Generators of modules in tropical geometry
（トロピカル幾何における加群の生成元）

吉冨 修平

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1 Introduction

1.1 Results

A tropical curve is a geometric object over the tropical semifield of real numbers $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, where the addition $\oplus$ is the max-operation in the real field $\mathbb{R}$, and the multiplication $\odot$ is the addition of $\mathbb{R}$. For a tropical curve C and a divisor D on C, the set $M = H^0(C, \mathcal{O}_C(D))$ of the sections of D has the structure of a $\mathbb{T}$-module that is defined as follows.

A $\mathbb{T}$-module M is defined as a module over a semifield. $(M, \oplus, \odot, -\infty)$ is said to be a $\mathbb{T}$-module if $(M, \oplus, -\infty)$ is a tropical semigroup, and $\odot$ is an additive semigroup action on M by $\mathbb{T}$. A tropical semigroup is a commutative semigroup with unity such that any element v satisfies the idempotent condition $v \oplus v = v$.

A $\mathbb{T}$-module M is analogous to a module over a field. A subset $S \subset M$ is said to be a basis if it is a minimal system of generators. But the number of elements of a basis of M is not necessarily equal to the topological dimension of it. We introduce straight $\mathbb{T}$-modules in section 2. This class is a generalization of lattice-preserving submodules of the free $\mathbb{T}$-module $\mathbb{T}^n$, where a lattice-preserving submodule is a submodule preserving the infimum of any two elements with respect to the canonical partial order relation on $\mathbb{T}^n$.

Theorem 1.1. *Let M be a finitely generated straight submodule of the free $\mathbb{T}$-module $\mathbb{T}^n$. Then M is generated by n elements.*

We have four corollaries (Theorem 2.1, 2.2, 2.3, 2.4). The semifield $\mathbb{T}$ is generalized to a quasi-complete totally ordered rational tropical semifield k. We find a sufficient condition to the existence of a left-inversion of an injective homomorphism of k-modules (Theorem 2.1). The dimension of a straight reflexive k-module is defined to be the number of elements of a basis. We show the inequality $\dim(M) \leq \dim(N)$ for a pair of straight reflexive k-modules $M \subset N$ (Theorem 2.2). We show that a finitely generated straight pre-reflexive k-module is reflexive (Theorem 2.3). Also we consider finiteness of a submodule of a k-module (Theorem 2.4). The proofs are given in section 3.7.

This result has an application to polytopes in a tropical projective space $\mathbb{TP}^n$. By Joswig and Kulas [3], a polytrope (it means a polytope in $\mathbb{TP}^n$ that is real convex) is a tropical simplex, and therefore it is the tropically convex hull of at most $n + 1$ points. We show a generalization of this result (Theorem 2.5). A polytope P is the tropically convex hull of at most $n + 1$ points if the corresponding submodule $M \subset \mathbb{T}^{n+1}$ is straight reflexive. Also M is straight reflexive if P is a polytrope.

Also we have an application to tropical curves. A Riemann-Roch theorem for tropical curves is proved by Gathmann and Kerber [1]. This theorem states an equality for an invariant $r(D)$ of the divisor. We see that $r(D)$ is not an invariant of the $\mathbb{T}$-module $M = H^0(C, \mathcal{O}_C(D))$ (Example 6.5), and show the inequality $r(D) \leq \dim(M) - 1$ (Theorem 2.7).

1.2 Background

A survey of tropical mathematics is found in [4]. Tropical varieties are introduced as follows. Let $K = \mathbb{C}[[\mathbb{R}]]$ be the group algebra of power series defined by the group $\mathbb{R}$. We have a multiplicative seminorm

$$||\cdot||\colon K \to \mathbb{R}_{\geq 0}$$

defined by

$$||x|| = \exp(-\operatorname{val}(x)),$$

where val means the canonical valuation on K. This seminorm induces the amoeba map

$$\mathcal{A}\colon (K^{\times})^n \longrightarrow \mathbb{R}^n$$

defined by

$$\mathcal{A}(x_1, \dots, x_n) = (\log ||x_1||, \dots, \log ||x_n||).$$

The image $\mathcal{A}(V)$ of a variety V in the algebraic torus $(K^{\times})^n$ is said to be a tropical variety in the tropical torus $\mathbb{R}^n$.

Tropical algebra is introduced by the map

$$\pi\colon K \longrightarrow \mathbb{R} \cup \{-\infty\}$$

defined by

$$\pi(x) = \log ||x||.$$

This map induces a hyperfield homomorphism

$$\pi\colon K \longrightarrow X,$$

where X is the tropical hyperfield with underlying set $\mathbb{R} \cup \{-\infty\}$, introduced in [7]. The power set 2^X is a semiring with operations induced by multi-operations of X.

Now we have the lower-saturation map

$$\nu\colon X \longrightarrow 2^X$$

defined by

$$\nu(a) = \{c \in X \mid c \leq a\}.$$

The power set 2^X has a subsemiring

$$\mathbb{I} = X \cup \nu(X),$$

which is isomorphic to Izhakian's extended tropical semiring introduced in [2]. The lower-saturation map ν means the ghost map in [2]. The image $\nu(X)$ means the ghost part, which is isomorphic to the tropical semifield of real numbers $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, where operations are defined as follows.

$$a \oplus b = \max\{a, b\},$$

$$a \odot b = a + b.$$

In this paper, the symbol $\mathbb{T}$ means the tropical semifield of real numbers. Under the identification $\mathbb{T} = \nu(X)$, the canonical homomorphism $\nu \colon \mathbb{I} \to \mathbb{T}$ is the lower-saturation map.

Section 2 contains definitions and theorems. Section 3 and 4 contain foundation of tropical modules, and the proof of Theorem 2.1, 2.2, 2.3, 2.4, and 2.5. Section 5 and 6 contain foundation of tropical matrices and tropical curves, and the proof of Theorem 2.7. Section 7 is an appendix for tropical plane curves.

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2 Definitions and theorems

A *semigroup* $(M, \oplus)$ is a set M with an associative operation $\oplus$.

Definition. $(M, \oplus, -\infty)$ is a *tropical semigroup* if it satisfies the following axioms.

(i) $(M, \oplus)$ is a semigroup.

(ii) $v \oplus w = w \oplus v$.

(iii) $v \oplus -\infty = v$.

(iv) $v \oplus v = v$.

The element $-\infty$ is called the zero element of M.

There is a unique partial order relation '$\leq$' on M such that for any $v, w \in M$ it implies

$$\sup\{v, w\} = v \oplus w.$$

The proof is given in section 3.1.

Definition. A tropical semigroup M is *quasi-complete* if any non-empty subset $S \subset M$ admits the infimum $\inf(S)$ (i.e. it admits the maximum element of the lower-bounds of S).

Definition. $(A, \oplus, \odot, -\infty, 0)$ is a *tropical semiring* if it satisfies the following axioms.

(i) $(A, \oplus, -\infty)$ is a tropical semigroup.

(ii) $(A, \odot)$ is a semigroup.

(iii) $a \odot b = b \odot a$.

(iv) $a \odot (b \oplus c) = a \odot b \oplus a \odot c$.

(v) $a \odot 0 = a$.

(vi) $a \odot -\infty = -\infty$.

The element $-\infty$ is called the zero element of A. The element 0 is called the unity of A.

Definition. $(k, \oplus, \odot, -\infty, 0)$ is a *tropical semifield* if it satisfies the following axioms.

(i) $(k, \oplus, \odot, -\infty, 0)$ is a tropical semiring.

(ii) For any $a \in k \setminus \{-\infty\}$ there is an element $\oslash a \in k$ such that $a \odot (\oslash a) = 0$.

Definition. A tropical semifield k is *rational* if it satisfies the following conditions.

(i) $a \in k,\ m \in \mathbb{N} \Rightarrow {}^{\exists}b \in k,\ a = b^{\odot m}$.

(ii) k has no maximum element.

The *tropical semifield of real numbers* $(\mathbb{T}, \oplus, \odot, -\infty, 0)$ is the set

$$\mathbb{T} = \mathbb{R} \cup \{-\infty\}$$

equipped with addition

$$a \oplus b = \max\{a, b\}$$

and multiplication

$$a \odot b = a + b$$

and zero element $-\infty$ and unity 0. $\mathbb{T}$ is a quasi-complete totally ordered rational tropical semifield.

Let k be a quasi-complete totally ordered rational tropical semifield.

Definition. $(M, \oplus, \odot, -\infty)$ is a *k-module* if it satisfies the following axioms.

(i) $(M, \oplus, -\infty)$ is a tropical semigroup.

(ii) $\odot$ is a semigroup action $k \times M \ni (a, v) \mapsto a \odot v \in M$, i.e.

i) $(a \odot b) \odot v = a \odot (b \odot v)$.

ii) $0 \odot v = v$.

(iii) $(a \oplus b) \odot v = (a \odot v) \oplus (b \odot v)$.

(iv) $a \odot (v \oplus w) = (a \odot v) \oplus (a \odot w)$.

(v) $-\infty \odot v = -\infty$.

(vi) $a \odot -\infty = -\infty$.

Definition. A *homomorphism* $\alpha\colon M \to N$ *of k-modules* is a map with the following conditions.

(i) $\alpha(-\infty) = -\infty$.

(ii) $\alpha(v \oplus w) = \alpha(v) \oplus \alpha(w)$.

(iii) $\alpha(a \odot v) = a \odot \alpha(v)$.

Let $\mathrm{Hom}(M, N)$ denote the k-module of homomorphisms from M to N.

The dual module $M^\vee$ is defined by $M^\vee = \mathrm{Hom}(M, k)$. We have the pairing map $\langle\cdot,\cdot\rangle \colon M \times M^\vee \to k$ defined by

$$\langle v, \xi \rangle = \xi(v).$$

Definition. M is *pre-reflexive* if the homomorphism $\iota_M \colon M \to (M^\vee)^\vee$ is injective. M is *reflexive* if ι_M is an isomorphism.

Definition. A k-module M is *straight* if it is a finitely distributive ordered lattice, i.e. it satisfies the following conditions.

(i) Any two elements $v, w \in M$ admit the infimum $\inf_M\{v, w\}$.

(ii) $v_1, v_2, w \in M \Rightarrow \inf_M\{v_1 \oplus v_2, w\} = \inf_M\{v_1, w\} \oplus \inf_M\{v_2, w\}$.

(iii) $v_1, v_2, w \in M \Rightarrow \inf_M\{v_1, v_2\} \oplus w = \inf_M\{v_1 \oplus w, v_2 \oplus w\}$.

Definition. A homomorphism $\alpha \colon M \to N$ is *lightly surjective* if for any $w \in N$ there is $v \in M$ such that $w \leq \alpha(v)$.

A homomorphism $\beta \colon N \to M$ is said to be a left-inversion of α if $\beta \circ \alpha = \mathrm{id}_M$.

Theorem 2.1. *Let $\alpha \colon M \to N$ be an injective lightly surjective homomorphism of k-modules such that M is straight reflexive. Then α has a left-inversion.*

Definition. A *basis* $\{e_\lambda \mid \lambda \in \Lambda\}$ of a k-module M is a minimal system of generators (i.e. there is no $\lambda_0 \in \Lambda$ such that the elements $\{e_\lambda \mid \lambda \in \Lambda \setminus \{\lambda_0\}\}$ generate M). A subset $S \subset M$ *generate* M if any element of M is written as a linear combination

$$a_1 \odot v_1 \oplus \cdots \oplus a_r \odot v_r$$

of elements of S over k.

Definition. An element $e \in M \setminus \{-\infty\}$ is *extremal* if for any $v_1, v_2 \in M$ such that $v_1 \oplus v_2 = e$ it implies $v_1 = e$ or $v_2 = e$. M is *extremally generated* if M is generated by extremal elements. An *extremal ray* of M is the submodule generated by an extremal element of M.

Definition. The *dimension* of a straight reflexive k-module M is the number of extremal rays.

The number of extremal rays of M is equal to the number of elements of any basis of M. The proof is given in section 3.3.

Theorem 2.2. *Let $\alpha \colon M \to N$ be an injective homomorphism of finitely generated straight reflexive k-modules. Then*

(1) $\dim(M) \leq \dim(N)$.

(2) *If* $\dim(M) = \dim(N)$*, then* α *is lightly surjective.*

Theorem 2.3. *Let M be a finitely generated straight pre-reflexive k-module. Then M is reflexive.*

Theorem 2.4. *Let $\alpha\colon M \to N$ be an injective homomorphism of straight pre-reflexive k-modules. Suppose that M has a basis, and that N is finitely generated. Then M is finitely generated.*

Let P be a polytope in $\mathbb{TP}^n$. P is the tropically convex hull of finitely many points $p_1, \ldots, p_r$. Let

$$\varphi\colon \mathbb{T}^{n+1} \setminus \{-\infty\} \longrightarrow \mathbb{TP}^n$$

be the canonical projection. Then the subset

$$M = \varphi^{-1}(P) \cup \{-\infty\} \subset \mathbb{T}^{n+1}$$

is a submodule generated by elements $v_1, \ldots, v_r$ such that $\varphi(v_i) = p_i$ $(1 \leq i \leq r)$. Also we have an injection

$$\iota\colon \mathbb{T}^n \longrightarrow \mathbb{TP}^n$$

defined by $(a_1, \ldots, a_n) \mapsto (0, a_1, \ldots, a_n)$. This map induces an embedding $\mathbb{R}^n \subset \mathbb{T}^n \subset \mathbb{TP}^n$. A polytope $P \subset \mathbb{TP}^n$ is said to be a polytrope if it is a real convex subset of $\mathbb{R}^n$.

Theorem 2.5. *Let P be a polytope in $\mathbb{TP}^n$ with the corresponding submodule $M \subset \mathbb{T}^{n+1}$.*

(1) *If P is a polytrope, then M is straight reflexive.*

(2) *If M is straight reflexive, then P is the tropically convex hull of at most $n+1$ points.*

Let C be a tropical curve. Let D be a divisor on C. Let $H^0(C, \mathcal{O}_C(D))$ be the set of the sections of D. (A section of D is a rational function $f\colon C \to \mathbb{T}$ such that either $f = -\infty$ or $(f) + D \geq 0$.) For $r \in \mathbb{Z}_{\geq 0}$, let

$$U(D, r) = C^r \setminus S(D, r),$$

$$S(D, r) = \{(P_1, \ldots, P_r) \in C^r \mid H^0(C, \mathcal{O}_C(D - \sum_{1 \leq i \leq r} P_i)) \neq -\infty\}.$$

Let $U(D, r) = \emptyset$ if $r = -1$. The following theorem is known.

Theorem 2.6 (Gathmann and Kerber [1]). *Let C be a compact tropical curve with first Betti number $b_1(C)$. Let D be a divisor on C. Let K be the canonical divisor on C. Then*

$$r(D) - r(K - D) = 1 - b_1(C) + \deg(D),$$

where

$$r(D) = \max\{r \in \mathbb{Z}_{\geq -1} \mid U(D, r) = \emptyset\}.$$

The set $M = H^0(C, \mathcal{O}_C(D))$ is a $\mathbb{T}$-module with addition

$$(f \oplus g)(P) = f(P) \oplus g(P)$$

and scalar multiplication

$$(a \odot f)(P) = a \odot f(P).$$

The dimension of M is defined as follows.

Definition. The *dimension* of a k-module M is the maximum dimension of the straight reflexive submodules of M.

This definition is compatible with the previous one. If M is straight reflexive, then the maximum dimension of the straight reflexive submodules of M equals the dimension of M by Theorem 2.2.

Theorem 2.7. *Let C be a tropical curve. Let D be a divisor on C. Then the inequality*

$$r(D) \leq \dim H^0(C, \mathcal{O}_C(D)) - 1$$

is fulfilled.

3 Tropical algebra

3.1 Tropical semigroups, semirings, and semifields

Proposition 3.1. *Let M be a tropical semigroup. Then there is a unique partial order relation '$\leq$' such that for any $v, w \in M$ it implies*

$$\sup\{v, w\} = v \oplus w.$$

Proof. We define a relation '$\leq$' on M as follows.

$$v \leq w \iff v \oplus w = w.$$

This is a partial order relation, because $v \oplus v = v$. The element $v \oplus w$ is the minimum element of the upper bounds of $\{v, w\}$. □

Let A be a tropical semiring.

Example 3.2. The *semiring of polynomials* $B = A[x_1, \ldots, x_n]$ is the set of polynomials

$$\begin{aligned} f &= \bigoplus_i a_i \odot x^{\odot i} \\ &= \bigoplus_{i_1, \ldots, i_n \geq 0} a_{i_1 \ldots i_n} \odot x_1^{\odot i_1} \odot \cdots \odot x_n^{\odot i_n} \end{aligned}$$

with coefficients $a_i \in A$, equipped with addition and multiplication of polynomials. B is a tropical semiring. An element $f \in B$ is said to be a *tropical polynomial over A*. The induced map

$$\begin{array}{rcl} \overline{f}\colon A^n & \longrightarrow & A \\ (a_1, \ldots, a_n) & \mapsto & f(a_1, \ldots, a_n) \end{array}$$

is said to be a tropical polynomial function.

Remark 3.3. We use the notation ma by the meaning of tropical m-th power $a^{\odot m}$. For example, $2(a \oplus b)$ means the second power of $(a \oplus b)$, so we have

$$\begin{aligned} 2(a \oplus b) &= 2a \oplus a \odot b \oplus a \odot b \oplus 2b \\ &= 2a \oplus a \odot b \oplus 2b. \end{aligned}$$

Also a tropical polynomial is written as

$$f = \bigoplus_i a_i \odot ix.$$

Proposition 3.4. *Let A be a tropical semiring. Let $f \in A[x_1, \ldots, x_n]$. Then for any $v, w \in A^n$,*

$$f(v \oplus w) \geq f(v) \oplus f(w).$$

Proof. Assume that

$$\begin{aligned} f &= i_1 x_1 \odot \cdots \odot i_n x_n, \\ v &= (a_1, \ldots, a_n), \\ w &= (b_1, \ldots, b_n). \end{aligned}$$

Then

$$\begin{aligned} f(v \oplus w) &= i_1(a_1 \oplus b_1) \odot \cdots \odot i_n(a_n \oplus b_n) \\ &\geq (i_1 a_1 \odot \cdots \odot i_n a_n) \oplus (i_1 b_1 \odot \cdots \odot i_n b_n) \\ &= f(v) \oplus f(w). \end{aligned}$$

□

Let k be a tropical semifield. Recall that k is said to be rational if it satisfies the following conditions.

(i) $a \in k,\ m \in \mathbb{N} \Rightarrow {}^{\exists}b \in k; a = b^{\odot m}$.

(ii) k has no maximum element.

Proposition 3.5. *Let k be a rational tropical semifield. Then for any $a \in k$ it implies*

$$\inf_k \{ b \in k \mid a < b \} = a.$$

Proof. The case of $a = -\infty$. Suppose that there is an element $c \in k \setminus \{-\infty\}$ such that $k_{\geq}(c) = k \setminus \{-\infty\}$. Then the element $0 \oslash c$ is the maximum element of k, which is contradiction.

The case of $a \neq -\infty$. The condition $a < b$ is fulfilled if and only if $0 < b \oslash a$. So we may assume $a = 0$. Suppose that there is an element $c \not\leq 0$ such that c is a lower-bound of the set $\{b \in k \mid 0 < b\}$. There is an element $c' \in k$ such that $c = (c')^{\odot 2} = 2c'$. Since $0 < 0 \oplus c'$, we have $c \leq 0 \oplus c'$. So we have

$$\begin{aligned} 2(0 \oplus c') &= 0 \oplus c' \oplus 2c' \\ &= 0 \oplus c' \oplus c \\ &= 0 \oplus c', \\ 0 \oplus c' &= 0. \end{aligned}$$

So we have $c \leq 0$, which is contradiction. □

3.2 Modules over a tropical semifield

Let k be a tropical semifield. Let M be a k-module.

Definition. A *submodule* N of M is a subset with the following conditions.

(i) $-\infty \in N$.

(ii) If $v, w \in N$ then $v \oplus w \in N$.

(iii) If $v \in N$ and $a \in k$ then $a \odot v \in N$.

Example 3.6. Suppose that k is totally ordered. Let $q \in k[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree m. Let $p\colon k^n \to k$ be a homomorphism of k-modules. Then the subset

$$M = \{v \in k^n \mid mp(v) \leq q(v)\}$$

is a submodule of k^n. Indeed, for $v, w \in M$ and $a \in k$,

$$\begin{aligned} mp(a \odot v) &= m(a \odot p(v)) \\ &= ma \odot mp(v) \\ &\leq ma \odot q(v) \\ &= q(a \odot v), \end{aligned}$$

$$\begin{aligned} mp(v \oplus w) &= m(p(v) \oplus p(w)) \\ &= \max\{mp(v), mp(w)\} \\ &= mp(v) \oplus mp(w) \\ &\leq q(v) \oplus q(w). \end{aligned}$$

By Proposition 3.4, we have $q(v) \oplus q(w) \leq q(v \oplus w)$.

Example 3.7. A free module $M = k^n$ of finite rank is reflexive. Indeed there is a pairing map $\langle \cdot, \cdot \rangle : k^n \times k^n \to k$ defined by

$$\langle (a_1, \dots, a_n), (b_1, \dots, b_n) \rangle = a_1 \odot b_1 \oplus \cdots \oplus a_n \odot b_n.$$

So we have $(k^n)^\vee \cong k^n$.

Recall that M is said to be pre-reflexive if the homomorphism $\iota_M : M \to (M^\vee)^\vee$ is injective.

Proposition 3.8. *M is pre-reflexive if and only if there is an injection $M \to F$ for some direct product $F = \prod_{\lambda \in \Lambda} k$.*

Proof. There is an injection $(M^\vee)^\vee \to \prod_{\lambda \in \Lambda} k$, where Λ is the set $M^\vee$. Conversely, if there is an injection $M \to F$ for some direct product F, then M is pre-reflexive, because F is pre-reflexive. □

Lemma 3.9. *Suppose that k is rational. Let M be a pre-reflexive k-module. Then for any $v \in M$ and any $a \in k$ it implies*

$$\inf_M \{ b \odot v \mid b \in k, a < b \} = a \odot v.$$

Proof. Let $w \in M$ be a lower-bound of the subset $\{ b \odot v \mid b \in k, a < b \}$. For $\xi \in M^\vee$ and $b \in k$ such that $a < b$, we have

$$\xi(w) \leq b \odot \xi(v).$$

By Proposition 3.5, we have

$$\xi(w) \leq a \odot \xi(v).$$

Since M is pre-reflexive, we have $w \leq a \odot v$. □

Lemma 3.10. *Suppose that k is totally ordered. Let M be a pre-reflexive k-module. Then for any $v, w \in M$ and any $a \in k$,*

$$v \not\leq w, a < 0 \Rightarrow v \not\leq w \oplus a \odot v.$$

Proof. Since M is pre-reflexive, there is an element $\xi \in M^\vee$ such that $\xi(v) \not\leq \xi(w)$. Since k is totally ordered, we have $\xi(w) < \xi(v)$. So

$$\max\{\xi(w), a \odot \xi(v)\} < \xi(v).$$

So we have the conclusion. □

Example 3.11. Let G be a tropical semigroup with at least two elements. Let $M = (G \times \mathbb{R}) \cup \{-\infty\}$ be the $\mathbb{T}$-module with addition

$$(v, a) \oplus (w, b) = (v \oplus w, a \oplus b)$$

and scalar multiplication

$$c \odot (v, a) = \begin{cases} (v, c \odot a) & \text{if } c \in \mathbb{R} \\ -\infty & \text{if } c = -\infty. \end{cases}$$

M is a $\mathbb{T}$-module generated by the subset $G \times \{0\}$. M is not pre-reflexive, because it does not satisfy Lemma 3.10. Let $v, w \in G$ be elements such that $v \not\leq w$. Then

$$(v, 0) \not\leq (w, 0),$$
$$(v, 0) \leq (w, 0) \oplus (-1) \odot (v, 0).$$

3.3 Basis and extremal rays

Let k be a totally ordered tropical semifield. Let M be a k-module. Recall that an element $e \in M \setminus \{-\infty\}$ is said to be extremal if for any $v_1, v_2 \in M$ such that $v_1 \oplus v_2 = e$ it implies $v_1 = e$ or $v_2 = e$.

Proposition 3.12. *Let M be a pre-reflexive k-module. Then the following are equivalent.*

(i) *There is a basis of M.*

(ii) *M is extremally generated.*

More precisely, a system of generators $E = \{e_\lambda \mid \lambda \in \Lambda\}$ is a basis if and only if each e_λ is extremal and it satisfies $k \odot e_\lambda \neq k \odot e_\mu$ $(\lambda \neq \mu)$.

Proof. Suppose that there is a basis E of M. Let e_1 be an element of the basis E. Let $v_1, v_2 \in M$ be elements such that $v_1 \oplus v_2 = e_1$. There are elements $e_2, e_3, \ldots, e_r$ of the basis E and elements $a_i, b_i \in k$ such that

$$v_1 = a_1 \odot e_1 \oplus a_2 \odot e_2 \oplus \cdots \oplus a_r \odot e_r,$$
$$v_2 = b_1 \odot e_1 \oplus b_2 \odot e_2 \oplus \cdots \oplus b_r \odot e_r.$$

Since k is totally ordered, we may assume $a_1 \leq b_1$. Then

$$e_1 = b_1 \odot e_1 \oplus w,$$

where

$$w = (a_2 \oplus b_2) \odot e_2 \oplus \cdots \oplus (a_r \oplus b_r) \odot e_r.$$

Since E is a basis, we have $w \neq e_1$. By Lemma 3.10, we have $b_1 = 0$. It means $v_2 \geq e_1$. So we have $v_2 = e_1$. Thus e_1 is extremal.

Conversely, let E be a system of generators that consists of extremal elements with different extremal rays. Suppose that E is not a basis. There are elements $e_1, e_2, \ldots, e_r$ of E and elements $a_i \in k$ such that

$$e_1 = a_2 \odot e_2 \oplus \cdots \oplus a_r \odot e_r.$$

Since e_1 is extremal, there is a number i such that $e_1 = a_i \odot e_i$, which is contradiction. □

Proposition 3.13. *Let $\alpha\colon M \to N$ be a homomorphism of k-modules. Let $w \in N$ be an extremal element. Then any minimal element of the subset $\alpha^{-1}(w)$ is extremal.*

Proof. Let $e \in M$ be a minimal element of $\alpha^{-1}(w)$. Let $v_1, v_2 \in M$ be elements such that $v_1 \oplus v_2 = e$. Then $\alpha(v_1) \oplus \alpha(v_2) = w$. Since w is extremal, we may assume $\alpha(v_1) = w$. Then v_1 is a lower-bound of e in $\alpha^{-1}(w)$. Since e is minimal, we have $v_1 = e$. □

3.4 Locators

Let k be a totally ordered tropical semifield. Let M be a k-module. For a subset $S \subset M$, the lower-saturation $M_{\leq}(S)$ is defined by

$$M_{\leq}(S) = \bigcup_{w \in S} \{v \in M \mid v \leq w\}.$$

The set of the lower-bounds $\mathrm{Low}_M(S)$ is defined by

$$\mathrm{Low}_M(S) = \bigcap_{w \in S} \{v \in M \mid v \leq w\}.$$

A subset $S \subset M$ is said to be lower-saturated if $M_{\leq}(S) = S$.

Definition. A *locator* S of M is a lower-saturated subsemigroup of the semigroup $(M, \oplus)$ that generates the k-module M.

Let $\mathrm{Loc}(M)$ denote the set of the locators of a k-module M, equipped with addition

$$S \overset{\vee}{\oplus} T = S \cap T$$

and scalar multiplication

$$a \overset{\vee}{\odot} S = \begin{cases} (\oslash a) \odot S & \text{if } a \in k \setminus \{-\infty\} \\ M & \text{if } a = -\infty. \end{cases}$$

Proposition 3.14. *$(\mathrm{Loc}(M), \overset{\vee}{\oplus}, \overset{\vee}{\odot})$ is a k-module with zero element M. There is a homomorphism*

$$i\colon M^{\vee} \longrightarrow \mathrm{Loc}(M)$$

defined by

$$i(\xi) = \{v \in M \mid \langle v, \xi \rangle \leq 0\}.$$

Proof. $\mathrm{Loc}(M)$ is a tropical semigroup. Indeed,

$$S \overset{\vee}{\oplus} S = S \cap S = S.$$

$\mathrm{Loc}(M)$ is a k-module. Indeed, for $a, b \in k$ such that $a \leq b$, since S is lower-saturated, we have

$$\oslash b \odot S \subset \oslash a \odot S.$$

So we have

$$\begin{aligned}(a \oplus b) \overset{\vee}{\odot} S &= \oslash b \odot S \\ &= (\oslash a \odot S) \cap (\oslash b \odot S) \\ &= a \overset{\vee}{\odot} S \oplus b \overset{\vee}{\odot} S.\end{aligned}$$

i is a homomorphism. Indeed, for $v \in M$,

$$\begin{aligned}v \in i(\xi_1 \oplus \xi_2) &\iff \langle v, \xi_1 \oplus \xi_2 \rangle \leq 0 \\ &\iff \langle v, \xi_1 \rangle \oplus \langle v, \xi_2 \rangle \leq 0 \\ &\iff v \in i(\xi_1) \cap i(\xi_2) \\ &\iff v \in i(\xi_1) \overset{\vee}{\oplus} i(\xi_2).\end{aligned}$$

So $i(\xi_1 \oplus \xi_2) = i(\xi_1) \overset{\vee}{\oplus} i(\xi_2)$.

$$\begin{aligned}v \in i(a \odot \xi) &\iff \langle v, a \odot \xi \rangle \leq 0 \\ &\iff \langle a \odot v, \xi \rangle \leq 0 \\ &\iff a \odot v \in i(\xi) \\ &\iff v \in a \overset{\vee}{\odot} i(\xi).\end{aligned}$$

So $i(a \odot \xi) = a \overset{\vee}{\odot} i(\xi)$. □

Lemma 3.15. *Suppose that k is quasi-complete and rational.*

(1) *For any locator $S \in \mathrm{Loc}(M)$ there is a unique element $\xi \in M^\vee$ that satisfies the following conditions.*

$$\begin{aligned}&\langle v, \xi \rangle \leq 0 \quad (v \in S), \\ &\langle v, \xi \rangle \geq 0 \quad (v \in M \setminus S).\end{aligned}$$

(2) *The mapping $S \mapsto \xi$ induces a homomorphism*

$$p \colon \mathrm{Loc}(M) \longrightarrow M^\vee$$

which satisfies $p \circ i = \mathrm{id}_{M^\vee}$.

Proof. (1) Let $\xi \colon M \to k$ be the map defined as follows.

$$\xi(v) = \inf_k \{a \in k \mid v \in a \odot S\}.$$

The set in right side is non-empty. (Since S generates the k-module M, there are $s_i \in S$ and $a_i \in k$ such that

$$v = a_1 \odot s_1 \oplus \cdots \oplus a_r \odot s_r.$$

Let a be the maximum element of $a_1, \dots, a_r$. Since S is lower-saturated, there are $s'_i \in S$ such that

$$v = a \odot (s'_1 \oplus \cdots \oplus s'_r).$$

Since S is a subsemigroup, we have $v \in a \odot S$.) For any $v \in M \setminus S$ we have $\xi(v) \geq 0$, because S is lower-saturated. For any $v \in S$, we have $\xi(v) \leq 0$.

We show that ξ is a homomorphism. Since S is lower-saturated, we have

$$\xi(v) \oplus \xi(w) \leq \xi(v \oplus w).$$

Suppose that $\xi(v) \oplus \xi(w) < \xi(v \oplus w)$. There are $a, b \in k$ such that $a \oplus b < \xi(v \oplus w)$ and $v \in a \odot S$ and $w \in b \odot S$. Then $v \oplus w \in (a \oplus b) \odot S$. So we have $\xi(v \oplus w) \leq a \oplus b$, which is contradiction.

We prove uniqueness. Let $\xi \in M^\vee$ be an element that satisfies the following conditions.

$$\langle v, \xi \rangle \leq 0 \quad (v \in S),$$
$$\langle v, \xi \rangle \geq 0 \quad (v \in M \setminus S).$$

Then

$$\langle v, \xi \rangle \leq \inf_k \{a \in k \mid v \in a \odot S\}$$
$$\leq \inf_k \{a \in k \mid \langle v, \xi \rangle < a\}.$$

By Proposition 3.5,

$$\inf_k \{a \in k \mid \langle v, \xi \rangle < a\} = \langle v, \xi \rangle.$$

So we have

$$\langle v, \xi \rangle = \inf_k \{a \in k \mid v \in a \odot S\}.$$

(2) We have

$$\langle v, p(S) \oplus p(T) \rangle \leq 0 \quad (v \in S \cap T),$$
$$\langle v, p(S) \oplus p(T) \rangle \geq 0 \quad (v \in M \setminus (S \cap T)).$$

It means $p(S) \oplus p(T) = p(S \overset{\vee}{\oplus} T)$. So p is a homomorphism. For $\xi \in M^\vee$, let

$$S = \{v \in M \mid \langle v, \xi \rangle \leq 0\}.$$

Then

$$\langle v, \xi \rangle \leq 0 \quad (v \in S),$$
$$\langle v, \xi \rangle \geq 0 \quad (v \in M \setminus S).$$

It means $\xi = p(S)$. □

3.5 Straight modules

Let k be a totally ordered tropical semifield. Recall that a k-module M is said to be straight if it satisfies the following conditions.

(i) Any two elements $v, w \in M$ admit the infimum $\inf_M\{v, w\}$.

(ii) $v_1, v_2, w \in M \Rightarrow \inf_M\{v_1 \oplus v_2, w\} = \inf_M\{v_1, w\} \oplus \inf_M\{v_2, w\}$.

(iii) $v_1, v_2, w \in M \Rightarrow \inf_M\{v_1, v_2\} \oplus w = \inf_M\{v_1 \oplus w, v_2 \oplus w\}$.

Proposition 3.16. *The above conditions* (ii), (iii) *are equivalent.*

Proof. (ii) $\Rightarrow$ (iii).

$$\begin{aligned}\inf_M\{v_1 \oplus w, v_2 \oplus w\} &= \inf_M\{v_1, v_2\} \oplus \inf_M\{v_1, w\} \oplus \inf_M\{w, v_2\} \oplus w \\ &= \inf_M\{v_1, v_2\} \oplus w.\end{aligned}$$

(iii) $\Rightarrow$ (ii) is similar. □

Definition. A homomorphism $\alpha\colon M \to N$ of k-modules is *lattice-preserving* if for any $v, w \in M$ and any lower-bound $x \in \mathrm{Low}_N(\alpha(v), \alpha(w))$ there is a lower-bound $y \in \mathrm{Low}_M(v, w)$ such that $x \leq \alpha(y)$.

If M, N are ordered lattices, α is lattice-preserving if and only if it preserves the infimum of any two elements.

Proposition 3.17. *Let $\alpha\colon M \to N$ be a lattice-preserving injective homomorphism of k-modules such that N is straight. Then M is straight.*

Proof. For $v, w \in M$, let $x = \inf_N\{\alpha(v), \alpha(w)\}$. There is a lower-bound y of $\{v, w\}$ such that $x \leq \alpha(y)$. Then $y = \inf_M\{v, w\}$. (Let $y' \in M$ be a lower-bound of $\{v, w\}$. Then $\alpha(y') \leq x \leq \alpha(y)$. Since α is injective, we have $y' \leq y$.) $\alpha(y)$ is a lower-bound of $\{\alpha(v), \alpha(w)\}$. So we have $x = \alpha(y)$. M is finitely distributive, because α preserves the infimum of any two elements. □

Proposition 3.18. *Suppose that k is quasi-complete and rational. Let M be a straight k-module. Then $M^\vee$ and* $\mathrm{Loc}(M)$ *are straight.*

Proof. We show that $\mathrm{Loc}(M)$ is straight. For $S, T \in \mathrm{Loc}(M)$, let

$$U = S \oplus T = \{s \oplus t \mid s \in S, t \in T\}.$$

U is lower-saturated. (Let $v \in M$ and $s \in S$ and $t \in T$ be elements such that $v \leq s \oplus t$. Then

$$\begin{aligned}v &= \inf_M\{v, s \oplus t\} \\ &= \inf_M\{v, s\} \oplus \inf_M\{v, t\}.\end{aligned}$$

So we have $v \in U$.) U is a locator of M, and we have

$$U = \inf_{\mathrm{Loc}(M)} \{S, T\}.$$

$\mathrm{Loc}(M)$ is finitely distributive. Indeed,

$$(S_1 \cap S_2) \oplus T = (S_1 \oplus T) \cap (S_2 \oplus T).$$

(Let v be an element of right side. There are $s_1 \in S_1$ and $s_2 \in S_2$ and $t_1, t_2 \in T$ such that

$$v = s_1 \oplus t_1 = s_2 \oplus t_2.$$

Then

$$\begin{aligned} v &= \inf_M \{s_1 \oplus t_1, s_2 \oplus t_2\} \\ &= \inf_M \{s_1, s_2\} \oplus \inf_M \{s_1, t_2\} \oplus \inf_M \{t_1, s_2\} \oplus \inf_M \{t_1, t_2\}. \end{aligned}$$

So we have $v \in (S_1 \cap S_2) \oplus T$.)

We show that $M^\vee$ is straight. For $\xi_1, \xi_2 \in M^\vee$, let $S_1, S_2 \in \mathrm{Loc}(M)$ be the induced element. There is a unique element $\eta \in M^\vee$ that satisfies the following conditions (Lemma 3.15).

$$\begin{aligned} \langle v, \eta \rangle &\leq 0 \quad (v \in S_1 \oplus S_2), \\ \langle v, \eta \rangle &\geq 0 \quad (v \in M \setminus (S_1 \oplus S_2)). \end{aligned}$$

We have $\eta = \inf_{M^\vee}\{\xi_1, \xi_2\}$. So the canonical injection $i\colon M^\vee \to \mathrm{Loc}(M)$ is lattice-preserving. Since $\mathrm{Loc}(M)$ is straight, $M^\vee$ is straight (Proposition 3.17). □

Proposition 3.19. *Let M be a k-module. Let $\eta\colon M \to k$ be a lattice-preserving homomorphism. Then η is an extremal element of $M^\vee$.*

Proof. Suppose that η is not extremal. There are elements $\xi_1, \xi_2 \in M^\vee$ and elements $v_1, v_2 \in M$ such that $\xi_1 \oplus \xi_2 = \eta$ and $\langle v_1, \xi_1 \rangle < \langle v_1, \eta \rangle$ and $\langle v_2, \xi_2 \rangle < \langle v_2, \eta \rangle$. We may assume $\langle v_1, \eta \rangle = \langle v_2, \eta \rangle = 0$. Since η is lattice-preserving, there is a lower-bound w of $\{v_1, v_2\}$ such that $\langle w, \eta \rangle = 0$. Then

$$\begin{aligned} 0 &= \langle w, \eta \rangle \\ &= \langle w, \xi_1 \oplus \xi_2 \rangle \\ &\leq \langle v_1, \xi_1 \rangle \oplus \langle v_2, \xi_2 \rangle \\ &< \langle v_1, \eta \rangle \oplus \langle v_2, \eta \rangle \\ &= 0, \end{aligned}$$

which is contradiction. □

Definition. A *dual element* $\eta \in M^\vee$ of an element $e \in M$ is an element with the following conditions.

(i) $\langle e, \eta \rangle = 0$.

(ii) $v \in M, \xi \in M^\vee \Rightarrow \langle v, \eta \rangle \odot \langle e, \xi \rangle \leq \langle v, \xi \rangle$.

Proposition 3.20. *The dual element of an element $e \in M$ is unique.*

Proof. Let η be a dual element of e. Then

$$\eta = \min\{\xi \in M^\vee \mid \langle e, \xi \rangle = 0\},$$

because

$$\eta \odot \langle e, \xi \rangle \leq \xi$$

for any $\xi \in M^\vee$. □

Proposition 3.21. *Let $e \in M$ be an element of a pre-reflexive k-module M. Suppose that e has the dual element $\eta \in M^\vee$. Then*

(1) *e is an extremal element.*

(2) *$\eta \colon M \to k$ is a lattice-preserving homomorphism (therefore is an extremal element of $M^\vee$).*

Proof. (1) Let $v_1, v_2 \in M$ be elements such that $v_1 \oplus v_2 = e$. Then

$$\langle v_1, \eta \rangle \oplus \langle v_2, \eta \rangle = \langle e, \eta \rangle = 0.$$

We may assume $\langle v_1, \eta \rangle = 0$. For $\xi \in M^\vee$,

$$\begin{aligned} \langle e, \xi \rangle &= \langle v_1, \eta \rangle \odot \langle e, \xi \rangle \\ &\leq \langle v_1, \xi \rangle. \end{aligned}$$

Since M is pre-reflexive, we have $e \leq v_1$. So we have $e = v_1$.

(2) Let $v_1, v_2 \in M$ be elements such that $\langle v_1, \eta \rangle \leq \langle v_2, \eta \rangle$. Let $w = \langle v_1, \eta \rangle \odot e$. For $\xi \in M^\vee$, we have $\langle w, \xi \rangle \leq \langle v_i, \xi \rangle$. Since M is pre-reflexive, we have $w \leq v_i$. So w is a lower-bound of $\{v_1, v_2\}$ such that $\langle w, \eta \rangle = \langle v_1, \eta \rangle$. Thus η is lattice-preserving. By Proposition 3.19, η is an extremal element of $M^\vee$. □

Lemma 3.22. *Suppose that k is quasi-complete and rational. Let M be a straight pre-reflexive k-module. Then any extremal element of M has the dual element.*

Proof. Let $e \in M$ be an extremal element. The subset

$$S = \{v \in M \mid e \not\leq v\}$$

is a subsemigroup. (Let $v_1, v_2 \in M$ be elements such that $e \leq v_1 \oplus v_2$. Then

$$\begin{aligned} e &= \inf_M \{e, v_1 \oplus v_2\} \\ &= \inf_M \{e, v_1\} \oplus \inf_M \{e, v_2\}. \end{aligned}$$

We may assume $e = \inf_M\{e, v_1\}$. Then $e \leq v_1$.) Also S generates the k module M. (Let $v \in M$ be any element. By Lemma 3.9, we have

$$\inf_M\{b \odot v \mid b \in k \setminus \{-\infty\}\} = -\infty.$$

So there is $b \in k \setminus \{-\infty\}$ such that $e \not\leq b \odot v$.) Thus S is a locator of M. By Lemma 3.15, there is a unique element $\eta \in M^\vee$ that satisfies the following conditions.

$$\begin{aligned} \langle v, \eta \rangle &\leq 0 \quad (v \in S), \\ \langle v, \eta \rangle &\geq 0 \quad (v \in M \setminus S). \end{aligned}$$

Then

$$\begin{aligned} \langle v, \eta \rangle &\leq \inf_k\{a \in k \mid a \odot e \not\leq v\} \\ &\leq \inf_k\{a \in k \mid \langle v, \eta \rangle < a\} \\ &= \langle v, \eta \rangle. \end{aligned}$$

So we have

$$\langle v, \eta \rangle = \inf_k\{a \in k \mid a \odot e \not\leq v\}.$$

So

$$\begin{aligned} \langle e, \eta \rangle &= \inf_k\{a \in k \mid 0 < a\} \\ &= 0. \end{aligned}$$

Also, for any $a \in k$ such that $0 < a$, we have

$$(\langle v, \eta \rangle \oslash a) \odot e \leq v.$$

By Lemma 3.9, we have

$$\langle v, \eta \rangle \odot e \leq v.$$

Thus η is the dual element of e. □

Lemma 3.23. *Let M be a finitely generated pre-reflexive k-module. Let $\beta \colon k^n \to M$ be the surjection defined by a basis $\{e_1, \ldots, e_n\}$ of M. Suppose that e_i has the dual element η_i $(1 \leq i \leq n)$. Then*

(1) *M is straight.*

(2) *The homomorphism $\alpha \colon M \to k^n$ defined by the elements $\eta_1, \ldots, \eta_n$ is a right-inversion of β, i.e. $\beta \circ \alpha = \mathrm{id}_M$.*

(3) *α is the unique right-inversion of β.*

Proof. For $v \in M$, we have

$$v \geq \langle v, \eta_1 \rangle \odot e_1 \oplus \cdots \oplus \langle v, \eta_n \rangle \odot e_n.$$

It means $\beta \circ \alpha \leq \mathrm{id}_M$. Also, for $1 \leq i \leq n$, we have

$$\beta \circ \alpha(e_i) \geq \langle e_i, \eta_i \rangle \odot e_i = e_i.$$

Since M is generated by $\{e_1, \ldots, e_n\}$, we have $\beta \circ \alpha = \mathrm{id}_M$. So α is injective. Also α is lattice-preserving (Proposition 3.21). Since k^n is straight, M is straight (Proposition 3.17).

We prove uniqueness. Let $\eta'_1, \ldots, \eta'_n \in M^\vee$ be elements such that the induced homomorphism $M \to k^n$ is a right-inversion of β. Then we have

$$v = \langle v, \eta'_1 \rangle \odot e_1 \oplus \cdots \oplus \langle v, \eta'_n \rangle \odot e_n.$$

So

$$e_i = \langle e_i, \eta'_i \rangle \odot e_i \oplus w_i,$$

where

$$w_i = \bigoplus_{j \neq i} \langle e_i, \eta'_j \rangle \odot e_j.$$

Since $\{e_1, \ldots, e_n\}$ is a basis, we have $w_i \neq e_i$ and

$$e_i = \langle e_i, \eta'_i \rangle \odot e_i$$

(Proposition 3.12). So we have $\langle e_i, \eta'_i \rangle = 0$. Thus η'_i is the dual element of e_i. □

3.6 Existence of inversions

Let k be a totally ordered tropical semifield. Let $\alpha \colon M \to N$ be a homomorphism of k-modules.

Definition. An element $\xi \in M^\vee$ *dominates* an element $w \in N$ if there is an element $v \in M$ such that $\langle v, \xi \rangle \leq 0$ and $w \leq \alpha(v)$.

Proposition 3.24. *Let $\xi_i \in M^\vee$ be an element that dominates $w_i \in N$ ($i = 1, 2$). Then any lower-bound $\xi \in \mathrm{Low}_{M^\vee}(\xi_1, \xi_2)$ dominates $w_1 \oplus w_2$.*

Proof. There are elements $v_1, v_2 \in M$ such that $\langle v_i, \xi_i \rangle \leq 0$ and $w_i \leq \alpha(v_i)$. Then

$$\begin{aligned} \langle v_1 \oplus v_2, \xi \rangle &\leq \langle v_1, \xi_1 \rangle \oplus \langle v_2, \xi_2 \rangle \\ &\leq 0. \end{aligned}$$

Also we have $w_1 \oplus w_2 \leq \alpha(v_1 \oplus v_2)$. □

Recall that a homomorphism $\alpha \colon M \to N$ is said to be lightly surjective if for any $w \in N$ there is $v \in M$ such that $w \leq \alpha(v)$.

Lemma 3.25. *Let $\alpha\colon M \to N$ be an injective lightly surjective homomorphism of k-modules. Suppose that $M^\vee$ is straight.*

(1) *There is a homomorphism $\gamma\colon N \to \operatorname{Loc}(M^\vee)$ that satisfies the following condition. For any $w \in N$ the locator $\gamma(w)$ is the subsemigroup of $M^\vee$ generated by the elements that dominates the element w.*

(2) *The diagram*

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & N \\ \downarrow & & \downarrow \gamma \\ (M^\vee)^\vee & \xrightarrow[i]{} & \operatorname{Loc}(M^\vee) \end{array}$$

commutes, i.e. for any $v \in M$ and any $\xi \in M^\vee$ the condition $\langle v, \xi \rangle \leq 0$ is fulfilled if and only if $\xi \in \gamma(\alpha(v))$.

Proof. (1) For $w \in N$, let $\gamma(w) \subset M^\vee$ be the subsemigroup of $M^\vee$ generated by the elements that dominates the element w. $\gamma(w)$ is lower-saturated. (Let $\xi \in M$ and $\xi' \in \gamma(w)$ be elements such that $\xi \leq \xi'$. There are elements $\xi_1, \dots, \xi_r \in M^\vee$ such that ξ_i dominates w and

$$\xi' = \xi_1 \oplus \cdots \oplus \xi_r.$$

Then

$$\begin{aligned} \xi &= \inf_{M^\vee}\{\xi, \xi_1 \oplus \cdots \oplus \xi_r\} \\ &= \inf_{M^\vee}\{\xi, \xi_1\} \oplus \cdots \oplus \inf_{M^\vee}\{\xi, \xi_r\}. \end{aligned}$$

So $\xi \in \gamma(w)$.) Also $\gamma(w)$ generates the k-module $M^\vee$. (Let $\xi \in M^\vee$ be any element. Since α is lightly surjective, there is $v \in M$ such that $w \leq \alpha(v)$. Let $a \in k \setminus \{-\infty\}$ be an element such that $\langle v, \xi \rangle \leq a$. Then $\oslash a \odot \xi$ dominates w.) So $\gamma(w)$ is a locator of $M^\vee$.

We show that γ is a homomorphism. For $w_1, w_2 \in N$, we have

$$\gamma(w_1 \oplus w_2) \subset \gamma(w_1) \cap \gamma(w_2).$$

Let ξ be an element of right side. There are elements $\xi_{i,j} \in M^\vee$ ($1 \leq i \leq 2$, $1 \leq j \leq r$) such that $\xi_{i,j}$ dominates w_i and

$$\xi = \xi_{1,1} \oplus \cdots \oplus \xi_{1,r} = \xi_{2,1} \oplus \cdots \oplus \xi_{2,r}.$$

Then

$$\begin{aligned} \xi &= \inf_{M^\vee}\{\xi_{1,1} \oplus \cdots \oplus \xi_{1,r}, \xi_{2,1} \oplus \cdots \oplus \xi_{2,r}\} \\ &= \bigoplus_{i,j} \eta_{i,j}, \end{aligned}$$

where

$$\eta_{i,j} = \inf_{M^\vee}\{\xi_{1,i}, \xi_{2,j}\}.$$

$\eta_{i,j}$ dominates $w_1 \oplus w_2$ (Proposition 3.24). So we have $\xi \in \gamma(w_1 \oplus w_2)$.

(2) Let $\xi \in M^\vee$ be an element that dominates $\alpha(v)$. There is an element $v' \in M$ such that $\langle v', \xi\rangle \leq 0$ and $\alpha(v) \leq \alpha(v')$. Since α is injective, we have $v \leq v'$. So we have

$$\langle v, \xi\rangle \leq \langle v', \xi\rangle \leq 0.$$

Let

$$T = \{\xi \in M^\vee \mid \langle v, \xi\rangle \leq 0\}.$$

Now we have $\xi \in T$. Since T is a subsemigroup, we have $\gamma(\alpha(v)) = T$. □

3.7 Straight reflexive modules

Let k be a quasi-complete totally ordered rational tropical semifield. Recall that the dimension of a straight reflexive k-module M is the number of extremal rays. By Proposition 3.12, the number of elements of any basis of M is $\dim(M)$.

Proof of Theorem 2.1. We have an isomorphism $\iota_M \colon M \to (M^\vee)^\vee$ and a homomorphism $\gamma \colon N \to \mathrm{Loc}(M^\vee)$ defined in Lemma 3.25. There is a left-inversion p of the homomorphism $i \colon (M^\vee)^\vee \to \mathrm{Loc}(M^\vee)$ (Lemma 3.15). By the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & N \\ {\scriptstyle \iota_M}\downarrow & & \downarrow{\scriptstyle \gamma} \\ (M^\vee)^\vee & \xrightarrow[i]{} & \mathrm{Loc}(M^\vee) \end{array}$$

we have ${\iota_M}^{-1} \circ p \circ \gamma \circ \alpha = \mathrm{id}_M$. □

Proof of Theorem 2.2. By Lemma 3.22 and Lemma 3.23, there is an injection $N \to k^n$, where $n = \dim(N)$. Let N' be the lower-saturation of the image of $M \to k^n$. N' is a free module of rank $n' \leq n$. If $n' = n$, then α is lightly surjective. Now we may assume that $N = k^n$ and that α is lightly surjective. By Theorem 2.1, α has a left-inversion $\beta \colon N \to M$. Since β is surjective, we have $\dim(M) \leq \dim(N)$. □

Proof of Theorem 2.3. By Lemma 3.22 and Lemma 3.23, there is a right-inversion $\alpha \colon M \to k^n$ of the surjection $\beta \colon k^n \to M$. By the commutative diagram

$$\begin{array}{ccc} k^n & \xrightarrow{\beta} & M \\ \wr\downarrow & & \downarrow{\scriptstyle \iota_M} \\ k^n & \xrightarrow[(\beta^\vee)^\vee]{} & (M^\vee)^\vee \end{array}$$

we have ${\iota_M}^{-1} = \beta \circ (\alpha^\vee)^\vee$. □

Proof of Theorem 2.4. By Theorem 2.3, N is reflexive. Similarly to the proof of Theorem 2.2, we may assume that $N = k^n$ and that α is lightly surjective. We have a homomorphism $\gamma\colon N \to \mathrm{Loc}(M^\vee)$ defined in Lemma 3.25. There is a left-inversion p of the homomorphism $i\colon (M^\vee)^\vee \to \mathrm{Loc}(M^\vee)$ (Lemma 3.15). There is a homomorphism $\delta\colon M^\vee \to N^\vee$ such that for any $w \in N$ and any $\xi \in M^\vee$ it implies

$$\langle w, \delta(\xi)\rangle = \langle p(\gamma(w)), \xi\rangle.$$

By the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & N \\ {\scriptstyle \iota_M}\downarrow & & \downarrow{\scriptstyle \gamma} \\ (M^\vee)^\vee & \xrightarrow[i]{} & \mathrm{Loc}(M^\vee) \end{array}$$

for any $v \in M$ we have

$$\langle \alpha(v), \delta(\xi)\rangle = \langle v, \xi\rangle.$$

So $\alpha^\vee \circ \delta = \mathrm{id}_{M^\vee}$. So we have

$$\dim(M^\vee) \leq \dim(N^\vee) = n.$$

By Lemma 3.22 and Proposition 3.21, there is an injection from the set of the extremal rays of M to the set of the extremal rays of $M^\vee$. So we have $\dim(M) \leq n$. □

Example 3.26. There is an example of straight submodule $M \subset \mathbb{T}^2$ that is not finitely generated. Let

$$M = \{(a, b) \in \mathbb{T}^2 \mid b \neq -\infty\} \cup \{-\infty\}.$$

M is a submodule of $\mathbb{T}^2$. M is straight, because it is lattice-preserving.

Example 3.27. There is an example of extremally generated submodule $M \subset \mathbb{T}^3$ that is not finitely generated. Let

$$M = \left\{ \begin{array}{ccc} (a, b, c) \in \mathbb{T}^3 & | & \begin{array}{c} (-1) \odot a \oplus c \leq b, \\ 2b \leq a \odot c \end{array} \end{array} \right\}.$$

M is a submodule of $\mathbb{T}^3$ (Example 3.6). For $0 \leq t \leq 1$, let

$$e(t) = (2t, t, 0) \in M.$$

$e(t)$ is extremal. (Proposition 3.13. Indeed $e(t)$ is a minimal element of the subset

$$S_t = \{(a, b, c) \in M \mid b = t\}.$$

So it is extremal.) So M is not finitely generated. $\{e(t) \mid 0 \leq t \leq 1\}$ is a basis of M. Indeed, for any $(a, b, c) \in M$,

$$(a, b, c) = c \odot e(b \oslash c) \oplus (2b \oslash a) \odot e(a \oslash b).$$

M is not straight. Indeed, let

$$\begin{aligned} v_1 &= (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ v_2 &= (1, \frac{1}{2}, 0), \\ w &= (1, 0, 0). \end{aligned}$$

Then we have

$$\begin{aligned} \inf_M\{v_1, v_2\} &= (\frac{1}{2}, \frac{1}{4}, 0), \\ \inf_M\{v_1, v_2\} \oplus w &= (1, \frac{1}{4}, 0), \\ \inf_M\{v_1 \oplus w, v_2 \oplus w\} &= (1, \frac{1}{2}, 0). \end{aligned}$$

So M is not straight.

3.8 Free modules

Let k be a totally ordered tropical semifield. Let $F = k^n$ be the free module with the basis $\{e_1, \dots, e_n\}$. Let F^* be the set of the linear combinations of $\{e_1, \dots, e_n\}$ with coefficients in $k^* = k \setminus \{-\infty\}$. Let $\{e_1^\vee, \dots, e_n^\vee\}$ be the dual basis in $F^\vee$. We have a bijective map

$$\psi \colon F^* \longrightarrow (F^\vee)^*$$

defined by

$$\psi(a_1 \odot e_1 \oplus \cdots \oplus a_n \odot e_n) = (\oslash a_1) \odot e_1^\vee \oplus \cdots \oplus (\oslash a_n) \odot e_n^\vee.$$

For $v, w \in F^*$, the condition $v \leq w$ is fulfilled if and only if

$$\langle v, \psi(w) \rangle \leq 0.$$

For $w \in F^*$ and $1 \leq i \leq n$, let

$$M(w,i) = \{v \in F \mid {}^\forall j, \langle v, e_j^\vee \rangle \odot \langle e_j, \psi(w) \rangle \geq \langle v, e_i^\vee \rangle \odot \langle e_i, \psi(w) \rangle\}.$$

$M(w,i)$ is a submodule of F (Example 3.6). It is easy to see that $M(w,i)$ is lattice-preserving in F, i.e. the inclusion $M(w,i) \to F$ preserves the infimum of any two elements. For $\eta \in F^\vee$ and $1 \leq i \leq n$, let

$$\begin{aligned} N(\eta, i) &= \{v \in F \mid \langle v, \eta \rangle = \langle v, e_i^\vee \rangle \odot \langle e_i, \eta \rangle\} \\ &= \{v \in F \mid {}^\forall j, \langle v, e_j^\vee \rangle \odot \langle e_j, \eta \rangle \leq \langle v, e_i^\vee \rangle \odot \langle e_i, \eta \rangle\}. \end{aligned}$$

$N(\eta, i)$ is also a lattice-preserving submodule of F.

Proposition 3.28. *Let M be a submodule of F with a basis $\{w_1, \ldots, w_r\}$. Suppose that $w_h \in F^*$ $(1 \leq h \leq r)$. Then the following are equivalent.*

(i) *M is lattice-preserving in F.*

(ii) *For any $i \in \{1, \ldots, n\}$, there is the minimum element of $M \cap V_i$, where*

$$V_i = \{v \in F \mid \langle v, e_i^\vee \rangle = 0\}.$$

(iii) *There is a surjective map*

$$s \colon \{1, \ldots, n\} \longrightarrow \{1, \ldots, r\}$$

such that

$$M = \bigcap_{1 \leq i \leq n} M(w_{s(i)}, i).$$

(iv) *There is a surjective map*

$$s \colon \{1, \ldots, n\} \longrightarrow \{1, \ldots, r\}$$

such that

$$M = \bigcap_{1 \leq i \leq n} N(\eta_{s(i)}, i),$$

where η_h is the dual element of w_h.

Proof. (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (i) are easy.

Since $F^\vee$ is also a free module, for $\eta \in (F^\vee)^*$ and $1 \leq i \leq n$ we have the lattice-preserving submodule $M(\eta, i)$ of $F^\vee$. The bijective map

$$\psi \colon F^* \longrightarrow (F^\vee)^*$$

induces bijective maps

$$\psi' \colon M \setminus \{-\infty\} \longrightarrow M^\vee \setminus \{-\infty\},$$

$$\psi'' \colon N(\eta, i) \setminus \{-\infty\} \longrightarrow M(\eta, i) \setminus \{-\infty\}.$$

So we have only to prove that conditions (i), (ii), (iii) are equivalent.

(i) $\Rightarrow$ (ii). Let

$$v_i = \inf_F \{\oslash a_{h,i} \odot w_h \mid 1 \leq h \leq r\},$$

where

$$a_{h,i} = \langle w_h, e_i^\vee \rangle.$$

Then v_i is the minimum element of $M \cap V_i$.

(ii) $\Rightarrow$ (iii). Let v_i be the minimum element of $M \cap V_i$. v_i is an extremal element of M. The extremal ray $k \odot v_i$ is generated by an element of the basis $\{w_1, \ldots, w_r\}$ (Proposition 3.12). There is a number $s(i)$ such that $k \odot v_i = k \odot w_{s(i)}$.

We show that s is surjective. For $h \in \{1, \ldots, r\}$, we have

$$w_h = a_{h,1} \odot v_1 \oplus \cdots \oplus a_{h,n} \odot v_n.$$

Since w_h is extremal (Proposition 3.12), there is a number i such that

$$w_h = a_{h,i} \odot v_i.$$

So we have $h = s(i)$.

We show the equality

$$M = \bigcap_{1 \le i \le n} M(w_{s(i)}, i).$$

For $v \in F$, let $x_i = \langle v, e_i^\vee \rangle$. The condition $v \in M(w_{s(i)}, i)$ is fulfilled if and only if for any j it implies

$$\oslash a_{s(i),j} \odot x_j \ge \oslash a_{s(i),i} \odot x_i.$$

For $1 \le h \le r$ and $1 \le i \le n$, we have

$$\begin{aligned} \oslash a_{s(i),i} \odot w_{s(i)} &= v_i \\ &\le \oslash a_{h,i} \odot w_h. \end{aligned}$$

For $1 \le j \le n$, we have

$$\oslash a_{s(i),i} \odot a_{s(i),j} \le \oslash a_{h,i} \odot a_{h,j}.$$

It means $w_h \in M(w_{s(i)}, i)$. Since M is generated by $\{w_1, \ldots, w_r\}$, we have

$$M \subset \bigcap_{1 \le i \le n} M(w_{s(i)}, i).$$

Let v be an element of right side. Then

$$v = \bigoplus_i \oslash a_{s(i),i} \odot x_i \odot w_{s(i)}.$$

(Indeed,

$$\begin{aligned} \langle v, e_i^\vee \rangle &= x_i \\ &= \langle \oslash a_{s(i),i} \odot x_i \odot w_{s(i)}, e_i^\vee \rangle. \end{aligned}$$

So

$$v \le \bigoplus_i \oslash a_{s(i),i} \odot x_i \odot w_{s(i)}.$$

The converse is easy.) So we have $v \in M$. □

4 Polytopes in a tropical projective space

Let $F = \mathbb{T}^{n+1}$ be the free module with coordinates $(x_1, \ldots, x_{n+1})$ over $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$. Let $F^* = \mathbb{R}^{n+1}$.

Proposition 4.1. *Let M be a submodule of F generated by finitely many elements of F^*. Then the following are equivalent.*

(i) *M is lattice-preserving in F.*

(ii) *$M \setminus \{-\infty\}$ is a real convex subset of $\mathbb{R}^{n+1}$.*

Proof. (i) $\Rightarrow$ (ii). By Proposition 3.28, M is defined by inequalities

$$x_j \geq x_i - c_{i,j} \quad (i, j \in \{1, \ldots, n+1\})$$

for some $c_{i,j} \in \mathbb{R}$. So $M \setminus \{-\infty\}$ is real convex.

(ii) $\Rightarrow$ (i). Let $\pi_1 : F \to \mathbb{T}^n$ and $\pi_2 : F \to \mathbb{T}$ be projections defined as follows.

$$\pi_1(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n),$$
$$\pi_2(x_1, \ldots, x_{n+1}) = x_{n+1}.$$

For $a \in \mathbb{R}$, let $N_i(a) \subset F$ be the submodule defined as follows.

$$N_i(a) = \{v = (x_1, \ldots, x_{n+1}) \in F \mid x_{n+1} = x_i + a\}.$$

By induction on n, we may assume that modules $\pi_1(M)$, $\pi_2(M)$, $M \cap N_i(a)$ are lattice-preserving. Suppose that M is not lattice-preserving. By Proposition 3.28, there is a number i such that there is no minimum element of $M \cap V_i$, where

$$V_i = \{v = (x_1, \ldots, x_{n+1}) \in F \mid x_i = 0\}.$$

We may assume $i \leq n$. Let w_1, w_2 be minimal elements of $M \cap V_i$ such that $\pi_1(w_1)$ is the minimum element of $\pi_1(M \cap V_i)$ and that $\pi_2(w_2)$ is the minimum element of $\pi_2(M \cap V_i)$. Let $a \in \mathbb{R}$ be an element such that

$$\pi_2(w_2) < a < \pi_2(w_1).$$

There is the minimum element $v(a)$ of $M \cap N_i(a) \cap V_i$. Since $M \cap V_i$ is real convex, $v(a)$ is a minimal element of $M \cap V_i$. (Let $v' \in M \cap V_i$ be an element such that $v' < v(a)$. The real line segment combining v' and w_1 contains an element $v'' \in M \cap N_i(a) \cap V_i$ such that $v'' \neq v'$. Since $\pi_1(w_1) < \pi_1(v') \leq \pi_1(v(a))$, we have $v'' < v(a)$.) So M has infinitely many extremal rays, which is contradiction. □

Let

$$\varphi \colon \mathbb{T}^{n+1} \setminus \{-\infty\} \longrightarrow \mathbb{TP}^n$$

be the canonical projection to the tropical projective space $\mathbb{TP}^n$. We identify $\varphi(\mathbb{R}^{n+1})$ with $\mathbb{R}^n$. A subset $P \subset \mathbb{TP}^n$ is said to be tropically convex if the subset

$$M = \varphi^{-1}(P) \cup \{-\infty\} \subset \mathbb{T}^{n+1}$$

is a submodule. A subset $P \subset \mathbb{TP}^n$ is said to be a tropical polytope if it is the tropically convex hull of finitely many points of $\mathbb{R}^n$.

Proof of Theorem 2.5. (1) Suppose that P is a polytrope. Then P is real convex. By Proposition 4.1, M is lattice-preserving in $\mathbb{T}^{n+1}$. So M is straight. By Theorem 2.3, M is reflexive.

(2) Suppose that M is straight reflexive. Let $\{v_1, \dots, v_r\}$ be a basis of M. By Theorem 2.2, we have $r \leq n+1$. Let $p_i = \varphi(v_i)$. Then P is the tropically convex hull of $\{p_1, \dots, p_r\}$. □

5 Square matrices over a tropical semifield

Let k be a totally ordered rational tropical semifield. A square matrix of order n over k is a homomorphism $A\colon k^n \to k^n$. Let $\{e_1, \dots, e_n\}$ be the basis of k^n. The coefficient $\langle A \odot e_j, e_i^\vee \rangle$ is simply written as A_{ij}. Let $E_n\colon k^n \to k^n$ be the identity.

Let $\Delta(A)$, $\overline{\Delta}(A)$ be square matrices of order n defined as follows.

$$\Delta(A)_{ij} = \delta_{ij} \odot A_{ij},$$

$$\overline{\Delta}(A)_{ij} = \overline{\delta}_{ij} \odot A_{ij},$$

where

$$\delta_{ij} = \begin{cases} 0 & \text{if } i = j \\ -\infty & \text{if } i \neq j \end{cases}$$

$$\overline{\delta}_{ij} = \begin{cases} -\infty & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

The determinant $\det(A)$ is the sum of elements $A_{1s(1)} \odot \cdots \odot A_{ns(n)}$ for all permutations $s \in S(n)$.

Lemma 5.1. *Let A be a square matrix of order n over k. Suppose that $\Delta(A) = E_n$ and $\det(A) = 0$. Then $A^{\odot n} = A^{\odot n-1}$.*

Proof. Since $E_n \leq A$, we have $A^{\odot r} \leq A^{\odot r+1}$ for any $r \geq 0$. $(A^{\odot n})_{ij}$ is the sum of elements

$$b = A_{h(0)h(1)} \odot A_{h(1)h(2)} \odot \cdots \odot A_{h(n-1)h(n)}$$

for all maps $h\colon \{0, \dots, n\} \to \{1, \dots, n\}$ such that $h(0) = i$ and $h(n) = j$. h is not injective. So there are numbers l, m and a cyclic permutation $s \in S(n)$ such that

$$s\colon h(l) \mapsto h(l+1) \mapsto \cdots \mapsto h(m-1) \mapsto h(m) = h(l).$$

Since $\Delta(A) = E_n$, we have

$$A_{h(l)h(l+1)} \odot \cdots \odot A_{h(m-1)h(m)} \leq \det(A).$$

So we have

$$A^{\odot n} \leq \det(A) \odot A^{\odot n-1}.$$

Since $\det(A) = 0$, we have the conclusion. □

Lemma 5.2. *Let A be a square matrix of order n over k. Then either* (i) *or* (ii) *is fulfilled.*

(i) *There are an element $v \in (k \setminus \{-\infty\})^n$ and an element $\varepsilon > 0$ such that*

$$(A \oplus \varepsilon \odot \overline{\Delta}(A)) \odot v = \Delta(A) \odot v.$$

(ii) *There is an element $v \in k^n \setminus \{-\infty\}$ such that*

$$A \odot v = \overline{\Delta}(A) \odot v.$$

Proof. Let $e(A)$ be the sum of elements $A_{1s(1)} \odot \cdots \odot A_{ns(n)}$ for all $s \in S(n) \setminus \{\mathrm{id}\}$. Let

$$c(A) = \det(\Delta(A)) = A_{11} \odot \cdots \odot A_{nn}.$$

We show that the condition (i) is fulfilled if $e(A) < c(A)$. Replacing A by $\oslash(\Delta(A)) \odot A$, we may assume $\Delta(A) = E_n$. There is an element $\varepsilon \in k$ such that $\varepsilon > 0$ and

$$e(A) \odot n\varepsilon \leq c(A).$$

Let

$$B = A \oplus \varepsilon \odot \overline{\Delta}(A).$$

Then we have $e(B) \leq c(B)$. By Lemma 5.1, we have $B^{\odot n} = B^{\odot n-1}$. Let $w \in (k \setminus \{-\infty\})^n$ be any element. Let $v = B^{\odot n-1} \odot w$. Then we have $B \odot v = v$.

We show that the condition (ii) is fulfilled if $c(A) \leq e(A)$. We may assume $\Delta(A) = E_n$. (If $A_{ii} = -\infty$, then the element $v = e_i$ satisfies the conclusion.) There is a cyclic permutation $s \in S(n) \setminus \{\mathrm{id}\}$ and a map $h\colon \{0, \ldots, l\} \to \{1, \ldots, n\}$ such that

$$s\colon h(0) \mapsto h(1) \mapsto \cdots \mapsto h(l-1) \mapsto h(l) = h(0),$$

$$A_{h(0)h(1)} \odot \cdots \odot A_{h(l-1)h(l)} \geq 0.$$

Let

$$v = \bigoplus_{1 \leq m \leq l} (A_{h(m)h(m+1)} \odot \cdots \odot A_{h(l-1)h(l)}) \odot e_{h(m)}.$$

Then

$$\overline{\Delta}(A) \odot v \geq v.$$

So we have the conclusion. □

6 Tropical curves

Let $A = \mathbb{T}[x_1, -x_1, \ldots, x_n, -x_n]$ be the semiring of Laurent polynomials over $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ (where $-x_i$ means $\oslash x_i$). Let

$$f = \bigoplus_{i_1 \ldots i_n \in \mathbb{Z}} c_{i_1 \ldots i_n} \odot i_1 x_1 \odot \cdots \odot i_n x_n$$

be any element of A. The induced map

$$\begin{array}{rcl} f\colon \mathbb{R}^n & \longrightarrow & \mathbb{T} \\ (a_1,\ldots,a_n) & \mapsto & f(a_1,\ldots,a_n) \end{array}$$

is said to be a Laurent polynomial function over $\mathbb{T}$. If f is a monomial, then f is a $\mathbb{Z}$-affine function, i.e. there are $c \in \mathbb{R}$ and $i_1,\ldots,i_n \in \mathbb{Z}$ such that

$$f = c + i_1x_1 + \cdots + i_nx_n.$$

In general case, f is the supremum of finitely many $\mathbb{Z}$-affine functions, which is a locally convex piecewise-$\mathbb{Z}$-affine function.

Let $\Gamma_n \subset \mathbb{R}^n$ be the subset defined as follows.

$$\Gamma_n = E_0 \cup E_1 \cup \cdots \cup E_n,$$

$$E_0 = \{(a_1,\ldots,a_n) \in \mathbb{R}^n \mid {}^{\forall}i, {}^{\forall}j, a_i = a_j \geq 0\},$$

for $1 \leq i \leq n$,

$$E_i = \{(a_1,\ldots,a_n) \in \mathbb{R}^n \mid a_i \leq 0, {}^{\forall}j \neq i, a_j = 0\}.$$

Γ_n has a $(n+1)$-valent vertex $P = (0,\ldots,0)$. Also Γ_n is equipped with Euclidean topology on $\mathbb{R}^n$.

Definition. A function $f\colon \Gamma_n \to \mathbb{T}$ is *regular* if it is induced by a locally Laurent polynomial function $f\colon \mathbb{R}^n \to \mathbb{T}$.

Let $\mathcal{O}_{\Gamma_n}$ be the sheaf of the regular functions on Γ_n. $\mathcal{O}_{\Gamma_n}$ is a sheaf of semirings. Let R be the stalk of $\mathcal{O}_{\Gamma_n}$ at the vertex P.

Proposition 6.1. *Let $f \in R \setminus \{-\infty\}$ be any element. Then there are a unique number $r \in \mathbb{Z}_{\geq 0}$ and a unique Laurent monomial $h \in R$ such that*

$$f = h \odot r(x_1 \oplus 0).$$

Proof. f is the sum of Laurent monomials $f_1,\ldots,f_m$. If $f_j(P) < f(P)$, then $f_j < f$ on a neighborhood of P. So we may assume $f_j(P) = f(P)$. Then f is $\mathbb{Z}$-affine on E_i $(1 \leq i \leq n)$. So there is $a_i \in \mathbb{Z}$ such that $f = f(P) \odot a_ix_i$ on E_i. Let

$$h = f(P) \odot a_1x_1 \odot \cdots \odot a_nx_n.$$

Then $f = h$ on $E_1 \cup \cdots \cup E_n$. $f \oslash h$ is the sum of monomials $g_1,\ldots,g_m$ such that $g_j(P) = 0$. There are $b_{ij} \in \mathbb{Z}_{\geq 0}$ such that

$$g_j = b_{1j}x_1 \odot \cdots \odot b_{nj}x_n.$$

Then

$$g_j = (b_{1j} + \cdots + b_{nj})x_1$$

on E_0. So we have $f \oslash h = rx_1$ on E_0, where

$$r = \bigoplus_{1\leq j\leq m} \sum_{1\leq i\leq n} b_{ij}.$$

□

The number r in the above statement is called the order of f at P, and denoted by $\mathrm{ord}(f, P)$.

For $0 \le i \le n$, let $X_i f$ be the partial differential of f at P with direction E_i. (i.e. $X_i f = a$ if and only if $f = f(P) - ax_i$ on E_i $(1 \le i \le n)$. $X_0 f = a$ if and only if $f = f(P) + ax_1$ on E_0.)

Proposition 6.2. *Let $f \in R \setminus \{-\infty\}$ be any element. Then*

$$\mathrm{ord}(f, P) = \sum_{0 \le i \le n} X_i f.$$

Proof. Let h be a Laurent monomial written as follows.

$$h = c \odot a_1 x_1 \odot \cdots \odot a_n x_n.$$

Then

$$X_i h = -a_i \quad (1 \le i \le n),$$
$$X_0 h = a_1 + \cdots + a_n.$$

So we have

$$\sum_{0 \le i \le n} X_i h = 0.$$

Also we have

$$\sum_{0 \le i \le n} X_i (x_1 \oplus 0) = 1.$$

So we have the conclusion. □

Proposition 6.3. *Let $f, g \in R \setminus \{-\infty\}$ be any elements.*

(1) $\mathrm{ord}(f \odot g, P) = \mathrm{ord}(f, P) + \mathrm{ord}(g, P)$.

(2) *If $g(P) \le f(P)$, then* $\mathrm{ord}(f, P) \le \mathrm{ord}(f \oplus g, P)$.

Proof. (1) is easy.

(2) If $g(P) < f(P)$, then $f \oplus g = f$. So we may assume $g(P) = f(P)$. Then we have

$$X_i(f \oplus g) = X_i f \oplus X_i g.$$

By Proposition 6.2, we have the conclusion. □

A function $f \colon \Gamma_n \to \mathbb{T}$ is said to be rational if locally

$$f = g_1 - g_2 = g_1 \oslash g_2$$

for regular functions g_1, g_2. By Proposition 6.1, there is a number $m \ge 0$ such that the function $m(x_1 \oplus 0) \odot f$ is regular at P. The order of f at P is defined as follows.

$$\mathrm{ord}(f, P) = \mathrm{ord}(m(x_1 \oplus 0) \odot f, P) - m.$$

Let $Q \in \Gamma_n$ be a point such that $Q \ne P$. Then a neighborhood of Q is embedded in $\Gamma_1 = \mathbb{R}$. So we can define the order of f at Q similarly.

Definition. $(C, \mathcal{O}_C)$ is a *tropical curve* if for any $P \in C$ there are a neighborhood U of P and a number $n \geq 1$ such that $(U, \mathcal{O}_U)$ is embedded in $(\Gamma_n, \mathcal{O}_{\Gamma_n})$.

A divisor D on a tropical curve C is an element of the free abelian group $\mathrm{Div}(C)$ generated by all the points of C. For a rational function $f\colon C \to \mathbb{T}$, the divisor $(f) \in \mathrm{Div}(C)$ is defined as follows.

$$(f) = \sum_{P \in C} \mathrm{ord}(f, P)P.$$

f is said to be a section of D if either $f = -\infty$ or $(f) + D \geq 0$. Let $\mathcal{O}_C(D)$ be the sheaf of the sections of D.

Proposition 6.4. *The set $M = H^0(C, \mathcal{O}_C(D))$ is a $\mathbb{T}$-module.*

Proof. Let $f, g \in M \setminus \{-\infty\}$ be any elements. By Proposition 6.3, for $P \in C$ we have

$$\mathrm{ord}(f \oplus g, P) \geq \min\{\mathrm{ord}(f, P), \mathrm{ord}(g, P)\}.$$

So

$$(f \oplus g) + D \geq \inf_{\mathrm{Div}(C)}\{(f), (g)\} + D \geq 0.$$

So we have $f \oplus g \in M$. □

Recall that

$$r(D) = \max\{r \in \mathbb{Z}_{\geq -1} \mid U(D, r) = \emptyset\}.$$

Proof of Theorem 2.7. Note that $r(D) = s(D) - 1$, where

$$s(D) = \min\{r \in \mathbb{Z}_{\geq 0} \mid U(D, r) \neq \emptyset\}.$$

Let $m = s(D)$. We show that there is a straight reflexive submodule $N \subset M = H^0(C, \mathcal{O}_C(D))$ with dimension m. Let $P_1, \ldots, P_m \in C$ be points such that

$$H^0(C, \mathcal{O}_C(D - E)) = -\infty,$$

where

$$E = P_1 + \cdots + P_m.$$

There is an element

$$f_i \in H^0(C, \mathcal{O}_C(D - E + P_i))$$

such that $f_i \neq -\infty$. Let

$$\alpha\colon \mathbb{T}^m \longrightarrow M$$

be the homomorphism defined by $\alpha(e_i) = f_i$. Let

$$\beta\colon M \longrightarrow \mathbb{T}^m$$

be the homomorphism defined by

$$\beta(g) = g(P_1) \odot e_1 \oplus \cdots \oplus g(P_m) \odot e_m.$$

Let A be the square matrix induced by $\beta \circ \alpha \colon \mathbb{T}^m \to \mathbb{T}^m$.

Now we suppose that there is an element

$$v = a_1 \odot e_1 \oplus \cdots \oplus a_m \odot e_m \in \mathbb{T}^m \setminus \{-\infty\}$$

such that $A \odot v = \overline{\Delta}(A) \odot v$. Then there is a map $h \colon \{1, \dots, m\} \to \{1, \dots, m\}$ such that $h(i) \neq i$ and

$$\alpha(v)(P_i) = a_{h(i)} \odot f_{h(i)}(P_i).$$

Then

$$\operatorname{ord}(\alpha(v), P_i) \geq \operatorname{ord}(f_{h(i)}, P_i)$$

(Proposition 6.3). So $\alpha(v)$ is a section of $D - E$ such that $\alpha(v) \neq -\infty$, which is contradiction.

So there is no element $v \in \mathbb{T}^m \setminus \{-\infty\}$ such that $A \odot v = \overline{\Delta}(A) \odot v$. By Lemma 5.2, there are an element $v \in \mathbb{R}^m$ and an element $\varepsilon > 0$ such that

$$(A \oplus \varepsilon \odot \overline{\Delta}(A)) \odot v = \Delta(A) \odot v.$$

Let $L(v, \varepsilon) \subset \mathbb{T}^m$ be the submodule defined as follows.

$$L(v, \varepsilon) = \mathbb{T} \odot \{w \in \mathbb{T}^m \mid v \leq w \leq \varepsilon \odot v\}.$$

$L(v, \varepsilon)$ is a straight reflexive $\mathbb{T}$-module with dimension m. We have

$$A|_{L(v,\varepsilon)} = \Delta(A)|_{L(v,\varepsilon)}.$$

So α is injective on $L(v, \varepsilon)$. The image $N = \alpha(L(v, \varepsilon))$ is a submodule of M such that $N \cong L(v, \varepsilon)$. □

Example 6.5. The mapping $D \mapsto r(D)$ is not an invariant of a $\mathbb{T}$-module. We show that there are tropical curves C, C' and divisors D, D' such that

$$H^0(C, \mathcal{O}_C(D)) \cong H^0(C', \mathcal{O}_{C'}(D')),$$

$$r(D) \neq r(D').$$

Let C be a tropical curve with genus 1 with a vertex V and an edge E. Let P be an interior point of E. Let $D = V + P$. Then $H^0(C, \mathcal{O}_C(D))$ is isomorphic to the submodule of $\mathbb{T}^2$ generated by $(0, 0)$ and $(0, \frac{a}{2})$, where a is the lattice length of E. We have $r(D) = 1$.

Let C' be a tropical curve with genus 2 with vertices V_1, V_2 and edges E_1, E_2, E_3 such that the boundary of E_i is $\{V_1, V_2\}$ $(1 \leq i \leq 3)$. Let P be an interior point of E_1. Let $D' = V_1 + P$. Then for any interior point Q of $E_2 \cup E_3$ we have

$$H^0(C', \mathcal{O}_{C'}(D' - Q)) = -\infty.$$

So $H^0(C', \mathcal{O}_{C'}(D'))$ is isomorphic to the submodule of $\mathbb{T}^2$ generated by $(0, 0)$ and $(0, \frac{b}{2})$, where b is the lattice length of the path from V_1 to P contained in E_1. We have $r(D') = 0$. In the case of $a = b$, the required condition is fulfilled.

7 Tropical plane curves

7.1 Tropicalization

It is well known that some example of tropical curve is given by tropicalization of a family of affine complex curves.

First, we define tropical plane curves. Let $f \in \mathbb{T}[x, -x, y, -y]$ be a Laurent polynomial over $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$. The subset

$$V(f) = \{(a, b) \in \mathbb{R}^2 \mid -f \text{ is not locally convex at } (a, b)\}$$

is called the algebraic subset defined by f. The morphism $C_f \to \mathbb{R}^2$ parametrizing $V(f)$ with a tropical curve C_f is called the tropical plane curve defined by f. The genus of C_f is defined to be the first Betti number $b_1(C_f)$.

A tropical plane curve is a dequantization of complex amoebas in following way. For $t > 1$, let

$$\mathcal{A}_t \colon (\mathbb{C}^\times)^2 \longrightarrow \mathbb{R}^2$$

be the homomorphism of groups defined by

$$\mathcal{A}_t(a, b) = (\frac{\log|a|}{\log(t)}, \frac{\log|b|}{\log(t)}).$$

$\mathcal{A}_t$ is called the complex amoeba map. Let

$$g_t \in \mathbb{C}[z_1, {z_1}^{-1}, z_2, {z_2}^{-1}] \quad (t > 1)$$

be a family of complex Laurent polynomials such that each coefficient is a Laurent polynomial of t^{-1}. This family is written as an element of a valuation field K. We use the group algebra $K = \mathbb{C}[[\mathbb{R}]]$ of power series defined by the group $\mathbb{R}$. The indeterminate is denoted by t^{-1}, and the valuation is defined to be the maximum index of t multiplied by -1. So, $\mathrm{val}(t^a) = -a$. The family $\{g_t \mid t > 1\}$ is written as an element

$$g \in K[z_1, {z_1}^{-1}, z_2, {z_2}^{-1}].$$

The amoeba map over K

$$\mathcal{A} \colon (K^\times)^2 \longrightarrow \mathbb{R}^2$$

is defined as follows.

$$\mathcal{A}(a, b) = (-\mathrm{val}(a), -\mathrm{val}(b)).$$

The affine curve $V(g) \subset (K^\times)^2$ is the family of affine complex curves $V(g_t) \subset (\mathbb{C}^\times)^2$. Taking $t \to +\infty$, the family of complex amoebas $\mathcal{A}_t(V(g_t))$ converges to the amoeba $\mathcal{A}(V(g))$ over K. Also, the amoeba over K is the algebraic subset defined by a tropical Laurent polynomial. Let

$$\mathcal{A} \colon K[z_1, {z_1}^{-1}, z_2, {z_2}^{-1}] \longrightarrow \mathbb{T}[x, -x, y, -y]$$

be the map defined as follows.

$$\mathcal{A}(g) = f;$$

$$g = \sum_{i,j\in\mathbb{Z}} c_{ij} z_1^i z_2^j,$$

$$f = \bigoplus_{i,j\in\mathbb{Z}} -\mathrm{val}(c_{ij}) \odot ix \odot jy.$$

Then we have

$$\mathcal{A}(V(g)) = V(f).$$

This construction is called the tropicalization of a family of affine complex curves.

7.2 Examples

Example 7.1. For $a, b, c \in \mathbb{C}^\times$, let

$$g = a + bz_1 + cz_2.$$

Then

$$f = \mathcal{A}(g) = 0 \oplus x \oplus y.$$

The tropical plane curve C_f is said to be a tropical projective line. We have $b_1(C_f) = 0$.

Example 7.2. For $r, s \in \mathbb{N}$ and $a_i, b_j \in \mathbb{R}$, let

$$f = f_1 \odot f_2,$$

$$f_1 = a_0 \oplus a_1 \odot x \oplus a_2 \odot 2x \oplus \cdots \oplus a_r \odot rx,$$

$$f_2 = b_0 \oplus b_1 \odot y \oplus b_2 \odot 2y \oplus \cdots \oplus b_s \odot sy.$$

Assume that

$$2a_i > a_{i-1} + a_{i+1},$$

$$2b_j > b_{j-1} + b_{j+1}.$$

Then $b_1(C_f) = (r-1)(s-1)$.

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