2\pi$, and we fix such σ. Let $\varepsilon > 0$, and we show that if

$$
\max(|z^\lambda(\sigma\lambda) - x_+|, |\zeta^\lambda(\sigma\lambda) - \xi_+|) > \varepsilon
$$

then λ is bounded from above, where $(x_+, \xi_+) = S_+(x, \xi)$, and $(z^\lambda, \zeta^\lambda)$ is as in the proof of the previous lemma. Our claim then follows from this assertion.

We first note

$$y^\lambda(t) = A(\tfrac{t}{\lambda})z^\lambda(t) + \lambda B(\tfrac{t}{\lambda})\zeta^\lambda(t).$$

For the moment, we suppose $|z^\lambda(t) - x_+|$, $|\zeta^\lambda(t) - \xi_+| < \varepsilon$. Then we have

$$|y^\lambda(t)| \geq 2\lambda \sin(\tfrac{t}{2\lambda})(|\xi_+| - \varepsilon) - (|x_+| + \varepsilon) \geq \delta_1 \lambda$$

with some $\delta_1 > 0$ provided $\delta\lambda \leq t \leq \sigma\lambda$, $\varepsilon < |\xi_+|/2$ and $\lambda \geq \lambda_0$, where δ_1 and λ_0 depend only on $|x_+|$ and $|\xi_+|$. Then, by using formulas (4.4) and (4.5), we learn

$$\left|\frac{d}{dt} z^\lambda(t)\right| \leq C\lambda^{-\mu}, \quad \left|\frac{d}{dt}\zeta^\lambda(t)\right| \leq C\lambda^{-\mu-1}.$$

Now we choose λ sufficiently large such that

$$\max(|z^\lambda(\delta\lambda) - x_+|, |\zeta^\lambda(\delta\lambda) - \xi_+|) \leq \frac{\varepsilon}{2},$$

and suppose

$$\max(|z^\lambda(\sigma\lambda) - x_+|, |\zeta^\lambda(\sigma\lambda) - \xi_+|) \geq \varepsilon.$$

Then there exists $t_0 \in (\delta\lambda, \sigma\lambda)$ such that

$$\max(|z^\lambda(t_0) - x_+|, |\zeta^\lambda(t_0) - \xi_+|) = \varepsilon$$

and

$$\max(|z^\lambda(t) - x_+|, |\zeta^\lambda(t) - \xi_+|) \leq \varepsilon \quad \text{for } \delta\lambda \leq t \leq t_0..$$

By the above observation, we learn

$$\begin{aligned}
|z^\lambda(t_0) - x_+| &= \left| z^\lambda(\delta\lambda) - x_+ + \int_{\delta\lambda}^{t_0} \frac{dz^\lambda}{dt}(t)dt \right| \\
&\leq |z^\lambda(\delta\lambda) - x_+| + C(t_0 - \delta\lambda)\lambda^{-\mu} \\
&\leq |z^\lambda(\delta\lambda) - x_+| + C(\sigma - \delta)\lambda^{-(\mu-1)}, \\
|\zeta^\lambda(t_0) - \xi_+| &= \left| \zeta^\lambda(\delta\lambda) - \xi_+ + \int_{\delta\lambda}^{t_0} \frac{d\zeta^\lambda}{dt}(t)dt \right| \\
&\leq |\zeta^\lambda(\delta\lambda) - \xi_+| + C(\sigma - \delta)\lambda^{-\mu}.
\end{aligned}$$

Thus we have

$$\varepsilon = \max(|z^\lambda(t_0) - x_+|, |\zeta^\lambda(t_0) - \xi_+|) \leq \frac{\varepsilon}{2} + C(\sigma - \delta)\lambda^{-(\mu-1)},$$

and hence $\lambda \leq (2C(\sigma-\delta)/\varepsilon)^{1/(\mu-1)}$, and this completes the proof of the assertion. □

The next theorem follows immediately from Lemma 3.7.

Theorem 3.8. (i) *Suppose (x_0, ξ_0) is forward nontrapping, and let $0 < \sigma < 2\pi$. Then there exists a neighborhood Ω of (x_0, ξ_0) such that*

$$\lim_{\lambda\to\infty} \pi_1\bigl(\exp(-\sigma H_{p_0}) \circ \exp(\sigma H_p)(x, \lambda\xi)\bigr) = \pi_1(S_+(x,\xi)),$$
$$\lim_{\lambda\to\infty} \lambda^{-1}\pi_2\bigl(\exp(-\sigma H_{p_0}) \circ \exp(\sigma H_p)(x, \lambda\xi)\bigr) = \pi_2(S_+(x,\xi))$$

for $(x,\xi) \in \Omega$, and the convergence is uniform in Ω.
(ii) *Suppose (x_0, ξ_0) be backward nontrapping, and let $-2\pi < \sigma < 0$. Then there exists a neighborhood Ω of (x_0, ξ_0) such that*

$$\lim_{\lambda\to\infty} \pi_1\bigl(\exp(-\sigma H_{p_0}) \circ \exp(\sigma H_p)(x, \lambda\xi)\bigr) = \pi_1(S_-(x,\xi)),$$
$$\lim_{\lambda\to\infty} \lambda^{-1}\pi_2\bigl(\exp(-\sigma H_{p_0}) \circ \exp(\sigma H_p)(x, \lambda\xi)\bigr) = \pi_2(S_-(x,\xi))$$

for $(x,\xi) \in \Omega$, and the convergence is uniform in Ω.

We introduce several notations as a preparation for the proof of our main results. We set

$$\begin{aligned}
\ell(t; x, \xi) &= \bigl(p \circ \exp(tH_{p_0})\bigr)(x,\xi) - p_0(x,\xi) \\
&= \frac{1}{2}\sum_{j,k=1}^{2} \Bigl(a_{jk}\bigl(A(t)x + B(t)\xi\bigr) - \delta_{jk}\Bigr) \\
&\qquad \times \Bigl\{-\bigl(B(t)x\bigr)_j + \bigl(A(t)\xi\bigr)_j - \Bigl(M\bigl(A(t)x + B(t)\xi\bigr)\Bigr)_j \\
&\qquad\quad - A_j\bigl(A(t)x + B(t)\xi\bigr)\Bigr\} \\
&\qquad \times \Bigl\{-\bigl(B(t)x\bigr)_k + \bigl(A(t)\xi\bigr)_k - \Bigl(M\bigl(A(t)x + B(t)\xi\bigr)\Bigr)_k \\
&\qquad\quad - A_k\bigl(A(t)x + B(t)\xi\bigr)\Bigr\}
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=1}^{2}\Big\{-B_{j\ell}(t)x_\ell + A_{j\ell}(t)\xi_\ell - \Big(M\big(A(t)x+B(t)\xi\big)\Big)_j \\
&\qquad -\frac{1}{2}A_j\big(A(t)x+B(t)\xi\big)\Big\} \\
&+V\big(A(t)x+B(t)\xi\big).
\end{aligned}
$$

Then it is easy to show that $\ell(t;x,\xi)$ generates the *scattering time evolution:*

$$S_t = \exp(-tH_{p_0})\circ\exp(tH_p).$$

Similarly,

$$\ell^\lambda(t;x,\xi) = \big(p^\lambda\circ\exp(tH_{p_0^\lambda})\big)(x,\xi) - p_0^\lambda(x,\xi)$$

generates the time evolution:

$$S_t^\lambda = \exp(-tH_{p_0^\lambda})\circ\exp(tH_{p^\lambda}).$$

We denote the scaling with respect to ξ by $\mathfrak{J}_\lambda$, i.e.,

$$\mathfrak{J}_\lambda(x,\xi) = (x,\lambda\xi) \quad \text{for } (x,\xi)\in\mathbb{R}^{2n}.$$

Then by (3.1) and (3.2), we have

$$\exp(tH_p)\circ\mathfrak{J}_\lambda = \mathfrak{J}_\lambda\circ\exp(\lambda tH_{p^\lambda}), \quad \exp(tH_{p_0})\circ\mathfrak{J}_\lambda = \mathfrak{J}_\lambda\circ\exp(\lambda tH_{p_0^\lambda}),$$

and hence we also have

$$S_t\circ\mathfrak{J}_\lambda = \mathfrak{J}_\lambda\circ S_{\lambda t}^\lambda. \tag{3.6}$$

3.3 Proof of main results

In this section, we mainly concern the case (x_0,ξ_0) is forward non-trapping, and prove the part (ii) of Theorem 3.1.

We first consider the property of $e^{itH_0}e^{-itH}$ for $t\le 0$. Let $v_0\in C_0^\infty(\mathbb{R}^2)$, and we consider

$$v(t) = e^{itH_0}e^{-itH}v_0.$$

Then it is easy to observe

$$
\begin{aligned}
\frac{d}{dt}v(t) &= -ie^{itH_0}(H-H_0)e^{-itH}v_0 \\
&= -i\big(e^{itH_0}He^{-itH_0} - H_0\big)v(t) = -iL(t)v(t),
\end{aligned}
$$

where $L(t) = e^{itH_0}He^{-itH_0} - H_0$. We recall, for any reasonable symbol $a(x,\xi)$, we have

$$e^{itH_0}a^w(x,D_x)e^{-itH_0} = \big(a \circ \exp(tH_{p_0})\big)^w(x,D_x),$$

without the remainder terms, since $p_0(x,\xi)$ is a quadratic form in (x,ξ), and we employ the Weyl quantization. Thus we have

$$L(t) = \big(p \circ \exp(tH_{p_0})\big)^w(x,D_x) - p_0(x,D_x) = \ell^w(t;x,D_x),$$

i.e., $\ell(t;x,\xi)$ is the Weyl-symbol of $L(t)$. This is in fact expected, since $e^{itH_0}e^{-itH}$ is the quantization of S_t.

Let Ω be a small neighborhood of (x_0,ξ_0) as in the last section, and let $f \in C_0^\infty(\Omega)$ be such that $f(x_0,\xi_0) > 0$, and $f(x,\xi) \geq 0$ on $\mathbb{R}^4$. We then set

$$f_h(x,\xi) = f(x,h\xi), \quad h = \lambda^{-1},$$

where $h > 0$ is our semiclassical parameter. We consider the behavior of

$$G(t) = e^{itH_0}e^{-itH}f^w(x,hD_x)e^{itH}e^{-itH_0}$$

as $h \to 0$. The operator valued function $G(t)$ satisfies the Heisenberg equation:

$$\frac{d}{dt}G(t) = -i[L(t),G(t)], \quad G(0) = f^w(x,hD_x). \tag{3.7}$$

The corresponding canonical equation of the classical mechanics is

$$\frac{\partial \psi_0}{\partial t}(t;x,\xi) = -\{\ell,\psi_0\}(t;x,\xi), \quad \psi(0;x,\xi) = f(x,h\xi)$$

and the solution is given by

$$\psi_0(t;x,\xi) = (f_h \circ S_t^{-1})(x,\xi)$$

since S_t is the Hamilton flow generated by $\ell(t;x,\xi)$. Now we note

$$f_h(x,\xi) = f(x,\xi/\lambda) = (f \circ \mathfrak{J}_\lambda^{-1})(x,\xi).$$

Hence, recalling (3.6), we learn

$$\begin{aligned} f_h \circ S_t^{-1} &= f \circ \mathfrak{J}_\lambda^{-1} \circ S_t^{-1} = f \circ (S_t \circ \mathfrak{J}_\lambda)^{-1} \\ &= f \circ (\mathfrak{J}_\lambda \circ S_{\lambda t}^\lambda)^{-1} = f \circ (S_{\lambda t}^\lambda)^{-1} \circ \mathfrak{J}_\lambda^{-1}. \end{aligned}$$

In other words, we have

$$\psi_0(t;x,\xi) = (f_h \circ S_t^{-1})(x,\xi) = \big(f \circ (S_{\lambda t}^{\lambda})^{-1}\big)(x,h\xi).$$

We expect

$$G(t) \sim \psi_0(t;x,D_x) = \big(f \circ (S_{\lambda t}^{\lambda})^{-1}\big)(x,hD_x)$$

for small $h > 0$, and we construct the asymptotic solution to the Heisenberg equation (3.7) with the principal symbol $\psi_0(t;x,\xi)$.

Lemma 3.9. *Let $-2\pi < t_0 < 0$, and set $I = [t_0, 0]$. There exists $\psi(t;x,\xi) \in C_0^\infty(\mathbb{R}^4)$ for $t \in I$ such that*

(i) $\psi(0;x,\xi) = f(x,h\xi)$.

(ii) *$\psi(t;x,\xi)$ is supported in $S_t \circ \mathfrak{J}_\lambda(\Omega) = \mathfrak{J}_\lambda \circ S_{\lambda t}^{\lambda}(\Omega)$.*

(iii) *For any $\alpha, \beta \in \mathbb{Z}_+^2$, there is $C_{\alpha\beta} > 0$ such that*

$$\big|\partial_x^\alpha \partial_\xi^\beta \psi(t;x,\xi)\big| \le C_{\alpha\beta} h^{|\beta|}, \quad t \in I, x, \xi \in \mathbb{R}^2.$$

(iv) *The principal symbol of ψ is given by ψ_0, i.e., for any $\alpha, \beta \in \mathbb{Z}_+^2$, there is $C_{\alpha\beta} > 0$ such that*

$$\big|\partial_x^\alpha \partial_\xi^\beta \big(\psi(t;x,\xi) - \psi_0(t;x,\xi)\big)\big| \le C_{\alpha\beta} h^{1+|\beta|}, \quad t \in I, x, \xi \in \mathbb{R}^2.$$

(v) *If we set $G(t) = \psi^w(t;x,D_x)$, then*

$$\left\| \frac{d}{dt} G(t) + i[L(t), G(t)] \right\|_{\mathcal{L}(\mathcal{H})} = O(h^\infty)$$

as $h \to 0$, uniformly in $t \in I$.

Proof. This can be proved by similar argument as in the previous chapter. Here we sketch it. Given the classical mechanical construction above, the construction of the asymptotic solution is quite similar to (or slightly simpler than) the proof of Lemma 4 of [24]. We note $\ell(t;x,\xi) \in S\big(\langle\xi\rangle^2, dx^2 + d\xi^2/\langle\xi\rangle^2\big)$ locally in x, and $\pi_1\big(\mathrm{supp}\,\big(\psi_0(t;\cdot,\cdot)\big)\big)$ is contained in a compact set by virtue of the asymptotic property: $S_{\lambda t}^{\lambda} \sim S_+$ as $\lambda \to \infty$. We omit the detail. ☐

And then as the same argument in the previous chapter, the proof of Theorem 3.1 is almost the same as the proof of Theorem 1 of [24], and we also simply refer the readers to the paper.

Finally, we show Theorem 3.3 follows from Theorem 3.1 by a similar argument.

Proof of Theorem 3.3. We prove the part (i) only. We note

$$WF\big(e^{-i\pi H_0}u\big) = WF\big(e^{i\pi H_0}u\big) = \Gamma\big(WF(u)\big). \tag{3.8}$$

where $\Gamma(x,\xi) = (-M\xi, Mx)$ is a symplectic transform in the phase space. (See also Remark 3.2.)

We set $(x',\xi') = S_+^{-1} \circ S_-(x_0,\xi_0)$ so that

$$(x_-,\xi_-) = (x'_+,\xi'_+), \quad \text{where } (x'_+,\xi'_+) = S_+(x',\xi'). \tag{3.9}$$

By Theorem 3.1 (i) with $t_0 = \pi$, and $u(\pi)$ as the initial condition, we have

$$\begin{aligned} (x_0,\xi_0) \in WF\big(u(2\pi)\big) \quad &\Longleftrightarrow \quad (x_-,\xi_-) \in WF\big(e^{-i\pi H_0}u(\pi)\big) \\ &\Longleftrightarrow \quad (x_-,\xi_-) \in \Gamma\big(WF\big(u(\pi)\big)\big). \end{aligned}$$

We have used (3.8) in the second step. On the other hand, by Theorem 3.1 (ii) with $t_0 = -\pi$, and $u(\pi)$ as the initial condition, and using (3.8) again, we also have

$$\begin{aligned} (x',\xi') \in WF(u_0) \quad \Longleftrightarrow \quad (x'_+,\xi'_+) &\in WF\big(e^{i\pi H_0}u(\pi)\big) \\ &= \Gamma\big(WF\big(u(\pi)\big)\big). \end{aligned}$$

By (3.9), this implies the claim of Theorem 3.3 (i). The part (ii) is proved similarly. □

3.4 Inhomogeneous magnetic fields operators

Here we consider the case when the magnetic vector potential is inhomogeneous, i.e.,

$$H_0 = \frac{1}{2}(D_x - Mx)^2 \qquad \text{on } \mathcal{H} = L^2(\mathbb{R}^{2n}),$$

and

$$H = \frac{1}{2}\sum_{j,k=1}^{2n} \big(D_{x_j} - (Mx)_j - A_j(x)\big) a_{jk}(x) \big(D_{x_k} - (Mx)_k - A_k(x)\big) + V(x)$$

where

$$M=\begin{pmatrix} \begin{pmatrix} 0 & \frac{\nu_1}{2} \\ -\frac{\nu_1}{2} & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & \frac{\nu_n}{2} \\ -\frac{\nu_n}{2} & 0 \end{pmatrix} \end{pmatrix},$$

and $\nu_1, \cdots, \nu_n$ are positive numbers. We assume $(a_{jk}(x))$ and $V(x)$ satisfy the analogous form of Assumption B. The behavior of the inhomogeneous magnetic operator depends on the number theoretical properties of $(\nu_j)_{j=1}^n$. If there exist no $t_0 > 0$ such that

$$t_0\nu_j \in 2\pi\mathbb{Z}, \quad j = 1, \ldots, n, \tag{3.10}$$

then it is well-known that the recurrence of the evolution operator does not occur, i.e., there are no $t_0 \neq 0$ such that $e^{-itH_0} = I$. In this case we have the following result:

Theorem 3.10. *Suppose (x_0, ξ_0) is backward nontrapping, and suppose that there are no $t_0 > 0$ such that (3.10) hold. Then for any $t > 0$,*

$$(x_0, \xi_0) \in WF(e^{-itH}u_0) \iff (x_-, \xi_-) \in WF(e^{-itH_0}u_0).$$

Obviously, an analogous result holds for $t < 0$, but we omit it here.

If there exists $t_0 > 0$ such that (3.10) holds, then we have the following result:

Theorem 3.11. *Let $t_0 > 0$ be the smallest positive number satisfying (3.10), and let $m_j = t_0\nu_j/2\pi \in \mathbb{Z}$. Suppose (x_0, ξ_0) is backward nontrapping. Then for $0 < t < t_0$,*

$$(x_0, \xi_0) \in WF(e^{-itH}u_0) \iff (x_-, \xi_-) \in WF(e^{-itH_0}u_0),$$

and

$$(x_0, \xi_0) \in WF(e^{-it_0H}u_0) \iff S_+^{-1} \circ S_-(x_0, \xi_0) \in WF(u_0).$$

The proofs of these theorems are similar to Theorems 3.1 and 3.3, and we omit the detail. We only note the fact that

$$\exp\left[-it_0\left(\frac{1}{2}(D_x - M_jx)^2\right)\right]u(x) = u(x)$$

for $u \in L^2(\mathbb{R}^2)$, with $M_j = \begin{pmatrix} 0 & \frac{\nu_j}{2} \\ -\frac{\nu_j}{2} & 0 \end{pmatrix}$, $j = 1, \ldots, n$.

4 Propagation of singularities for harmonic oscillators with long-range perturbations

4.1 Introduction

In this chapter we consider a Schrödinger operator with variable coefficients and the harmonic potential:

$$H = -\frac{1}{2}\sum_{j,k=1}^{n} \partial_{x_j} a_{jk}(x)\partial_{x_k} + \frac{1}{2}|x|^2 + V(x)$$

on $\mathcal{H} = L^2(\mathbb{R}^n)$, $n \geq 1$. We denote the unperturbed harmonic oscillator by H_0:

$$H_0 = -\frac{1}{2}\triangle + \frac{1}{2}|x|^2 \qquad \text{on } \mathcal{H},$$

and we suppose H is a long-range perturbation of H_0 in the following sense:

Assumption C. $a_{jk}(x), V(x) \in C^\infty(\mathbb{R}^n;\mathbb{R})$ for $j,k = 1,\dots,n$, and $(a_{jk}(x))_{j,k}$ is positive symmetric for each $x \in \mathbb{R}^n$. Moreover, there exists $\mu > 0$ such that for any $\alpha \in \mathbb{Z}_+^n$,

$$\left|\partial_x^\alpha(a_{jk}(x) - \delta_{jk})\right| \leq C_\alpha \langle x\rangle^{-\mu-|\alpha|},$$
$$\left|\partial_x^\alpha V(x)\right| \leq C_\alpha \langle x\rangle^{2-\mu-|\alpha|}$$

for $x \in \mathbb{R}^n$ with some $C_\alpha > 0$, where $\langle x\rangle = \sqrt{1+|x|^2}$.

Then it is well-known that H is essentially self adjoint on $C_0^\infty(\mathbb{R}^n)$, and we denote the unique self-adjoint extension by the same symbol H. As has done in the previous chapters, we denote the symbols of H, H_0, the kinetic energy and the free Schrödinger operator by p, p_0, k and k_0, respectively. Namely, we denote

$$p(x,\xi) = \frac{1}{2}\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k + \frac{1}{2}|x|^2 + V(x),$$
$$p_0(x,\xi) = \frac{1}{2}|\xi|^2 + \frac{1}{2}|x|^2,$$
$$k(x,\xi) = \frac{1}{2}\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k, \quad k_0(x,\xi) = \frac{1}{2}|\xi|^2.$$

And also as has done in the previous two chapters, we denote the Hamilton flow generated by a symbol $a(x,\xi)$ on $\mathbb{R}^{2n}$ by $\exp(tH_a):\mathbb{R}^{2n}\to\mathbb{R}^{2n}$. Then we can denote

$$\ell(t,x,\xi)=p\circ\exp(tH_{p_0})(x,\xi)-p_0(x,\xi),$$

which is the most important quantity in our work and we shall devote much effort to estimating it. We also use

$$\pi_1(X)=x,\quad \pi_2(X)=\xi \quad \text{for } X=(x,\xi)\in\mathbb{R}^{2n}$$

to denote the projection to the first and the second coordinate of the phase space $T^*\mathbb{R}^n$ respectively.

Let $(x_0,\xi_0)\in\mathbb{R}^{2n}$ and recall that (x_0,ξ_0) is called *forward (backward, resp.) nontrapping* (with respect to k) if

$$|\pi_1(\exp(tH_k)(x_0,\xi_0)|\to\infty$$

as $t\to+\infty$ ($t\to-\infty$, resp.). If (x_0,ξ_0) is forward/backward nontrapping, then it is well-known that the modified classical scattering data exists, and $S_\pm$: $(x_0,\xi_0)\mapsto(z_\pm(x_0,\xi_0),\xi_\pm(x_0,\xi_0))$ are locally diffeomorphic (see, e.g., Nakamura [25], Section 2.). Specifically, if we let

$$(\tilde{y}(t,x_0,\xi_0),\tilde{\eta}(t,x_0,\xi_0))=\exp(tH_k)(x_0,\xi_0)$$

denote the Hamilton flow associated with the kinetic energy $k(x,\xi)$, then from [25] section 2, we know that

$$\xi_\pm(x_0,\xi_0)=\lim_{t\to\pm\infty}\tilde{\eta}(t,x_0,\xi_0)$$

exist and $\xi_\pm(\pm R\frac{\xi}{|\xi|},\xi)$ are diffeomorphisms from $\{\xi\in\mathbb{R}^n;\ |\xi|\geq cR\}$ to the image, for some $c>0$, when $R>0$ is chosen to be sufficiently large. Since $\xi_\pm(x,\xi)$ and $\xi_\pm(\pm R\frac{\xi}{|\xi|},\xi)$ are homogeneous of degree 1 w.r.t. ξ, we can choose $\tilde{P}(x_0,\xi_0)$ such that

$$\xi_\pm(x_0,\xi_0)=\xi_\pm(\pm R\tfrac{\tilde{P}}{|\tilde{P}|},\tilde{P}).$$

Then from the argument in [25] section 2, we also know

$$z_\pm(x_0,\xi_0)=\lim_{t\to\pm\infty}(\tilde{y}(t,x_0,\xi_0)-\tilde{y}(t,\pm R\tfrac{\tilde{P}}{|\tilde{P}|},\tilde{P})).$$

In order to characterize the singularities, we will construct a solution $W(t,\xi)$ to the time-dependent Hamilton-Jacobi equation in

Section 4.2, i.e., we will construct a solution to the following equation:

$$\frac{\partial}{\partial t}W(t,\xi) = \ell(t, \frac{\partial W}{\partial \xi}(t,\xi),\xi), \quad |\xi| \geq c_4 R,$$

for $\pm t \in [0,t_0]$, and any $t_0 \in [0,\pi)$.

Our result concerns the evolution by H up to time π. We denote

$$u(t) = e^{-itH}u_0, \quad u_0 \in \mathcal{H}.$$

If the wave front set of a distribution f is denoted by $WF(f)$, then we have

Theorem 4.1. *(i) Suppose* (x_0,ξ_0) *is backward nontrapping, and let* $0 < t_0 < \pi$, $u_0 \in \mathcal{H}$. *Then*

$$(x_0,\xi_0) \in WF(u(t_0)) \iff (z_-,\xi_-) \in WF(e^{iW(-t_0,D_x)}e^{-it_0H_0}u_0).$$

(ii) Suppose (x_0,ξ_0) *is forward nontrapping, and let* $-\pi < t_0 < 0$, $u_0 \in \mathcal{H}$. *Then*

$$(x_0,\xi_0) \in WF(u(t_0)) \iff (z_+,\xi_+) \in WF(e^{iW(-t_0,D_x)}e^{-it_0H_0}u_0).$$

Remark 4.2. We note that microlocally e^{-itH_0} is a rotation in the phase space. More precisely, for any reasonable symbol $a = a(x,\xi)$,

$$e^{-itH_0}a^w(x,D_x)e^{itH_0} = a^w\big(\cos(t)x + \sin(t)D_x, -\sin(t)x + \cos(t)D_x\big),$$

where $a^w(x,D_x)$ denotes the Weyl-quantization of a. Hence, in particular, $(x_0,\xi_0) \notin WF(e^{-itH_0}u_0)$ if and only if there exists a symbol: $a \in C_0^\infty(\mathbb{R}^{2n})$ such that $a(x_0,\xi_0) \neq 0$ and

$$\big\|a^w(\cos(t)x - \sin(t)D_x, h(\sin(t)x + \cos(t)D_x))u_0\big\| = O(h^\infty)$$

as $h \to 0$.

Our method and formulation has the advantage of having more clear structure and can be used in a unified way to deal with long-range perturbation with harmonic or without harmonic potential, compared to the traditional method where only one term of the modified propagator was constructed in order to deal with long-range perturbations. In fact we can use our method to construct a new modified propagator in the form $e^{i(-\frac{1}{2}t\Delta+\tilde{W}(t,D_x))}$ to the long-range perturbation of the free Schrödinger operator in [25], and by the constructions, we see that for large ξ, we have the relationship $-\frac{1}{2}t|\xi|^2+\tilde{W}(t,\xi) = W(t,\xi)$ between the modified propagator $W(t,\xi)$ in [25] and our $\tilde{W}(t,\xi)$. Also it is more clear and more natural in our method to see that we can regard the long-range perturbation of the free Schrödinger operator as the "limit" case of the perturbation of the harmonic oscillator when the angular frequency ω in the term $\frac{1}{2}\omega^2x^2$ tends to 0. Indeed in this case the period $T = 2\pi/\omega$ tends to ∞ and H_0 tends to $-\frac{1}{2}\Delta$, then theorem 4.1 "tends to" the corresponding theorem for perturbation of the free Schrödinger operator.

The main idea for the study of the problem is simple. If we let $v(t) = e^{itH_0}e^{-itH}v_0$ as in the short-range perturbations, then we know $v(t)$ satisfies the evolution equation:

$$\frac{d}{dt}v(t) = -i(e^{itH_0}He^{-itH_0} - H_0)v(t)$$

with $v_0 \in L^2(\mathbb{R}^n)$. Namely, $v(t)$ is a solution to a Schrödinger equation with time-dependent Hamiltonian:

$$L(t) = \frac{d}{dt}v(t) = -i(e^{itH_0}He^{-itH_0} - H_0)v(t).$$

Since the perturbation has been assumed to be long-range perturbation, by the calculus of symbols we see that the speed of propagation of singularities is not necessarily finite for $t \neq 0$. But carefully checking gives us the information that in $L(t)$ we have separated the quadratic part H_0 from H, thus we can try to construct a modified propagator as in [25] for the evolution equation for $v(t)$ with the Hamiltonian $L(t)$. Then we can combine the method used in short-range perturbation in chapter 2 with the argument in [25] to complete the proof.

Since the propagation of singularities is closely related to the corresponding classical Hamilton flow, we study the high energy asymptotic behavior of the classical flow in Section 4.2, and prove the main theorems in Section 4.3.

4.2 High energy asymptotics of the classical flow

In this section, we study the high energy behavior of the classical flow generated by $p(x,\xi)$. More precisely, we consider the properties of $\exp(tH_p)(x_0, \lambda\xi_0)$ as $\lambda \to +\infty$. Thoughout this section, we suppose (x_0, ξ_0) is forward nontrapping, and consider the case $t > 0$. The case $t \leq 0$ can be considered similarly.

For $\lambda > 0$, we write

$$p^\lambda(x,\xi) = \frac{1}{2}\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k + \frac{|x|^2}{2\lambda^2} + \frac{1}{\lambda^2}V(x),$$
$$p_0^\lambda(x,\xi) = \frac{1}{2}|\xi|^2 + \frac{1}{2\lambda^2}|x|^2.$$

Then, by direct computations, we learn

$$\pi_1(\exp(tH_p)(x,\lambda\xi)) = \pi_1(\exp(\lambda t H_{p^\lambda})(x,\xi)), \tag{4.1}$$
$$\pi_2(\exp(tH_p)(x,\lambda\xi)) = \lambda\cdot\pi_2(\exp(\lambda t H_{p^\lambda})(x,\xi)). \tag{4.2}$$

Hence, it suffices to consider $\exp(tH_{p^\lambda})(x,\xi)$ for $0 \leq t \leq \lambda t_0$, instead of $\exp(tH_p)(x,\lambda\xi)$ for $0 \leq t \leq t_0$.

We note, for each fixed $t \in \mathbb{R}$,

$$\lim_{\lambda\to\infty} \exp(tH_{p^\lambda})(x,\xi) = \exp(tH_k)(x,\xi) \tag{4.3}$$

by the continuity of the solutions to ODEs with respect to the coefficients. Hence, if $t > 0$ is large and then $\lambda > 0$ is taken sufficiently large (after fixing t), $\pi_1(\exp(tH_{p^\lambda})(x_0,\xi_0))$ is far away from the origin by virtue of the nontrapping condition. The next lemma claims that this statement holds for $0 \leq t \leq \lambda\delta$ with sufficiently small $\delta > 0$.

Lemma 4.3. *There exists $\delta > 0$ and a small neighborhood Ω of (x_0,ξ_0) such that*

$$|\pi_1(\exp(tH_{p^\lambda})(x,\xi))| \geq c_1 t - c_2 \quad \textit{for } 0 \leq t \leq \lambda\delta,\ (x,\xi)\in\Omega$$

with some $c_1, c_2 > 0$.

Proof. In the following, we denote

$$\exp(tH_{p^\lambda})(x,\xi) = (y^\lambda(t;x,\xi), \eta^\lambda(t;x,\xi)).$$

By the conservation of the energy: $p^\lambda(y^\lambda(t),\eta^\lambda(t)) = \text{const.}$, and the ellipticity of the principal symbol, we easily see

$$\frac{1}{\lambda^2}|y^\lambda(t;x,\xi)|^2 + |\eta^\lambda(t;x,\xi)|^2 \leq C, \quad (x,\xi)\in\Omega, t\in\mathbb{R},$$

where Ω is a small neighborhood of (x_0, ξ_0). Hence, in particular, we have

$$|y^\lambda(t;x,\xi)| \le C\langle t\rangle, \quad |\eta^\lambda(t;x,\xi)| \le C$$

for $t > 0$ with some $C > 0$ by the Hamilton equations. On the other hand, by direct computations, we have

$$\begin{aligned}\frac{d^2}{dt^2}|y^\lambda(t)|^2 &= 2\frac{d^2}{dt^2}\left(y^\lambda \cdot \frac{dy^\lambda}{dt}\right) = 2\frac{d}{dt}\left(\sum_{j,k} a_{jk}(y^\lambda)y_j^\lambda\eta_k^\lambda\right)\\ &= 4p^\lambda(y^\lambda,\eta^\lambda) + 2W(y^\lambda,\eta^\lambda),\end{aligned}$$

where

$$\begin{aligned}W(y^\lambda,\eta^\lambda) = &\sum_{j,k,\ell} a_{jk}(y^\lambda)(a_{k\ell}(y^\lambda) - \delta_{k\ell})\eta_j^\lambda\eta_k^\lambda\\ &+ \sum_{j,k,\ell,m} \frac{\partial a_{jk}}{\partial x_\ell}(y^\lambda)\, a_{\ell m}(y^\lambda)\eta_m^\lambda y_j^\lambda\eta_k^\lambda\\ &- \sum_{j,k,\ell,m} a_{jk}(y^\lambda)\frac{\partial a_{\ell m}}{\partial x_k}(y^\lambda)y_j^\lambda\eta_\ell^\lambda\eta_m^\lambda - \frac{1}{\lambda^2}\sum_{j,k} a_{jk}(y^\lambda)y_j^\lambda y_k^\lambda\\ &- \frac{1}{\lambda^2}\sum_{j,k} a_{jk}(y^\lambda)\frac{\partial V}{\partial x_k}(y^\lambda)y_j^\lambda - \frac{1}{\lambda^2}|y^\lambda|^2 - \frac{2}{\lambda^2}V(y^\lambda).\end{aligned}$$

Combining these, we learn

$$\frac{d^2}{dt^2}|y^\lambda(t)|^2 \ge 4p^\lambda(y^\lambda,\eta^\lambda) - c_4(\langle y^\lambda\rangle^{-\mu} + \lambda^{-2}\langle y^\lambda\rangle^2).$$

We note $p^\lambda(x_0,\xi_0) = k(x_0,\xi_0) + O(\lambda^{-2})$ and $k(x_0,\xi_0) > 0$, and hence $p^\lambda(x_0,\xi_0) > 0$ for large λ. Since $\lambda^{-2}\langle y^\lambda\rangle^2 = O(\langle t\rangle^2/\lambda^2)$, if $0 \le t \le \delta\lambda$ with sufficiently small $\delta > 0$, the last term is small and

$$\frac{d^2}{dt^2}|y^\lambda(t)|^2 \ge 3p^\lambda(y^\lambda,\eta^\lambda) - c_4\langle y^\lambda\rangle^{-\mu}$$

for the initial conditon $(x,\xi) \in \Omega$. By the nontrapping condition and (2.3), if $T_0 > 0$ is sufficiently large and λ is large (depending on T_0), then

$$c_4\langle y^\lambda(T_0)\rangle^{-\mu} \le p^\lambda(x,\xi) \quad \text{for } (x,\xi) \in \Omega, \text{ and } \quad \frac{d}{dt}|y^\lambda(T_0)| > 0.$$

Then by the standard convexity argument, we learn

$$|y^\lambda(t)|^2 \geq |y^\lambda(T_0)|^2 + p^\lambda(x,\xi)(t-T_0)^2 \quad \text{for } t \in [T_0, \delta\lambda],$$

and this implies the assertion. □

Lemma 4.4. *Let $\delta > 0$ and Ω as in the previous lemma, and let $\sigma \in (0,\delta)$. Then*

$$\lim_{\lambda\to\infty} \pi_2\big(\exp(-\sigma\lambda H_{p_0^\lambda}) \circ \exp(\sigma\lambda H_{p^\lambda})(x,\xi)\big) = \pi_2\big(S_+(x,\xi)\big),$$

for $(x,\xi) \in \Omega$.

Proof. We denote

$$(z^\lambda(t;x,\xi), \zeta^\lambda(t;x,\xi)) = \exp(-tH_{p_0^\lambda}) \circ \exp(tH_{p^\lambda})(x,\xi),$$

and we show the convergence of $\zeta^\lambda(\sigma\lambda)$ to $\pi_2(S_+(x,\xi))$ for $(x,\xi) \in \Omega$. We recall

$$\exp(-tH_{p_0^\lambda})(x,\xi) = \big(\cos(\tfrac{t}{\lambda})x - \lambda\sin(\tfrac{t}{\lambda})\xi, \tfrac{1}{\lambda}\sin(\tfrac{t}{\lambda})x + \cos(\tfrac{t}{\lambda})\xi\big)$$

since p_0^λ is the scaled harmonic oscillator. Thus we have

$$z^\lambda(t) = \cos(\tfrac{t}{\lambda})y^\lambda(t) - \lambda\sin(\tfrac{t}{\lambda})\eta^\lambda(t),$$

and

$$\zeta^\lambda = \tfrac{1}{\lambda}\sin(\tfrac{t}{\lambda})y^\lambda(t) + \cos(\tfrac{t}{\lambda})\eta^\lambda(t).$$

By direct computations, we have

(4.4)

$$\begin{aligned}
\frac{d}{dt}z_k^\lambda &= -\frac{1}{\lambda}\sin(\tfrac{t}{\lambda})y_k^\lambda + \cos(\tfrac{t}{\lambda})\frac{dy_k^\lambda}{dt} - \cos(\tfrac{t}{\lambda})\eta_k^\lambda - \lambda\sin(\tfrac{t}{\lambda})\frac{d\eta_k^\lambda}{dt} \\
&= -\frac{1}{\lambda}\sin(\tfrac{t}{\lambda})y_k^\lambda + \cos(\tfrac{t}{\lambda})\sum_j a_{jk}(y^\lambda)\eta_j^\lambda - \cos(\tfrac{t}{\lambda})\eta_k^\lambda \\
&\quad + \frac{1}{\lambda}\sin(\tfrac{t}{\lambda})y_k^\lambda + \sin(\tfrac{t}{\lambda})\left(\frac{\lambda}{2}\sum_{i,j}\frac{\partial a_{ij}}{\partial x_k}(y^\lambda)\eta_i^\lambda\eta_j^\lambda + \frac{1}{\lambda}\frac{\partial V}{\partial x_k}(y^\lambda)\right) \\
&= \cos(\tfrac{t}{\lambda})\sum_j (a_{jk}(y^\lambda) - \delta_{jk})\eta_j^\lambda \\
&\quad + \sin(\tfrac{t}{\lambda})\left(\frac{\lambda}{2}\sum_{i,j}\frac{\partial a_{ij}}{\partial x_k}(y^\lambda)\eta_i^\lambda\eta_j^\lambda + \frac{1}{\lambda}\frac{\partial V}{\partial x_k}(y^\lambda)\right)
\end{aligned}$$

$$
\begin{aligned}
&= O(\langle y^\lambda\rangle^{-\mu}) + O(\lambda\langle y^\lambda\rangle^{-\mu-1} + \lambda^{-1}\langle y^\lambda\rangle^{1-\mu}) \\
&= O(\langle t\rangle^{-\mu})
\end{aligned}
$$

for $0 \le t \le \delta\lambda$. Similarly, we have

(4.5)

$$
\begin{aligned}
\frac{d}{dt}\zeta_k^\lambda &= \frac{1}{\lambda^2}\cos(\tfrac{t}{\lambda})y_k^\lambda + \frac{1}{\lambda}\sin(\tfrac{t}{\lambda})\frac{dy_k^\lambda}{dt} - \frac{1}{\lambda}\sin(\tfrac{t}{\lambda})\eta_k^\lambda + \cos(\tfrac{t}{\lambda})\frac{d\eta_k^\lambda}{dt} \\
&= \frac{1}{\lambda^2}\cos(\tfrac{t}{\lambda})y_k^\lambda + \frac{1}{\lambda}\sin(\tfrac{t}{\lambda})\sum_j a_{jk}(y^\lambda)\eta_j^\lambda - \frac{1}{\lambda}\sin(\tfrac{t}{\lambda})\eta_k^\lambda \\
&\quad - \frac{1}{\lambda^2}\cos(\tfrac{t}{\lambda})y_k^\lambda - \cos(\tfrac{t}{\lambda})\left(\frac{1}{2}\sum_{i,j}\frac{\partial a_{ij}}{\partial x_k}(y^\lambda)\eta_i^\lambda\eta_j^\lambda + \frac{1}{\lambda^2}\frac{\partial V}{\partial x_k}(y^\lambda)\right) \\
&= \frac{1}{\lambda}\sin(\tfrac{t}{\lambda})\sum_j (a_{jk}(y^\lambda) - \delta_{jk})\eta_j^\lambda \\
&\quad - \cos(\tfrac{t}{\lambda})\left(\frac{1}{2}\sum_{i,j}\frac{\partial a_{ij}}{\partial x_k}(y^\lambda)\eta_i^\lambda\eta_j^\lambda + \frac{1}{\lambda^2}\frac{\partial V}{\partial x_k}(y^\lambda)\right) \\
&= O(\lambda^{-1}\langle y^\lambda\rangle^{-\mu}) + O(\langle y^\lambda\rangle^{-\mu-1} + \lambda^{-2}\langle y^\lambda\rangle^{-(\mu-1)}) \\
&= O(\langle t\rangle^{-\mu-1})
\end{aligned}
$$

for $0 \le t \le \delta\lambda$. Moreover, for each $t \in \mathbb{R}$, we have

$$
\begin{aligned}
\lim_{\lambda\to\infty}\frac{d}{dt}z_k^\lambda(t) &= \sum_j a_{jk}(\tilde{y})\tilde{\eta}_j - \tilde{\eta}_k + \frac{t}{2}\sum_{i,j}\frac{\partial a_{ij}}{\partial x_k}(\tilde{y})\tilde{y}_j\tilde{\eta}_j \\
&= \frac{d}{dt}(\tilde{y}_k - t\tilde{\eta}_k), \\
\lim_{\lambda\to\infty}\frac{d}{dt}\zeta_k^\lambda(t) &= -\frac{1}{2}\sum_{i,j}\frac{\partial a_{ij}}{\partial x_k}(\tilde{y})\tilde{\eta}_i\tilde{\eta}_j = \frac{d}{dt}\tilde{\eta}_k,
\end{aligned}
$$

where $(\tilde{y}(t), \tilde{\eta}(t)) = \exp(tH_k)(x,\xi)$. By using the dominated convergence theorem, we have the following conclusion

$$
\begin{aligned}
\lim_{\lambda\to\infty}\zeta^\lambda(\sigma\lambda) &= \xi + \lim_{\lambda\to\infty}\int_0^{\sigma\lambda}\frac{d\zeta^\lambda}{dt}dt = \xi + \int_0^\infty \frac{d}{dt}\tilde{\eta}(t)dt \\
&= \lim_{t\to+\infty}\tilde{\eta}(t) = \pi_2(S_+(x,\xi)).
\end{aligned}
$$

This completes the proof of the lemma. □

From the proof of the above lemma, we see that $\lim_{\lambda\to\infty} z^\lambda(\sigma\lambda)$ does not necessarily exist, although $\lim_{\lambda\to\infty} \zeta^\lambda(\sigma\lambda)$ does exist. However we have the estimate

$$z^\lambda(\delta\lambda) = O(\lambda^{1-\mu}), \text{ as } \lambda \to \infty.$$

In fact we can extend these estimates to larger time as follows:

Lemma 4.5. *Let $0 < \sigma < \pi$, and Ω be as in the above lemma, then*

$$\lim_{\lambda\to\infty} \zeta^\lambda(\sigma\lambda) = \xi_+(x,\xi),$$

and for any $0 < \mu_0 < \mu < 1$,

$$\lim_{\lambda\to\infty} \frac{z^\lambda(\sigma\lambda)}{\lambda^{1-\mu_0}} = 0,$$

for $(x,\xi) \in \Omega$.

Proof. By above lemma, we need only to consider the case $\delta < \sigma < \pi$, and we fix such a σ. Let $\varepsilon_0 > 0$, we will show that if

$$\max\Big\{\frac{|z^\lambda(\sigma\lambda)|}{\lambda^{1-\mu_0}},\ |\zeta^\lambda(\sigma\lambda) - \xi_+|\Big\} > \varepsilon_0,$$

then λ is bounded from above. This will implies the result.

In the following argument, we may take ε_0 small, such that $\varepsilon_0 < \frac{|\xi_+|}{2}$. We choose $\lambda > 0$ sufficiently large such that

$$\max\Big\{\frac{|z^\lambda(\delta\lambda)|}{\lambda^{1-\mu_0}},\ |\zeta^\lambda(\delta\lambda) - \xi_+|\Big\} \le \frac{\varepsilon_0}{2},$$

and suppose

$$\max\Big\{\frac{|z^\lambda(\sigma\lambda)|}{\lambda^{1-\mu_0}},\ |\zeta^\lambda(\sigma\lambda) - \xi_+|\Big\} \ge \varepsilon_0,$$

then there exists $t_0 \in (\delta\lambda, \sigma\lambda)$ such that

$$\max\Big\{\frac{|z^\lambda(t_0)|}{\lambda^{1-\mu_0}},\ |\zeta^\lambda(t_0) - \xi_+|\Big\} = \varepsilon_0,$$

and

$$\max\Big\{\frac{|z^\lambda(t)|}{\lambda^{1-\mu_0}},\ |\zeta^\lambda(t) - \xi_+|\Big\} \le \varepsilon_0,$$

for $\delta\lambda \le t \le t_0$.

Since

$$y^\lambda(t) = \cos(\tfrac{t}{\lambda})z^\lambda(t) + \lambda\sin(\tfrac{t}{\lambda})\zeta^\lambda(t),$$

we have

$$|y^\lambda(t)| \ge \lambda\sin(\tfrac{t}{\lambda})(|\xi_+| - \varepsilon_0) - \lambda^{1-\mu_0}\varepsilon_0 \ge \delta_1\lambda$$

with some $\delta_1 > 0$ provided $\delta\lambda \le t \le t_0 < \sigma\lambda < \pi\lambda$, $\varepsilon_0 < \frac{|\xi_+|}{2}$ and $\lambda \ge \lambda_0$. Then by the formulas (4.4) and (4.5), we learn that

$$\left|\frac{d}{dt}z^\lambda(t)\right| \le C\lambda^{-\mu}, \ \left|\frac{d}{dt}\zeta^\lambda(t)\right| \le C\lambda^{-\mu-1}$$

for $t \in (\delta\lambda, t_0]$. Thus

$$\begin{aligned}\left|\frac{z^\lambda(t_0)}{\lambda^{1-\mu_0}}\right| &= \left|\frac{z^\lambda(\delta\lambda)}{\lambda^{1-\mu_0}} + \frac{1}{\lambda^{1-\mu_0}}\int_{\delta\lambda}^{t_0}\frac{dz^\lambda}{dt}(t)dt\right| \\ &\le \left|\frac{z^\lambda(\delta\lambda)}{\lambda^{1-\mu_0}}\right| + \frac{(t_0 - \delta\lambda)}{\lambda^{1-\mu_0}}\lambda^{-\mu} \\ &\le \frac{\varepsilon_0}{2} + C(\sigma - \delta)\lambda^{-(\mu-\mu_0)},\end{aligned}$$

and

$$\begin{aligned}|\zeta^\lambda(t_0) - \xi_+| &= \left|\zeta^\lambda(\delta\lambda) - \xi_+ + \int_{\delta\lambda}^{t_0}\frac{d\zeta^\lambda}{dt}(t)dt\right| \\ &\le |\zeta^\lambda(\delta\lambda) - \xi_+| + C(t_0 - \delta\lambda)\lambda^{-\mu-1} \\ &\le \frac{\varepsilon_0}{2} + C(\sigma - \delta)\lambda^{-\mu}.\end{aligned}$$

These estimates imply

$$\varepsilon_0 = \max\left\{\frac{|z^\lambda(t_0)|}{\lambda^{1-\mu_0}}, \ |\zeta^\lambda(t_0) - \xi_+|\right\} \le \frac{\varepsilon_0}{2} + C(\sigma - \delta)\lambda^{-(\mu-\mu_0)}$$

and hence $\lambda \le (2C(\sigma - \delta)/\varepsilon_0)^{1/(\mu-\mu_0)}$, and this proves the assertion.

□

From the above lemma, we learn that for any $0 < \sigma < \pi$, there exist $C_1, C_2 > 0$ such that

$$|y^\lambda(t)| \ge C_1|t| - C_2$$

for $t \in [0, \sigma\lambda]$. And then by (4.4), we also have

$$\frac{d}{dt} z^\lambda(t) = O(\langle t \rangle^{-\mu}),$$

for $t \in [0, \sigma\lambda]$. This implies that

$$z^\lambda(t) = O(\langle t \rangle^{1-\mu}).$$

as $\lambda \to \infty$.

Proposition 4.6. *Let $\delta_1 > 0$, then there exist $R_0 > 0,\ c_0 > 0,\ C > 0$ such that*

$$\left| \frac{\partial}{\partial x} \zeta(t, x, \xi) \right| \leq C R^{-1-\mu} |\xi|,$$
$$\left| \frac{\partial}{\partial \xi} (\zeta(t, x, \xi) - \xi) \right| \leq C R^{-\mu}$$

for $0 \leq t \leq \sigma < \pi$,

$$(x, \xi) \in \Omega_{R,\delta_1} := \left\{ (x, \xi) \in \mathbb{R}^{2n} \;\middle|\; \big||x| - R\big| \leq 1,\ |\xi| \geq \lambda,\ x \cdot \xi \geq \delta_1 |x||\xi| \right\}$$

with $R \geq R_0$, and $\lambda \geq c_0 R$. Moreover, for any $\alpha, \beta \in \mathbb{Z}^n_+$, there is $C_{\alpha\beta} > 0$ such that

$$\left| \left(\frac{\partial}{\partial x} \right)^\alpha \left(\frac{\partial}{\partial \xi} \right)^\beta (z(t, x, \xi) - x) \right| \leq C_{\alpha\beta} |t| \langle \xi \rangle^{1-\mu-|\beta|},$$
$$\left| \left(\frac{\partial}{\partial x} \right)^\alpha \left(\frac{\partial}{\partial \xi} \right)^\beta (\zeta(t, x, \xi) - \xi) \right| \leq C_{\alpha\beta} \langle \xi \rangle^{1-|\beta|}$$

for $(x, \xi) \in \Omega_{R,\delta_1}$ and $0 \leq t \leq \sigma < \pi$.

Remark 4.7. By similar calculation as in [25], we know that for large $R > 0$, the points in Ω_{R,δ_1} are forward nontrapping.

Proof. By setting $\lambda = |\xi|$, we need only to show the above estimates for z^λ and ζ^λ with $|\xi| = 1$, $\lambda \geq \lambda_0$ and $t \in [0, \sigma\lambda]$.

We mimic the argument in [25]. Let $s = \xi_j$ or x_j, $j = 1, ..., n$. We note the fact that

$$(y^\lambda(t), \eta^\lambda(t)) = \exp(t H_{p^\lambda})(z^\lambda, \zeta^\lambda).$$

From the formulas (4.4) and (4.5), we see that

$$\frac{d}{dt}\Big(\frac{\partial z_k^\lambda}{\partial s}\Big) = \cos(\tfrac{t}{\lambda})\sum_{j=1}^{n}(a_{jk}(y^\lambda)-\delta_{jk})[-\frac{1}{\lambda}\sin(\tfrac{t}{\lambda})\frac{\partial z_j^\lambda}{\partial s}+\cos(\tfrac{t}{\lambda})\frac{\partial \zeta_j^\lambda}{\partial s}]$$
$$+\cos(\tfrac{t}{\lambda})\sum_{j,\ell=1}^{n}\frac{\partial a_{jk}}{\partial x_\ell}(y^\lambda)\eta_j^\lambda[\cos(\tfrac{t}{\lambda})\frac{\partial z_\ell^\lambda}{\partial s}+\lambda\sin(\tfrac{t}{\lambda})\frac{\partial \zeta_\ell^\lambda}{\partial s}]$$
$$+\sin(\tfrac{t}{\lambda})\Big\{\lambda\sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial x_k}(y^\lambda)\eta_j^\lambda[-\frac{1}{\lambda}\sin(\tfrac{t}{\lambda})\frac{\partial z_i^\lambda}{\partial s}+\cos(\tfrac{t}{\lambda})\frac{\partial \zeta_i^\lambda}{\partial s}]$$
$$+\frac{\lambda}{2}\sum_{i,j,\ell=1}^{n}\frac{\partial^2 a_{ij}}{\partial x_k\partial x_\ell}(y^\lambda)\eta_i^\lambda\eta_j^\lambda[\cos(\tfrac{t}{\lambda})\frac{\partial z_\ell^\lambda}{\partial s}+\lambda\sin(\tfrac{t}{\lambda})\frac{\partial \zeta_\ell^\lambda}{\partial s}]$$
$$+\frac{1}{\lambda}\sum_{\ell=1}^{n}\frac{\partial^2 V}{\partial x_k\partial x_\ell}(y^\lambda)[\cos(\tfrac{t}{\lambda})\frac{\partial z_\ell^\lambda}{\partial s}+\lambda\sin(\tfrac{t}{\lambda})\frac{\partial \zeta_\ell^\lambda}{\partial s}]\Big\},$$

$$\frac{d}{dt}\Big(\frac{\partial \zeta_k^\lambda}{\partial s}\Big) = \frac{1}{\lambda}\sin(\tfrac{t}{\lambda})\sum_{j=1}^{n}(a_{jk}(y^\lambda)-\delta_{jk})[-\frac{1}{\lambda}\sin(\tfrac{t}{\lambda})\frac{\partial z_j^\lambda}{\partial s}+\cos(\tfrac{t}{\lambda})\frac{\partial \zeta_j^\lambda}{\partial s}]$$
$$+\frac{1}{\lambda}\sin(\tfrac{t}{\lambda})\sum_{j,\ell=1}^{n}\frac{\partial a_{jk}}{\partial x_\ell}(y^\lambda)\eta_j^\lambda[\cos(\tfrac{t}{\lambda})\frac{\partial z_\ell^\lambda}{\partial s}+\lambda\sin(\tfrac{t}{\lambda})\frac{\partial \zeta_\ell^\lambda}{\partial s}]$$
$$-\cos(\tfrac{t}{\lambda})\Big\{\sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial x_k}(y^\lambda)\eta_j^\lambda[-\frac{1}{\lambda}\sin(\tfrac{t}{\lambda})\frac{\partial z_i^\lambda}{\partial s}+\cos(\tfrac{t}{\lambda})\frac{\partial \zeta_i^\lambda}{\partial s}]$$
$$+\frac{1}{2}\sum_{i,j,\ell=1}^{n}\frac{\partial^2 a_{ij}}{\partial x_k\partial x_\ell}(y^\lambda)\eta_i^\lambda\eta_j^\lambda[\cos(\tfrac{t}{\lambda})\frac{\partial z_\ell^\lambda}{\partial s}+\lambda\sin(\tfrac{t}{\lambda})\frac{\partial \zeta_\ell^\lambda}{\partial s}]$$
$$+\frac{1}{\lambda^2}\sum_{\ell=1}^{n}\frac{\partial^2 V}{\partial x_k\partial x_\ell}(y^\lambda)[\cos(\tfrac{t}{\lambda})\frac{\partial z_\ell^\lambda}{\partial s}+\lambda\sin(\tfrac{t}{\lambda})\frac{\partial \zeta_\ell^\lambda}{\partial s}]\Big\}$$

Then $\left|\frac{\partial z^\lambda}{\partial s}\right|$ and $\left|\frac{\partial \zeta^\lambda}{\partial s}\right|$ are majorized by a solution to

$$\frac{d}{dt}Z \geq c_1(R+\delta|t|)^{-1-\mu}Z + c_1(R+\delta|t|)^{-\mu}\Xi$$
$$\frac{d}{dt}\Xi \geq c_1(R+\delta|t|)^{-2-\mu}Z + c_1(R+\delta|t|)^{-1-\mu}\Xi$$

for $0 \leq t \leq \lambda\sigma$, with

$$Z(0) \geq 0, \ \Xi(0) \geq 1, \ \text{if } s = \xi_j,$$
$$Z(0) \geq 1, \ \Xi(0) \geq 0, \ \text{if } s = x_j.$$

Let

$$Z = c_2(R+\delta t)^{1-\mu}, \ \Xi = c_3(1-(R+\delta t)^{-\mu'}), \ 0 \leq t \leq \lambda\sigma$$

with $0 < \mu' < \mu$. Then the differential inequalities for the majorants are satisfied if

$$\delta c_2(1-\mu) \geq c_1 c_2 R^{-\mu} + c_1 c_3,$$
$$\delta c_3 \mu' \geq R^{-(\mu-\mu')}(c_1 c_3 + c_1 c_2 R^{-2\mu+\mu'}),$$

and $R^{-\mu'} \leq \frac{1}{2}$ so that $\Xi > 0$. Now we choose c_2, c_3 and R_0 so that

$$\max\{\gamma := \frac{c_3}{c_2}, R^{-\mu}\} < \frac{\delta(1-\mu)}{2c_1},$$
$$R^{-\mu'} \leq \frac{1}{2},$$

for $R \geq R_0$, then the above conditions are satisfied. Thus we learn

$$\left|\frac{\partial z_j^\lambda}{\partial s}(t)\right| \leq c_2(R+\delta t)^{1-\mu}, \ \left|\frac{\partial \zeta_j^\lambda}{\partial s}(t)\right| \leq c_2\gamma,$$

for $R \geq R_0$, $\lambda \geq c_0 R$ and $t \in [0, \lambda\sigma]$, provided

$$\left|\frac{\partial z_j^\lambda}{\partial s}(0)\right| \leq c_2 R, \ \left|\frac{\partial \zeta_j^\lambda}{\partial s}(0)\right| \leq \frac{1}{2}c_2\gamma.$$

We now consider the case $s = x_k$. Then we may set $c_2 = R^{-1}$ and we have

$$\left|\frac{\partial z_j^\lambda}{\partial x_k}(t)\right| \leq \frac{1}{R}(R+\delta t)^{1-\mu}, \ \left|\frac{\partial \zeta_j^\lambda}{\partial x_k}(t)\right| \leq \frac{\gamma}{R}.$$

We integrate the above equation for $\frac{d}{dt}\left(\frac{\partial \zeta_j^\lambda}{\partial x_k}\right)$ to obtain

$$\begin{aligned}\left|\frac{\partial \zeta_j^\lambda}{\partial x_k}\right| &\leq \frac{c_1}{R}\int_0^t (R+\delta r^{-1-2\mu} dr + \frac{c_1\gamma}{R}\int_0^t (R+\delta r)^{-1-\mu} dr \\ &= \frac{c_1\gamma}{2\delta\mu} R^{-1-2\mu}(1-(1+\frac{\delta}{R}t)^{-2\mu}) \\ &\quad + \frac{c_1\gamma}{\delta\mu} R^{-1-\mu}(1-(1+\frac{\delta}{R}t)^{-\mu}) \\ &\leq CR^{-1-\mu},\end{aligned}$$

if $t \in [0, \lambda t_0]$. Similarly, if $s = \xi_k$, we may set $c_2 = \frac{2}{\gamma}$ and we have

$$\left|\frac{\partial z_j^\lambda}{\partial \xi_k}(t)\right| \le \frac{2}{\gamma}(R+\delta t)^{1-\mu}, \quad \left|\frac{\partial \zeta_j^\lambda}{\partial \xi_k}(t)\right| \le 2.$$

We integrate the above equation for $\frac{d}{dt}\left(\frac{\partial \zeta_j^\lambda}{\partial \xi_k}\right)$ again to obtain

$$\begin{aligned}\left|\frac{\partial \zeta_j^\lambda}{\partial \xi_k} - \delta_{jk}\right| &\le \frac{2c_1}{\gamma}\int_0^t (R+\delta r^{-1-2\mu} dr + 2c_1 \int_0^t (R+\delta r)^{-1-\mu} dr \\ &= \frac{c_1}{\gamma\delta\mu} R^{-2\mu}(1-(1+\frac{\delta}{R}t)^{-2\mu}) \\ &\quad + \frac{2c_1}{\delta\mu} R^{-\mu}(1-(1+\frac{\delta}{R}t)^{-\mu}) \\ &\le CR^{-\mu}.\end{aligned}$$

For higher derivatives, we can prove it by induction. Since the argument is similar to that in [25], we omit the details and the interesting reader may refer to that paper. □

Now we consider the map:

$$\Lambda : \xi \mapsto \zeta(t, R\xi/|\xi|, \xi).$$

The above proposition implies $\left\|\frac{\partial \Lambda}{\partial \xi} - I\right\| = O(R^{-\mu})$ uniformly for $|\xi| \ge c_0 R$. We choose R so large that $\frac{\partial \Lambda}{\partial \xi}$ is invertible for $|\xi| \ge c_0 R$. It is also easy to see that $|\Lambda - \xi| = O(R^{-\mu}|\xi|)$ for $|\xi| \ge c_0 R$, and hence

$$Ran(\Lambda) \supset \{\xi \in \mathbb{R}^n \bigm| |\xi| \ge c_4 R\}$$

with some $c_4 > 0$. Then we set

$$P(t, \cdot)^{-1} \ : \ \{\xi \in \mathbb{R}^n \bigm| |\xi| \ge c_4 R\} \to \mathbb{R}^n,$$

i.e.,

$$\zeta(t, RP(t,\xi)/|P(t,\xi)|, P(t,\xi)) = \xi, \ \text{for } |\xi| \ge c_4 R.$$

By the above proposition, we learn

$$\left|\left(\frac{\partial}{\partial \xi}\right)^\alpha P(t,\xi)\right| \le C\langle \xi \rangle^{1-|\alpha|}, \quad t \in [0,\sigma], \ |\xi| \ge c_4 R. \tag{4.6}$$

Then we set

$$W_1(t,\xi) = \int_0^t \Big(\ell(s,z(s),\zeta(s)) + z(s)\cdot\partial_t\zeta(s)\Big)ds + R|\xi|, \text{ for } |\xi| \geq c_4R,$$

where

$$z(s) = z(s, RP(t,\xi)/|P(t,\xi)|, P(t,\xi)),$$
$$\zeta(s) = \zeta(s, RP(t,\xi)/|P(t,\xi)|, P(t,\xi)).$$

It is well-known that $W_1(t,\xi)$ satisfies the Hamilton-Jacobi equation:

$$\frac{\partial}{\partial t}W_1(t,\xi) = \ell(t, \frac{\partial W_1}{\partial \xi}(t,\xi),\xi), \quad |\xi| \geq c_4R. \tag{4.7}$$

and

$$\partial_\xi W_1(t,\xi) = z(t, RP(t,\xi)/|P(t,\xi)|, P(t,\xi)). \tag{4.8}$$

By the construction, we also know

$$|\partial_\xi^\alpha W_1(t,\xi)| \leq C_\alpha \langle \xi \rangle^{2-\mu-|\alpha|}, \text{ for } t \in [0,\sigma],\ |\xi| \geq c_4R.$$

This can be seen as follows: for $|\alpha| \leq 1$, we use proposition 4.6, (4.8) and (4.6) to estimate $\partial_\xi^\alpha W_1(t,\xi)$; and for $|\alpha| = 0$, we use (4.8) and (4.7) to estimate it, by noticing that

$$\begin{aligned}\ell(t,x,\xi) =& \frac{1}{2}\sum_{jk=1}^n (a_{jk}(x\cos t + \xi\sin t) - \delta_{jk}) \\ & \times (-x_j \sin t + \xi_j \cos t)(-x_k \sin t + \xi_k \cos t) \\ & + V(x\cos t + \xi \sin t).\end{aligned}$$

Finally we use a partition of unity to construct $W(t,\xi)$ such that

$$W(t,\xi) = \begin{cases} W_1(t,\xi), & |\xi| \geq c_4R+1, \\ R|\xi| + t|\xi|^{2-\mu}/(2-\mu), & |\xi| \leq c_4R. \end{cases}$$

Lemma 4.8. *([25], Appendix) Suppose* $n \geq 2$, $f \in C^1(\mathbb{R}^n)$, *and suppose*

$$\big|\partial_x f(x)\big| \leq C\langle x\rangle^\beta, \quad x \in \mathbb{R}^n,$$

with some $C > 0$, $\beta \in \mathbb{R}$. *Then*

$$\big|f(x) - f(y)\big| \leq C\frac{\pi}{2}\max\{\langle x\rangle^\beta, \langle y\rangle^\beta\}\,|x-y|.$$

The same estimate holds for $n = 1$ *if* $x \cdot y > 0$.

Lemma 4.9. *Suppose (x_0, ξ_0) is forward/backward nontrapping, and Ω is a small neighborhood of (x_0, ξ_0), then*

$$\tilde{y}(t,x,\xi) - \tilde{y}(t, \pm R\tilde{P}(x,\xi)/|\tilde{P}(x,\xi)|, \tilde{P}(x,\xi)) = O(1),$$
$$\tilde{\eta}(t,x,\xi) - \tilde{\eta}(t, \pm R\tilde{P}(x,\xi)/|\tilde{P}(x,\xi)|, \tilde{P}(x,\xi)) = O(\langle t\rangle^{-1-\mu}),$$

as $t \to \pm\infty$, for $(x,\xi) \in \Omega$.

Proof. For simplify the notations we let

$$\tilde{y}(t) = \tilde{y}(t,x,\xi),\ \tilde{y}'(t) = \tilde{y}(t, \pm R\tilde{P}(x,\xi)/|\tilde{P}(x,\xi)|, \tilde{P}(x,\xi)),$$
$$\tilde{\eta}(t) = \tilde{\eta}(t,x,\xi),\ \tilde{\eta}'(t) = \tilde{\eta}(t, \pm R\tilde{P}(x,\xi)/|\tilde{P}(x,\xi)|, \tilde{P}(x,\xi)).$$

Then by the Hamilton equation and the above lemma, we have

$$\left|\frac{d}{dt}(\tilde{y}(t) - \tilde{y}'(t))\right| \le C\langle t\rangle^{-1-\mu}\big|\tilde{y}(t) - \tilde{y}'(t)\big| + C\big|\tilde{\eta}(t) - \tilde{\eta}'(t)\big|,$$
$$\left|\frac{d}{dt}(\tilde{\eta}(t) - \tilde{\eta}'(t))\right| \le C\langle t\rangle^{-2-\mu}\big|\tilde{y}(t) - \tilde{y}'(t)\big| + C\langle t\rangle^{-1-\mu}\big|\tilde{\eta}(t) - \tilde{\eta}'(t)\big|.$$

Since we initially know

$$\big|\tilde{y}(t)\big| = O(\langle t\rangle),\ \big|\tilde{\eta}(t) - \xi_\pm\big| = O(\langle t\rangle^{-\mu}),$$
$$\big|\tilde{y}'(t)\big| = O(\langle t\rangle),\ \big|\tilde{\eta}'(t) - \xi_\pm\big| = O(\langle t\rangle^{-\mu}),$$

we have

$$\big|\tilde{y}(t) - \tilde{y}'(t)\big| = O(\langle t\rangle),\quad \big|\tilde{\eta}(t) - \tilde{\eta}'(t)\big| = O(\langle t\rangle^{-\mu}).$$

We get $\big|\tilde{y}(t) - \tilde{y}'(t)\big| = O(\langle t\rangle^{1-\mu})$ by integrating the first differential inequality, and then $\big|\tilde{\eta}(t) - \tilde{\eta}'(t)\big| = O(\langle t\rangle^{-2\mu})$ by integrating the second differential inequality and using the new estimate for $\big|\tilde{\eta}(t) - \tilde{\eta}'(t)\big|$ and also noting that $\tilde{\eta}(\pm\infty) - \tilde{\eta}'(\pm\infty) = 0$. And then we repeat and iterate it to obtain

$$\big|\tilde{y}(t) - \tilde{y}'(t)\big| = O(1),\quad \big|\tilde{\eta}(t) - \tilde{\eta}'(t)\big| = O(\langle t\rangle^{-1-\mu}),$$

which completes the proof. □

Proposition 4.10. *Suppose (x_0, ξ_0) is backward nontrapping, and let $0 < t_0 < \pi$, then there exists a neighborhood Ω of (x_0, ξ_0) such that*

$$\xi_+(x,\xi) = \lim_{\lambda\to\infty} \lambda^{-1}\zeta(t_0, x, \lambda\xi),$$
$$z_+(x,\xi) = \lim_{\lambda\to\infty} \Big\{z(t_0,x,\lambda\xi) - \partial_\xi W(t_0, \zeta(t_0,x,\lambda\xi))\Big\},$$

for $(x,\xi) \in \Omega$.

Proof. We need only to show

$$\hat{z}^\lambda(t,x,\xi) = z^\lambda(t,x,\xi) - \partial_\xi W^\lambda(t,\zeta^\lambda(t_0,x,\xi))$$

converges as $\lambda \to \infty$, where $W^\lambda(t,\xi) = W(t/\lambda, \lambda\xi)$ and $t = \lambda t_0$. For (x,ξ) near (x_0,ξ_0), we choose $P^\lambda \in \mathbb{R}^n$ such that

$$\zeta^\lambda(t,x,\xi) = \zeta(t, RP^\lambda/|P^\lambda|, P^\lambda),$$

and we set

$$v^\lambda(s) = z^\lambda(s, RP^\lambda/|P^\lambda|, P^\lambda), \quad w^\lambda(s) = \zeta^\lambda(s, RP^\lambda/|P^\lambda|, P^\lambda)$$

for $s \in [0,t]$. We also set

$$\begin{aligned} a(s) &= z^\lambda(s,x,\xi) - v^\lambda(s), \\ b(s) &= \zeta^\lambda(s,x,\xi) - w^\lambda(s). \end{aligned}$$

Then $\big|a(0)\big| = \big|x - RP^\lambda/|P^\lambda|\big| \leq |x| + R$, $b(t) = 0$, and a, b satisfy the following differential equations:

$$\begin{aligned} \frac{d}{ds}a(s) &= \frac{\partial \ell^\lambda}{\partial \xi}(s, z^\lambda, \zeta^\lambda) - \frac{\partial \ell^\lambda}{\partial \xi}(s, v^\lambda, w^\lambda), \\ \frac{d}{ds}b(s) &= -\left(\frac{\partial \ell^\lambda}{\partial x}(s, z^\lambda, \zeta^\lambda) - \frac{\partial \ell^\lambda}{\partial x}(s, v^\lambda, w^\lambda)\right). \end{aligned}$$

where $\ell^\lambda(t,x,\xi) = p^\lambda \circ \exp(-tH_{p_0^\lambda})(x,\xi) - p_0^\lambda(x,\xi)$. Then by using the Hamilton equation and the above lemmas, we have the following differential inequalities

$$\begin{aligned} \big|a'(s)\big| &\leq c_1\langle s\rangle^{-1-\mu}|a(s)| + c_1\langle s\rangle^{-\mu}|b(s)|, \\ \big|b'(s)\big| &\leq c_1\langle s\rangle^{-2-\mu}|a(s)| + c_1\langle s\rangle^{-1-\mu}|b(s)|, \end{aligned}$$

for $s \in [0,t]$ with some $c_1 > 0$. Since we initially know that $a(s) = O(\langle s\rangle^{1-\mu})$, $b(s) = O(1)$, we get

$$|b(s)| = \left|\int_s^t b'(u)du\right| \leq c_2\langle s\rangle^{-\mu},$$

by integrating the second differential inequality, and then

$$|a(s)| = \left|a(0) + \int_0^s a'(u)du\right| \leq |x| + R + c_3\langle s\rangle^{1-2\mu} = O(\langle s\rangle^{1-2\mu}),$$

by integrating the first differential inequality and using the new estimate for $|b(s)|$. Repeating these steps, we have

$$|b(s)| = O(\langle s\rangle^{-2\mu}), \quad |a(s)| = O(\langle s\rangle^{1-3\mu}).$$

Then by iterating it, we finally get

$$|b(s)| = O(\langle s\rangle^{-1-\mu}), \quad |a(s)| = O(1).$$

Moreover, we have the following

$$|a'(s)| = O(\langle s\rangle^{-1-\mu})$$

which implies $a'(s) \in L^1(\mathbb{R})$.

Since $\zeta^\lambda(t,x,\xi) \to \xi_+(x,\xi)$ as $\lambda \to \infty$, for $t = \lambda t_0$, $0 < t_0 < \pi$, we have $P^\lambda \to \tilde{P}$ as $\lambda \to \infty$, where $\tilde{P}$ is given by the equation: $\xi_+(x,\xi) = \xi_+(R\tilde{P}/|\tilde{P}|, \tilde{P})$. Hence, in particular, $v(s) \to \tilde{y}(s, R\tilde{P}/|\tilde{P}|, \tilde{P})$ for each s. Then by the Lebesgue convergence theorem, we learn

$$\begin{aligned}
&\lim_{\lambda\to\infty}\Big\{z(t_0,x,\lambda\xi) - \partial_\xi W(t_0,\zeta(t_0,x,\lambda\xi))\Big\}\\
&= \lim_{\lambda\to\infty}\Big\{z^\lambda(t,x,\lambda\xi) - v^\lambda(t)\Big\}\\
&= \lim_{\lambda\to\infty}\Big\{x - R\tilde{P}/|\tilde{P}| + \int_0^t \frac{d}{ds}(z^\lambda(s,x,\xi) - v^\lambda(s))ds\Big\}\\
&= x - R\tilde{P}/|\tilde{P}| + \int_0^\infty \frac{d}{ds}\Big(\tilde{y}(s) - \tilde{y}(s, R\tilde{P}/|\tilde{P}|, \tilde{P})\Big)ds\\
&\quad - \int_0^\infty \frac{d}{ds}\Big\{s\Big(\tilde{\eta}(s) - \tilde{\eta}(s, R\tilde{P}/|\tilde{P}|, \tilde{P})\Big)\Big\}ds\\
&= \lim_{s\to+\infty}\Big(\tilde{y}(s) - \tilde{y}(s, R\tilde{P}/|\tilde{P}|, \tilde{P})\Big)\\
&\quad + \lim_{s\to+\infty}\Big\{s\Big(\tilde{\eta}(s) - \tilde{\eta}(s, R\tilde{P}/|\tilde{P}|, \tilde{P})\Big)\Big\}\\
&= z_+(x,\xi).
\end{aligned}$$

in the last step we used the fact that

$$\lim_{s\to+\infty}\Big(\tilde{y}(s) - \tilde{y}(s, R\tilde{P}/|\tilde{P}|, \tilde{P})\Big) = z_+(x,\xi),$$
$$\tilde{\eta}(s) - \tilde{\eta}(s, R\tilde{P}/|\tilde{P}|, \tilde{P}) = O(\langle s\rangle^{-1-\mu}),$$

which is proved in lemma 4.9. This implies the result. □

4.3 Proof of the main theorems

In this section, we consider the case $0 < t_0 < \pi$, the case for $-\pi < t_0 < 0$ is the same.

Lemma 4.11. *Let $\nu, \rho > 0$ and suppose $a \in S(\langle x\rangle^\nu \langle\xi\rangle^\rho, dx^2/\langle x\rangle^2 + d\xi^2/\langle\xi\rangle^2)$ and the terms involving ξ are homogeneous polynomials w.r.t ξ. Let*

$$Q = e^{iW(t,D_x)} e^{itH_0} a(x, D_x) e^{-itH_0} e^{-iW(t,D_x)}.$$

Then $Q \in OPS_K(\langle t\xi\rangle^\nu \langle\xi\rangle^\rho, dx^2/\langle t\xi\rangle^2 + d\xi^2/\langle\xi\rangle^2)$ with any $K \subset\subset \mathbb{R}^n$. Let $g(t,x,\xi) = \sigma(Q)$ be the Weyl symbol of Q. Then the principal symbol of Q is given by

$$g_0 = a\Big((x + \partial_\xi W(t,\xi)) \cos t + \xi \sin t, -(x + \partial_\xi W(t,\xi)) \sin t + \xi \cos t\Big),$$

and

$$g(t,x,\xi) - g_0 \in S_K(\langle t\xi\rangle^{\nu-2} \langle\xi\rangle^{\rho-2}, dx^2/\langle t\xi\rangle^2 + d\xi^2/\langle\xi\rangle^2).$$

Here $t \in [-t_0, 0)$.

Proof. The proof is similar to the one in [25], so we just sketch it here. Since the symbol of H_0 is a polynomial of order 2, we have

$$e^{itH_0} a(x, D_x) e^{-itH_0} u = a(x \cos t + D_x \sin t, -x \sin t + D_x \cos t) u,$$

for $u \in \mathcal{S}(\mathbb{R}^n)$. And we write

$$b(t,x,\xi) = a(x \cos t + \xi \sin t, -x \sin t + \xi \cos t).$$

Since the Weyl quantization has the same symbol representation in the Fourier space as in the configuration space, we may write

$$\begin{aligned}\hat{A}u &:= \mathcal{F}(b(t,x,D_x)\mathcal{F}^{-1}u) \\ &= (2\pi)^{-n} \int\int e^{-i(\xi-\eta)\cdot x} b(t,x,\frac{\xi+\eta}{2}) u(\eta) d\eta dx\end{aligned}$$

for $u \in \mathcal{S}(\mathbb{R}^n)$. Then by direct computations, we have

$$\begin{aligned}&e^{iW(t,\xi)} \hat{A} e^{-iW(t,\xi)} u(\xi) \\ &= (2\pi)^{-n} \int\int e^{i(W(t,\xi)-W(t,\eta)) - i(\xi-\eta)\cdot x} b(t,x,\frac{\xi+\eta}{2}) u(\eta) d\eta dx \\ &= (2\pi)^{-n} \int\int e^{i(\xi-\eta)\cdot(x - \tilde{W}(t,\xi,\eta))} b(t,x,\frac{\xi+\eta}{2}) u(\eta) d\eta dx\end{aligned}$$

$$= (2\pi)^{-n} \int\int e^{i(\xi-\eta)\cdot x} b(t, x + \tilde{W}(t,\xi,\eta), \frac{\xi+\eta}{2}) u(\eta) d\eta dx,$$

where

$$\tilde{W}(t,\xi,\eta) = \int_0^1 \partial_\xi W(t, s\xi + (1-s)\eta) ds$$
$$= \frac{1}{2}\int_{-1}^1 \partial_\xi W(t, \frac{\xi+\eta}{2} + \tau\frac{\xi-\eta}{2}) d\tau.$$

We easily see that $\tilde{W}(t,\xi,\xi) = \partial_\xi W(t,\xi)$ and

$$\Big|\partial_\xi^\alpha \partial_\eta^\beta \Big\langle \Big(\frac{\xi+\eta}{2}\Big) + \tau\Big(\frac{\xi-\eta}{2}\Big)\Big\rangle^{1-\mu}\Big|$$
$$\le C_{\alpha\beta}\Big\langle\frac{\xi+\eta}{2}\Big\rangle^{1-\mu-|\alpha+\beta|} \langle\xi-\eta\rangle^{1-\mu+|\alpha+\beta|},$$

for any $\alpha, \beta \in \mathbb{Z}_+^n$. By the definition of $W(t,\xi)$, we learn

$$|\partial_\xi^\alpha W(t,\xi)| \le C_\alpha\Big(\langle\xi\rangle^{1-|\alpha|} + |t|\langle\xi\rangle^{2-\mu-|\alpha|}\Big),$$

and hence

$$|\partial_\xi^\alpha \partial_\eta^\beta \tilde{W}(t,\xi,\eta)|$$
$$\le C_{\alpha\beta}\Big(\Big\langle\frac{\xi+\eta}{2}\Big\rangle^{-|\alpha+\beta|}\langle\xi-\eta\rangle^{|\alpha+\beta|}$$
$$+ |t|\Big\langle\frac{\xi+\eta}{2}\Big\rangle^{1-\mu-|\alpha+\beta|}\langle\xi-\eta\rangle^{1-\mu+|\alpha+\beta|}\Big)$$
$$\le C_{\alpha\beta}\Big\langle t\Big(\frac{\xi+\eta}{2}\Big)\Big\rangle^{1-\mu}\Big\langle\frac{\xi+\eta}{2}\Big\rangle^{-|\alpha+\beta|}\langle\xi-\eta\rangle^{1-\mu+|\alpha+\beta|}.$$

Since $\sin t \ne 0$ when $t \in (0, t_0]$, we learn, from the definition of $\tilde{W}(t,\xi,\eta)$,

$$C_1\Big\langle t\Big(\frac{\xi+\eta}{2}\Big)\Big\rangle \le \Big|(x + \tilde{W}(t,\xi,\eta))\cos t + \Big(\frac{\xi+\eta}{2}\Big)\sin t\Big|$$
$$\le C_2\Big\langle t\Big(\frac{\xi+\eta}{2}\Big)\Big\rangle,$$

if $|\xi-\eta| \le \frac{1}{2}|\xi+\eta|$ and for $x \in K \subset\subset \mathbb{R}^n$. Now since

$$b\Big(t, x + \tilde{W}(t,\xi,\eta), \frac{\xi+\eta}{2}\Big) = a(X(t), \Xi(t))$$

where

$$X(t) = \big(x + \tilde{W}(t,\xi,\eta)\big)\cos t + \Big(\frac{\xi+\eta}{2}\Big)\sin t,$$
$$\Xi(t) = -\big(x + \tilde{W}(t,\xi,\eta)\big)\sin t + \Big(\frac{\xi+\eta}{2}\Big)\cos t,$$

we consider two cases:

(i) $\cos t \neq 0$. In this case, we also have

$$C_1\Big\langle\frac{\xi+\eta}{2}\Big\rangle \leq \Big|-\big(x + \tilde{W}(t,\xi,\eta)\big)\sin t + \Big(\frac{\xi+\eta}{2}\Big)\cos t\Big|$$
$$\leq C_2\Big\langle\frac{\xi+\eta}{2}\Big\rangle,$$

if $|\xi-\eta| \leq \frac{1}{2}|\xi+\eta|$ and for $x \in K \subset\subset \mathbb{R}^n$. Then we can show

$$\Big|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma b\Big(t, x + \tilde{W}(t,\xi,\eta), \frac{\xi+\eta}{2}\Big)\Big|$$
$$\leq C_{\alpha\beta\gamma}\Big\langle t\Big(\frac{\xi+\eta}{2}\Big)\Big\rangle^{\nu-|\alpha|}\Big\langle\frac{\xi+\eta}{2}\Big\rangle^{\rho-|\beta+\gamma|}\langle\xi-\eta\rangle^{|\nu|+|\rho|+|\alpha+\beta+\gamma|}$$

for $x \in K \subset\subset \mathbb{R}^n$.

(ii) $\cos t = 0$. In this case, we have

$$b\Big(t, x + \tilde{W}(t,\xi,\eta), \frac{\xi+\eta}{2}\Big) = a\Big(\frac{\xi+\eta}{2}, -\big(x + \tilde{W}(t,\xi,\eta)\big)\Big).$$

Since $a(x,\xi)$ is supposed to be a polynomial with respect to ξ, we need only to consider the number of derivatives with respect to the second variable of $a(x,\xi)$ small than ρ.

First we consider derivatives with respect to x, we have

$$\Big|\partial_x^\alpha b\Big(t, x + \tilde{W}(t,\xi,\eta), \frac{\xi+\eta}{2}\Big)\Big|$$
$$\leq C_\alpha\Big\langle x + \tilde{W}(t,\xi,\eta)\Big\rangle^{\rho-|\alpha|}\Big\langle\frac{\xi+\eta}{2}\Big\rangle^{\nu}$$
$$\leq C_\alpha\Big\langle\frac{\xi+\eta}{2}\Big\rangle^{\nu}\Big\langle\frac{\xi+\eta}{2}\Big\rangle^{(1-\mu)(\rho-|\alpha|)}$$
$$= C_\alpha\Big\langle\frac{\xi+\eta}{2}\Big\rangle^{\nu+\rho}\Big\langle\frac{\xi+\eta}{2}\Big\rangle^{-(1-\mu)|\alpha|-\mu\rho}$$
$$\leq C_\alpha\Big\langle\frac{\xi+\eta}{2}\Big\rangle^{\nu+\rho}\Big\langle\frac{\xi+\eta}{2}\Big\rangle^{-|\alpha|},$$

in the last step we used the fact that we need only to consider $|\alpha| \le \rho$.

The derivatives for ξ is treated by the same trick as for x, by noting that one derivative with respect to the first variable of $a(x,\xi)$ we get a decreasing factor of the form $\left\langle \frac{\xi+\eta}{2} \right\rangle^{-1}$, and one derivative with respect to the second variable leads to a extra decreasing factor of the form $\left\langle \frac{\xi+\eta}{2} \right\rangle^{-\mu}$, and also noting that one derivative of $\tilde{W}(t,\xi,\eta)$ leads to a decreasing term $\left\langle \frac{\xi+\eta}{2} \right\rangle^{-1}$. Thus we need only to consider the term which can be estimated by a form

$$C_\beta \langle x + \tilde{W}(t,\xi,\eta) \rangle^{\rho-|\beta|} \big| \partial_\xi \tilde{W}(t,\xi,\eta) \big|^{|\beta|},$$

which in turn can be estimated by

$$\begin{aligned} &C_\beta \langle x + \tilde{W}(t,\xi,\eta) \rangle^{\rho-|\beta|} \big| \partial_\xi \tilde{W}(t,\xi,\eta) \big|^{|\beta|} \\ &\quad \le C_\beta \Big\langle \frac{\xi+\eta}{2} \Big\rangle^{(\rho-|\beta|)(1-\mu)} \Big\langle \frac{\xi+\eta}{2} \Big\rangle^{-\mu|\beta|} \\ &\quad \le C_\beta \Big\langle \frac{\xi+\eta}{2} \Big\rangle^{(1-\rho)\mu-|\beta|}. \end{aligned}$$

Combining the above estimates we can show that

$$\begin{aligned} &\Big| \partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma b\Big(t, x + \tilde{W}(t,\xi,\eta), \frac{\xi+\eta}{2}\Big) \Big| \\ &\quad \le C_{\alpha\beta\gamma} \Big\langle t\Big(\frac{\xi+\eta}{2}\Big) \Big\rangle^{\nu-|\alpha|} \Big\langle \frac{\xi+\eta}{2} \Big\rangle^{\rho-|\beta+\gamma|} \langle \xi - \eta \rangle^{|\nu|+|\rho|+|\alpha+\beta+\gamma|} \end{aligned}$$

for $x \in K \subset\subset \mathbb{R}^n$. Then by the asymptotic expansion formula for the simplified symbol, we learn that the principle symbol is given by

$$b\Big(t, x + \tilde{W}(t,\xi,\eta), \frac{\xi+\eta}{2}\Big),$$

and we have

$$|\partial_x^\alpha \partial_\xi^\beta g(t,x,\xi)| \le C_{\alpha\beta} \langle t\xi \rangle^{\nu-|\alpha|} \langle \xi \rangle^{\rho-|\beta|}$$

for $x \in K$, $\xi \in \mathbb{R}^n$. The other claims follow from the asymptotic expansion formula. □

Now we will prove Theorem 4.1(i). Let $a \in C_0^\infty(\mathbb{R}^{2n})$ such that $a(x_0, \xi_0) \neq 0$ and supported in a small neighborhood of (x_0, ξ_0), and set

$$a_h(x, \xi) = a(x, h\xi).$$

We also set

$$A(t) = e^{iW(t,D_x)} e^{itH_0} e^{-itH} a_h(x, D_x) e^{itH} e^{-itH_0} e^{-iW(t,D_x)}$$

for $t \in [-t_0, 0]$. Then $A(t)$ satisfies the Heisenberg equation:

$$\frac{d}{dt} A(t) = -i[\tilde{L}(t), A(t)],$$

with the initial condition:

$$A(0) = e^{iW(0,D_x)} a_h(x, D_x) e^{-iW(0,D_x)} = \tilde{a}_h(x, D_x),$$

where

$$\tilde{L}(t) = e^{iW(t,D_x)} (e^{itH_0} H e^{-itH_0} - H_0) e^{-iW(t,D_x)} - \frac{\partial W}{\partial t}(t, D_x).$$

We also note that the principal symbol of $\tilde{a}_h(x, \xi)$ is given by $a(x - R\xi/|\xi|, h\xi)$, and $\tilde{a}_h(x, \xi)$ is supported in a neighborhood of $(x_0 - R\xi_0/|\xi_0|, \xi_0/h)$ modulo $O(h^\infty)$-terms.

We denote

$$\begin{aligned} S_t : \ (x, \xi) &\mapsto (\hat{z}(t, x, \xi), \zeta(t, x, \xi)), \\ S_t^\lambda : \ (x, \xi) &\mapsto (\hat{z}^\lambda(t, x, \xi), \zeta^\lambda(t, x, \xi)). \end{aligned}$$

S_t (resp. S_t^λ) is the Hamilton flow generated by

$$\tilde{\ell}(t, x, \xi) = \ell(t, x + \partial_\xi W(t, \xi), \xi) - \partial_t W(t, \xi)$$

$(\tilde{\ell}^\lambda(t, x, \xi) = \lambda^{-2} \tilde{\ell}(t/\lambda, x, \lambda\xi)$, resp.) with the initial condition:

$$\hat{z}(0, x, \xi) = x + R\xi/|\xi|, \ \ \zeta(0, x, \xi) = \xi.$$

By virtue of the Hamilton-Jacobi equation, we have

$$\tilde{\ell}(t, x, \xi) = \ell(t, x + \partial_\xi W(t, \xi), \xi) - \ell(t, \partial_\xi W(t, \xi), \xi) \tag{4.9}$$

for sufficiently large $|\xi|$.

By the above lemma, we learn that the principle symbol of $\tilde{L}(t)$ is given by $\tilde{\ell}(t,x,\xi)$, and the remainder symbol $r(t,x,\xi)$ satisfies

$$|\partial_x^\alpha \partial_\xi^\beta r(t,x,\xi)| \le C_{\alpha\beta}\big(\langle t\xi\rangle^{-\mu-2-|\alpha|}\langle\xi\rangle^{-|\beta|} + \langle t\xi\rangle^{-\mu-|\alpha|}\langle\xi\rangle^{-2-|\beta|}\big)$$

for $x \in K \subset\subset \mathbb{R}^n$, $t \in [-t_0, 0]$.

If we denote the scaling with respect to ξ by $\mathfrak{d}_\lambda$, i.e.,

$$\mathfrak{d}_\lambda(x,\xi) = (x, \lambda\xi) \quad \text{for } (x,\xi) \in \mathbb{R}^{2n},$$

then we can show that

$$S_t \circ \mathfrak{d}_\lambda = \mathfrak{d}_\lambda \circ S_{\lambda t}^\lambda,$$

as in the argument for short-range perturbation of harmonic oscillators. Thus we get

$$\begin{aligned} f_h \circ S_t^{-1} &= f \circ \mathfrak{d}_\lambda^{-1} \circ S_t^{-1} = f \circ (S_t \circ \mathfrak{d}_\lambda)^{-1} \\ &= f \circ (\mathfrak{d}_\lambda \circ S_{\lambda t}^\lambda)^{-1} = f \circ (S_{\lambda t}^\lambda)^{-1} \circ \mathfrak{d}_\lambda^{-1} \end{aligned}$$

with $h = \frac{1}{\lambda}$. Then we can construct an asymptotic solution to the Heisenberg equation by the same argument as in [25] in the following lemma and therefore complete the proof of the theorem 4.1.

Lemma 4.12. *Let $0 < t_0 < \pi$, and set $I = [-t_0, 0]$. There exists $\psi(t;x,\xi) \in C_0^\infty(\mathbb{R}^{2n})$ for $t \in I$ such that*

(i) If we write $G(t) = \psi(t; x, D_x)$, then

$$G(0) = e^{iW(0,D_x)} a_h(x, D_x) e^{-iW(0,D_x)}$$

modulo $O(h^\infty)$-terms.

(ii) $\psi(t;x,\xi)$ is supported in $S_t \circ \mathfrak{d}_\lambda(\Omega) = \mathfrak{d}_\lambda \circ S_{\lambda t}^\lambda(\Omega)$.

(iii) For any $\alpha, \beta \in \mathbb{Z}_+^n$, there is $C_{\alpha\beta} > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta \psi(t;x,\xi)\right| \le C_{\alpha\beta} h^{|\beta|}, \quad t \in I, x, \xi \in \mathbb{R}^n.$$

(iv) The principal symbol of ψ is given by ψ_0, i.e., for any $\alpha, \beta \in \mathbb{Z}_+^n$, there is $C_{\alpha\beta} > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta (\psi(t;x,\xi) - \psi_0(t;x,\xi))\right| \le C_{\alpha\beta} h^{1+|\beta|}, \quad t \in I, x, \xi \in \mathbb{R}^n.$$

(v) For $t \in I$, we have

$$\left\| \frac{d}{dt} G(t) + i[L(t), G(t)] \right\|_{\mathcal{L}(\mathcal{H})} = O(h^\infty)$$

as $h \to 0$.

Proof. Sketch: We denote

$$c(x,\xi) = \frac{1}{2} \sum_{j,k=1}^{n} (a_{jk}(x) - \delta_{jk})\xi_j \xi_k + V(x).$$

Then

$$\ell(t,x,\xi) = c(x\cos t + \xi \sin t, -x\sin t + \cos t).$$

By lemma 4.11 with a replaced by c, we see that

$$\ell(t, x + \partial_\xi W(t,\xi), \xi) \in S(\langle t\xi\rangle^{-\mu}\langle\xi\rangle^2 + \langle t\xi\rangle^{2-\mu}, dx^2/\langle t\xi\rangle^2 + d\xi^2/\langle\xi\rangle^2)$$

for $x \in K \subset\subset \mathbb{R}^n$. Then by (4.9), we have

$$\tilde{\ell}(t, x + \partial_\xi W(t,\xi), \xi) \in S(\langle t\xi\rangle^{-\mu-1}\langle\xi\rangle^2 + \langle t\xi\rangle^{1-\mu}, dx^2/\langle t\xi\rangle^2 + d\xi^2/\langle\xi\rangle^2)$$

for $x \in K \subset\subset \mathbb{R}^n$. Now the remaining argument is similar to [25], so we omit the detail and refer the reader to that paper. □

Proof of Theorem 4.1(i). By the above lemma and the construction of $\tilde{L}(t)$, we have

$$\left\| \frac{d}{dt} \left(e^{itH} e^{-itH_0} e^{-iW(t,D_x)} G(t) e^{iW(t,D_x)} e^{itH_0} e^{-itH} \right) \right\| = O(h^\infty),$$

as $h \to 0$, for $t \in I$. This implies

$$\Big\| e^{-it_0H} e^{it_0H_0} e^{-iW(-t_0,D_x)} G(-t_0) e^{iW(-t_0,D_x)} e^{-it_0H_0} e^{it_0H} u \\ - e^{-iW(0,D_x)} G(0) e^{iW(0,D_x)} u \Big\| = O(h^\infty).$$

By (i) of the above lemma, we have

$$\Big| \left\| G(-t_0) e^{iW(-t_0,D_x)} e^{-it_0H_0} u_0 \right\| - \left\| a_h(x,D_x) u(t_0) \right\| \Big| = O(h^\infty),$$

where $u(t) = e^{-itH}u_0$ with $u_0 \in L^2(\mathbb{R}^n)$. We note that the principal symbol of $\psi(-t_0, x, \xi)$ is given by $\psi_0(-t_0, x, \xi) = a_h \circ S_{-t_0}^{-1}$. Then by (ii) of the above lemma, we learn

$$|\psi(-t_0, x, \xi)| \geq \varepsilon > 0$$

for $|x - z_-(x_0, \xi_0)| \leq \delta$, $|\xi/h - \xi_-(x_0, \xi_0)| \leq \delta$ with some $\delta, \varepsilon > 0$.

Now we suppose $(x_0, \xi_0) \notin WF(u(t_0))$. Then by choosing $a(x, \xi)$ supported in a sufficiently small neighborhood of (x_0, ξ_0), we may suppose

$$\|a_h(x, D_x)u(t_0)\| = O(h^\infty).$$

Then we obtain

$$\|G(-t_0)e^{iW(-t_0, D_x)}e^{-it_0H_0}u_0\| = O(h^\infty)$$

and this implies

$$(z_-(x_0, \xi_0), \xi_-(x_0, \xi_0)) \notin WF(e^{iW(-t_0, D_x)}e^{-it_0H_0}u_0).$$

Conversely, if $(z_-(x_0, \xi_0), \xi_-(x_0, \xi_0)) \notin WF(e^{iW(-t_0, D_x)}e^{-it_0H_0}u_0)$, then also by taking $a(x, \xi)$ supported in a sufficiently small neighborhood of (x_0, ξ_0), we have $|\psi(-t_0, x, \xi)| \geq \varepsilon > 0$ for $|x - z_-(x_0, \xi_0)| \leq \delta$, $|\xi/h - \xi_-(x_0, \xi_0)| \leq \delta$ with some $\delta, \varepsilon > 0$, since $\psi(-t_0, x, \xi)$ is supported in $S_{-t_0}(supp\ a_h)$ modulo $O(h^\infty)$-terms, and it is very close to $S_-(supp\ a_h)$ if h is small enough. Then we have $\|a_h(x, D_x)u(t_0)\| = O(h^\infty)$, and hence $(x_0, \xi_0) \notin WF(u(t_0))$. □

Remark 4.13.

(i) We can also study the long-range perturbation of the inhomogeneous Harmonic oscillators as in short-range perturbations in Chapter 2.

(ii) We can also study the long-range perturbation of Magnetic fields as in Chapter 3.

The argument is almost the same as the above proof, so we omit the details.

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