Solvability and irreducibility of difference equations

（差分方程式の可解性と既約性）

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Contents

Chapter 1

Introduction

The elementary functions are basic functions on the differential and integral calculus. For example, the exponential function, the trigonometric functions and their inverses. Some authors have studied them on the differential algebra. Liouville introduced the concept of the elementary extension, which is constructed from functions having an explicit representation in terms of a finite number of algebraic operations, logarithms, and exponentials (cf. [25, 26]). In this paper we study solvability and irreducibility of difference equations in the same direction as Liouville. We introduce Rosenlicht's definition of the elementary extension to see the concept precisely.

Define a differential field to be a field K, together with a derivation on K, that is, a map of K to itself, usually denoted $a \mapsto a'$, such that $(a+b)' = a'+b'$ and $(ab)' = a'b + ab'$ for all $a, b \in K$. We call a an exponential of b, or b a logarithm of a, if $b' = a'/a$. By a differential overfield of a differential field K we mean a differential field which is an overfield of K whose derivation extends the derivation on K. An elementary extension of a differential field K is defined to be a differential overfield of the form $K(t_1, \dots, t_n)$, where for each $i = 1, \dots, n$, the element t_i is either algebraic over the field $K(t_1, \dots, t_{i-1})$, or the logarithm or exponential of an element of $K(t_1, \dots, t_{i-1})$.

Although we are familiar with the elementary functions, it has not been clarified what kind of functions are elementary on the difference algebra. On this problem Karr defined the $\Pi\Sigma$-extension and introduced a difference analogue of Liouville's theorem on elementary integrals (see [13]). The $\Pi\Sigma$-extension is constructed from solutions of difference equations of the form, $y_1 = \alpha y + \beta$, $\alpha \neq 0$. The gamma function for example satisfies $\Gamma(x+1) =$

$x\Gamma(x)$, the case $\alpha = x$ and $\beta = 0$, and

$$\log \Gamma(x+1) = \log \Gamma(x) + \log x,$$

the case $\alpha = 1$ and $\beta = \log x$.

Other transforming operators than $x \mapsto x+1$ can be considered. We introduce some examples with elementary difference equations. For example, $\cos x$ satisfies the multiplication formula, $\cos 2x = 2\cos^2 x - 1$, where the transforming operator is $x \mapsto 2x$. On this operator, $\log x$ satisfies $y_1 = y + \log 2$, the case $\alpha = 1$ and $\beta = \log 2$ of the preceding form. The q-gamma function Γ_q tells us another example (cf. [7]). It is defined for $0 < q < 1$ and satisfies

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x), \quad \Gamma_q(1) = 1.$$

If we put $t = q^x$, then q-gamma function is a solution of the difference equation,

$$y_1 = \frac{1-t}{1-q}y, \quad (\alpha = \frac{1-t}{1-q}, \beta = 0),$$

with the transforming operator $t \mapsto qt$. In his paper [15] Koornwinder says that $\Gamma_q(x+1)$ tends to the gamma function $\Gamma(x+1)$ for all complex $x \neq -1, -2, \ldots$ as $q \to 1^-$, which should be one of the reasons why the q-gamma function is said to be a q-analogue of the gamma function. Difference equations with the transforming operator $t \mapsto qt$, $q \in \mathbb{C}^\times$, which are multiplication formulas, are especially called q-difference equations.

The transforming operator $z \mapsto z^2$ is the last example. The function $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ is called the Mahler function and satisfies $f(z^2) = f(z) - z$. Mahler proved that $f(\alpha)$ is a transcendental number for any algebraic number α with $0 < |\alpha| < 1$ (see [17]).

Before we define the solvability of difference equations, we introduce a solvable equation to see the notion. The difference equation,

$$y_2 - y_1 - x^2 y = 0, \tag{1.1}$$

with the transforming operator $x \mapsto x+1$ can be reduced to two equations of the form $y_1 = \alpha y + \beta$, $\alpha \neq 0$. In fact let f be one of its solutions. Then f satisfies

$$f_2 - (x+1)f_1 = -xf_1 + x^2 f,$$

where f_i denotes the i-th transform of f. If we put $g = f_1 - xf$ then g is a solution of $y_1 = -xy$ and f is a solution of $y_1 = xy + g$, where the field of coefficients in the latter equation should be extended to include g. This equation is derived from the function $F(x)$ of Miscellaneous Example 46 in Whittaker and Watson [32], chapter XII.

In his papers [4, 5, 6] Franke introduced the theory of solvability of linear difference equations. A linear homogeneous difference equation is said to be solvable (by elementary operations) if some fundamental system for the equation is contained in a Liouville-Franke extension (LFE).

LFE defined by Franke is a sort of finite chain of field extensions, which is a difference analogue of Liouville extension (see [5, 27]). A Liouville extension of a differential field K is a differential overfield of the form $K(t_1, \ldots, t_n)$, where for each $i = 1, \ldots, n$ either t_i' is in $K(t_1, \ldots, t_{i-1})$, or t_i'/t_i is in $K(t_1, \ldots, t_{i-1})$, or t_i is algebraic over $K(t_1, \ldots, t_{i-1})$.

LFE is defined as follows. A pair $\mathcal{K} = (K, \tau)$ is called a difference field if K is a field and τ is an isomorphism of K into K. The pairs $(\mathbb{C}(x), x \mapsto x+1)$, $(\mathbb{C}(x), x \mapsto 2x)$ and $(\mathbb{C}(x), x \mapsto x^2)$ are difference fields. A difference field $\mathcal{K}$ is said to be inversive if $\tau K = K$. For difference fields $\mathcal{K} = (K, \tau)$ and $\mathcal{K}' = (K', \tau')$ we say that $\mathcal{K}'/\mathcal{K}$ is a difference field extension if K' is an overfield of K and $\tau'|_K = \tau$. An LFE $\mathcal{N}/\mathcal{K}$, which is also called kLE, is a finite chain of difference field extensions,

$$\mathcal{K} = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \cdots \subset \mathcal{K}_n = \mathcal{N} = (N, \tau),$$

such that for some identical $k \in \mathbb{Z}_{\geq 1}$ each of the extensions $\mathcal{K}_i/\mathcal{K}_{i-1}$ is "generated" by an element a_i satisfying $\tau^k a_i = y + \beta$, $\beta \in K_{i-1}$ or $\tau^k a_i = \alpha y$, $\alpha \in K_{i-1}$, or an element algebraic over K_{i-1}. It is easily seen that any solution of $y_1 = \alpha y + \beta$, $\alpha \neq 0$ is contained in an LFE, and so we find that the equation (1.1) is solvable by elementary operations.

In the differential case it is well known that the Airy equation is not solvable (see [12]). Like the gamma function, there is a q-analogue of the Airy function, which satisfies the following linear homogeneous q-difference equation,

$$y_2 + qty_1 - y = 0 \quad (t \mapsto qt). \tag{1.2}$$

In Chapter 3 we construct a general theory of solvability of difference Riccati equations and show that this equation is not solvable for transcendental number q. If $f \neq 0$ is a solution of the equation (1.2) then $g = f_1/f$ satisfies the Riccati equation,

$$y_1 = \frac{-qty+1}{y} \quad (t \mapsto qt). \tag{1.3}$$

We define

Definition 3.1 (difference field extensions of valuation ring type). Let $\mathcal{N}/\mathcal{K}$ be a difference field extension, and $\mathcal{N} = (N, \tau)$. We say $\mathcal{N}/\mathcal{K}$ is a *difference field extension of valuation ring type* if there is a chain of difference fields,

$$\mathcal{K} = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \cdots \subset \mathcal{K}_{n-1} \subset \mathcal{K}_n = \mathcal{N},$$

such that for each $1 \leq i \leq n$ the extension $\mathcal{K}_i/\mathcal{K}_{i-1}$ satisfies one of the following.

(i) The extension K_i/K_{i-1} is algebraic.

(ii) $\mathcal{K}_i$ and $\mathcal{K}_{i-1}$ are inversive, K_i/K_{i-1} is an algebraic function field of one variable, and there is a valuation ring $\mathcal{O}$ of K_i/K_{i-1} such that $\tau^j\mathcal{O} \subset \mathcal{O}$ for some $j \in \mathbb{Z}_{>0}$.

Then we prove

Theorem 3.2. *Let $\mathcal{K} = (K, \tau_K)$ be a difference field, and $a, b, c, d \in K$. Define the matrices $A = A_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $A_i = (\tau_K A_{i-1})A$ $(i \geq 2)$, and put $A_i = \left(\begin{smallmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{smallmatrix}\right)$. Suppose $b^{(i)} \neq 0$ and $c^{(i)} \neq 0$ for all $i \geq 1$. Let $k \geq 1$, and suppose the equation over $\mathcal{K}$, $y_k(c^{(k)}y + d^{(k)}) = a^{(k)}y + b^{(k)}$ has a solution in a difference field extension $\mathcal{N}/\mathcal{K}$ of valuation ring type. Let $\overline{\mathcal{N}}$ be an algebraic closure of $\mathcal{N}$ and $\overline{\mathcal{K}}$ the algebraic closure of $\mathcal{K}$ in $\overline{\mathcal{N}}$. Then there exists $i \geq 1$ such that the equation over $\mathcal{K}$, $y_{ki}(c^{(ki)}y + d^{(ki)}) = a^{(ki)}y + b^{(ki)}$, has a solution in $\overline{\mathcal{K}}$.*

As a corollary we obtain

Proposition 3.8. *Let $\mathcal{K}$ be a inversive difference field, $a, b, c, d \in K$, and $k \in \mathbb{Z}_{>0}$. Define the matrices A_i as in the preceding theorem. Suppose $b^{(ki)} \neq 0$ and $c^{(ki)} \neq 0$ for all $i \geq 1$, and the equation over $\mathcal{K}$, $y_1(cy + d) = ay + b$, has a solution f in a kLE $\mathcal{L}/\mathcal{K}$. Let $\overline{\mathcal{L}} = (\overline{L}, \tau)$ be an algebraic closure of $\mathcal{L}$,* *and $\overline{\mathcal{K}}$ be the algebraic closure of $\mathcal{K}$ in $\overline{\mathcal{L}}$. Then there exists $i \geq 1$ such that the equation over $\mathcal{K}$, $y_{ki}(c^{(ki)}y + d^{(ki)}) = a^{(ki)}y + b^{(ki)}$, has a solution in $\overline{\mathcal{K}}$.*

This implies that if a difference Riccati equation which does not turn out to be linear by the iterations has a solution in LFE, then one of the iterated Riccati equations has an algebraic solution.

Next we introduce several results on irreducibility of difference equations. A solution of the Riccati equation (1.3) satisfies the q-Painlevé equation of type $A_6^{(1)}$,

$$q\text{-}P(A_6): \quad (y_2y_1 - 1)(y_1y - 1)(y_1 + qt) = aq^2t^2y_1,$$

with $a = q$. q-$P(A_6)$ is said to be a q-analogue of the Painlevé equation of type II and is also called q-P_{II}. When q is not an algebraic unit, we find that any solution of q-$P(A_6)$ contained in a decomposable extension, which is generated by solutions of linear difference equations, solutions of algebraic difference equations of order 1, algebraic elements, etc., can be represented rationally by a solution of the equation (1.3) (see Chapter 4, 6). A similar proposition for Painlevé equation of type II is proved by Noumi and Okamoto in their paper [23].

This kind of study is called study of irreducibility of differential or difference equations. On irreducibility of differential equations Umemura developed an analytic explanation of the Painlevé's irreducibility (cf. [31]). He defined his extension of sets of meromorphic functions over a domain of the complex plane $\mathbb{C}$ by the following six permissible operations.

(O) Let $f(x)$ be a known function. Then the derived function $f'(x)$ is a new known function.

(P1) Let f and g be known functions. Then $f+g$, $f-g$, fg, and f/g, $g \neq 0$ are new known functions.

(P2) Let $a_1, a_2, \ldots, a_n$ be n known functions. Then algebroid function f or any solution f of an algebraic equation,

$$f^n + a_1f^{n-1} + a_2f^{n-2} + \cdots + a_n = 0,$$

is a new known function.

(P3) Let f be a known function. Then the quadrature $\int f(x)\,dx$ is a new known function.

(P4) Let $a_1, a_2, \ldots, a_n$ be known functions. Then any solution f of a linear differential equation,

$$\frac{d^n f}{dx^n} + a_1 \frac{d^{n-1} f}{dx^{n-1}} + a_2 \frac{d^{n-2} f}{dx^{n-2}} + \cdots + a_n f = 0,$$

is a new known function.

(P5) Let $\Gamma \subset \mathbb{C}^n$ be a lattice such that the quotient $\mathbb{C}^n/\Gamma$ is an abelian variety. Let $p : \mathbb{C}^n \to \mathbb{C}^n/\Gamma$ be the projection. Let $f_1, f_2, \ldots, f_n$ be holomorphic known functions. We denote a holomorphic map $D \to \mathbb{C}^n$ sending x to $(f_1(x), f_2(x), \ldots, f_n(x))$ by F. Then the function $\phi \circ p \circ F$ is a new known function for any meromorphic function ϕ on the Abelian variety $\mathbb{C}^n/\Gamma$. Here we have to avoid the constant function taking the value infinity.

In terms of the differential algebra this extension is a finite chain of Kolchin's strongly normal extensions and algebraic extensions (for the strongly normal extension see [1, 14]).

By Bialynicki-Birula [1] a generalization of the strongly normal extension was studied, which includes the difference case. The author defined the decomposable extension of difference field mentioned above and proved that a finite chain of Bialynicki-Birula's strongly normal extensions and algebraic extensions, where mostly taking algebraic closures, is decomposable (cf. Chapter 4). Here is the definition of the decomposable extension.

Definition 4.1 (decomposable extension)**.** Let $\mathcal{K}$ be a difference field, and $\mathcal{L}$ an algebraically closed difference overfield of $\mathcal{K}$ satisfying $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} < \infty$. We define decomposable extensions by induction on $\operatorname{tr.deg} \mathcal{L}/\mathcal{K}$.

(i) If $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} \leq 1$, then $\mathcal{L}/\mathcal{K}$ is decomposable.

(ii) When $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} \geq 2$, $\mathcal{L}/\mathcal{K}$ is decomposable if there exist a difference overfield $\mathcal{U}$ of $\mathcal{L}$, a difference overfield $\mathcal{E}$ of $\mathcal{K}$ in $\mathcal{U}$ of finite transcendence degree which is free from $\mathcal{L}$ over $\mathcal{K}$, and a difference intermediate field $\mathcal{M}$ of $\mathcal{L}\mathcal{E}/\mathcal{E}$ satisfying $\operatorname{tr.deg} \mathcal{L}\mathcal{E}/\mathcal{M} \geq 1$ and $\operatorname{tr.deg} \mathcal{M}/\mathcal{E} \geq 1$, such that $\overline{\mathcal{L}\mathcal{E}}/\mathcal{M}$ and $\overline{\mathcal{M}}/\mathcal{E}$ are decomposable, where $\overline{\mathcal{L}\mathcal{E}}$ is an algebraic closure of $\mathcal{L}\mathcal{E}$ and $\overline{\mathcal{M}}$ the algebraic closure of $\mathcal{M}$ in $\overline{\mathcal{L}\mathcal{E}}$.

On linear difference equations we prove

Corollary 4.8. *Let $\mathcal{K}$ be a difference field,*

$$y_n + a_{n-1}y_{n-1} + \cdots + a_0 y = b \tag{4.3}$$

be a linear difference equation over $\mathcal{K}$, where $n \geq 1$, and f a solution of (4.3). *Then $\overline{\mathcal{K}\langle f\rangle}/\mathcal{K}$ is decomposable for any algebraic closure $\overline{\mathcal{K}\langle f\rangle}$ of $\mathcal{K}\langle f\rangle$.*

To obtain irreducibility theorems the following Lemma will be useful.

Lemma 4.10. *Let $\mathcal{K}$ be an almost inversive difference field, $\mathcal{D}$ a decomposable extension of $\mathcal{K}$ and $B \subset D$. Suppose that if $\mathcal{L}$ is an inversive difference overfield of $\mathcal{K}$ and $\mathcal{U}$ a difference overfield of $\mathcal{L}$ with $\mathcal{K}\langle B\rangle_{\mathcal{D}} \subset \mathcal{U}$, then the following holds,*

$$\operatorname{tr.deg} \mathcal{L}\langle B\rangle_{\mathcal{U}}/\mathcal{L} \leq 1 \Rightarrow \textit{any } f \in B \textit{ is algebraic over } L.$$

Then any $f \in B$ is algebraic over K.

We say that a difference field $\mathcal{K} = (K, \tau)$ is almost inversive if the field extension $K/\tau K$ is algebraic. For example, using this Lemma, we obtain Theorem 7.3 as a corollary of Proposition 7.2.

Theorem 7.3. *Let $\mathcal{K}$ be an almost inversive difference field, $\mathcal{N}$ a decomposable extension of $\mathcal{K}$ and $f \in N$ a solution in $\mathcal{N}$ of the equation over $\mathcal{K}$,*

$$B(y_1)y_2 y = A(y_1),$$

where $A, B \in K[X] \setminus \{0\}$ are polynomials over K such that A and B are relatively prime, B monic and $\max\{\deg A, \deg B\} > 2$. *Then f is algebraic over K.*

Proposition 7.2. *Let $\mathcal{L} = (L, \tau)$ be an inversive difference field and f a solution of the equation over $\mathcal{L}$,*

$$B(y_1)y_2 y = A(y_1),$$

where $A, B \in L[X] \setminus \{0\}$ are polynomials over L such that A and B are relatively prime, B monic and $\max\{\deg A, \deg B\} > 2$. *Then it follows that*

$$\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} \leq 1 \Rightarrow f \textit{ is algebraic over } L.$$

This paper is organized as follows. In Chapter 2 we introduce a lemma on algebraic solutions of difference equations and terms on the difference algebra. In Chapter 3 we generalize Karr's $\Pi\Sigma$-extension and 1LE, Franke's generalized Liouvillian extension, and construct a general theory on solvability of differece Riccati equations. In addition we prove unsolvability of the q-Airy equation and the q-Bessel equation. In Chapter 4 we define the decomposable extensions of difference fields. We find that every strongly normal extension or LFE satisfies that its appropriate algebraic closure is a decomposable extension. We also introduce lemmas on the irreduciblity of difference equation. The object in Chapter 5 is the q-Painlevé equation of type $A_7^{(1)'}$,

$$\begin{cases} y_1 y = z_1^2, \\ z_1 z = \dfrac{y(1-ty)}{t(y-1)}. \end{cases}$$

We prove that if q is not a root of unity and (f, g) a solution in a decomposable extension of $(\mathbb{C}(t), t \mapsto qt)$, then f and g are algebraic fucntions of the form $c/\sqrt{t}$, $c \in \mathbb{C}$. In Chapter 6 we study irreducibility and transcendence of solutions of the q-Painlevé equation of type $A_6^{(1)}$,

$$(y_2 y_1 - 1)(y_1 y - 1)(y_1 + qt) = aq^2 t^2 y_1.$$

On the transcendence of solutions we used discrete valuations and the Hankel determinant. In Chapter 7 we study irreducibility of systems of difference equations of birational form, such as

$$y_2 y = \frac{A(y_1)}{B(y_1)}$$

and

$$\begin{cases} y_1 y = \dfrac{A(z)}{B(z)}, \\ z_1 z = \dfrac{C(y_1)}{D(y_1)}, \end{cases}$$

where A, B, C and D are polynomials.

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Chapter 2

Preliminaries

Notation. Throughout this chapter a field is of characteristic zero. Terms used here will be seen in [11, 30] and [3, 16].

We use the following to study transcendence of solutions of difference equations.

Lemma 2.1. *Let C an algebraically closed field, $q \in C^\times$ not a root of unity, t transcendental over C, $F/C(t)$ a finite extension of degree n, and τ an isomorphism of F into F over C sending t to qt. Then $F = C(x)$, $x^n = t$.*

Proof. Put $\mathbb{P}$ and $\mathbb{P}'$ be the sets of all prime divisors of $C(t)/C$ and F/C respectively. As in [30] we identify a prime divisor with the maximal ideal of the valuation ring associated with it. Define the following valuation rings of $C(t)/C$,

$$\mathcal{O}_\alpha = \{f/g \mid f, g \in C[t],\, t - \alpha \nmid g\} \quad \text{for each } \alpha \in C,$$
$$\mathcal{O}_\infty = \{f/g \mid f, g \in C[t],\, \deg g - \deg f \geq 0\},$$

and let $P_\alpha = \mathcal{O}_\alpha \setminus \mathcal{O}_\alpha^\times$ be the prime divisor associated with $\mathcal{O}_\alpha$ for each $\alpha \in C \cup \{\infty\}$.

We show that if $\alpha \in C^\times$ then P_α is unramified in $F/C(t)$. Let $\alpha \in C^\times$ and assume that P_α is ramified in $F/C(t)$. Then there is $P' \in \mathbb{P}'$ such that $e(P'|P_\alpha) > 1$, where $e(P'|P_\alpha)$ is the ramification index of P' over P_α. Let $\mathcal{O}'$ be the valuation ring associated with P'. We find that for any $i \in \mathbb{Z}_{\geq 0}$, $\tau^i P_\alpha = P_{\alpha/q^i} \in \mathbb{P}$ and $\tau^i P'$ is the prime divisor associated with the valuation ring $\tau^i \mathcal{O}'$ of $\tau^i F/C$. We also find that $e(\tau^i P'|\tau^i P_\alpha) > 1$ for all $i \geq 0$. For any

$i \geq 0$ there is $Q_i \in \mathbb{P}'$ such that $Q_i \cap \tau^i F = \tau^i P'$, and we have

$$e(Q_i|\tau^i P_\alpha) = e(Q_i|\tau^i P')e(\tau^i P'|\tau^i P_\alpha) \geq e(\tau^i P'|\tau^i P_\alpha) > 1,$$

which implies $\tau^i P_\alpha = P_{\alpha/q^i}$ is ramified in $F/C(t)$ for any $i \geq 0$. Since $q \in C^\times$ is not a root of unity, the prime divisors P_{α/q^i} ($i \geq 0$) are distinct, a contradiction. Therefore P_α is unramified in $F/C(t)$.

Let g be the genus of F/C. By the Riemann-Hurwitz Genus Formula we obtain

$$\begin{aligned} 2g-2 &= -2n + \sum_{\alpha=0,\infty} \left(\sum_{P' \in \mathbb{P}',\, P' \cap C(t) = P_\alpha} (e(P'|P_\alpha) - 1) \right) \\ &\leq -2n + 2(n-1) = -2, \end{aligned}$$

which implies $g = 0$. Therefore $F = C(y)$ for some $y \in F$.

Again by the Riemann-Hurwitz Genus Formula we obtain

$$\sum_{\alpha=0,\infty} \left(\sum_{P' \in \mathbb{P}',\, P' \cap C(t) = P_\alpha} (e(P'|P_\alpha) - 1) \right) = 2(n-1),$$

which implies

$$\sum_{P' \in \mathbb{P}',\, P' \cap C(t) = P_\alpha} (e(P'|P_\alpha) - 1) = n - 1$$

for $\alpha = 0, \infty$. Therefore P_α ($\alpha = 0, \infty$) has only one extension P'_α in $\mathbb{P}'$, which satisfies $e(P'_\alpha|P_\alpha) = n$.

$t \in C(y)$ yields the expression,

$$t = c\prod_{i=1}^{m}(y - \alpha_i)^{k_i}, \quad c \in C^\times,\ m \in \mathbb{Z}_{\geq 1},\ \alpha_i \in C,\ k_i \in \mathbb{Z},$$

where α_i ($1 \leq i \leq m$) are distinct. Let Q'_i be the prime divisor of $C(y)/C$ associated with the prime element $y - \alpha_i$, and put $Q_i = Q'_i \cap C(t)$ for each $1 \leq i \leq m$. We obtain

$$k_i = v_{Q'_i}(t) = e(Q'_i|Q_i)v_{Q_i}(t) = \begin{cases} 0 & \text{if } Q_i = P_\alpha,\ \alpha \in C^\times, \\ n & \text{if } Q_i = P_0, \\ -n & \text{if } Q_i = P_\infty, \end{cases}$$

where $v_{Q_i'}$ and v_{Q_i} are the normalized discrete valuations associated with Q_i' and Q_i respectively, which implies $n \mid k_i$ for all $1 \leq i \leq m$. Put $x = c^{1/n} \prod_{i=1}^{m} (y - \alpha_i)^{k_i/n} \in C(y)$. We have $x^n = t$, and so $[C(t,x) : C(t)] = n$, which implies $F = C(t,x) = C(x)$. □

When R is a commutative ring and τ is an injective homomorphism of R into R, the pair $\mathcal{R} = (R, \tau)$ is called a difference ring. We call τ the (transforming) operator and R the underlying ring of $\mathcal{R}$. If R is a field, $\mathcal{R}$ is called a difference field and R the underlying field. If R is a field, τ is an isomorphism of R into R.

Let $\mathcal{R} = (R, \tau)$ be a difference ring. For $a \in R$, an element $\tau^n a \in R$, $n \in \mathbb{Z}$ is called the n-th transform of a and is denoted by a_n if exists. If $\tau R = R$, we say that $\mathcal{R}$ is inversive. For a difference field $\mathcal{K} = (K, \tau_K)$, $\mathcal{K}$ is inversive if and only if τ_K is an isomorphism of K onto K. If the field extension $K/\tau_K K$ is algebraic, we say that $\mathcal{K}$ is almost inversive.

Let $\mathcal{R} = (R, \tau)$ and $\mathcal{R}' = (R', \tau')$ be difference rings. A mapping ϕ of R to R' is called a difference homomorphism of $\mathcal{R}$ to $\mathcal{R}'$ if ϕ is a ring homomorphism and $\phi\tau = \tau'\phi$. Let ϕ be a difference homomorphism of $\mathcal{R}$ to $\mathcal{R}'$. If ϕ is surjective, we say that ϕ is a difference homomorphism of $\mathcal{R}$ onto $\mathcal{R}'$. If ϕ is a ring isomorphism of R to R', we say that ϕ is a difference isomorphism of $\mathcal{R}$ to $\mathcal{R}'$. If there exists a difference isomorphism of $\mathcal{R}$ to $\mathcal{R}'$, they are said to be isomorphic. Let $\mathcal{K} = (K, \tau)$ and $\mathcal{K}' = (K', \tau')$ be difference fields. An isomorphism ϕ of K into (onto) K' is called a difference isomorphism of $\mathcal{K}$ into (onto, respectively) $\mathcal{K}'$ if $\phi\tau = \tau'\phi$. $\mathcal{K}$ and $\mathcal{K}'$ are isomorphic if and only if there exists a difference isomorphism of $\mathcal{K}$ onto $\mathcal{K}'$.

Let $\mathcal{R} = (R, \tau)$ and $\mathcal{R}' = (R', \tau')$ be difference rings. $\mathcal{R}$ is called a difference subring of $\mathcal{R}'$ if R is a subring of R' and $\tau'|_R = \tau$. We then call $\mathcal{R}'$ a difference overring of $\mathcal{R}$, and say that $\mathcal{R}'/\mathcal{R}$ is a difference ring extension. A difference ring $\mathcal{S}$ is called a difference intermediate ring of a difference ring extension $\mathcal{R}'/\mathcal{R}$ if $\mathcal{S}$ is a difference subring of $\mathcal{R}'$ and a difference overring of $\mathcal{R}$. We define a difference overfield, a difference subfield, a difference field extension and a difference intermediate field by replacing ring with field. We also use the expression $\mathcal{R}' \supset \mathcal{R}$ when $\mathcal{R}'/\mathcal{R}$ is a difference ring extension. If $\mathcal{R}$ and $\mathcal{R}'$ are difference field, $\mathcal{R}'/\mathcal{R}$ is a difference field extension if and only if $\mathcal{R}' \supset \mathcal{R}$.

Let $\mathcal{R} = (R, \tau)$ be a difference ring. An element $a \in R$ is called an invariant element of $\mathcal{R}$ if a satisfies $\tau a = a$. We let $C_{\mathcal{R}}$ denote the set of invariant elements of $\mathcal{R}$. $C_{\mathcal{R}}$ is a difference subring of $\mathcal{R}$ and if $\mathcal{R}$ is a difference field, a difference subfield of $\mathcal{R}$.

Let $\mathcal{R}$ be a difference ring, $\mathcal{S} = (S, \tau)$ a difference overring of $\mathcal{R}$ and B a subset of S. The difference subring of $\mathcal{S}$ whose underlying ring is the intersection of all difference subring of $\mathcal{S}$ containing R and B is denoted by $\mathcal{R}\{B\}_{\mathcal{S}}$. $\mathcal{R}\{B\}_{\mathcal{S}}$ is the minimum difference subring of $\mathcal{S}$ containing R and B. Put

$$B' = \{\tau^n b \in S \mid n \in \mathbb{Z}_{\geq 0} \text{ and } b \in B\}.$$

Then the underlying ring of $\mathcal{R}\{B\}_{\mathcal{S}}$ is equal to $R[B']$. For brevity we use $\mathcal{R}\{B\}$ instead of $\mathcal{R}\{B\}_{\mathcal{S}}$.

Let $\mathcal{K}$ be a difference field, $\mathcal{L} = (L, \tau)$ a difference overfield of $\mathcal{K}$ and B a subset of L. The difference subfield of $\mathcal{L}$ whose underlying field is the field of fractions of the underlying ring of $\mathcal{K}\{B\}_{\mathcal{L}}$ in L is denoted by $\mathcal{K}\langle B\rangle_{\mathcal{L}}$. $\mathcal{K}\langle B\rangle_{\mathcal{L}}$ is the minimum difference subfield of $\mathcal{L}$ containing K and B. For brevity we use $\mathcal{K}\langle B\rangle$ instead of $\mathcal{K}\langle B\rangle_{\mathcal{L}}$.

Let $\mathcal{K}$ be an almost inversive difference field and $\overline{\mathcal{K}}$ an algebraic closure of $\mathcal{K}$, which is defined to be a difference overfield of K whose underlying field is an algebraic closure of K. We easily find that $\overline{\mathcal{K}}$ exists and it is inversive. We call the minimum inversive difference subfield of $\overline{\mathcal{K}}$ containing K an inversive closure of $\mathcal{K}$.

Let $\mathcal{R} = (R, \tau)$ be a difference ring, $n \in \mathbb{Z}_{\geq 1}$ and $R[\{y_i^{(k)} \mid 1 \leq k \leq n,\ i \geq 0\}]$ a polynomial ring with indeterminates $y_i^{(k)}$'s. Extend τ to the injective homomorphism τ' of $R[\{y_i^{(k)}\}_{ki}]$ to itself sending $a \in R$ to τa and $y_i^{(k)}$ to $y_{i+1}^{(k)}$. Then the pair $(R[\{y_i^{(k)}\}_{ki}], \tau')$ is a difference overring of $\mathcal{R}$. If we put $y^{(k)} = y_0^{(k)}$, $(R[\{y_i^{(k)}\}_{ki}], \tau')$ can be expressed as $\mathcal{R}\{y^{(1)}, y^{(2)}, \ldots, y^{(n)}\}$. We call $\mathcal{R}\{y^{(1)}, y^{(2)}, \ldots, y^{(n)}\}$ an n-fold difference polynomial ring over $\mathcal{R}$. Any two n-fold difference polynomial rings over $\mathcal{R}$ are isomorphic. Let $A, B \in \mathcal{R}\{y^{(1)}, y^{(2)}, \ldots, y^{(n)}\}$. A is called an n-fold difference polynomial over $\mathcal{R}$ and "$A = B$" is called an n-fold difference equation over $\mathcal{R}$.

Let $\mathcal{K}$ be a difference field, $\mathcal{P} = \mathcal{K}\{y^{(1)}, y^{(2)}, \ldots, y^{(n)}\}$ a difference polynomial ring and $\Phi \subset P$ a subset. For a difference overfield $\mathcal{L}$ of $\mathcal{K}$, an n-tuple $a = (a^{(1)}, a^{(2)}, \ldots, a^{(n)}) \in L^n$ is called a zero of Φ in $\mathcal{L}$ if $\phi A = 0$ for any $A \in \Phi$, where ϕ is the difference homomorphism of $\mathcal{P}$ to $\mathcal{L}$ over K sending $y_i^{(k)}$ to $a_i^{(k)}$. Let $\{A_\lambda\}_{\lambda\in\Lambda}, \{B_\lambda\}_{\lambda\in\Lambda} \subset P$ be subsets, where Λ is a set of indexes. For a difference overfield $\mathcal{L}$ of $\mathcal{K}$, an n-tuple $a \in L^n$ is called a solution of the system of difference equations over $\mathcal{K}$, $\{A_\lambda = B_\lambda\}_{\lambda\in\Lambda}$, in $\mathcal{L}$ if a is a zero of $\{A_\lambda - B_\lambda\}_{\lambda\in\Lambda} \subset P$ in $\mathcal{L}$. We say that an n-tuple a is a solution of $\{A_\lambda = B_\lambda\}_{\lambda\in\Lambda}$ if for some difference overfield $\mathcal{L}$ of $\mathcal{K}$, $a \in L^n$ and a is a solution of $\{A_\lambda = B_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{L}$.

Chapter 3

Solvability

We generalize Karr's $\Pi\Sigma$-extension and Franke's generalized Liouvillian extension, and study solvability of difference Riccati equations. We prove the following. If a difference Riccati equation which does not turn out to be linear by the iterations has a solvable solution, then one of the iterated Riccati equations has an algebraic solution. In addition we prove unsolvability of the q-Airy equation and the q-Bessel equation.

Notation. Throughout this chapter a field is of characteristic zero and C denotes an algebraically closed field.

3.1 Introduction

In his [4, 5] Franke studied the solvability of linear homogeneous difference equations by elementary operations. He defined qLE $\mathcal{N}/\mathcal{K}$ ($q \in \mathbb{Z}_{>0}$), a difference field extension having the chain of inversive difference fields,

$$\mathcal{K} = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \cdots \subset \mathcal{K}_n = \mathcal{N} = (N, \tau), \quad \mathcal{K}_i = \mathcal{K}_{i-1}\langle a_i \rangle^*,$$

where * denotes the inversive closure, and a_i satisfies one of the following.

(i) $\tau^q a_i = a_i + \beta$ for some $\beta \in K_{i-1}$.

(ii) $\tau^q a_i = \alpha a_i$ for some $\alpha \in K_{i-1}$.

(iii) a_i is algebraic over K_{i-1}.

qLE is called generalized Liouvillian extension (GLE) when $q = 1$. Note that for a qLE $(N, \tau)/(K, \tau)$, the extension $(N, \tau^q)/(K, \tau^q)$ is a GLE (see [5]). We may say that a solution f of a difference equation over $\mathcal{K}$ is solvable by elementary operations if $\mathcal{K}\langle f \rangle$ is contained in a qLE of $\mathcal{K}$. We call qLEs Liouville-Franke extensions.

In his [13] Karr defined $\Pi\Sigma$-extensions, and introduced results on the computation of symbolic solutions to first order linear difference equations and an analogue to Liouville's theorem on elementary integrals. The following is a generalization of the $\Pi\Sigma$-extension and GLE.

Definition 3.1 (difference field extensions of valuation ring type). Let $\mathcal{N}/\mathcal{K}$ be a difference field extension, and $\mathcal{N} = (N, \tau)$. We say $\mathcal{N}/\mathcal{K}$ is a *difference field extension of valuation ring type* if there is a chain of difference fields,

$$\mathcal{K} = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \cdots \subset \mathcal{K}_{n-1} \subset \mathcal{K}_n = \mathcal{N},$$

such that for each $1 \leq i \leq n$ the extension $\mathcal{K}_i/\mathcal{K}_{i-1}$ satisfies one of the following.

(i) The extension K_i/K_{i-1} is algebraic.

(ii) $\mathcal{K}_i$ and $\mathcal{K}_{i-1}$ are inversive, K_i/K_{i-1} is an algebraic function field of one variable, and there is a valuation ring $\mathcal{O}$ of K_i/K_{i-1} such that $\tau^j \mathcal{O} \subset \mathcal{O}$ for some $j \in \mathbb{Z}_{>0}$.

Proposition 3.5 shows that the gamma function is contained in a difference field extension of valuation ring type over the rational function field. The idea to use valuation rings for investigating differential equations originated with Rosenlicht [27].

We prove

Theorem 3.2. *Let $\mathcal{K} = (K, \tau_K)$ be a difference field, and $a, b, c, d \in K$. Define the matrices $A = A_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $A_i = (\tau_K A_{i-1})A$ $(i \geq 2)$, and put $A_i = \left(\begin{smallmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{smallmatrix}\right)$. Suppose $b^{(i)} \neq 0$ and $c^{(i)} \neq 0$ for all $i \geq 1$. Let $k \geq 1$, and suppose the equation over $\mathcal{K}$, $y_k(c^{(k)}y + d^{(k)}) = a^{(k)}y + b^{(k)}$ has a solution in a difference field extension $\mathcal{N}/\mathcal{K}$ of valuation ring type. Let $\overline{\mathcal{N}}$ be an algebraic closure of $\mathcal{N}$ and $\overline{\mathcal{K}}$ the algebraic closure of $\mathcal{K}$ in $\overline{\mathcal{N}}$. Then there exists $i \geq 1$ such that the equation over $\mathcal{K}$, $y_{ki}(c^{(ki)}y + d^{(ki)}) = a^{(ki)}y + b^{(ki)}$, has a solution in $\overline{\mathcal{K}}$.*

The application to solvability is written in §3.3.

3.2 Proof of Theorem

Lemma 3.3. *Let $\mathcal{L}/\mathcal{K}$ be a difference field extension, $\mathcal{L} = (L, \tau)$, and $a, b, c, d \in K$. Define the matrices $A = A_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $A_i = (\tau A_{i-1})A$ $(i \geq 2)$, and put $A_i = \left(\begin{smallmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{smallmatrix}\right)$. Then we have*

(a) *$A_i = (\tau^{i-1}A)(\tau^{i-2}A)\dots(\tau A)A$.*

Let $k \geq 1$.

(b) *Define the matrices $B = B_1 = A_k$, $B_i = (\tau^k B_{i-1})B$ $(i \geq 2)$. Then $B_i = A_{ki}$.*

(c) *Let $f \in \mathcal{L}$ be a solution of $y_k(c^{(k)}y + d^{(k)}) = a^{(k)}y + b^{(k)}$. Then $f \in \mathcal{L}$ is a solution of $y_{ki}(c^{(ki)}y + d^{(ki)}) = a^{(ki)}y + b^{(ki)}$ for all $i \geq 1$.*

Proof. (a) Let $i \geq 2$, and suppose the statement is true for $i - 1$. Then we have

$$\begin{aligned} A_i &= (\tau A_{i-1})A = \tau((\tau^{i-2}A)(\tau^{i-3}A)\dots(\tau A)A)A \\ &= (\tau^{i-1}A)(\tau^{i-2}A)\dots(\tau^2 A)(\tau A)A. \end{aligned}$$

Therefore the proof can be completed by induction.

(b) Let $i \geq 2$, and suppose the statement is true for $i - 1$. Then we have

$$\begin{aligned} B_i &= (\tau^k B_{i-1})B = \tau^k(A_{k(i-1)})A_k \\ &= \tau^k((\tau^{k(i-1)-1}A)(\tau^{k(i-1)-2}A)\dots(\tau A)A)A_k \\ &= (\tau^{ki-1}A)(\tau^{ki-2}A)\dots(\tau^{k+1}A)(\tau^k A)A_k \\ &= A_{ki}. \end{aligned}$$

Therefore the proof can be completed by induction.

(c) *Case 1.* Firstly we deal with the case $k = 1$. There is nothing to prove in case $i = 1$. Let $i \geq 2$, and suppose the statement is true for $i - 1$. By definition we have

$$\begin{aligned} \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix} &= \begin{pmatrix} a_1^{(i-1)} & b_1^{(i-1)} \\ c_1^{(i-1)} & d_1^{(i-1)} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} aa_1^{(i-1)} + cb_1^{(i-1)} & ba_1^{(i-1)} + db_1^{(i-1)} \\ ac_1^{(i-1)} + cd_1^{(i-1)} & bc_1^{(i-1)} + dd_1^{(i-1)} \end{pmatrix}, \end{aligned}$$

and so $f_{i-1}(c^{(i-1)}f + d^{(i-1)}) = a^{(i-1)}f + b^{(i-1)}$ yields

$$f_i(c_1^{(i-1)}f_1 + d_1^{(i-1)}) = a_1^{(i-1)}f_1 + b_1^{(i-1)},$$
$$f_i(c_1^{(i-1)}(af + d) + d_1^{(i-1)}(cf + d)) = a_1^{(i-1)}(af + b) + b_1^{(i-1)}(cf + d),$$
$$f_i(c^{(i)}f + d^{(i)}) = a^{(i)}f + b^{(i)}.$$

Case 2. Secondly we deal with the other case, $k \geq 2$. Define the matrices B_i as in (b). From (b) we obtain $B_i = A_{ki}$. Since $f \in (L, \tau^k)$ is a solution of $y_1(c^{(k)}y + d^{(k)}) = a^{(k)}y + b^{(k)}$, Case 1 implies that $f \in (L, \tau^k)$ is a solution of $y_i(c^{(ki)}y + d^{(ki)}) = a^{(ki)}y + b^{(ki)}$ for all $i \geq 1$. Therefore we find that $\tau^{ki}(f)(c^{(ki)}f + d^{(ki)}) = a^{(ki)}f + b^{(ki)}$ for all $i \geq 1$. □

Lemma 3.4. *Let $\mathcal{L}/\mathcal{K}$ be a differece field extension, both $\mathcal{L} = (L, \tau_L)$ and $\mathcal{K}$ inversive, and L/K an algebraic function field of one variable. Suppose there exists a valuation ring $\mathcal{O}$ of L/K such that $\tau_L^j\mathcal{O} \subset \mathcal{O}$ for some $j \in \mathbb{Z}_{>0}$. Let $\overline{\mathcal{L}} = (\overline{L}, \tau)$ be an algebraic closure of $\mathcal{L}$ and $\overline{\mathcal{K}}$ the algebraic closure of $\mathcal{K}$ in $\overline{\mathcal{L}}$. Let $a, b, c, d \in K$, and define the matrices A_i as in Lemma 3.3. Suppose $b^{(i)} \neq 0$ and $c^{(i)} \neq 0$ for all $i \geq 1$, and the equation over $\mathcal{K}$, $y_1(cy + d) = ay + b$, has a solution f in $\overline{\mathcal{L}}$. Then for some $i \geq 1$ the equation over $\mathcal{K}$, $y_i(c^{(i)}y + d^{(i)}) = a^{(i)}y + b^{(i)}$, has a solution in $\overline{\mathcal{K}}$.*

Proof. It is enough to prove this for $f \notin \overline{K}$. The additional assumption implies $cf + d \neq 0$, and so we obtain

$$f_1 = \frac{af + b}{cf + d}.$$

Put $\mathcal{M} = \mathcal{L}\langle f \rangle \subset \overline{\mathcal{L}}$. We find $\mathcal{M}$ is inversive. In fact, since $cf_1 - a = 0$ implies $f = \tau^{-1}(a/c) \in K$, we have

$$f = -\frac{df_1 - b}{cf_1 - a} = \tau\left(-\frac{\tau^{-1}(d)f - \tau^{-1}b}{\tau^{-1}(c)f - \tau^{-1}a}\right) \in \tau M.$$

As a field, $M = L(f)$ is an algebraic function field of one variable over K, and so $M\overline{K}$ is an algebraic function field of one variable over $\overline{K}$.

Choose $j \in \mathbb{Z}_{>0}$ such that $\tau^j\mathcal{O} \subset \mathcal{O}$, and choose valuation ring $\mathcal{O}'$ of $M\overline{K}/\overline{K}$ such that $\mathcal{O}' \cap L = \mathcal{O}$. Note that $\tau^j\mathcal{O} \subset \mathcal{O}$ implies $\tau^j\mathcal{O} = \mathcal{O}$. Therefore for any $i \geq 0$ the following holds.

$$\tau^{ij}\mathcal{O}' \cap L = \tau^{ij}(\mathcal{O}' \cap L) = \tau^{ij}\mathcal{O} = \mathcal{O}.$$

From this we obtain $\#\{\tau^{ij}\mathcal{O}' \mid i \geq 0\} < \infty$, which implies $\tau^k\mathcal{O}' = \mathcal{O}'$ for some $k \geq 1$. Let v be the normalized discrete valuation associated with $\mathcal{O}'$, and $t \in M\overline{K}$ a prime element of $\mathcal{O}'$. Then we have $v(\tau^k t) = 1$, and so $v(\tau^k x) = v(x)$ for any $x \in M\overline{K}$.

By Lemma 3.3 we find that f satisfies

$$f_k(c^{(k)}f + d^{(k)}) = a^{(k)}f + b^{(k)}, \tag{3.1}$$

which yields $v(f) = 0$. In fact, firstly assume $v(f) > 0$. Then we have $v(f_k) = v(f) > 0$. This contradicts $v(f_k) = -v(c^{(k)}f + d^{(k)}) \leq 0$ obtained from the above equation (3.1). Secondly assume $v(f) < 0$. Then $v(f_k) = v(f) < 0$ contradicts

$$v(f_k) = v(a^{(k)}f + b^{(k)}) - v(f) \geq 0.$$

Let ϕ be the embedding of $M\overline{K}$ into $\overline{K}((t))$, and express f and $\tau^k t$ as

$$\phi(f) = \sum_{i=0}^{\infty} h_i t^i, \quad h_i \in \overline{K},\ h_0 \neq 0,$$
$$\phi(\tau^k t) = \sum_{i=1}^{\infty} e_i t^i, \quad e_i \in \overline{K},\ e_1 \neq 0.$$

Then

$$\phi(f_k) = \sum_{i=0}^{\infty} \tau^k(h_i)\left(\sum_{l=1}^{\infty} e_l t^l\right)^i .$$

Note that ϕ is a difference isomorphism of $(M\overline{K}, (\tau|_{M\overline{K}})^k)$ into $(\overline{K}((t)), \sigma)$, where σ sends $\sum_{i=0}^{\infty} \alpha_i t^i$ to $\sum_{i=0}^{\infty} \tau^k(\alpha_i)(\sum_{l=1}^{\infty} e_l t^l)^i$. Comparing the coefficients of t^0 of the equation (3.1), we obtain

$$\tau^k(h_0)(c^{(k)}h_0 + d^{(k)}) = a^{(k)}h_0 + b^{(k)}.$$

Therefore $h_0 \in \overline{\mathcal{K}}$ is a solution of the equation, $y_k(c^{(k)}y + d^{(k)}) = a^{(k)}y + b^{(k)}$. $\square$

Proof of Theorem 3.2. We prove this by induction on $\operatorname{tr.deg} N/K$. When $\operatorname{tr.deg} N/K = 0$, the equation, $y_k(c^{(k)}y + d^{(k)}) = a^{(k)}y + b^{(k)}$, has a solution in $\overline{\mathcal{K}}$. Suppose $\operatorname{tr.deg} N/K \geq 1$, and the theorem is true for the transcendence degree $< \operatorname{tr.deg} N/K$.

Let $\overline{\mathcal{N}} = (\overline{N}, \tau)$. There is a chain of difference fields,

$$\mathcal{K} = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \cdots \subset \mathcal{K}_{n-1} \subset \mathcal{K}_n = \mathcal{N}, \quad n \geq 1,$$

such that for each $1 \leq i \leq n$ the extension $\mathcal{K}_i/\mathcal{K}_{i-1}$ satisfies one of the conditions (i), (ii) in Definition 3.1. Put

$$n_0 = \min\{0 \leq i \leq n \mid K_n/K_i \text{ is algebraic}\}.$$

We find $n_0 \geq 1$, and that the extension $\mathcal{K}_{n_0}/\mathcal{K}_{n_0-1}$ satisfies the condition (ii). Choose a valuation ring $\mathcal{O}$ of K_{n_0}/K_{n_0-1} such that $\tau^j\mathcal{O} \subset \mathcal{O}$ for some $j \in \mathbb{Z}_{>0}$. We have $(\tau^k)^j\mathcal{O} \subset \mathcal{O}$.

Let $\overline{\mathcal{K}}_{n_0-1}$ be the algebraic closure of $\mathcal{K}_{n_0-1}$ in $\overline{\mathcal{N}}$, and put $\overline{\mathcal{N}}^{(k)} = (\overline{N}, \tau^k)$, $\mathcal{K}_{n_0}^{(k)} = (K_{n_0}, \tau^k|_{K_{n_0}})$, $\mathcal{K}_{n_0-1}^{(k)} = (K_{n_0-1}, \tau^k|_{K_{n_0-1}})$ and $\overline{\mathcal{K}}_{n_0-1}^{(k)} = (\overline{K}_{n_0-1}, \tau^k|_{\overline{K}_{n_0-1}})$. By the hypothesis we find that the equation over $\mathcal{K}_{n_0}^{(k)}$, $y_1(c^{(k)}y + d^{(k)}) = a^{(k)}y + b^{(k)}$, has a solution in $\mathcal{N}^{(k)}$.

Define the matrices $B = B_1 = A_k$, $B_i = (\tau^k B_{i-1})B$ $(i \geq 2)$. By Lemma 3.3 we obtain $B_i = A_{ki}$. Therefore by Lemma 3.4 we find that there exists $i_0 \geq 1$ such that the equation over $\mathcal{K}_{n_0-1}^{(k)}$, $y_{i_0}(c^{(ki_0)}y + d^{(ki_0)}) = a^{(ki_0)}y + b^{(ki_0)}$, has a solution in $\overline{\mathcal{K}}_{n_0-1}^{(k)}$. Let $f \in \overline{K}_{n_0-1}$ be such a solution. It satisfies

$$\tau^{ki_0}(f)(c^{(ki_0)}f + d^{(ki_0)}) = a^{(ki_0)}f + b^{(ki_0)},$$

which implies that the equation over $\mathcal{K}$, $y_{ki_0}(c^{(ki_0)}y + d^{(ki_0)}) = a^{(ki_0)}y + b^{(ki_0)}$, has a solution in $\overline{\mathcal{K}}_{n_0-1}$.

Since $\overline{\mathcal{K}}_{n_0-1}/\mathcal{K}$ is a difference field extension of valuation ring type whose transcendence degree is less than $\operatorname{tr.deg} N/K$, we find by the induction hypothesis that there exists $i_1 \geq 1$ such that the equation over $\mathcal{K}$, $y_{ki_0i_1}(c^{(ki_0i_1)}y + d^{(ki_0i_1)}) = a^{(ki_0i_1)}y + b^{(ki_0i_1)}$, has a solution in $\overline{\mathcal{K}}$. □

The following is concerned with the case that the matrix turns out to be triangular by the iterations.

Proposition 3.5. *Let $\mathcal{K}$ be an inversive difference field, and $a, b, c, d \in K$ satisfy $ad - bc \neq 0$. Define the matrices A_i as in Lemma 3.3, and suppose $b^{(k)} = 0$ or $c^{(k)} = 0$ for some $k \geq 1$. Let f be a solution transcendental over K of the equation over $\mathcal{K}$, $y_1(cy + d) = ay + b$, and put $\mathcal{L} = \mathcal{K}\langle f\rangle$. Then the following hold.*

(i) *$\mathcal{L}$ is inversive.*

(ii) *L/K is an algebraic function field of one variable.*

(iii) *There is a valuation ring $\mathcal{O}$ of L/K such that $\tau^k\mathcal{O} \subset \mathcal{O}$.*

(iv) *$\mathcal{L}/\mathcal{K}$ is of valuation ring type.*

Proof. Let $\mathcal{L} = (L, \tau)$. Since $cf_1 - a = 0$ implies $f = \tau^{-1}(a/c) \in K$, we obtain

$$f = -\frac{df_1 - b}{cf_1 - a} = \tau\left(-\frac{\tau^{-1}(d)f - \tau^{-1}b}{\tau^{-1}(c)f - \tau^{-1}a}\right) \in \tau L.$$

Therefore $\mathcal{L}$ is inversive, which is the result (i). Since $cf + d = 0$ implies $f = -d/c \in K$, we obtain $f_1 \in K(f)$, which yields $L = K(f)$. This proves (ii).

By Lemma 3.3 we have $f_k(c^{(k)}f + d^{(k)}) = a^{(k)}f + b^{(k)}$. Put

$$g = \begin{cases} f & \text{if } c^{(k)} = 0, \\ 1/f & \text{if } c^{(k)} \neq 0. \end{cases}$$

We find that $g_k = \alpha g + \beta$ for some $\alpha, \beta \in K$, $\alpha \neq 0$. In fact, if $c^{(k)} = 0$, we have

$$g_k = f_k = \frac{a^{(k)}}{d^{(k)}} f + \frac{b^{(k)}}{d^{(k)}}.$$

Note that we obtain $\det A_k \neq 0$ from $\det A \neq 0$ by Lemma 3.3. If $c^{(k)} \neq 0$, we have $b^{(k)} = 0$ and

$$g_k = \frac{1}{f_k} = \frac{d^{(k)}}{a^{(k)}} \cdot \frac{1}{f} + \frac{c^{(k)}}{a^{(k)}}.$$

For the algebraic function field $L = K(g)$ of one variable over K, let $\mathcal{O}$ be the following valuation ring.

$$\mathcal{O} = \{p/q \in L \mid p, q \in K[g],\ \deg q - \deg p \geq 0\}.$$

For any $p \in K[g]$, the k-th transform $\tau^k p$ has the same degree as p. Therefore we obtain $\tau^k\mathcal{O} \subset \mathcal{O}$, which is the result (iii).

(i),(ii) and (iii) yield (iv). □

As a remark we introduce the following example.

Example 3.6. Let $\mathcal{K} = (K, \tau)$ be a difference field, and put $a = 1$, $b = 2$, $c = 1$ and $d = 0$, which are associated with the equation over $\mathcal{K}$, $y_1 y = y + 2$. This equation has solutions -1 and 2. Define the matrices A_i as in Lemma 3.3. We find that $b^{(i)} \neq 0$ and $c^{(i)} \neq 0$ for all $i \geq 1$.

3.3 Application to solvability

3.3.1 Preliminaries

Lemma 3.7. *Let $\mathcal{L}/\mathcal{K}$ be a GLE. Then $\mathcal{L}/\mathcal{K}$ is of valuation ring type.*

Proof. We prove this by induction on the transcendence degree of $\mathcal{L}/\mathcal{K}$. There is nothing to prove in case $\mathrm{tr.deg}\, L/K = 0$. Suppose $\mathrm{tr.deg}\, L/K > 0$, and the lemma is true for the transcendence degree $< \mathrm{tr.deg}\, L/K$. Let $\mathcal{L} = (L, \tau)$. There is a chain of inversive difference fields,

$$\mathcal{K} = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \cdots \subset \mathcal{K}_n = \mathcal{L}, \quad \mathcal{K}_i = \mathcal{K}_{i-1}\langle a_i \rangle^*,$$

such that a_i satisfies one of the following.

(i) $\tau a_i = a_i + \beta$ for some $\beta \in K_{i-1}$.

(ii) $\tau a_i = \alpha a_i$ for some $\alpha \in K_{i-1}$.

(iii) a_i is algebraic over K_{i-1}.

Put $m = \min\{1 \leq i \leq n \mid \mathrm{tr.deg}\, K_i/K > 0\}$. The chain $\mathcal{K}_m \subset \cdots \subset \mathcal{K}_n = \mathcal{L}$ is a GLE and satisfies $\mathrm{tr.deg}\, L/K_m < \mathrm{tr.deg}\, L/K$. Therefore by the induction hypothesis we find that $\mathcal{L}/\mathcal{K}_m$ is of valuation ring type.

Since a_m is transcendental over K_{m-1} because of $\mathrm{tr.deg}\, K_{m-1}/K = 0$, there are $\alpha, \beta \in K_{m-1}$, $\alpha \neq 0$ such that $\tau a_m = \alpha a_m + \beta$. By Proposition 3.5 we find that $\mathcal{K}_{m-1}\langle a_m \rangle/\mathcal{K}_{m-1}$ is of valuation ring type. Note that we have $\mathcal{K}_m = \mathcal{K}_{m-1}\langle a_m \rangle$. Therefore the chain

$$\mathcal{K} \subset \mathcal{K}_{m-1} \subset \mathcal{K}_m \subset \mathcal{L}$$

implies $\mathcal{L}/\mathcal{K}$ is of valuation ring type. □

Proposition 3.8. *Let $\mathcal{K}$ be a inversive difference field, $a, b, c, d \in K$, and $q \in \mathbb{Z}_{>0}$. Define the matrices A_i as in Lemma 3.3. Suppose $b^{(qi)} \neq 0$ and $c^{(qi)} \neq 0$ for all $i \geq 1$, and the equation over $\mathcal{K}$, $y_1(cy + d) = ay + b$, has a solution f in a qLE $\mathcal{L}/\mathcal{K}$. Let $\overline{\mathcal{L}} = (\overline{L}, \tau)$ be an algebraic closure of $\mathcal{L}$, and $\overline{\mathcal{K}}$ be the algebraic closure of $\mathcal{K}$ in $\overline{\mathcal{L}}$. Then there exists $i \geq 1$ such that the equation over $\mathcal{K}$, $y_{qi}(c^{(qi)}y + d^{(qi)}) = a^{(qi)}y + b^{(qi)}$, has a solution in $\overline{\mathcal{K}}$.*

Proof. Put $\overline{\mathcal{L}}^{(q)} = (\overline{L}, \tau^q)$, $\mathcal{L}^{(q)} = (L, \tau^q|_L)$, $\overline{\mathcal{K}}^{(q)} = (\overline{K}, \tau^q|_{\overline{K}})$, and $\mathcal{K}^{(q)} = (K, \tau^q|_K)$. Since $\mathcal{L}/\mathcal{K}$ is a qLE, $\mathcal{L}^{(q)}/\mathcal{K}^{(q)}$ is a GLE. By Lemma 3.7 we find that $\mathcal{L}^{(q)}/\mathcal{K}^{(q)}$ is of valuation ring type.

Since we have $f_q(c^{(q)}f + d^{(q)}) = a^{(q)}f + b^{(q)}$ by Lemma 3.3, $f \in \overline{\mathcal{L}}^{(q)}$ is a solution of the equation over $\mathcal{K}^{(q)}$, $y_1(c^{(q)}y + d^{(q)}) = a^{(q)}y + b^{(q)}$. Therefore by Theorem 3.2 we conclude that there exists $i \geq 1$ such that the equation over $\mathcal{K}^{(q)}$, $y_i(c^{(qi)}y + d^{(qi)}) = a^{(qi)}y + b^{(qi)}$, has a solution g in $\overline{\mathcal{K}}^{(q)}$, which implies $g \in \overline{\mathcal{K}}$ is a solution of the equation over $\mathcal{K}$, $y_{qi}(c^{(qi)}y + d^{(qi)}) = a^{(qi)}y + b^{(qi)}$. □

3.3.2 q-Airy equation

In their [8], Hamamoto, Kajiwara and Witte introduced that each of the basic hypergeometric solutions of the q-difference equation, $y(qt) + ty(t) = y(t/q)$, has a limit to the Airy function. Let $f \in \mathcal{K}^\times$ be a solution of the equation over $(C(t), t \mapsto qt)$, $y_2 + qty_1 - y = 0$, and put $g = f_1/f$. Then $g \in \mathcal{K}$ is a solution of the equation over $(C(t), t \mapsto qt)$, $y_1 y + qty - 1 = 0$, the object here.

The outline of the proof of the unsolvability is the following. *Step 1.* Define the matrices A_i as in Lemma 3.3, and find that they are not triangular. *Step 2.* Prove that there is no algebraic solution of the equation associated with A_i for all $i \geq 1$. *Step 3.* Apply Proposition 3.8.

Proposition 3.9. *Let $q \in C$ be transcendental over $\mathbb{Q}$, and t transcendental over C. Put $\mathcal{K} = (C(t), t \mapsto qt)$, and let $\overline{\mathcal{K}} = (\overline{C(t)}, \tau)$ be an algebraic closure of $\mathcal{K}$. Put $a = -qt$, $b = 1$, $c = 1$ and $d = 0$, and define the matrices A_i as in Lemma 3.3. Then the following hold.*

(i) $b^{(i)} \neq 0$ *and* $c^{(i)} \neq 0$ *for all* $i \geq 1$.

(ii) *For any $i \geq 1$ the equation over $\mathcal{K}$, $y_i(c^{(i)}y + d^{(i)}) = a^{(i)}y + b^{(i)}$, has no solution in $\overline{\mathcal{K}}$.*

Proof. We have

$$A = \begin{pmatrix} -qt & 1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = (\tau A)A = \begin{pmatrix} q^3t^2 + 1 & -q^2t \\ -qt & 1 \end{pmatrix},$$

and for any $i \geq 2$,

$$A_i = (\tau A_{i-1})A = \begin{pmatrix} -qta_1^{(i-1)} + b_1^{(i-1)} & a_1^{(i-1)} \\ -qtc_1^{(i-1)} + d_1^{(i-1)} & c_1^{(i-1)} \end{pmatrix},$$

$$A_i = (\tau^{i-1}A)A_{i-1} = \begin{pmatrix} -q^ita^{(i-1)} + c^{(i-1)} & -q^itb^{(i-1)} + d^{(i-1)} \\ a^{(i-1)} & b^{(i-1)} \end{pmatrix},$$

which imply $b^{(i)} = a_1^{(i-1)}$ and $c^{(i)} = a^{(i-1)}$ for all $i \geq 2$, and $d^{(i)} = a_1^{(i-2)}$ for all $i \geq 3$. From these we obtain

$$a^{(i)} = -q^ita^{(i-1)} + c^{(i-1)} = -q^ita^{(i-1)} + a^{(i-2)}, \quad \text{for any } i \geq 3.$$

Note $A_i \in M_2(C[t])$. We find

$$a^{(i)} = (-1)^i q^{\frac{i(i+1)}{2}} t^i + (\text{a polynomial of deg} \leq i-2) \tag{3.2}$$

by induction, and so $\deg a^{(i)} = i$. This implies $a^{(i)} \neq 0$, by which we conclude $b^{(i)} \neq 0$ and $c^{(i)} \neq 0$ for all $i \geq 1$, the result (i).

Assume that there exists $i_0 \geq 1$ such that the equation over $\mathcal{K}$, $y_{i_0}(c^{(i_0)}y + d^{(i_0)}) = a^{(i_0)}y + b^{(i_0)}$, has a solution f in $\overline{\mathcal{K}}$. Put $k = 3i_0 \geq 3$. By Lemma 3.3, $f \in \overline{\mathcal{K}}$ is a solution of the equation over $\mathcal{K}$, $y_k(c^{(k)}y + d^{(k)}) = a^{(k)}y + b^{(k)}$. Put $\mathcal{L} = \mathcal{K}\langle f \rangle \subset \overline{\mathcal{K}}$. Since both of the assumptions, $c^{(k)}f_k - a^{(k)} = 0$ and $c^{(k)}f + d^{(k)} = 0$, yield $\det A_k = 0$, which contradicts $\det A = -1$ by Lemma 3.3, we find that $\mathcal{L}$ is inversive, and $L = C(t)(f, f_1, \ldots, f_{k-1})$. Put $n = [L : C(t)] < \infty$. Then from Lemma 2.1 we obtain $L = C(x)$ with $x^n = t$. Note that x is transcendental over C, $f \in C(x)$, $A_i \in M_2(C[x^n])$, and $(\frac{\tau x}{x})^n = q \in C$, which implies $\frac{\tau x}{x} \in C$. Put $r = \frac{\tau x}{x} \in C^{\times}$.

Express $f = P/Q$, where $P, Q \in C[x]$ are relatively prime. The equation $f_k(c^{(k)}f + d^{(k)}) = a^{(k)}f + b^{(k)}$ yields

$$P_k(c^{(k)}P + d^{(k)}Q) = Q_k(a^{(k)}P + b^{(k)}Q) \quad (\neq 0), \tag{3.3}$$

where both sides of this are not equal to 0. We find by induction that $a^{(i)}P + b^{(i)}Q$ and $c^{(i)}P + d^{(i)}Q$ are relatively prime. In fact we obtain that $aP + bQ = -qtP + Q$ and $cP + dQ = P$ are relatively prime from the hypothesis, P and Q are relatively prime, the case $i = 1$. Let $i \geq 2$ and suppose the statement is true for $i - 1$. Since we have

$$\begin{aligned} a^{(i)}P + b^{(i)}Q &= (-q^ita^{(i-1)} + c^{(i-1)})P + (-q^itb^{(i-1)} + d^{(i-1)})Q \\ &= -q^it(a^{(i-1)}P + b^{(i-1)}Q) + (c^{(i-1)}P + d^{(i-1)}Q) \end{aligned}$$

and $a^{(i-1)}P + b^{(i-1)}Q = c^{(i)}P + d^{(i)}Q$, we conclude that $a^{(i)}P + b^{(i)}Q$ and $c^{(i)}P + d^{(i)}Q$ are relatively prime by the induction hypothesis.

Therefore $a^{(k)}P + b^{(k)}Q$ and $c^{(k)}P + d^{(k)}Q$ are relatively prime. From the equation (3.3) we obtain $\deg_x(a^{(k)}P + b^{(k)}Q) = \deg_x P_k = \deg_x P$. Since $\deg_x a^{(k)}P = nk + \deg_x P > \deg_x P$, we find $\deg_x a^{(k)}P = \deg_x b^{(k)}Q$, which implies $\deg_x Q - \deg_x P = n$.

Express

$$f = \sum_{i=n}^{\infty} e_i \left(\frac{1}{x}\right)^i, \quad e_i \in C,\ e_n \neq 0.$$

We will show $f \in C(t)$. Assume there exists $i \geq n$ such that $n \nmid i$ and $e_i \neq 0$, and put $ln + m$ $(0 < m < n)$ be the minimum number of them. Note

$$\deg_x a^{(k)} = kn, \quad \deg_x b^{(k)} = \deg_x c^{(k)} = (k-1)n, \quad \deg_x d^{(k)} = (k-2)n.$$

The first term of

$$\begin{aligned} & a^{(k)} f + b^{(k)} \\ & = a^{(k)} \left(e_n \left(\frac{1}{x}\right)^n + \cdots + e_{ln} \left(\frac{1}{x}\right)^{ln} + e_{ln+m} \left(\frac{1}{x}\right)^{ln+m} + \cdots \right) + b^{(k)} \end{aligned}$$

whose exponent is not divisible by n has the exponent, $-kn + (ln+m)$. The first term of

$$\begin{aligned} & f_k(c^{(k)} f + d^{(k)}) \\ & = \left\{ \frac{e_n}{r^{kn}} \left(\frac{1}{x}\right)^n + \cdots + \frac{e_{ln}}{r^{kln}} \left(\frac{1}{x}\right)^{ln} + \frac{e_{ln+m}}{r^{k(ln+m)}} \left(\frac{1}{x}\right)^{ln+m} + \cdots \right\} \\ & \times \left\{ c^{(k)} \left(e_n \left(\frac{1}{x}\right)^n + \cdots + e_{ln} \left(\frac{1}{x}\right)^{ln} + e_{ln+m} \left(\frac{1}{x}\right)^{ln+m} + \cdots \right) + d^{(k)} \right\} \end{aligned}$$

whose exponent is not divisible by n has the exponent $\geq (2-k)n + (ln+m)$, which is impossible. Therefore we obtain $f = \sum_{i=1}^{\infty} e_{ni}(1/x^n)^i$, and so $f \in C(1/x^n) = C(t)$.

Then we have $L = C(t)(f, f_1, \ldots, f_{k-1}) \subset C(t)$, which implies $n = [L : C(t)] = 1$, $x = t$ and $r = q$. We find $a^{(i)} \in \mathbb{Z}[q,t]$ by induction, and so $b^{(i)}, c^{(i)}, d^{(i)} \in \mathbb{Z}[q,t]$. We will show $e_j \in \mathbb{Z}[q, 1/q]$ for any $j \geq 1$ by induction. We have

$$\text{(3.4)} \qquad f_k(c^{(k)} f + d^{(k)}) = \left(\sum_{i=1}^{\infty} \frac{e_i}{q^{ki}} \left(\frac{1}{t}\right)^i \right) \left(c^{(k)} \sum_{i=1}^{\infty} e_i \left(\frac{1}{t}\right)^i + d^{(k)} \right)$$

and

$$a^{(k)}f + b^{(k)} = a^{(k)} \sum_{i=1}^{\infty} e_i \left(\frac{1}{t}\right)^i + b^{(k)}. \tag{3.5}$$

Note that the equation (3.2) yields

$$a^{(k)} = (-1)^k q^{\frac{k(k+1)}{2}} t^k + (\text{a polynomial of deg} \le k-2),$$
$$b^{(k)} = a_1^{(k-1)} = (-1)^{k-1} q^{\frac{(k-1)(k+2)}{2}} t^{k-1} + (\text{a polynomial of deg} \le k-3).$$

Comparing the terms of exponent $-k+1$ of the equation (3.4) = (3.5), we obtan

$$0 = (-1)^k q^{\frac{k(k+1)}{2}} e_1 + (-1)^{k-1} q^{\frac{(k-1)(k+2)}{2}},$$

which implies $e_1 = q^{-1} \in \mathbb{Z}[q, 1/q]$.

Let $j \ge 2$ and suppose the statement is true for the numbers $\le j-1$. On the one hand the term of exponent $-k+j$ of (3.5) has the coefficient,

$$(-1)^k q^{\frac{k(k+1)}{2}} e_j + (\text{an element of } \mathbb{Z}[q][e_1, e_2, \dots, e_{j-1}])$$
$$\in (-1)^k q^{\frac{k(k+1)}{2}} e_j + \mathbb{Z}[q, 1/q].$$

On the other hand the term of exponent $-k+j$ of (3.4) is the same one of

$$\left(\sum_{i=1}^{j-1} \frac{e_i}{q^{ki}} \left(\frac{1}{t}\right)^i\right) \left(c^{(k)} \sum_{i=1}^{j-1} e_i \left(\frac{1}{t}\right)^i + d^{(k)}\right) \in \mathbb{Z}[q, 1/q]((1/t)) \subset C((1/t)).$$

Therefore we obtain

$$(-1)^k q^{\frac{k(k+1)}{2}} e_j \in \mathbb{Z}[q, 1/q],$$

which implies $e_j \in \mathbb{Z}[q, 1/q]$.

Let $\phi\colon \mathbb{Q}[q, 1/q] \mapsto \mathbb{Q}$ be a ring homomorphism sending q to 1, and extend it to the ring homomorphism $\overline{\phi}\colon \mathbb{Q}[q, 1/q]((1/t)) \mapsto C((1/t))$ sending $\sum_{i=m}^{\infty} h_i (1/t)^i$ to $\sum_{i=m}^{\infty} \phi(h_i)(1/t)^i$. This $\overline{\phi}$ is a difference homomorphism of $(\mathbb{Q}[q, 1/q]((1/t)), t \mapsto qt)$ to $(C((1/t)), id)$, and so we obtain

$$\overline{\phi}(f)(\overline{\phi}(c^{(k)})\overline{\phi}(f) + \overline{\phi}(d^{(k)})) = \overline{\phi}(a^{(k)})\overline{\phi}(f) + \overline{\phi}(b^{(k)}).$$

We find $\overline{\phi}(f) \in C(t)$. In fact since $f \in C(1/t)$, there are $s \in \mathbb{Z}_{\ge 0}$ and $m_0 \in \mathbb{Z}_{\ge 0}$ such that $F_f(m, s) = 0$ for all $m \ge m_0$, where $F_f(m, s)$ is the Hankel determinant $\det(e_{m+i+j})_{0\le i,j\le s}$ of f (refer to [2] for the Hankel determinant). Therefore for any $m \ge m_0$ we obtain

$$\begin{aligned} F_{\overline{\phi}(f)}(m,s) &= \det(\phi(e_{m+i+j}))_{0\le i,j\le s} = \phi(\det(e_{m+i+j})_{0\le i,j\le s}) \\ &= \phi(F_f(m,s)) = 0, \end{aligned}$$

which implies $\overline{\phi}(f) \in C(1/t) = C(t)$.

Express $\overline{\phi}(f) = P'/Q'$, where $P', Q' \in C[t]$ are relatively prime, and put $a' = \overline{\phi}(a^{(k)})$, $b' = \overline{\phi}(b^{(k)})$, $c' = \overline{\phi}(c^{(k)})$ and $d' = \overline{\phi}(d^{(k)})$. Note

$$\begin{aligned} c' &= \overline{\phi}(c^{(k)}) = \overline{\phi}(a^{(k-1)}) = \overline{\phi}(a_1^{(k-1)}) = \overline{\phi}(b^{(k)}) = b', \\ d' &= \overline{\phi}(d^{(k)}) = \overline{\phi}(a_1^{(k-2)}) = \overline{\phi}(a^{(k-2)}) = \overline{\phi}(a^{(k)} + q^k t a^{(k-1)}) = a' + tb', \end{aligned}$$

and $b' = (-1)^{k-1}t^{k-1} +$ (a polynomial of deg $\le k-3$) $\ne 0$. Then we obtain the following from $P'(c'P' + d'Q') = Q'(a'P' + b'Q')$,

$$P'^2 + tP'Q' = Q'^2. \tag{3.6}$$

This equation yields $P' \mid Q'^2$ and $Q' \mid P'^2$, which imply $\deg P' = \deg Q' = 0$. Comparing the degree of the equation (3.6), we find $1 = 0$, a contradiction. Therefore we obtain (ii). □

Corollary 3.10. *Let $q \in C$ be transcendental over $\mathbb{Q}$, t transcendental over C, $\mathcal{K} = (C(t), t \mapsto qt)$, and $k \in \mathbb{Z}_{>0}$. Then the equation over $\mathcal{K}$, $y_1 y + qty - 1 = 0$, has no solution in a kLE of $\mathcal{K}$.*

Proof. Assume the equation has a solution in a kLE $\mathcal{N}/\mathcal{K}$. Put $a = -qt$, $b = c = 1$ and $d = 0$. Define the matrices A_i as in Lemma 3.3. By Proposition 3.9 we have $b^{(i)} \ne 0$ and $c^{(i)} \ne 0$ for all $i \ge 1$.

Let $\overline{\mathcal{N}}$ be an algebraic closure of $\mathcal{N}$, and $\overline{\mathcal{K}}$ the algebraic closure of $\mathcal{K}$ in $\overline{\mathcal{N}}$. By Proposition 3.8 we find that there exists $i \ge 1$ such that the equation over $\mathcal{K}$, $y_{ki}(c^{(ki)}y + d^{(ki)}) = a^{(ki)}y + b^{(ki)}$, has a solution in $\overline{\mathcal{K}}$, which contradicts Proposition 3.9. □

3.3.3 q-Bessel equation

Seeing [7], we find one of the q-Bessel functions, $J_\nu^{(3)}(x;q)$, and the equation,

$$g_\nu(qx) + (x^2/4 - q^\nu - q^{-\nu})g_\nu(x) + g_\nu(xq^{-1}) = 0,$$

where $g_\nu(x) = J_\nu^{(3)}(xq^{\nu/2}; q^2)$. This section deals with the Riccati equation associated with it.

Proposition 3.11. *Let $q \in C$ be transcendental over $\mathbb{Q}$, and t transcendental over C. Put $\mathcal{K} = (C(t), t \mapsto qt)$, and let $\overline{\mathcal{K}} = (\overline{C(t)}, \tau)$ be an algebraic closure of $\mathcal{K}$. Put $a = -(t^2/4 - q^\nu - q^{-\nu})$, $b = -1$, $c = 1$ and $d = 0$, where $\nu \in \mathbb{Q}$, and define the matrices A_i as in Lemma 3.3. Then the following hold.*

(i) $b^{(i)} \neq 0$ *and* $c^{(i)} \neq 0$ *for all* $i \geq 1$.

(ii) *For any* $i \geq 1$ *the equation over* $\mathcal{K}$, $y_i(c^{(i)}y + d^{(i)}) = a^{(i)}y + b^{(i)}$, *has no solution in* $\overline{\mathcal{K}}$.

Proof. Put $p = q^\nu + q^{-\nu} \in C$. We have

$$A = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_1 a - 1 & -a_1 \\ a & -1 \end{pmatrix},$$

and for any $i \geq 2$,

$$A_i = (\tau A_{i-1})A = \begin{pmatrix} a a_1^{(i-1)} + b_1^{(i-1)} & -a_1^{(i-1)} \\ a c_1^{(i-1)} + d_1^{(i-1)} & -c_1^{(i-1)} \end{pmatrix},$$

$$A_i = (\tau^{i-1}A)A_{i-1} = \begin{pmatrix} a_{i-1}a^{(i-1)} - c^{(i-1)} & a_{i-1}b^{(i-1)} - d^{(i-1)} \\ a^{(i-1)} & b^{(i-1)} \end{pmatrix},$$

which imply $b^{(i)} = -a_1^{(i-1)}$ and $c^{(i)} = a^{(i-1)}$ for all $i \geq 2$, and $d^{(i)} = -a_1^{(i-2)}$ for all $i \geq 3$. From these we obtain

$$a^{(i)} = a_{i-1}a^{(i-1)} - c^{(i-1)} = a_{i-1}a^{(i-1)} - a^{(i-2)}, \quad \text{for any } i \geq 3.$$

Note $A_i \in M_2(C[t])$. We find

$$a^{(i)} = (-1)^i \frac{q^{(i-1)i}}{4^i} t^{2i} + (\text{a polynomial of deg} \leq 2i - 2) \tag{3.7}$$

by induction, and so $\deg a^{(i)} = 2i$. This implies $a^{(i)} \neq 0$, by which we conclude $b^{(i)} \neq 0$ and $c^{(i)} \neq 0$ for all $i \geq 1$, the result (i).

Assume that there exists $i_0 \geq 1$ such that the equation over $\mathcal{K}$, $y_{i_0}(c^{(i_0)}y + d^{(i_0)}) = a^{(i_0)}y + b^{(i_0)}$, has a solution f in $\overline{\mathcal{K}}$. Put $k = 3i_0 \geq 3$. By Lemma 3.3, $f \in \overline{\mathcal{K}}$ is a solution of the equation over $\mathcal{K}$, $y_k(c^{(k)}y + d^{(k)}) = a^{(k)}y + b^{(k)}$. Put $\mathcal{L} = \mathcal{K}\langle f \rangle \subset \overline{\mathcal{K}}$. We find that $\mathcal{L}$ is inversive, and $L = C(t)(f, f_1, \ldots, f_{k-1})$. Put $n = [L : C(t)] < \infty$. Then from Lemma 2.1 we obtain $L = C(x)$ with

$x^n = t$. Note that x is transcendental over C, $f \in C(x)$, $A_i \in M_2(C[x^n])$, and $(\frac{\tau x}{x})^n = q \in C$, which implies $\frac{\tau x}{x} \in C$. Put $r = \frac{\tau x}{x} \in C^\times$.

Express $f = P/Q$, where $P, Q \in C[x]$ are relatively prime. The equation $f_k(c^{(k)} f + d^{(k)}) = a^{(k)} f + b^{(k)}$ yields

$$P_k(c^{(k)} P + d^{(k)} Q) = Q_k(a^{(k)} P + b^{(k)} Q) \quad (\neq 0). \tag{3.8}$$

We find by induction that $a^{(i)}P + b^{(i)}Q$ and $c^{(i)}P + d^{(i)}Q$ are relatively prime. In fact we obtain that $aP + bQ = aP - Q$ and $cP + dQ = P$ are relatively prime, the case $i = 1$. Let $i \geq 2$ and suppose the statement is true for $i - 1$. Since we have

$$\begin{aligned} a^{(i)}P + b^{(i)}Q &= (a_{i-1}a^{(i-1)} - c^{(i-1)})P + (a_{i-1}b^{(i-1)} - d^{(i-1)})Q \\ &= a_{i-1}(a^{(i-1)}P + b^{(i-1)}Q) - (c^{(i-1)}P + d^{(i-1)}Q) \end{aligned}$$

and $a^{(i-1)}P + b^{(i-1)}Q = c^{(i)}P + d^{(i)}Q$, we conclude that $a^{(i)}P + b^{(i)}Q$ and $c^{(i)}P + d^{(i)}Q$ are relatively prime by the induction hypothesis.

Therefore $a^{(k)}P + b^{(k)}Q$ and $c^{(k)}P + d^{(k)}Q$ are relatively prime. From the equation (3.8) we obtain $\deg_x(a^{(k)}P + b^{(k)}Q) = \deg_x P_k = \deg_x P$. Since $\deg_x a^{(k)}P = 2kn + \deg_x P > \deg_x P$, we find that $\deg_x a^{(k)}P = \deg_x b^{(k)}Q$, which implies $\deg_x Q - \deg_x P = 2n$.

Express

$$f = \sum_{i=2n}^{\infty} e_i \left(\frac{1}{x}\right)^i, \quad e_i \in C,\ e_{2n} \neq 0.$$

We obtain $f \in C(t)$ by the same way as in the proof of Proposition 3.9, and so $L = C(t)$, $n = 1$, $x = t$ and $r = q$. Note $a^{(i)}, b^{(i)}, c^{(i)}, d^{(i)} \in \mathbb{Q}[q, p, t]$. We will show $e_j \in \mathbb{Q}[q, 1/q, p]$ for any $j \geq 2$ by induction. We have

$$f_k(c^{(k)} f + d^{(k)}) = \left(\sum_{i=2}^{\infty} \frac{e_i}{q^{ki}} \left(\frac{1}{t}\right)^i\right) \left(c^{(k)} \sum_{i=2}^{\infty} e_i \left(\frac{1}{t}\right)^i + d^{(k)}\right) \tag{3.9}$$

and

$$a^{(k)} f + b^{(k)} = a^{(k)} \sum_{i=2}^{\infty} e_i \left(\frac{1}{t}\right)^i + b^{(k)}. \tag{3.10}$$

The equation (3.7) yields

$$\begin{aligned} a^{(k)} &= (-1)^k \frac{q^{(k-1)k}}{4^k} t^{2k} + (\text{a polynomial of deg} \leq 2k-2), \\ b^{(k)} &= (-1)^k \frac{q^{(k-1)k}}{4^{k-1}} t^{2(k-1)} + (\text{a polynomial of deg} \leq 2k-4). \end{aligned}$$

Comparing the terms of exponent $-2k+2$ of the equation (3.9) = (3.10), we obtain

$$0 = (-1)^k \frac{q^{(k-1)k}}{4^k} e_2 + (-1)^k \frac{q^{(k-1)k}}{4^{k-1}},$$

which implies $e_2 = -4$.

Let $j \geq 3$ and suppose the statement is true for the numbers $\leq j-1$. On the one hand the term of exponent $-2k+j$ of (3.10) has the coefficient,

$$\begin{aligned} &(-1)^k \frac{q^{(k-1)k}}{4^k} e_j + (\text{an element of } \mathbb{Q}[q, p, e_2, e_3, \ldots, e_{j-1}]) \\ &\in (-1)^k \frac{q^{(k-1)k}}{4^k} e_j + \mathbb{Q}[q, 1/q, p]. \end{aligned}$$

On the other hand the term of exponent $-2k+j$ of (3.9) is the same one of

$$\left(\sum_{i=2}^{j-1} \frac{e_i}{q^{ki}} \left(\frac{1}{t}\right)^i\right) \left(c^{(k)} \sum_{i=2}^{j-1} e_i \left(\frac{1}{t}\right)^i + d^{(k)}\right) \in \mathbb{Q}[q, 1/q, p]((1/t)) \subset C((1/t)).$$

Therefore we obtain

$$(-1)^k \frac{q^{(k-1)k}}{4^k} e_j \in \mathbb{Q}[q, 1/q, p],$$

which implies $e_j \in \mathbb{Q}[q, 1/q, p]$.

Let $\nu = \nu_1/\nu_2$, where $\nu_1 \in \mathbb{Z}$ and $\nu_2 \in \mathbb{Z}_{>0}$ are relatively prime. Then we have

$$\mathbb{Q}[q, 1/q, p] \subset \mathbb{Q}[q^{\frac{1}{\nu_2}}, 1/q^{\frac{1}{\nu_2}}].$$

Let $\phi\colon \mathbb{Q}[q^{(1/\nu_2)}, 1/q^{(1/\nu_2)}] \mapsto \mathbb{Q}$ be a ring homomorphism sending $q^{(1/\nu_2)}$ to 1, and extend it to the ring homomorphism $\overline{\phi}\colon \mathbb{Q}[q^{(1/\nu_2)}, 1/q^{(1/\nu_2)}]((1/t)) \mapsto \mathbb{Q}((1/t))$ sending $\sum_{i=m}^{\infty} h_i (1/t)^i$ to $\sum_{i=m}^{\infty} \phi(h_i)(1/t)^i$. This $\overline{\phi}$ is a difference homomorphism of $(\mathbb{Q}[q^{(1/\nu_2)}, 1/q^{(1/\nu_2)}]((1/t)), t \mapsto qt)$ to $(\mathbb{Q}((1/t)), id)$, and so we obtain

$$\overline{\phi}(f)(\overline{\phi}(c^{(k)})\overline{\phi}(f) + \overline{\phi}(d^{(k)})) = \overline{\phi}(a^{(k)})\overline{\phi}(f) + \overline{\phi}(b^{(k)}).$$

We find $\overline{\phi}(f) \in C(t)$ by seeing the Hankel determinant. Express $\overline{\phi}(f) = P'/Q'$, where $P', Q' \in C[t]$ are relatively prime, and put $a' = \overline{\phi}(a^{(k)})$, $b' = \overline{\phi}(b^{(k)})$, $c' = \overline{\phi}(c^{(k)})$ and $d' = \overline{\phi}(d^{(k)})$. Note

$$c' = \overline{\phi}(c^{(k)}) = \overline{\phi}(a^{(k-1)}) = \overline{\phi}(a_1^{(k-1)}) = -\overline{\phi}(b^{(k)}) = -b',$$

$$\begin{aligned} d' &= \overline{\phi}(d^{(k)}) = \overline{\phi}(-a_1^{(k-2)}) = \overline{\phi}(-a^{(k-2)}) = \overline{\phi}(a^{(k)} - a_{k-1}a^{(k-1)}) \\ &= a' + \left(-\frac{t^2}{4} + 2\right) b', \end{aligned}$$

and $b' \neq 0$. Then we obtain the following from $P'(c'P' + d'Q') = Q'(a'P' + b'Q')$,

$$-P'^2 + \left(-\frac{t^2}{4} + 2\right) P'Q' = Q'^2. \tag{3.11}$$

This equation yields $P' \mid Q'^2$ and $Q' \mid P'^2$, which imply $\deg P' = \deg Q' = 0$. Comparing the degree of the equation (3.11), we find $2 = 0$, a contradiction. Therefore we obtain (ii). □

Corollary 3.12. *Let $q \in C$ be transcendental over $\mathbb{Q}$, t transcendental over C, $\mathcal{K} = (C(t), t \mapsto qt)$, and $k \in \mathbb{Z}_{>0}$. Then the equation over $\mathcal{K}$, $y_1 y = -(t^2/4 - q^{\nu} - q^{-\nu})y - 1$, where $\nu \in \mathbb{Q}$, has no solution in a kLE of $\mathcal{K}$.*

Chapter 4

General theory of decomposable extensions

We define the decomposable extensions of difference fields. Every strongly normal extension or Liouville-Franke extension, the latter of which is a difference analogue of the Liouvillian extension, satisfies that its appropriate algebraic closure is a decomposable extension.

Notation. Throughout this chapter a field is of characteristic zero.

4.1 Introduction

In [21] the author introduced the definition and some examples of the $\mathcal{U}$-decomposable extensions of difference fields. In this chapter we define the decomposable extensions of difference fields, which do not require the fixed difference field $\mathcal{U}$, and study the irreducibility of q-Painlevé equation of type $A_7^{(1)'}$.

We show that some algebraic closure of any $\mathcal{U}$-decomposable extension is decomposable in Proposition 4.4. Therefore some algebraic closure of Bialynicki-Birula's strongly normal extension or Infante's is decomposable (see [1, 9, 10, 21, 22]). Moreover Corollary 4.8 implies that any algebraic closure of the Liouville-Franke extension, a difference analogue of the Liouvillian extension, is decomposable (see [4, 5]).

We define the decomposable extensions and the $\mathcal{U}$-decomposable extensions.

Definition 4.1 (decomposable extension)**.** Let $\mathcal{K}$ be a difference field, and $\mathcal{L}$ an algebraically closed difference overfield of $\mathcal{K}$ satisfying $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} < \infty$. We define decomposable extensions by induction on $\operatorname{tr.deg} \mathcal{L}/\mathcal{K}$.

(i) If $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} \leq 1$, then $\mathcal{L}/\mathcal{K}$ is decomposable.

(ii) When $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} \geq 2$, $\mathcal{L}/\mathcal{K}$ is decomposable if there exist a difference overfield $\mathcal{U}$ of $\mathcal{L}$, a difference overfield $\mathcal{E}$ of $\mathcal{K}$ in $\mathcal{U}$ of finite transcendence degree which is free from $\mathcal{L}$ over $\mathcal{K}$, and a difference intermediate field $\mathcal{M}$ of $\mathcal{L}\mathcal{E}/\mathcal{E}$ satisfying $\operatorname{tr.deg} \mathcal{L}\mathcal{E}/\mathcal{M} \geq 1$ and $\operatorname{tr.deg} \mathcal{M}/\mathcal{E} \geq 1$, such that $\overline{\mathcal{L}\mathcal{E}}/\mathcal{M}$ and $\overline{\mathcal{M}}/\mathcal{E}$ are decomposable, where $\overline{\mathcal{L}\mathcal{E}}$ is an algebraic closure of $\mathcal{L}\mathcal{E}$ and $\overline{\mathcal{M}}$ the algebraic closure of $\mathcal{M}$ in $\overline{\mathcal{L}\mathcal{E}}$.

Definition 4.2 ($\mathcal{U}$-decomposable extension)**.** Let $\mathcal{U}$ be a difference field and $\mathcal{L}/\mathcal{K}$ a difference field extension in $\mathcal{U}$ of finite transcendence degree. We define $\mathcal{U}$-decomposable extensions by induction on $\operatorname{tr.deg} L/K$.

(i) If $\operatorname{tr.deg} L/K \leq 1$ then $\mathcal{L}/\mathcal{K}$ is $\mathcal{U}$-decomposable.

(ii) When $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} \geq 2$, $\mathcal{L}/\mathcal{K}$ is $\mathcal{U}$-decomposable if there exist a difference overfield $\mathcal{E}$ of $\mathcal{K}$ in $\mathcal{U}$ of finite transcendence degree which is free from $\mathcal{L}$ over $\mathcal{K}$, and a difference intermediate field $\mathcal{M}$ of $\mathcal{L}\mathcal{E}/\mathcal{E}$ such that $\operatorname{tr.deg} LE/M \geq 1$, $\operatorname{tr.deg} M/E \geq 1$, $\mathcal{L}\mathcal{E}/\mathcal{M}$ is $\mathcal{U}$-decomposable, and $\mathcal{M}/\mathcal{E}$ is $\mathcal{U}$-decomposable.

4.2 Decomposable extension

Proposition 4.3. *Let $\mathcal{K}$ be a difference field, and $\mathcal{L}/\mathcal{K}$ and $\mathcal{N}/\mathcal{L}$ be decomposable extensions. Then $\mathcal{N}/\mathcal{K}$ is decomposable.*

Proof. (i) If $\operatorname{tr.deg} N/K \leq 1$, then we find that $\mathcal{N}/\mathcal{K}$ is decomposable by the definition.

(ii) Suppose $\operatorname{tr.deg} N/K \geq 2$.

(ii-1) If $\operatorname{tr.deg} N/L = 0$, then we obtain $\mathcal{N} = \mathcal{L}$ because $\mathcal{L}$ is algebraically closed. Therefore $\mathcal{N}/\mathcal{K}$ is decomposable.

(ii-2) Suppose $\operatorname{tr.deg} L/K = 0$. Since $\mathcal{N}/\mathcal{L}$ is a decomposable extension of $\operatorname{tr.deg} N/L \geq 2$, there exist a difference overfield $\mathcal{U}$ of $\mathcal{N}$, a difference overfield $\mathcal{E}$ of $\mathcal{L}$ in $\mathcal{U}$ of finite transcendence degree which is free from $\mathcal{N}$ over $\mathcal{L}$, and a difference intermediate field $\mathcal{M}$ of $\mathcal{N}\mathcal{E}/\mathcal{E}$ satisfying $\operatorname{tr.deg} \mathcal{N}\mathcal{E}/\mathcal{M} \geq 1$ and

$\operatorname{tr.deg} \mathcal{M}/\mathcal{E} \geq 1$, such that $\overline{\mathcal{N}\mathcal{E}}/\mathcal{M}$ and $\overline{\mathcal{M}}/\mathcal{E}$ are decomposable, where $\overline{\mathcal{N}\mathcal{E}}$ is an algebraic closure of $\mathcal{N}\mathcal{E}$ and $\overline{\mathcal{M}}$ the algebraic closure of $\mathcal{M}$ in $\overline{\mathcal{N}\mathcal{E}}$.

Note $\operatorname{tr.deg} E/K = \operatorname{tr.deg} E/L < \infty$ and that N and E are free over K. Then we find that $\mathcal{N}/\mathcal{K}$ is decomposable.

(ii-3) Suppose $\operatorname{tr.deg} N/L \geq 1$ and $\operatorname{tr.deg} L/K \geq 1$. Putting $\mathcal{U} = \mathcal{N}$, $\mathcal{E} = \mathcal{K}$ and $\mathcal{M} = \mathcal{L}$, we find that $\mathcal{N}/\mathcal{K}$ is decomposable by the definition. □

Therefore chains of decomposable extensions are decomposable.

Proposition 4.4. *Let $\mathcal{L}/\mathcal{K}$ be a $\mathcal{U}$-decomposable extension, $\overline{\mathcal{U}}$ an algebraic closure of $\mathcal{U}$, and $\overline{\mathcal{L}}$ the algebraic closure of $\mathcal{L}$ in $\overline{\mathcal{U}}$. Then $\overline{\mathcal{L}}/\mathcal{K}$ is decomposable.*

Proof. We prove this by induction on $\operatorname{tr.deg} L/K$. If $\operatorname{tr.deg} L/K \leq 1$, then $\operatorname{tr.deg} \overline{L}/K \leq 1$, and so $\overline{\mathcal{L}}/\mathcal{K}$ is decomposable.

Suppose $\operatorname{tr.deg} L/K \geq 2$ and that the statement is true for ones of less transcendence degree. Since $\mathcal{L}/\mathcal{K}$ and $\overline{\mathcal{L}}/\mathcal{L}$ are $\overline{\mathcal{U}}$-decomposable, we find that $\overline{\mathcal{L}}/\mathcal{K}$ is $\overline{\mathcal{U}}$-decomposable (see [21]). Therefore there exist a difference overfield $\mathcal{E} \subset \overline{\mathcal{U}}$ of $\mathcal{K}$ satisfying $\operatorname{tr.deg} E/K < \infty$ and that E is free from $\overline{L}$ over K, and a difference intermediate field $\mathcal{M}$ of $\overline{\mathcal{L}}\mathcal{E}/\mathcal{E}$ satisfying $\operatorname{tr.deg} \overline{L}E/M \geq 1$ and $\operatorname{tr.deg} M/E \geq 1$, such that $\overline{\mathcal{L}}\mathcal{E}/\mathcal{M}$ and $\mathcal{M}/\mathcal{E}$ are $\overline{\mathcal{U}}$-decomposable.

Let $\overline{\overline{\mathcal{L}}\mathcal{E}}$ and $\overline{\mathcal{M}}$ be the algebraic closures of $\overline{\mathcal{L}}\mathcal{E}$ and $\mathcal{M}$ in $\overline{\mathcal{U}}$ respectively. By the induction hypothesis we find that $\overline{\overline{\mathcal{L}}\mathcal{E}}/\mathcal{M}$ and $\overline{\mathcal{M}}/\mathcal{E}$ are decomposable, which implies that $\overline{\mathcal{L}}/\mathcal{K}$ is decomposable. □

The remaining results in this section are on a linear difference equation. We include the following Lemma for readers convenience.

Lemma 4.5. *Let $\mathcal{K}$ be a difference field, $C = C_{\mathcal{K}}$, $n \in \mathbb{Z}_{\geq 1}$, and $b^{(1)}, \ldots, b^{(n)} \in K$. Then the following are equivalent.*

(i) *$b^{(1)}, \ldots, b^{(n)}$ are linearly dependent over C.*

(ii) $\operatorname{Cas}(b^{(1)}, \ldots, b^{(n)}) = 0$.

Proof. Let $\mathcal{K} = (K, \tau)$. If $b^{(1)}, \ldots, b^{(n)}$ are linearly dependent over C, there are $c_1, \ldots, c_n \in C$ such that $(c_1, \ldots, c_n) \neq 0$ and $\sum_{i=1}^{n} c_i b^{(i)} = 0$. Then we obtain $\sum_{i=1}^{n} c_i b_j^{(i)} = 0$ for all $0 \leq j \leq n-1$, which implies $\operatorname{Cas}(b^{(1)}, \ldots, b^{(n)}) = 0$.

Suppose $\mathrm{Cas}\,(b^{(1)}, \dots, b^{(n)}) = 0$. We prove (i) by induction on n. The statement is true in the case $n = 1$. Suppose $n \geq 2$ and the statement is true for $n - 1$. There are $c_1, \dots, c_n \in K$ such that $(c_1, \dots, c_n) \neq 0$ and

$$\begin{pmatrix} b^{(1)} & \cdots & b^{(n)} \\ \vdots & \ddots & \vdots \\ b^{(1)}_{n-1} & \cdots & b^{(n)}_{n-1} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = 0.$$

We may suppose $c_1 = 1$. From $\sum_{i=1}^{n} c_i b^{(i)}_j = 0$ for any $0 \leq j \leq n-1$, we obtain $\sum_{i=1}^{n} \tau(c_i) b^{(i)}_j = 0$ for any $1 \leq j \leq n$. Therefore it follows that for any $1 \leq j \leq n-1$,

$$\sum_{i=2}^{n} (\tau(c_i) - c_i) b^{(i)}_j = \sum_{i=1}^{n} (\tau(c_i) - c_i) b^{(i)}_j = 0,$$

which implies

$$\begin{pmatrix} b^{(2)}_1 & \cdots & b^{(n)}_1 \\ \vdots & \ddots & \vdots \\ b^{(2)}_{n-1} & \cdots & b^{(n)}_{n-1} \end{pmatrix} \begin{pmatrix} \tau(c_2) - c_2 \\ \vdots \\ \tau(c_n) - c_n \end{pmatrix} = 0.$$

Case 1. The case $\mathrm{Cas}\,(b^{(2)}_1, \dots, b^{(n)}_1) \neq 0$. In this case we find that $\tau(c_i) = c_i$ for all $2 \leq i \leq n$, which implies $c_i \in C$ for all $1 \leq i \leq n$. Since we have $\sum_{i=1}^{n} c_i b^{(i)} = 0$, we conclude that $b^{(1)}, \dots, b^{(n)}$ are linearly dependent over C.

Case 2. The case $\mathrm{Cas}\,(b^{(2)}_1, \dots, b^{(n)}_1) = 0$. We obtain $\mathrm{Cas}\,(b^{(2)}, \dots, b^{(n)}) = 0$. By the induction hypothesis we find that $b^{(2)}, \dots, b^{(n)}$ are linearly dependent over C, which implies $b^{(1)}, b^{(2)}, \dots, b^{(n)}$ are linearly dependent over C. □

Lemma 4.6. *Let $\mathcal{K}$ be a difference field,*

$$y_n + a_{n-1} y_{n-1} + \cdots + a_0 y = 0 \tag{4.1}$$

a linear homogeneous difference equation over $\mathcal{K}$, where $n \geq 1$, f a solution of (4.1), and $\mathcal{L}$ an algebraic difference overfield of $\mathcal{K}\langle f \rangle$. Then $\mathcal{L}/\mathcal{K}$ is $\mathcal{U}$-decomposable for some difference overfield $\mathcal{U}$ of $\mathcal{L}$.

Proof. We may suppose $\mathrm{tr.deg}\,\mathcal{K}\langle f \rangle / \mathcal{K} \geq 2$. Let $\mathcal{L} = (L, \tau_L)$ and choose $b^{(i)}_j$, $1 \leq i \leq n$, $0 \leq j \leq n-1$ to be algebraically independent over L. Put

$B = \{b_j^{(i)} \mid 1 \le i \le n,\ 0 \le j \le n-1\}$, $b^{(i)} = b_0^{(i)}$ and $b_n^{(i)} = -a_{n-1}b_{n-1}^{(i)} - \cdots - a_0 b^{(i)}$. Define the isomorphism τ of $L(B)$ into $L(B)$ sending $b_j^{(i)}$ to $b_{j+1}^{(i)}$ for all $1 \le i \le n$ and $0 \le j \le n-1$, and $x \in L$ to $\tau_L x \in L$. Put $\mathcal{N} = (L(B), \tau)$. Then $\mathcal{N}$ is difference overfield of $\mathcal{L}$. Note the following,

$$n^2 = \mathrm{tr.deg}\,\mathcal{N}/\mathcal{L} = \mathrm{tr.deg}\,\mathcal{K}\langle f, B\rangle/\mathcal{K}\langle f\rangle = \mathrm{tr.deg}\,\mathcal{K}\langle B\rangle/\mathcal{K},$$

which implies $\mathcal{K}\langle f\rangle$ and $\mathcal{L}$ are free from $\mathcal{K}\langle B\rangle$ over $\mathcal{K}$.

Since we have

$$\begin{pmatrix} f & f_1 & \cdots & f_n \\ b^{(1)} & b_1^{(1)} & \cdots & b_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ b^{(n-1)} & b_1^{(n-1)} & \cdots & b_n^{(n-1)} \\ b^{(n)} & b_1^{(n)} & \cdots & b_n^{(n)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

we obtain $\mathrm{Cas}\,(f, b^{(1)}, \ldots, b^{(n)}) = 0$. Put $C = C_{\mathcal{K}\langle f,B\rangle}$. By Lemma 4.5 we find that $f, b^{(1)}, \ldots, b^{(n)}$ are linearly dependent over C. On the other hand we have $\mathrm{Cas}\,(b^{(1)}, \ldots, b^{(n)}) \neq 0$, which implies that $b^{(1)}, \ldots, b^{(n)}$ are linearly independent over C. Therefore we find that there are $c_1, \ldots, c_n \in C$ such that $f = \sum_{i=1}^n c_i b^{(i)}$.

Put $\mathcal{M}_i = \mathcal{K}\langle B, c_1, \ldots, c_i\rangle \subset \mathcal{K}\langle f, B\rangle$ for all $1 \le i \le n$ and $\mathcal{M}_0 = \mathcal{K}\langle B\rangle$. Note $\mathcal{M}_n = \mathcal{K}\langle f, B\rangle$ and $M_i = M_0(c_1, \ldots, c_i)$. Then we obtain $\mathrm{tr.deg}\,M_i/M_{i-1} \le 1$ for any $1 \le i \le n$, and so

$$\mathrm{tr.deg}\,\mathcal{K}\langle f, B\rangle/\mathcal{K}\langle B\rangle = \mathrm{tr.deg}\,\mathcal{K}\langle f\rangle/\mathcal{K} \ge 2$$

implies that there is some $1 \le k \le n-1$ such that $\mathrm{tr.deg}\,M_k/M_0 = 1$. We also find that $\mathcal{M}_i/\mathcal{M}_{i-1}$ is $\mathcal{N}$-decomposable for any $1 \le i \le n$, and so $\mathcal{M}_k/\mathcal{M}_0$ and $\mathcal{M}_n/\mathcal{M}_k$ are $\mathcal{N}$-decomposable. Since $\mathcal{N}/\mathcal{K}\langle f, B\rangle$ is algebraic, $\mathcal{N}/\mathcal{M}_k$ is $\mathcal{N}$-decomposable of $\mathrm{tr.deg} \ge 1$.

Note $\mathcal{N} = \mathcal{L}\mathcal{M}_0$, and it follows that $\mathcal{L}/\mathcal{K}$ is $\mathcal{N}$-decomposable. □

Proposition 4.7. *Let $\mathcal{K}$ be a difference field,*

$$y_n + a_{n-1}y_{n-1} + \cdots + a_0 y = b \tag{4.2}$$

a linear difference equation over $\mathcal{K}$, where $n \ge 1$, f a solution of (4.2), and $\mathcal{L}$ an algebraic difference overfield of $\mathcal{K}\langle f\rangle$. Then $\mathcal{L}/\mathcal{K}$ is $\mathcal{U}$-decomposable for some difference overfield $\mathcal{U}$ of $\mathcal{L}$.

Proof. We may suppose $b \neq 0$. Let $\mathcal{L} = (L, \tau)$, and put $a_n = 1$. The solution f satisfies $\sum_{i=0}^{n} a_i f_i = b$ and $\sum_{i=1}^{n+1} \tau(a_{i-1}) f_i = b_1$. Then we have

$$\begin{aligned} 0 &= \sum_{i=1}^{n+1} \tau(a_{i-1}) f_i - \frac{b_1}{b} \sum_{i=0}^{n} a_i f_i \\ &= f_{n+1} + \sum_{i=1}^{n} (\tau(a_{i-1}) - \frac{b_1}{b} a_i) f_i - \frac{b_1}{b} a_0 f. \end{aligned}$$

Therefore by Proposition 4.6 there is a difference overfield $\mathcal{U}$ of $\mathcal{L}$ such that $\mathcal{L}/\mathcal{K}$ is $\mathcal{U}$-decomposable. □

Corollary 4.8. *Let $\mathcal{K}$ be a difference field,*

$$y_n + a_{n-1} y_{n-1} + \cdots + a_0 y = b \tag{4.3}$$

be a linear difference equation over $\mathcal{K}$, where $n \geq 1$, and f a solution of (4.3). Then $\overline{\mathcal{K}\langle f \rangle}/\mathcal{K}$ is decomposable for any algebraic closure $\overline{\mathcal{K}\langle f \rangle}$ of $\mathcal{K}\langle f \rangle$.

Proof. Let $\mathcal{L} = \overline{\mathcal{K}\langle f \rangle}$ be an algebraic closure of $\mathcal{K}\langle f \rangle$. By Proposition 4.7 we find that $\mathcal{L}/\mathcal{K}$ is $\mathcal{U}$-decomposable for some difference overfield $\mathcal{U}$ of $\mathcal{L}$. Let $\overline{\mathcal{U}}$ be an algebraic closure of $\mathcal{U}$. The algebraic closure of $\mathcal{L}$ in $\overline{\mathcal{U}}$ equals $\mathcal{L}$ because $\mathcal{L}$ is algebraically closed. Therefore by Proposition 4.4 we conclude that $\mathcal{L}/\mathcal{K}$ is decomposable. □

4.3 Irreducibility

Lemma 4.9. *Let $\mathcal{K}$ be a difference field, $\mathcal{D}$ a decomposable extension of $\mathcal{K}$ and $B \subset D$. Suppose that if $\mathcal{L}$ is a difference overfield of $\mathcal{K}$ of finite transcendence degree and $\mathcal{U}$ a difference overfield of $\mathcal{L}$ such that $\mathcal{K}\langle B \rangle_{\mathcal{D}} \subset \mathcal{U}$, then the following holds,*

$$\text{tr.deg}\, \mathcal{L}\langle B \rangle_{\mathcal{U}}/\mathcal{L} \leq 1 \Rightarrow \textit{any } f \in B \textit{ is algebraic over } L.$$

Then any $f \in B$ is algebraic over K.

Proof. Assume that there exists $f \in B$ being transcendental over K. Put

$$\begin{aligned} S = \{(\mathcal{L}, \mathcal{N}) \mid\ & \mathcal{K} \subset \mathcal{L} \subset \mathcal{N},\ \text{tr.deg}\, L/K < \infty, \\ & \mathcal{N}/\mathcal{L} \text{ is decomposable},\ \mathcal{K}\langle B \rangle_{\mathcal{D}} \subset \mathcal{N}, \\ & \text{and there exists } f \in B \text{ being transcendental over } L\}. \end{aligned}$$

$S \neq \{\}$ because $(\mathcal{K}, \mathcal{D}) \in S$. Choose $(\mathcal{L}, \mathcal{N}) \in S$ which has the minimum transcendence degree $\operatorname{tr.deg} N/L$. If we assume $\operatorname{tr.deg} N/L \leq 1$, we obtain $\operatorname{tr.deg} \mathcal{L}\langle B\rangle_{\mathcal{N}}/\mathcal{L} \leq 1$, which implies that any $f \in B$ is algebraic over L by the hypothesis, a contradiction. Therefore it follows that $\operatorname{tr.deg} N/L \geq 2$.

Since $\mathcal{N}/\mathcal{L}$ is decomposable, there exist a difference overfield $\mathcal{U}$ of $\mathcal{N}$, a difference overfield $\mathcal{E}$ of $\mathcal{L}$ in $\mathcal{U}$ of finite transcendence degree which is free from $\mathcal{N}$ over $\mathcal{L}$, and a difference intermediate field $\mathcal{M}$ of $\mathcal{N}\mathcal{E}/\mathcal{E}$ satisfying $\operatorname{tr.deg} \mathcal{N}\mathcal{E}/\mathcal{M} \geq 1$ and $\operatorname{tr.deg} \mathcal{M}/\mathcal{E} \geq 1$, such that $\overline{\mathcal{N}\mathcal{E}}/\mathcal{M}$ and $\overline{\mathcal{M}}/\mathcal{E}$ are decomposable, where $\overline{\mathcal{N}\mathcal{E}}$ is an algebraic closure of $\mathcal{N}\mathcal{E}$ and $\overline{\mathcal{M}}$ the algebraic closure of $\mathcal{M}$ in $\overline{\mathcal{N}\mathcal{E}}$.

From $\mathcal{K}\langle B\rangle_{\mathcal{D}} \subset \mathcal{N} \subset \overline{\mathcal{N}\mathcal{E}}$ and $\operatorname{tr.deg} \overline{\mathcal{N}\mathcal{E}}/\mathcal{M} < \operatorname{tr.deg} \mathcal{N}/\mathcal{L}$ we find that any $f \in B$ is algebraic over M, namely $B \subset \overline{M}$. Note that

$$\mathcal{K}\langle B\rangle_{\mathcal{D}} = \mathcal{K}\langle B\rangle_{\overline{\mathcal{N}\mathcal{E}}} = \mathcal{K}\langle B\rangle_{\overline{\mathcal{M}}} \subset \overline{\mathcal{M}}.$$

Then from $\operatorname{tr.deg} \overline{\mathcal{M}}/\mathcal{E} < \operatorname{tr.deg} \mathcal{N}/\mathcal{L}$ we find that any $f \in B$ is algebraic over E.

Since N and E are free over L, we find that any $f \in B$ is algebraic over L, a contradiction. Therefore any $f \in B$ is algebraic over K. □

Lemma 4.10. *Let $\mathcal{K}$ be an almost inversive difference field, $\mathcal{D}$ a decomposable extension of $\mathcal{K}$ and $B \subset D$. Suppose that if $\mathcal{L}$ is an inversive difference overfield of $\mathcal{K}$ and $\mathcal{U}$ a difference overfield of $\mathcal{L}$ with $\mathcal{K}\langle B\rangle_{\mathcal{D}} \subset \mathcal{U}$, then the following holds,*

$$\operatorname{tr.deg} \mathcal{L}\langle B\rangle_{\mathcal{U}}/\mathcal{L} \leq 1 \Rightarrow \textit{any } f \in B \textit{ is algebraic over } L.$$

Then any $f \in B$ is algebraic over K.

Proof. Let $\mathcal{L}$ be a difference overfield of $\mathcal{K}$ of finite transcendence degree and $\mathcal{U}$ a difference overfield of $\mathcal{L}$ with $\mathcal{K}\langle B\rangle_{\mathcal{D}} \subset \mathcal{U}$.

We show

$$\operatorname{tr.deg} \mathcal{L}\langle B\rangle_{\mathcal{U}}/\mathcal{L} \leq 1 \Rightarrow \text{any } f \in B \text{ is algebraic over } L.$$

Suppose $\operatorname{tr.deg} \mathcal{L}\langle B\rangle_{\mathcal{U}}/\mathcal{L} \leq 1$. Let $\overline{\mathcal{U}}$ an algebraic closure of $\mathcal{U}$ and $\overline{\mathcal{L}}$ the algebraic closure of $\mathcal{L}$ in $\overline{\mathcal{U}}$. Note that $\overline{\mathcal{L}}$ is inversive. We find

$$\operatorname{tr.deg} \overline{\mathcal{L}}\langle B\rangle_{\overline{\mathcal{U}}}/\overline{\mathcal{L}} = \operatorname{tr.deg} \mathcal{L}\langle B\rangle_{\mathcal{U}}/\mathcal{L} \leq 1,$$

which implies that any $f \in B$ is algebraic over $\overline{L}$. Therefore any $f \in B$ is algebraic over L.

By Lemma 4.9 we conclude that any $f \in B$ is algebraic over K. □

Chapter 5

q-Painlevé equation of type $A_7^{(1)'}$

Notation. Throughout this chapter a field is of characteristic zero, C denotes an algebraically closed field, $C(t)$ a rational function field over C and $q \in C^\times$.

The q-Painlevé equation of type $A_7^{(1)'}$, the object here, appears in Sakai's paper [29]. The system over $(C(t), t \mapsto qt)$ is the following,

$$y_1 y = z_1^2,$$
$$z_1 z = \frac{y(1-ty)}{t(y-1)}.$$

We prove that if q is not a root of unity and (f, g) a solution in a decomposable extension of $(\mathbb{C}(t), t \mapsto qt)$, then f and g are algebraic functions of the form $c/\sqrt{t}$, $c \in \mathbb{C}$.

5.1 Irreducibility

Lemma 5.1. *Let $q \in C^\times$ be not a root of unity, $\mathcal{K}$ an inversive difference overfield of $(C(t), t \mapsto qt)$, $\mathcal{U} = (U, \tau)$ a difference overfield of $\mathcal{K}$, $\mathcal{L} \subset \mathcal{U}$ a difference overfield of $\mathcal{K}$ satisfying* $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} < \infty$, *and $f \in \mathcal{U}$ a solution of the equation over $\mathcal{K}$,*

$$q^2t^2(y_1-1)^2 y_2 y = (1-qty_1)^2.$$

Then we obtain

$$\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} \le 1 \Rightarrow f \textit{ is algebraic over } L.$$

Proof. We may suppose that L is algebraically closed. Then $\mathcal{L}$ is inversive. Assume $\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} = 1$. We find that f and f_1 are transcendental over L. Choose an irreducible polynomial over L,

$$F = \sum_{i=0}^{n_0}\sum_{j=0}^{n_1} a_{ij} Y^i Y_1^j \in L[Y, Y_1] \setminus \{0\}, \quad n_0 = \deg_Y F,\ n_1 = \deg_{Y_1} F,$$

such that $F(f, f_1) = 0$, and $a_{n_0 n_1} = 0$ or 1. Define the following three polynomials,

$$\begin{aligned}
F^* &= \sum_{i=0}^{n_0}\sum_{j=0}^{n_1} \tau(a_{ij}) Y^i Y_1^j, \\
F_1 &= (q^2t^2Y(Y_1-1)^2)^{n_1} F^*\left(Y_1, \frac{(1-qtY_1)^2}{q^2t^2Y(Y_1-1)^2}\right) \in L[Y, Y_1] \setminus \{0\}, \\
F_0 &= (q^2t^2Y_1(Y-1)^2)^{n_0} F\left(\frac{(1-qtY)^2}{q^2t^2Y_1(Y-1)^2}, Y\right) \in L[Y, Y_1] \setminus \{0\}.
\end{aligned}$$

Since the solution f satisfies

$$q^2t^2(f_1-1)^2 f_2 f = (1-qtf_1)^2,$$

we obtain $F_1(f, f_1) = F_0(f_1, f_2) = 0$, and so $F \mid F_1$ and $F^* \mid F_0$. These imply

$$n_1 = \deg_{Y_1} F^* \le \deg_{Y_1} F_0 \le n_0 = \deg_Y F \le \deg_Y F_1 \le n_1.$$

Therefore we obtain $n_0 = n_1$. Put $n = n_0 = n_1 \ge 1$. Let $P \in L[Y, Y_1] \setminus \{0\}$ be the polynomial such that $F_1 = PF$. We find $P \in L[Y_1]$ because $\deg_Y P = \deg_Y F_1 - \deg_Y F = 0$.

We have

$$\begin{aligned}
F_1 &= (q^2t^2Y(Y_1-1)^2)^n \sum_{i=0}^{n}\sum_{j=0}^{n} \tau(a_{ij}) Y_1^i \left(\frac{(1-qtY_1)^2}{q^2t^2Y(Y_1-1)^2}\right)^j \\
&= \sum_{j=0}^{n} (q^2t^2Y(Y_1-1)^2)^{n-j}(1-qtY_1)^{2j} \sum_{i=0}^{n} \tau(a_{ij}) Y_1^i \\
&= \sum_{j=0}^{n} (q^2t^2Y(Y_1-1)^2)^{j}(1-qtY_1)^{2(n-j)} \sum_{i=0}^{n} \tau(a_{i,n-j}) Y_1^i \\
&= \sum_{j=0}^{n} \left\{ (qt)^{2j}(Y_1-1)^{2j}(1-qtY_1)^{2(n-j)} \sum_{i=0}^{n} \tau(a_{i,n-j}) Y_1^i \right\} Y^j
\end{aligned}$$

and

$$PF = P\sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij}Y^iY_1^j = P\sum_{j=0}^{n}\sum_{i=0}^{n} a_{ji}Y^jY_1^i = \sum_{j=0}^{n}\left\{P\sum_{i=0}^{n} a_{ji}Y_1^i\right\}Y^j.$$

Therefore for all $k \in \{0, 1, \dots, n\}$ we obtain

$$(*k) \qquad (qt)^{2k}(Y_1-1)^{2k}(1-qtY_1)^{2(n-k)}\sum_{i=0}^{n}\tau(a_{i,n-k})Y_1^i = P\sum_{i=0}^{n} a_{ki}Y_1^i.$$

The equation $(*n)$ and $(*0)$ are the following,

$$(*n) \qquad (qt)^{2n}(Y_1-1)^{2n}\sum_{i=0}^{n}\tau(a_{i0})Y_1^i = P\sum_{i=0}^{n} a_{ni}Y_1^i \quad (\neq 0),$$

$$(*0) \qquad (1-qtY_1)^{2n}\sum_{i=0}^{n}\tau(a_{in})Y_1^i = P\sum_{i=0}^{n} a_{0i}Y_1^i \quad (\neq 0).$$

Note that $\sum_{i=0}^{n} a_{ni}Y_1^i \neq 0$ and $\sum_{i=0}^{n}\tau(a_{in})Y_1^i \neq 0$.

By $(*n)$ we find $(Y_1-1)^n \mid P$, and so by $(*0)$, $(Y_1-1)^n \mid \sum_{i=0}^{n}\tau(a_{in})Y_1^i$. Therefore we obtain $\sum_{i=0}^{n}\tau(a_{in})Y_1^i = \tau(a_{nn})(Y_1-1)^n$, which implies $a_{nn} = 1$. Comparing the terms of degree 0 of the equation

$$(5.1) \qquad \sum_{i=0}^{n}\tau(a_{in})Y_1^i = (Y_1-1)^n,$$

we find

$$(5.2) \qquad a_{0n} = (-1)^n \neq 0.$$

By this the equation $(*0)$ yields $\deg P = 2n$, and so

$$P = p(Y_1-1)^n(1-qtY_1)^n, \quad p \in L^\times.$$

Then from $(*0)$ we obtain

$$(1-qtY_1)^n = p\sum_{i=0}^{n} a_{0i}Y_1^i,$$

which implies $1 = pa_{00}$ and $(-qt)^n = pa_{0n}$. By (5.2) we find $p = (qt)^n$ and $a_{00} = (qt)^{-n}$.

Since we have $(1 - qtY_1)^n \mid P$, we obtain $(1 - qtY_1)^n \mid \sum_{i=0}^n \tau(a_{i0})Y_1^i$ by the equation $(*n)$, and so

$$\sum_{i=0}^{n} \tau(a_{i0})Y_1^i = \tau(a_{00})(1 - qtY_1)^n = (q^2t)^{-n}(1 - qtY_1)^n.$$

Then from $(*n)$ we obtain

$$(Y_1 - 1)^n q^{-n} = \sum_{i=0}^{n} a_{ni}Y_1^i.$$

Comparing the terms of degree n, we find $q^{-n} = a_{nn} = 1$, a contradiction.

Therefore we conclude that $\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} \neq 1$, which implies

$$\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} \leq 1 \Rightarrow \operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} = 0 \Rightarrow f \text{ is algebraic over } L,$$

the required. □

Theorem 5.2. *Let $q \in C^\times$ be not a root of unity, $\mathcal{K}$ an inversive difference overfield of $(C(t), t \mapsto qt)$, $\mathcal{D}$ a decomposable extension of $\mathcal{K}$, and $f, g \in \mathcal{D}$ satisfy two equations,*

$$f_1 f = g_1^2, \quad g_1 g = \frac{f(1 - tf)}{t(f - 1)}.$$

Then f and g are algebraic over K.

Proof. We may suppose $f \neq 0$ and $g \neq 0$. The two equations yield

$$f_2 f_1^2 f = (f_2 f_1)(f_1 f) = g_2^2 g_1^2 = (g_2 g_1)^2 = \frac{f_1^2(1 - qtf_1)^2}{q^2t^2(f_1 - 1)^2},$$
$$q^2t^2(f_1 - 1)^2 f_2 f = (1 - qtf_1)^2.$$

If we let $\mathcal{L}$ be a difference overfield of $\mathcal{K}$ satisfying $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} < \infty$, and $\mathcal{U}$ a difference overfield of $\mathcal{L}$ satisfying $\mathcal{K}\langle f\rangle_{\mathcal{D}} \subset \mathcal{U}$, by Lemma 5.1 we obtain the following,

$$\operatorname{tr.deg} \mathcal{L}\langle f\rangle_{\mathcal{U}}/\mathcal{L} \leq 1 \Rightarrow f \text{ is algebraic over } L.$$

Therefore by Lemma 4.9 we find that f is algebraic over K, which implies g is also algebraic over K. □

5.2 Algebraic solutions

It remains to find the algebraic solutions.

Theorem 5.3. *Let $q \in C^\times$ be not a root of unity, put $\mathcal{K} = (C(t), t \mapsto qt)$, and let $\overline{\mathcal{K}} = (\overline{C(t)}, \tau)$ be an algebraic closure of $\mathcal{K}$. Suppose that $f, g \in \overline{\mathcal{K}}$ satisfy the following two equations,*

$$f_1 f = g_1^2, \tag{5.3}$$

$$g_1 g = \frac{f(1 - tf)}{t(f-1)}. \tag{5.4}$$

Then one of the following holds.

(i) $(f, g) = (0, 0)$.

(ii) $(f, g) = (-1/x, -\alpha/x)$, $(-1/x, \alpha/x)$, $(1/x, -\alpha/x)$ *or* $(1/x, \alpha/x)$, *where* $\alpha \in C^\times$ *satisfies* $\alpha^4 = q$ *and* $x \in \overline{C(t)}$ *satisfies* $x^2 = t$ *and* $\tau x = \alpha^2 x$.

Proof. We may suppose $f \neq 0$ and $g \neq 0$. Put $\mathcal{L} = \mathcal{K}\langle f, g\rangle \subset \overline{\mathcal{K}}$. Then we have $L = C(t)(f, g)$. Put $n = [L : C(t)] < \infty$. By Lemma 2.1 we find $L = C(x)$, $x^n = t$. Since we have $(\tau x/x)^n = q \in C^\times$, we obtain $\tau x/x \in C^\times$. Put $r = \tau x/x \in C^\times$, which satisfies $r^n = q$ and $\tau x = rx$. Note that $f, g \in L = C(x)$ and $\mathcal{L}$ is inversive.

Express f and g as $f = P/Q$ and $g = R/S$, where $P, Q, R, S \in C[x] \setminus \{0\}$, P and Q are relatively prime, R and S are relatively prime, and Q and S are monic. From the equation (5.3) we obtain

$$P_1 P S_1^2 = Q_1 Q R_1^2 \quad (\neq 0), \tag{5.5}$$

and from the equation (5.4),

$$x^n (P - Q) Q R_1 R = P(Q - x^n P) S_1 S \quad (\neq 0). \tag{5.6}$$

By these equations we find $x \mid P(Q - x^n P)S_1 S$, and so $x \mid P$ or $x \mid Q$.

Let v_0 be the normalized discrete valuation of $C(x)/C$ with the prime element x. We prove $x \mid Q$ in $C[x]$. Assume $x \mid P$. Put $m = v_0(P) \in \mathbb{Z}_{>0}$, namely $x^m \mid P$ and $x^{m+1} \nmid P$. We obtain $x \mid R$ from (5.5), and so $x \nmid S$. Then it follows that

$$2m = v_0(P_1 P S_1^2) = v_0(Q_1 Q R_1^2) = v_0(R_1^2) = 2v_0(R_1),$$

which implies $v_0(R) = m$. Therefore by (5.6) we find $n + 2m = m$, a contradiction.

Put $m = v_0(Q) \in \mathbb{Z}_{>0}$. From the equation (5.5) we obtain $x \mid S$ and $x \nmid R$, and so $v_0(S) = m$. Then from the equation (5.6) we obtain $v_0(Q - x^n P) = n - m$. Since we have $0 \leq n - m < n$, we find $v_0(Q) = n - m$, which implies $n = 2m$.

Express f and g as $f = \sum_{i=-m}^{\infty} a_i x^i$, $a_{-m} \neq 0$ and $g = \sum_{i=-m}^{\infty} b_i x^i$, $b_{-m} \neq 0$. Seeing the first terms of the equation (5.3), we obtain $a_{-m}^2 = b_{-m}^2 r^{-m}$. On the other hand from the equation (5.4) we obtain $b_{-m}^2 r^{-m} = 1$. Then it follows that $a_{-m}^2 = 1$.

Combining the equations (5.3) and (5.4) as

$$f_2 f_1^2 f = (f_2 f_1)(f_1 f) = g_2^2 g_1^2 = (g_2 g_1)^2 = \frac{f_1^2 (1 - qtf_1)^2}{q^2 t^2 (f_1 - 1)^2},$$

we obtain

$$q^2 t^2 (f_1 - 1)^2 f_2 f = (1 - qtf_1)^2. \tag{5.7}$$

We prove that for any $i \geq -m$,

$$m \nmid i \Rightarrow a_i = 0,$$

which yields $f \in C(x^m)$. Assume that there is $i \geq -m$ such that $m \nmid i$ and $a_i \neq 0$. Let

$$km + l = \min\{i \geq -m \mid m \nmid i \text{ and } a_i \neq 0\}, \quad 0 < l < m.$$

The left side of the equation (5.7) is

$$\begin{aligned}
&q^2 x^{4m} (-1 + a_{-m} r^{-m} x^{-m} + \cdots + a_{km} r^{km} x^{km} + a_{km+l} r^{km+l} x^{km+l} + \cdots)^2 \\
&\times (a_{-m} r^{-2m} x^{-m} + \cdots + a_{km} r^{2km} x^{km} + a_{km+l} r^{2(km+l)} x^{km+l} + \cdots) \\
&\times (a_{-m} x^{-m} + \cdots + a_{km} x^{km} + a_{km+l} x^{km+l} + \cdots)
\end{aligned}$$

and the right side is

$$(-1 + q a_{-m} r^{-m} x^m + \cdots + q a_{km} r^{km} x^{(k+2)m} + q a_{km+l} r^{km+l} x^{(k+2)m+l} + \cdots)^2.$$

On the one hand the first term of the right side whose exponent is not divisible by m is $2(-1) q a_{km+l} r^{km+l} x^{(k+2)m+l}$. On the other hand the term of degree

$(k+1)m+l$ of the left side is

$$\begin{aligned}
&q^2x^{4m}(2a_{km+l}r^{km+l}x^{km+l}\cdot a_{-m}r^{-m}x^{-m}\cdot a_{-m}r^{-2m}x^{-m}\cdot a_{-m}x^{-m}\\
&\quad + a_{km+l}r^{2(km+l)}x^{km+l}(a_{-m}r^{-m}x^{-m})^2a_{-m}x^{-m}\\
&\quad + a_{km+l}x^{km+l}(a_{-m}r^{-m}x^{-m})^2a_{-m}r^{-2m}x^{-m})\\
&= q^2x^{(k+1)m+l}a_{km+l}a_{-m}^3(2r^{(k-3)m+l}+r^{2((k-1)m+l)}+r^{-4m}).
\end{aligned}$$

Therefore it follows that

$$(r^{(k-1)m+l}+r^{-2m})^2 = r^{2((k-1)m+l)}+2r^{(k-3)m+l}+r^{-4m}=0,$$

which implies $q^{2((k+1)m+l)}=1$, a contradiction.

Put $z=x^m$. Then we have $f=\sum_{i=-1}^{\infty}a_{mi}z^i$. The left side of the equation (5.7) is

$$\begin{aligned}
&q^2z^4(a_{-m}r^{-m}z^{-1}+(a_0-1)+a_mr^mz+\cdots)^2\\
&\times(a_{-m}r^{-2m}z^{-1}+a_0+a_mr^{2m}z+\cdots)\\
&\times(a_{-m}z^{-1}+a_0+a_mz+\cdots)
\end{aligned}$$

and the right side is

$$(-1+qa_{-m}r^{-m}z+qa_0z^2+qa_mr^mz^3+\cdots)^2.$$

Comparing the terms of degree 1, we find $a_0(r^m+1)^2=0$. Since $r^m+1=0$ implies $q=1$, we obtain $a_0=0$.

We prove that $a_{mi}=0$ for all $i\geq 1$ by induction. Firstly we deal with the case $i=1$. Comparing the terms of degree 2 of the above two expansions, we find $a_m(r^{-2m}+1)^2=0$, which implies $a_m=0$. Secondly we suppose $i\geq 2$ and the statement is true for the numbers $<i$. Comparing the terms of degree $i+1$, we find $a_{mi}(r^{m(i+1)}+1)^2=0$, which implies $a_{mi}=0$.

Therefore we obtain $f=a_{-m}/z=a_{-m}/x^m\in C(x^m)$. The equation (5.3) yields $S^2=r^{-m}x^{2m}R^2$. Since S is monic, we find $S^2=x^{2m}$, and so $S=x^m$. Then we have $R^2=r^m\in C^\times$, which implies $R\in C^\times$. Therefore we obtain $g=R/S\in C(x^m)$.

By $L=C(t)(f,g)\subset C(x^m)\subset C(x)=L$ we find $L=C(x^m)$. Then we have

$$2\leq 2m=n=[L:C(t)]=[C(x^m):C(x^{2m})]\leq 2,$$

which implies $n=2$ and $m=1$. Let $\alpha\in C^\times$ be a root of the polynomial $X^2-r\in C[X]$. We have $f=a_{-1}/x$, $a_{-1}=-1$ or 1, and $g=R/x$, $R=-\alpha$ or α. Note that $\alpha^4=r^2=q$. $\square$

Chapter 6

q-Painlevé equation of type $A_6^{(1)}$

In this chapter we will study the irreducibility of q-$P(A_6)$, q-Painlevé equation of type $A_6^{(1)}$, in the sense of order using the notion of decomposable extensions. q-$P(A_6)$ is one of the special non-linear q-difference equations of order 2 with symmetry $(A_1 + A_1)^{(1)}$ and is also called q-P_{II}.

6.1 Introduction

q-$P(A_6)$, q-Painlevé equation of type $A_6^{(1)}$ is expressed as

$$(\overline{F}F - 1)(F\underline{F} - 1) = \frac{at^2F}{F+t},$$

where t is a variable, a a parameter, $\overline{t} = qt$, $\underline{t} = t/q$, $F = F(t)$, $\overline{F} = F(qt)$ and $\underline{F} = F(t/q)$.

q-$P(A_6)$ is one of the discrete Painlevé equations Ramani and Grammaticos first studied in their paper [24]. The notation q-$P(A_6)$ is based on the Sakai's classification of discrete Painlevé equations by rational surfaces (see [28]).

In their [8] Hamamoto, Kajiwara and Witte constructed hypergeometric solutions to q-$P(A_6)$ by applying Bäcklund transformations to the "seed" solution which satisfies a Riccati equation. Their solutions have a determinantal form with basic hypergeometric function elements whose continuous limits are showed by them to be Airy functions, the hypergeometric solutions of the Painlevé II equation.

In Section 6.2 we prove that transcendental solutions of q-$P(A_6)$ in a decomposable extension may exist only for special parameters, and that each of them satisfies the Riccati equation mentioned above if we apply the Bäcklund transformations to it appropriate times.

In his [19, 20] the author proved Proposition 6.13 which dealt with algebraic solutions of q-$P(A_6)$. In Section 6.3 we prove the further result, non-existence of algebraic solutions.

Notation. Throughout this chapter a field is of characteristic zero, C denotes an algebraically closed field and $C(t)$ a rational function field over C. For $q \in C^{\times}$, q-Painlevé equation of type $A_6^{(1)}$ with (parameter) $a \in C$ is the difference equation over $(C(t), t \mapsto qt)$,

$$q\text{-}P(A_6)_a\colon \quad (y_2y_1 - 1)(y_1y - 1)(y_1 + qt) = aq^2t^2y_1,$$

where y_i ($y_0 = y$) is the i-th transform of y.

6.2 Irreducibility

In this section we prove

Theorem 6.1. *Let $q \in C^{\times}$ be not a root of unity, $\mathcal{K}$ an inversive difference overfield of $(C(t), t \mapsto qt)$. Furthermore let $\mathcal{D}/\mathcal{K}$ be a decomposable extension and $f \in \mathcal{D}$ a solution of q-$P(A_6)_a$ with $a \in C^{\times}$. Then the following hold:*

(i) *If $a \neq q^{2i+1}$ for all $i \in \mathbb{Z}$, the solution f is algebraic over K.*

(ii) *If $a = q^{2i+1}$ for some $i \in \mathbb{Z}$, the solution f is algebraic over K or $T_{\mathcal{D}}^{-i}f \in \mathcal{D}$ satisfies the equation $y_1y + qty - 1 = 0$ over $(C(t), t \mapsto qt)$, where $T_{\mathcal{D}}$ is the map defined in Definition 6.5 and regarded as Bäcklund transformation of q-$P(A_6)$ in [8].*

6.2.1 Bäcklund transformations

We define Bäcklund transformations algebraically.

Lemma 6.2. *Let $\mathcal{K} = (K, \tau)$ be a difference field lying over $(C(t), t \mapsto qt)$ and $f \in \mathcal{K}$ a solution of q-$P(A_6)_a$ with $a \neq 0$. Then there is a unique f_{-1} in $\mathcal{K}$ such that $\tau f_{-1} = f$.*

Proof. f satisfies

$$(f_2 f_1 - 1)(f_1 f - 1)(f_1 + qt) = aq^2 t^2 f_1.$$

Since $f_1 \neq 0$, $f_2 f_1 - 1 \neq 0$ and $f_1 + qt \neq 0$, we obtain $f \in C(t)(f_1, f_2)$. Thus we conclude that there is $g \in C(t)(f, f_1) \subset \mathcal{K}$ such that $\tau g = f$. ($C(t)(f_1, f_2)$ is the field of quotients of $C(t)[f_1, f_2]$.) The uniqueness follows from the injectivity of τ. □

Lemma 6.3. *Let $\mathcal{K} = (K, \tau)$ be a difference field lying over $(C(t), t \mapsto qt)$ and $f \in \mathcal{K}$ a solution of q-$P(A_6)_a$ with $a \neq 0$. Then*

$$g = \begin{cases} t\dfrac{qatf_1 + f_1 f - 1}{(f_1 f - 1)(tf_1 + f_1 f - 1)} & \text{if } a \neq q^{-1}, \\[2ex] \dfrac{t}{f_1 f - 1} & \text{if } a = q^{-1} \end{cases}$$

is well-defined and a solution of q-$P(A_6)_{q^2 a}$ in $\mathcal{K}$.

Proof. (i) Case $a = q^{-1}$. f satisfies

$$(f_2 f_1 - 1)(f_1 f - 1)(f_1 + qt) = qt^2 f_1,$$

and so $f \neq 0$ and $f_1 f - 1 \neq 0$, which implies g is well-defined.

Put $h = f_1 f - 1$ and then $g = t/h$. Note $h_1 h(f_1 + qt) = qt^2 f_1$ and $f \neq 0$. Then we obtain the followings:

$$g_2 g_1 - 1 = \frac{q^3 t^2}{h_2 h_1} - 1 = \frac{q^3 t^2}{\dfrac{q^3 t^2 f_2}{f_2 + q^2 t}} - 1 = \frac{f_2 + q^2 t}{f_2} - 1 = \frac{q^2 t}{f_2},$$

$$g_1 g - 1 = \frac{qt}{f_1},$$

$$g_1 + qt = \frac{qt}{h_1} + qt = qt\left(\frac{1}{h_1} + 1\right) = qt\frac{f_2 f_1}{h_1}.$$

Thus we find

$$(g_2 g_1 - 1)(g_1 g - 1)(g_1 + qt) = \frac{q^2 t}{f_2} \cdot \frac{qt}{f_1} \cdot qt\frac{f_2 f_1}{h_1} = \frac{q^4 t^3}{h_1} = q^3 t^2 g_1,$$

which implies g is a solution of q-$P(A_6)_q$.

(ii) Case $a \neq q^{-1}$. f satisfies

$$(f_2 f_1 - 1)(f_1 f - 1)(f_1 + qt) = aq^2 t^2 f_1,$$

and so $f \neq 0$ and $f_1 f - 1 \neq 0$. We will show $tf_1 + f_1 f - 1 \neq 0$. Assume $tf_1 + f_1 f - 1 = 0$. Then it follows that $f_1 f - 1 = -tf_1$ and $f_1(f + t) = 1$. From the above equation we obtain

$$(-qtf_2)(-tf_1)(f_1 + qt) = aq^2 t^2 f_1,$$

$$f_2(f_1 + qt) = aq,$$

$$1 = aq, \quad a = q^{-1},$$

which is a contradiction. Thus $tf_1 + f_1 f - 1$ is not equal to zero, which implies g is well-defined.

We will prove that g satisfies q-$P(A_6)_{q^2 a}$. Put $h = f_1 f - 1$. Note

$$g = t\frac{qatf_1 + h}{h(tf_1 + h)},$$

$h_1 h(f_1 + qt) = aq^2t^2 f_1$ and $f \neq 0$. Then we obtain

$$
\begin{aligned}
g_2 - \frac{qh_1}{f_2} &= q^2t\frac{q^3atf_3 + h_2}{h_2(q^2tf_3 + h_2)} - \frac{qh_1}{f_2} \\
&= \frac{q^5at^2f_3f_2 + q^2th_2f_2 - qh_1h_2(q^2tf_3 + h_2)}{h_2(q^2tf_3f_2 + h_2f_2)} \\
&= \frac{q^5at^2(h_2+1) + q^2th_2f_2 - q\dfrac{aq^4t^2f_2}{f_2 + q^2t}(q^2tf_3 + h_2)}{h_2(q^2t(h_2+1) + h_2f_2)} \\
&= \frac{(f_2 + q^2t)(q^5at^2(h_2+1) + q^2th_2f_2) - aq^5t^2f_2(q^2tf_3 + h_2)}{h_2(q^2t(h_2+1) + h_2f_2)(f_2 + q^2t)} \\
&= \frac{(f_2 + q^2t)(q^5at^2(h_2+1) + q^2th_2f_2) - aq^7t^3(h_2+1) - aq^5t^2f_2h_2}{h_2(q^2t(h_2+1) + h_2f_2)(f_2 + q^2t)} \\
&= \frac{q^5at^2f_2 + q^2th_2f_2^2 + q^4t^2h_2f_2}{h_2(q^2t(h_2+1) + h_2f_2)(f_2 + q^2t)} \\
&= \frac{q^5at^2f_2 + q^2th_2f_2\dfrac{aq^4t^2f_2}{h_2h_1}}{\dfrac{1}{h_1}\cdot\dfrac{aq^4t^2f_2}{f_2 + q^2t}\left(q^2t + \dfrac{aq^4t^2f_2}{h_1}\right)(f_2 + q^2t)} \\
&= \frac{h_1(q^5at^2f_2h_1 + aq^6t^3f_2^2)}{aq^4t^2f_2(q^2th_1 + aq^4t^2f_2)} \\
&= \frac{h_1(h_1 + qtf_2)}{qt(h_1 + aq^2tf_2)} \\
&= g_1^{-1},
\end{aligned}
$$

and so it follows that

$$
g_2g_1 - 1 = \frac{qh_1}{f_2}g_1 = \frac{q^2t(h_1 + aq^2tf_2)}{f_2(h_1 + qtf_2)}.
$$

Moreover we obtain

$$g_1 g - 1 = \frac{qh}{f_1} g = \frac{qh}{f_1} \cdot t \cdot \frac{qatf_1 + h}{h(tf_1 + h)} = \frac{qt}{f_1} \cdot \frac{qatf_1 + \dfrac{1}{h_1} \cdot \dfrac{aq^2t^2f_1}{f_1 + qt}}{tf_1 + \dfrac{1}{h_1} \cdot \dfrac{aq^2t^2f_1}{f_1 + qt}}$$
$$= \frac{qt}{f_1} \cdot \frac{qah_1(f_1 + qt) + aq^2t}{h_1(f_1 + qt) + aq^2t}$$
$$= \frac{qt}{f_1} \cdot \frac{qa(f_2f_1 - 1)(f_1 + qt) + aq^2t}{h_1(f_1 + qt) + aq^2t}$$
$$= \frac{qt}{f_1} \cdot \frac{qaf_2f_1^2 + q^2atf_2f_1 - qaf_1}{h_1(f_1 + qt) + aq^2t}$$
$$= q^2at\frac{f_2f_1 + qtf_2 - 1}{h_1(f_1 + qt) + aq^2t}$$

and

$$g_1 + qt = qt\frac{q^2atf_2 + h_1}{h_1(qtf_2 + h_1)} + qt = qt\frac{q^2atf_2 + h_1(qtf_2 + h_1 + 1)}{h_1(qtf_2 + h_1)}$$
$$= qt\frac{q^2atf_2 + h_1(qtf_2 + f_2f_1)}{h_1(qtf_2 + h_1)}$$
$$= qt\frac{f_2(q^2at + h_1(qt + f_1))}{h_1(qtf_2 + h_1)}.$$

Thus we find

$$(g_2g_1 - 1)(g_1g - 1)(g_1 + qt)$$
$$= \frac{q^2t(h_1 + aq^2tf_2)}{f_2(h_1 + qtf_2)} \cdot q^2at\frac{f_2f_1 + qtf_2 - 1}{h_1(f_1 + qt) + aq^2t} \cdot qt\frac{f_2(q^2at + h_1(qt + f_1))}{h_1(qtf_2 + h_1)}$$
$$= aq^5t^3\frac{h_1 + aq^2tf_2}{h_1(qtf_2 + h_1)}$$
$$= aq^4t^2g_1,$$

which implies g is a solution of $q\text{-}P(A_6)_{q^2a}$. □

Lemma 6.4. *Let $\mathcal{K} = (K, \tau)$ be a difference field lying over $(C(t), t \mapsto qt)$*

and $f \in \mathcal{K}$ a solution of q-$P(A_6)_a$ with $a \neq 0$. Then

$$g = \begin{cases} t\dfrac{a(t/q)f_{-1} + ff_{-1} - 1}{(ff_{-1} - 1)(tf_{-1} + ff_{-1} - 1)} & \text{if } a \neq q, \\[2ex] \dfrac{t}{ff_{-1} - 1} & \text{if } a = q \end{cases}$$

is well-defined and a solution of q-$P(A_6)_{q^{-2}a}$ in $\mathcal{K}$.

Proof. (i) Case $a = q$. f satisfies

$$(f_1 f - 1)(ff_{-1} - 1)(f + t) = qt^2 f,$$

and so $f \neq 0$ and $ff_{-1} - 1 \neq 0$, which implies g is well-defined.
Put $h = f_1 f - 1$. Then we have $g_1 = qt/h$ and we obtain

$$g_2 g_1 - 1 = \frac{q^3t^2}{h_1 h} - 1 = \frac{q^3t^2}{\dfrac{q^3t^2 f_1}{f_1 + qt}} - 1 = \frac{f_1 + qt}{f_1} - 1 = \frac{qt}{f_1},$$

$$g_3 g_2 - 1 = \frac{q^2 t}{f_2}$$

and

$$g_2 + q^2 t = \frac{q^2 t}{h_1} + q^2 t = q^2 t\frac{f_2 f_1}{h_1}.$$

Thus we find

$$(g_3 g_2 - 1)(g_2 g_1 - 1)(g_2 + q^2 t) = \frac{q^2 t}{f_2} \cdot \frac{qt}{f_1} \cdot q^2 t\frac{f_2 f_1}{h_1} = \frac{q^5 t^3}{h_1} = q^3 t^2 g_2,$$

and so it follows that

$$(g_2 g_1 - 1)(g_1 g - 1)(g_1 + qt) = qt^2 g_1,$$

which implies g is a solution of q-$P(A_6)_{q^{-1}}$.
(ii) Case $a \neq q$. f satisfies

$$(f_1 f - 1)(ff_{-1} - 1)(f + t) = at^2 f,$$

and so $f \neq 0$ and $ff_{-1} - 1 \neq 0$. We will show $tf_{-1} + ff_{-1} - 1 \neq 0$. Assume $tf_{-1}+ff_{-1}-1 = 0$. Then it follows that $ff_{-1}-1 = -tf_{-1}$ and $f_{-1}(f+t) = 1$. From the above equation we obtain

$$(-qtf)(-tf_{-1})(f+t) = at^2 f,$$

$$qf_{-1}(f+t) = a,$$

$$q = a,$$

which is a contradiction. Thus $tf_{-1} + ff_{-1} - 1$ is not equal to zero, which implies g is well-defined.

We will prove that g satisfies q-$P(A_6)_{q^{-2}a}$. Put $h = f_1 f - 1$, and then we have

$$g = t\frac{a(t/q)f_{-1} + h_{-1}}{h_{-1}(tf_{-1} + h_{-1})}.$$

The first transform of g is

$$g_1 = qt\frac{atf + h}{h(qtf + h)}.$$

Note $h_1h(f_1+qt)=aq^2t^2f_1$ and $f\neq 0$. We obtain

$$
\begin{aligned}
g_1-\frac{h_1}{qf_1} &= qt\frac{atf+h}{h(qtf+h)}-\frac{h_1}{qf_1}=\frac{q^2at^2f_1f+q^2tf_1h-h_1h(qtf+h)}{qf_1h(qtf+h)}\\
&= \frac{q^2at^2f_1f+q^2tf_1h-\dfrac{aq^2t^2f_1}{f_1+qt}(qtf+h)}{qf_1h(qtf+h)}\\
&= \frac{(f_1+qt)(q^2at^2f_1f+q^2tf_1h)-aq^2t^2f_1(qtf+h)}{qf_1h(qtf+h)(f_1+qt)}\\
&= \frac{q^2at^2f_1^2f+q^2tf_1^2h+q^3t^2f_1h-aq^2t^2f_1h}{qf_1h(qtf+h)(f_1+qt)}\\
&= \frac{q^2tf_1^2h+q^3t^2f_1h+aq^2t^2f_1}{qf_1h(qtf+h)(f_1+qt)}\\
&= \frac{aq^2t^2f_1+q^2tf_1h(f_1+qt)}{qh(qt(h+1)+f_1h)(f_1+qt)}\\
&= \frac{aq^2t^2f_1+q^2tf_1h(f_1+qt)}{qh(qt+h(qt+f_1))(f_1+qt)}\\
&= \frac{a^2q^2t^2f_1+q^2tf_1h\dfrac{aq^2t^2f_1}{h_1h}}{qh(qt+h\dfrac{aq^2t^2f_1}{h_1h})(f_1+qt)}\\
&= h_1\frac{aq^2t^2f_1h_1+q^2tf_1\cdot aq^2t^2f_1}{qh_1h(qth_1+aq^2t^2f_1)(f_1+qt)}\\
&= \frac{h_1(aq^2t^2f_1h_1+aq^4t^3f_1^2)}{aq^3t^2f_1(qth_1+aq^2t^2f_1)}\\
&= \frac{h_1(h_1+q^2tf_1)}{q^2t(h_1+aqtf_1)}\\
&= g_2^{-1},
\end{aligned}
$$

and so it follows that

$$g_2g_1-1=\frac{h_1g_2}{qf_1}=\frac{qt(aqtf_1+h_1)}{f_1(q^2tf_1+h_1)}.$$

Moreover we obtain

$$
\begin{aligned}
g_3g_2 - 1 &= \frac{h_2}{qf_2} \cdot q^3t\frac{aq^2tf_2 + h_2}{h_2(q^3tf_2 + h_2)} = \frac{q^2t}{f_2} \cdot \frac{aq^2tf_2 + \frac{1}{h_1} \cdot \frac{aq^4t^2f_2}{f_2 + q^2t}}{q^3tf_2 + \frac{1}{h_1} \cdot \frac{aq^4t^2f_2}{f_2+q^2t}} \\
&= \frac{q^2t}{f_2} \cdot \frac{aq^2t(f_2f_1 - 1)(f_2 + q^2t) + aq^4t^2}{q^3t(f_2f_1 - 1)(f_2 + q^2t) + aq^4t^2} \\
&= \frac{q^2t}{f_2} \cdot \frac{aq^2tf_2^2f_1 + aq^4t^2f_2f_1 - aq^2tf_2}{q^3tf_2^2f_1 + q^5t^2f_2f_1 - q^3tf_2 - q^5t^2 + aq^4t^2} \\
&= \frac{aqt(f_2f_1 + q^2tf_1 - 1)}{f_2^2f_1 + q^2tf_2f_1 - f_2 - q^2t + aqt}
\end{aligned}
$$

and

$$
\begin{aligned}
g_2 + q^2t &= q^2t\frac{aqtf_1 + h_1}{h_1(q^2tf_1 + h_1)} + q^2t = q^2t\frac{aqtf_1 + h_1(q^2tf_1 + h_1 + 1)}{h_1(q^2tf_1 + h_1)} \\
&= q^2t\frac{aqtf_1 + h_1(q^2tf_1 + f_2f_1)}{h_1(q^2tf_1 + h_1)} \\
&= q^2tf_1\frac{aqt + h_1(q^2t + f_2)}{h_1(q^2tf_1 + h_1)}.
\end{aligned}
$$

Thus we find

$$
\begin{aligned}
&(g_3g_2 - 1)(g_2g_1 - 1)(g_2 + q^2t) \\
&= \frac{aqt(f_2f_1 + q^2tf_1 - 1)}{f_2^2f_1 + q^2tf_2f_1 - f_2 - q^2t + aqt} \cdot \frac{qt(aqtf_1 + h_1)}{f_1(q^2tf_1 + h_1)} \cdot q^2tf_1\frac{aqt + h_1(q^2t + f_2)}{h_1(q^2tf_1 + h_1)} \\
&= aq^4t^3\frac{aqtf_1 + h_1}{h_1(q^2tf_1 + h_1)} \\
&= aq^2t^2g_2,
\end{aligned}
$$

and so we conclude

$$(g_2g_1 - 1)(g_1g - 1)(g_1 + qt) = at^2g_1,$$

which implies g is a solution of q-$P(A_6)_{q^{-2}a}$. □

Definition 6.5. Let $\mathcal{K}$ be a difference overfield of $(C(t), t \mapsto qt)$, and define the sets of solutions as $S_a(\mathcal{K}) = \{f \in K \mid f \text{ is a solution of } q\text{-}P(A_6)_a \text{ in } \mathcal{K}\}$

for any $a \in C$ and $S(\mathcal{K}) = \bigcup_{a\neq 0} S_a(\mathcal{K})$. $S(\mathcal{K})$ is the disjoint union of $S_a(\mathcal{K})$'s. (In fact let $f \in \mathcal{K}$ be a solution of q-$P(A_6)_a$ and q-$P(A_6)_b$. Then we obtain

$$aq^2t^2f_1 = (f_2f_1 - 1)(f_1f - 1)(f_1 + qt) = bq^2t^2f_1,$$

which implies $a = b$.) We define maps $T_{\mathcal{K}}$ and $T_{\mathcal{K}}^{\triangleleft}$ as follows:

$$\begin{array}{rccc} T_{\mathcal{K}}: & S(\mathcal{K}) & \to & S(\mathcal{K}) \\ & f \in S_a(\mathcal{K}) & \mapsto & \begin{cases} t\frac{qatf_1+f_1f-1}{(f_1f-1)(tf_1+f_1f-1)} & \text{if } a \neq q^{-1} \\ \frac{t}{f_1f-1} & \text{if } a = q^{-1} \end{cases} \in S_{q^2a}(\mathcal{K}) \end{array}$$

and

$$\begin{array}{rccc} T_{\mathcal{K}}^{\triangleleft}: & S(\mathcal{K}) & \to & S(\mathcal{K})\ . \\ & f \in S_a(\mathcal{K}) & \mapsto & \begin{cases} t\frac{a(t/q)f_{-1}+ff_{-1}-1}{(ff_{-1}-1)(tf_{-1}+ff_{-1}-1)} & \text{if } a \neq q \\ \frac{t}{ff_{-1}-1} & \text{if } a = q \end{cases} \in S_{q^{-2}a}(\mathcal{K}) \end{array}$$

Those are well-defined because of Lemma 6.3 and Lemma 6.4. In their [8] Hamamoto, Kajiwara and Witte regard $T_{\mathcal{K}}$ and $T_{\mathcal{K}}^{\triangleleft}$ as Bäcklund transformations of q-$P(A_6)$.

Proposition 6.6. *Let $\mathcal{K} = (K, \tau)$ be a difference overfield of $(C(t), t \mapsto qt)$. Then $T_{\mathcal{K}}^{\triangleleft} = T_{\mathcal{K}}^{-1}$.*

Proof. First we prove $T_{\mathcal{K}}^{\triangleleft}T_{\mathcal{K}} = id$. Let $f \in S(\mathcal{K})$. There is a unique $a \neq 0$ such that $f \in S_a(\mathcal{K})$. (i) Case $a = q^{-1}$. Put $g = T_{\mathcal{K}}(f)$ and $h = f_1f - 1$. Note $g = t/h$ and $h_1h(f_1 + qt) = qt^2f_1$. Since $g \in S_q(\mathcal{K})$, we obtain

$$T_{\mathcal{K}}^{\triangleleft}(g) = \frac{t}{gg_{-1} - 1},$$

$$\tau(T_{\mathcal{K}}^{\triangleleft}g) = \frac{qt}{g_1g - 1} = \frac{qt}{\dfrac{qt^2}{h_1h} - 1} = \frac{qt}{\dfrac{f_1 + qt}{f_1} - 1} = \frac{qtf_1}{f_1 + qt - f_1} = f_1,$$

which implies $T_{\mathcal{K}}^{\triangleleft}g = f$.

(ii) Case $a \neq q^{-1}$. Put $g = T_{\mathcal{K}}(f)$ and $h = f_1f - 1$. From the proof of Lemma 6.3, we have

$$g_1g - 1 = \frac{qh}{f_1}g.$$

Since $T_{\mathcal{K}}(f) \in S_{q^2a}(\mathcal{K})$ with $q^2a \neq q$, we obtain

$$\begin{aligned}
\tau(T_{\mathcal{K}}^{\triangleleft}g) &= qt\frac{q^2atg + \frac{qh}{f_1}g}{\frac{qh}{f_1}g\left(qtg + \frac{qh}{f_1}g\right)} = qtf_1\frac{q^2atf_1 + qh}{qhg(qtf_1 + qh)} \\
&= qtf_1\frac{q^2atf_1 + qh}{qht\frac{qatf_1 + h}{h(tf_1 + h)}(qtf_1 + qh)} \\
&= f_1,
\end{aligned}$$

which implies $T_{\mathcal{K}}^{\triangleleft}g = f$.

Next we prove $T_{\mathcal{K}}T_{\mathcal{K}}^{\triangleleft} = id$. Let $f \in S(\mathcal{K})$. There is a unique $a \neq 0$ such that $f \in S_a(\mathcal{K})$. (i) Case $a = q$. Put $g = T_{\mathcal{K}}^{\triangleleft}f$ and $h = f_1f - 1$. Note $g = t/h_{-1}$ and $hh_{-1}(f+t) = qt^2f$. Since $q \in S_{q^{-1}}(\mathcal{K})$, we obtain

$$T_{\mathcal{K}}g = \frac{t}{g_1g - 1} = \frac{t}{\frac{qt^2}{hh_{-1}} - 1} = \frac{t}{\frac{f+t}{f} - 1} = f,$$

which implies $T_{\mathcal{K}}T_{\mathcal{K}}^{\triangleleft}f = f$.

(ii) Case $a \neq q$. Put $g = T_{\mathcal{K}}^{\triangleleft}(f)$ and $h = f_1f - 1$. From the proof of Lemma 6.4, we have

$$g_2g_1 - 1 = \frac{h_1g_2}{qf_1}.$$

Since $g \in S_{q^{-2}a}(\mathcal{K})$ with $q^{-2}a \neq q^{-1}$, we obtain

$$\begin{aligned}
\tau(T_{\mathcal{K}}g) &= qt\frac{atg_2 + g_2g_1 - 1}{(g_2g_1 - 1)(qtg_2 + g_2g_1 - 1)} = qt\frac{atg_2 + \frac{h_1g_2}{qf_1}}{\frac{h_1g_2}{qf_1}\left(qtg_2 + \frac{h_1g_2}{qf_1}\right)} \\
&= q^2tf_1\frac{qatf_1 + h_1}{h_1q^2t\frac{aqtf_1 + h_1}{h_1(q^2tf_1 + h_1)}(q^2tf_1 + h_1)} \\
&= f_1,
\end{aligned}$$

which implies $T_{\mathcal{K}}(g) = f$. □

Proposition 6.7. *Let $\mathcal{K} = (K, \tau_K)$ be a difference overfield of $(C(t), t \mapsto qt)$ and $\mathcal{L} = (L, \tau)$ a difference overfield of $\mathcal{K}$. Then we obtain $S_a(\mathcal{K}) \subset S_a(\mathcal{L})$ for all $a \in C$, and so $S(\mathcal{K}) \subset S(\mathcal{L})$, moreover it follows that $T_{\mathcal{L}}|_{S(\mathcal{K})} = T_{\mathcal{K}}$ and $T_{\mathcal{L}}^{-1}|_{S(\mathcal{K})} = T_{\mathcal{K}}^{-1}$.*

Proof. Let $a \in C$ and $f \in S_a(\mathcal{K})$, which means $f \in K \subset L$ is a solution of q-$P(A_6)_a$ in $\mathcal{K}$. We have

$$\begin{aligned}
&(\tau^2(f)\tau(f) - 1)(\tau(f)f - 1)(\tau f + qt)\\
&= (\tau_K^2(f)\tau_K(f) - 1)(\tau_K(f)f - 1)(\tau_K f + qt)\\
&= aq^2t^2\tau_K f\\
&= aq^2t^2\tau f,
\end{aligned}$$

which implies $f \in S_a(\mathcal{L})$. Thus we conclude that $S_a(\mathcal{K}) \subset S_a(\mathcal{L})$ and $S(\mathcal{K}) \subset S(\mathcal{L})$.

Let $f \in S(\mathcal{K})$. There is a unique $a \in C^\times$ such that $f \in S_a(\mathcal{K}) \subset S_a(\mathcal{L})$. Then we have

$$\begin{aligned}
T_{\mathcal{L}} f &= \left\{ \begin{array}{ll} t\frac{qat\tau(f)+\tau(f)f-1}{(\tau(f)f-1)(t\tau(f)+\tau(f)f-1)} & \text{if } a \neq q^{-1} \\ \frac{t}{\tau(f)f-1} & \text{if } a = q^{-1} \end{array} \right\}\\
&= \left\{ \begin{array}{ll} t\frac{qat\tau_K(f)+\tau_K(f)f-1}{(\tau_K(f)f-1)(t\tau_K(f)+\tau_K(f)f-1)} & \text{if } a \neq q^{-1} \\ \frac{t}{\tau_K(f)f-1} & \text{if } a = q^{-1} \end{array} \right\}\\
&= T_{\mathcal{K}} f,
\end{aligned}$$

and so $T_{\mathcal{L}}|_{S(\mathcal{K})} = T_{\mathcal{K}}$. From this we find that

$$T_{\mathcal{L}}^{-1}|_{S(\mathcal{K})} \circ T_{\mathcal{K}} = T_{\mathcal{L}}^{-1}|_{S(\mathcal{K})} \circ T_{\mathcal{L}}|_{S(\mathcal{K})} = id|_{S(\mathcal{K})},$$

which implies $T_{\mathcal{L}}^{-1}|_{S(\mathcal{K})} = T_{\mathcal{K}}^{-1}$. □

6.2.2 Proof of Theorem

We need several Lemmas for the proof of Theorem.

Lemma 6.8. *Let $q \in C^\times$ be not a root of unity, $\mathcal{U} = (U, \tau)$ a difference overfield of $(C(t), t \mapsto qt)$, $\mathcal{L} \subset \mathcal{U}$ a difference overfield of $(C(t), t \mapsto qt)$ whose operator is surjective and $f \in S_a(\mathcal{U})$ with $a \in C^\times$ satisfy* $\operatorname{tr.deg} \mathcal{L}\langle f \rangle / \mathcal{L} = 1$. *Then we obtain one of the following:*

(i) $a = q$ *and* f *is a solution of the equation* $y_1 y + qty - 1 = 0$ *over* $(C(t), t \mapsto qt)$ *in* $\mathcal{U}$.

(ii) $a = q^{-1}$.

(iii) *There are* $m, n \in \mathbb{Z}$ *such that* $a^{m-n} = q^n$, $0 < m < 2n$, $m \neq n$ *and* $n \geq 1$.

Proof. For a polynomial $F = \sum_{i,j} a_{ij} Y^i Y_1^j \in U[Y, Y_1]$, we define $F^* \in U[Y, Y_1]$ as $F^* = \sum_{i,j} \tau(a_{ij}) Y^i Y_1^j$. For any $i \geq 0$ we find f_i is transcendental over L, which is inductively proved. In fact from the assumption $\operatorname{tr.deg} \mathcal{L}\langle f \rangle / \mathcal{L} = 1$ we have f is transcendental over L. Let $i \geq 1$ and assume it is true for $i-1$ and f_i is algebraic over L. There is $F \in L[X] \setminus \{0\}$ such that $F(f_i) = 0$. Since $\tau|_L \in \operatorname{Aut} L$ there exists $F_* \in L[X]$ such that $F_*^* = F$. Though f_{i-1} is not algebraic over L, it satisfies $F_*(f_{i-1}) = 0$. Hence we obtain $F_* = 0$, which implies $F = 0$. This is a contradiction, and so f_i is transcendental over L.

Step 1. Because of $\operatorname{tr.deg} \mathcal{L}\langle f \rangle / \mathcal{L} = 1$ there is an irreducible polynomial

$$F = \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} a_{ij} Y^i Y_1^j \in L[Y, Y_1] \setminus \{0\}, \quad n_0 = \deg_Y F,\ n_1 = \deg_{Y_1} F$$

such that $F(f, f_1) = 0$ and $a_{n_0 n_1} = 0$ or 1. Choose such F using the above expression, and put

$$F_1 = \{Y_1(Y_1 Y - 1)(Y_1 + qt)\}^{n_1} F^* \left(Y_1, \frac{1}{Y_1} \left(\frac{aq^2t^2 Y_1}{(Y_1 Y - 1)(Y_1 + qt)} + 1 \right) \right)$$

and

$$F_0 = \{Y(Y_1 Y - 1)(Y + qt)\}^{n_0} F \left(\frac{1}{Y} \left(\frac{aq^2t^2 Y}{(Y_1 Y - 1)(Y + qt)} + 1 \right), Y \right).$$

They satisfy $F_0, F_1 \in L[Y, Y_1] \setminus \{0\}$ because both $\{Y_1, \frac{1}{Y_1}(\dots)\}$ and $\{\frac{1}{Y}(\dots), Y\}$ are sets of elements algebraically independent over L.

$f \in S_a(\mathcal{U})$ means $f \in \mathcal{U}$ satisfies

$$(f_2 f_1 - 1)(f_1 f - 1)(f_1 + qt) = aq^2t^2 f_1.$$

From $f \neq 0$ and $a \neq 0$, we have

$$f_2 = \frac{1}{f_1} \left(\frac{aq^2t^2 f_1}{(f_1 f - 1)(f_1 + qt)} + 1 \right)$$

and

$$f = \frac{1}{f_1}\left(\frac{aq^2t^2f_1}{(f_2f_1-1)(f_1+qt)}+1\right).$$

Hence it follows that

$$\begin{aligned}F_1(f, f_1) &= \{f_1(f_1f-1)(f_1+qt)\}^{n_1}F^*\left(f_1, \frac{1}{f_1}\left(\frac{aq^2t^2f_1}{(f_1f-1)(f_1+qt)}+1\right)\right)\\ &= \{f_1(f_1f-1)(f_1+qt)\}^{n_1}F^*(f_1, f_2)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}F_0(f_1, f_2) &= \{f_1(f_2f_1-1)(f_1+qt)\}^{n_0}F\left(\frac{1}{f_1}\left(\frac{aq^2t^2f_1}{(f_2f_1-1)(f_1+qt)}+1\right), f_1\right)\\ &= \{f_1(f_2f_1-1)(f_1+qt)\}^{n_0}F(f, f_1)\\ &= 0.\end{aligned}$$

Note that F and F^* are irreducible polynomial over L and that f_i is transcendental over L for all $i \geq 0$, and we find $F \mid F_1$ and $F^* \mid F_0$. Then we obtain

$$n_1 = \deg_{Y_1} F^* \leq \deg_{Y_1} F_0 \leq n_0$$

and

$$n_0 = \deg_Y F \leq \deg_Y F_1 \leq n_1,$$

which implies $n_0 = n_1$. Put $n = n_0 = n_1 \geq 1$.

Step 2. Express the non-zero polynomial F_1 as $F_1 = PF$, $P \in L[Y, Y_1] \setminus \{0\}$. We find $P \in L[Y_1]$ from

$$\deg_Y P = \deg_Y F_1 - \deg_Y F = 0.$$

Put $X = Y_1Y - 1$. Note that Y_1 and X are algebraically independent over L. We use

$$\sum_{i=0}^{m}\sum_{k=0}^{i} c_{ik} = \sum_{k=0}^{m}\sum_{i=k}^{m} c_{ik} \quad \text{for all } m \geq 0$$

in the following.

We calculate F_1 and PF independently:

$$
\begin{aligned}
F_1 &= \{Y_1X(Y_1+qt)\}^n \sum_{i,j} \tau(a_{ij})Y_1^i \left\{\frac{1}{Y_1}\left(\frac{aq^2t^2Y_1}{X(Y_1+qt)}+1\right)\right\}^j \\
&= \sum_{i,j} \tau(a_{ij})Y_1^i\{Y_1X(Y_1+qt)\}^{n-j}\{aq^2t^2Y_1+X(Y_1+qt)\}^j \\
&= \sum_{i,j} \tau(a_{ij})Y_1^{n+i-j}X^{n-j}(Y_1+qt)^{n-j}\sum_{k=0}^{j}\binom{j}{k}(aq^2t^2Y_1)^kX^{j-k}(Y_1+qt)^{j-k} \\
&= \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{i=0}^{n}\binom{j}{k}(aq^2t^2)^k\tau(a_{ij})Y_1^{n+i-j+k}(Y_1+qt)^{n-k}X^{n-k} \\
&= \sum_{k=0}^{n}\sum_{j=k}^{n}\sum_{i=0}^{n}\binom{j}{k}(aq^2t^2)^k\tau(a_{ij})Y_1^{n+i-j+k}(Y_1+qt)^{n-k}X^{n-k} \\
&= \sum_{k=0}^{n}\sum_{j=n-k}^{n}\sum_{i=0}^{n}\binom{j}{n-k}(aq^2t^2)^{n-k}\tau(a_{ij})Y_1^{2n-k+i-j}(Y_1+qt)^kX^k \\
&= \sum_{k=0}^{n}\left\{(aq^2t^2)^{n-k}Y_1^{n-k}(Y_1+qt)^k\sum_{j=n-k}^{n}\sum_{i=0}^{n}\binom{j}{n-k}\tau(a_{ij})Y_1^{n+i-j}\right\}X^k,
\end{aligned}
$$

(6.1)

$$
\begin{aligned}
PF &= P\sum_{i,j} a_{ij}Y_iY_1^j \\
&= P\sum_{i,j} a_{ij}\left(\frac{X+1}{Y_1}\right)^iY_1^j \\
&= P\sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij}\left(\sum_{k=0}^{i}\binom{i}{k}X^k\right)Y_1^{j-i} \\
&= P\sum_{i=0}^{n}\sum_{k=0}^{i}\sum_{j=0}^{n} a_{ij}\binom{i}{k}X^kY_1^{j-i} \\
&= P\sum_{k=0}^{n}\sum_{i=k}^{n}\sum_{j=0}^{n}\binom{i}{k}a_{ij}Y_1^{j-i}X^k \\
&= \sum_{k=0}^{n}\left\{P\sum_{i=k}^{n}\sum_{j=0}^{n}\binom{i}{k}a_{ij}Y_1^{j-i}\right\}X^k.
\end{aligned}
$$

Hence

$$(*k)\quad \begin{aligned}&(aq^2t^2)^{n-k}Y_1^{2n-k}(Y_1+qt)^k\sum_{j=n-k}^{n}\sum_{i=0}^{n}\binom{j}{n-k}\tau(a_{ij})Y_1^{n+i-j}\\ &=P\sum_{i=k}^{n}\sum_{j=0}^{n}\binom{i}{k}a_{ij}Y_1^{n+j-i},\end{aligned}$$

for $k=0,\dots,n$. Let $(*k)_l$ denote the left side of the equation $(*k)$ and $(*k)_r$ the right side. The equations $(*n)$ and $(*0)$ are

$$(*n)\qquad Y_1^n(Y_1+qt)^n\sum_{j=0}^{n}\sum_{i=0}^{n}\tau(a_{ij})Y_1^{n+i-j}=P\sum_{j=0}^{n}a_{nj}Y_1^j,$$

$$(*0)\qquad (aq^2t^2)^nY_1^{2n}\sum_{i=0}^{n}\tau(a_{in})Y_1^i=P\sum_{i=0}^{n}\sum_{j=0}^{n}a_{ij}Y_1^{n+j-i}.$$

We will show $\sum_{j=0}^{n}a_{nj}Y_1^j\neq 0$ and $\sum_{i=0}^{n}\tau(a_{in})Y_1^i\neq 0$ to find both sides of the above two equations are not equal to 0. Assuming $\sum_{j=0}^{n}a_{nj}Y_1^j=0$, we obtain

$$F=\sum_{i=0}^{n}\sum_{j=0}^{n}a_{ij}Y^iY_1^j=\sum_{i=0}^{n-1}\sum_{j=0}^{n}a_{ij}Y^iY_1^j,$$

which implies $\deg_Y F\leq n-1<n$, a contradiction. Assuming $\sum_{i=0}^{n}\tau(a_{in})Y_1^i=0$, we obtain $a_{in}=0$ for all $0\leq i\leq n$, and so

$$F=\sum_{j=0}^{n}\sum_{i=0}^{n}a_{ij}Y^iY_1^j=\sum_{j=0}^{n-1}\sum_{i=0}^{n}a_{ij}Y^iY_1^j,$$

which implies $\deg_{Y_1}F\leq n-1<n$, a contradiction.

Choose $m\in\mathbb{Z}_{\geq 0}$ satisfying

$$Y_1^m\parallel\sum_{i,j}\tau(a_{ij})Y_1^{n+i-j}(\neq 0),$$

which means $Y_1^m\mid\sum_{i,j}\tau(a_{ij})Y_1^{n+i-j}$ and $Y_1^{m+1}\nmid\sum_{i,j}\tau(a_{ij})Y_1^{n+i-j}$. Note

$$(6.2)\qquad \sum_{n+i-j=h}\tau(a_{ij})=0\iff\sum_{n+j-i=2n-h}a_{ij}=0.$$

Then from the above condition we find

$$(6.3) \qquad \sum_{n+j-i=h} a_{ij} = 0 \quad \text{for all } 2n - m < h \leq 2n$$

and

$$\sum_{n+j-i=2n-m} a_{ij} \neq 0,$$

which imply $\deg \sum_{i,j} a_{ij} Y_1^{n+j-i} = 2n - m$.

We will show $a_{nn} = 1$. Assume $a_{nn} = 0$. From eq.$(*0)$ and

$$\deg(*0)_l \leq 2n + (n-1) = 3n - 1,$$

we obtain

$$\deg P = \deg(*0)_l - (2n - m) \leq n + m - 1.$$

Hence it follows that

$$\deg(*n)_r \leq (n + m - 1) + (n - 1) = 2n + m - 2 < 2n + m,$$

which contradicts $\deg(*n)_l \geq 2n + m$. Since we have chosen F satisfying $a_{nn} = 0$ or 1, we conclude $a_{nn} = 1$.

From eq.$(*0)$ we have

$$\deg P = (2n + n) - (2n - m) = n + m,$$

and so from eq.$(*n)$ we obtain

$$\deg \sum_{i,j} \tau(a_{ij}) Y_1^{n+i-j} = (n + m) + n - 2n = m.$$

Since the sum satisfies $Y_1^m \parallel \sum_{i,j} \tau(a_{ij}) Y_1^{n+i-j}$, it follows that

$$\sum_{i,j} \tau(a_{ij}) Y_1^{n+i-j} = \left(\sum_{n+i-j=m} \tau(a_{ij}) \right) Y_1^m.$$

Then the equation eq.$(*n)$ is

$$(*n) \qquad Y_1^n (Y_1 + qt)^n \left(\sum_{n+i-j=m} \tau(a_{ij}) \right) Y_1^m = P \sum_{j=0}^{n} a_{nj} Y_1^j,$$

from which we find $Y_1^m \mid P$. Put $l \in \mathbb{Z}_{\geq 0}$ such that $Y_1^l \parallel P$. We have $m \leq l \leq n+m$.

Step 3. Express the polynomial $P \in L[Y_1]$ as $P = pY_1^l(Y_1+qt)^{n+m-l}$, $p \in L^\times$. Comparing the terms of highest degree of eq.($*n$), we obtain $\sum_{n+i-j=m} \tau(a_{ij}) = p$. From $\sum_{i,j} \tau(a_{ij}) Y_1^{n+i-j} = pY_1^m$ we have

$$\sum_{n+i-j=h} \tau(a_{ij}) = 0, \quad \text{for all } m < h \leq 2n,$$

and so from (6.2) we obtain

$$\sum_{n+j-i=h} a_{ij} = 0, \quad \text{for all } 0 \leq h < 2n-m.$$

Since we have already found (6.3), it follows that

$$\begin{aligned}\sum_{i,j} a_{ij} Y_1^{n+j-i} &= \left(\sum_{n+j-i=2n-m} a_{ij}\right) Y_1^{2n-m} = \left(\sum_{n+i-j=m} a_{ij}\right) Y_1^{2n-m}\\ &= (\tau^{-1}p) Y_1^{2n-m}.\end{aligned}$$

Then from the equations ($*n$) and ($*0$) we obtain the following two equations:

$$(*n.1) \qquad Y_1^n (Y_1+qt)^{l-m} = Y_1^{l-m} \sum_{j=0}^{n} a_{nj} Y_1^j,$$

$$(*0.1) \qquad (aq^2t^2)^n \sum_{i=0}^{n} \tau(a_{in}) Y_1^i = p(\tau^{-1}p) Y_1^{l-m} (Y_1+qt)^{n+m-l}.$$

Comparing the terms of degree n of eq.($*n.1$), we obtain

$$(6.4) \qquad (qt)^{l-m} = a_{n,n+m-l}.$$

Comparing the terms of degree $l-m$ of eq.($*0.1$) we obtain

$$(6.5) \qquad (aq^2t^2)^n \tau(a_{l-m,n}) = p(\tau^{-1}p)(qt)^{n+m-l}.$$

Since we have $0 \le l-m \le n$, we can consider the equation $(*l-m)$,

$(*l-m)$

$$(aq^2t^2)^{n+m-l}Y_1^{2n+m-l}(Y_1+qt)^{l-m}\sum_{j=n+m-l}^{n}\sum_{i=0}^{n}\binom{j}{n+m-l}\tau(a_{ij})Y_1^{n+i-j}$$
$$=pY_1^l(Y_1+qt)^{n+m-l}\sum_{i=l-m}^{n}\sum_{j=0}^{n}\binom{i}{l-m}a_{ij}Y_1^{n+j-i}.$$

From (6.4) and (6.5) we obtain $a_{n,n+m-l} \neq 0$ and $a_{l-m,n} \neq 0$. Comparing the terms of highest degree of eq.$(*l-m)$, we find

$$(aq^2t^2)^{n+m-l}Y_1^{2n+m-l}Y_1^{l-m}\tau(a_{n,n+m-l})Y_1^{n-m+l} = pY_1^lY_1^{n+m-l}a_{l-m,n}Y_1^{2n-l+m}.$$

Hence it follows that $2l = 3m$ and

$$(aq^2t^2)^{n-\frac{m}{2}}\tau(a_{n,n-\frac{m}{2}}) = pa_{\frac{m}{2},n}. \tag{6.6}$$

Comparing the terms of degree n of eq.$(*0.1)$, we obtain $p(\tau^{-1}p) = (aq^2t^2)^n$. Then from (6.5) we have $a_{\frac{m}{2},n} = t^{n-\frac{m}{2}}$, and so from (6.4) and (6.6), we also have $(aq^2t^2)^{n-\frac{m}{2}}(q^2t)^{\frac{m}{2}} = pt^{n-\frac{m}{2}}$, which implies $p = (aq^2t)^{n-\frac{m}{2}}(q^2t)^{\frac{m}{2}}$. Hence we obtain

$$(aq^2t)^{n-\frac{m}{2}}(q^2t)^{\frac{m}{2}} \cdot (aqt)^{n-\frac{m}{2}}(qt)^{\frac{m}{2}} = (aq^2t^2)^n,$$

from which we conclude $a^{m-n} = q^n$. Consequently

$$\begin{cases} P = pY_1^{\frac{3}{2}m}(Y_1+qt)^{n-\frac{m}{2}}, \\ p = (aq^2t)^{n-\frac{m}{2}}(q^2t)^{\frac{m}{2}}, \\ a_{n,n-\frac{m}{2}} = (qt)^{\frac{m}{2}}, \\ a_{\frac{m}{2},n} = t^{n-\frac{m}{2}}, \\ a^{m-n} = q^n. \end{cases} \tag{6.7}$$

Note $0 \le m \le 2n$.

Step 4. We divide the problem into 3 cases according to the value of m. Firstly we deal with the case $m = 2n$. In this case, from (6.7) we have

$$\begin{cases} P = pY_1^{3n}, \\ p = (q^2t)^n, \\ a_{n0} = (qt)^n, \\ a^n = q^n. \end{cases}$$

By $(*n.1)$ and $(*0.1)$ we obtain the following two equations:

$$(*n.2) \qquad (Y_1 + qt)^n = \sum_{j=0}^{n} a_{nj} Y_1^j,$$

$$(*0.2) \qquad \sum_{i=0}^{n} \tau(a_{in}) Y_1^i = Y_1^n.$$

Since we have $P = pY_1^{3n}$ in this case, it follows from eq.$(*k)$ that

$$Y_1^{n+k} \mid \sum_{j=n-k}^{n} \sum_{i=0}^{n} \binom{j}{n-k} \tau(a_{ij}) Y_1^{n+i-j} \quad \text{for all } 0 \leq k \leq n,$$

which implies

$$\sum_{j=n-k}^{n} \sum_{i=0}^{n} \binom{j}{n-k} \tau(a_{ij}) Y_1^{n+i-j} = \tau(a_{n,n-k}) Y_1^{n+k},$$

where $0 \leq k \leq n$, and we find $a_{n,n-k} = \binom{n}{n-k}(qt)^k \neq 0$ from eq.$(*n.2)$. Then from eq.$(*k)$ we obtain

$$(6.8) \qquad (aq^2t^2)^{n-k}(Y_1 + qt)^k \tau(a_{n,n-k}) = (q^2t)^n \sum_{i=k}^{n} \sum_{j=0}^{n} \binom{i}{k} a_{ij} Y_1^{n+j-i}$$

for all $0 \leq k \leq n$. Comparing the terms of degree 0 of the above equations, we find

$$(aq^2t^2)^{n-k}(qt)^k \tau(a_{n,n-k}) = (q^2t)^n \binom{n}{k} a_{n0} \quad \text{for all } 0 \leq k \leq n.$$

Hence for all $0 \leq k \leq n$ we obtain

$$(aq^2t^2)^{n-k}(qt)^k \binom{n}{n-k} (q^2t)^k = (q^2t)^n \binom{n}{k} (qt)^n,$$

which implies $a^{n-k} = q^{n-k}$, especially $a = q$.

We will calculate the polynomial F using (6.1) and (6.8):

$$\begin{aligned}
F &= \sum_{k=0}^{n}\left\{\sum_{i=k}^{n}\sum_{j=0}^{n}\binom{i}{k}a_{ij}Y_1^{j-i}\right\}X^k \\
&= \sum_{k=0}^{n}\left\{\frac{(aq^2t^2)^{n-k}(Y_1+qt)^k}{(q^2t)^nY_1^n}\binom{n}{k}(q^2t)^k\right\}X^k \\
&= \frac{1}{Y_1^n}\sum_{k=0}^{n}\binom{n}{k}\{(Y_1+qt)X\}^k(at)^{n-k} \\
&= \frac{1}{Y_1^n}\{(Y_1+qt)(Y_1Y-1)+qt\}^n \\
&= \{Y_1Y+qtY-1\}^n.
\end{aligned}$$

Since we have chosen F satisfying $F(f,f_1)=0$, we obtain $(f_1f+qtf-1)^n=0$, which implies $f_1f+qtf-1=0$. Thus in this case the condition (i) is satisfied.

Step 5. Secondly we deal with the case $m=0$. In this case, by (6.7) we have

$$\begin{cases} P = p(Y_1+qt)^n, \\ p = (aq^2t)^n, \\ a_{0n} = t^n, \\ a^{-n} = q^n. \end{cases}$$

From ($*n$.1) and ($*0$.1) we obtain the following two equations:

$$(*n.3) \qquad Y_1^n = \sum_{j=0}^{n} a_{nj}Y_1^j,$$

$$(*0.3) \qquad \sum_{i=0}^{n}\tau(a_{in})Y_1^i = (Y_1+qt)^n.$$

Since we have $P=p(Y_1+qt)^n$, it follows from ($*k$) that

$$Y_1^{2n-k} \mid \sum_{i=k}^{n}\sum_{j=0}^{n}\binom{i}{k}a_{ij}Y_1^{n+j-i} \quad \text{for all } 0\le k\le n,$$

which implies

$$\sum_{i=k}^{n}\sum_{j=0}^{n}\binom{i}{k}a_{ij}Y_1^{n+j-i} = a_{kn}Y_1^{2n-k} \quad \text{for all } 0\le k\le n.$$

Then from eq.($*k$) and eq.($*0.3$) we obtain

$$(aq^2t^2)^{n-k}(Y_1+qt)^k \sum_{j=n-k}^{n} \sum_{i=0}^{n} \binom{j}{n-k} \tau(a_{ij}) Y_1^{n+i-j}$$
$$= (aq^2t)^n (Y_1+qt)^n \binom{n}{k} t^{n-k}$$

for all $0 \le k \le n$. Comparing the terms of degree 0 of the above equations, we find

$$(aq^2t^2)^{n-k}(qt)^k \binom{n}{n-k} \tau(a_{0n}) = (aq^2t)^n (qt)^n \binom{n}{k} t^{n-k} \quad \text{for all } 0 \le k \le n.$$

Hence for any $0 \le k \le n$ we obtain

$$(aq^2t^2)^{n-k}(qt)^k(qt)^n = (aq^2t)^n(qt)^n t^{n-k},$$

which implies $a^k = q^{-k}$, especially $a = q^{-1}$, the condition (ii).

Step 6. Finally we deal with the case $0 < m < 2n$. If we assume $m = n$, it follows that $q^n = 1$, which contradicts the assumption, q is not a root of unity. Thus we conclude $m \neq n$, which implies the condition (iii). □

Lemma 6.9. *Let $\mathcal{U} = (U, \tau)$ be a difference overfield of $(C(t), t \mapsto qt)$, $\mathcal{L} \subset \mathcal{U}$ a difference overfield of $(C(t), t \mapsto qt)$ whose operator is surjective and $f \in S_a(\mathcal{U})$ with $a \in C^\times$. If $\operatorname{tr.deg} \mathcal{L}\langle f \rangle/\mathcal{L} = 1$ then $\operatorname{tr.deg} \mathcal{L}\langle T_{\mathcal{U}}^i f \rangle/\mathcal{L} = 1$ for all $i \in \mathbb{Z}$.*

Proof. We prove this by induction. Let $k \ge 1$ and suppose $\operatorname{tr.deg} \mathcal{L}\langle T_{\mathcal{U}}^i f \rangle/\mathcal{L} = 1$ for all i satisfying $|i| < k$.

Since we find $T_{\mathcal{U}}^k f \in \mathcal{L}\langle T_{\mathcal{U}}^{k-1} f \rangle$ by the definition of $T_{\mathcal{U}}$, it follows that

$$\operatorname{tr.deg} \mathcal{L}\langle T_{\mathcal{U}}^k f \rangle/\mathcal{L} \le \operatorname{tr.deg} \mathcal{L}\langle T_{\mathcal{U}}^{k-1} f \rangle/\mathcal{L} = 1.$$

By Lemma 6.2 there is $\tau^{-1}(T_{\mathcal{U}}^k f)$ in $\mathcal{U}$. Assuming $T_{\mathcal{U}}^k f$ is algebraic over L, we obtain $\tau^{-1}(T_{\mathcal{U}}^k f)$ is also algebraic over L, which implies $T_{\mathcal{U}}^{k-1} f$ is algebraic over L because

$$T_{\mathcal{U}}^{k-1} f = T_{\mathcal{U}}^{-1} T_{\mathcal{U}}^k f \in L(\tau^{-1}(T_{\mathcal{U}}^k f), T_{\mathcal{U}}^k f),$$

a contradiction. Thus $T_{\mathcal{U}}^k f$ is transcendental over L, and so we conclude $\operatorname{tr.deg} \mathcal{L}\langle T_{\mathcal{U}}^k f \rangle/\mathcal{L} = 1$.

By Lemma 6.2 there is $\tau^{-1}(T_{\mathcal{U}}^{-k+1}f) \in \mathcal{L}\langle T_{\mathcal{U}}^{-k+1}f\rangle$. Since we find $T_{\mathcal{U}}^{-k}f \in \mathcal{L}\langle T_{\mathcal{U}}^{-k+1}f\rangle$ by the definition of $T_{\mathcal{U}}^{-1}$, it follows that

$$\operatorname{tr.deg} \mathcal{L}\langle T_{\mathcal{U}}^{-k}f\rangle/\mathcal{L} \leq \operatorname{tr.deg} \mathcal{L}\langle T_{\mathcal{U}}^{-k+1}f\rangle/\mathcal{L} = 1.$$

Assuming $T_{\mathcal{U}}^{-k}f$ is algebraic over L, we obtain $T_{\mathcal{U}}^{-k+1}f = T_{\mathcal{U}}T_{\mathcal{U}}^{-k}f$ is also algebraic over L, a contradiction. Thus $T_{\mathcal{U}}^{-k}f$ is transcendental over L, and so we conclude $\operatorname{tr.deg} \mathcal{L}\langle T_{\mathcal{U}}^{-k}f\rangle/\mathcal{L} = 1$. □

Lemma 6.10. *Let $q \in C^{\times}$ be not a root of unity, $\mathcal{K}$ a difference overfield of $(C(t), t \mapsto qt)$ whose operator is surjective, $\mathcal{U} = (U, \tau)$ a difference overfield of $\mathcal{K}$ and $\mathcal{L} \subset \mathcal{U}$ a difference overfield of $\mathcal{K}$ satisfying* $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} < \infty$. *Then for any $a \in C^{\times}$ and $f \in S_a(\mathcal{U})$ satisfying* $\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} = 1$, *there is $i \in \mathbb{Z}$ such that $a = q^{2i+1}$ and $T_{\mathcal{U}}^{-i}f \in U$ is a solution of the equation $y_1 y + qty - 1 = 0$ over $(C(t), t \mapsto qt)$ in $\mathcal{U}$.*

Proof. It is enough to prove this for algebraically closed L. In fact suppose this is proved for the case, and let $a \in C^{\times}$ and $f \in S_a(\mathcal{U})$ satisfy $\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} = 1$. The operator τ of $\mathcal{U}$ can be extended to an isomorphism $\overline{\tau}$ of an algebraic closure $\overline{U}$ of U into $\overline{U}$. Put $\overline{\mathcal{U}} = (\overline{U}, \overline{\tau})$. For any difference subfield $\mathcal{F}$ of $\overline{\mathcal{U}}$, we find that $\overline{\mathcal{F}} = (\overline{F}, \overline{\tau}|_{\overline{F}})$ is also a difference subfield of $\overline{\mathcal{U}}$, where $\overline{F}$ is the algebraic closure of F in $\overline{U}$. It follows from $\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} = 1$ that $\operatorname{tr.deg} \overline{\mathcal{L}}\langle f\rangle/\overline{\mathcal{L}} = 1$. Since we obtain $f \in S_a(\overline{\mathcal{U}})$ by Proposition 6.7, there is $i \in \mathbb{Z}$ such that $a = q^{2i+1}$ and $g = T_{\overline{\mathcal{U}}}^{-i}f \in \overline{U}$ is a solution of the equation $y_1 y + qty - 1 = 0$ over $(C(t), t \mapsto qt)$ in $\overline{\mathcal{U}}$. We find $g = T_{\mathcal{U}}^{-i}f \in U$ by Proposition 6.7. We have

$$\tau(g)g + qtg - 1 = \overline{\tau}(g)g + qtg - 1 = 0,$$

which implies g is a solution of $y_1 y + qty - 1 = 0$ in $\mathcal{U}$.

Suppose L is algebraically closed. Then the operator $\tau|_L$ of $\mathcal{L}$ is surjective. In fact note $\operatorname{tr.deg} \mathcal{L}/\mathcal{K} < \infty$ and $\tau|_K \in \operatorname{Aut} K$. Since $\operatorname{tr.deg} L/K = \operatorname{tr.deg} \tau L/K$, it follows that

$$\operatorname{tr.deg} L/\tau L = \operatorname{tr.deg} L/K - \operatorname{tr.deg} \tau L/K = 0,$$

which implies $L/\tau L$ is an algebraic extension. By theorem of Steinitz there exists an isomorphism $\tilde{\tau}$ of L onto L such that $\tilde{\tau} = \tilde{\tau}|_L = \tau|_L$. Thus $\tau|_L$ is a surjective operator.

Let

$$\Phi = \{(a,j) \in C^\times \times \mathbb{Z}_{\geq 0} \mid a \neq q^i \text{ for all } i \in \mathbb{Z}, a^i = q^j \text{ for some } |i| \in \mathbb{Z}_{\geq 2}, \\ \text{and there is } f \in S_a(\mathcal{U}) \text{ satisfying } \operatorname{tr.deg} \mathcal{L}\langle f \rangle / \mathcal{L} = 1\},$$

and assume $\Phi \neq \{\}$. We will derive a contradiction. Choose $(a,j) \in \Phi$ whose j is minimam, and then choose $i \in \mathbb{Z}$ and $f \in S_a(\mathcal{U})$ such that $|i| \geq 2$, $a^i = q^j$ and $\operatorname{tr.deg} \mathcal{L}\langle f \rangle / \mathcal{L} = 1$. Since we have $a \neq q, q^{-1}$, it follows from Lemma 6.8 that there are $m, n \in \mathbb{Z}$ such that $a^{m-n} = q^n$, $0 < m < 2n$, $m \neq n$ and $n \geq 1$. We find $q^{in} = a^{i(m-n)} = q^{j(m-n)}$, which implies $in = j(m-n)$, and so $j > 0$.

Firstly we deal with the case $0 < m < n$. From

$$(q^2 a)^i = q^{2i} a^i = q^{2i} q^j = q^{2i+j},$$

we obtain $(q^2 a)^i = q^{|2i+j|}$ or $(q^2 a)^{-i} = q^{|2i+j|}$. Since we find $\operatorname{tr.deg} \mathcal{L}\langle T_{\mathcal{U}} f \rangle / \mathcal{L} = 1$ by Lemma 6.9, it follows that $(q^2 a, |2i+j|) \in \Phi$. From $0 < m < n$ we obtain $|2i+j| < j$ by the following calculation:

$$\begin{aligned} -n < m - n &< 0, \\ -j < \frac{j(m-n)}{n} &< 0, \\ -j < i &< 0, \\ -2j < 2i &< 0, \\ -j < 2i + j &< j. \end{aligned}$$

This contradicts the minimality of j.

Secondly we deal with the case $n < m < 2n$. From

$$(q^{-2} a)^i = q^{-2i} a^i = q^{-2i} q^j = q^{-2i+j},$$

we obtain $(q^{-2} a)^i = q^{|-2i+j|}$ or $(q^{-2} a)^{-i} = q^{|-2i+j|}$. By Lemma 6.9 we find $\operatorname{tr.deg} \mathcal{L}\langle T_{\mathcal{U}}^{-1} f \rangle / \mathcal{L} = 1$, and so $(q^{-2} a, |-2i+j|) \in \Phi$. From $n < m < 2n$ we obtain $|-2i+j| < j$ by a similar calculation to the above. This contradicts the minimality of j. Thus we conclude $\Phi = \{\}$.

Let $a \in C^\times$ and $f \in S_a(\mathcal{U})$ satisfy $\operatorname{tr.deg} \mathcal{L}\langle f \rangle / \mathcal{L} = 1$. Assuming $a \neq q^i$ for any $i \in \mathbb{Z}$, we find by Lemma 6.8 that there are $m, n \in \mathbb{Z}$ such that $a^{m-n} = q^n$, $0 < m < 2n$, $m \neq n$ and $n \geq 1$, which implies $(a, n) \in \Phi$ because of $|m - n| \geq 2$. Hence there is some $j \in \mathbb{Z}$ such that $a = q^j$.

Assume $j = 2i$ for some $i \in \mathbb{Z}$. Then we have $T_{\mathcal{U}}^{-i} f \in S_1(\mathcal{U})$. Since we obtain $\operatorname{tr.deg} \mathcal{L}\langle T_{\mathcal{U}}^{-i} f\rangle/\mathcal{L} = 1$ by Lemma 6.9, it follows by Lemma 6.8 that $1 = q^k$ for some $k \in \mathbb{Z} \setminus \{0\}$, a contradiction. Thus j is odd.

Choose $i \in \mathbb{Z}$ such that $j = 2i + 1$. Then we have $a = q^{2i+1}$ and $T_{\mathcal{U}}^{-i} f \in S_q(\mathcal{U})$. Since we obtain $\operatorname{tr.deg} \mathcal{L}\langle T_{\mathcal{U}}^{-i} f\rangle/\mathcal{L} = 1$ by Lemma 6.9, it follows by Lemma 6.8 that $T_{\mathcal{U}}^{-i} f$ is a solution of $y_1 y + qty - 1 = 0$ in $\mathcal{U}$. Note that $q^{m-n} = q^n$ denotes $m = 2n$. □

Proof of Theorem 6.1. (i) Suppose $a \neq q^{2i+1}$ for all $i \in \mathbb{Z}$. By Lemma 6.10 we find that for any difference overfield $\mathcal{L}$ of $\mathcal{K}$ with $\operatorname{tr.deg} L/K < \infty$ and a difference overfield $\mathcal{U}$ of $\mathcal{L}$ with $\mathcal{K}\langle f\rangle_{\mathcal{D}} \subset \mathcal{U}$, $\operatorname{tr.deg} \mathcal{L}\langle f\rangle_{\mathcal{U}}/\mathcal{L} \leq 1$ implies that f is algebraic over L. Then we conclude that f is algebraic over K by Lemma 4.9.

(ii) Suppose $a = q^{2i+1}$ for some $i \in \mathbb{Z}$. Choose such i, and suppose f is transcendental over K. By Lemma 4.9 there are a defference overfield $\mathcal{L}$ of $\mathcal{K}$ with $\operatorname{tr.deg} L/K < \infty$ and a difference overfield $\mathcal{U}$ of $\mathcal{L}$ with $\mathcal{K}\langle f\rangle_{\mathcal{D}} \subset \mathcal{U}$ such that $\operatorname{tr.deg} \mathcal{L}\langle f\rangle_{\mathcal{U}}/\mathcal{L} = 1$. By Lemma 6.10 there is $j \in \mathbb{Z}$ such that $a = q^{2j+1}$ and $T_{\mathcal{U}}^{-j} f$ is a solution of $y_1 y + qty - 1 = 0$ in $\mathcal{U}$. Note that $q^{2i+1} = q^{2j+1}$ implies $i = j$, and we conclude that $T_{\mathcal{U}}^{-i} f = T_{\mathcal{D}}^{-i} f$ is a solution of $y_1 y + qty - 1 = 0$ in $\mathcal{D}$. □

6.3 Transcendence of solutions

Notation. Throughout this section we define $\sum_{i=m}^{n} a_i = 0$ for $n < m$.

We prove

Theorem 6.11. *Let $q, a \in C^{\times}$. Put $K = \mathbb{Q}(q, a)$, and suppose there is a normalized discrete valuation $v \colon K \to \mathbb{Z} \cup \{\infty\}$ such that $v(q) > 0$. Then q-$P(A_6)_a$ has no solution algebraic over $C(t)$.*

A discrete valuation v is a mapping of a field K to $\mathbb{Z} \cup \{\infty\}$ such that

(i) $v(0) = \infty$ and $v(a) \in \mathbb{Z}$ for $a \neq 0$.

(ii) $v(a) + v(b) = v(ab)$.

(iii) $v(a + b) \geq \min\{v(a), v(b)\}$.

(iv) $v(a) \neq 0, \infty$ for some a.

A discrete valuation v is called to be normalized additionally if $v(K^\times) = \mathbb{Z}$.

We also prove

Corollary 6.12. *Let $q \in C^\times$ transcendental over $\mathbb{Q}$ and $a \in C^\times$. Then q-$P(A_6)_a$ has no solution algebraic over $C(t)$.*

Remark. We obtain the same result for any algebraic number q which is not a unit because a solution of q-$P(A_6)_a$ is also a solution of $(1/q)$-$P(A_6)_a$.

We have

Proposition 6.13 (Proposition in [19, 20]). *Let $q \in C^\times$ be not a root of unity. Let f be a solution of q-$P(A_6)_a$ with $a \in C^\times$. If f is algebraic over $C(t)$ then f can be expressed as*

$$f = \frac{tP}{Q}, \quad P, Q \in C[t^2],\ t \nmid P \text{ and } t \nmid Q.$$

We need the following two lemmas.

Lemma 6.14. *Let K be a field and $q, a \in K^\times$. Let $\{a_i\}_{i\geq 0} \subset K$ be the sequence such that $a_0 = -1$ and*

$$a_i = aa_{i-1} - \sum_{l=1}^{i-1} \left\{ \sum_{j=0}^{l-2} \left(\sum_{k=0}^{j} a_k a_{j-k} q^{4k-2j} \right) \left(\sum_{k=0}^{l-2-j} a_k a_{l-2-j-k} \right) - \sum_{k=0}^{l-1} a_{l-1-k} a_k (q^{2k+1} + q^{-(2k+1)}) \right\} a_{i-l} \quad \text{for all } i \geq 1.$$

Let $v\colon K \to \mathbb{Z} \cup \{\infty\}$ be a discrete valuation such that $v(q) > 0$ and $-v(q) \leq v(a) < v(q)$. Put

$$n_i = \begin{cases} -\frac{i(i-1)}{2} v(q) + \frac{i}{2} v(a) & \text{if } 2 \mid i, \\ -\frac{i(i-1)}{2} v(q) + \frac{i+1}{2} v(a) & \text{if } 2 \nmid i \end{cases} \quad \text{for all } i \geq 0.$$

Then for any $i \geq 0$ we obtain

$$\begin{cases} v(a_i) \geq n_i & \text{if } 2 \mid i, \\ v(a_i) = n_i & \text{if } 2 \nmid i. \end{cases}$$

Proof. We prove this by induction on i. We have $v(a_0) = 0 = n_0$, the case $i = 0$. Since we have $a_1 = -a$, it follows that $v(a_1) = v(a) = n_1$, the case $i = 1$. When $i = 2$, we have $a_2 = -a^2 - (q + q^{-1})a$, and so

$$v(a_2) \geq \min\{2v(a), -v(q) + v(a)\} \geq -v(q) + v(a) = n_2.$$

Suppose $i \geq 3$, and the lemma is true for numbers $\leq i - 1$. We show that the following holds for all $1 \leq l \leq i - 1$,

$$v(\sum_{k=0}^{l-1} a_{l-1-k}a_k(q^{2k+1} + q^{-(2k+1)})) \begin{cases} = -\frac{1}{2}v(q)(l^2 + l) + \frac{l}{2}v(a) & \text{if } 2 \mid l \\ \geq -\frac{1}{2}v(q)(l^2 + l) + \frac{l-1}{2}v(a) & \text{if } 2 \nmid l \end{cases} < -\frac{1}{2}v(q)l^2.$$

Firstly we deal with the case $2 \mid l$. For all $0 \leq k \leq l - 2$, we obtain

$$\begin{aligned} & v(a_{l-1-k}a_k(q^{2k+1} + q^{-(2k+1)})) \\ & \geq -\frac{(l-1-k)(l-1-k-1)}{2}v(q) + \left\{ \begin{array}{ll} \frac{l-1-k+1}{2}v(a) & \text{if } 2 \mid k \\ \frac{l-1-k}{2}v(a) & \text{if } 2 \nmid k \end{array} \right\} \\ & \quad - \frac{k(k-1)}{2}v(q) + \left\{ \begin{array}{ll} \frac{k}{2}v(a) & \text{if } 2 \mid k \\ \frac{k+1}{2}v(a) & \text{if } 2 \nmid k \end{array} \right\} - (2k+1)v(q) \\ & = -\frac{1}{2}v(q)\{2k^2 - (2l-6)k + l^2 - 3l + 4\} + \frac{l}{2}v(a) \\ & = -\frac{1}{2}v(q)\left\{2\left(k - \frac{l-3}{2}\right)^2 + \frac{1}{2}(l^2 - 1)\right\} + \frac{l}{2}v(a) \\ & \geq -\frac{1}{2}v(q)(l^2 - l) + \frac{l}{2}v(a), \end{aligned}$$

and for $k = l - 1$,

$$v(a_0 a_{l-1}(q^{2(l-1)+1} + q^{-(2(l-1)+1)})) = -\frac{1}{2}v(q)(l^2 + l) + \frac{l}{2}v(a),$$

which imply

$$\begin{aligned} & v(\sum_{k=0}^{l-1} a_{l-1-k}a_k(q^{2k+1} + q^{-(2k+1)})) = -\frac{1}{2}v(q)(l^2 + l) + \frac{l}{2}v(a) \\ & < -\frac{1}{2}v(q)(l^2 + l) + \frac{l}{2}v(q) = -\frac{1}{2}v(q)l^2. \end{aligned}$$

Secondly we deal with the case $2 \nmid l$. For all $0 \le k \le l-1$, we have

$$\begin{aligned}
&v(a_{l-1-k}a_k(q^{2k+1}+q^{-(2k+1)})) \\
&\ge -\frac{(l-1-k)(l-2-k)}{2}v(q) + \left\{ \begin{array}{ll} \frac{l-1-k}{2}v(a) & \text{if } 2 \mid k \\ \frac{l-k}{2}v(a) & \text{if } 2 \nmid k \end{array} \right\} \\
&\quad -\frac{k(k-1)}{2}v(q) + \left\{ \begin{array}{ll} \frac{k}{2}v(a) & \text{if } 2 \mid k \\ \frac{k+1}{2}v(a) & \text{if } 2 \nmid k \end{array} \right\} - (2k+1)v(q) \\
&= -\frac{1}{2}v(q)\left\{2\left(k-\frac{l-3}{2}\right)^2 + \frac{1}{2}(l^2-1)\right\} + \left\{ \begin{array}{ll} \frac{l-1}{2}v(a) & \text{if } 2 \mid k \\ \frac{l+1}{2}v(a) & \text{if } 2 \nmid k \end{array} \right\}.
\end{aligned}$$

Therefore we obtain that for all $0 \le k \le l-2$,

$$\begin{aligned}
&v(a_{l-1-k}a_k(q^{2k+1}+q^{-(2k+1)})) \ge -\frac{1}{2}v(q)(l^2-l) + \left\{ \begin{array}{ll} \frac{l-1}{2}v(a) & \text{if } 2 \mid k \\ \frac{l+1}{2}v(a) & \text{if } 2 \nmid k \end{array} \right\} \\
&\ge -\frac{1}{2}v(q)(l^2-l) - \left\{ \begin{array}{ll} \frac{l-1}{2}v(q) & \text{if } 2 \mid k \\ \frac{l+1}{2}v(q) & \text{if } 2 \nmid k \end{array} \right\} \ge -\frac{1}{2}v(q)(l^2+1),
\end{aligned}$$

and for $k = l-1$,

$$\begin{aligned}
&v(a_0a_{l-1}(q^{2(l-1)+1}+q^{-(2(l-1)+1)})) \\
&\ge -\frac{1}{2}v(q)(l^2+l) + \frac{l-1}{2}v(a) \\
&\le -\frac{1}{2}v(q)(l^2+l) + \frac{l-1}{2}v(q) = -\frac{1}{2}v(q)(l^2+1),
\end{aligned}$$

which imply

$$\begin{aligned}
&v(\sum_{k=0}^{l-1} a_{l-1-k}a_k(q^{2k+1}+q^{-(2k+1)})) \\
&\ge -\frac{1}{2}v(q)(l^2+l) + \frac{l-1}{2}v(a) \le -\frac{1}{2}v(q)(l^2+1) < -\frac{1}{2}v(q)l^2.
\end{aligned}$$

For all $0 \le j \le i-3$, we obtain the following,

$$v(\sum_{k=0}^{j} a_k a_{j-k} q^{4k-2j}) \ge -\frac{1}{2}v(q)(j^2+4j+2).$$

In fact, let $0 \leq j \leq i-3$. For all $0 \leq k \leq j$, we find

$$\begin{aligned}
&v(a_k a_{j-k} q^{4k-2j}) \\
&\geq -\frac{k(k-1)}{2}v(q) - \frac{k+1}{2}v(q) \\
&\quad - \frac{(j-k)(j-k-1)}{2}v(q) - \frac{j-k+1}{2}v(q) + (4k-2j)v(q) \\
&= -\frac{1}{2}v(q)\left\{2\left(k - \frac{j+4}{2}\right)^2 + \frac{1}{2}(j^2-12)\right\} \\
&\geq -\frac{1}{2}v(q)(j^2+4j+2),
\end{aligned}$$

which yields the above. In the same way, for all $0 \leq h \leq i-3$, we obtain

$$v(\sum_{k=0}^{h} a_k a_{h-k}) \geq -\frac{1}{2}v(q)(h^2+2).$$

Therefore, for all $2 \leq l \leq i-1$, we conclude

$$v(\sum_{j=0}^{l-2}\left(\sum_{k=0}^{j} a_k a_{j-k} q^{4k-2j}\right)\left(\sum_{k=0}^{l-2-j} a_k a_{l-2-j-k}\right)) \geq -\frac{1}{2}v(q)l^2,$$

since it follows that each element of the sum of the left side has the value greater than or equal to

$$\begin{aligned}
&-\frac{1}{2}v(q)(j^2+4j+2) - \frac{1}{2}v(q)((l-2-j)^2+2) \\
&= -\frac{1}{2}v(q)\left\{2\left(j - \frac{l-4}{2}\right)^2 + \frac{l^2}{2}\right\} \\
&\geq -\frac{1}{2}v(q)l^2.
\end{aligned}$$

Put

$$\begin{aligned}
b_l =& \sum_{j=0}^{l-2}\left(\sum_{k=0}^{j} a_k a_{j-k} q^{4k-2j}\right)\left(\sum_{k=0}^{l-2-j} a_k a_{l-2-j-k}\right) \\
&- \sum_{k=0}^{l-1} a_{l-1-k} a_k (q^{2k+1} + q^{-(2k+1)}) \quad \text{for all } l \geq 1.
\end{aligned}$$

Then we find that for all $1 \leq l \leq i-1$,

$$v(b_l)\begin{cases} = -\frac{1}{2}v(q)(l^2+l)+\frac{l}{2}v(a) & \text{if } 2 \mid l, \\ \geq -\frac{1}{2}v(q)(l^2+l)+\frac{l-1}{2}v(a) & \text{if } 2 \nmid l. \end{cases}$$

We show

$$v(\sum_{l=1}^{i-1} b_l a_{i-l})\begin{cases} \geq -\frac{1}{2}v(q)(i^2-i)+\frac{i}{2}v(a) & \text{if } 2 \mid i, \\ = -\frac{1}{2}v(q)(i^2-i)+\frac{i+1}{2}v(a) & \text{if } 2 \nmid i. \end{cases}$$

Firstly we deal with the case $2 \mid i$. For all $1 \leq l \leq i-1$, we obtain

$$\begin{aligned} & v(b_l a_{i-l}) \\ & \geq -\frac{1}{2}v(q)(l^2+l)+\left\{ \begin{array}{ll} \frac{l}{2}v(a) & \text{if } 2 \mid l \\ \frac{l-1}{2}v(a) & \text{if } 2 \nmid l \end{array} \right\} \\ & \quad -\frac{(i-l)(i-l-1)}{2}v(q)+\left\{ \begin{array}{ll} \frac{i-l}{2}v(a) & \text{if } 2 \mid l \\ \frac{i-l+1}{2}v(a) & \text{if } 2 \nmid l \end{array} \right\} \\ & = -\frac{1}{2}v(q)\left\{2\left(l-\frac{i-1}{2}\right)^2+\frac{1}{2}(i^2-1)\right\}+\frac{i}{2}v(a) \\ & \geq -\frac{1}{2}v(q)(i^2-i)+\frac{i}{2}v(a), \end{aligned}$$

which yields the required. Secondly we deal with the case $2 \nmid i$. Note that

$$v(b_{i-1}a_1) = -\frac{1}{2}v(q)(i^2-i)+\frac{i+1}{2}v(a).$$

For all $1 \leq l \leq i-2$, we obtain

$$\begin{aligned} & v(b_l a_{i-l}) \\ & \geq -\frac{1}{2}v(q)(l^2+l)+\left\{ \begin{array}{ll} \frac{l}{2}v(a) & \text{if } 2 \mid l \\ \frac{l-1}{2}v(a) & \text{if } 2 \nmid l \end{array} \right\} \\ & \quad -\frac{(i-l)(i-l-1)}{2}v(q)+\left\{ \begin{array}{ll} \frac{i-l+1}{2}v(a) & \text{if } 2 \mid l \\ \frac{i-l}{2}v(a) & \text{if } 2 \nmid l \end{array} \right\} \\ & = -\frac{1}{2}v(q)\left\{2\left(l-\frac{i-1}{2}\right)^2+\frac{1}{2}(i^2-1)\right\}+\left\{ \begin{array}{ll} \frac{i+1}{2}v(a) & \text{if } 2 \mid l \\ \frac{i-1}{2}v(a) & \text{if } 2 \nmid l \end{array} \right\} \\ & \geq -\frac{1}{2}v(q)(i^2-3i+4)+\left\{ \begin{array}{ll} \frac{i+1}{2}v(a) & \text{if } 2 \mid l \\ \frac{i-1}{2}v(a) & \text{if } 2 \nmid l \end{array} \right\}, \end{aligned}$$

which implies

$$\begin{aligned}
&v(b_l a_{i-l}) - v(b_{i-1}a_1)\\
&\geq -\frac{1}{2}v(q)(i^2-3i+4) + \left\{ \begin{array}{ll} \frac{i+1}{2}v(a) & \text{if } 2 \mid l \\ \frac{i-1}{2}v(a) & \text{if } 2 \nmid l \end{array} \right\}\\
&\quad + \frac{1}{2}v(q)(i^2-i) - \frac{i+1}{2}v(a)\\
&= \frac{1}{2}v(q)(2i-4) + \left\{ \begin{array}{ll} 0 & \text{if } 2 \mid l \\ -v(a) & \text{if } 2 \nmid l \end{array} \right\}\\
&\geq v(q) + \min\{0, -v(a)\} > 0.
\end{aligned}$$

Therefore we conclude that

$$v(\sum_{l=1}^{i-1} b_l a_{i-l}) = v(b_{i-1}a_1) = -\frac{1}{2}v(q)(i^2-i) + \frac{i+1}{2}v(a).$$

Since we have

$$v(aa_{i-1}) \geq -\frac{1}{2}v(q)(i^2-3i+2) + \left\{ \begin{array}{ll} \frac{i+2}{2}v(a) & \text{if } 2 \mid i \\ \frac{i+1}{2}v(a) & \text{if } 2 \nmid i \end{array} \right\},$$

we find

$$\begin{aligned}
&v(aa_{i-1}) - \left(-\frac{1}{2}v(q)(i^2-i) + \left\{ \begin{array}{ll} \frac{i}{2}v(a) & \text{if } 2 \mid i \\ \frac{i+1}{2}v(a) & \text{if } 2 \nmid i \end{array} \right\} \right)\\
&\geq \frac{1}{2}v(q)(2i-2) + \left\{ \begin{array}{ll} v(a) & \text{if } 2 \mid i \\ 0 & \text{if } 2 \nmid i \end{array} \right\}\\
&\geq 2v(q) + \min\{v(a), 0\} \geq v(q) > 0,
\end{aligned}$$

which implies

$$v(a_i) \begin{cases} \geq -\frac{1}{2}v(q)(i^2-i) + \frac{i}{2}v(a) = n_i & \text{if } 2 \mid i, \\ = -\frac{1}{2}v(q)(i^2-i) + \frac{i+1}{2}v(a) = n_i & \text{if } 2 \nmid i. \end{cases}$$

□

Lemma 6.15. *Let K be a field and $q, a \in K^\times$. Let $v\colon K \to \mathbb{Z} \cup \{\infty\}$ be a normalized discrete valuation such that $v(q) > 0$ and $-v(q) \leq v(a) < v(q)$. Put*

$$n_i = \begin{cases} -\frac{i(i-1)}{2}v(q) + \frac{i}{2}v(a) & \text{if } 2 \mid i, \\ -\frac{i(i-1)}{2}v(q) + \frac{i+1}{2}v(a) & \text{if } 2 \nmid i \end{cases} \quad \textit{for all } i \geq 0,$$

and choose $\{b_i\}_{i\geq 0} \subset K^\times$ such that $v(b_i) = n_i$. Let S_i be the symmetric group of degree i. Then for any $s \geq 0$, $m \geq 0$ satisfying $2 \nmid m$ and $\sigma \in S_{s+1} \setminus \{id\}$, we find

$$v(\prod_{i=0}^{s} b_{m+i+\sigma(i)}) > v(\prod_{i=0}^{s} b_{m+2i}).$$

Proof. We prove this by induction on s. When $s = 0$, we have $S_{s+1} = \{id\}$. Suppose $s \geq 1$ and the lemma is true for $s-1$. Let $m \geq 0$ satisfy $2 \nmid m$ and $\sigma \in S_{s+1} \setminus \{id\}$. Choose $\sigma' \in S_s$ which is associated with $(\sigma(s), s) \circ \sigma \in S_{s+1}$.

Firstly we deal with the case $\sigma(s) = s$. We obtain $\sigma' \neq id$ from $\sigma \neq id$. By the induction hypothesis we find

$$\begin{aligned} v(\prod_{i=0}^{s} b_{m+i+\sigma(i)}) &= v(\prod_{i=0}^{s-1} b_{m+i+\sigma'(i)}) + v(b_{m+2s}) \\ &> v(\prod_{i=0}^{s-1} b_{m+2i}) + v(b_{m+2s}) = v(\prod_{i=0}^{s} b_{m+2i}). \end{aligned}$$

Secondly we deal with the case $\sigma(s) \neq s$. We have $\sigma'(i) = \sigma(i)$ for any $0 \leq i \leq s-1$ satisfying $\sigma(i) \neq s$. Noting

$$\sigma' \circ \sigma^{-1}(s) = (\sigma(s), s) \circ \sigma \circ \sigma^{-1}(s) = \sigma(s),$$

we obtain

$$\begin{aligned} & v(\prod_{i=0}^{s} b_{m+i+\sigma(i)}) \\ &= v(\prod_{i=0}^{s-1} b_{m+i+\sigma'(i)}) - v(b_{m+\sigma^{-1}(s)+\sigma'\circ\sigma^{-1}(s)}) + v(b_{m+\sigma^{-1}(s)+s}) + v(b_{m+s+\sigma(s)}) \\ &\geq v(\prod_{i=0}^{s-1} b_{m+2i}) - v(b_{m+\sigma^{-1}(s)+\sigma(s)}) + v(b_{m+\sigma^{-1}(s)+s}) + v(b_{m+s+\sigma(s)}), \end{aligned}$$

which yields

$$\begin{aligned} & v(\prod_{i=0}^{s} b_{m+i+\sigma(i)}) - v(\prod_{i=0}^{s} b_{m+2i}) \\ &\geq (-v(b_{m+2s}) + v(b_{m+s+\sigma(s)})) - (-v(b_{m+\sigma^{-1}(s)+s}) + v(b_{m+\sigma^{-1}(s)+\sigma(s)})). \end{aligned}$$

The right side of the above is equal to the following,

$$
\begin{aligned}
(6.9)\qquad & \Bigg[\frac{(m+2s)(m+2s-1)}{2}v(q) - \frac{m+2s+1}{2}v(a) \\
& \quad -\frac{(m+s+\sigma(s))(m+s+\sigma(s)-1)}{2}v(q) \\
& \quad +\left\{\begin{array}{ll} \frac{m+s+\sigma(s)+1}{2}v(a) & \text{if } 2 \mid s+\sigma(s) \\ \frac{m+s+\sigma(s)}{2}v(a) & \text{if } 2 \nmid s+\sigma(s) \end{array}\right\}\Bigg] \\
& -\Bigg[\frac{(m+\sigma^{-1}(s)+s)(m+\sigma^{-1}(s)+s-1)}{2}v(q) \\
& \quad -\left\{\begin{array}{ll} \frac{m+\sigma^{-1}(s)+s+1}{2}v(a) & \text{if } 2 \mid \sigma^{-1}(s)+s \\ \frac{m+\sigma^{-1}(s)+s}{2}v(a) & \text{if } 2 \nmid \sigma^{-1}(s)+s \end{array}\right\} \\
& \quad -\frac{(m+\sigma^{-1}(s)+\sigma(s))(m+\sigma^{-1}(s)+\sigma(s)-1)}{2}v(q) \\
& \quad +\left\{\begin{array}{ll} \frac{m+\sigma^{-1}(s)+\sigma(s)+1}{2}v(a) & \text{if } 2 \mid \sigma^{-1}(s)+\sigma(s) \\ \frac{m+\sigma^{-1}(s)+\sigma(s)}{2}v(a) & \text{if } 2 \nmid \sigma^{-1}(s)+\sigma(s) \end{array}\right\}\Bigg].
\end{aligned}
$$

Using

$$(x+y)(x+y-1)-(x+z)(x+z-1)=(y-z)(2x+y+z-1),$$

we find that equation (6.9) is equal to the following,

$$
\begin{aligned}
& \frac{1}{2}v(q)(s-\sigma(s))(2(m+s)+s+\sigma(s)-1) - \left\{\begin{array}{c} \frac{s-\sigma(s)}{2}v(a) \\ \text{or} \\ \frac{s-\sigma(s)+1}{2}v(a) \end{array}\right\} \\
& -\frac{1}{2}v(q)(s-\sigma(s))(2(m+\sigma^{-1}(s))+s+\sigma(s)-1) + \left\{\begin{array}{c} \frac{s-\sigma(s)}{2}v(a), \\ \frac{s-\sigma(s)+1}{2}v(a) \\ \text{or} \\ \frac{s-\sigma(s)-1}{2}v(a) \end{array}\right\} \\
& = v(q)(s-\sigma(s))(s-\sigma^{-1}(s)) + \left\{\begin{array}{c} 0, \\ -\frac{1}{2}v(a), \\ \frac{1}{2}v(a) \\ \text{or} \\ -v(a) \end{array}\right\}.
\end{aligned}
$$

Therefore we obtain

$$(\text{equation } (6.9)) \geq v(q) + \min\{0, -(1/2)v(a), (1/2)v(a), -v(a)\} > 0,$$

which implies

$$v(\prod_{i=0}^{s} b_{m+i+\sigma(i)}) - v(\prod_{i=0}^{s} b_{m+2i}) > 0,$$

the required. □

Proof of Theorem 6.11. We may suppose $-v(q) \leq v(a) < v(q)$ because the Bäcklund transformations defined in Definition 6.5 shift the parameter from a to $q^{2k}a$ for any $k \in \mathbb{Z}$. Assume q-$P(A_6)_a$ has a solution f algebraic over $C(t)$. Note that q is not a root of unity, which is obtained from $v(q) > 0$. Then it follows from Proposition 6.13 that

$$f = \frac{tP}{Q}, \quad P, Q \in C[t^2], \ t \nmid P \text{ and } t \nmid Q.$$

Put $g = f/t = P/Q$ and $x = t^2$. We have $P, Q \in C[x]$, $x \nmid P$, $x \nmid Q$ and

$$(qxg_1g - 1)((x/q)gg_{-1} - 1)(g+1) = axg. \tag{6.10}$$

Express g as

$$g = \sum_{i=0}^{\infty} a_i x^i, \quad a_i \in C, a_0 \neq 0.$$

Comparing the terms of degree 0 of equation (6.10), we obtain $a_0 = -1$.

We show the following for all $i \geq 1$,

$$a_i = aa_{i-1} - \sum_{l=1}^{i-1} \left\{ \sum_{j=0}^{l-2} \left(\sum_{k=0}^{j} a_k a_{j-k} q^{4k-2j} \right) \left(\sum_{k=0}^{l-2-j} a_k a_{l-2-j-k} \right) - \sum_{k=0}^{l-1} a_{l-1-k} a_k (q^{2k+1} + q^{-(2k+1)}) \right\} a_{i-l}.$$

Firstly we calculate

$$(qxg_1g - 1)((x/q)gg_{-1} - 1) = x^2 g_1 g^2 g_{-1} - xg(qg_1 + (1/q)g_{-1}) + 1.$$

We have

$$g_1 g_{-1} = \sum_{i=0}^{\infty} \left(\sum_{k=0}^{i} a_k a_{i-k} q^{4k-2i} \right) x^i,$$
$$g^2 = \sum_{i=0}^{\infty} \left(\sum_{k=0}^{i} a_k a_{i-k} \right) x^i,$$

and so

$$g_1 g^2 g_{-1} = \sum_{i=0}^{\infty} \left\{ \sum_{j=0}^{i} \left(\sum_{k=0}^{j} a_k a_{j-k} q^{4k-2j} \right) \left(\sum_{k=0}^{i-j} a_k a_{i-j-k} \right) \right\} x^i.$$

We also obtain

$$g(qg_1 + (1/q)g_{-1}) = \sum_{i=0}^{\infty} \left(\sum_{k=0}^{i} a_{i-k} a_k (q^{2k+1} + q^{-(2k+1)}) \right) x^i.$$

Secondly express

$$(qxg_1 g - 1)((x/q)gg_{-1} - 1) = \sum_{i=0}^{\infty} c_i x^i, \quad c_i \in C, c_0 = 1.$$

Then we obtain

$$(qxg_1 g - 1)((x/q)gg_{-1} - 1)(g+1) = \sum_{i=1}^{\infty} \left(a_i + \sum_{l=1}^{i-1} c_l a_{i-l} \right) x^i.$$

Therefore if we compare the terms of degree $i \geq 1$ of equation (6.10), we find

$$\begin{aligned} a_i &= a a_{i-1} - \sum_{l=1}^{i-1} c_l a_{i-l} \\ &= a a_{i-1} - \sum_{l=1}^{i-1} \left\{ \sum_{j=0}^{l-2} \left(\sum_{k=0}^{j} a_k a_{j-k} q^{4k-2j} \right) \left(\sum_{k=0}^{l-2-j} a_k a_{l-2-j-k} \right) \right. \\ &\qquad \left. - \sum_{k=0}^{l-1} a_{l-1-k} a_k (q^{2k+1} + q^{-(2k+1)}) \right\} a_{i-l}. \end{aligned}$$

Note that $a_i \in \mathbb{Q}[q, 1/q, a] \subset K$ for any $i \geq 0$. Put

$$n_i = \begin{cases} -\frac{i(i-1)}{2}v(q) + \frac{i}{2}v(a) & \text{if } 2 \mid i, \\ -\frac{i(i-1)}{2}v(q) + \frac{i+1}{2}v(a) & \text{if } 2 \nmid i \end{cases} \quad \text{for all } i \geq 0.$$

Then it follows from Lemma 6.14 that for any $i \geq 0$,

$$\begin{cases} v(a_i) \geq n_i & \text{if } 2 \mid i, \\ v(a_i) = n_i & \text{if } 2 \nmid i. \end{cases}$$

Choose a sequence $\{b_i\}_{i \geq 0} \subset K^\times$ such that $v(b_i) = n_i$. Let S_i denote the symmetric group of degree i. By Lemma 6.15 we find that for any $s \geq 0$, $m \geq 0$ satisfying $2 \nmid m$ and $\sigma \in S_{s+1} \setminus \{id\}$,

$$v(\prod_{i=0}^{s} a_{m+i+\sigma(i)}) \geq v(\prod_{i=0}^{s} b_{m+i+\sigma(i)}) > v(\prod_{i=0}^{s} b_{m+2i}) = v(\prod_{i=0}^{s} a_{m+2i}).$$

Therefore we obtain that for any $s \geq 0$ and $m \geq 0$ satisfying $2 \nmid m$,

$$\begin{aligned} v(F_g(m,s)) &= v\left(\begin{vmatrix} a_m & a_{m+1} & \cdots & a_{m+s} \\ a_{m+1} & a_{m+2} & \cdots & a_{m+1+s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m+s} & a_{m+s+1} & \cdots & a_{m+2s} \end{vmatrix}\right) \\ &= v\left(\sum_{\sigma \in S_{s+1}} (\operatorname{sgn} \sigma) \prod_{i=0}^{s} a_{m+i+\sigma(i)}\right) \\ &= v(\prod_{i=0}^{s} a_{m+2i}) \\ &= \sum_{i=0}^{s} n_{m+2i} \in \mathbb{Z}, \end{aligned}$$

where $F_g(m,s)$ is the Hankel determinant of g (refer to [2] for the Hankel determinant). This implies that for any $s \geq 0$ and $m \geq 0$ satisfying $2 \nmid m$, $F_g(m,s) \neq 0$, which contradicts $g \in C(x)$. $\square$

Proof of Corollary 6.12. Firstly we deal with the case, a is algebraic over $\mathbb{Q}(q)$. Put

$$\mathcal{O} = \{f/g \mid f, g \in \mathbb{Q}(q) \text{ and } q \nmid g\},$$

and let v be the normalized discrete valuation of $\mathbb{Q}(q)/\mathbb{Q}$ associated with $\mathcal{O}$. Then we have $v(q) = 1$. Choose a valuation ring $\mathcal{O}'$ of $\mathbb{Q}(q, a)/\mathbb{Q}$ such that $\mathcal{O}' \cap \mathbb{Q}(q) = \mathcal{O}$. Let v' be the normalized discrete valuation associated with $\mathcal{O}'$ and e the ramification index of $\mathcal{O}'$ over $\mathcal{O}$. We obtain $v'(q) = ev(q) = e \geq 1$. Therefore by Theorem 6.11 we conclude that q-$P(A_6)_a$ has no solution algebraic over $C(t)$.

Secondly we deal with the other case, a is transcendental over $\mathbb{Q}(q)$. In this case, q is transcendental over $\mathbb{Q}(a)$. Put

$$\mathcal{O} = \{f/g \mid f, g \in \mathbb{Q}(a)[q] \text{ and } q \nmid g\},$$

and let v be the normalized discrete valuation of $\mathbb{Q}(a, q)/\mathbb{Q}(a)$ associated with $\mathcal{O}$. Then we have $v(q) = 1$. By Theorem 6.11 we conclude that q-$P(A_6)_a$ has no solution algebraic over $C(t)$. □

Chapter 7

System of equations of birational form

In this chapter we study irreducibility of systems of difference equations of birational form, such as

$$y_2 y = \frac{A(y_1)}{B(y_1)}$$

and

$$\begin{cases} y_1 y = \dfrac{A(z)}{B(z)}, \\ z_1 z = \dfrac{C(y_1)}{D(y_1)}, \end{cases}$$

where A, B, C and D are polynomials.

Notation. Throughout this chapter a field is of characteristic zero.

7.1 Single equation of birational form

Lemma 7.1. *Let L be a field, $m, n \in \mathbb{Z}_{\geq 1}$ and $A, B, P, R, R', S, S' \in L[X] \setminus \{0\}$ polynomials over L such that A and B are relatively prime,*

$$\max\{\deg R, \deg R', \deg S, \deg S'\} \leq n,$$

$A^m R = PS$ and $B^m R' = PS'$. Then $\deg A \leq 2n/m$ *and* $\deg B \leq 2n/m$.

Proof. It is sufficient to prove that $\deg A \le 2n/m$. For polynomials $C, D \in L[X] \setminus \{0\}$ we let (C, D) denote the monic greatest common divisor of C and D. Put $C = (A^m, S)$. From $A^m R = PS$ we obtain

$$(A^m/C)R = P(S/C), \quad A^m/C, S/C \in L[X].$$

Since A^m/C and S/C are relatively prime, we find $(A^m/C) \mid P$, which implies

$$\begin{aligned}\deg(A^m, P) &\ge \deg \frac{A^m}{C} = m \deg A - \deg C \\ &\ge m \deg A - \deg S \ge m \deg A - n.\end{aligned}$$

We obtain $(A^m, P) \mid B^m R'$ from $B^m R' = PS'$ and $(A^m, P) \mid P$. Since (A^m, P) and B^m are relatively prime, we find $(A^m, P) \mid R'$, which implies

$$\deg(A^m, P) \le \deg R' \le n.$$

Therefore we conclude that $\deg A \le 2n/m$. □

Proposition 7.2. *Let $\mathcal{L} = (L, \tau)$ be an inversive difference field and f a solution of the equation over $\mathcal{L}$,*

$$B(y_1)y_2y = A(y_1),$$

where $A, B \in L[X] \setminus \{0\}$ are polynomials over L such that A and B are relatively prime, B monic and $\max\{\deg A, \deg B\} > 2$. *Then it follows that*

$$\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} \le 1 \Rightarrow f \text{ is algebraic over } L.$$

Proof. To obtain $\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} \ne 1$ we assume $\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} = 1$. Then f_i is transcendental over L for any $i \ge 0$. Since it follows that $\operatorname{tr.deg} L(f, f_1)/L = 1$, there exists an irreducible polynomial F over L,

$$F = \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} a_{ij} Y^i Y_1^j \in L[Y, Y_1] \setminus \{0\}, \quad a_{ij} \in L,$$

such that $F(f, f_1) = 0$, $n_0 = \deg_Y F \ge 1$, $n_1 = \deg_{Y_1} F \ge 1$ and $a_{n_0 n_1} \in \{0, 1\}$. Put

$$\begin{aligned}F_1 &= (YB(Y_1))^{n_1} F^* \left(Y_1, \frac{A(Y_1)}{YB(Y_1)}\right), \\ F_0 &= (Y_1 B(Y))^{n_0} F \left(\frac{A(Y)}{Y_1 B(Y)}, Y\right),\end{aligned}$$

where $F^* = \sum_{i=0}^{n_0}\sum_{j=0}^{n_1}\tau(a_{ij})Y^iY_1^j$. It is seen that $F_1, F_0 \in L[Y, Y_1]\setminus\{0\}$. We find

$$\begin{aligned} F_1(f, f_1) &= (fB(f_1))^{n_1}F^*\left(f_1, \frac{A(f_1)}{fB(f_1)}\right) \\ &= (fB(f_1))^{n_1}F^*(f_1, f_2) = 0 \end{aligned}$$

and

$$\begin{aligned} F_0(f_1, f_2) &= (f_2B(f_1))^{n_0}F\left(\frac{A(f_1)}{f_2B(f_1)}, f_1\right) \\ &= (f_2B(f_1))^{n_0}F(f, f_1) = 0, \end{aligned}$$

which imply $F \mid F_1$ and $F^* \mid F_0$. Therefore we obtain

$$n_0 = \deg_Y F \le \deg_Y F_1 \le n_1 = \deg_{Y_1} F^* \le \deg_{Y_1} F_0 \le n_0,$$

and so $n_0 = n_1$. Put $n = n_0 = n_1 \ge 1$.

Let $P \in L[Y, Y_1]\setminus\{0\}$ be a polynomial satisfying $F_1 = PF$. We find $P \in L[Y_1]$ by $\deg_Y P = \deg_Y F_1 - \deg_Y F = 0$. We have

$$\begin{aligned} F_1 &= (YB(Y_1))^n\sum_{i=0}^{n}\sum_{j=0}^{n}\tau(a_{ij})Y_1^i\left(\frac{A(Y_1)}{YB(Y_1)}\right)^j \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n}\tau(a_{ij})Y_1^i(YB(Y_1))^{n-j}A(Y_1)^j \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n}\tau(a_{i,n-j})Y_1^iA(Y_1)^{n-j}B(Y_1)^jY^j \\ &= \sum_{j=0}^{n}\left\{A(Y_1)^{n-j}B(Y_1)^j\sum_{i=0}^{n}\tau(a_{i,n-j})Y_1^i\right\}Y^j \end{aligned}$$

and

$$PF = P\sum_{i=0}^{n}\sum_{j=0}^{n}a_{ij}Y^iY_1^j = P\sum_{j=0}^{n}\sum_{i=0}^{n}a_{ji}Y^jY_1^i = \sum_{j=0}^{n}\left\{P\sum_{i=0}^{n}a_{ji}Y_1^i\right\}Y^j.$$

From $F_1 = PF$ we obtain

$$A(Y_1)^n \sum_{i=0}^{n} \tau(a_{in}) Y_1^i = P \sum_{i=0}^{n} a_{0i} Y_1^i \quad (\neq 0), \tag{7.1}$$

$$B(Y_1)^n \sum_{i=0}^{n} \tau(a_{i0}) Y_1^i = P \sum_{i=0}^{n} a_{ni} Y_1^i \quad (\neq 0). \tag{7.2}$$

By Lemma 7.1 we find $\deg A \leq 2$ and $\deg B \leq 2$, which imply

$$\max\{\deg A, \deg B\} \leq 2,$$

a contradiction. Therefore we conclude that $\operatorname{tr.deg} \mathcal{L}\langle f \rangle / \mathcal{L} \neq 1$, which yields

$$\operatorname{tr.deg} \mathcal{L}\langle f \rangle / \mathcal{L} \leq 1 \Rightarrow f \text{ is algebraic over } L,$$

the required. □

Theorem 7.3. *Let $\mathcal{K}$ be an almost inversive difference field, $\mathcal{N}$ a decomposable extension of $\mathcal{K}$ and $f \in N$ a solution in $\mathcal{N}$ of the equation over $\mathcal{K}$,*

$$B(y_1) y_2 y = A(y_1),$$

where $A, B \in K[X] \setminus \{0\}$ are polynomials over K such that A and B are relatively prime, B monic and $\max\{\deg A, \deg B\} > 2$. *Then f is algebraic over K.*

Proof. Let $\mathcal{L}$ be an inversive difference overfield of $\mathcal{K}$ and $\mathcal{U}$ a difference overfield of $\mathcal{L}$ with $\mathcal{K}\langle f \rangle_{\mathcal{N}} \subset \mathcal{U}$. Then by Proposition 7.2 we obtain

$$\operatorname{tr.deg} \mathcal{L}\langle f \rangle_{\mathcal{U}} / \mathcal{L} \leq 1 \Rightarrow f \text{ is algebraic over } L.$$

Therefore we find that f is algebraic over K by Lemma 4.10. □

7.2 System of two equations of birational form

Lemma 7.4. *Let $\mathcal{L} = (L, \tau)$ be an inversive difference field and $(y, z) = (f, g)$ a solution of the system of equations over $\mathcal{L}$,*

$$\begin{cases} B(z) y_1 y = A(z), \\ D(y_1) z_1 z = C(y_1), \end{cases}$$

where $A, B, C, D \in L[X] \setminus \{0\}$ *are polynomials over* L *such that* A *and* B *are relatively prime,* C *and* D *relatively prime,* B *and* D *monic,* $\deg AB \geq 1$ *and* $\deg CD \geq 1$. *Then*

$$\mathrm{tr.deg}\, \mathcal{L}\langle f\rangle/\mathcal{L} = 1 \Leftrightarrow \mathrm{tr.deg}\, \mathcal{L}\langle g\rangle/\mathcal{L} = 1 \Leftrightarrow \mathrm{tr.deg}\, \mathcal{L}\langle f, g\rangle/\mathcal{L} = 1.$$

If we suppose $\mathrm{tr.deg}\, \mathcal{L}\langle f, g\rangle/\mathcal{L} = 1$ *then we find that there are polynomials over* L *with indeterminates* Y, Z,

$$F = \sum_{i=0}^{n_0}\sum_{j=0}^{n_1} \alpha_{ij} Y^i Z^j \in L[Y, Z] \setminus \{0\}, \quad \alpha_{ij} \in L,$$
$$G = \sum_{i=0}^{n_0}\sum_{j=0}^{n_1} \beta_{ij} Y^i Z^j \in L[Y, Z] \setminus \{0\}, \quad \beta_{ij} \in L,$$

$P \in L[Z] \setminus \{0\}$ *and* $Q \in L[Y] \setminus \{0\}$ *such that* $F(f, g) = G(f_1, g) = 0$, *both* F *and* G *irreducible,*

$$n_0 = \deg_Y F = \deg_Y G \geq 1,$$
$$n_1 = \deg_Z F = \deg_Z G \geq 1,$$
$$\alpha_{n_0 n_1}, \beta_{n_0 n_1} \in \{0, 1\},$$

$$\sum_{i=0}^{n_0}\left\{A(Z)^{n_0-i}B(Z)^i \sum_{j=0}^{n_1} \alpha_{n_0-i,j} Z^j\right\} Y^i = \sum_{i=0}^{n_0}\left\{P \sum_{j=0}^{n_1} \beta_{ij} Z^j\right\} Y^i \tag{7.3}$$

and

$$\sum_{j=0}^{n_1}\left\{C(Y)^{n_1-j}D(Y)^j \sum_{i=0}^{n_0} \beta_{i,n_1-j} Y^i\right\} Z^j = \sum_{j=0}^{n_1}\left\{Q \sum_{i=0}^{n_0} \tau(\alpha_{ij}) Y^i\right\} Z^j. \tag{7.4}$$

Proof. (1) Firstly, we prove

$$\mathrm{tr.deg}\, \mathcal{L}\langle f\rangle/\mathcal{L} = 1 \Leftrightarrow \mathrm{tr.deg}\, \mathcal{L}\langle f, g\rangle/\mathcal{L} = 1.$$

g is a zero of the polynomial $f_1 f B(X) - A(X) \in L(f, f_1)[X]$ because $B(g) f_1 f = A(g)$. If we assume $f_1 f B(X) - A(X) = 0$ then we find that $A(X)$ and $B(X)$ has a common divisor in $L(f, f_1)[X]$, a contradiction. Therefore we have

$f_1 f B(X) - A(X) \neq 0$, which implies that g is algebraic over $L(f, f_1)$. We obtain the required from

$$\begin{aligned}\operatorname{tr.deg} \mathcal{L}\langle f, g\rangle/\mathcal{L} &= \operatorname{tr.deg} \mathcal{L}\langle f, g\rangle/\mathcal{L}\langle f\rangle + \operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} \\ &= \operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L}.\end{aligned}$$

(2) Secondly, we prove

$$\operatorname{tr.deg} \mathcal{L}\langle g\rangle/\mathcal{L} = 1 \Leftrightarrow \operatorname{tr.deg} \mathcal{L}\langle f, g\rangle/\mathcal{L} = 1.$$

f_1 is a zero of the polynomial $g_1 g D(X) - C(X) \in L(g, g_1)[X]$ because $D(f_1) g_1 g = C(f_1)$. If we assume $g_1 g D(X) - C(X) = 0$ then we find that $C(X)$ and $D(X)$ has a common divisor in $L(g, g_1)[X]$, a contradiction. Therefore we have $g_1 g D(X) - C(X) \neq 0$, which implies that f_1 is algebraic over $L(g, g_1)$.

We may suppose that f is transcendental over L because we have

$$\operatorname{tr.deg} \mathcal{L}\langle f, g\rangle/\mathcal{L} = \operatorname{tr.deg} \mathcal{L}\langle f, g\rangle/\mathcal{L}\langle g\rangle + \operatorname{tr.deg} \mathcal{L}\langle g\rangle/\mathcal{L}.$$

Since $\mathcal{L}$ is inversive, we find that f_1 is also transcendental over L, which implies that g is transcendental over L. Then from $B(g) f_1 f = A(g)$ we obtain

$$f = \frac{A(g)}{B(g) f_1} \in L(f_1, g),$$

and so

$$\begin{aligned}\operatorname{tr.deg} \mathcal{L}\langle f, g\rangle/\mathcal{L} &= \operatorname{tr.deg} \mathcal{L}\langle f_1, g\rangle/\mathcal{L} \\ &= \operatorname{tr.deg} \mathcal{L}\langle f_1, g\rangle/\mathcal{L}\langle g\rangle + \operatorname{tr.deg} \mathcal{L}\langle g\rangle/\mathcal{L} \\ &= \operatorname{tr.deg} \mathcal{L}\langle g\rangle/\mathcal{L},\end{aligned}$$

which yields the required.

(3) Finally we suppose $\operatorname{tr.deg} \mathcal{L}\langle f, g\rangle/\mathcal{L} = 1$. By (1) and (2) we find that f_i and g_i are transcendental over L for all $i \geq 0$, where note that $\mathcal{L}$ is inversive. Since it follows that $\operatorname{tr.deg} L(f, g)/L = 1$, we find that there exists an irreducible polynomial $F \in L[Y, Z] \setminus \{0\}$ over L,

$$F = \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \alpha_{ij} Y^i Z^j, \quad \alpha_{ij} \in L,$$

such that $F(f,g) = 0$, $n_0 = \deg_Y F \geq 1$, $n_1 = \deg_Z F \geq 1$, and $\alpha_{n_0 n_1} \in \{0,1\}$. By $\operatorname{tr.deg} L(f_1, g)/L = 1$ there exists an irreducible polynomial $G \in L[Y,Z] \setminus \{0\}$,

$$G = \sum_{i=0}^{n_2} \sum_{j=0}^{n_3} \beta_{ij} Y^i Z^j, \quad \beta_{ij} \in L,$$

such that $G(f_1, g) = 0$, $n_2 = \deg_Y G$, $n_3 = \deg_Z G$ and $\beta_{n_2 n_3} \in \{0,1\}$.

For any $P = \sum_i p_i X^i \in L[X]$ we define P^* as $P^* = \sum_i \tau(p_i) X^i$, and for any $P = \sum_{i,j} p_{ij} Y^i Z^j$, we define P^* as $P^* = \sum_{i,j} \tau(p_{ij}) Y^i Z^j$. Put

$$F_1 = \{ZD(Y)\}^{n_1} F^* \left(Y, \frac{C(Y)}{ZD(Y)}\right) \in L[Y,Z] \setminus \{0\},$$
$$G_1 = \{YB^*(Z)\}^{n_2} G^* \left(\frac{A^*(Z)}{YB^*(Z)}, Z\right) \in L[Y,Z] \setminus \{0\}.$$

Then we have

$$\begin{aligned} F_1(f_1, g) &= \{gD(f_1)\}^{n_1} F^* \left(f_1, \frac{C(f_1)}{gD(f_1)}\right) \\ &= \{gD(f_1)\}^{n_1} F^*(f_1, g_1) = 0 \end{aligned}$$

and

$$\begin{aligned} G_1(f_1, g_1) &= \{f_1 B^*(g_1)\}^{n_2} G^* \left(\frac{A^*(g_1)}{f_1 B^*(g_1)}, g_1\right) \\ &= \{f_1 B^*(g_1)\}^{n_2} G^*(f_2, g_1) = 0, \end{aligned}$$

which imply $G \mid F_1$ and $F^* \mid G_1$ respectively. Put

$$F_0 = \{YB(Z)\}^{n_0} F \left(\frac{A(Z)}{YB(Z)}, Z\right) \in L[Y,Z] \setminus \{0\},$$
$$G_0 = \{ZD(Y)\}^{n_3} G \left(Y, \frac{C(Y)}{ZD(Y)}\right) \in L[Y,Z] \setminus \{0\}.$$

Then we have

$$\begin{aligned} F_0(f_1, g) &= \{f_1 B(g)\}^{n_0} F \left(\frac{A(g)}{f_1 B(g)}, g\right) \\ &= \{f_1 B(g)\}^{n_0} F(f, g) = 0 \end{aligned}$$

and

$$G_0(f_1, g_1) = \{g_1 D(f_1)\}^{n_3} G\left(f_1, \frac{C(f_1)}{g_1 D(f_1)}\right)$$
$$= \{g_1 D(f_1)\}^{n_3} G(f_1, g) = 0,$$

which imply $G \mid F_0$ and $F^* \mid G_0$ respectively. Therefore we find $n_0 = n_2$ and $n_1 = n_3$ by

$$n_0 = \deg_Y F^* \leq \deg_Y G_1 \leq n_2 = \deg_Y G \leq \deg_Y F_0 \leq n_0$$

and

$$n_1 = \deg_Z F^* \leq \deg_Z G_0 \leq n_3 = \deg_Z G \leq \deg_Z F_1 \leq n_1.$$

Let $P, Q \in L[Y, Z] \setminus \{0\}$ be polynomials such that $F_0 = PG$ and $G_0 = QF^*$. Since we have

$$\deg_Y P = \deg_Y F_0 - \deg_Y G = 0,$$

and

$$\deg_Z Q = \deg_Z G_0 - \deg_Z F^* = 0,$$

we obtain $P \in L[Z]$ and $Q \in L[Y]$. Calculate F_0 and PG as follows,

$$\begin{aligned}
F_0 &= \{YB(Z)\}^{n_0} \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \alpha_{ij} \left(\frac{A(Z)}{YB(Z)}\right)^i Z^j \\
&= \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \alpha_{ij} A(Z)^i Z^j (YB(Z))^{n_0-i} \\
&= \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \alpha_{n_0-i,j} A(Z)^{n_0-i} Z^j (YB(Z))^i \\
&= \sum_{i=0}^{n_0} \left\{ A(Z)^{n_0-i} B(Z)^i \sum_{j=0}^{n_1} \alpha_{n_0-i,j} Z^j \right\} Y^i,
\end{aligned}$$

$$PG = P \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \beta_{ij} Y^i Z^j = \sum_{i=0}^{n_0} \left\{ P \sum_{j=0}^{n_1} \beta_{ij} Z^j \right\} Y^i.$$

Then we obtain the equation (7.3). To obtain the equation (7.4) we calculate G_0 and QF^* as follows,

$$\begin{aligned}
G_0 &= \{ZD(Y)\}^{n_1} \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \beta ij Y^i \left(\frac{C(Y)}{ZD(Y)} \right)^j \\
&= \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \beta_{ij} Y^i C(Y)^j (ZD(Y))^{n_1 - j} \\
&= \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \beta_{i,n_1-j} Y^i C(Y)^{n_1-j} (ZD(Y))^j \\
&= \sum_{j=0}^{n_1} \left\{ C(Y)^{n_1-j} D(Y)^j \sum_{i=0}^{n_0} \beta_{i,n_1-j} Y^i \right\} Z^j,
\end{aligned}$$

$$QF^* = Q \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \tau(\alpha_{ij}) Y^i Z^j = \sum_{j=0}^{n_1} \left\{ Q \sum_{i=0}^{n_0} \tau(\alpha_{ij}) Y^i \right\} Z^j.$$

□

Proposition 7.5. *Let $\mathcal{L} = (L, \tau)$ be an inversive difference field and $(y, z) = (f, g)$ be a solution of the system of equations over $\mathcal{L}$,*

$$\begin{cases} B(z) y_1 y = A(z), \\ D(y_1) z_1 z = C(y_1), \end{cases}$$

where $A, B, C, D \in L[X] \setminus \{0\}$ are polynomials over L such that A and B are relatively prime, C and D relatively prime, B and D monic and

$$\max\{\deg A, \deg B\} \cdot \max\{\deg C, \deg D\} > 4.$$

Then it follows that

$$\operatorname{tr.deg} \mathcal{L}\langle f, g \rangle / \mathcal{L} \leq 1 \Rightarrow f \text{ and } g \text{ are algebraic over } L.$$

Proof. To obtain $\operatorname{tr.deg} \mathcal{L}\langle f, g \rangle / \mathcal{L} \neq 1$ we assume $\operatorname{tr.deg} \mathcal{L}\langle f, g \rangle / \mathcal{L} = 1$. By Lemma 7.4 there exist polynomials $F, G, P, Q \in L[Y, Z] \setminus \{0\}$ over L such that

$$\begin{aligned}
F &= \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \alpha_{ij} Y^i Z^j, \quad \alpha_{ij} \in L, \\
G &= \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \beta_{ij} Y^i Z^j, \quad \beta_{ij} \in L,
\end{aligned}$$

$P \in L[Z]$, $Q \in L[Y]$, $F(f, g) = G(f_1, g) = 0$, F and G are irreducible,

$$n_0 = \deg_Y F = \deg_Y G \geq 1,$$
$$n_1 = \deg_Z F = \deg_Z G \geq 1,$$

$\alpha_{n_0 n_1}, \beta_{n_0 n_1} \in \{0, 1\}$,

$$(7.5) \qquad \sum_{i=0}^{n_0} \left\{ A(Z)^{n_0 - i} B(Z)^i \sum_{j=0}^{n_1} \alpha_{n_0 - i, j} Z^j \right\} Y^i = \sum_{i=0}^{n_0} \left\{ P \sum_{j=0}^{n_1} \beta_{ij} Z^j \right\} Y^i$$

and

$$(7.6) \quad \sum_{j=0}^{n_1} \left\{ C(Y)^{n_1 - j} D(Y)^j \sum_{i=0}^{n_0} \beta_{i, n_1 - j} Y^i \right\} Z^j = \sum_{j=0}^{n_1} \left\{ Q \sum_{i=0}^{n_0} \tau(\alpha_{ij}) Y^i \right\} Z^j.$$

From the equation (7.5) we obtain the following two equations,

$$(7.7) \qquad A(Z)^{n_0} \sum_{j=0}^{n_1} \alpha_{n_0 j} Z^j = P \sum_{j=0}^{n_1} \beta_{0j} Z^j \quad (\neq 0),$$

$$(7.8) \qquad B(Z)^{n_0} \sum_{j=0}^{n_1} \alpha_{0j} Z^j = P \sum_{j=0}^{n_1} \beta_{n_0 j} Z^j \quad (\neq 0).$$

From the equation (7.6) we obtain the following two equations,

$$(7.9) \qquad C(Y)^{n_1} \sum_{i=0}^{n_0} \beta_{i n_1} Y^i = Q \sum_{i=0}^{n_0} \tau(\alpha_{i0}) Y^i \quad (\neq 0),$$

$$(7.10) \qquad D(Y)^{n_1} \sum_{i=0}^{n_0} \beta_{i0} Y^i = Q \sum_{i=0}^{n_0} \tau(\alpha_{i n_1}) Y^i \quad (\neq 0).$$

By Lemma 7.1 we find that

$$\deg A \leq \frac{2n_1}{n_0}, \qquad \deg B \leq \frac{2n_1}{n_0},$$
$$\deg C \leq \frac{2n_0}{n_1}, \qquad \deg D \leq \frac{2n_0}{n_1},$$

which imply

$$\max\{\deg A, \deg B\} \cdot \max\{\deg C, \deg D\} \leq \frac{2n_1}{n_0} \cdot \frac{2n_0}{n_1} = 4,$$

a contradiction. Therefore we conclude $\operatorname{tr.deg} \mathcal{L}\langle f, g\rangle/\mathcal{L} \neq 1$, which yields

$$\operatorname{tr.deg} \mathcal{L}\langle f, g\rangle/\mathcal{L} \leq 1 \Rightarrow f \text{ and } g \text{ are algebraic over } L,$$

the required. □

Theorem 7.6. *Let $\mathcal{K}$ be an almost inversive difference field, $\mathcal{N}$ a decomposable extension of $\mathcal{K}$ and $(y, z) = (f, g)$ a solution in $\mathcal{N}$ of the system of equations over $\mathcal{K}$,*

$$\begin{cases} B(z)y_1 y = A(z), \\ D(y_1)z_1 z = C(y_1), \end{cases}$$

where $A, B, C, D \in K[X] \setminus \{0\}$ are polynomials over K such that A and B are relatively prime, C and D relatively prime, B and D monic and

$$\max\{\deg A, \deg B\} \cdot \max\{\deg C, \deg D\} > 4.$$

Then f and g are algebraic over K.

Proof. Let $\mathcal{L}$ be an inversive difference overfield of $\mathcal{K}$ and $\mathcal{U}$ a difference overfield of $\mathcal{L}$ with $\mathcal{K}\langle f, g\rangle_{\mathcal{N}} \subset \mathcal{U}$. By Proposition 7.5 we obtain

$$\operatorname{tr.deg} \mathcal{L}\langle f, g\rangle_{\mathcal{U}}/\mathcal{L} \leq 1 \Rightarrow f \text{ and } g \text{ are algebraic over } L.$$

Therefore by Lemma 4.10 we conclude that f and g are algebraic over K. □

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