On the moduli spaces of finite flat models of Galois representations

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Introduction

This paper is a doctor thesis of the author. In this paper, we study the moduli spaces of finite flat models of 2-dimensional local Galois representations over finite fields.

First, we explain the moduli space of finite flat models. Let K be a p-adic field for $p > 2$. We consider a two-dimensional continuous representation $V_{\mathbb{F}}$ of the absolute Galois group G_K over a finite field $\mathbb{F}$ of characteristic p. By a finite flat model of $V_{\mathbb{F}}$, we mean a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$, equipped with an action of $\mathbb{F}$, and an isomorphism $V_{\mathbb{F}} \stackrel{\sim}{\longrightarrow} \mathcal{G}(\overline{K})$ that respects the action of G_K and $\mathbb{F}$. We assume that $V_{\mathbb{F}}$ has at least one finite flat model. Then there exists a moduli space of finite flat models of $V_{\mathbb{F}}$, which is projective scheme over $\mathbb{F}$, and we denoted it by $\mathscr{GR}_{V_{\mathbb{F}},0}$.

In the section 1, we recall the moduli space of finite flat models, and explain a relationship between a local deformation ring and the moduli space of finite flat models. Then we explain a conjecture by Kisin on the connected component of the moduli space of finite flat models.

In the section 2, we study the connected components of the moduli space of finite flat models. The projective scheme $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ over $\mathbb{F}$ is the moduli of finite flat models of $V_{\mathbb{F}}$ with some determinant condition. From the viewpoint of the application to the modularity problem, we are interested in the connected components of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$.

In the subsention 1, we prove some preliminary lemmas. In the subsection 2, we prove the Kisin conjecture. The statement is the following.

Theorem. *Let $\mathbb{F}'$ be a finite extension of $\mathbb{F}$. Suppose $x_1, x_2 \in \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F}')$ correspond to objects $\mathfrak{M}_{1,\mathbb{F}'}, \mathfrak{M}_{2,\mathbb{F}'}$ of $(\mathrm{Mod}/\mathfrak{S})_{\mathbb{F}'}$ respectively. If $\mathfrak{M}_{1,\mathbb{F}'}$ and $\mathfrak{M}_{2,\mathbb{F}'}$ are both non-ordinary, then x_1 and x_2 lie on the same connected component of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$.*

When K is totally ramified over $\mathbb{Q}_p$, this was proved in [Kis]. If the residue field of K is bigger than $\mathbb{F}_p$, the situation changes greatly because $\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathbb{F}$ can be split into a direct product. When K is a general p-adic field, the case of $V_{\mathbb{F}}$ being the trivial representation was treated in [Gee].

In the subsection 3, as an application to global Galois representations, we prove a theorem on the modularity, which states that a deformation ring is isomorphic to a Hecke ring up to p-power torsion kernel. This completes Kisin's theory for GL_2.

In the section 2, we study the dimension of moduli space of finite flat models. Let e be the ramification index of K over $\mathbb{Q}_p$, and k be the residue field of K. We consider a two-dimensional continuous representation $V_{\mathbb{F}}$ of the absolute Galois group G_K over a finite field $\mathbb{F}$ of characteristic p. We assume that $V_{\mathbb{F}}$ has at least one finite flat model. If $e < p-1$, the finite flat model of $V_{\mathbb{F}}$ is unique by Raynaud's result [Ray, Theorem 3.3.3]. In general, there are finitely many finite flat models of $V_{\mathbb{F}}$, and these appear as the $\mathbb{F}$-rational points of $\mathscr{GR}_{V_{\mathbb{F}},0}$. It is natural to ask about the dimension of $\mathscr{GR}_{V_{\mathbb{F}},0}$. In this section, we determine the type of the zeta functions and the range of the dimensions of the moduli spaces. The main theorem of this section is the following.

Theorem. *Let* $d_{V_{\mathbb{F}}} = \dim \mathscr{GR}_{V_{\mathbb{F}},0}$, *and* $Z(\mathscr{GR}_{V_{\mathbb{F}},0}; T)$ *be the zeta function of* $\mathscr{GR}_{V_{\mathbb{F}},0}$. *We put* $n = [k : \mathbb{F}_p]$. *Then followings are true.*

1. *After extending the field* $\mathbb{F}$ *sufficiently, we have*

$$Z(\mathscr{GR}_{V_{\mathbb{F}},0}; T) = \prod_{i=0}^{d_{V_{\mathbb{F}}}} (1 - |\mathbb{F}|^i T)^{-m_i}$$

   for some $m_i \in \mathbb{Z}$ *such that* $m_{d_{V_{\mathbb{F}}}} > 0$.

2. *If* $n = 1$, *we have*

$$0 \leq d_{V_{\mathbb{F}}} \leq \left[\frac{e+2}{p+1}\right].$$

   If $n \geq 2$, *we have*

$$0 \leq d_{V_{\mathbb{F}}} \leq \left[\frac{n+1}{2}\right]\left[\frac{e}{p+1}\right] + \left[\frac{n-2}{2}\right]\left[\frac{e+1}{p+1}\right] + \left[\frac{e+2}{p+1}\right].$$

   Here, $[x]$ *is the greatest integer less than or equal to* x *for* $x \in \mathbb{R}$.

   Furthermore, each equality in the above inequalities can happen for any finite extension K *of* $\mathbb{Q}_p$.

Raynaud's result says that if $e < p-1$ then $\mathscr{GR}_{V_{\mathbb{F}},0}$ is one point, that is, zero-dimensional and connected. If $e < p-1$, the above theorem also implies that $\mathscr{GR}_{V_{\mathbb{F}},0}$ is zero-dimensional. So it gives a dimensional generalization of Raynaud's result for two-dimensional Galois representations. The connectedness of $\mathscr{GR}_{V_{\mathbb{F}},0}$ is completely false in general. For example, we can check that if $K = \mathbb{Q}_p(\zeta_p)$ and $V_{\mathbb{F}}$ is trivial representations then $\mathscr{GR}_{V_{\mathbb{F}},0}$ consists of $\mathbb{P}^1_{\mathbb{F}}$ and two points (c.f. [Kis, Proposition 2.5.15(2)]). Here $\mathbb{P}^1_{\mathbb{F}}$ denotes the 1-dimensional projective space over $\mathbb{F}$.

In the subsection 1, we prove some Lemmas, and give an example for any K where the moduli space of finite flat models is one point.

A proof of the main theorem separates into two cases, that is, the case where $V_{\mathbb{F}}$ is not absolutely irreducible and the case where $V_{\mathbb{F}}$ is absolutely irreducible. In the subsection 2, we treat the case where $V_{\mathbb{F}}$ is not absolutely irreducible. In this case, we decompose $\mathscr{GR}_{V_{\mathbb{F}},0}$ into affine spaces in the level of rational points. Then we express the dimensions of these affine spaces explicitly and bound it by combinatorial arguments. In the subsection 3, we treat the case where $V_{\mathbb{F}}$ is absolutely irreducible. A proof is similar to the case where $V_{\mathbb{F}}$ is not absolutely irreducible, but, in this case, we have to decompose $\mathscr{G}\mathscr{R}_{V_{\mathbb{F}},0}$ into $\mathbb{A}_{\mathbb{F}}^{d}$ and $\mathbb{A}_{\mathbb{F}}^{d-1} \times \mathbb{G}_m$ and $\mathbb{A}_{\mathbb{F}}^{d-2} \times \mathbb{G}_m^2$ in the level of rational points. Here $\mathbb{A}_{\mathbb{F}}^{d}$ denotes the d-dimensional affine space over $\mathbb{F}$, and $\mathbb{G}_m$ is $\mathbb{A}_{\mathbb{F}}^{1} - \{0\}$.

In the subsection 4, we state the main theorem and prove it by collecting the results of former sections.

In the section 4, we study the rational points of moduli space of finite flat models. In this section, we assume that K is totally ramified of degree e over $\mathbb{Q}_p$, and $V_{\mathbb{F}}$ is the two-dimensional trivial representation of G_K over F.

We consider the constant group scheme $C_{\mathbb{F}}$ over $\operatorname{Spec} K$ of the two-dimensional vector space over $\mathbb{F}$. Let $M(C_{\mathbb{F}}, K)$ be the set of the isomorphism class of the finite flat models of $C_{\mathbb{F}}$. If $e < p-1$, then $M(C_{\mathbb{F}}, K)$ is one-point set by [Ray, Theorem 3.3.3]. However, if the ramification is big, there are surprisingly many finite flat models. In this section, we calculate the number of the isomorphism class of the finite flat models of $C_{\mathbb{F}}$, that is, $|M(C_{\mathbb{F}}, K)|$. The main theorem of this section is the following.

Theorem. *Let q be the cardinality of $\mathbb{F}$. Then we have*

$$|M(C_{\mathbb{F}}, K)| = \sum_{n \geq 0} (a_n + a'_n) q^n.$$

Here a_n and a'_n are defined as in the following.

We express e and n by

$$e = (p-1)e_0 + e_1, \;\; n = (p-1)n_0 + n_1 = (p-1)n'_0 + n'_1 + e_1$$

such that $e_0, n_0, n'_0 \in \mathbb{Z}$ and $0 \leq e_1, n_1, n'_1 \leq p-2$. Then

$$\begin{aligned} a_n &= \max\bigl\{e_0 - (p+1)n_0 - n_1 - 1, 0\bigr\} && \textit{if } n_1 \neq 0, 1, \\ a_n &= \max\bigl\{e_0 - (p+1)n_0 - n_1 - 1, 0\bigr\} \\ &\quad + \max\bigl\{e_0 - (p+1)n_0 - n_1 + 1, 0\bigr\} && \textit{if } n_1 = 0, 1, \end{aligned}$$

and

$$\begin{aligned} a'_n &= \max\bigl\{e_0 - e_1 - (p+1)n'_0 - n'_1 - 2, 0\bigr\} && \textit{if } n'_1 \neq 0, 1, \\ a'_n &= \max\bigl\{e_0 - e_1 - (p+1)n'_0 - n'_1 - 2, 0\bigr\} \\ &\quad + \max\bigl\{e_0 - e_1 - (p+1)n'_0 - n'_1, 0\bigr\} && \textit{if } n'_1 = 0, 1 \end{aligned}$$

except in the case where $n = 0$ and $e_1 = p-2$, in which case we put $a'_0 = e_0$.

In the above theorem, we can easily check that $|M(C_{\mathbb{F}}, K)| = 1$ if $e < p-1$.

Notation

Throughout this paper, we use the following notation. Let $p > 2$ be a prime number, and k be the finite field of cardinality $q = p^n$. For a positive number m, the finite field of cardinality p^m is denoted by $\mathbb{F}_{p^m}$. For a ring R, the ring of Witt vectors over R with respect to p is denoted by $W(R)$. Let K_0 be the quotient field of $W(k)$, and K be a totally ramified extension of K_0 of degree e. The ring of integers of K is denoted by $\mathcal{O}_K$, and the absolute Galois group of K is denoted by G_K. Let I_K be the inertia group of the absolute Galois group G_K, and Fr_q be the q-th power Frobenius of the absolute Galois group G_k. Let $\mathbb{F}$ be a finite field of characteristic p. For a ring A, the formal power series ring of u over A is denoted by $A[[u]]$, and we put $A((u)) = A[[u]](1/u)$. For a field F, the algebraic closure of F is denoted by $\overline{F}$ and the separable closure of F is denoted by F^{sep}. Let v_u be the valuation of $\mathbb{F}((u))$ normalized by $v_u(u) = 1$, and we put $v_u(0) = \infty$. For a local ring A, the maximal ideal of A is denoted by $\mathfrak{m}_A$. For a topological space X, the set of connected components of X is denoted by $\pi_0(X)$. For $x \in \mathbb{R}$, the greatest integer less than or equal to x is denoted by $[x]$. For a positive integer d, the d-dimensional affine space over $\mathbb{F}$ is denoted by $\mathbb{A}^d_{\mathbb{F}}$. Let $\mathbb{G}_m$ be $\mathbb{A}^1_{\mathbb{F}} - \{0\}$.

1 Deformation ring and moduli space of finite flat models

In this section, we explain the relationship between a deformation ring and a moduli space of finite flat models.

First, we are going to introduce a deformation ring. Let $V_{\mathbb{F}}$ be a two-dimensional continuous G_K-representation over $\mathbb{F}$ with a fixed ordered basis. A G_K-representation over a finite ring is said to be flat if and only if it is isomorphic to the generic fiber of a finite flat group scheme over $\mathcal{O}_K$ as a G_K-module. We assume that $V_{\mathbb{F}}$ is flat. Let $\mathfrak{AR}_{W(\mathbb{F})}$ be the category of Artin local finite $W(\mathbb{F})$-algebra A whose residue field is isomorphic to $\mathbb{F}$ as a $W(\mathbb{F})$-algebra. To define a deformation, we use a notion of groupoids. For the notion of groupoids, please consult [Kis, Appendix on groupoids]. The framed flat deformation $D^{\mathrm{fl},\square}_{V_{\mathbb{F}}}$ of $V_{\mathbb{F}}$ over $\mathfrak{AR}_{W(\mathbb{F})}$ is a groupoid $D^{\mathrm{fl},\square}_{V_{\mathbb{F}}}$ over $\mathfrak{AR}_{W(\mathbb{F})}$ determined as in the followings:

- For an object A in $\mathfrak{AR}_{W(\mathbb{F})}$, an object of $D^{\mathrm{fl},\square}_{V_{\mathbb{F}}}(A)$ is a triple (V_A, ψ, β), where V_A is a flat continuous G_K-representation that is a free A-module of rank 2 with an ordered basis β over A, and $\psi : V_A \otimes_A \mathbb{F} \xrightarrow{\sim} V_{\mathbb{F}}$ is an $\mathbb{F}$-linear G_K-isomorphism sending β to the fixed ordered basis of $V_{\mathbb{F}}$.
- A morphism $(V_A, \psi, \beta) \to (V_{A'}, \psi', \beta')$ covering a given morphism $A \to A'$ in $\mathfrak{AR}_{W(\mathbb{F})}$ is an equivalence class $[\alpha]$, where $\alpha : V_A \otimes_A A' \xrightarrow{\sim} V_{A'}$ is an A'-linear G_K-isomorphism that is compatible with the morphisms ψ, ψ' and sending β to β', and two morphisms are equivalent if they differ by an element of $A'^{\times}$.

Then the framed flat deformation $D^{\mathrm{fl},\square}_{V_{\mathbb{F}}}$ is pro-represented by a complete local $W(\mathbb{F})$-algebra $R^{\mathrm{fl},\square}_{V_{\mathbb{F}}}$.

We are going to define a deformation ring with the condition that the p-adic Hodge type $\mathbf{v} = (1)$, which is denoted by $R^{\mathrm{fl},\square,\mathbf{v}}_{V_{\mathbb{F}}}$. Let $\bigl(R^{\mathrm{fl},\square}_{V_{\mathbb{F}}}[1/p]\bigr)^{\mathbf{v}}$ be the quotient of $R^{\mathrm{fl},\square}_{V_{\mathbb{F}}}[1/p]$ corresponding to the connected components of $\operatorname{Spec} R^{\mathrm{fl},\square}_{V_{\mathbb{F}}}[1/p]$ whose closed points ξ satisfy the following:

> If V_ξ is the deformation corresponding to ξ, then $\mathrm{Fil}^0 D_{\mathrm{crys}}\bigl(V_\xi[1/p]\bigr)_K$ is free of rank 1 over $k(\xi) \otimes_{\mathbb{Q}_p} K$. Here, $k(\xi)$ is the residue field of ξ.

We note that $V_\xi[1/p]$ is Barsotti-Tate representation, since we are considering a flat deformation. Then we define $R^{\mathrm{fl},\square,\mathbf{v}}_{V_{\mathbb{F}}}$ by the image of $R^{\mathrm{fl},\square}_{V_{\mathbb{F}}}$ in $\bigl(R^{\mathrm{fl},\square}_{V_{\mathbb{F}}}[1/p]\bigr)^{\mathbf{v}}$.

The information of the connected components of $\operatorname{Spec} R^{\mathrm{fl},\square,\mathbf{v}}_{V_{\mathbb{F}}}[1/p]$ is very important for an application to a theorem comparing a deformation ring and a Hecke ring ([Kis, Theorem 3.4.11]). So we want to know $\pi_0(\operatorname{Spec} R^{\mathrm{fl},\square,\mathbf{v}}_{V_{\mathbb{F}}}[1/p])$.

Next, we are going to explain the Kisin module and the moduli space of finite flat models of $V_{\mathbb{F}}$. By a finite flat model of $V_{\mathbb{F}}$, we mean a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$, equipped with an action of $\mathbb{F}$, and an isomorphism $V_{\mathbb{F}} \xrightarrow{\sim} \mathcal{G}(\overline{K})$ that respects the action of G_K and $\mathbb{F}$.

Let $\mathfrak{S} = W(k)[[u]]$, and $\mathcal{O}_{\mathcal{E}}$ be the p-adic completion of $\mathfrak{S}[1/u]$. We consider the action of ϕ on $\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F} \cong k((u)) \otimes_{\mathbb{F}_p} \mathbb{F}$ defined by p-th power on $k((u))$. Let $\Phi\mathrm{M}_{\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F}}$ be the category of finite $(\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F})$-modules M with ϕ-semi-linear map $\phi : M \to M$ such that the induced linear map $\phi^* M \to M$ is bijective.

We take and fix a uniformizer π of $\mathcal{O}_K$. We choose a system $(\pi_m)_{m \geq 1}$ of elements in $\overline{K}$ such that $\pi_1^p = \pi$ and $\pi_{m+1}^p = \pi_m$ for $m \geq 1$, and put $K_\infty = \bigcup_{m \geq 1} K(\pi_m)$. Let $\mathrm{Rep}_{\mathbb{F}}(G_{K_\infty})$ be the category of finite-dimensional continuous G_{K_∞}-representations over $\mathbb{F}$.

Then the functor

$$T : \Phi\mathrm{M}_{\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F}} \to \mathrm{Rep}_{\mathbb{F}}(G_{K_\infty});\ M \mapsto \bigl(k((u))^{\mathrm{sep}} \otimes_{k((u))} M\bigr)^{\phi=1}$$

is an equivalence of abelian categories. We take $M_{\mathbb{F}} \in \Phi\mathrm{M}_{\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F}}$ such that $T(M_{\mathbb{F}})$ is isomorphic to $V_{\mathbb{F}}(-1)|_{G_{K_\infty}}$. Here (-1) denotes the inverse of the Tate twist. Then $M_{\mathbb{F}}$ is a free $(\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F})$-module of rank 2.

We put $\mathfrak{S}_{\mathbb{F}} = \mathfrak{S} \otimes_{\mathbb{Z}_p} \mathbb{F}$. Let $(\mathrm{Mod}/\mathfrak{S}_{\mathbb{F}})$ be the category of finite free $\mathfrak{S}_{\mathbb{F}}$-modules $\mathfrak{M}$ with ϕ-semi-linear map $\phi : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of the induced linear map $\phi^* \mathfrak{M} \to \mathfrak{M}$ is killed by u^e. An object of $(\mathrm{Mod}/\mathfrak{S}_{\mathbb{F}})$ is called a Kisin module with coefficients in $\mathbb{F}$. Let $(\mathbb{F}\text{-}\mathrm{Gr}/\mathcal{O}_K)$ be the category of finite flat group schemes over $\mathcal{O}_K$ with a structure of an $\mathbb{F}$-vector space.

Theorem 1.1. *There exists an equivalence of categories*

$$\mathrm{Gr} : (\mathrm{Mod}/\mathfrak{S}_{\mathbb{F}}) \to (\mathbb{F}\text{-}\mathrm{Gr}/\mathcal{O}_K).$$

Proof. This follows from [Br, Théorèm 4.2.1.6] and [Kis, Lemma 1.2.5]. □

Proposition 1.2 ([Kis, Proposition 1.1.13])**.** *For an object $\mathfrak{M}$ of $(\mathrm{Mod}/\mathfrak{S}_{\mathbb{F}})$, there exists a canonical isomorphism*

$$T(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M})(1) \xrightarrow{\sim} \mathrm{Gr}(\mathfrak{M})(\overline{K})|_{G_{K_\infty}}$$

as G_{K_∞}-representations. Here (1) denotes the Tate twist.

By this proposition, we see that a Kisin module which is a sublattice of $M_{\mathbb{F}}$ corresponds to a finite flat model of $V_{\mathbb{F}}$. Here and in the sequel, a sublattices means a finite free $\mathfrak{S}_{\mathbb{F}}$-submodule of M_F that spans $M_{\mathbb{F}}$ over $\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F}$. In the above, we have defined a Kisin module with coefficients in $\mathbb{F}$. More generally, we can define a Kisin module with coefficients in a $\mathbb{Z}_p$-algebra (cf. [Kis, (1.2)]). Using this general Kisin module, we can construct a moduli space of Kisin modules, which is denoted by $\mathscr{GR}_{V_{\mathbb{F}}}$ and projective over $\operatorname{Spec} R^{\mathrm{fl},\square}_{V_{\mathbb{F}}}$ (cf. [Kis, (2.1)]). The closed fiber of $\mathscr{GR}_{V_{\mathbb{F}}}$ over $\operatorname{Spec} R^{\mathrm{fl},\square}_{V_{\mathbb{F}}}$ is denoted by $\mathscr{GR}_{V_{\mathbb{F}},0}$. The scheme $\mathscr{GR}_{V_{\mathbb{F}},0}$ is a moduli space of finite flat models of $V_{\mathbb{F}}$ in the sense of the following proposition.

Proposition 1.3 ([Kis, Corollary 2.1.13])**.** *For any finite extension $\mathbb{F}'$ of $\mathbb{F}$, there is a natural bijection between the set of isomorphism classes of finite flat models of $V_{\mathbb{F}'} = V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ and $\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$.*

A closed subscheme $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}}} \subset \mathscr{GR}_{V_{\mathbb{F}}}$ is defined by the condition that p-adic Hodge type $\mathbf{v} = (1)$ as in [Kis, (2.4.2)]. The closed fiber of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}}}$ over $\operatorname{Spec} R^{\mathrm{fl},\square}_{V_{\mathbb{F}}}$ is denoted by $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$.

Then there is the following relation between the deformation ring $R^{\mathrm{fl},\square,\mathbf{v}}_{V_{\mathbb{F}}}$ and the moduli space $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$.

Proposition 1.4. *There exists a natural bijection*

$$\pi_0(\operatorname{Spec} R^{\mathrm{fl},\square,\mathbf{v}}_{V_{\mathbb{F}}}[1/p]) \stackrel{\sim}{=} \pi_0(\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}).$$

Proof. This follows from [Kis, Corollary 2.4.10], since $\mathscr{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} = \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ by [Kis, Proposition 2.4.6] if the p-adic Hodge type $\mathbf{v} = (1)$. □

So the problem has been reduced to study $\pi_0(\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0})$. The connected components $\mathscr{GR}^{\mathbf{v},\mathrm{ord}}_{V_{\mathbb{F}},0} \subset \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ is defined by the points corresponding to the ordinary finite flat group schemes. We can easily determine the set $\pi_0(\mathscr{GR}^{\mathbf{v},\mathrm{ord}}_{V_{\mathbb{F}},0})$ as in the following:

Proposition 1.5 ([Kis, Proposition 2.5.15])**.** *If $\mathscr{GR}^{\mathbf{v},\mathrm{ord}}_{V_{\mathbb{F}},0}$ is non-empty, then it consist of a single point, unless $V_{\mathbb{F}} \sim \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$ where χ_1 and χ_2 are unramified characters of G_K. In the latter case, we have the followings:*

1. *If $\chi_1 \neq \chi_2$, then $\mathscr{GR}^{\mathbf{v},\mathrm{ord}}_{V_{\mathbb{F}},0}$ consists of two points.*
2. *If $\chi_1 = \chi_2$, then $\mathscr{GR}^{\mathbf{v},\mathrm{ord}}_{V_{\mathbb{F}},0} \stackrel{\sim}{=} \mathbb{P}^1_{\mathbb{F}}$.*

Next, we consider the non-ordinary part. We put

$$\mathscr{GR}^{\mathbf{v},\text{non-ord}}_{V_{\mathbb{F}},0} = \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0} \setminus \mathscr{GR}^{\mathbf{v},\mathrm{ord}}_{V_{\mathbb{F}},0}.$$

Then Kisin conjectured that $\mathscr{GR}^{\mathbf{v},\text{non-ord}}_{V_{\mathbb{F}},0}$ is connected.

2 Connected components

2.1 Preliminaries

We assume $\mathrm{Spec}\, R^{\mathrm{fl},\square}_{V_{\mathbb{F}}} \neq \emptyset$, and this assumption assures that the action of I_K on $\det V_{\mathbb{F}}$ is the reduction mod p of the cyclotomic character.

The fundamental character of level m is given by

$$\omega_m : I_K \to \overline{k}^{\times};\ g \mapsto \frac{g(\sqrt[p^m-1]{\pi})}{\sqrt[p^m-1]{\pi}} \bmod m_{\mathcal{O}_{\overline{K}}}.$$

Here $m_{\mathcal{O}_{\overline{K}}}$ is the maximal ideal of $\mathcal{O}_{\overline{K}}$. If K'/K is a finite unramified extension that contains the (p^m-1)-st roots of unity, then the same formula as above defines a character of $G_{K'}$, which is again denoted by ω_m. Note that this extension depends on the choice of the uniformizer π.

Lemma 2.1. *If $V_{\mathbb{F}}$ is absolutely irreducible and $\mathbb{F}_{q^2} \subset \mathbb{F}$, then*

$$V_{\mathbb{F}}|_{I_K} \sim \omega_{2n}^{s} \oplus \omega_{2n}^{qs}$$

for a positive integer s such that $(q+1) \nmid s$.

Proof. Let $I_P \subset I_K$ be the wild inertia group. Then $V_{\mathbb{F}}^{I_P} \neq 0$ and $V_{\mathbb{F}}^{I_P}$ is G_K-stable, so $V_{\mathbb{F}}^{I_P} = V_{\mathbb{F}}$. As the action of I_K on $V_{\mathbb{F}}$ factors through the tame inertia group, we get $V_{\mathbb{F}}|_{I_K} \sim \omega_{m_1}^{s_1} \oplus \omega_{m_2}^{s_2}$ for some non-negative integers s_1, s_2 and some positive integers m_1, m_2. Now we fix a lifting $\widetilde{\mathrm{Fr}}_q \in G_K$ of the q-th Frobenius Fr_q. For every $\sigma \in I_K$ and every positive integer m, we have $\omega_m\big(\widetilde{\mathrm{Fr}}_q \circ \sigma \circ (\widetilde{\mathrm{Fr}}_q)^{-1}\big) = \omega_m(\sigma)^q$. Changing the above basis by the action of $(\widetilde{\mathrm{Fr}}_q)^{-1}$, we obtain $V_{\mathbb{F}}|_{I_K} \sim \omega_{m_1}^{qs_1} \oplus \omega_{m_2}^{qs_2}$.

If $\omega_{m_1}^{s_1} = \omega_{m_2}^{s_2}$, we get $\omega_{m_1}^{s_1} = \omega_{m_1}^{qs_1}$. So we may assume $m_1 = n$. As ω_n is defined over G_K, we can consider the representation $V_{\mathbb{F}} \otimes \omega_n^{-s_1}$ of G_K. Then this representation is absolutely irreducible and factors through G_k. This is a contradiction.

So we may assume $\omega_{m_1}^{s_1} \neq \omega_{m_2}^{s_2}$. As $V_{\mathbb{F}}$ is an irreducible representation, $\omega_{m_1}^{s_1} = \omega_{m_2}^{qs_2}$ and $\omega_{m_2}^{s_2} = \omega_{m_1}^{qs_1}$. Hence $\omega_{m_1}^{s_1} = \omega_{m_1}^{q^2 s_1}$ and we may assume $m_1 = 2n$. Thus we get $V_{\mathbb{F}}|_{I_K} \sim \omega_{2n}^{s} \oplus \omega_{2n}^{qs}$.

If $(q+1) \mid s$, then $V_{\mathbb{F}}|_{I_K} \sim \omega_n^{s'} \oplus \omega_n^{s'}$ where $s' = s/(q+1)$. This contradicts the absolutely irreducibility of $V_{\mathbb{F}}$ by considering $V_{\mathbb{F}} \otimes \omega_n^{-s'}$. So we get $(q+1) \nmid s$. □

From now on, in this section, we assume $\mathbb{F}_{q^2} \subset \mathbb{F}$ and fix an embedding $k \hookrightarrow \mathbb{F}$. This assumption does not matter, because we may extend $\mathbb{F}$ to prove the Kisin conjecture. We consider the isomorphism

$$\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F} \cong k((u)) \otimes_{\mathbb{F}_p} \mathbb{F} \xrightarrow{\sim} \prod_{\sigma \in \mathrm{Gal}(k/\mathbb{F}_p)} \mathbb{F}((u))\ ;\ \Big(\sum a_i u^i\Big) \otimes b \mapsto \Big(\sum \sigma(a_i) b u^i\Big)_{\sigma}$$

and let $\epsilon_\sigma \in k((u)) \otimes_{\mathbb{F}_p} \mathbb{F}$ be the primitive idempotent corresponding to σ. Take $\sigma_1, \cdots, \sigma_n \in \mathrm{Gal}(k/\mathbb{F}_p)$ such that $\sigma_{i+1} = \sigma_i \circ \phi^{-1}$. Here we regard ϕ as the p-th power Frobenius, and use the convention that $\sigma_{n+i} = \sigma_i$. In the following, we often use such conventions. Then we have $\phi(\epsilon_{\sigma_i}) = \epsilon_{\sigma_{i+1}}$, and $\phi : M_{\mathbb{F}} \to M_{\mathbb{F}}$ determines $\phi : \epsilon_{\sigma_i} M_{\mathbb{F}} \to \epsilon_{\sigma_{i+1}} M_{\mathbb{F}}$. For $(A_i)_{1\le i\le n} \in GL_2\big(\mathbb{F}((u))\big)^n$, we write

$$M_{\mathbb{F}} \sim (A_1, A_2, \dots, A_n) = (A_i)_i$$

if there is a basis $\{e_1^i, e_2^i\}$ of $\epsilon_{\sigma_i} M_{\mathbb{F}}$ over $\mathbb{F}((u))$ such that $\phi\begin{pmatrix} e_1^i \\ e_2^i \end{pmatrix} = A_i \begin{pmatrix} e_1^{i+1} \\ e_2^{i+1} \end{pmatrix}$. We use the same notation for any sublattice $\mathfrak{M}_{\mathbb{F}} \subset M_{\mathbb{F}}$ similarly. Here and in the following, we consider only sublattices that are $(\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathbb{F})$-modules.

Finally, for any sublattice $\mathfrak{M}_{\mathbb{F}} \subset M_{\mathbb{F}}$ with a chosen basis $\{e_1^i, e_2^i\}_{1\le i\le n}$ and $B = (B_i)_{1\le i\le n} \in GL_2\big(\mathbb{F}((u))\big)^n$, the module generated by the entries of $\left\langle B_i \begin{pmatrix} e_1^i \\ e_2^i \end{pmatrix} \right\rangle$ with the basis given by these entries is denoted by $B\cdot\mathfrak{M}_{\mathbb{F}}$. Note that $B\cdot\mathfrak{M}_{\mathbb{F}}$ depends on the choice of the basis of $\mathfrak{M}_{\mathbb{F}}$.

Lemma 2.2. *Suppose $V_{\mathbb{F}}$ is absolutely irreducible. If $\mathbb{F}'$ is the quadratic extension of $\mathbb{F}$, then*

$$M_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}' \sim \left(\begin{pmatrix} 0 & \alpha_1 \\ \alpha_1 u^s & 0 \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \dots, \begin{pmatrix} \alpha_n & 0 \\ 0 & \alpha_n \end{pmatrix} \right)$$

for some $\alpha_i \in (\mathbb{F}')^\times$ and a positive integer s such that $(q+1) \nmid s$. Conversely, for each positive integer s such that $(q+1) \nmid s$, there exists an absolutely irreducible representation $V_{\mathbb{F}}$ as above.

Proof. Let K' be the quadratic unramified extension of K, and k' be the residue field of K'. Then

$$V_{\mathbb{F}}(-1)|_{G_{K'}} \sim \lambda' \omega_{2n}^{-s} \oplus \lambda' \omega_{2n}^{-qs}$$

for an unramified character $\lambda' : G_{K'} \to \mathbb{F}^\times$ and a positive integer s such that $(q+1) \nmid s$ by applying Lemma 2.1 to $V_{\mathbb{F}}(-1)^*$. By taking the quadratic extension $\mathbb{F}'$ of $\mathbb{F}$, we can extend λ' to $\lambda : G_K \to (\mathbb{F}')^\times$. We take a lifting $\widetilde{\mathrm{Fr}}_q \in G_{K_\infty}$ of the q-th Frobenius Fr_q. Now we fix a (q^2-1)-st root of π, which is denoted by $\sqrt[q^2-1]{\pi}$. Then we put $\widetilde{\alpha} = \widetilde{\mathrm{Fr}}_q(\sqrt[q^2-1]{\pi})/\sqrt[q^2-1]{\pi} \in \mathcal{O}_{\overline{K}}$, and let α be the reduction of $\widetilde{\alpha}$ in $\overline{k}$. We have $\alpha \in \mathbb{F}'$, because $\alpha^{q^2-1} = 1$. Considering $V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$, we may assume $\mathbb{F} = \mathbb{F}'$.

We put $K'_\infty = K' \cdot K_\infty$. Then $(\widetilde{\mathrm{Fr}}_q)^2$ is in $G_{K'_\infty}$. Now we have

$$\frac{(\widetilde{\mathrm{Fr}}_q)^2(\sqrt[q^2-1]{\pi})}{\sqrt[q^2-1]{\pi}} = \frac{(\widetilde{\mathrm{Fr}}_q)^2(\sqrt[q^2-1]{\pi})}{\widetilde{\mathrm{Fr}}_q(\sqrt[q^2-1]{\pi})} \cdot \frac{\widetilde{\mathrm{Fr}}_q(\sqrt[q^2-1]{\pi})}{\sqrt[q^2-1]{\pi}} = \frac{\widetilde{\mathrm{Fr}}_q(\widetilde{\alpha}\sqrt[q^2-1]{\pi})}{\widetilde{\mathrm{Fr}}_q(\sqrt[q^2-1]{\pi})}\widetilde{\alpha} = \widetilde{\mathrm{Fr}}_q(\widetilde{\alpha})\widetilde{\alpha}$$

and $\omega_{2n}\big((\widetilde{\mathrm{Fr}}_q)^2\big) = \alpha^{q+1}$. Hence we can take $v_1, v_2 \in V_{\mathbb{F}}(-1)$ so that

$$\widetilde{\mathrm{Fr}}_q(v_1) = \lambda(\widetilde{\mathrm{Fr}}_q)\alpha^{-qs} v_2,\ \widetilde{\mathrm{Fr}}_q(v_2) = \lambda(\widetilde{\mathrm{Fr}}_q)\alpha^{-s} v_1$$

and

$$g(v_1) = \lambda(g)\omega_{2n}^{-s}(g) v_1,\ g(v_2) = \lambda(g)\omega_{2n}^{-qs}(g) v_2$$

for all $g \in G_{K'_\infty}$. We take an element w_λ of $(\overline{k} \otimes_{\mathbb{F}_p} \mathbb{F})^\times$ so that $g(w_\lambda) = \big(1 \otimes \lambda(g)\big) w_\lambda$ for all $g \in G_K$. By this condition, w_λ is determined up to $(k \otimes_{\mathbb{F}_p} \mathbb{F})^\times$.

By the definition of the action of G_{K_∞} on $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$, we can choose an element u_{2n} of $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/p\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ so that $u_{2n}^{q^2-1} = u$ and $\widetilde{\mathrm{Fr}}_q(u_{2n}) = \alpha u_{2n}$. We consider the isomorphism

$$k' \otimes_{\mathbb{F}_p} \mathbb{F} \xrightarrow{\sim} \prod_{\sigma \in \mathrm{Gal}(k'/\mathbb{F}_p)} \mathbb{F} \; ; \; a \otimes b \mapsto \big(\sigma(a) b\big)_\sigma$$

and let $\epsilon_0 \in k' \otimes_{\mathbb{F}_p} \mathbb{F}$ be the primitive idempotent corresponding to $\mathrm{id}_{k'}$. For $0 \leq r \leq 2n-1$, we put $\epsilon_r = \phi^r \epsilon_0$. Note that $(a^{p^r} \otimes 1)\epsilon_r = (1 \otimes a)\epsilon_r$ for all $a \in k'$.

We put

$$\begin{aligned} e_1 = w_\lambda^{-1} \big\{ &(u_{2n}^s \otimes 1)(\epsilon_0 v_1 + \epsilon_n v_2) + (u_{2n}^{ps} \otimes 1)(\epsilon_1 v_1 + \epsilon_{n+1} v_2) + \\ &\cdots + (u_{2n}^{p^{n-1}s} \otimes 1)(\epsilon_{n-1} v_1 + \epsilon_{2n-1} v_2) \big\}, \end{aligned}$$

$$\begin{aligned} e_2 = w_\lambda^{-1} \big\{ &(u_{2n}^{p^n s} \otimes 1)(\epsilon_n v_1 + \epsilon_0 v_2) + (u_{2n}^{p^{n+1}s} \otimes 1)(\epsilon_{n+1} v_1 + \epsilon_1 v_2) + \\ &\cdots + (u_{2n}^{p^{2n-1}s} \otimes 1)(\epsilon_{2n-1} v_1 + \epsilon_{n-1} v_2) \big\} \end{aligned}$$

in $(\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/p\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}) \otimes_{\mathbb{F}_p} V_{\mathbb{F}}(-1)$. Then e_1 and e_2 are fixed by $g \in G_{K'_\infty}$ and $\widetilde{\mathrm{Fr}}_q$. Hence e_1, e_2 are fixed by G_{K_∞}, and these are a basis of $\Phi\mathrm{M}_{\mathcal{O}_{\mathcal{E}},\mathbb{F}}$ over $\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F}$. We put $\alpha_\lambda = w_\lambda/\phi(w_\lambda)$. As $\phi(w_\lambda)$ satisfies the condition determining w_λ, the element α_λ of $(\overline{k} \otimes_{\mathbb{F}_p} \mathbb{F})^\times$ is in $(k \otimes_{\mathbb{F}_p} \mathbb{F})^\times$. Now we have

$$\begin{aligned} \phi(e_1) &= \alpha_\lambda \big\{ (\epsilon_1 + \epsilon_{n+1}) + \cdots + (\epsilon_{n-1} + \epsilon_{2n-1}) \big\} e_1 + \alpha_\lambda (\epsilon_0 + \epsilon_n) e_2, \\ \phi(e_2) &= \alpha_\lambda u^s (\epsilon_0 + \epsilon_n) e_1 + \alpha_\lambda \big\{ (\epsilon_1 + \epsilon_{n+1}) + \cdots + (\epsilon_{n-1} + \epsilon_{2n-1}) \big\} e_2. \end{aligned}$$

If we put

$$\sigma_1 = \phi, \; \sigma_2 = \mathrm{id}_k, \; \sigma_3 = \phi^{-1}, \ldots, \; \sigma_n = \phi^2,$$

then we have

$$\epsilon_{\sigma_1} = \epsilon_{n-1} + \epsilon_{2n-1}, \; \epsilon_{\sigma_2} = \epsilon_0 + \epsilon_n, \ldots, \; \epsilon_{\sigma_n} = \epsilon_{n-2} + \epsilon_{2n-2}$$

and

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} 0 & \alpha_1 \\ \alpha_1 u^s & 0 \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \ldots, \begin{pmatrix} \alpha_n & 0 \\ 0 & \alpha_n \end{pmatrix} \right).$$

Here α_i is the σ_{i+1}-th component of α_λ in $\prod_{\sigma \in \mathrm{Gal}(k/\mathbb{F}_p)} \mathbb{F}$.

We can check the last statement easily. □

2.2 Kisin conjecture

Lemma 2.3 ([Gee, Lemma 2.2]). *If $\mathbb{F}'$ is a finite extension of $\mathbb{F}$, the elements of $\mathscr{GR}^{\mathrm{v}}_{V_{\mathbb{F}},0}(\mathbb{F}')$ naturally correspond to free $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}'$-submodules $\mathfrak{M}_{\mathbb{F}'} \subset M_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ of rank 2 that satisfy the following:*

1. $\mathfrak{M}_{\mathbb{F}'}$ *is* ϕ*-stable.*

2. *For some (so any) choice of* $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}'$*-basis for* $\mathfrak{M}_{\mathbb{F}'}$*, and for each* $\sigma \in \mathrm{Gal}(k/\mathbb{F}_p)$*, the map*
$$\phi : \epsilon_\sigma \mathfrak{M}_{\mathbb{F}'} \to \epsilon_{\sigma\circ\phi^{-1}} \mathfrak{M}_{\mathbb{F}'}$$
has determinant αu^e *for some* $\alpha \in \mathbb{F}'[[u]]^\times$*.*

Lemma 2.4 ([Gee, Lemma 2.4])**.** *Suppose* $x_1, x_2 \in \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F})$ *correspond to objects* $\mathfrak{M}_{1,\mathbb{F}}, \mathfrak{M}_{2,\mathbb{F}}$ *of* $(\mathrm{Mod}/\mathfrak{S})_{\mathbb{F}}$ *respectively. Let* $N = (N_i)_{1\le i\le n}$ *be a nilpotent element of* $M_2\big(\mathbb{F}((u))\big)^n$ *such that* $\mathfrak{M}_{2,\mathbb{F}} = (1+N)\cdot\mathfrak{M}_{1,\mathbb{F}}$*, and* $A = (A_i)_{1\le i\le n}$ *be an element of* $GL_2\big(\mathbb{F}((u))\big)^n$ *such that* $\mathfrak{M}_{1,\mathbb{F}} \sim A$*. If* $\phi(N_i)A_iN_{i+1} \in M_2\big(\mathbb{F}[[u]]\big)$ *for all* i*, then there is a morphism* $\mathbb{P}^1 \to \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ *sending* 0 *to* x_1 *and* 1 *to* x_2*.*

Lemma 2.5. *Suppose* $n \ge 2$*. Let* $\mathfrak{M}_{\mathbb{F}}$ *be the object of* $(\mathrm{Mod}/\mathfrak{S})_{\mathbb{F}}$ *corresponding to a point* $x \in \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F})$*. Fix a basis of* $\mathfrak{M}_{\mathbb{F}}$ *over* $k[[u]]\otimes_{\mathbb{F}_p}\mathbb{F}$*. Consider* $U^{(i)} = (U^{(i)}_j)_{1\le j\le n} \in GL_2\big(\mathbb{F}((u))\big)^n$ *such that* $U^{(i)}_i = \begin{pmatrix} u & 0 \\ 0 & u^{-1}\end{pmatrix}$ *and* $U^{(i)}_j = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$ *for all* $j \ne i$*. If* $U^{(i)}\cdot\mathfrak{M}_{\mathbb{F}}$ *is* ϕ*-stable, it corresponds to a point* $x' \in \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F})$*, and* x' *lies on the same connected component of* $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ *as* x*.*

Proof. First, $U^{(i)}\cdot\mathfrak{M}_{\mathbb{F}}$ corresponds to a point $x' \in \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F})$, because it satisfies the conditions of Lemma 2.3.

Next, we consider $N^{(i)} = (N^{(i)}_j)_{1\le j\le n} \in M_2\big(\mathbb{F}((u))\big)^n$ such that
$$N^{(i)}_i = \begin{pmatrix} 1 & -u \\ u^{-1} & -1\end{pmatrix} \text{ and } N^{(i)}_j = 0 \text{ for all } j \ne i.$$
Then $U^{(i)}\cdot\mathfrak{M}_{\mathbb{F}} = (1+N^{(i)})\cdot\mathfrak{M}_{\mathbb{F}}$, because $\begin{pmatrix} u^{-1} & 0 \\ 0 & u\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2u\end{pmatrix}\begin{pmatrix} 2 & -u \\ u^{-1} & 0\end{pmatrix}$. So we can apply Lemma 2.4. □

Theorem 2.6. *Let* $\mathbb{F}'$ *be a finite extension of* $\mathbb{F}$*. Suppose* $x_1, x_2 \in \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F}')$ *correspond to objects* $\mathfrak{M}_{1,\mathbb{F}'}, \mathfrak{M}_{2,\mathbb{F}'}$ *of* $(\mathrm{Mod}/\mathfrak{S})_{\mathbb{F}'}$ *respectively. If* $\mathfrak{M}_{1,\mathbb{F}'}$ *and* $\mathfrak{M}_{2,\mathbb{F}'}$ *are both non-ordinary, then* x_1 *and* x_2 *lie on the same connected component of* $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$*.*

Proof. When $n=1$, this was proved in [Kis]. If $e < p-1$, then $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F}')$ is one point by [Ray, Theorem 3.3.3]. So we may assume $n \ge 2$ and $e \ge p-1$. Furthermore, replacing $V_{\mathbb{F}}$ by $V_{\mathbb{F}}\otimes_{\mathbb{F}}\mathbb{F}'$, we may assume $\mathbb{F} = \mathbb{F}'$.

Suppose first that $V_{\mathbb{F}}$ is reducible. We can choose a basis so that $\mathfrak{M}_{1,\mathbb{F}} \sim A = (A_i)_{1\le i\le n} \in M_2\big(\mathbb{F}[[u]]\big)^n$ where $A_i = \begin{pmatrix} a_i & b_i \\ 0 & c_i\end{pmatrix}$ for $a_i, b_i, c_i \in \mathbb{F}[[u]]$, because $M_{\mathbb{F}}$ is reducible and $\mathfrak{M}_{1,\mathbb{F}}$ is ϕ-stable. By the Iwasawa decomposition and the determinant conditions, we can take $B = (B_i)_{1\le i\le n} \in GL_2\big(\mathbb{F}((u))\big)^n$ such that $\mathfrak{M}_{2,\mathbb{F}} = B\cdot\mathfrak{M}_{1,\mathbb{F}}$ and $B_i = \begin{pmatrix} u^{-s_i} & v_i \\ 0 & u^{s_i}\end{pmatrix}$ for $s_i \in \mathbb{Z}$ and $v_i \in \mathbb{F}((u))$. Then $\mathfrak{M}_{2,\mathbb{F}} \sim \big(\phi(B_i)A_iB_{i+1}^{-1}\big)_i$, and we have

$$\begin{aligned}\phi(B_i)A_iB_{i+1}^{-1} &= \begin{pmatrix} u^{-ps_i} & \phi(v_i) \\ 0 & u^{ps_i} \end{pmatrix} \begin{pmatrix} a_i & b_i \\ 0 & c_i \end{pmatrix} \begin{pmatrix} u^{s_{i+1}} & -v_{i+1} \\ 0 & u^{-s_{i+1}} \end{pmatrix} \\ &= \begin{pmatrix} a_iu^{-ps_i+s_{i+1}} & -a_iv_{i+1}u^{-ps_i} + b_iu^{-ps_i-s_{i+1}} + c_i\phi(v_i)u^{-s_{i+1}} \\ 0 & c_iu^{ps_i-s_{i+1}} \end{pmatrix}.\end{aligned}$$

In the last matrix, every component is integral because $\mathfrak{M}_{2,\mathbb{F}}$ is ϕ-stable.

First of all, we want to reduce the problem to the case where $s_i = 0$ for all i. When $e = p-1$, we have $0 \leq v_u(c_i) \leq p-1$ and $0 \leq v_u(c_i) + ps_i - s_{i+1} \leq p-1$ for all i by the determinant conditions. From the second set of inequalities, we obtaine

$$0 \leq \sum_{j=0}^{n-1} \bigl\{ v_u(c_{i-1-j}) + ps_{i-1-j} - s_{i-j} \bigr\} p^j \leq p^n - 1,$$

and we have

$$\sum_{j=0}^{n-1} \bigl\{ v_u(c_{i-1-j}) + ps_{i-1-j} - s_{i-j} \bigr\} p^j = (p^n-1)s_i + \sum_{j=0}^{n-1} v_u(c_{i-1-j})p^j.$$

Combining these with $0 \leq v_u(c_i) \leq p-1$, we get $-1 \leq s_i \leq 1$. If $s_i = 1$ for some i, the second sign of the above inequality must be the equality sign. So we get $v_u(c_j) = 0$ for all j. This contradicts the non-ordinarity of $\mathfrak{M}_{1,\mathbb{F}}$. If $s_i = -1$ for some i, the first sign of the above inequality must be the equality sign. So we get $v_u(c_j) + ps_j - s_{j+1} = 0$ for all j. This contradicts the non-ordinarity of $\mathfrak{M}_{2,\mathbb{F}}$. Hence, we have $s_i = 0$ for all i. So we may assume $e \geq p$.

We consider $U^{(i)}$ as in Lemma 2.5. If $s_i > 0$ and $U^{(i)} \cdot \mathfrak{M}_{2,\mathbb{F}}$ is ϕ-stable, we may replace $\mathfrak{M}_{2,\mathbb{F}}$ with $U^{(i)} \cdot \mathfrak{M}_{2,\mathbb{F}}$ by Lemma 2.5. This replacement changes s_i into $s_i - 1$ and v_i into uv_i. If $s_i < 0$, switching $\mathfrak{M}_{1,\mathbb{F}}$ with $\mathfrak{M}_{2,\mathbb{F}}$ so that we have $s_i > 0$, we consider the same replacement as above. Note that these replacements decrease $|s_i|$ by 1. We prove that we can continue these replacements until we get to the case where $s_i = 0$ for all i. Suppose that we cannot continue the replacements and there is some nonzero s_i. Take an index i_0 such that $|s_{i_0}|$ is the greatest. By switching $\mathfrak{M}_{1,\mathbb{F}}$ with $\mathfrak{M}_{2,\mathbb{F}}$, we may assume $s_{i_0} > 0$. As we cannot continue the replacements, we cannot decrease s_{i_0} keeping the ϕ-stability, that is,

$$v_u(c_{i_0}) + ps_{i_0} - s_{i_0+1} \leq p-1 \text{ or } v_u(a_{i_0-1}) - ps_{i_0-1} + s_{i_0} = 0.$$

If $v_u(c_{i_0}) + ps_{i_0} - s_{i_0+1} \leq p-1$, we have $s_{i_0} = 1$, $v_u(c_{i_0}) = 0$ and $s_{i_0+1} = 1$, because $v_u(c_{i_0}) + (p-1)s_{i_0} + (s_{i_0} - s_{i_0+1}) \leq p-1$. Now we have $v_u(a_{i_0}) - ps_{i_0} + s_{i_0+1} \geq 1$, because $e \geq p$ and $v_u(c_{i_0}) + ps_{i_0} - s_{i_0+1} \leq p-1$. As s_{i_0+1} cannot be decreased, $v_u(c_{i_0+1}) + ps_{i_0+1} - s_{i_0+2} \leq p-1$. The same argument shows that $v_u(c_i) = 0$ and $s_i = 1$ for all i. This contradicts the non-ordinarity of $\mathfrak{M}_{1,\mathbb{F}}$.

If $v_u(a_{i_0-1}) - ps_{i_0-1} + s_{i_0} = 0$, then $s_{i_0-1} > 0$ and $v_u(c_{i_0-1}) + ps_{i_0-1} - s_{i_0} = e \geq p$. As s_{i_0-1} cannot be decreased, $v_u(a_{i_0-2}) - ps_{i_0-2} + s_{i_0-1} = 0$. The same argument shows that $v_u(a_i) - ps_i + s_{i+1} = 0$ for all i. So we have that $\mathfrak{M}_{2,\mathbb{F}}$ is an extension of a multiplicative module by an étale module. We show that such an extension splits. Now we have $\mathfrak{M}_{2,\mathbb{F}} \sim \left(\begin{pmatrix} a_i' & b_i' \\ 0 & u^e c_i' \end{pmatrix} \right)_i$ for $a_i', c_i' \in \mathbb{F}[[u]]^\times$ and $b_i' \in \mathbb{F}[[u]]$. Then

$$\left(\begin{pmatrix} 1 & v_i' \\ 0 & 1 \end{pmatrix} \right)_i \cdot \mathfrak{M}_{2,\mathbb{F}} \sim \left(\begin{pmatrix} a_i' & -a_i' v_{i+1}' + b_i' + u^e c_i' \phi(v_i') \\ 0 & u^e c_i' \end{pmatrix} \right)_i$$

for $v_i' \in \mathbb{F}[[u]]$. It suffices to show that there is $(v_i')_{1 \leq i \leq n} \in \mathbb{F}[[u]]^n$ such that $a_i' v_{i+1}' = b_i' + u^e c_i' \phi(v_i')$ for all i, and we can solve the system of equations by finding v_i' successively in ascending order of their degrees. Hence we have that $\mathfrak{M}_{2,\mathbb{F}}$ is ordinary, and this is a contradiction.

Thus we may assume $s_i = 0$ for all i. Consider $N = (N_i)_{1 \leq i \leq n} \in M_2\big(\mathbb{F}((u))\big)^n$ such that $N_i = \begin{pmatrix} 0 & v_i \\ 0 & 0 \end{pmatrix}$ for $v_i \in \mathbb{F}((u))$. Then we have $\mathfrak{M}_{2,\mathbb{F}} = (1+N) \cdot \mathfrak{M}_{1,\mathbb{F}}$ and $\phi(N_i) \begin{pmatrix} a_i & b_i \\ 0 & c_i \end{pmatrix} N_{i+1} = 0$. Hence x_1 and x_2 lie on the same connected component by Lemma 2.4. This completes the proof in the case where $V_{\mathbb{F}}$ is reducible.

From now on, we consider the case where $V_{\mathbb{F}}$ is irreducible. If $V_{\mathbb{F}}$ is reducible after extending the base field $\mathbb{F}$, we can reduce this case to the reducible case. So we may assume $V_{\mathbb{F}}$ is absolutely irreducible. Extending the field $\mathbb{F}$, we have

$$M_{\mathbb{F}} \sim \left(\alpha_1 \begin{pmatrix} 0 & 1 \\ u^s & 0 \end{pmatrix}, \alpha_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ldots, \alpha_n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

for some $\alpha_i \in \mathbb{F}^\times$ and a positive integer s by Lemma 2.2. This basis gives a sublattice $\mathfrak{M}_{\mathbb{F}}$. By the Iwasawa decomposition, we can take $s_i', t_i' \in \mathbb{Z}$ and $v_i' \in \mathbb{F}((u))$ so that $\mathfrak{M}_{1,\mathbb{F}} = \left(\begin{pmatrix} u^{s_i'} & v_i' \\ 0 & u^{t_i'} \end{pmatrix} \right)_i \cdot \mathfrak{M}_{\mathbb{F}}$. Changing the basis by $\left(\begin{pmatrix} u^{s_i'} & 0 \\ 0 & u^{t_i'} \end{pmatrix} \right)_i$, we get

$$M_{\mathbb{F}} \sim \left(\alpha_1 \begin{pmatrix} 0 & u^{s_1} \\ u^{t_1} & 0 \end{pmatrix}, \alpha_2 \begin{pmatrix} u^{s_2} & 0 \\ 0 & u^{t_2} \end{pmatrix}, \ldots, \alpha_n \begin{pmatrix} u^{s_n} & 0 \\ 0 & u^{t_n} \end{pmatrix} \right).$$

Here we have $0 \leq t_1$, $0 \leq s_i, t_i \leq e$ for $2 \leq i \leq n$, and $s_i + t_i = e$ for all i by the ϕ-stability and the determinant conditions of $\mathfrak{M}_{1,\mathbb{F}}$.

We are going to change the basis so that we have moreover $t_1 \leq e$. Changing the basis of the i-th component by $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$, we get the following transformations:

$$\begin{aligned} &T_i : t_i \rightsquigarrow t_i - p,\ t_{i-1} \rightsquigarrow t_{i-1} + 1 \text{ for } i \neq 2, \\ &T_2 : t_2 \rightsquigarrow t_2 - p,\ t_1 \rightsquigarrow t_1 - 1. \end{aligned}$$

If $t_1 > e$, we put

$$m = \max\{\ 1 \leq i \leq n \mid t_i \neq e\ \},$$

and carry out T_1 when $m = n$, and $T_{m+1}, T_{m+2}, \dots, T_n, T_1$ when $m \neq n$. Then $0 \leq s_i, t_i \leq e$ for $2 \leq i \leq n$, and t_1 decrease by p when $m \neq 1$, by $p+1$ when $m = 1$. Repeat this until we get to the situation where $t_1 \leq e$. If $e \geq p$, we get to the situation where $0 \leq s_1, t_1 \leq e$. If $e = p-1$ and we do not get to the situation where $0 \leq s_1, t_1 \leq p-1$, then we have

$$M_{\mathbb{F}} \sim \left(\alpha_1 \begin{pmatrix} 0 & u^{-1} \\ u^p & 0 \end{pmatrix}, \alpha_2 \begin{pmatrix} 1 & 0 \\ 0 & u^{p-1} \end{pmatrix}, \dots, \alpha_n \begin{pmatrix} 1 & 0 \\ 0 & u^{p-1} \end{pmatrix} \right).$$

In this case, changing the basis by $\left(\begin{pmatrix} 1 & u^{-1} \\ 0 & 1 \end{pmatrix} \right)_i$, we get

$$M_{\mathbb{F}} \sim \left(\alpha_1 \begin{pmatrix} 1 & 0 \\ u^p & -u^{p-1} \end{pmatrix}, \alpha_2 \begin{pmatrix} 1 & 0 \\ 0 & u^{p-1} \end{pmatrix}, \dots, \alpha_n \begin{pmatrix} 1 & 0 \\ 0 & u^{p-1} \end{pmatrix} \right).$$

This contradicts that $M_{\mathbb{F}}$ is irreducible. Hence we obtain a basis such that

$$M_{\mathbb{F}} \sim \left(\alpha_1 \begin{pmatrix} 0 & u^{s_1} \\ u^{t_1} & 0 \end{pmatrix}, \alpha_2 \begin{pmatrix} u^{s_2} & 0 \\ 0 & u^{t_2} \end{pmatrix}, \dots, \alpha_n \begin{pmatrix} u^{s_n} & 0 \\ 0 & u^{t_n} \end{pmatrix} \right)$$

for some s_i and t_i satisfying $s_i + t_i = e$ and $0 \leq s_i, t_i \leq e$ for all i. Let $\mathfrak{M}_{0,\mathbb{F}}$ be the sublattice of $M_{\mathbb{F}}$ determined by this basis. Note that $\mathfrak{M}_{0,\mathbb{F}}$ satisfies the conditions of Lemma 2.3, and let x_0 be the point of $\mathscr{GR}^{\mathrm{v}}_{V_{\mathbb{F}},0}$ corresponding to $\mathfrak{M}_{0,\mathbb{F}}$.

We prove that we can change $(t_i)_{1 \leq i \leq n}$ furthermore by T_i's or T_i^{-1}'s keeping $0 \leq t_i \leq e$ for all i, and get to the situation where $|s_i - t_i| \leq p+1$ for all i. By Lemma 2.5, these changes do not affect which of the connected components x_0 lies on. If $e \leq p+1$, this is satisfied automatically. So we may assume $e \geq p+2$. We prove that if there is an index j such that $|s_j - t_j| \geq p+2$, then there is an index j_0 such that $|s_{j_0} - t_{j_0}| \geq p+2$ and we can change t_{j_0} by T_{j_0} or $T_{j_0}^{-1}$ so that $|s_{j_0} - t_{j_0}|$ decreases keeping $0 \leq t_i \leq e$ for all i. We put $h_i = (-1)^{[(i-2)/n]}(s_i - t_i)$ for $i \in \mathbb{Z}$. By assumption, there is an integer j_0 such that $1 \leq j_0 \leq 2n$, $h_{j_0} \geq p+2$ and $h_{j_0-1} < e$. If $2 \leq j_0 \leq n+1$, we can change t_{j_0} by $T_{j_0}^{-1}$, otherwise by T_{j_0}, so that $|s_{j_0} - t_{j_0}|$ decreases keeping $0 \leq t_i \leq e$ for all i. Thus we have proved the claim. Hence if $|s_j - t_j| \geq p+2$ for an index j, we can carry out T_{j_0} or $T_{j_0}^{-1}$ for an index j_0 as above and this operation decreases $\sum_{i=1}^{n} |s_i - t_i|$ by at least 2. So after finitely many operations, we get to the situation where $|s_i - t_i| \leq p+1$ for all i.

Hence we may assume that s_i and t_i satisfy $s_i + t_i = e$, $0 \leq s_i, t_i \leq e$ and $|s_i - t_i| \leq p+1$ for all i. We are going to prove that x_0 and x_1 lie on the same connected component. We can prove that x_0 and x_2 lie on the same connected component by the same argument.

By the Iwasawa decomposition and the determinant conditions, we can take $B = (B_i)_{1 \leq i \leq n} \in GL_2\big(\mathbb{F}((u))\big)^n$ such that $\mathfrak{M}_{1,\mathbb{F}} = B \cdot \mathfrak{M}_{0,\mathbb{F}}$ and $B_i = \begin{pmatrix} u^{-a_i} & v_i \\ 0 & u^{a_i} \end{pmatrix}$

for $a_i \in \mathbb{Z}$ and $v_i \in \mathbb{F}((u))$. Then we put $r_i = v_u(v_i)$. Now we have

$$\phi(B_1)\begin{pmatrix} 0 & u^{s_1} \\ u^{t_1} & 0 \end{pmatrix} B_2^{-1} = \begin{pmatrix} \phi(v_1)u^{t_1+a_2} & u^{s_1-pa_1-a_2} - \phi(v_1)v_2u^{t_1} \\ u^{t_1+pa_1+a_2} & -v_2u^{t_1+pa_1} \end{pmatrix},$$

$$\phi(B_i)\begin{pmatrix} u^{s_i} & 0 \\ 0 & u^{t_i} \end{pmatrix} B_{i+1}^{-1} = \begin{pmatrix} u^{s_i-pa_i+a_{i+1}} & \phi(v_i)u^{t_i-a_{i+1}} - v_{i+1}u^{s_i-pa_i} \\ 0 & u^{t_i+pa_i-a_{i+1}} \end{pmatrix}$$

for $2 \leq i \leq n$. On the right-hand sides, every component of the matrices is integral because $\mathfrak{M}_{1,\mathbb{F}}$ is ϕ-stable.

First, we consider the case $t_1 + pa_1 + a_2 > e$. In this case,

$$(pr_1 + t_1 + a_2) + (r_2 + t_1 + pa_1) = e, \ s_1 - pa_1 - a_2 = pr_1 + r_2 + t_1 < 0$$

by the ϕ-stability and the determinant conditions of $\mathfrak{M}_{1,\mathbb{F}}$. We have $a_1 > r_1$, because $t_1 + pa_1 + a_2 > e \geq pr_1 + t_1 + a_2$. Similarly, we have $a_2 > r_2$, because $t_1 + pa_1 + a_2 > e \geq r_2 + t_1 + pa_1$.

We consider the following operations:

$$a_i \rightsquigarrow a_i - 1, \ v_i \rightsquigarrow uv_i, \ \text{if it preserves the } \phi\text{-stability of } B \cdot \mathfrak{M}_{0,\mathbb{F}}.$$

These operations replace x_1 by a point that lies on the same connected component as x_1 by Lemma 2.5. We prove that we can continue these operations until we get to the situation where $t_1 + pa_1 + a_2 \leq e$. In other words, we reduce the problem to the case $t_1 + pa_1 + a_2 \leq e$. If we can continue the operations endlessly, we get to the situation where $t_1 + pa_1 + a_2 \leq e$, because the conditions $s_i - pa_i + a_{i+1} \geq 0$ for $2 \leq i \leq n$ exclude that both a_1 and a_2 remain bounded below. Suppose we cannot continue the operations. This is equivalent to the following condition:

$$\begin{aligned} s_n - pa_n + a_1 = 0 &\text{ or } r_2 + t_1 + pa_1 \leq p - 1, \\ pr_1 + t_1 + a_2 = 0 &\text{ or } t_2 + pa_2 - a_3 \leq p - 1, \\ s_{i-1} - pa_{i-1} + a_i = 0 &\text{ or } t_i + pa_i - a_{i+1} \leq p - 1 \text{ for each } 3 \leq i \leq n. \end{aligned}$$

If $e \geq p$, there are only the following two cases, because $(pr_1 + t_1 + a_2) + (r_2 + t_1 + pa_1) = e$ and $(s_i - pa_i + a_{i+1}) + (t_i + pa_i - a_{i+1}) = e$ for $2 \leq i \leq n$.

$$\begin{aligned} &\text{Case 1} : pr_1 + t_1 + a_2 = 0, \ s_i - pa_i + a_{i+1} = 0 \text{ for } 2 \leq i \leq n. \\ &\text{Case 2} : r_2 + t_1 + pa_1 \leq p - 1, \ t_i + pa_i - a_{i+1} \leq p - 1 \text{ for } 2 \leq i \leq n. \end{aligned}$$

If $e = p - 1$, clearly it is in Case 2.

In the Case 1. Suppose that there is an index i such that $2 \leq i \leq n$ and $pr_i + t_i - a_{i+1} \neq r_{i+1} + s_i - pa_i$. Then both sides are non-negative, because $v_u(\phi(v_i)u^{t_i-a_{i+1}} - v_{i+1}u^{s_i-pa_i}) \geq 0$. Comparing $r_{i+1} + s_i - pa_i \geq 0$ with $s_i - pa_i + a_{i+1} = 0$, we get $r_{i+1} \geq a_{i+1}$. Then $pr_{i+1} + t_{i+1} - a_{i+2} \geq pa_{i+1} + t_{i+1} - a_{i+2} \geq 0$, and $r_{i+2} + s_{i+1} - pa_{i+1} \geq 0$ because $v_u(\phi(v_{i+1})u^{t_{i+1}-a_{i+2}} - v_{i+2}u^{s_{i+1}-pa_{i+1}}) \geq 0$. Comparing $r_{i+2} + s_{i+1} - pa_{i+1} \geq 0$ with $s_{i+1} - pa_{i+1} + a_{i+2} = 0$, we get $r_{i+2} \geq a_{i+2}$. The same argument goes on and shows $r_1 \geq a_1$. This is a contradiction. Thus $pr_i + t_i - a_{i+1} = r_{i+1} + s_i - pa_i$ for all $2 \leq i \leq n$. Now we change the basis of

$$M_{\mathbb{F}} \sim \left(\alpha_1 \begin{pmatrix} 0 & u^{s_1} \\ u^{t_1} & 0 \end{pmatrix}, \alpha_2 \begin{pmatrix} u^{s_2} & 0 \\ 0 & u^{t_2} \end{pmatrix}, \ldots, \alpha_n \begin{pmatrix} u^{s_n} & 0 \\ 0 & u^{t_n} \end{pmatrix}\right)$$

by $\left(\begin{pmatrix} u^{-a_i} & u^{r_i} \\ 0 & u^{a_i} \end{pmatrix}\right)_i$. Then we have

$$M_{\mathbb{F}} \sim \left(\alpha_1 \begin{pmatrix} 1 & 0 \\ u^{t_1+pa_1+a_2} & -u^e \end{pmatrix}, \alpha_2 \begin{pmatrix} 1 & 0 \\ 0 & u^e \end{pmatrix}, \dots, \alpha_n \begin{pmatrix} 1 & 0 \\ 0 & u^e \end{pmatrix}\right),$$

and this contradicts that $M_{\mathbb{F}}$ is irreducible.

In the Case 2. Suppose that there is an index i such that $2 \le i \le n$ and $pr_i + t_i - a_{i+1} \neq r_{i+1} + s_i - pa_i$. Then both sides are non-negative, because $v_u(\phi(v_i)u^{t_i-a_{i+1}} - v_{i+1}u^{s_i-pa_i}) \ge 0$. Comparing $pr_i + t_i - a_{i+1} \ge 0$ with $t_i + pa_i - a_{i+1} \le p-1$, we get $r_i \ge a_i$. Then $r_i + s_{i-1} - pa_{i-1} \ge s_{i-1} - pa_{i-1} + a_i \ge 0$, and $pr_{i-1} + t_{i-1} - a_i \ge 0$ because $v_u(\phi(v_{i-1})u^{t_{i-1}-a_i} - v_i u^{s_{i-1}-pa_{i-1}}) \ge 0$. Comparing $pr_{i-1} + t_{i-1} - a_i \ge 0$ with $t_{i-1} + pa_{i-1} - a_i \le p-1$, we get $r_{i-1} \ge a_{i-1}$. The same argument goes on and shows that $r_2 \ge a_2$. This is a contradiction.

The above argument shows that

$$r_i < a_i,\ pr_i + t_i - a_{i+1} = r_{i+1} + s_i - pa_i < 0 \text{ for } 2 \le i \le n.$$

Combining these equations with $s_1 - pa_1 - a_2 = pr_1 + r_2 + t_1$, we get

$$\begin{aligned} -(p^n+1)r_1 = (p^n+1)a_1 + (s_n - t_n) + p(s_{n-1} - t_{n-1}) + \\ \cdots + p^{n-3}(s_3 - t_3) + p^{n-2}(s_2 - t_2) - p^{n-1}(s_1 - t_1), \end{aligned}$$

$$\begin{aligned} -(p^n+1)r_2 = (p^n+1)a_2 - (s_1 - t_1) - p(s_n - t_n) - \\ \cdots - p^{n-3}(s_4 - t_4) - p^{n-2}(s_3 - t_3) - p^{n-1}(s_2 - t_2), \end{aligned}$$

$$\begin{aligned} -(p^n+1)r_3 = (p^n+1)a_3 + (s_2 - t_2) - p(s_1 - t_1) - \\ \cdots - p^{n-3}(s_5 - t_5) - p^{n-2}(s_4 - t_4) - p^{n-1}(s_3 - t_3), \end{aligned}$$

$$\vdots$$

$$\begin{aligned} -(p^n+1)r_n = (p^n+1)a_n + (s_{n-1} - t_{n-1}) + p(s_{n-2} - t_{n-2}) + \\ \cdots + p^{n-3}(s_2 - t_2) - p^{n-2}(s_1 - t_1) - p^{n-1}(s_n - t_n). \end{aligned}$$

As $|s_i - t_i| \le p+1$ and

$$(p+1) + p(p+1) + \cdots + p^{n-1}(p+1) = \left(\frac{p^n-1}{p-1}\right)(p+1) < 2(p^n+1),$$

we get $-a_i - 1 \le r_i \le -a_i + 1$. When $e = p-1$, as $|s_i - t_i| \le p-1$ and

$$(p-1) + p(p-1) + \cdots + p^{n-1}(p-1) = \left(\frac{p^n-1}{p-1}\right)(p-1) < (p^n+1),$$

we get $r_i = -a_i$.

As $r_2 + t_1 + pa_1 \le p-1$, we have

$$pa_1 \le t_1 + pa_1 \le p - 1 - r_2 \le p + a_2.$$

For $2 \le i \le n$, as $t_i + pa_i - a_{i+1} \le p-1$, we have

$$pa_i \le t_i + pa_i \le p - 1 + a_{i+1}.$$

Take an index i_0 such that a_{i_0} is the greatest. As $pa_{i_0} \le a_{i_0+1} + p \le a_{i_0} + p$, we get $a_{i_0} \le \frac{p}{p-1} < 2$. Combining $-a_i - 1 \le r_i$ and $r_i < a_i$, we get $a_i \ge 0$. Hence

$$a_i = 0,\ r_i = -1,\ \text{or } a_i = 1,\ -2 \le r_i \le 0$$

for every i.

In the case $a_2 = 0$, we have $r_2 = -1$. Comparing $t_1 + pa_1 + a_2 > e$ with $r_2 + t_1 + pa_1 \le p - 1$, we get $e < p$. When $e = p-1$, we have $r_2 = -a_2$. This is a contradiction.

In the case $a_2 = 1$. As $0 \le t_i + pa_i - a_{i+1} \le p-1$ for $2 \le i \le n$, we have $a_i = 1$ for all i and $t_i = 0$ for $2 \le i \le n$. As $r_2 + pa_1 + t_1 \le p-1$, we have $r_2 \le -1$. As $pr_2 + t_2 - a_3 = r_3 + s_2 - pa_2$, we have $r_3 = pr_2 + p - 1 - e \le -e - 1 \le -3$. This is a contradiction.

Thus we may assume $t_1 + pa_1 + a_2 \le e$. We put $\mathfrak{M}_{3,\mathbb{F}} = \left(\begin{pmatrix} u^{-a_i} & 0 \\ 0 & u^{a_i} \end{pmatrix} \right)_i \cdot \mathfrak{M}_{0,\mathbb{F}}$, then

$$\mathfrak{M}_{3,\mathbb{F}} \sim \left(\alpha_1 \begin{pmatrix} 0 & u^{s_1 - pa_1 - a_2} \\ u^{t_1 + pa_1 + a_2} & 0 \end{pmatrix}, \alpha_2 \begin{pmatrix} u^{s_2 - pa_2 + a_3} & 0 \\ 0 & u^{t_2 + pa_2 - a_3} \end{pmatrix}, \ldots, \alpha_n \begin{pmatrix} u^{s_n - pa_n + a_1} & 0 \\ 0 & u^{t_n + pa_n - a_1} \end{pmatrix} \right)$$

and $\mathfrak{M}_{1,\mathbb{F}} = \left(\begin{pmatrix} 1 & v_i u^{-a_i} \\ 0 & 1 \end{pmatrix} \right)_i \cdot \mathfrak{M}_{3,\mathbb{F}}$. Note that $\mathfrak{M}_{3,\mathbb{F}}$ satisfies the conditions of Lemma 2.3, and let x_3 be the point of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ corresponding to $\mathfrak{M}_{3,\mathbb{F}}$. If we put $N_i = \begin{pmatrix} 0 & v_i u^{-a_i} \\ 0 & 0 \end{pmatrix}$, then

$$\phi(N_1) \begin{pmatrix} 0 & u^{s_1 - pa_1 - a_2} \\ u^{t_1 + pa_1 + a_2} & 0 \end{pmatrix} N_2 = \begin{pmatrix} 0 & \phi(v_1) v_2 u^{t_1} \\ 0 & 0 \end{pmatrix},$$

$$\phi(N_i) \begin{pmatrix} u^{s_i - pa_i + a_{i+1}} & 0 \\ 0 & u^{t_i + pa_i - a_{i+1}} \end{pmatrix} N_{i+1} = 0$$

for $2 \le i \le n$. Here we have $v_u\big(\phi(v_1) v_2 u^{t_1}\big) \ge 0$, because $s_1 - pa_1 - a_2 \ge 0$ and $v_u\big(u^{s_1 - pa_1 - a_2} - \phi(v_1) v_2 u^{t_1}\big) \ge 0$. Hence x_1 and x_3 lie on the same connected component by Lemma 2.4.

We are going to compare $\mathfrak{M}_{0,\mathbb{F}}$ and $\mathfrak{M}_{3,\mathbb{F}}$. Recall the previous operations on the basis of $\mathfrak{M}_{0,\mathbb{F}}$ that changed $(t_i)_{1\leq i\leq n}$ so that $|s_i - t_i| \leq p+1$ keeping $0 \leq t_i \leq e$ for all i. Apply the same operations to the basis of $\mathfrak{M}_{3,\mathbb{F}}$. By Lemma 2.5, these operations do not affect which of the connected components x_3 lies on. So we may assume that

$$s_1 - pa_1 - a_2,\ s_2 - pa_2 + a_3, \ldots,\ s_n - pa_n + a_1$$

are all in $\big[(e-p-1)/2, (e+p+1)/2\big]$. As $(e-p-1)/2 \leq s_i \leq (e+p+1)/2$, we have that

$$|pa_1 + a_2| \leq p+1,\ |pa_2 - a_3| \leq p+1, \ldots,\ |pa_n - a_1| \leq p+1.$$

Summing up the above inequalities after multiplying some p-powers so that we can eliminate a_j for $j \neq i$, we get $|(p^n+1)a_i| \leq \big\{(p^n-1)/(p-1)\big\}(p+1)$. So we have $|a_i| \leq 1$ for all i.

In the case $e \geq p$. We consider the operations that decrease $|a_i|$ by 1 for an index i keeping the condition of ϕ-stability. By Lemma 2.5, these operations do not affect which of the connected components x_3 lies on. We prove that we can continue the operations until we have $a_i = 0$ for all i, that is, x_0 and x_3 lie on the same connected component. Suppose that we cannot continue the operations and there is some nonzero a_i. The condition of ϕ-stability is equivalent to

$$C_1 : 0 \leq s_1 - pa_1 - a_2 \leq e,\ C_2 : 0 \leq s_2 - pa_2 + a_3 \leq e,$$
$$\ldots,\ C_n : 0 \leq s_n - pa_n + a_1 \leq e.$$

Note that if $a_i \neq 0$ or $a_{i+1} \neq 0$, we can decrease $|a_i|$ or $|a_{i+1}|$ keeping C_i.

We put

$$c_i = \sharp\big\{i \leq j \leq i+1 \;\big|\; \text{we can decrease } |a_j| \text{ keeping } C_i\big\},$$

and claim that $\sharp\{j \mid a_j \neq 0\} = \sum_{i=1}^n c_i$. First, if $a_i \neq 0$, we have $c_{i-1} \geq 1$ and $c_i \geq 1$ from the above remark. So we have $\sharp\{j \mid a_j \neq 0\} \leq \sum_{i=1}^n c_i$. Second, we count $a_i \neq 0$ in not both of C_{i-1} and C_i, because we cannot continue the operations. So we have $\sharp\{j \mid a_j \neq 0\} \geq \sum_{i=1}^n c_i$. Hence we have equality. From this equality, we have $a_i \neq 0$ and $c_i = 1$ for all i. For $2 \leq i \leq n$, we have $a_i = a_{i+1} \neq 0$ because $c_i = 1$. So we have $a_1 = a_2 \neq 0$, but this contradicts $c_1 = 1$.

In the case $e = p-1$. We have $|pa_1 + a_2| \leq p-1$ by C_1, and $|pa_i - a_{i+1}| \leq p-1$ by C_i for $2 \leq i \leq n$. Summing up these inequalities after multiplying some p-powers so that we can eliminate a_j for $j \neq i$, we get $|(p^n+1)a_i| \leq p^n - 1$. So we have $a_i = 0$ for all i.

Hence x_0 and x_3 lie on the same connected component. This completes the proof. □

2.3 Application

As an application of Theorem 2.6, we can improve a theorem in [Kis] comparing a deformation ring and a Hecke ring. We recall some notation from [Kis], and the interested reader should consult [Kis] for more detailed definitions.

Let F be a totally real field, and D be a totally definite quaternion algebra with center F. Let Σ be the set of finite primes where D is ramified. We assume that Σ does not contain any primes dividing p. We put $\Sigma_p = \Sigma \cup \{\mathfrak{p}\}_{\mathfrak{p}|p}$, and fix a maximal order $\mathcal{O}_D$ of D. Let $U = \prod_v U_v \subset (D \otimes_F \mathbb{A}_F^f)^\times$ be a compact open subgroup contained in $\prod_v (\mathcal{O}_D)_v^\times$, and we assume that $U_v = (\mathcal{O}_D)_v^\times$ for all $v \in \Sigma_p$. Let $\mathcal{O}$ be the ring of integers of a p-adic field. We fix a continuous character $\psi : (\mathbb{A}_F^f)^\times / F^\times \to \mathcal{O}^\times$ such that ψ is trivial on $U_v \cap \mathcal{O}_{F_v}^\times$ for any finite place v of F. Let S be a finite set of primes containing the infinite primes, Σ_p, and the finite primes v of F such that $U_v \subset D_v^\times$ is not maximal compact. We fix a decomposition group $G_{F_v} \subset G_{F,S}$ for each $v \in S$. Let $\mathbb{T}'_{\psi,\mathcal{O}}(U)$ (resp. $\mathbb{T}_{\psi,\mathcal{O}}(U)$) denote the image of $\mathbb{T}^{\mathrm{univ}}_{S,\mathcal{O}}$ (resp. $\mathbb{T}^{\mathrm{univ}}_{S^p,\mathcal{O}}$) in the endomorphism ring of $S_{2,\psi}(U,\mathcal{O})$. Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_{\psi,\mathcal{O}}(U)$ that induces a non-Eisenstein maximal ideal of $\mathbb{T}^{\mathrm{univ}}_{S^p,\mathcal{O}}$, and put $\mathfrak{m}' = \mathfrak{m} \cap \mathbb{T}'_{\psi,\mathcal{O}}(U)$. Then there exists a continuous representation $\rho_{\mathfrak{m}'} : G_{F,S} \to GL_2(\mathbb{T}'_{\psi,\mathcal{O}}(U)_{\mathfrak{m}'})$ such that the characteristic polynomial of $\rho_{\mathfrak{m}'}(\mathrm{Frob}_v)$ is $X^2 - T_v X + \mathbf{N}(v) S_v$ for $v \notin S$. Here $\mathbf{N}(v)$ denotes the order of the residue field at v. Let $\mathbb{F}$ be the residue field of $\mathbb{T}'_{\psi,\mathcal{O}}(U)_{\mathfrak{m}'}$. Let $\bar{\rho}_{\mathfrak{m}'} : G_{F,S} \to GL_2(\mathbb{F})$ denote the representation obtained by reducing $\rho_{\mathfrak{m}'}$ modulo $\mathfrak{m}'$.

Now we suppose that $\bar{\rho}_{\mathfrak{m}'}$ satisfies the following conditions.

1. $\bar{\rho}_{\mathfrak{m}'}$ is unramified outside the primes of F dividing p.
2. The restriction of $\bar{\rho}_{\mathfrak{m}'}$ to $G_{F(\zeta_p)}$ is absolutely irreducible.
3. If $p = 5$, and $\bar{\rho}_{\mathfrak{m}'}$ has projective image isomorphic to $PGL_2(\mathbb{F}_5)$, then the kernel of $\mathrm{proj}\,\bar{\rho}_{\mathfrak{m}'}$ does not fix $F(\zeta_5)$.
4. For each finite prime $v \in S \setminus \Sigma_p$, we have
$$\big(1 - \mathbf{N}(v)\big)\Big(\big(1 + \mathbf{N}(v)\big)^2 \det \bar{\rho}_{\mathfrak{m}'}(\mathrm{Frob}_v) - \big(\mathbf{N}(v)\big)\big(\mathrm{tr}\,\bar{\rho}_{\mathfrak{m}'}(\mathrm{Frob}_v)\big)^2\Big) \in \mathbb{F}^\times.$$

Let $R_{F,S}$ (resp. $R^\square_{F,S}$) be the universal deformation $\mathcal{O}$-algebra (resp. the universal framed deformation $\mathcal{O}$-algebra) of $\bar{\rho}_{\mathfrak{m}'}$, and put $\mathbb{T}^\square = R^\square_{F,S} \otimes_{R_{F,S}} \mathbb{T}_{\psi,\mathcal{O}}(U)_{\mathfrak{m}}$. We take a subset σ' of the set of primes of F dividing p, and an unramified character $\chi_{\mathfrak{p}}$ of $G_{F_{\mathfrak{p}}}$ for each $\mathfrak{p} \in \sigma'$, such that $\mathfrak{m}$ is σ-ordinary when we put $\sigma = (\sigma', \{\chi_{\mathfrak{p}}\}_{\mathfrak{p} \in \sigma'})$. Now we can define a deformation ring $\tilde{R}^{\sigma,\psi}_{F,S}$ and a map $\tilde{R}^{\sigma,\psi}_{F,S} \to \mathbb{T}^\square$ as in (3.4) of [Kis].

Theorem 2.7. *With the above notation and the assumptions, $\tilde{R}^{\sigma,\psi}_{F,S} \to \mathbb{T}^\square$ is an isomorphism up to p-power torsion kernel.*

Proof. Applying the Theorem 2.6, the proof goes on as in the proof of [Kis, Theorem 3.4.11]. □

3 Dimension

3.1 Preliminaries

The moduli space $\mathscr{GR}_{V_{\mathbb{F}},0}$ is described via the Kisin modules as in the following.

Proposition 3.1. *For any $\mathbb{F}$-algebra A, the elements of $\mathscr{GR}_{V_{\mathbb{F}},0}(A)$ naturally correspond to finite projective $(k[[u]]\otimes_{\mathbb{F}_p}A)$-submodules $\mathfrak{M}_A\subset M_{\mathbb{F}}\otimes_{\mathbb{F}}A$ that satisfy the followings:*

1. *$\mathfrak{M}_A$ generate $M_{\mathbb{F}}\otimes_{\mathbb{F}}A$ over $\big(k((u))\otimes_{\mathbb{F}_p}A\big)$.*
2. *$u^e\mathfrak{M}_A\subset(1\otimes\phi)\big(\phi^*(\mathfrak{M}_A)\big)\subset\mathfrak{M}_A$.*

Proof. This follows from the construction of $\mathscr{GR}_{V_{\mathbb{F}},0}$ in [Kis, Corollary 2.1.13]. $\square$

By Proposition 3.1, we often identify a point of $\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$ with the corresponding finite free $(k[[u]]\otimes_{\mathbb{F}_p}\mathbb{F}')$-module.

From now on, in this section, we assume $\mathbb{F}_{q^2}\subset\mathbb{F}$ and fix an embedding $k\hookrightarrow\mathbb{F}$. This assumption does not matter, because we may extend $\mathbb{F}$ to prove the main theorem of this section. We consider the isomorphism

$$\mathcal{O}_{\mathcal{E}}\otimes_{\mathbb{Z}_p}\mathbb{F}\cong k((u))\otimes_{\mathbb{F}_p}\mathbb{F}\overset{\sim}{\to}\prod_{\sigma\in\mathrm{Gal}(k/\mathbb{F}_p)}\mathbb{F}((u))\ ;\ \Big(\sum a_iu^i\Big)\otimes b\mapsto\Big(\sum\sigma(a_i)bu^i\Big)_\sigma$$

and let $\epsilon_\sigma\in k((u))\otimes_{\mathbb{F}_p}\mathbb{F}$ be the primitive idempotent corresponding to σ. Take $\sigma_1,\cdots,\sigma_n\in\mathrm{Gal}(k/\mathbb{F}_p)$ such that $\sigma_{i+1}=\sigma_i\circ\phi^{-1}$. Here we regard ϕ as the p-th power Frobenius, and use the convention that $\sigma_{n+i}=\sigma_i$. In the following, we often use such conventions. Then we have $\phi(\epsilon_{\sigma_i})=\epsilon_{\sigma_{i+1}}$ and $\phi: M_{\mathbb{F}}\to M_{\mathbb{F}}$ determines $\phi:\epsilon_{\sigma_i}M_{\mathbb{F}}\to\epsilon_{\sigma_{i+1}}M_{\mathbb{F}}$. For $(A_i)_{1\le i\le n}\in GL_2\big(\mathbb{F}((u))\big)^n$, we write

$$M_{\mathbb{F}}\sim(A_1,A_2,\dots,A_n)=(A_i)_i$$

if there is a basis $\{e_1^i,e_2^i\}$ of $\epsilon_{\sigma_i}M_{\mathbb{F}}$ over $\mathbb{F}((u))$ such that $\phi\begin{pmatrix}e_1^i\\e_2^i\end{pmatrix}=A_i\begin{pmatrix}e_1^{i+1}\\e_2^{i+1}\end{pmatrix}$. We use the same notation for any sublattice $\mathfrak{M}_{\mathbb{F}}\subset M_{\mathbb{F}}$ similarly. Here and in the following, we consider only sublattices that are $(\mathfrak{S}\otimes_{\mathbb{Z}_p}\mathbb{F})$-modules.

Let A be an $\mathbb{F}$-algebra, and $\mathfrak{M}_A$ be a finite free $(k[[u]]\otimes_{\mathbb{F}_p}A)$-submodules of $M_{\mathbb{F}}\otimes_{\mathbb{F}}A$ that generate $M_{\mathbb{F}}\otimes_{\mathbb{F}}A$ over $k((u))\otimes_{\mathbb{F}_p}A$. We choose a basis $\{e_1^i,e_2^i\}_i$ of $\mathfrak{M}_A$ over $k[[u]]\otimes_{\mathbb{F}_p}A$. For $B=(B_i)_{1\le i\le n}\in GL_2\big(\mathbb{F}((u))\otimes_{\mathbb{F}_p}A\big)^n$, the $(\mathfrak{S}\otimes_{\mathbb{Z}_p}A)$-module generated by the entries of $\left\langle B_i\begin{pmatrix}e_1^i\\e_2^i\end{pmatrix}\right\rangle$ for $1\le i\le n$ with the basis given by these entries is denoted by $B\cdot\mathfrak{M}_A$. Note that $B\cdot\mathfrak{M}_A$ depends on the choice of the basis of $\mathfrak{M}_A$. We can see that if $\mathfrak{M}_{\mathbb{F}}\sim(A_i)_i$ for $(A_i)_{1\le i\le n}\in GL_2\big(\mathbb{F}((u))\big)^n$ with respect to a given basis, then we have

$$B\cdot\mathfrak{M}_{\mathbb{F}}\sim\big(\phi(B_i)A_i(B_{i+1})^{-1}\big)_i$$

with respect to the induced basis.

Lemma 3.2. *Suppose $\mathbb{F}'$ is a finite extension of $\mathbb{F}$, and $x\in\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$ corresponds to $\mathfrak{M}_{\mathbb{F}'}$. Put $\mathfrak{M}_{j,\mathbb{F}'}=\left(\begin{pmatrix}u^{s_{j,i}}&v_{j,i}\\0&u^{t_{j,i}}\end{pmatrix}\right)_i\cdot\mathfrak{M}_{\mathbb{F}'}$ for $1\le j\le 2$, $s_{j,i},t_{j,i}\in\mathbb{Z}$ and $v_{j,i}\in$*

$\mathbb{F}'((u))$. *Assume* $\mathfrak{M}_{1,\mathbb{F}'}$ *and* $\mathfrak{M}_{2,\mathbb{F}'}$ *correspond to* $x_1, x_2 \in \mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$ *respectively. Then* $x_1 = x_2$ *if and only if*

$$s_{1,i} = s_{2,i},\ t_{1,i} = t_{2,i} \text{ and } v_{1,i} - v_{2,i} \in u^{t_{1,i}}\mathbb{F}'[[u]] \text{ for all } i.$$

Proof. The equality $x_1 = x_2$ is equivalent to the existence of $B = (B_i)_{1 \leq i \leq n} \in GL_2(\mathbb{F}'[[u]])^n$ such that

$$B_i \begin{pmatrix} u^{s_{1,i}} & v_{1,i} \\ 0 & u^{t_{1,i}} \end{pmatrix} = \begin{pmatrix} u^{s_{2,i}} & v_{2,i} \\ 0 & u^{t_{2,i}} \end{pmatrix}$$

for all i. It is further equivalent to the condition that

$$\begin{pmatrix} u^{s_{2,i}-s_{1,i}} & v_{2,i}u^{-t_{1,i}} - u^{s_{2,i}-s_{1,i}-t_{1,i}}v_{1,i} \\ 0 & u^{t_{2,i}-t_{1,i}} \end{pmatrix} \in GL_2(\mathbb{F}'[[u]])$$

for all i. The last condition is equivalent to the desired condition. □

Proposition 3.3. *If* $M_{\mathbb{F}} \sim \left(\begin{pmatrix} u^e & u \\ 0 & 1 \end{pmatrix} \right)_i$, *then* $\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$ *is one point for any finite extension* $\mathbb{F}'$ *of* $\mathbb{F}$.

Proof. Let $\mathfrak{M}_{0,\mathbb{F}}$ be the lattice of $M_{\mathbb{F}}$ generated by the basis giving

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} u^e & u \\ 0 & 1 \end{pmatrix} \right)_i,$$

and let $\mathfrak{M}_{0,\mathbb{F}'} = \mathfrak{M}_{0,\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ for finite extensions $\mathbb{F}'$ of $\mathbb{F}$. Then $\mathfrak{M}_{0,\mathbb{F}'}$ gives a point of $\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$. By the Iwasawa decomposition, any point $\mathfrak{M}_{\mathbb{F}'}$ of $\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$ is written as $\left(\begin{pmatrix} u^{-s_i} & v_i \\ 0 & u^{t_i} \end{pmatrix} \right)_i \cdot \mathfrak{M}_{0,\mathbb{F}'}$ for $s_i, t_i \in \mathbb{Z}$ and $v_i \in \mathbb{F}((u))$. Then we have

$$\begin{aligned} \mathfrak{M}_{\mathbb{F}'} &\sim \left(\begin{pmatrix} u^{-ps_i} & \phi(v_i) \\ 0 & u^{pt_i} \end{pmatrix} \begin{pmatrix} u^e & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^{s_{i+1}} & -v_{i+1}u^{s_{i+1}-t_{i+1}} \\ 0 & u^{-t_{i+1}} \end{pmatrix} \right)_i \\ &= \left(\begin{pmatrix} u^{e-ps_i+s_{i+1}} & u^{1-ps_i-t_{i+1}} + \phi(v_i)u^{-t_{i+1}} - v_{i+1}u^{e-ps_i+s_{i+1}-t_{i+1}} \\ 0 & u^{pt_i-t_{i+1}} \end{pmatrix} \right)_i \end{aligned}$$

with respect to the basis induced from the given basis of $\mathfrak{M}_{0,\mathbb{F}'}$. We put $r_i = -v_u(v_i)$.

By $u^e\mathfrak{M}_{\mathbb{F}'} \subset (1 \otimes \phi)\big(\phi^*(\mathfrak{M}_{\mathbb{F}'})\big) \subset \mathfrak{M}_{\mathbb{F}'}$, we have $e - ps_i + s_{i+1} \leq e$ and $pt_i - t_{i+1} \geq 0$ for all i, so we get $s_i, t_i \geq 0$ for all i.

We are going to show that $1 - ps_i - t_{i+1} \geq 0$ for all i. We assume that $1 - ps_{i_0} - t_{i_0+1} < 0$ for some i_0. Then $v_u(v_{i_0+1}u^{e-ps_{i_0}+s_{i_0+1}-t_{i_0+1}}) \leq 1 - ps_{i_0} - t_{i_0+1}$, because $\phi(v_{i_0})u^{-t_{i_0+1}}$ has no term of degree $1 - ps_{i_0} - t_{i_0+1}$. So we get $r_{i_0+1} - s_{i_0+1} \geq e - 1 \geq 0$. Take an index i_1 such that $r_{i_1} - s_{i_1}$ is the maximum. We note that $r_{i_1} - s_{i_1} \geq 0$. Then we have $v_u\big(\phi(v_{i_1})u^{-t_{i_1+1}}\big) = v_u(v_{i_1+1}u^{e-ps_{i_1}+s_{i_1+1}-t_{i_1+1}})$, because $v_u\big(\phi(v_{i_1})u^{-t_{i_1+1}}\big) \leq -ps_{i_1} - t_{i_1+1}$. So we get $r_{i_1+1} - s_{i_1+1} = p(r_{i_1} - s_{i_1}) + e > r_{i_1} - s_{i_1}$. This is a contradiction. Thus we have proved that $1 - ps_i - t_{i+1} \geq 0$ for all i, and this is equivalent to that $s_i = 0$ and $0 \leq t_i \leq 1$ for all i.

We assume $t_i = 1$ for some i. Then we have $t_i = 1$ for all i, because $pt_{i-1} - t_i \geq 0$ for all i. We show that $r_i \leq -1$ for all i. We take an index i_2 such that r_{i_2} is the maximum, and assume that $r_{i_2} \geq 0$. Then we have $r_{i_2+1} = pr_{i_2} + e > r_{i_2}$, because $v_u\big(1 + \phi(v_{i_2})u^{-1} - v_{i_2+1}u^{e-1}\big) \geq 0$. This is a contradiction. So we have $r_i \leq -1$ for all i. Then we may assume $v_i = 0$ for all i by Lemma 3.2. Now we have $\mathfrak{M}_{\mathbb{F}'} \sim \left(\begin{pmatrix} u^e & 1 \\ 0 & u^{p-1} \end{pmatrix}\right)_i$, but this contradicts $u^e\mathfrak{M}_{\mathbb{F}'} \subset (1 \otimes \phi)\big(\phi^*(\mathfrak{M}_{\mathbb{F}'})\big)$.

Thus we have proved $s_i = t_i = 0$ for all i. Then we have $r_i \leq 0$, because $v_u(u + \phi(v_i) - v_{i+1}u^e) \geq 0$. So we may assume $v_i = 0$ for all i by Lemma 3.2, and we have $\mathfrak{M}_{\mathbb{F}'} = \mathfrak{M}_{0,\mathbb{F}'}$. This shows that $\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$ is one point. □

3.2 The case where $V_{\mathbb{F}}$ is not absolutely irreducible

In this section, we give the maximum of the dimensions of the moduli spaces in the case where $V_{\mathbb{F}}$ is not absolutely irreducible. We put $d_{V_{\mathbb{F}}} = \dim \mathscr{GR}_{V_{\mathbb{F}},0}$. In the proof of the following Proposition, three Lemmas appear.

Proposition 3.4. *We assume $V_{\mathbb{F}}$ is not absolutely irreducible, and write $e = (p+1)e_0 + e_1$ for $e_0 \in \mathbb{Z}$ and $0 \leq e_1 \leq p$. Then the followings are true.*

1. *There are $m_i \in \mathbb{Z}$ for $0 \leq i \leq d_{V_{\mathbb{F}}}$ such that $m_i \geq 0$, $m_{d_{V_{\mathbb{F}}}} > 0$ and*

$$|\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')| = \sum_{i=0}^{d_{V_{\mathbb{F}}}} m_i |\mathbb{F}'|^i$$

   for all sufficiently large extensions $\mathbb{F}'$ of $\mathbb{F}$.

2. (a) *In the case $0 \leq e_1 \leq p-2$, we have $d_{V_{\mathbb{F}}} \leq ne_0$. In this case, if*

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} u^{e_0} & 0 \\ 0 & u^{pe_0} \end{pmatrix}\right)_i,$$

   then $d_{V_{\mathbb{F}}} = ne_0$.

   (b) *In the case $e_1 = p-1$, we have $d_{V_{\mathbb{F}}} \leq ne_0 + 1$. In this case, if*

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} u^{e_0} & 0 \\ 0 & u^{pe_0+p-1} \end{pmatrix}\right)_i,$$

   then $d_{V_{\mathbb{F}}} = ne_0 + 1$.

   (c) *In the case $e_1 = p$, we have $d_{V_{\mathbb{F}}} \leq ne_0 + \max\big\{[n/2], 1\big\}$. In this case, if $n = 1$ and*

$$M_{\mathbb{F}} \sim \begin{pmatrix} u^{e_0} & 0 \\ 0 & u^{pe_0+p-1} \end{pmatrix},$$

then $d_{V_{\mathbb{F}}} = e_0 + 1$*, and if* $n \geq 2$ *and*

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} u^{e_{0,i}} & 0 \\ 0 & u^{p(2e_0+1-e_{0,i})} \end{pmatrix} \right)_i ,$$

then $d_{V_{\mathbb{F}}} = ne_0 + [n/2]$*. Here,* $e_{0,i} = e_0$ *if* i *is odd, and* $e_{0,i} = e_0 + 1$ *if* i *is even.*

Proof. Extending the field $\mathbb{F}$, we may assume that $V_{\mathbb{F}}$ is reducible. Let $\mathfrak{M}_{0,\mathbb{F}}$ be a lattice of $M_{\mathbb{F}}$ corresponding to a point of $\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F})$. Then we take and fix a basis of $\mathfrak{M}_{0,\mathbb{F}}$ over $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$ such that $\mathfrak{M}_{0,\mathbb{F}} \sim \left(\begin{pmatrix} \alpha_i u^{a_{0,i}} & w_{0,i} \\ 0 & \beta_i u^{b_{0,i}} \end{pmatrix} \right)_i$ for $\alpha_i, \beta_i \in \mathbb{F}^{\times}$, $0 \leq a_{0,i}, b_{0,i} \leq e$ and $w_{0,i} \in \mathbb{F}[[u]]$. For any finite extension $\mathbb{F}'$ of $\mathbb{F}$, we put $\mathfrak{M}_{0,\mathbb{F}'} = \mathfrak{M}_{0,\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ and $M_{\mathbb{F}'} = M_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$. By the Iwasawa decomposition, any sublattice of $M_{\mathbb{F}'}$ can be written as $\left(\begin{pmatrix} u^{s_i} & v'_i \\ 0 & u^{t_i} \end{pmatrix} \right)_i \cdot \mathfrak{M}_{0,\mathbb{F}'}$ for $s_i, t_i \in \mathbb{Z}$ and $v'_i \in \mathbb{F}'((u))$.

We put

$$I = \left\{ (\underline{a}, \underline{b}) \in \mathbb{Z}^n \times \mathbb{Z}^n \;\middle|\; \underline{a} = (a_i)_{1 \leq i \leq n},\ \underline{b} = (b_i)_{1 \leq i \leq n},\ 0 \leq a_i, b_i \leq e \right\},$$

and

$$\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') = \left\{ \left(\begin{pmatrix} u^{s_i} & v'_i \\ 0 & u^{t_i} \end{pmatrix} \right)_i \cdot \mathfrak{M}_{0,\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}') \;\middle|\; s_i, t_i \in \mathbb{Z}, v'_i \in \mathbb{F}'((u)), \right.$$
$$\left. a_i = a_{0,i} + ps_i - s_{i+1},\ b_i = b_{0,i} + pt_i - t_{i+1} \right\}$$

for $(\underline{a}, \underline{b}) = \big((a_i)_{1 \leq i \leq n}, (b_i)_{1 \leq i \leq n} \big) \in I$. Then we have

$$\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}') = \bigcup_{(\underline{a},\underline{b}) \in I} \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}'),$$

and this is a disjoint union by Lemma 3.2.

Take $\mathfrak{M}_{\mathbb{F}'} = \left(\begin{pmatrix} u^{s_i} & v'_i \\ 0 & u^{t_i} \end{pmatrix} \right)_i \cdot \mathfrak{M}_{0,\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$ with the basis induced from the basis of $\mathfrak{M}_{0,\mathbb{F}'}$, then $\mathfrak{M}_{\mathbb{F}'} \sim \left(\begin{pmatrix} \alpha_i u^{a_i} & w_i \\ 0 & \beta_i u^{b_i} \end{pmatrix} \right)_i$ for some $(w_i)_{1 \leq i \leq n} \in \mathbb{F}'[[u]]^n$. We note that $a_i + b_i - v_u(w_i) \leq e$ for all i by the condition $u^e \mathfrak{M}_{\mathbb{F}'} \subset (1 \otimes \phi)\big(\phi^*(\mathfrak{M}_{\mathbb{F}'})\big)$.

Now, any $\mathfrak{M}'_{\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$ can be written as $\left(\begin{pmatrix} 1 & v_i \\ 0 & 1 \end{pmatrix} \right)_i \cdot \mathfrak{M}_{\mathbb{F}'}$ for some

$(v_i)_{1\leq i\leq n} \in \mathbb{F}'((u))^n$. With the basis induced from $\mathfrak{M}_{\mathbb{F}'}$, we have

$$\mathfrak{M}'_{\mathbb{F}'} \sim \left(\begin{pmatrix} 1 & \phi(v_i) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_i u^{a_i} & w_i \\ 0 & \beta_i u^{b_i} \end{pmatrix} \begin{pmatrix} 1 & -v_{i+1} \\ 0 & 1 \end{pmatrix} \right)_i$$
$$= \left(\begin{pmatrix} \alpha_i u^{a_i} & w_i - \alpha_i u^{a_i} v_{i+1} + \beta_i u^{b_i} \phi(v_i) \\ 0 & \beta_i u^{b_i} \end{pmatrix} \right)_i .$$

We are going to examine the condition for $(v_i)_{1\leq i\leq n} \in \mathbb{F}'((u))^n$ to give a point of $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$ as $\left(\begin{pmatrix} 1 & v_i \\ 0 & 1 \end{pmatrix} \right)_i \cdot \mathfrak{M}_{\mathbb{F}'}$. Extending the field $\mathbb{F}$, we may assume that $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}) = \emptyset$ if and only if $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') = \emptyset$ for each $(\underline{a},\underline{b}) \in I$ and any finite extension $\mathbb{F}'$ of $\mathbb{F}$.

For $(v_i)_{1\leq i\leq n} \in \mathbb{F}'((u))^n$, we have $\mathfrak{M}'_{\mathbb{F}'} = \left(\begin{pmatrix} 1 & v_i \\ 0 & 1 \end{pmatrix} \right)_i \cdot \mathfrak{M}_{\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$ if and only if

$$v_u\bigl(w_i - \alpha_i u^{a_i} v_{i+1} + \beta_i u^{b_i}\phi(v_i)\bigr) \geq 0 \text{ and}$$
$$v_u(\alpha_i u^{a_i}) + v_u(\beta_i u^{b_i}) - v_u\bigl(w_i - \alpha_i u^{a_i} v_{i+1} + \beta_i u^{b_i}\phi(v_i)\bigr) \leq e \text{ for all } i,$$

by the condition $u^e \mathfrak{M}'_{\mathbb{F}'} \subset (1\otimes\phi)\bigl(\phi^*(\mathfrak{M}'_{\mathbb{F}'})\bigr) \subset \mathfrak{M}'_{\mathbb{F}'}$. This is further equivalent to

$$v_u\bigl(\alpha_i u^{a_i} v_{i+1} - \beta_i u^{b_i}\phi(v_i)\bigr) \geq \max\{0, a_i + b_i - e\},$$

because $v_u(w_i) \geq \max\{0, a_i + b_i - e\}$. We put $r_i = -v_u(v_i)$, and note that

$$v_u(\alpha_{i-1} u^{a_{i-1}} v_i) \geq \max\{0, a_{i-1} + b_{i-1} - e\} \Leftrightarrow r_i \leq \min\{a_{i-1}, e - b_{i-1}\},$$
$$v_u\bigl(\beta_i u^{b_i}\phi(v_i)\bigr) \geq \max\{0, a_i + b_i - e\} \Leftrightarrow r_i \leq \min\left\{ \frac{e - a_i}{p}, \frac{b_i}{p} \right\}.$$

We define an $\mathbb{F}'$-vector space $\widetilde{N}_{\underline{a},\underline{b},\mathbb{F}'}$ by

$$\widetilde{N}_{\underline{a},\underline{b},\mathbb{F}'} = \bigl\{ (v_1,\ldots,v_n) \in \mathbb{F}'((u))^n \bigm| v_u\bigl(\alpha_i u^{a_i} v_{i+1} - \beta_i u^{b_i}\phi(v_i)\bigr) \geq \max\{0, a_i + b_i - e\} \text{ for all } i \bigr\}.$$

We note that $\widetilde{N}_{\underline{a},\underline{b},\mathbb{F}'} \supset \mathbb{F}'[[u]]^n$, and put $N_{\underline{a},\underline{b},\mathbb{F}'} = \widetilde{N}_{\underline{a},\underline{b},\mathbb{F}'} / \mathbb{F}'[[u]]^n$. Then we have a bijection $N_{\underline{a},\underline{b},\mathbb{F}'} \to \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$ by Lemma 3.2. We put $d_{\underline{a},\underline{b}} = \dim_{\mathbb{F}'} N_{\underline{a},\underline{b},\mathbb{F}'}$, and note that $\dim_{\mathbb{F}'} N_{\underline{a},\underline{b},\mathbb{F}'}$ is independent of finite extensions $\mathbb{F}'$ of $\mathbb{F}$.

We take a basis $(\mathbf{v}_j)_{1\leq j\leq d_{\underline{a},\underline{b}}}$ of $N_{\underline{a},\underline{b},\mathbb{F}}$ over $\mathbb{F}$, where $\mathbf{v}_j = (v_{j,1},\ldots,v_{j,n}) \in \mathbb{F}((u))^n$. Then, by Proposition 3.1, an $(\mathbb{F}[[u]] \otimes_{\mathbb{F}} \mathbb{F}[X_1,\ldots,X_{d_{\underline{a},\underline{b}}}])$-module

$$\mathfrak{M}'_{\mathbb{F}[X_1,\ldots,X_{d_{\underline{a},\underline{b}}}]} = \left(\begin{pmatrix} 1 & \sum_j v_{j,i} X_j \\ 0 & 1 \end{pmatrix} \right)_i \cdot \bigl(\mathfrak{M}_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}[X_1,\ldots,X_{d_{\underline{a},\underline{b}}}]\bigr)$$

gives a morphism $f_{\underline{a},\underline{b}} : \mathbb{A}_{\mathbb{F}}^{d_{\underline{a},\underline{b}}} \to \mathscr{GR}_{V_{\mathbb{F}},0}$ such that $f_{\underline{a},\underline{b}}(\mathbb{F}')$ is injective and the image of $f_{\underline{a},\underline{b}}(\mathbb{F}')$ is $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$. Then we have (1) and

$$d_{V_{\mathbb{F}}} = \max_{(\underline{a},\underline{b})\in I,\ \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F})\neq\emptyset} \{d_{\underline{a},\underline{b}}\}.$$

Before going into a proof of (2), we will examine $d_{\underline{a},\underline{b}}$ to evaluate $d_{V_{\mathbb{F}}}$. We put

$$S_{\underline{a},\underline{b},i} = \left\{ (0,\dots,0,v_i,0,\dots,0) \in \mathbb{F}((u))^n \;\middle|\; v_i = u^{-r_i}, \right.$$
$$\left. 1 \leq r_i \leq \min\left\{ a_{i-1}, e-b_{i-1}, \frac{e-a_i}{p}, \frac{b_i}{p} \right\} \right\}$$

for $1 \leq i \leq n$,

$$S_{\underline{a},\underline{b},i,j} = \left\{ (0,\dots,0,v_i,v_{i+1},\dots,v_{i+j},0,\dots,0) \in \mathbb{F}((u))^n \;\middle|\; v_i = u^{-r_i}, \right.$$
$$1 \leq r_i \leq \min\{a_{i-1}, e-b_{i-1}\},\ \alpha_{i+l}u^{a_{i+l}}v_{i+l+1} = \beta_{i+l}u^{b_{i+l}}\phi(v_{i+l})$$
$$\text{and } -v_u(v_{i+l+1}) > \min\{a_{i+l}, e-b_{i+l}\} \text{ for } 0 \leq l \leq j-1,$$
$$\left. -v_u(v_{i+j}) \leq \min\left\{ \frac{e-a_{i+j}}{p}, \frac{b_{i+j}}{p} \right\} \right\}$$

for $1 \leq i \leq n$ and $1 \leq j \leq n-1$, and

$$S_{\underline{a},\underline{b}} = \left\{ (v_1,\dots,v_n) \in \mathbb{F}((u))^n \;\middle|\; \alpha_i u^{a_i} v_{i+1} = \beta_i u^{b_i}\phi(v_i),\ v_1 = u^{v_u(v_1)} \right.$$
$$\left. \text{and } -v_u(v_{i+1}) > \min\{a_i, e-b_i\} \text{ for all } i \right\}.$$

In the above definitions, v_i is on the i-th component. Clearly, all elements of $\bigcup_i S_{\underline{a},\underline{b},i} \cup \bigcup_{i,j} S_{\underline{a},\underline{b},i,j} \cup S_{\underline{a},\underline{b}}$ are in $\widetilde{N}_{\underline{a},\underline{b},\mathbb{F}}$.

Lemma 3.5. *The image of $\bigcup_i S_{\underline{a},\underline{b},i} \cup \bigcup_{i,j} S_{\underline{a},\underline{b},i,j} \cup S_{\underline{a},\underline{b}}$ in $N_{\underline{a},\underline{b},\mathbb{F}}$ forms an $\mathbb{F}$-basis of $N_{\underline{a},\underline{b},\mathbb{F}}$.*

Proof. It is clear that the image of $\bigcup_i S_{\underline{a},\underline{b},i} \cup \bigcup_{i,j} S_{\underline{a},\underline{b},i,j} \cup S_{\underline{a},\underline{b}}$ in $N_{\underline{a},\underline{b},\mathbb{F}}$ are linearly independent over $\mathbb{F}$. So it suffices to show that $\bigcup_i S_{\underline{a},\underline{b},i} \cup \bigcup_{i,j} S_{\underline{a},\underline{b},i,j} \cup S_{\underline{a},\underline{b}}$ and $\mathbb{F}[[u]]^n$ generates $\widetilde{N}_{\underline{a},\underline{b},\mathbb{F}}$. We take $(v_1,\dots,v_n) \in \widetilde{N}_{\underline{a},\underline{b},\mathbb{F}}$. We want to write $(v_1,\dots,v_n)$ as a linear combination of elements of $\bigcup_i S_{\underline{a},\underline{b},i} \cup \bigcup_{i,j} S_{\underline{a},\underline{b},i,j} \cup S_{\underline{a},\underline{b}}$ and $\mathbb{F}[[u]]^n$.

First, we consider the case where there exsits an index i_0 such that $-v_u(v_{i_0}) > \min\{a_{i_0-1}, e-b_{i_0-1}, (e-a_{i_0})/p, b_{i_0}/p\}$. Then there are following two cases:

(i) There are $1 \leq i_1 \leq n$ and $1 \leq j_1 \leq n-1$ such that
$i_0 \in [i_1, i_1+j_1]$, $1 \leq -v_u(v_{i_1}) \leq \min\{a_{i_1-1}, e-b_{i_1-1}\}$,
$a_{i_1+l} + v_u(v_{i_1+l+1}) = b_{i_1+l} + pv_u(v_{i_1+l})$
and $-v_u(v_{i_1+l+1}) > \min\{a_{i+l}, e-b_{i+l}\}$ for $0 \leq l \leq j_1-1$
and $-v_u(v_{i_1+j_1}) \leq \min\{(e-a_{i_1+j_1})/p, (b_{i_1+j_1})/p\}$.

(ii) $a_i + v_u(v_{i+1}) = b_i + pv_u(v_i)$ and $-v_u(v_{i+1}) > \min\{a_i, e - b_i\}$ for all i.

In the case (i), we can subtract a linear multiple of an element of $S_{\underline{a},\underline{b},i_1,j_1}$ from $(v_1, \dots, v_n)$ so that the u-valuations of the i-th component increase for all $i \in [i_1, i_1 + j_1]$. In the case (ii), we can subtract a linear multiple of an element of $S_{\underline{a},\underline{b}}$ from $(v_1, \dots, v_n)$ so that the u-valuations of the i-th component increase for all i.

Repeating such subtractions, we may assume that $-v_u(v_i) \leq \min\{a_{i-1}, e - b_{i-1}, (e-a_i)/p, b_i/p\}$ for all i. Then we can write $(v_1, \dots, v_n)$ as a linear combination of elements of $\bigcup_i S_{\underline{a},\underline{b},i}$ and $\mathbb{F}[[u]]^n$. □

By Lemma 3.5, we have $d_{\underline{a},\underline{b}} = \sum_i |S_{\underline{a},\underline{b},i}| + \sum_{i,j} |S_{\underline{a},\underline{b},i,j}| + |S_{\underline{a},\underline{b}}|$. We note that $0 \leq |S_{\underline{a},\underline{b}}| \leq 1$ by the definition, and put $d'_{\underline{a},\underline{b}} = \sum_i |S_{\underline{a},\underline{b},i}| + \sum_{i,j} |S_{\underline{a},\underline{b},i,j}|$.

We put

$$T_{\underline{a},\underline{b},i} = \left\{ m \in \mathbb{Z} \;\middle|\; \min\{a_{i-1}, e - b_{i-1}\} < pm + a_{i-1} - b_{i-1} \leq \min\left\{ \frac{e - a_i}{p}, \frac{b_i}{p} \right\} \right\},$$

and consider the map

$$\bigcup_{i+j=h} S_{\underline{a},\underline{b},i,j} \to T_{\underline{a},\underline{b},h};\ (v_{i'})_{1 \leq i' \leq n} \mapsto -v_u(v_{h-1}).$$

We can easily check that this map is injective. So we have $\sum_{i+j=h} |S_{\underline{a},\underline{b},i,j}| \leq |T_{\underline{a},\underline{b},h}|$ and $d'_{\underline{a},\underline{b}} \leq \sum_{1 \leq i \leq n} \bigl(|S_{\underline{a},\underline{b},i}| + |T_{\underline{a},\underline{b},i}|\bigr)$.

We take $(\underline{a}', \underline{b}') \in I$ such that $\sum_{1 \leq i \leq n} \bigl(|S_{\underline{a}',\underline{b}',i}| + |T_{\underline{a}',\underline{b}',i}|\bigr)$ is the maximum.

Lemma 3.6. *$|T_{\underline{a}',\underline{b}',i}| \leq 1$ for all i.*

Proof. We assume there is an index i_0 such that $|T_{\underline{a}',\underline{b}',i_0}| \geq 2$. We note that

$$\min\{a'_{i_0-1}, e - b'_{i_0-1}\} + p + 1 \leq \min\left\{ \frac{e - a'_{i_0}}{p}, \frac{b'_{i_0}}{p} \right\} \tag{$*$}$$

by $|T_{\underline{a}',\underline{b}',i_0}| \geq 2$. We are going to show that we can replace a'_{i_0-1}, b'_{i_0-1} so that $\sum_{1 \leq i \leq n} \bigl(|S_{\underline{a}',\underline{b}',i}| + |T_{\underline{a}',\underline{b}',i}|\bigr)$ increases. This contradicts the maximality of $\sum_{1 \leq i \leq n} \bigl(|S_{\underline{a}',\underline{b}',i}| + |T_{\underline{a}',\underline{b}',i}|\bigr)$. We divide the problem into three cases.

Firstly, if $a'_{i_0-1} + 2 \leq e - b'_{i_0-1}$, we replace a'_{i_0-1} by $a'_{i_0-1} + p$, and note that $a'_{i_0-1} + p \leq e$ by $(*)$. Then there is no change except for $S_{\underline{a}',\underline{b}',i_0-1}$, $S_{\underline{a}',\underline{b}',i_0}$, $T_{\underline{a}',\underline{b}',i_0-1}$ and $T_{\underline{a}',\underline{b}',i_0}$. We can see that $|S_{\underline{a}',\underline{b}',i_0}|$ increases by at least 2. The condition that there exists $m \in \mathbb{Z}$ such that

$$\min\{a'_{i_0-1}, e - b'_{i_0-1}\} < pm + a'_{i_0-1} - b'_{i_0-1} \leq \min\{a'_{i_0-1} + p, e - b'_{i_0-1}\},$$

is equivalent to the condition that there exists $m \in \mathbb{Z}$ such that

$$\min\left\{ \frac{e - a'_{i_0-1}}{p}, \frac{b'_{i_0-1}}{p} \right\} < m \leq \min\left\{ \frac{e - a'_{i_0-1}}{p}, \frac{b'_{i_0-1}}{p} + 1 \right\},$$

and further equivalent to the condition that there does not exists $m \in \mathbb{Z}$ such that

$$\min\left\{\frac{e-a'_{i_0-1}}{p}-1, \frac{b'_{i_0-1}}{p}\right\} < m \leq \min\left\{\frac{e-a'_{i_0-1}}{p}, \frac{b'_{i_0-1}}{p}\right\}.$$

If the above condition is satisfied, then $|S_{\underline{a}',\underline{b}',i_0-1}|$, $|T_{\underline{a}',\underline{b}',i_0-1}|$ do not change and $|T_{\underline{a}',\underline{b}',i_0}|$ decreases by 1. Otherwise, $|S_{\underline{a}',\underline{b}',i_0-1}|+|T_{\underline{a}',\underline{b}',i_0-1}|$ decreases by at most 1 and $|T_{\underline{a}',\underline{b}',i_0}|$ does not change. In both cases, we have that $\sum_{1\leq i\leq n}\bigl(|S_{\underline{a}',\underline{b}',i}|+|T_{\underline{a}',\underline{b}',i}|\bigr)$ increases by at least 1.

Secondly, if $a'_{i_0-1} \geq e-b'_{i_0-1}+2$, we replace b'_{i_0-1} by $b'_{i_0-1}-p$. Then, by the same arguments, we have that $\sum_{1\leq i\leq n}\bigl(|S_{\underline{a}',\underline{b}',i}|+|T_{\underline{a}',\underline{b}',i}|\bigr)$ increases by at least 1.

In the remaining case, that is the case where $a'_{i_0-1}-1 \leq e-b'_{i_0-1} \leq a'_{i_0-1}+1$, we replace a'_{i_0-1}, b'_{i_0-1} by $a'_{i_0-1}+p$, $b'_{i_0-1}-p$ respectively, and note that $a'_{i_0-1}+p \leq e$ and $b'_{i_0-1}-p \geq 0$ by $(*)$. Then there is no change except for $S_{\underline{a}',\underline{b}',i_0-1}$, $S_{\underline{a}',\underline{b}',i_0}$, $T_{\underline{a}',\underline{b}',i_0-1}$ and $T_{\underline{a}',\underline{b}',i_0}$. We can see that $|S_{\underline{a}',\underline{b}',i_0-1}|+|T_{\underline{a}',\underline{b}',i_0-1}|$ decreases by at most 1, $|S_{\underline{a}',\underline{b}',i_0}|$ increases by p and $|T_{\underline{a}',\underline{b}',i_0}|$ decreases by 1. Hence $\sum_{1\leq i\leq n}\bigl(|S_{\underline{a}',\underline{b}',i}|+|T_{\underline{a}',\underline{b}',i}|\bigr)$ increases by at least $p-2>0$.

Thus we have proved that $|T_{\underline{a}',\underline{b}',i}| \leq 1$ for all i. □

Lemma 3.7. *For all i, we have the followings:*

(A_i) *If $|S_{\underline{a}',\underline{b}',i}|+|T_{\underline{a}',\underline{b}',i}| = e_0+l$ for $l \geq 1$,*
then $|S_{\underline{a}',\underline{b}',i+1}|+|T_{\underline{a}',\underline{b}',i+1}| \leq e_0+e_1-pl+1$.

(B_i) *If $|S_{\underline{a}',\underline{b}',i}|+|T_{\underline{a}',\underline{b}',i}| = e_0+1$*
and $|S_{\underline{a}',\underline{b}',i+1}|+|T_{\underline{a}',\underline{b}',i+1}| = e_0+e_1-p+1$,
then $|S_{\underline{a}',\underline{b}',i+2}|+|T_{\underline{a}',\underline{b}',i+2}| \leq e_0-(p-1)e_1+1$.

Proof. By the definition of $T_{\underline{a},\underline{b},i}$, we have

$$|T_{\underline{a},\underline{b},i}| \leq \max\left\{\min\left\{\frac{e-a_i}{p}, \frac{b_i}{p}\right\} - \min\{a_{i-1}, e-b_{i-1}\}, 0\right\}.$$

Combining this with the definition of $S_{\underline{a},\underline{b},i}$, we get

$$|S_{\underline{a},\underline{b},i}|+|T_{\underline{a},\underline{b},i}| \leq \min\left\{\left[\frac{e-a_i}{p}\right], \left[\frac{b_i}{p}\right]\right\}, \tag{$\star$}$$

and equality happens if and only if in the following two cases:

- $\min\left\{\left[\frac{e-a_i}{p}\right], \left[\frac{b_i}{p}\right]\right\} - \min\{a_{i-1}, e-b_{i-1}\} \leq 0$.
- $\min\left\{\left[\frac{e-a_i}{p}\right], \left[\frac{b_i}{p}\right]\right\} - \min\{a_{i-1}, e-b_{i-1}\} = 1$
  and $p \mid \bigl(\min\{e-a_{i-1}, b_{i-1}\}+1\bigr)$.

We assume $|S_{\underline{a}',\underline{b}',i_1}| + |T_{\underline{a}',\underline{b}',i_1}| = e_0 + l$ for some i_1 and $l \geq 1$. Then we have $p(e_0 + l) \leq \min\{e - a'_{i_1}, b'_{i_1}\}$ by $(\star)$. By this inequality, we have

$$\begin{aligned} |S_{\underline{a}',\underline{b}',i_1+1}| &\leq \min\{a'_{i_1}, e - b'_{i_1}\} \leq \max\{a'_{i_1}, e - b'_{i_1}\} \\ &= e - \min\{e - a'_{i_1}, b'_{i_1}\} \leq e - p(e_0 + l) = e_0 + e_1 - pl. \end{aligned}$$

Combining this with $|T_{\underline{a}',\underline{b}',i_1+1}| \leq 1$, we get

$$|S_{\underline{a}',\underline{b}',i_1+1}| + |T_{\underline{a}',\underline{b}',i_1+1}| \leq e_0 + e_1 - pl + 1.$$

This shows (A_i) for all i.

Further, we examine the case where equality holds in the above inequality, assuming $l = 1$. In this case, we have that $\min\{a'_{i_1}, e - b'_{i_1}\} = e_0 + e_1 - p$, $\min\{e - a'_{i_1}, b'_{i_1}\} = p(e_0 + 1)$ and $|T_{\underline{a}',\underline{b}',i_1+1}| = 1$. Let m be the unique element of $T_{\underline{a}',\underline{b}',i_1+1}$. Then, by the definition of $T_{\underline{a}',\underline{b}',i_1+1}$, we have

$$\min\left\{\frac{e - a'_{i_1+1}}{p}, \frac{b'_{i_1+1}}{p}\right\} - \min\{a'_{i_1}, e - b'_{i_1}\} \geq pm - \min\{e - a'_{i_1}, b'_{i_1}\} \geq p,$$

because $\min\{e - a'_{i_1}, b'_{i_1}\} = p(e_0 + 1)$ and $pm - \min\{e - a'_{i_1}, b'_{i_1}\} > 0$. Combining this with $\min\{a'_{i_1}, e - b'_{i_1}\} = e_0 + e_1 - p$, we get $p(e_0 + e_1) \leq \min\{e - a'_{i_1+1}, b'_{i_1+1}\}$. By the previous argument, we have

$$|S_{\underline{a}',\underline{b}',i_1+2}| + |T_{\underline{a}',\underline{b}',i_1+2}| \leq e_0 - (p - 1)e_1 + 1.$$

Thus we have proved (B_i) for all i. $\square$

We are going to show (2). Firstly, we treat (a). We note that $e_0 + e_1 - pl + 1 \leq e_0 - p(l - 1) - 2$ in the case where $0 \leq e_1 \leq p - 3$, and that $e_0 + e_1 - pl + 1 \leq e_0 - p(l - 1) - 1$ and $e_0 - (p - 1)e_1 + 1 \leq e_0 - 1$ in the case where $e_1 = p - 2$. Then (A_i) and (B_i) for all i implies that $\sum_{1 \leq i \leq n} \bigl(|S_{\underline{a}',\underline{b}',i}| + |T_{\underline{a}',\underline{b}',i}|\bigr) \leq ne_0$. It further implies that

$$d'_{\underline{a},\underline{b}} \leq \sum_{1 \leq i \leq n} \bigl(|S_{\underline{a},\underline{b},i}| + |T_{\underline{a},\underline{b},i}|\bigr) \leq ne_0$$

for all $(\underline{a}, \underline{b}) \in I$, and that $d'_{\underline{a},\underline{b}} = ne_0$ only if $|S_{\underline{a},\underline{b},i}| + |T_{\underline{a},\underline{b},i}| = e_0$ for all i. To prove $d_{\underline{a},\underline{b}} \leq ne_0$, it suffice to show that $d'_{\underline{a},\underline{b}} = ne_0$ implies $S_{\underline{a},\underline{b}} = \emptyset$, because $|S_{\underline{a},\underline{b}}| \leq 1$ for all $(\underline{a}, \underline{b}) \in I$.

We assume that $d'_{\underline{a},\underline{b}} = ne_0$ and $S_{\underline{a},\underline{b}} \neq \emptyset$. By the maximality of $\sum_{1 \leq i \leq n} \bigl(|S_{\underline{a},\underline{b},i}| + |T_{\underline{a},\underline{b},i}|\bigr)$, we have $|T_{\underline{a},\underline{b},i}| \leq 1$ for all i. Let $(v_{0,i})_{1 \leq i \leq n}$ be the unique element of $S_{\underline{a},\underline{b}}$, and we put $r_{0,i} = -v_u(v_{0,i})$. Then we have

$$a_i - r_{0,i+1} = b_i - pr_{0,i} < \max\{0, a_i + b_i - e\}$$

for all i, by the definition of $S_{\underline{a},\underline{b}}$. By $(\star)$ and $e_0 - 1 \leq |S_{\underline{a},\underline{b},i}|$ for all i, we have

$$e_0 - 1 \leq a_i \leq e_0 + e_1,\ pe_0 \leq b_i \leq pe_0 + e_1 + 1$$

for all i. Take an index i_2 such that r_{0,i_2} is the maximum. Then we have

$$\begin{aligned}(p-1)r_{0,i_2} \leq pr_{0,i_2} - r_{0,i_2+1} = b_{i_2} - a_{i_2} &\leq (pe_0 + e_1 + 1) - (e_0 - 1) \\ &= (p-1)e_0 + e_1 + 2 \leq (p-1)e_0 + p.\end{aligned}$$

So we get $r_{0,i} \leq e_0 + 1$ for all i.

If $a_i + b_i - e \leq 0$, we have $r_{0,i} \geq e_0 + 1$ by $b_i - pr_{0,i} < 0$ and $pe_0 \leq b_i$. If $a_i + b_i - e > 0$, we have $r_{0,i} \geq e_0 + 1$ by $b_i - pr_{0,i} < a_i + b_i - e$ and $a_i \leq e_0 + e_1$. So we have $r_{0,i} = e_0 + 1$ for all i.

By $a_i - r_{0,i+1} = b_i - pr_{0,i}$, we have $(p-1)(e_0+1) = b_i - a_i$ for all i. By the range of a_i and b_i, we have the following two possibilities for each i:

$$(a_i, b_i) = (e_0 - 1, pe_0 + p - 2) \text{ or } (e_0, pe_0 + p - 1).$$

In both cases, we have $|S_{\underline{a},\underline{b},i+1}| = e_0 - 1$.

Now we must have equality in $(\star)$. So we must have $p \mid (\min\{e - a_{i-1}, b_{i-1}\} + 1)$, noting that $|T_{\underline{a},\underline{b},i}| = 1$. This contradicts the possibilities of a_{i-1}, b_{i-1}. Thus we have proved $d_{V_{\mathbb{F}}} \leq ne_0$.

For $\underline{a} = (e_0)_{1 \leq i \leq n}$ and $\underline{b} = (pe_0)_{1 \leq i \leq n}$, we have $d_{\underline{a},\underline{b}} \geq \sum_{1 \leq i \leq n} |S_{\underline{a},\underline{b},i}| = ne_0$. This shows that $d_{V_{\mathbb{F}}} = ne_0$, if

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} u^{e_0} & 0 \\ 0 & u^{pe_0} \end{pmatrix} \right)_i .$$

Secondly, we treat (b). In this case, we note that $e_0 + e_1 - pl + 1 = e_0 - p(l-1)$ and $e_0 - (p-1)e_1 + 1 \leq e_0 - 3$. Then (A_i) and (B_i) for all i implies $d'_{\underline{a},\underline{b}} \leq ne_0$, and further implies $d_{\underline{a},\underline{b}} \leq ne_0 + 1$, because $|S_{\underline{a},\underline{b}}| \leq 1$. Thus we have proved $d_{V_{\mathbb{F}}} \leq ne_0 + 1$.

For $\underline{a} = (e_0)_{1 \leq i \leq n}$ and $\underline{b} = (pe_0 + p - 1)_{1 \leq i \leq n}$, we have $d_{\underline{a},\underline{b}} \geq \sum_{1 \leq i \leq n} |S_{\underline{a},\underline{b},i}| + |S_{\underline{a},\underline{b}}| = ne_0 + 1$, because $(u^{-(e_0+1)})_{1 \leq i \leq n} \in S_{\underline{a},\underline{b}}$. This shows that $d_{V_{\mathbb{F}}} = ne_0 + 1$, if

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} u^{e_0} & 0 \\ 0 & u^{pe_0+p-1} \end{pmatrix} \right)_i .$$

At last, we treat (c). In this case, we note that $e_0 + e_1 - pl + 1 = e_0 - p(l-1) + 1$ and $e_0 - (p-1)e_1 + 1 \leq e_0 - 5$. Then (A_i) and (B_i) for all i implies $d'_{\underline{a},\underline{b}} \leq ne_0 + [n/2]$, and that $d'_{\underline{a},\underline{b}} = ne_0 + [n/2]$ only if $e_0 \leq |S_{\underline{a},\underline{b},i}| + |T_{\underline{a},\underline{b},i}| \leq e_0 + 1$ for all i.

If $n = 1$, then $d'_{\underline{a},\underline{b}} \leq e_0$ implies $d_{\underline{a},\underline{b}} \leq e_0 + 1$, and the given example for $d_{V_{\mathbb{F}}} = e_0 + 1$ is the same as in (b). So we may assume $n \geq 2$ in the following.

To prove $d_{\underline{a},\underline{b}} \leq ne_0 + [n/2]$, it suffices to show that $d'_{\underline{a},\underline{b}} = ne_0 + [n/2]$ implies $S_{\underline{a},\underline{b}} = \emptyset$, because $|S_{\underline{a},\underline{b}}| \leq 1$ for all $(\underline{a}, \underline{b}) \in I$.

We assume that $d'_{\underline{a},\underline{b}} = ne_0 + [n/2]$ and $S_{\underline{a},\underline{b}} \neq \emptyset$. By the maximality of $\sum_{1 \leq i \leq n} \bigl(|S_{\underline{a},\underline{b},i}| + |T_{\underline{a},\underline{b},i}|\bigr)$, we have $|T_{\underline{a},\underline{b},i}| \leq 1$ for all i. Let $(v_{1,i})_{1 \leq i \leq n}$ be the unique element of $S_{\underline{a},\underline{b}}$, and we put $r_{1,i} = -v_u(v_{1,i})$. Then we have

$$a_i - r_{1,i+1} = b_i - pr_{1,i} < \max\{0, a_i + b_i - e\}$$

for all i by the definition of $S_{\underline{a},\underline{b}}$. By ($\star$) and $e_0 - 1 \le |S_{\underline{a},\underline{b},i}|$, we have

$$e_0 - 1 \le a_i \le e_0 + p,\ pe_0 \le b_i \le pe_0 + p + 1$$

for all i. Take an index i_3 such that r_{1,i_3} is the maximum. Then we have

$$\begin{aligned}(p-1)r_{1,i_3} &\le pr_{1,i_3} - r_{1,i_3+1} = b_{i_3} - a_{i_3} \\ &\le (pe_0 + p + 1) - (e_0 - 1) = (p-1)e_0 + p + 2.\end{aligned}$$

So we get $r_{1,i} \le e_0 + 2$ for all i.

If $a_i + b_i - e \le 0$, we have $r_{1,i} \ge e_0 + 1$ by $b_i - pr_{1,i} < 0$ and $pe_0 \le b_i$. If $a_i + b_i - e > 0$, we have $r_{1,i} \ge e_0 + 1$ by $b_i - pr_{1,i} < a_i + b_i - e$ and $a_i \le e_0 + p$. So we have $e_0 + 1 \le r_{1,i} \le e_0 + 2$ for all i.

By $n \ge 2$, there is an index i_4 such that $|S_{\underline{a},\underline{b},i_4}| + |T_{\underline{a},\underline{b},i_4}| = e_0 + 1$. Then we have $e_0 + 1 \le \min\bigl\{(e - a_{i_4})/p, b_{i_4}/p\bigr\}$ by ($\star$). We are going to prove that if $e_0 + 1 \le \min\bigl\{(e - a_i)/p, b_i/p\bigr\}$, then $|S_{\underline{a},\underline{b},i+1}| + |T_{\underline{a},\underline{b},i+1}| = e_0$ and $e_0 + 1 \le \min\bigl\{(e - a_{i+1})/p, b_{i+1}/p\bigr\}$. If we have proved this claim, we have a contradiction by considering i_4.

We assume that $e_0 + 1 \le \min\bigl\{(e - a_i)/p, b_i/p\bigr\}$. Then we have $e_0 - 1 \le a_i \le e_0$, $pe_0 + p \le b_i \le pe_0 + p + 1$ and $e_0 - 1 \le |S_{\underline{a},\underline{b},i+1}| \le e_0$. If $|S_{\underline{a},\underline{b},i+1}| = e_0$, we have $a_i = e_0$ and $b_i = pe_0 + p$. However, this contradicts $pr_i - r_{i+1} = b_i - a_i$, because $pr_i - r_{i+1} \ne (p-1)e_0 + p$ by $e_0 + 1 \le r_i, r_{i+1} \le e_0 + 2$. So we have $|S_{\underline{a},\underline{b},i+1}| = e_0 - 1$ and $|T_{\underline{a},\underline{b},i+1}| = 1$. Let m be the unique element of $T_{\underline{a},\underline{b},i+1}$. By the definition of $T_{\underline{a},\underline{b},i+1}$, we have

$$\min\left\{\frac{e - a_{i+1}}{p}, \frac{b_{i+1}}{p}\right\} - \min\{a_i, e - b_i\} \ge pm - \min\{e - a_i, b_i\} \ge p - 1 \ge 2,$$

because $pe_0 + p \le \min\{e - a_i, b_i\} \le pe_0 + p + 1$ and $pm - \min\{e - a_i, b_i\} > 0$. This shows $e_0 + 1 \le \min\bigl\{(e - a_{i+1})/p, b_{i+1}/p\bigr\}$. Thus we have proved that $d_{V_{\mathbb{F}}} \le ne_0 + [n/2]$.

For $\underline{a} = (e_{0,i})_{1 \le i \le n}$ and $\underline{b} = \bigl(p(2e_0 + 1 - e_{0,i})\bigr)_{1 \le i \le n}$, we have

$$d_{\underline{a},\underline{b}} \ge \sum_{1 \le i \le n} |S_{\underline{a},\underline{b},i}| = ne_0 + [n/2],$$

where $e_{0,i}$ is defined in the statement of Proposition 3.4(2)(c). This shows that $d_{V_{\mathbb{F}}} = ne_0 + [n/2]$, if

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} u^{e_{0,i}} & 0 \\ 0 & u^{p(2e_0+1-e_{0,i})} \end{pmatrix}\right)_i.$$

□

3.3 The case where $V_{\mathbb{F}}$ is absolutely irreducible

In this section, we give the maximum of the dimensions of the moduli spaces in the case where $V_{\mathbb{F}}$ is absolutely irreducible. In the proof of the following Proposition, three Lemmas appear.

Proposition 3.8. *We assume $V_{\mathbb{F}}$ is absolutely irreducible, and write $e=(p+1)e_0+e_1$ for $e_0 \in \mathbb{Z}$ and $0 \le e_1 \le p$. Then the followings are true.*

1. *There are $m_i \in \mathbb{Z}$ for $0 \le i \le d_{V_{\mathbb{F}}}$ such that $m_{d_{V_{\mathbb{F}}}} > 0$ and*

$$|\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')| = \sum_{i=0}^{d_{V_{\mathbb{F}}}} m_i |\mathbb{F}'|^i$$

for all sufficiently large extensions $\mathbb{F}'$ of $\mathbb{F}$.

2. (a) *In the case $e_1=0$, we have $d_{V_{\mathbb{F}}} \le ne_0-1$. In this case, if*

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} 0 & 1 \\ u^{(p+1)e_0-1} & 0 \end{pmatrix}, \begin{pmatrix} u^{e_0} & 0 \\ 0 & u^{pe_0} \end{pmatrix}, \dots, \begin{pmatrix} u^{e_0} & 0 \\ 0 & u^{pe_0} \end{pmatrix} \right),$$

then $d_{V_{\mathbb{F}}} = ne_0-1$.

(b) *In the case $1 \le e_1 \le p-1$, we have $d_{V_{\mathbb{F}}} \le ne_0$. In this case, if*

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} 0 & 1 \\ u^{(p+1)e_0+1} & 0 \end{pmatrix}, \begin{pmatrix} u^{e_0} & 0 \\ 0 & u^{pe_0} \end{pmatrix}, \dots, \begin{pmatrix} u^{e_0} & 0 \\ 0 & u^{pe_0} \end{pmatrix} \right),$$

we have $d_{V_{\mathbb{F}}} = ne_0$.

(c) *In the case $e_1=p$, we have $d_{V_{\mathbb{F}}} \le ne_0+[n/2]$. In this case, if*

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} 0 & 1 \\ u^{(p+1)e_0+1} & 0 \end{pmatrix}, \begin{pmatrix} u^{2e_0+1-e_{0,i}} & 0 \\ 0 & u^{pe_{0,i}} \end{pmatrix}_{2 \le i \le n} \right),$$

then $d_{V_{\mathbb{F}}} = ne_0+[n/2]$. Here, $e_{0,i}=e_0$ if i is odd, and $e_{0,i}=e_0+1$ if i is even.

Proof. Extending the field $\mathbb{F}$, we may assume that

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} 0 & \alpha_1 \\ \alpha_1 u^m & 0 \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \dots, \begin{pmatrix} \alpha_n & 0 \\ 0 & \alpha_n \end{pmatrix} \right)$$

for some $\alpha_i \in \mathbb{F}^\times$ and a positive integer m such that $(q+1) \nmid m$, by Lemma 2.2. Let $\mathfrak{M}_{0,\mathbb{F}}$ be the lattice of $M_{\mathbb{F}}$ generated by the basis giving the above matrix expression.

For any finite extension $\mathbb{F}'$ of $\mathbb{F}$, we put $\mathfrak{M}_{0,\mathbb{F}'} = \mathfrak{M}_{0,\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ and $M_{\mathbb{F}'} = M_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$. By the Iwasawa decomposition, any sublattice of $M_{\mathbb{F}'}$ can be written as $\left(\begin{pmatrix} u^{s_i} & v_i' \\ 0 & u^{t_i} \end{pmatrix} \right)_i \cdot \mathfrak{M}_{0,\mathbb{F}'}$ for $s_i, t_i \in \mathbb{Z}$ and $v_i' \in \mathbb{F}'((u))$.

We put

$$\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') = \left\{ \left(\begin{pmatrix} u^{s_i} & v_i' \\ 0 & u^{t_i} \end{pmatrix} \right)_i \cdot \mathfrak{M}_{0,\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}') \;\middle|\; \begin{array}{l} s_i, t_i \in \mathbb{Z},\ v_i' \in \mathbb{F}'((u)), \\ ps_1-t_2=a_1,\ m+pt_1-s_2=b_1, \\ ps_j-s_{j+1}=a_j,\ pt_j-t_{j+1}=b_j \text{ for } 2 \le j \le n \end{array} \right\}$$

for $(\underline{a},\underline{b})=\big((a_i)_{1\leq i\leq n},(b_i)_{1\leq i\leq n}\big)\in\mathbb{Z}^n\times\mathbb{Z}^n$. Then we have

$$\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')=\bigcup_{(\underline{a},\underline{b})\in\mathbb{Z}^n\times\mathbb{Z}^n}\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$$

and this is a disjoint union by Lemma 3.2. Later, in Lemma 3.9, we will show that there are only finitely many $(\underline{a},\underline{b})$ such that $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')\neq\emptyset$.

We take

$$\left(\begin{pmatrix}u^{s_i} & v_i'\\ 0 & u^{t_i}\end{pmatrix}\right)_i\cdot\mathfrak{M}_{0,\mathbb{F}'}\in\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}'),$$

and put

$$\mathfrak{M}_{\underline{a},\underline{b},\mathbb{F}'}=\left(\begin{pmatrix}u^{s_i} & 0\\ 0 & u^{t_i}\end{pmatrix}\right)_i\cdot\mathfrak{M}_{0,\mathbb{F}'}.$$

Then we have

$$\mathfrak{M}_{\underline{a},\underline{b},\mathbb{F}'}\sim\left(\alpha_1\begin{pmatrix}0 & u^{a_1}\\ u^{b_1} & 0\end{pmatrix},\alpha_2\begin{pmatrix}u^{a_2} & 0\\ 0 & u^{b_2}\end{pmatrix},\dots,\alpha_n\begin{pmatrix}u^{a_n} & 0\\ 0 & u^{b_n}\end{pmatrix}\right)$$

with respect to the basis induced from $\mathfrak{M}_{0,\mathbb{F}'}$.

Now, any $\mathfrak{M}_{\mathbb{F}'}\in\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$ can be written as $\left(\begin{pmatrix}1 & v_i\\ 0 & 1\end{pmatrix}\right)_i\cdot\mathfrak{M}_{\underline{a},\underline{b},\mathbb{F}'}$ for some $(v_i)_{1\leq i\leq n}\in\mathbb{F}'((u))^n$, and we put $r_i=-v_u(v_i)$. Then we have

$$\mathfrak{M}_{\mathbb{F}'}\sim\left(\alpha_1\begin{pmatrix}\phi(v_1)u^{b_1} & u^{a_1}-\phi(v_1)v_2u^{b_1}\\ u^{b_1} & -v_2u^{b_1}\end{pmatrix},\alpha_i\begin{pmatrix}u^{a_i} & \phi(v_i)u^{b_i}-v_{i+1}u^{a_i}\\ 0 & u^{b_i}\end{pmatrix}_{2\leq i\leq n}\right)$$

with respect to the induced basis, and

$$\begin{aligned}\begin{pmatrix}\phi(v_1)u^{b_1} & u^{a_1}-\phi(v_1)v_2u^{b_1}\\ u^{b_1} & -v_2u^{b_1}\end{pmatrix}&=\begin{pmatrix}\phi(v_1)u^{b_1} & u^{a_1}\\ u^{b_1} & 0\end{pmatrix}\begin{pmatrix}1 & -v_2\\ 0 & 1\end{pmatrix}\\&=\begin{pmatrix}v_2^{-1}u^{a_1} & u^{a_1}-\phi(v_1)v_2u^{b_1}\\ 0 & -v_2u^{b_1}\end{pmatrix}\begin{pmatrix}1 & 0\\ -v_2^{-1} & 1\end{pmatrix}.\end{aligned}$$

Naturally, we consider the second equality only in the case $v_2\neq 0$.

If $r_2\geq 0$, the condition $u^e\mathfrak{M}_{\mathbb{F}'}\subset(1\otimes\phi)\big(\phi^*(\mathfrak{M}_{\mathbb{F}'})\big)\subset\mathfrak{M}_{\mathbb{F}'}$ is equivalent to

$$\begin{aligned}&0\leq a_1+r_2\leq e,\ 0\leq b_1-r_2\leq e,\\&\qquad v_u\big(u^{a_1}-\phi(v_1)v_2u^{b_1}\big)\geq\max\{0,a_1+b_1-e\},\end{aligned}\qquad (C_{1,+})$$

$$\begin{aligned}&0\leq a_i\leq e,\,0\leq b_i\leq e,\\&\qquad v_u\big(\phi(v_i)u^{b_i}-v_{i+1}u^{a_i}\big)\geq\max\{0,a_i+b_i-e\}\text{ for }2\leq i\leq n.\end{aligned}\qquad (C_2)$$

If $r_2<0$, it is equivalent to

$$0\leq a_1\leq e,\ 0\leq b_1\leq e,\ pr_1\leq\min\{e-a_1,b_1\},\qquad (C_{1,-})$$

and (C_2).

We show the following fact:

$$\begin{array}{l}\text{If } \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') \neq \emptyset, \text{ there does not exist } (r'_i)_{1\leq i\leq n} \in \mathbb{Z}^n \\ \text{such that } a_1 = b_1 - pr'_1 - r'_2 \text{ and } a_i - r'_{i+1} = b_i - pr'_i \text{ for } 2 \leq i \leq n.\end{array} \qquad (\diamondsuit)$$

We assume that there exists $(r'_i)_{1\leq i\leq n} \in \mathbb{Z}^n$ satisfying this condition. Changing the basis of $\mathfrak{M}_{\underline{a},\underline{b},\mathbb{F}'}$ by $\left(\begin{pmatrix} 1 & u^{-r'_i} \\ 0 & 1 \end{pmatrix}\right)_i$, we get

$$M_{\mathbb{F}'} \sim \left(\alpha_1 \begin{pmatrix} u^{b_1-pr'_1} & 0 \\ u^{b_1} & -u^{b_1-r'_2} \end{pmatrix}, \alpha_i \begin{pmatrix} u^{a_i} & 0 \\ 0 & u^{b_i} \end{pmatrix}_{2\leq i\leq n}\right).$$

This contradicts that $V_{\mathbb{F}}$ is absolutely irreducible.

Lemma 3.9. *If $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') \neq \emptyset$, then*

$$-\frac{e}{p-1} \leq a_1 \leq e,\ 0 \leq b_1 \leq \frac{pe}{p-1} \text{ and } 0 \leq a_i, b_i \leq e \text{ for } 2 \leq i \leq n.$$

Proof. We take $\mathfrak{M}_{\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$ and write it as $\left(\begin{pmatrix} 1 & v_i \\ 0 & 1 \end{pmatrix}\right)_i \cdot \mathfrak{M}_{\underline{a},\underline{b},\mathbb{F}'}$ for some $(v_i)_{1\leq i\leq n} \in \mathbb{F}'((u))^n$. We put $r_i = -v_u(v_i)$.

If $r_2 > e/(p-1)$, we have that $a_i - r_{i+1} = b_i - pr_i < 0$ for $2 \leq i \leq n$ and $r_i > e/(p-1)$ for all i by the condition (C_2), and that $a_1 = b_1 - pr_1 - r_2 < 0$ by the condition $(C_{1,+})$. This contradicts $(\diamondsuit)$, and we have $r_2 \leq e/(p-1)$.

Then $(C_{1,+})$, $(C_{1,-})$ and (C_2) shows the claim. □

To examine $|\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')|$, we consider the case where $0 \leq a_1 \leq e$ and $0 \leq b_1 \leq e$, and the case where $\max\{-a_1, b_1 - e\} > 0$.

First, we treat the case where $0 \leq a_1 \leq e$ and $0 \leq b_1 \leq e$. In this case, the condition $u^e\mathfrak{M}_{\mathbb{F}'} \subset (1\otimes\phi)\big(\phi^*(\mathfrak{M}_{\mathbb{F}'})\big) \subset \mathfrak{M}_{\mathbb{F}'}$ is equivalent to the condition that $\max\{pr_1 + r_2, pr_1, r_2\} \leq \min\{e - a_1, b_1\}$ and (C_2). We put

$$I_{\underline{a},\underline{b}} = \big\{(R_1, R_2) \in \mathbb{Z}\times\mathbb{Z} \;\big|\; pR_1 + R_2 \leq \min\{e - a_1, b_1\},\ R_1, R_2 \geq 0\big\}$$

and

$$\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}') = \left\{\left(\begin{pmatrix} 1 & v_i \\ 0 & 1 \end{pmatrix}\right)_i \cdot \mathfrak{M}_{\underline{a},\underline{b},\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') \;\middle|\; v_i \in \mathbb{F}'((u)),\ r_1 = R_1,\ r_2 = R_2\right\}$$

for $(R_1, R_2) \in I_{\underline{a},\underline{b}}$. Then we have a disjoint union

$$\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') = \bigcup_{(R_1,R_2)\in I_{\underline{a},\underline{b}}} \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$$

by Lemma 3.2, because if $\left(\begin{pmatrix} 1 & v_i \\ 0 & 1 \end{pmatrix}\right)_i \cdot \mathfrak{M}_{\underline{a},\underline{b},\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$ for $v_i \in \mathbb{F}'((u))$ then we may replace v_i so that $v_i \notin u\mathbb{F}'[[u]]$ without changing the $(k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}')$-module $\left(\begin{pmatrix} 1 & v_i \\ 0 & 1 \end{pmatrix}\right)_i \cdot \mathfrak{M}_{\underline{a},\underline{b},\mathbb{F}'}$ again by Lemma 3.2.

We fix $(R_1, R_2) \in I_{\underline{a},\underline{b}}$. Then the condition that $r_1 = R_1$ and $r_2 = R_2$ implies $\max\{pr_1 + r_2, pr_1, r_2\} \leq \min\{e - a_1, b_1\}$. So $\left(\begin{pmatrix} 1 & v_i \\ 0 & 1 \end{pmatrix}\right)_i \cdot \mathfrak{M}_{\underline{a},\underline{b},\mathbb{F}'}$ gives a point of $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$ if and only if

$$\max\{r_1, 0\} = R_1,\ \max\{r_2, 0\} = R_2 \text{ and } (C_2).$$

We assume $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}') \neq \emptyset$. Considering $-v_u(v_i)$ for $(v_i)_{1\leq i\leq n}$ that gives a point of $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$, we have the following two cases:

(i) There are $2 \leq n_2 < n_1 \leq n+1$ and $R_i \in \mathbb{Z}$ for $3 \leq i \leq n_2$ and $n_1 \leq i \leq n$ such that
$$a_i - R_{i+1} = b_i - pR_i < \max\{0, a_i + b_i - e\}$$
for $2 \leq i \leq n_2 - 1$ and $n_1 \leq i \leq n$, and
$$R_{n_1} \leq \min\{a_{n_1-1}, e - b_{n_1-1}\},\ R_{n_2} \leq \min\left\{\frac{e - a_{n_2}}{p}, \frac{b_{n_2}}{p}\right\}.$$

(ii) There are $R_i \in \mathbb{Z}$ for $3 \leq i \leq n$ such that
$$a_i - R_{i+1} = b_i - pR_i < \max\{0, a_i + b_i - e\}$$
for $2 \leq i \leq n$.

We note that (ii) includes the case $n = 1$.

We define an $\mathbb{F}'$-vector space $\widetilde{N}_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}$ by

$$\widetilde{N}_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} = \left\{(v_i)_{1\leq i\leq n} \in \mathbb{F}'((u))^n \;\middle|\; r_1 \leq R_1,\ r_2 \leq R_2 \text{ and } (C_2)\right\}.$$

We note that $\widetilde{N}_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} \supset \mathbb{F}'[[u]]^n$. We put $N_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} = \widetilde{N}_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}/\mathbb{F}'[[u]]^n$ and $d_{\underline{a},\underline{b},R_1,R_2} = \dim_{\mathbb{F}'} N_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}$. We note that $\dim_{\mathbb{F}'} N_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}$ is independent of finite extensions $\mathbb{F}'$ of $\mathbb{F}$. We put

$$\widetilde{N}^{\circ}_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} = \left\{(v_i)_{1\leq i\leq n} \in \widetilde{N}_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} \;\middle|\; r_1 = R_1,\ r_2 = R_2\right\}.$$

Let $N^{\circ}_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}$ be the image of $\widetilde{N}^{\circ}_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}$ in $N_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}$. Then we have a bijection
$$N^{\circ}_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} \to \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$$
by Lemma 3.2. By choosing a basis of $N_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}}$ over $\mathbb{F}$, we have a morphism

$$f_{\underline{a},\underline{b},R_1,R_2} : \mathbb{A}_{\mathbb{F}}^{d_{\underline{a},\underline{b},R_1,R_2}} \to \mathscr{GR}_{V_{\mathbb{F}},0}$$

in the case $R_1 = R_2 = 0$,

$$f_{\underline{a},\underline{b},R_1,R_2} : \mathbb{A}_{\mathbb{F}}^{(d_{\underline{a},\underline{b},R_1,R_2}-2)} \times \mathbb{G}_{m,\mathbb{F}}^2 \to \mathscr{GR}_{V_{\mathbb{F}},0}$$

in the case where $R_1 > 0$, $R_2 > 0$ and (i) holds true, and

$$f_{\underline{a},\underline{b},R_1,R_2} : \mathbb{A}_{\mathbb{F}}^{(d_{\underline{a},\underline{b},R_1,R_2}-1)} \times \mathbb{G}_{m,\mathbb{F}} \to \mathscr{GR}_{V_{\mathbb{F}},0}$$

in the other case, such that $f_{\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$ is injective and the image of $f_{\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$ is $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$.

Lemma 3.10. *If $0 \leq a_1 \leq e$ and $0 \leq b_1 \leq e$, the followings hold:*

(a) *In the case $e_1 = 0$, we have $d_{\underline{a},\underline{b},R_1,R_2} \leq ne_0 - 1$. In this case, if $a_1 = 0$, $b_1 = (p+1)e_0 - 1$, $a_i = e_0$ and $b_i = pe_0$ for $2 \leq i \leq n$, then there exists $(R_1, R_2) \in I_{\underline{a},\underline{b}}$ such that $d_{\underline{a},\underline{b},R_1,R_2} = ne_0 - 1$.*

(b) *In the case $1 \leq e_1 \leq p-1$, we have $d_{\underline{a},\underline{b},R_1,R_2} \leq ne_0$. In this case, if $a_1 = 0$, $b_1 = (p+1)e_0 + 1$, $a_i = e_0$ and $b_i = pe_0$ for $2 \leq i \leq n$, then there exists $(R_1, R_2) \in I_{\underline{a},\underline{b}}$ such that $d_{\underline{a},\underline{b},R_1,R_2} = ne_0$.*

(c) *In the case $e_1 = p$, we have $d_{\underline{a},\underline{b},R_1,R_2} \leq ne_0 + [n/2]$. In this case, if $a_1 = 0$, $b_1 = (p+1)e_0 + 1$, $a_i = 2e_0 + 1 - e_{0,i}$ and $b_i = pe_{0,i}$ for $2 \leq i \leq n$, then there exists $(R_1, R_2) \in I_{\underline{a},\underline{b}}$ such that $d_{\underline{a},\underline{b},R_1,R_2} = ne_0 + [n/2]$. Here, $e_{0,i} = e_0$ if i is odd, and $e_{0,i} = e_0 + 1$ if i is even.*

Proof. First, we treat the case $n = 1$. In this case, we have

$$R_1 = R_2 \leq \left[\frac{\min\{e - a_1, b_1\}}{p+1} \right] \leq e_0.$$

So we get $d_{\underline{a},\underline{b},R_1,R_2} \leq e_0$ for $(\underline{a}, \underline{b}) \in \mathbb{Z}^n \times \mathbb{Z}^n$ and $(R_1, R_2) \in I_{\underline{a},\underline{b}}$ such that $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}') \neq \emptyset$ and $0 \leq a_1, b_1 \leq e$. We have to eliminate the possibility of equality in the case $e_1 = 0$. In this case, if we have $d_{\underline{a},\underline{b},R_1,R_2} = e_0$, then $a_1 = 0$ and $b_1 = (p+1)e_0$. This contradicts ($\diamondsuit$).

We can check that if $e_1 = 0$, $a_1 = 0$, $b_1 = e - 1$ and $R_1 = R_2 = e_0 - 1$, then $d_{\underline{a},\underline{b},R_1,R_2} = e_0 - 1$, and that if $e_1 \neq 0$, $a_1 = 0$, $b_1 = (p+1)e_0 + 1$ and $R_1 = R_2 = e_0$, then $d_{\underline{a},\underline{b},R_1,R_2} = e_0$.

So we may assume $n \geq 2$. We put

$$S_{\underline{a},\underline{b},R_1,R_2,1} = \left\{ (u^{-r_1}, 0, \ldots, 0) \in \mathbb{F}((u))^n \;\middle|\; 1 \leq r_1 \leq \min\{R_1, a_n, e - b_n\} \right\},$$

$$S_{\underline{a},\underline{b},R_1,R_2,2} = \left\{ (0, u^{-r_2}, 0, \ldots, 0) \in \mathbb{F}((u))^n \;\middle|\; 1 \leq r_2 \leq \min\left\{ R_2, \frac{e - a_2}{p}, \frac{b_2}{p} \right\} \right\},$$

$$S_{\underline{a},\underline{b},R_1,R_2,i} = \left\{ (0, \ldots, 0, v_i, 0, \ldots, 0) \in \mathbb{F}((u))^n \;\middle|\; v_i = u^{-r_i}, \; 1 \leq r_i \leq \min\left\{ a_{i-1}, e - b_{i-1}, \frac{e - a_i}{p}, \frac{b_i}{p} \right\} \right\}$$

for $3 \le i \le n$, and

$$S_{\underline{a},\underline{b},R_1,R_2,i,j} = \Bigg\{ (0,\dots,0,v_i,v_{i+1},\dots,v_{j+1},0,\dots,0) \in \mathbb{F}((u))^n \;\Bigg|\; v_i = u^{-r_i},$$
$$r_i \le \min\{a_{i-1}, e-b_{i-1}\} \text{ if } i \ne 2,\ r_2 \le R_2 \text{ if } i=2,$$
$$u^{a_l} v_{l+1} = u^{b_l}\phi(v_l) \text{ and } -v_u(v_{l+1}) > \min\{a_l, e-b_l\} \text{ for } i \le l \le j,$$
$$-v_u(v_{j+1}) \le \min\left\{\frac{e-a_{j+1}}{p}, \frac{b_{j+1}}{p}\right\} \text{ if } j \ne n,\ -v_u(v_1) \le R_1 \text{ if } j = n \Bigg\}$$

for $2 \le i \le j \le n$. In the above definitions, v_i is on the i-th component. Then, as in the proof of Lemma 3.5, we can check that $\bigcup_i S_{\underline{a},\underline{b},R_1,R_2,i} \cup \bigcup_{i,j} S_{\underline{a},\underline{b},R_1,R_2,i,j}$ is an $\mathbb{F}$-basis of $N_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}}$. So we have $d_{\underline{a},\underline{b},R_1,R_2} = \sum_i |S_{\underline{a},\underline{b},R_1,R_2,i}| + \sum_{i,j} |S_{\underline{a},\underline{b},R_1,R_2,i,j}|$.

We put

$$T_{\underline{a},\underline{b},R_1,R_2,1} = \big\{ m \in \mathbb{Z} \;\big|\; \min\{a_n, e-b_n\} < pm + a_n - b_n \le R_1 \big\},$$

$T_{\underline{a},\underline{b},R_1,R_2,2} = \emptyset$ and

$$T_{\underline{a},\underline{b},R_1,R_2,i} = \Bigg\{ m \in \mathbb{Z} \;\Bigg|\; \min\{a_{i-1}, e-b_{i-1}\} < pm + a_{i-1} - b_{i-1}$$
$$\le \min\left\{\frac{e-a_i}{p}, \frac{b_i}{p}\right\}\Bigg\}$$

for $3 \le i \le n$. We consider the map

$$\bigcup_{2 \le i \le h-1} S_{\underline{a},\underline{b},R_1,R_2,i,h-1} \to T_{\underline{a},\underline{b},R_1,R_2,h};\ (v_{i'})_{1 \le i' \le n} \mapsto -v_u(v_{h-1})$$

for $3 \le h \le n+1$. We can easily check that this map is injective. So we have $\sum_{2 \le i \le h-1} |S_{\underline{a},\underline{b},R_1,R_2,i,h-1}| \le |T_{\underline{a},\underline{b},R_1,R_2,h}|$ and $d_{\underline{a},\underline{b},R_1,R_2} \le \sum_{1 \le i \le n} \big(|S_{\underline{a},\underline{b},R_1,R_2,i}| + |T_{\underline{a},\underline{b},R_1,R_2,i}|\big)$.

We take $(\underline{a}', \underline{b}') \in \mathbb{Z}^n \times \mathbb{Z}^n$ and $(R'_1, R'_2) \in I_{\underline{a}',\underline{b}'}$ such that $0 \le a'_1, b'_1 \le e$ and $\sum_{1 \le i \le n} \big(|S_{\underline{a}',\underline{b}',R'_1,R'_2,i}| + |T_{\underline{a}',\underline{b}',R'_1,R'_2,i}|\big)$ is the maximum. We can prove that $|T_{\underline{a}',\underline{b}',R'_1,R'_2,i}| \le 1$ for all i as in the proof of Lemma 3.6.

We can also show that

(A_i) if $|S_{\underline{a}',\underline{b}',R'_1,R'_2,i}| + |T_{\underline{a}',\underline{b}',R'_1,R'_2,i}| = e_0 + l$ for $l \ge 1$,
then $|S_{\underline{a}',\underline{b}',R'_1,R'_2,i+1}| + |T_{\underline{a}',\underline{b}',R'_1,R'_2,i+1}| \le e_0 + e_1 - pl + 1$

for $i \ne 1$, and that

(B_i) if $|S_{\underline{a}',\underline{b}',R'_1,R'_2,i}| + |T_{\underline{a}',\underline{b}',R'_1,R'_2,i}| = e_0 + 1$
and $|S_{\underline{a}',\underline{b}',R'_1,R'_2,i+1}| + |T_{\underline{a}',\underline{b}',R'_1,R'_2,i+1}| = e_0 + e_1 - p + 1$,
then $|S_{\underline{a}',\underline{b}',R'_1,R'_2,i+2}| + |T_{\underline{a}',\underline{b}',R'_1,R'_2,i+2}| \le e_0 - (p-1)e_1 + 1$

for $2 \leq i \leq n-1$ as in the proof of Lemma 3.7. By the same argument, we can show that

(A_1) if $|S_{\underline{a}',\underline{b}',R'_1,R'_2,1}| + |T_{\underline{a}',\underline{b}',R'_1,R'_2,1}| = e_0 + l$ for $l \geq 1$,
then $|S_{\underline{a}',\underline{b}',R'_1,R'_2,2}| + |T_{\underline{a}',\underline{b}',R'_1,R'_2,2}| \leq e_0 + e_1 - pl$,

and that

(B_n) if $|S_{\underline{a}',\underline{b}',R'_1,R'_2,n}| + |T_{\underline{a}',\underline{b}',R'_1,R'_2,n}| = e_0 + 1$
and $|S_{\underline{a}',\underline{b}',R'_1,R'_2,1}| + |T_{\underline{a}',\underline{b}',R'_1,R'_2,1}| = e_0 + e_1 - p + 1$,
then $|S_{\underline{a}',\underline{b}',R'_1,R'_2,2}| + |T_{\underline{a}',\underline{b}',R'_1,R'_2,2}| \leq e_0 - (p-1)e_1$,

using the followings:

$$|S_{\underline{a}',\underline{b}',R'_1,R'_2,1}| + |T_{\underline{a}',\underline{b}',R'_1,R'_2,1}| \leq R_1,\ pR_1 + R_2 \leq e,$$
$$|S_{\underline{a}',\underline{b}',R'_1,R'_2,2}| \leq R_2 \text{ and } T_{\underline{a}',\underline{b}',R'_1,R'_2,2} = \emptyset.$$

Firstly, we treat the case where $0 \leq e_1 \leq p-1$, that is, (a) or (b). We note that $e_0 + e_1 - pl + 1 \leq e_0 - p(l-1) - 1$ in the case $0 \leq e_1 \leq p-2$, and that $e_0 + e_1 - pl + 1 = e_0 - p(l-1)$ and $e_0 - (p-1)e_1 + 1 \leq e_0 - 3$ in the case $e_1 = p-1$. Then (A_i) for all i and (B_i) for $i \neq 1$ implies

$$\begin{aligned} d_{\underline{a},\underline{b},R_1,R_2} &\leq \sum_{1 \leq i \leq n} \big(|S_{\underline{a},\underline{b},R_1,R_2,i}| + |T_{\underline{a},\underline{b},R_1,R_2,i}|\big) \\ &\leq \sum_{1 \leq i \leq n} \big(|S_{\underline{a}',\underline{b}',R'_1,R'_2,i}| + |T_{\underline{a}',\underline{b}',R'_1,R'_2,i}|\big) \leq ne_0 \end{aligned}$$

for $(\underline{a},\underline{b}) \in \mathbb{Z}^n \times \mathbb{Z}^n$ and $(R_1,R_2) \in I_{\underline{a},\underline{b}}$ such that $\mathscr{GR}_{\mathrm{V}_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}') \neq \emptyset$ and $0 \leq a_1, b_1 \leq e$. So we get the desired bound, if $1 \leq e_1 \leq p-1$. In the case $e_1 = 0$, we have to eliminate the possibility of equality. In this case, if we have equality, we get that $\sum_{1 \leq i \leq n}\big(|S_{\underline{a},\underline{b},R_1,R_2,i}| + |T_{\underline{a},\underline{b},R_1,R_2,i}|\big)$ is the maximum and $\big(|S_{\underline{a},\underline{b},R_1,R_2,i}| + |T_{\underline{a},\underline{b},R_1,R_2,i}|\big) = e_0$ for all i by (A_i) for all i. Then we have

$$R_1 = R_2 = e_0,\ e_0 - 1 \leq a_i \leq e_0,\ pe_0 \leq b_i \leq pe_0 + 1 \text{ for } 2 \leq i \leq n$$

by the followings:

$$\begin{aligned} &pR_1 + R_2 = e, |S_{\underline{a},\underline{b},R_1,R_2,1}| + |T_{\underline{a},\underline{b},R_1,R_2,1}| \leq R_1, |S_{\underline{a},\underline{b},R_1,R_2,2}| \leq R_2, \\ &|S_{\underline{a},\underline{b},R_1,R_2,i}| + |T_{\underline{a},\underline{b},R_1,R_2,i}| \leq \min\{(e-a_i)/p, b_i/p\} \text{ for } 2 \leq i \leq n \\ &\text{and } |S_{\underline{a},\underline{b},R_1,R_2,i}| \geq e_0 - 1 \text{ for } i \neq 2. \end{aligned}$$

Now we have $a_1 = 0$ and $b_1 = (p+1)e_0$ by $R_1 = R_2 = e_0$. We show that $|T_{\underline{a},\underline{b},R_1,R_2,i}| = 0$ for $3 \leq i \leq n$. We assume that $|T_{\underline{a},\underline{b},R_1,R_2,i_0}| = 1$ for some $i_0 \neq 1, 2$, and let m be the unique element of $T_{\underline{a},\underline{b},R_1,R_2,i_0}$. Then, by the definition of $T_{\underline{a},\underline{b},R_1,R_2,i_0}$, we have

$$\begin{aligned} \min\left\{\frac{e-a_{i_0}}{p}, \frac{b_{i_0}}{p}\right\} - \min\{a_{i_0-1}, e - b_{i_0-1}\} &\geq pm - \min\{e - a_{i_0-1}, b_{i_0-1}\} \\ &\geq p - 1 \geq 2, \end{aligned}$$

because $pe_0 \leq \min\{e - a_{i_0-1}, b_{i_0-1}\} \leq pe_0 + 1$ and $pm - \min\{e - a_{i_0-1}, b_{i_0-1}\} > 0$. This contradicts the possibilities of a_{i_0-1}, a_{i_0}, b_{i_0-1} and b_{i_0}. The same argument shows that $|T_{\underline{a},\underline{b},R_1,R_2,1}| = 0$. Now we have $|S_{\underline{a},\underline{b},R_1,R_2,i}| = e_0$ for all i, and that

$$a_1 = 0,\ b_1 = (p+1)e_0,\ a_i = e_0,\ b_i = pe_0 \text{ for } 2 \leq i \leq n.$$

Then we have

$$a_1 = b_1 - pr'_1 - r'_2 \text{ and } a_i - r'_{i+1} = b_i - pr'_i \text{ for } 2 \leq i \leq n$$

for $(r'_i)_{1\leq i\leq n} = (e_0)_{1\leq i\leq n}$. This contradicts ($\diamondsuit$). So we have $d_{\underline{a},\underline{b},R_1,R_2} \leq ne_0 - 1$, if $e_1 = 0$.

We can check that if $e_1 = 0$, $a_1 = 0$, $b_1 = (p+1)e_0 - 1$, $R_1 = e_0$, $R_2 = e_0 - 1$, $a_i = e_0$ and $b_i = pe_0$ for $2 \leq i \leq n$, then $d_{\underline{a},\underline{b},R_1,R_2} \geq \sum_{1\leq i\leq n} |S_{\underline{a},\underline{b},R_1,R_2,i}| = ne_0 - 1$. We can check also that if $1 \leq e_1 \leq p-1$, $a_1 = 0$, $b_1 = (p+1)e_0 + 1$, $R_1 = e_0$, $R_2 = e_0 + 1$, $a_i = e_0$ and $b_i = pe_0$ for $2 \leq i \leq n$, then $d_{\underline{a},\underline{b},R_1,R_2} \geq \sum_{1\leq i\leq n} |S_{\underline{a},\underline{b},R_1,R_2,i}| = ne_0$.

Secondly, we treat (c). In this case, we note that $e_0 + e_1 - pl + 1 = e_0 - p(l-1) + 1$ and $e_0 - (p-1)e_1 + 1 \leq e_0 - 5$. Then (A_i) for all i and (B_i) for $i \neq 1$ implies

$$\begin{aligned} d_{\underline{a},\underline{b},R_1,R_2} &\leq \sum_{1\leq i\leq n} \big(|S_{\underline{a},\underline{b},R_1,R_2,i}| + |T_{\underline{a},\underline{b},R_1,R_2,i}|\big) \\ &\leq \sum_{1\leq i\leq n} \big(|S_{\underline{a}',\underline{b}',R'_1,R'_2,i}| + |T_{\underline{a}',\underline{b}',R'_1,R'_2,i}|\big) \leq ne_0 + \left[\frac{n}{2}\right] \end{aligned}$$

for $(\underline{a},\underline{b}) \in \mathbb{Z}^n \times \mathbb{Z}^n$ and $(R_1, R_2) \in I_{\underline{a},\underline{b}}$ such that $0 \leq a_1, b_1 \leq e$. So we get the desired bound.

We can check that if $e_1 = p$, $a_1 = 0$, $b_1 = (p+1)e_0 + 1$, $R_1 = e_0$, $R_2 = e_0 + 1$, $a_i = 2e_0 + 1 - e_{0,i}$ and $b_i = pe_{0,i}$ for $2 \leq i \leq n$, then $d_{\underline{a},\underline{b},R_1,R_2} \geq \sum_{1\leq i\leq n} |S_{\underline{a},\underline{b},R_1,R_2,i}| = ne_0 + [n/2]$. □

Next, we consider the remaining case, that is, the case where $\max\{-a_1, b_1 - e\} > 0$. In this case, $v_u\big(u^{a_1} - \phi(v_1)v_2u^{b_1}\big) \geq \max\{0, a_1 + b_1 - e\}$ implies $pr_1 + r_2 = b_1 - a_1$, because $a_1 < \max\{0, a_1 + b_1 - e\}$. So the condition $u^e\mathfrak{M}_{\mathbb{F}'} \subset (1\otimes\phi)\big(\phi^*(\mathfrak{M}_{\mathbb{F}'})\big) \subset \mathfrak{M}_{\mathbb{F}'}$ implies

$$pr_1 + r_2 = b_1 - a_1,\ \max\{-a_1, b_1 - e\} \leq r_2 \leq \min\{e - a_1, b_1\}.$$

We note that if $n = 1$, then $pr_1 + r_2 = b_1 - a_1$ contradicts ($\diamondsuit$) because $r_1 = r_2$. So we may assume $n \geq 2$. We put

$$I_{\underline{a},\underline{b}} = \Big\{(R_1, R_2) \in \mathbb{Z} \times \mathbb{Z} \ \Big|\ pR_1 + R_2 = b_1 - a_1, \\ \max\{-a_1, b_1 - e\} \leq R_2 \leq \min\{e - a_1, b_1\}\Big\}$$

and $m_{\underline{a},\underline{b}} = \big[\big(\max\{-a_1, b_1 - e\} - 1\big)/p\big]$. We note that $R_1 \geq m_{\underline{a},\underline{b}} + 1 > 0$ and

$R_2 \geq \max\{-a_1, b_1 - e\} > 0$. We put

$$\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}') = \left\{ \left(\begin{pmatrix} 1 & v_i \\ 0 & 1 \end{pmatrix} \right)_i \cdot \mathfrak{M}_{\underline{a},\underline{b},\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') \;\middle|\; v_i \in \mathbb{F}'((u)), \right.$$
$$\left. v_u(v_1) = -R_1,\ v_u(v_2) = -R_2 \right\}$$

for $(R_1, R_2) \in I_{\underline{a},\underline{b}}$. Then we have a disjoint union

$$\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') = \bigcup_{(R_1,R_2)\in I_{\underline{a},\underline{b}}} \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$$

by Lemma 3.2. Extending the field $\mathbb{F}$, we may assume that $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}') \neq \emptyset$ if and only if $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}) \neq \emptyset$ for each $(R_1, R_2) \in I_{\underline{a},\underline{b}}$, $(\underline{a}, \underline{b}) \in \mathbb{Z}^n \times \mathbb{Z}^n$ and any finite extension $\mathbb{F}'$ of $\mathbb{F}$.

We fix $(R_1, R_2) \in I_{\underline{a},\underline{b}}$, and assume $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}) \neq \emptyset$. If $v_u(v_1) = -R_1$ and $v_u(v_2) = -R_2$, the condition $v_u\bigl(u^{a_1} - \phi(v_1)v_2u^{b_1}\bigr) \geq \max\{0, a_1 + b_1 - e\}$ is equivalent to the following:

There uniquely exist $\gamma_{1,0}, \gamma_{2,0} \in (\mathbb{F}')^\times$ and $\gamma_{1,i}, \gamma_{2,i} \in \mathbb{F}'$ for $1 \leq i \leq m_{\underline{a},\underline{b}}$ such that

$$-v_u\left(v_1 - \sum_{0\leq i\leq m_{\underline{a},\underline{b}}} \gamma_{1,i} u^{-R_1+i} \right) \leq R_1 - m_{\underline{a},\underline{b}} - 1,$$
$$-v_u\left(v_2 - \sum_{0\leq i\leq m_{\underline{a},\underline{b}}} \gamma_{2,i} u^{-R_2+pi} \right) \leq R_2 - \max\{-a_1, b_1 - e\},$$
$$\gamma_{1,0}\gamma_{2,0} = 1,\ \sum_{0\leq i\leq l} \gamma_{1,i}\gamma_{2,l-i} = 0 \text{ for } 1 \leq l \leq m_{\underline{a},\underline{b}}.$$

We note that $(\gamma_{1,i})_{0\leq i\leq m_{\underline{a},\underline{b}}}$ determines $(\gamma_{1,i}, \gamma_{2,i})_{0\leq i\leq m_{\underline{a},\underline{b}}}$.

We prove that for $0 \leq i \leq m_{\underline{a},\underline{b}}$ there uniquely exist $2 \leq n_{2,i} < n_{1,i} \leq n + 1$, $r_{1,i,j} \in \mathbb{Q}$ for $n_{1,i} \leq j \leq n + 1$ and $r_{2,i,j} \in \mathbb{Z}$ for $2 \leq j \leq n_{2,i}$ such that $r_{1,0,j} \in \mathbb{Z}$ for $n_{1,0} \leq j \leq n + 1$ and

$$\begin{aligned} a_j - r_{1,i,j+1} &= b_j - pr_{1,i,j} < \max\{0, a_j + b_j - e\} \text{ for } n_{1,i} \leq j \leq n, \\ r_{1,i,n+1} &= R_1 - i,\ r_{1,i,n_{1,i}} \leq \min\{a_{n_{1,i}-1}, e - b_{n_{1,i}-1}\}, \\ a_j - r_{2,i,j+1} &= b_j - pr_{2,i,j} < \max\{0, a_j + b_j - e\} \text{ for } 2 \leq j \leq n_{2,i} - 1, \\ r_{2,i,2} &= R_2 - pi,\ r_{2,i,n_{2,i}} \leq \min\left\{ \frac{e - a_{n_{2,i}}}{p}, \frac{b_{n_{2,i}}}{p} \right\}. \end{aligned}$$

Define $r_{1,i,j} \in \mathbb{Q}$ for $2 \leq j \leq n + 1$ and $r_{2,i,j} \in \mathbb{Z}$ for $2 \leq j \leq n + 1$ such that

$$\begin{aligned} r_{1,i,n+1} &= R_1 - i,\ a_j - r_{1,i,j+1} = b_j - pr_{1,i,j} \text{ for } 2 \leq j \leq n, \\ r_{2,i,2} &= R_2 - pi,\ a_j - r_{2,i,j+1} = b_j - pr_{2,i,j} \text{ for } 2 \leq j \leq n. \end{aligned}$$

We put

$$n_{1,i} = \max\Big\{\{3 \le j \le n+1 \mid r_{1,i,j} \le \min\{a_{j-1}, e - b_{j-1}\}\} \cup \{2\}\Big\},$$
$$n_{2,i} = \min\left\{\left\{2 \le j \le n \;\middle|\; r_{2,i,j} \le \min\Big\{\frac{e-a_j}{p}, \frac{b_j}{p}\Big\}\right\} \cup \{n+1\}\right\}.$$

We consider $(v_i)_{1\le i\le n}$ that gives a point of $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F})$. Then we have $r_{1,0,j} = -v_u(v_j) \in \mathbb{Z}$ for $n_{1,0} \le j \le n+1$ and $r_{2,0,j} = -v_u(v_j) \in \mathbb{Z}$ for $2 \le j \le n_{2,0}$. It remains to show that $n_{2,i} < n_{1,i}$. We have $n_{2,i} \le n_{2,0}$ and $n_{1,0} \le n_{1,i}$, because $r_{1,i,j} \le r_{1,0,j}$ and $r_{2,i,j} \le r_{2,0,j}$ for $2 \le j \le n+1$. So it suffices to show $n_{2,0,j} < n_{1,0,j}$. If $n_{2,0,j} \ge n_{1,0,j}$, we have

$$a_1 = b_1 - pv_u(v_1) - v_u(v_2) \text{ and } a_j - v_u(v_{j+1}) = b_j - v_u(v_j) \text{ for } 2 \le j \le n,$$

and this contradicts $(\diamondsuit)$.

We put

$$M_{\underline{a},\underline{b},R_1,R_2} = \big\{0 \le i \le m_{\underline{a},\underline{b}} \mid r_{1,i,j} \in \mathbb{Z} \text{ for } n_{1,i} \le j \le n+1\big\}.$$

For $(v_i)_{1\le i\le n}$ that gives a point of $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$, we take $\gamma_{1,i}$, $\gamma_{2,i}$ and $n_{1,i}$, $n_{2,i}$, $r_{1,i,j}$, $r_{2,i,j}$ as above. We note that $\gamma_{1,i} = 0$ if $i \notin M_{\underline{a},\underline{b},R_1,R_2}$. We put

$$M_{1,\underline{a},\underline{b},R_1,R_2,j} = \big\{0 \le i \le m_{\underline{a},\underline{b}} \mid n_{1,i} \le j \le n+1\big\},$$
$$M_{2,\underline{a},\underline{b},R_1,R_2,j} = \big\{0 \le i \le m_{\underline{a},\underline{b}} \mid 2 \le j \le n_{2,i}\big\}$$

for $2 \le j \le n+1$, and define $(v_i^*)_{1\le i\le n} \in \mathbb{F}'((u))^n$ by

$$v_j^* = v_j - \sum_{i \in M_{1,\underline{a},\underline{b},R_1,R_2,j}} \gamma_{1,i} u^{-r_{1,i,j}} - \sum_{i \in M_{2,\underline{a},\underline{b},R_1,R_2,j}} \gamma_{2,i} u^{-r_{2,i,j}}$$

for $2 \le j \le n+1$. This is well-defined by the above remark. We put

$$\widetilde{N}^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} = \big\{(v_i^*)_{1\le i\le n} \in \mathbb{F}'((u))^n \mid (v_i)_{1\le i\le n} \in \mathbb{F}'((u))^n \text{ gives a point of } \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')\big\}.$$

Then we have

$$\widetilde{N}^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} = \big\{(v_i)_{1\le i\le n} \in \mathbb{F}'((u))^n \mid -v_u(v_1) \le R_1 - m_{\underline{a},\underline{b}} - 1, -v_u(v_2) \le R_2 - \max\{-a_1, b_1 - e\}, (C_2)\big\}$$

by the construction of $(v_i^*)_{1\le i\le n}$ and the conditions $(C_{1,+})$ and (C_2). This implies that $\widetilde{N}^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} \subset \mathbb{F}'((u))^n$ is an $\mathbb{F}'$-vector subspace, and $\widetilde{N}^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} \supset \mathbb{F}'[[u]]^n$.

We put

$$N^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} = \widetilde{N}^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}/\mathbb{F}'[[u]]^n$$

and $d^*_{\underline{a},\underline{b},R_1,R_2} = \dim_{\mathbb{F}'} N^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}$. We note that $\dim_{\mathbb{F}'} N^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}$ is independent of finite extensions $\mathbb{F}'$ of $\mathbb{F}$. By Lemma 3.2, giving an element of $N^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}$ and

$(\gamma_{1,i})_{0 \leq i \leq m_{\underline{a},\underline{b}}}$ such that $\gamma_{1,0} \neq 0$ and $\gamma_{1,i} = 0$ if $i \notin M_{\underline{a},\underline{b},R_1,R_2}$ is equivalent to giving a point of $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$. By choosing a basis of $N_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}^*}$ over $\mathbb{F}$, we have a morphism

$$f_{\underline{a},\underline{b},R_1,R_2} : \mathbb{A}_{\mathbb{F}}^{\left(d^*_{\underline{a},\underline{b},R_1,R_2}+|M_{\underline{a},\underline{b},R_1,R_2}|-1\right)} \times \mathbb{G}_{m,\mathbb{F}} \to \mathscr{GR}_{V_{\mathbb{F}},0}$$

such that $f_{\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$ is injective and the image of $f_{\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$ is equal to $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$. We put $d_{\underline{a},\underline{b},R_1,R_2} = d^*_{\underline{a},\underline{b},R_1,R_2} + |M_{\underline{a},\underline{b},R_1,R_2}|$. Then we have (1) and

$$d_{V_{\mathbb{F}}} = \max_{\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}) \neq \emptyset} \{d_{\underline{a},\underline{b},R_1,R_2}\}.$$

In this maximum, we consider all $(\underline{a},\underline{b}) \in \mathbb{Z}^n \times \mathbb{Z}^n$. We have already examined $d_{\underline{a},\underline{b},R_1,R_2}$ for $(\underline{a},\underline{b})$ such that $a_1 \geq 0$ and $b_1 \leq e$. So it suffices to bound $d_{\underline{a},\underline{b},R_1,R_2}$ for $(\underline{a},\underline{b})$ such that $\max\{-a_1, b_1 - e\} > 0$.

Lemma 3.11. *If* $\max\{-a_1, b_1 - e\} > 0$*, the followings hold:*

(a) In the case $e_1 = 0$*, we have* $d_{\underline{a},\underline{b},R_1,R_2} \leq ne_0 - 1$*.*

(b) In the case $1 \leq e_1 \leq p-1$*, we have* $d_{\underline{a},\underline{b},R_1,R_2} \leq ne_0$*.*

(c) In the case $e_1 = p$*, we have* $d_{\underline{a},\underline{b},R_1,R_2} \leq ne_0 + [n/2]$*.*

Proof. We put

$$S_{\underline{a},\underline{b},R_1,R_2,1} = \big\{(v_1, 0, \ldots, 0) \in \mathbb{F}((u))^n \;\big|\; v_1 = u^{-r_1},\; 1 \leq r_1 \leq \min\{R_1 - m_{\underline{a},\underline{b}} - 1, a_n, e - b_n\}\big\},$$

$$S_{\underline{a},\underline{b},R_1,R_2,2} = \left\{(0, v_2, 0, \ldots, 0) \in \mathbb{F}((u))^n \;\middle|\; v_2 = u^{-r_2},\; 1 \leq r_2 \leq \min\left\{R_2 - \max\{-a_1, b_1 - e\}, \frac{e-a_2}{p}, \frac{b_2}{p}\right\}\right\},$$

$$S_{\underline{a},\underline{b},R_1,R_2,i} = \left\{(0, \ldots, 0, v_i, 0, \ldots, 0) \in \mathbb{F}((u))^n \;\middle|\; v_i = u^{-r_i},\; 1 \leq r_i \leq \min\left\{a_{i-1}, e - b_{i-1}, \frac{e-a_i}{p}, \frac{b_i}{p}\right\}\right\}$$

for $3 \leq i \leq n$, and

$$S_{\underline{a},\underline{b},R_1,R_2,i,j} = \left\{(0, \ldots, 0, v_i, v_{i+1}, \ldots, v_{j+1}, 0, \ldots, 0) \in \mathbb{F}((u))^n \;\middle|\; v_i = u^{-r_i},\right.$$
$$r_i \leq \min\{a_{i-1}, e - b_{i-1}\} \text{ if } i \neq 2,\; r_2 \leq R_2 - \max\{-a_1, b_1 - e\} \text{ if } i = 2,$$
$$u^{a_l} v_{l+1} = u^{b_l} \phi(v_l) \text{ and } -v_u(v_{l+1}) > \min\{a_l, e - b_l\} \text{ for } i \leq l \leq j,$$
$$\left. -v_u(v_{j+1}) \leq \min\left\{\frac{e - a_{j+1}}{p}, \frac{b_{j+1}}{p}\right\} \text{ if } j \neq n,\; -v_u(v_1) \leq R_1 - m_{\underline{a},\underline{b}} - 1 \text{ if } j = n\right\}$$

for $2 \le i \le j \le n$. In the above definitions, v_i is on the i-th component. Then, as in the proof of Lemma 3.5, we can check that $\bigcup_i S_{\underline{a},\underline{b},R_1,R_2,i} \cup \bigcup_{i,j} S_{\underline{a},\underline{b},R_1,R_2,i,j}$ is an $\mathbb{F}$-basis of $N^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}}$. So we have $d^*_{\underline{a},\underline{b},R_1,R_2} = \sum_i |S_{\underline{a},\underline{b},R_1,R_2,i}| + \sum_{i,j} |S_{\underline{a},\underline{b},R_1,R_2,i,j}|$.

We put

$$T_{\underline{a},\underline{b},R_1,R_2,1} = \bigl\{ m \in \mathbb{Z} \bigm| \min\{a_n, e - b_n\} < pm + a_n - b_n \le R_1 - m_{\underline{a},\underline{b}} - 1 \bigr\},$$

$$T_{\underline{a},\underline{b},R_1,R_2,2} = \left\{ m \in \mathbb{Z} \;\middle|\; R_2 - \max\{-a_1, b_1 - e\} < R_2 - pm \le \min\left\{ R_2, \frac{e - a_2}{p}, \frac{b_2}{p} \right\} \right\}$$

and

$$T_{\underline{a},\underline{b},R_1,R_2,i} = \left\{ m \in \mathbb{Z} \;\middle|\; \min\{a_{i-1}, e - b_{i-1}\} < pm + a_{i-1} - b_{i-1} \le \min\left\{ \frac{e - a_i}{p}, \frac{b_i}{p} \right\} \right\}$$

for $3 \le i \le n$. We note that these definitions for $S_{\underline{a},\underline{b},R_1,R_2,i}$, $S_{\underline{a},\underline{b},R_1,R_2,i,j}$ and $T_{\underline{a},\underline{b},R_1,R_2,i}$ in the case $\max\{-a_1, b_1 - e\} > 0$ are compatible with the definitions in the case $\max\{-a_1, b_1 - e\} \le 0$, if $\max\{-a_1, b_1 - e\} = 0$. So in the following, we can consider also the case $\max\{-a_1, b_1 - e\} = 0$. We need to consider this case in the following arguments.

We consider the map

$$\bigcup_{2 \le j \le h-1} S_{\underline{a},\underline{b},R_1,R_2,j,h-1} \cup \bigl\{ 0 \le i \le m_{\underline{a},\underline{b}} \bigm| n_{2,i} = h \bigr\} \to T_{\underline{a},\underline{b},R_1,R_2,h};$$
$$(v_i)_{1 \le i \le n} \mapsto -v_u(v_{h-1}),\ i \mapsto r_{2,i,h-1}$$

for $3 \le h \le n + 1$. We can easily check that this map is injective and that

$$\bigl\{ 0 \le i \le m_{\underline{a},\underline{b}} \bigm| n_{2,i} = 2 \bigr\} = T_{\underline{a},\underline{b},R_1,R_2,2}.$$

So we have $\left(\sum_{2 \le 1 \le j \le n} |S_{\underline{a},\underline{b},R_1,R_2,i,j}| \right) + m_{\underline{a},\underline{b}} + 1 \le \sum_{1 \le i \le n} |T_{\underline{a},\underline{b},R_1,R_2,i}|$ and

$$d_{\underline{a},\underline{b},R_1,R_2} \le d^*_{\underline{a},\underline{b},R_1,R_2} + m_{\underline{a},\underline{b}} + 1 \le \sum_{1 \le i \le n} \bigl(|S_{\underline{a},\underline{b},R_1,R_2,i}| + |T_{\underline{a},\underline{b},R_1,R_2,i}| \bigr).$$

We take $(\underline{a}'', \underline{b}'') \in \mathbb{Z}^n \times \mathbb{Z}^n$ and $(R''_1, R''_2) \in I_{\underline{a}'',\underline{b}''}$ such that $\max\{-a''_1, e - b''_1\} \ge 0$ and $\sum_{1 \le i \le n} \bigl(|S_{\underline{a}'',\underline{b}'',R''_1,R''_2,i}| + |T_{\underline{a}'',\underline{b}'',R''_1,R''_2,i}| \bigr)$ is the maximum. We can prove that $|T_{\underline{a}'',\underline{b}'',R''_1,R''_2,i}| \le 1$ for all $i \ne 2$ as in the proof of Lemma 3.6.

We show that we may take $(\underline{a}'', \underline{b}'') \in \mathbb{Z}^n \times \mathbb{Z}^n$ and $(R''_1, R''_2) \in I_{\underline{a}'',\underline{b}''}$ such that $0 \le -a''_1 = b''_1 - e \le p - 1$. If $-a''_1 > b''_1 - e$, then we replace b''_1 by $b''_1 + 1$

and R''_2 by R''_2+1. We again have $(R''_1,R''_2)\in I_{\underline{a}'',\underline{b}''}$ after the replacement. This replacement increases $\sum_{1\le i\le n}\bigl(|S_{\underline{a}'',\underline{b}'',R''_1,R''_2,i}|+|T_{\underline{a}'',\underline{b}'',R''_1,R''_2,i}|\bigr)$ by 0 or 1, but by the maximality there is no case where it increases by 1. Similarly, if $-a''_1<b''_1-e$, we may replace a''_1 by a''_1-1 and R''_2 by R''_2+1. So we may assume $-a''_1=b''_1-e$.

If $-a''_1\ge p$ and $\min\{b''_2/p,(e-a''_2)/p\}\ge R''_2$, we replace R''_1 by R''_1-1 and R''_2 by R''_2+p. By

$$R''_2+p\le\frac{e}{p}+p<e+p\le e-a''_1=b''_1,$$

we again have $(R''_1,R''_2)\in I_{\underline{a}'',\underline{b}''}$ after the replacement. This replacement increases $\sum_{1\le i\le n}\bigl(|S_{\underline{a}'',\underline{b}'',R''_1,R''_2,i}|+|T_{\underline{a}'',\underline{b}'',R''_1,R''_2,i}|\bigr)$ by at least $p-2$. This is a contradiction. So if $-a''_1\ge p$, we have $\min\{b''_2/p,(e-a''_2)/p\}<R''_2$. If $-a''_1\ge p$, we replace a''_1 by a''_1+p, b''_1 by b''_1-p, R''_1 by R''_1-1 and R''_2 by R''_2-p. We again have $(R''_1,R''_2)\in I_{\underline{a}'',\underline{b}''}$ after the replacement. This replacement does not change $\sum_{1\le i\le n}\bigl(|S_{\underline{a}'',\underline{b}'',R''_1,R''_2,i}|+|T_{\underline{a}'',\underline{b}'',R''_1,R''_2,i}|\bigr)$. Iterating these replacements, we may assume $0\le -a''_1=b''_1-e\le p-1$. We already treated the case where $-a''_1=b''_1-e=0$. So we may assume $1\le -a''_1=b''_1-e\le p-1$. We note that $|T_{\underline{a}'',\underline{b}'',R''_1,R''_2,2}|\le 1$ in this case.

Now we can show that

(A'_i) if $|S_{\underline{a}'',\underline{b}'',R''_1,R''_2,i}|+|T_{\underline{a}'',\underline{b}'',R''_1,R''_2,i}|=e_0+l$ for $l\ge 1$,
then $|S_{\underline{a}'',\underline{b}'',R''_1,R''_2,i+1}|+|T_{\underline{a}'',\underline{b}'',R''_1,R''_2,i+1}|\le e_0+e_1-pl+1$

for $i\neq 1$, and that

(B'_i) if $|S_{\underline{a}'',\underline{b}'',R''_1,R''_2,i}|+|T_{\underline{a}'',\underline{b}'',R''_1,R''_2,i}|=e_0+1$
and $|S_{\underline{a}'',\underline{b}'',R''_1,R''_2,i+1}|+|T_{\underline{a}'',\underline{b}'',R''_1,R''_2,i+1}|=e_0+e_1-p+1$,
then $|S_{\underline{a}'',\underline{b}'',R''_1,R''_2,i+2}|+|T_{\underline{a}'',\underline{b}'',R''_1,R''_2,i+2}|\le e_0-(p-1)e_1+1$

for $2\le i\le n-1$ as in the proof of Lemma 3.7. By the same argument, we can show that

(A'_1) if $|S_{\underline{a}'',\underline{b}'',R''_1,R''_2,1}|+|T_{\underline{a}'',\underline{b}'',R''_1,R''_2,1}|=e_0+l$ for $l\ge 0$,
then $|S_{\underline{a}'',\underline{b}'',R''_1,R''_2,2}|+|T_{\underline{a}'',\underline{b}'',R''_1,R''_2,2}|\le e_0+e_1-pl$,

and that

(B'_n) if $|S_{\underline{a}'',\underline{b}'',R''_1,R''_2,n}|+|T_{\underline{a}'',\underline{b}'',R''_1,R''_2,n}|=e_0+1$
and $|S_{\underline{a}'',\underline{b}'',R''_1,R''_2,1}|+|T_{\underline{a}'',\underline{b}'',R''_1,R''_2,1}|=e_0+e_1-p+1$,
then $|S_{\underline{a}'',\underline{b}'',R''_1,R''_2,2}|+|T_{\underline{a}'',\underline{b}'',R''_1,R''_2,2}|\le e_0-(p-1)e_1$,

using the followings:

$$|S_{\underline{a}'',\underline{b}'',R''_1,R''_2,1}|+|T_{\underline{a}'',\underline{b}'',R''_1,R''_2,1}|\le R_1-1,\ pR_1+R_2=e-2a''_1,$$
$$|S_{\underline{a}'',\underline{b}'',R''_1,R''_2,2}|\le R_2+a''_1,\ 1\le -a''_1\le p-1 \text{ and } |T_{\underline{a}'',\underline{b}'',R''_1,R''_2,2}|\le 1.$$

Then (A'_i) for all i and (B'_i) for $i\neq 1$ implies that

$$\sum_{1\le i\le n}\bigl(|S_{\underline{a}'',\underline{b}'',R''_1,R''_2,i}|+|T_{\underline{a}'',\underline{b}'',R''_1,R''_2,i}|\bigr)\le ne_0$$

in the case $0 \leq e_1 \leq p-2$, and that

$$\sum_{1 \leq i \leq n} \bigl(|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}|\bigr) \leq ne_0 + \left[\frac{n}{2}\right]$$

in the case $e_1 = p-1$. It remains to eliminate the possibility of equality in the case $e_1 = 0$.

We assume that $e_1 = 0$ and $\sum_{1 \leq i \leq n} \bigl(|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}|\bigr) = ne_0$. Then (A_i') for all i implies that $|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}| = e_0$ for all i. Now we have

$$e_0 = |S_{\underline{a}'',\underline{b}'',R_1'',R_2'',1}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',1}| \leq R_1 - 1$$

and

$$e_0 - 1 \leq |S_{\underline{a}'',\underline{b}'',R_1'',R_2'',2}| \leq R_2 + a_1''.$$

This implies $e+p-1-a_1'' \leq pR_1 + R_2$. Because $pR_1 + R_2 = e - 2a_1''$, this inequality happens only in the case $-a_1'' = p-1$, and in this case the above inequalities become equality. So we have $e_0 - 1 = |S_{\underline{a}'',\underline{b}'',R_1'',R_2'',2}|$ and $R_2 = e_0 + p - 2$. By $|T_{\underline{a}'',\underline{b}'',R_1'',R_2'',2}| = 1$, we have $R_2 \leq \min\{(e-a_2'')/p, b_2''/p\}$. So we get $a_2'' \leq e_0 - p(p-2) \leq e_0 - 3$, but this contradicts $|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',3}| \geq e_0 - 1$. Thus we have eliminated the possibility of equality in the case $e_1 = 0$. □

The claim (2) follows from Lemma 3.10 and Lemma 3.11. □

Remark 3.12. *By Lemma 2.2, we can check that there is $V_{\mathbb{F}}$ satisfying the conditions for $M_{\mathbb{F}}$ in Proposition 3.8.*

3.4 Bounds of dimensions

To fix the notation, we recall the definition of the zeta function of a scheme of finite type over a finite field.

Definition 3.13. *Let X be a scheme of finite type over $\mathbb{F}$. We put $q_{\mathbb{F}} = |\mathbb{F}|$. The zeta function $Z(X;T)$ of X is defined by*

$$Z(X;T) = \exp\left(\sum_{m=1}^{\infty} \frac{\bigl|X(\mathbb{F}_{q_{\mathbb{F}}^m})\bigr|}{m} T^m\right).$$

Here,

$$\exp\bigl(f(T)\bigr) = \sum_{m=0}^{\infty} \frac{1}{m!} f(T)^m \in \mathbb{Q}[[T]]$$

for $f(T) \in T\mathbb{Q}[[T]]$.

Theorem 3.14. *Let $Z(\mathscr{GR}_{V_{\mathbb{F}},0};T)$ be the zeta function of $\mathscr{GR}_{V_{\mathbb{F}},0}$. Then the followings are true.*

1. *After extending the field $\mathbb{F}$ sufficiently, we have*

$$Z(\mathscr{GR}_{V_{\mathbb{F}},0};T) = \prod_{i=0}^{d_{V_{\mathbb{F}}}} (1 - |\mathbb{F}|^i T)^{-m_i}$$

   for some $m_i \in \mathbb{Z}$ such that $m_{d_{V_{\mathbb{F}}}} > 0$.

2. *If $n = 1$, we have*

$$0 \leq d_{V_{\mathbb{F}}} \leq \left[\frac{e+2}{p+1}\right].$$

   If $n \geq 2$, we have

$$0 \leq d_{V_{\mathbb{F}}} \leq \left[\frac{n+1}{2}\right]\left[\frac{e}{p+1}\right] + \left[\frac{n-2}{2}\right]\left[\frac{e+1}{p+1}\right] + \left[\frac{e+2}{p+1}\right].$$

   Furthermore, each equality in the above inequalities can happen for any finite extension K of $\mathbb{Q}_p$.

Proof. This follows from Proposition 3.3, Proposition 3.4, Proposition 3.8 and Remark 3.12. □

4 Rational points

Let $C_{\mathbb{F}}$ be the constant group scheme over $\operatorname{Spec} K$ of the two-dimensional vector space over $\mathbb{F}$. To calculate the number of finite flat models of $C_{\mathbb{F}}$, we use the moduli spaces of finite flat models.

Let $V_{\mathbb{F}}$ be the two-dimensional trivial representation of G_K over $\mathbb{F}$. By Proposition 1.3, to calculate the number of finite flat models, it suffices to count the number of the $\mathbb{F}$-rational points of $\mathscr{GR}_{V_{\mathbb{F}},0}$.

For $A \in GL_2\big(\mathbb{F}((u))\big)$, we write $M_{\mathbb{F}} \sim A$ if there is a basis $\{e_1, e_2\}$ of $M_{\mathbb{F}}$ over $\mathbb{F}((u))$ such that $\phi \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = A \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$. We use the same notation for any sublattice $\mathfrak{M}_{\mathbb{F}} \subset M_{\mathbb{F}}$ similarly.

Finally, for any sublattice $\mathfrak{M}_{\mathbb{F}} \subset M_{\mathbb{F}}$ with a chosen basis $\{e_1, e_2\}$ and $B \in GL_2\big(\mathbb{F}((u))\big)$, the module generated by the entries of $\left\langle B \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \right\rangle$ with the basis given by these entries is denoted by $B \cdot \mathfrak{M}_{\mathbb{F}}$. Note that $B \cdot \mathfrak{M}_{\mathbb{F}}$ depends on the choice of the basis of $\mathfrak{M}_{\mathbb{F}}$. We can see that if $\mathfrak{M}_{\mathbb{F}} \sim A$ for $A \in GL_2\big(\mathbb{F}((u))\big)$ with respect to a given basis, then we have

$$B \cdot \mathfrak{M}_{\mathbb{F}} \sim \phi(B)AB^{-1}$$

with respect to the induced basis.

Theorem 4.1. *Let q be the cardinality of $\mathbb{F}$. Then we have*

$$|M(C_{\mathbb{F}}, K)| = \sum_{n \geq 0} (a_n + a'_n) q^n.$$

Here a_n and a'_n are defined as in the introduction.

Proof. Since $V_{\mathbb{F}}$ is the trivial representation, $M_{\mathbb{F}} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for some basis. Let $\mathfrak{M}_{\mathbb{F},0}$ be the lattice of $M_{\mathbb{F}}$ generated by the basis giving $M_{\mathbb{F}} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. By the Iwasawa decomposition, any sublattice of $M_{\mathbb{F}}$ can be written as $\begin{pmatrix} u^s & v \\ 0 & u^t \end{pmatrix} \cdot \mathfrak{M}_{\mathbb{F},0}$ for $s, t \in \mathbb{Z}$ and $v \in \mathbb{F}((u))$. We put

$$\mathscr{GR}_{V_{\mathbb{F}},0,s,t}(\mathbb{F}) = \left\{ \begin{pmatrix} u^s & v \\ 0 & u^t \end{pmatrix} \cdot \mathfrak{M}_{\mathbb{F},0} \in \mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}) \;\middle|\; v \in \mathbb{F}((u)) \right\}.$$

Then

$$\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}) = \bigcup_{s,t \in \mathbb{Z}} \mathscr{GR}_{V_{\mathbb{F}},0,s,t}(\mathbb{F})$$

and this is a disjoint union by Lemma 3.2.

We put

$$\mathfrak{M}_{\mathbb{F},s,t} = \begin{pmatrix} u^s & 0 \\ 0 & u^t \end{pmatrix} \cdot \mathfrak{M}_{\mathbb{F},0}.$$

Then we have $\mathfrak{M}_{\mathbb{F},s,t} \sim \begin{pmatrix} u^{(p-1)s} & 0 \\ 0 & u^{(p-1)t} \end{pmatrix}$ with respect to the basis induced from $\mathfrak{M}_{\mathbb{F},0}$. Any $\mathfrak{M}_{\mathbb{F}}$ in $\mathscr{GR}_{V_{\mathbb{F}},0,s,t}(\mathbb{F})$ can be written as $\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \cdot \mathfrak{M}_{\mathbb{F},s,t}$ for v in $\mathbb{F}((u))$. Then we have

$$\mathfrak{M}_{\mathbb{F}} \sim \begin{pmatrix} u^{(p-1)s} & -vu^{(p-1)s} + \phi(v)u^{(p-1)t} \\ 0 & u^{(p-1)t} \end{pmatrix}$$

with respect to the induced basis. The condition $u^e \mathfrak{M}_{\mathbb{F}} \subset (1 \otimes \phi)\bigl(\phi^*(\mathfrak{M}_{\mathbb{F}})\bigr) \subset \mathfrak{M}_{\mathbb{F}}$ is equivalent to the following:

$$0 \leq (p-1)s \leq e,\ 0 \leq (p-1)t \leq e,$$
$$v_u(vu^{(p-1)s} - \phi(v)u^{(p-1)t}) \geq \max\bigl\{0, (p-1)(s+t) - e\bigr\}.$$

Conversely, $s, t \in \mathbb{Z}$ and $v \in \mathbb{F}((u))$ satisfying this condition gives a point of $\mathscr{GR}_{V_{\mathbb{F}},0,s,t}(\mathbb{F})$ as $\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \cdot \mathfrak{M}_{\mathbb{F},s,t}$. We put $r = -v_u(v)$.

We fix $s, t \in \mathbb{Z}$ such that $0 \leq s, t \leq e_0$. The lowest degree term of $vu^{(p-1)s}$ is equal to that of $\phi(v)u^{(p-1)t}$ if and only if $v_u(v) = s - t$, in which case $v_u(vu^{(p-1)s}) = ps - t$.

In the case where $ps - t \geq \max\bigl\{0, (p-1)(s+t) - e\bigr\}$, the condition $v_u(vu^{(p-1)s} - \phi(v)u^{(p-1)t}) \geq \max\bigl\{0, (p-1)(s+t) - e\bigr\}$ is equivalent to

$$\min\bigl\{v_u(vu^{(p-1)s}), v_u\bigl(\phi(v)u^{(p-1)t}\bigr)\bigr\} \geq \max\bigl\{0, (p-1)(s+t) - e\bigr\},$$

and further equivalent to

$$r \leq \min\left\{(p-1)s, \frac{e - (p-1)s}{p}, e - (p-1)t, \frac{(p-1)t}{p}\right\}.$$

We put

$$r_{s,t} = \min\left\{(p-1)s, \left[\frac{e-(p-1)s}{p}\right], e-(p-1)t, \left[\frac{(p-1)t}{p}\right]\right\}.$$

In this case, the number of the points of $\mathscr{GR}_{V_{\mathbb{F}},0,s,t}(\mathbb{F})$ is equal to $q^{r_{s,t}}$ by Lemma 3.2.

Next, we consider the case where $ps - t < \max\{0, (p-1)(s+t) - e\}$. We note that

$$r_{s,t} \leq \min\{(p-1)s, e-(p-1)t\} < t - s$$

in this case. We claim that the condition $v_u(vu^{(p-1)s} - \phi(v)u^{(p-1)t}) \geq \max\{0, (p-1)(s+t) - e\}$ is satisfied if and only if

$$v = \alpha u^{s-t} + v_+ \text{ for } \alpha \in \mathbb{F} \text{ and } v_+ \in \mathbb{F}((u)) \text{ such that } -v_u(v_+) \leq r_{s,t}.$$

Clearly, the latter implies the former. We prove the converse. We assume that the former condition. If

$$\min\{v_u(vu^{(p-1)s}), v_u\big(\phi(v)u^{(p-1)t}\big)\} \geq \max\{0, (p-1)(s+t) - e\},$$

we may take $\alpha = 0$. So we may assume that

$$\min\{v_u(vu^{(p-1)s}), v_u\big(\phi(v)u^{(p-1)t}\big)\} < \max\{0, (p-1)(s+t) - e\}.$$

Then the lowest degree term of $vu^{(p-1)s}$ is equal to that of $\phi(v)u^{(p-1)t}$, and the lowest degree term of v can be writen as αu^{s-t} for $\alpha \in \mathbb{F}^{\times}$. We put $v_+ = v - \alpha u^{s-t}$. We can see $-v_u(v_+) \leq r_{s,t}$, because $v_u(v_+u^{(p-1)s} - \phi(v_+)u^{(p-1)t}) \geq \max\{0, (p-1)(s+t) - e\}$ and the lowest degree term of $v_+u^{(p-1)s}$ cannot be equal to that of $\phi(v_+)u^{(p-1)t}$. Thus the claim has been proved, and the number of the points of $\mathscr{GR}_{V_{\mathbb{F}},0,s,t}(\mathbb{F})$ is equal to $q^{r_{s,t}+1}$ by Lemma 3.2.

We put $h_{s,t} = \log_q |\mathscr{GR}_{V_{\mathbb{F}},0,s,t}(\mathbb{F})|$. Collecting the above results, we get the followings:

- If $s + t \leq e_0$ and $ps - t \geq 0$, then $h_{s,t} = [(p-1)t/p]$.
- If $s + t \leq e_0$ and $ps - t < 0$, then $h_{s,t} = (p-1)s + 1$.
- If $s + t > e_0$ and $ps - t \geq (p-1)(s+t) - e$, then $h_{s,t} = \big[\big(e - (p-1)s\big)/p\big]$.
- If $s + t > e_0$ and $ps - t < (p-1)(s+t) - e$, then $h_{s,t} = e - (p-1)t + 1$.

Now we have

$$|M(C_{\mathbb{F}}, K)| = \sum_{0 \leq s,t \leq e_0} q^{h_{s,t}}.$$

We put

$$S_n = \big\{(s,t) \in \mathbb{Z}^2 \bigm| 0 \leq s,t \leq e_0,\ h_{s,t} = n\big\},$$

and

$$\begin{aligned}
S_{n,1} &= \bigl\{(s,t) \in S_n \bigm| s+t \le e_0,\ ps-t \ge 0\bigr\},\\
S_{n,2} &= \bigl\{(s,t) \in S_n \bigm| s+t \le e_0,\ ps-t < 0\bigr\},\\
S'_{n,1} &= \bigl\{(s,t) \in S_n \bigm| s+t > e_0,\ ps-t \ge (p-1)(s+t)-e\bigr\},\\
S'_{n,2} &= \bigl\{(s,t) \in S_n \bigm| s+t > e_0,\ ps-t < (p-1)(s+t)-e\bigr\}.
\end{aligned}$$

It suffices to show that $|S_{n,1}| + |S_{n,2}| = a_n$ and $|S'_{n,1}| + |S'_{n,2}| = a'_n$.

Firstly, we calculate $|S_{n,1}|$. We assume $(s,t) \in S_{n,1}$. In the case $n_1 \neq 0$, we have $t = pn_0 + n_1 + 1$ by $[(p-1)t/p] = (p-1)n_0 + n_1$. Then $ps \ge t = pn_0 + n_1 + 1$ implies $s \ge n_0 + 1$, and we have

$$n_0 + 1 \le s \le e_0 - pn_0 - n_1 - 1.$$

We note that if $t > e_0$, we have

$$(e_0 - pn_0 - n_1 - 1) - (n_0 + 1) + 1 = e_0 - (p+1)n_0 - n_1 - 1 < 0.$$

So we get

$$|S_{n,1}| = \max\{e_0 - (p+1)n_0 - n_1 - 1, 0\}.$$

In the case $n_1 = 0$, we have $t = pn_0$ or $t = pn_0 + 1$ by $[(p-1)t/p] = (p-1)n_0$. If $t = pn_0$, we have $n_0 \le s \le e_0 - pn_0$. If $t = pn_0 + 1$, we have $n_0 + 1 \le s \le e_0 - pn_0 - 1$. So we get

$$|S_{n,1}| = \max\{e_0 - (p+1)n_0 + 1, 0\} + \max\{e_0 - (p+1)n_0 - 1, 0\}.$$

Secondly, we calculate $|S_{n,2}|$. In the case $n_1 \neq 1$, we have $S_{n,2} = \emptyset$. In the case $n_1 = 1$, we assume $(s,t) \in S_{n,2}$. Then $s = n_0$, and we have $pn_0 + 1 \le t \le e_0 - n_0$. So we get

$$|S_{n,2}| = \max\{e_0 - (p+1)n_0, 0\}.$$

Collecting these results, we have $|S_{n,1}| + |S_{n,2}| = a_n$.

Next, we calculate $|S'_{n,1}|$. We assume $(s,t) \in S'_{n,1}$. In the case $n'_1 \neq 0$, we have $s = e_0 - e_1 - pn'_0 - n'_1 - 1$ by $\bigl[\bigl(e - (p-1)s\bigr)/p\bigr] = (p-1)n'_0 + n'_1 + e_1$. We note that $\bigl[\bigl(e - (p-1)s\bigr)/p\bigr] = n \ge 0$ shows $s \le e_0$. Then $ps - t \ge (p-1)(s+t) - e$ implies $pt \le pe_0 - pn'_0 - n'_1 - 1$, and further implies $t \le e_0 - n'_0 - 1$. So we have

$$e_1 + pn'_0 + n'_1 + 2 \le t \le e_0 - n'_0 - 1.$$

We note that $e_1 + pn'_0 + n'_1 + 2 = n + n'_0 + 2 \ge 1$ and $e_0 - n'_0 - 1 \le e_0$, because $n'_0 \ge -1$. We note also that if $s < 0$, then

$$(e_0 - n'_0 - 1) - (e_1 + pn'_0 + n'_1 + 2) + 1 = e_0 - e_1 - (p+1)n'_0 - n'_1 - 2 < 0.$$

So we get

$$|S'_{n,1}| = \max\{e_0 - e_1 - (p+1)n'_0 - n'_1 - 2, 0\}.$$

In the case $n'_1 = 0$, we have $s = e_0 - e_1 - pn'_0 - 1$ or $s = e_0 - e_1 - pn'_0$ by $\big[\big(e-(p-1)s\big)/p\big] = (p-1)n'_0 + e_1$. If $s = e_0 - e_1 - pn'_0 - 1$, we have $e_1 + pn'_0 + 2 \leq t \leq e_0 - n'_0 - 1$. If $s = e_0 - e_1 - pn'_0$, we have $e_1 + pn'_0 + 1 \leq t \leq e_0 - n'_0$. We note that $n'_0 \geq 0$, because $n'_1 = 0$. So we get

$$|S'_{n,1}| = \max\{e_0 - e_1 - (p+1)n'_0 - 2, 0\} + \max\{e_0 - e_1 - (p+1)n'_0, 0\}.$$

At last, we calculate $|S'_{n,2}|$. In the case $n'_1 \neq 1$, we have $S'_{n,2} = \emptyset$. In the case $n'_1 = 1$, we assume $(s,t) \in S'_{n,2}$. Then $t = e_0 - n'_0$, and we have $n'_0 + 1 \leq s \leq e_0 - e_1 - pn'_0 - 1$. Here we need some care, because there is the case $n'_0 = -1$, in which case $t > e_0$. Now $n'_0 = -1$ is equivalent to $n = 0$ and $e_1 = p-2$. So we get

$$|S'_{n,2}| = \max\{e_0 - e_1 - (p+1)n'_0 - 1, 0\}$$

except in the case where $n = 0$ and $e_1 = p - 2$, in which case $S'_{n,2} = \emptyset$. Collecting these results, we have $|S'_{n,1}| + |S'_{n,2}| = a'_n$. This completes the proof. $\square$

Example 4.2. *If $K = \mathbb{Q}_p(\zeta_p)$ and $\mathbb{F} = \mathbb{F}_p$, we have $\big|M\big(C_{\mathbb{F}_p}, \mathbb{Q}_p(\zeta_p)\big)\big| = p + 3$ by Theorem 4.1. We know that $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, $\mathbb{Z}/p\mathbb{Z} \oplus \mu_p$ and $\mu_p \oplus \mu_p$ over $\mathcal{O}_{\mathbb{Q}_p(\zeta_p)}$ have the generic fibers that are isomorphic to $C_{\mathbb{F}_p}$. We can see $\big|\mathrm{Aut}(C_{\mathbb{F}_p})\big| = p(p+1)(p-1)^2$. On the other hand, we have*

$$\mathrm{Aut}(\mathbb{Z}/p\mathbb{Z} \oplus \mu_p) \cong \mathrm{Aut}(\mathbb{Z}/p\mathbb{Z}) \times \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, \mu_p) \times \mathrm{Aut}(\mu_p),$$

because $\mathrm{Hom}(\mu_p, \mathbb{Z}/p\mathbb{Z}) = 0$. In particular, we have $\big|\mathrm{Aut}(\mathbb{Z}/p\mathbb{Z} \oplus \mu_p)\big| = p(p-1)^2$. Hence, there are $(p+1)$-choices of an isomorphism $C_{\mathbb{F}_p} \xrightarrow{\sim} \big(\mathbb{Z}/p\mathbb{Z} \oplus \mu_p\big)_{\mathbb{Q}_p(\zeta_p)}$ that give the different elements of $M\big(C_{\mathbb{F}_p}, \mathbb{Q}_p(\zeta_p)\big)$. So the equation $\big|M\big(C_{\mathbb{F}_p}, \mathbb{Q}_p(\zeta_p)\big)\big| = 1 + (p+1) + 1$ shows that there does not exist any other isomorphism class of finite flat models of $C_{\mathbb{F}_p}$.

Remark 4.3. *Theorem 4.1 is equivalent to an explicit calculation of the zeta function of $\mathscr{GR}_{V_{\mathbb{F}},0}$, and we can see that $\dim \mathscr{GR}_{V_{\mathbb{F}},0} = \max\big\{n \geq 0 \;\big|\; a_n + a'_n \neq 0\big\}$.*

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