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Non-commutative Reidemeister torsion, Morse-Novikov theory and homology cylinders of higher-order

（非可換ライデマイスタートーション，モース‐ノビコフ理論及び高次のホモロジーシリンダー）

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NON-COMMUTATIVE REIDEMEISTER TORSION, MORSE-NOVIKOV THEORY AND HOMOLOGY CYLINDERS OF HIGHER-ORDER

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1. INTRODUCTION

This paper is divided into two parts. One of them presents a Morse theoretical and dynamical description of non-commutative Reidemeister torsion, which is a generalization of the results of Hutchings and Lee [21, 22] and Pazhitnov [39, 40] on abelian coefficients to the case of skew fields. This content is the theme of [26]. The aim of the other part is to study algebraic structures of certain submonoids of homology cylinders and the homology cobordism groups of them as an application of non-commutative Reidemeister torsion. Here we consider a non-commutative extension of the work of Cha, Friedl and Kim [3], and obtain an analogous result of Goda and Sakasai [15].

Let X be a closed connected oriented Riemannian d-manifold with $\chi(X) = 0$ and $f: X \to S^1$ a Morse function such that the stable and unstable manifolds of the critical points of f transversely intersect and the closed orbits of flows of ∇f are all nondegenerate. (See Section 2.1.2 and 2.2.1.)

For a generic closed 1-form, for instance df, we can define the Lefschetz-type zeta function which counts closed orbits of flows induced by the 1-form. In [20], [21], [22] Hutchings and Lee showed that the product of the zeta function and the algebraic torsion of the abelian Novikov complex associated to the 1-form is a topological invariant and is equal to the abelian Reidemeister torsion of X. In [39], [40] Pazhitnov also proved a similar theorem in terms of the torsion of a canonical chain homotopy equivalence map between the abelian Novikov complex and the completed simplicial chain complex of the maximal abelian covering of X. In the case where X is a fiber bundle over a circle and f is the projection these results give Milnor's theorem in [30] which claims that the Lefschetz zeta function of a self map is equal to the abelian Reidemeister torsion of the mapping torus of the map.

In fixed point theory there is a non-commutative substitute for the Lefschetz zeta function which is called the total Lefschetz-Nielsen invariant, and in [12] Geoghegan and Nicas showed that the invariant has similar properties to these of torsion and determines the Reidemeister traces of iterates of a self map. In [36] Pajitnov considered the eta function associated to $-\nabla f$ which lies in a suitable quotient of the Novikov ring of $\pi_1 X$ and whose abelianization coincides with the logarithm of the Lefschetz-type zeta function. He also proved a formula expressing the eta function in terms of the torsion of a chain homotopy equivalence map between the Novikov complex and the completed simplicial chain complex of the universal covering of X. These works were generalized to the case of generic closed 1-forms by Schütz in [46] and [47].

Non-commutative Alexander polynomials which are called the higher-order Alexander polynomials were introduced, in particular for 3-manifolds, by Cochran in [4] and Harvey in [19], and are known by Friedl in [9] to be essentially equal to Reidemeister torsion over certain skew fields. We call it higher-order Reidemeister torsion. The main aim of this paper is to give a generalization of Hutchings and Lee's theorem to the case where the coefficients are skew fields by using Dieudonné determinant and to obtain a Morse theoretical and dynamical description of higher-order Reidemeister torsion. Note that it is known by Goda and Pajitnov in [13] that the torsion of a chain homotopy equivalence between the twisted Novikov complex and the twisted simplicial complex by a linear representation equals the twisted Lefschetz zeta function which was introduced by Jiang and Wang in [23]. This work is closely related to twisted Alexander polynomials which were introduced first by Lin in [28] and later generally by Wada in [51]. Our objects and approach considered here are different from theirs.

Let Λ_f be the Novikov completion of $\mathbb{Z}[\pi_1 X]$ associated to $f_*\colon \pi_1 X \to \langle t \rangle$, where t is the "downward" generator of $\pi_1 S^1$. We denote by W the subgroup of the unit group of Λ_f consisting of elements of the form $1 + \sum_{\gamma \in \pi_1 X, \deg f_*(\gamma) > 0} a_\gamma \gamma$. We first consider a certain quotient group $\overline{W}_{ab}$ of the abelianization of W and introduce a non-commutative Lefschetz-type zeta function $\zeta_f \in \overline{W}_{ab}$ of f. Taking a poly-torsion-free-abelian group G and group homomorphisms $\rho\colon \pi_1 X \to G$, $\alpha\colon G \to \pi_1 S^1$ such that $\alpha \circ \rho = f_*$, we construct a certain Novikov-type skew field $\mathcal{K}_\theta((t^l))$. Similar to $\overline{W}_{ab}$ we define a certain quotient group $\overline{\mathcal{K}_\theta((t^l))_{ab}^\times}$ of the abelianization $\mathcal{K}_\theta((t^l))_{ab}^\times$ of $\mathcal{K}_\theta((t^l))^\times$. We can check that ρ naturally extends to a ring homomorphism $\Lambda_f \to \mathcal{K}_\theta((t^l))$ and also denote it by ρ. There is a naturally induced homomorphism $\rho_*\colon \overline{W}_{ab} \to \overline{\mathcal{K}_\theta((t^l))_{ab}^\times}$ by ρ. If the twisted homology group $H_*^\rho(X; \mathcal{K}_\theta((t^l)))$ of X associated to ρ vanishes, then we can define the Reidemeister torsion $\tau_\rho(X)$ of X associated to ρ and the algebraic torsion $\tau_\rho^{Nov}(f)$ of the Novikov complex over $\mathcal{K}_\theta((t^l))$ as elements in $\mathcal{K}_\theta((t^l))_{ab}^\times / \pm\rho(\pi_1 X)$. Here is the main theorem which can be applied for the higher-order Reidemeister torsion.

Theorem 1.0.1 (Theorem 2.2.7). *For a given pair (ρ, α) as above, if $H_*^\rho(X; \mathcal{K}_\theta((t^l))) = 0$, then*

$$\tau_\rho(X) = \rho_*(\zeta_f)\tau_\rho^{Nov}(f) \in \overline{\mathcal{K}_\theta((t^l))_{ab}^\times} / \pm\rho(\pi_1 X).$$

To prove the theorem we use a similar approach to that of Hutchings and Lee in [22], but we need more subtle argument because of the non-commutative nature, especially in the second half, which is the heart of the proof. We can check that the non-commutative zeta function ζ_f can be seen as a certain reduction of the eta function associated to $-\nabla f$, and this theorem can also be deduced from the results of Pajitnov in [36] by a purely algebraic functoriality argument. In [14, Theorem 5.4] Goda and Sakasai showed another splitting formula for Reidemeister torsion over skew fields.

The background of the second part of this paper is as follows. Let $\Sigma_{g,n}$ be a compact oriented surface of genus g with (possibly empty) n boundary components. We denote by $\mathcal{M}_{g,n}$ the mapping class group of $\Sigma_{g,n}$ which is defined to be the group of isotopy classes of orientation preserving homeomorphisms of $\Sigma_{g,n}$, where these isotopies are understood to fix $\partial\Sigma_{g,n}$ pointwise.

Homology cylinders were first introduced by Goussarov [16] and Habiro [17], where these were referred to as *homology cobordisms*, in their works on so-called *clover* or *clasper* surgery of 3-manifolds developed for the study of finite-type invariants. The set $C_{g,n}$ of isomorphism classes of homology cylinders over $\Sigma_{g,n}$ naturally has a monoid structure by "stacking". We denote by $\overline{C}_{g,n}$ the submonoid consisting of isomorphism classes of irreducible ones as 3-manifolds. In [11, 27] Garoufalidis and Levine introduced the group $\mathcal{H}_{g,n}$ of smooth homology cobordism classes of homology cylinders over $\Sigma_{g,n}$, which can be seen as an enlargement of $\mathcal{M}_{g,n}$. (See also [3, Proposition 2. 4].) These sets naturally act on $H_1(\Sigma_{g,n};\mathbb{Z})$, and we can consider substitutes $\mathcal{I}C_{g,n}, \mathcal{I}\overline{C}_{g,n}, \mathcal{I}\mathcal{H}_{g,n}$ of the Torelli subgroup $\mathcal{I}_{g,n}$ which are defined as the kernels of the actions.

It is a natural question which properties of $\mathcal{M}_{g,n}$ are carried over to $C_{g,n}, \mathcal{H}_{g,n}$. The following results contrast with the well-known facts that $\mathcal{M}_{g,n}$ is finitely presented, that $\mathcal{M}_{g,n}$ is perfect for $g \geq 3$ [41] and that $\mathcal{I}_{g,0}$ and $\mathcal{I}_{g,1}$ are finitely generated for $g \geq 3$ [24]. Morita [32] showed by using his "trace maps" in [31] that the abelianization of $\mathcal{I}\mathcal{H}_{g,1}$ has infinite rank. Goda and Sakasai [15] showed by using sutured Floer homology theory that $\overline{C}_{g,1}$ is not finitely generated if $g \geq 1$. Cha, Friedl and Kim [3] showed by using abelian Reidemeister torsion that the abelianization of $\mathcal{H}_{g,n}$ contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty$ if $(g,n) \neq (0,0),(0,1)$ and one isomorphic to $\mathbb{Z}^\infty$ if $n > 1$ and that the abelianization of $\mathcal{I}\mathcal{H}_{g,n}$ contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty$ if $(g,n) \neq (0,0),(0,1)$ and one isomorphic to $\mathbb{Z}^\infty$ if $g > 1$ or $n > 1$.

We set $\Gamma_m := \pi_1\Sigma_{g,n}/(\pi_1\Sigma_{g,n})^{(m+1)}$ for each $m \geq 0$, where $(\pi_1\Sigma_{g,n})^{(m)}$ is the derived series of $\pi_1\Sigma_{g,n}$. The derived series $G^{(m)}$ of a group G is defined inductively by $G^{(0)} := G$ and $G^{(m+1)} := [G^{(m)}, G^m]$. In this paper for given m, we introduce *homology cylinders of order m* over $\Sigma_{g,n}$, which are characterized as homology cylinders over $\Sigma_{g,n}$ satisfying that the marking embeddings from $\Sigma_{g,n}$ to the boundary of the underlying manifold M induce isomorphisms $\Gamma_m \to \pi_1 M/(\pi_1 M)^{(m+1)}$. We denote by $C_{g,n}^{(m)}$ and $\overline{C}_{g,n}^{(m)}$ the submonoids of $C_{g,n}$ and $\overline{C}_{g,n}$ consisting of isomorphism classes of homology cylinders of order m. These naturally give filtrations of $C_{g,n}^{(m)}$ and $\overline{C}_{g,n}^{(m)}$. We also define an appropriate smooth homology cobordism group $\mathcal{H}_{g,n}^{(m)}$ in this context, which can be also seen as an enlargement of $\mathcal{M}_{g,n}$. There are a natural homomorphism $C_{g,n}^{(m)} \to \mathrm{Out}(\Gamma_m)$ and the induced homomorphisms $\overline{C}_{g,n}^{(m)} \to \mathrm{Out}(\Gamma_m), \mathcal{H}_{g,n}^{(m)} \to \mathrm{Out}(\Gamma_m)$. We use the notation $\mathcal{I}C_{g,n}^{(m)}, \mathcal{I}\overline{C}_{g,n}^{(m)}, \mathcal{I}\mathcal{H}_{g,n}^{(m)}$ for the kernels, which are substitutes of $\mathrm{Ker}(\mathcal{M}_{g,n} \to \mathrm{Out}(\Gamma_m))$. The filtration

$$\cdots \subset \mathcal{I}C_{g,n}^{(m+1)} \subset \mathcal{I}C_{g,n}^{(m)} \subset \cdots \subset \mathcal{I}C_{g,n}^{(1)} \subset \mathcal{I}C_{g,n}$$

and the sequence of the homomorphisms

$$\cdots \to \mathcal{I}\mathcal{H}_{g,n}^{(m+1)} \to \mathcal{I}\mathcal{H}_{g,n}^{(m)} \to \cdots \to \mathcal{I}\mathcal{H}_{g,n}^{(1)} \to \mathcal{I}\mathcal{H}_{g,n}$$

can be seen as alternatives for the derived series of the Johnson filtrations of $C_{g,n}$, $\mathcal{H}_{g,n}$ [17, 11] for the lower central series.

Our purpose is to investigate the algebraic structures of these objects by using non-commutative Reidemeister torsion as an analogue of the work of Cha, Friedl and Kim [3]. We first construct the Reidemeister torsion homomorphisms

$$C_{g,n}^{(m)} \to (\mathbb{Q}(\Gamma_m)_{ab}^{\times}/\pm\Gamma_m) \rtimes \mathrm{Out}(\Gamma_m),$$
$$\mathcal{H}_{g,n}^{(m)} \to (\mathbb{Q}(\Gamma_m)_{ab}^{\times}/\pm\Gamma_m \cdot \langle q\bar{q}\rangle) \rtimes \mathrm{Out}(\Gamma_m),$$

where $\mathbb{Q}(\Gamma_m)$ is the classical right ring of quotients $\mathbb{Q}[\Gamma_m](\mathbb{Q}[\Gamma_m] \setminus 0)^{-1}$ of $\mathbb{Q}[\Gamma_m]$ and $\bar{\cdot}\colon \mathbb{Q}(\Gamma_m)_{ab}^{\times} \to \mathbb{Q}(\Gamma_m)_{ab}^{\times}$ is the induced involution by $\gamma \mapsto \gamma^{-1}$ for $\gamma \in \Gamma_m$. (See Corollaries 3.2.6, 3.2.7, 3.2.9, 3.2.10.) Moreover, we prove the following theorems, which establish own interests of the objects. (See also Lemmas 3.1.4, 3.1.10.)

Theorem 1.0.2 (Theorem 3.3.4). *(i)* $\overline{\mathcal{IC}}_{0,2}^{(1)} \neq \overline{\mathcal{IC}}_{0,2}$.

(ii) $\overline{\mathcal{IC}}_{1,0}^{(1)} \neq \overline{\mathcal{IC}}_{1,0}$.

(iii) If $(g,n) \neq (0,0),(0,1),(0,2),(1,0)$, *then* $\overline{\mathcal{IC}}_{g,n}^{(m+1)} \neq \overline{\mathcal{IC}}_{g,n}^{(m)}$ *for all* m.

Theorem 1.0.3 (Theorem 3.2.11). *If* $(g,n) \neq (0,0),(0,1)$, *then the homomorphisms* $\mathcal{H}_{g,n}^{(m)} \to \mathcal{H}_{g,n}$, $\mathcal{IH}_{g,n}^{(m)} \to \mathcal{IH}_{g,n}$ *are not surjective for* $m > 0$.

Finally, we prove the following theorem, and give an observation on an approach for whether $\mathcal{IH}_{g,n}^{(m)}$ is in general finitely generated or not.

Theorem 1.0.4 (Corollary 3.4.6). *If* $n > 0$ *and* $(g,n) \neq (0,1),(0,2)$, *then* $\overline{\mathcal{IC}}_{g,n}^{(m)}$ *is not finitely generated for all* m.

These can be regarded as an analogue of the question whether $\mathrm{Ker}(\mathcal{M}_{g,n} \to \mathrm{Out}(\Gamma_m))$ is finitely generated or not. It is worth pointing out that the technique in Section 3.4 to detect nontriviality of elements in $\mathbb{Q}(\Gamma_m)_{ab}^{\times}/\pm\Gamma_m$ has multiplicity of use and could be useful also in other applications of non-commutative Reidemeister torsion.

This paper is organized as follows. In Section 2.1 we review some of the standard facts of Reidemeister torsion and the Novikov complex of f. In Section 2.2 we introduce the non-commutative Lefschetz-type zeta function ζ_f and construct the skew field $\mathcal{K}_\theta((t^l))$. There we also set up notation for higher-order Reidemeister torsion. Section 2.3 is devoted to the proof of Theorem 1.0.1. In Section 3.1 we define homology cylinders of order m and smooth homology cobordisms of them. Section 3.2 establishes the Reidemeister torsion homomorphisms of $C_{g,n}^{(m)}$, $\mathcal{H}_{g,n}^{(m)}$ and contains a proof of Theorem 1.0.3. Section 3.3 provides a way to construct homology cylinders of order m from knots in S^3 by performing surgery and computations of Reidemeister torsion of them. Here we prove Theorem 1.0.2. Finally, we prove Theorem 1.0.4 and discuss an approach for $\mathcal{IH}_{g,n}^{(m)}$ in Section 3.4.

In this paper all homology groups and cohomology groups are with respect to integral coefficients unless specifically noted.

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2. Non-commutative Reidemeister torsion and Morse-Novikov theory

2.1. Preliminaries.

2.1.1. *Reidemeister torsion.* We begin with the definition of Reidemeister torsion over a skew field $\mathbb{K}$. See [29] and [49] for more details.

For a matrix over $\mathbb{K}$, we mean by an elementary row operation the addition of a left multiple of one row to another row. After elementary row operations we can turn any matrix $A \in GL_k(\mathbb{K})$ into a diagonal matrix $(d_{i,j})$. Then the *Dieudonné determinant* $\det A$ is defined to be $[\prod_{i=1}^{k} d_{i,i}] \in \mathbb{K}_{ab}^{\times} := \mathbb{K}^{\times}/[\mathbb{K}^{\times}, \mathbb{K}^{\times}]$.

Let $C_* = (C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_0)$ be a chain complex of finite dimensional right $\mathbb{K}$-vector spaces. If we have bases b_i of $\operatorname{Im}\partial_{i+1}$ and h_i of $H_i(C_*)$ for $i = 0, 1, \ldots n$, we can take a basis $b_ih_ib_{i-1}$ of C_i as follows. Picking a lift of h_i in $\operatorname{Ker}\partial_i$ and combining it with b_i, we first obtain a basis b_ih_i of C_i. Then picking a lift of b_{i-1} in C_i and combining it with b_ih_i, we can obtain a basis $b_ih_ib_{i-1}$ of C_i.

Definition 2.1.1. For given bases $\boldsymbol{c} = \{c_i\}$ of C_* and $\boldsymbol{h} = \{h_i\}$ of $H_*(C_*)$, we choose a basis $\{b_i\}$ of $\operatorname{Im}\partial_*$ and define

$$\tau(C_*, \boldsymbol{c}, \boldsymbol{h}) := \prod_{i=0}^{n} [b_ih_ib_{i-1}/c_i]^{(-1)^{i+1}} \in \mathbb{K}_{ab}^{\times},$$

where $[b_ih_ib_{i-1}/c_i]$ is the Dieudonné determinant of the base change matrix from c_i to $b_ih_ib_{i-1}$. If C_* is acyclic, then we write $\tau(C_*, \boldsymbol{c})$.

It can be easily checked that $\tau(C_*, \boldsymbol{c}, \boldsymbol{h})$ does not depend on the choices of b_i and $b_ih_ib_{i-1}$.

Torsion has the following multiplicative property. Let

$$0 \to C'_* \to C_* \to C''_* \to 0$$

be a short exact sequence of finite chain complexes of finite dimensional right $\mathbb{K}$-vector spaces and let $\boldsymbol{c} = \{c_i\}, \boldsymbol{c}' = \{c'_i\}, \boldsymbol{c}'' = \{c''_i\}$ and $\boldsymbol{h} = \{h_i\}, \boldsymbol{h}' = \{h'_i\}, \boldsymbol{h}'' = \{h''_i\}$ be bases of C_*, C'_*, C''_* and $H_*(C_*), H_*(C'_*), H_*(C''_*)$. Picking a lift of c''_i in C_i and combining it with the image of c'_i in C_i, we obtain a basis $c'_ic''_i$ of C_i. We denote by $\mathcal{H}_*$ the corresponding long exact sequence in homology and by $\boldsymbol{d}$ the basis of $\mathcal{H}_*$ obtained by combining $\boldsymbol{h}, \boldsymbol{h}', \boldsymbol{h}''$.

Lemma 2.1.2. ([29, Theorem 3. 1]) *If* $[c'_ic''_i/c_i] = 1$ *for all i, then*

$$\tau(C_*, \boldsymbol{c}, \boldsymbol{h}) = \tau(C'_*, \boldsymbol{c}', \boldsymbol{h}')\tau(C''_*, \boldsymbol{c}'', \boldsymbol{h}'')\tau(\mathcal{H}_*, \boldsymbol{d}).$$

The following lemma is a certain non-commutative version of [49, Theorem 2.2]. Turaev's proof can be easily applied to this setting.

Lemma 2.1.3. *If C_* is acyclic and we find a decomposition $C_* = C'_* \oplus C''_*$ such that C'_i and C''_i are spanned by subbases of c_i and the induced map $pr_{C''_{i-1}} \circ \partial_i|_{C'_i} \colon C'_i \to C''_{i-1}$ is an isomorphism for each i, then*

$$\tau(C_*, c) = \pm \prod_{i=0}^{n} (\det pr_{C''_{i-1}} \circ \partial|_{C'_i})^{(-1)^i}.$$

Let (X, Y) be a connected finite CW-pair and let $\varphi\colon \mathbb{Z}[\pi_1 X] \to \mathbb{K}$ be a ring homomorphism. We define the twisted homology group associated to φ as follows:

$$H_i^{\varphi}(X, Y; \mathbb{K}) := H_i(C_*(\widetilde{X}, \widetilde{Y}) \otimes_{\mathbb{Z}[\pi_1 X]} \mathbb{K}),$$

where $p\colon \widetilde{X} \to X$ is the universal covering and $\widetilde{Y} := p^{-1}(Y)$.

Definition 2.1.4. If $H_*^{\varphi}(X, Y; \mathbb{K}) = 0$, then we define the *Reidemeister torsion* $\tau_\varphi(X, Y)$ associated to φ as follows. We choose a lift $\tilde{e}$ in $\widetilde{X}$ for each cell $e \subset X \setminus Y$. Then

$$\tau_\varphi(X, Y) := [\tau(C_*(\widetilde{X}, \widetilde{Y}) \otimes_{\mathbb{Z}[\pi_1 X]} \mathbb{K}, \langle \tilde{e} \otimes 1 \rangle_e)] \in \mathbb{K}^\times_{ab} / \pm \varphi(\pi_1 X).$$

We can check that $\tau_\varphi(X, Y)$ does not depend on the choice of $\tilde{e}$. It is known that Reidemeister torsion is a simple homotopy invariant of a finite CW-pair.

2.1.2. *The Novikov complex.* Let X be a closed connected oriented Riemannian d-manifold and $f\colon X \to S^1$ a Morse function. Here we review the Novikov complex of f, which is the simplest version of Novikov's construction for closed 1-forms in [34]. See also [37] and [38].

We can lift f to a function $\tilde{f}\colon \widetilde{X} \to \mathbb{R}$. If p is a critical point of f or $\tilde{f}$, the *unstable manifold* $\mathcal{D}(p)$ is the set of all points x such that the upward gradient flow starting at x converges to p. Similarly, the *stable manifold* $\mathcal{A}(p)$ is the set of all points x such that the downward gradient flow starting at x converges to p. We choose a Riemann metric such that $\mathcal{D}(p) \pitchfork \mathcal{A}(p)$ for any critical points p, q of f.

We take the "downward" generator t of $\pi_1 S^1$.

Definition 2.1.5. We define the *Novikov completion* Λ_f of $\mathbb{Z}[\pi_1 X]$ associated to $f_*\colon \pi_1 X \to \langle t \rangle$ to be the set of a formal sum $\sum_{\gamma \in \pi_1 X} a_\gamma \gamma$ such that $a_\gamma \in \mathbb{Z}$ and for any $k \in \mathbb{Z}$, the number of γ such that $a_\gamma \neq 0$ and $\deg f_*(\gamma) \leq k$ is finite.

Definition 2.1.6. The *Novikov complex* $(C_*^{Nov}(f), \partial_*^f)$ of f is defined as follows. For each critical point p of f, we choose a lift $\tilde{p} \in \widetilde{X}$. Then we define $C_i^{Nov}(f)$ to be the free right Λ_f-module generated by the lifts $\tilde{p}$ of index i. If p is a critical point of index i, then

$$\partial_i^f(\tilde{p} \cdot \gamma) := \sum_{q \text{ of index } i-1, \gamma' \in \pi_1 X} n(\tilde{p} \cdot \gamma, \tilde{q} \cdot \gamma') \tilde{q} \cdot \gamma',$$

where $n(\tilde{p} \cdot \gamma, \tilde{q} \cdot \gamma')$ is the algebraic intersection number of $\mathcal{D}(\tilde{p} \cdot \gamma)$, $\mathcal{A}(\tilde{q} \cdot \gamma')$ and an appropriate level set, which can be seen as the signed number of negative gradient flow lines from $\tilde{p} \cdot \gamma$ to $\tilde{q} \cdot \gamma'$. By the linear extension we obtain the differential $\partial_i^f\colon C_i^{Nov}(f) \to C_{i-1}^{Nov}(f)$.

Obviously, the definition dose not depend on the choices of $\tilde{f}$ and $\tilde{p}$. It is known that appropriate orientations of stable and unstable manifolds ensure that $\partial_{i-1}^f \circ \partial_i^f = 0$.

Theorem 2.1.7 ([38]). *The Novikov complex $C_*^{Nov}(f)$ is chain homotopic to $C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1 X]} \Lambda_f$.*

2.2. The main theorem.

2.2.1. *Non-commutative zeta functions.* First we introduce a non-commutative zeta function ζ_f associated to f, which is closely related to the *total Lefschetz-Nielsen invariant* of a self map in [12].

Let Λ_f^+ be the subring of Λ_f whose element $\sum_{\gamma\in\pi_1 X} a_\gamma \gamma \in \Lambda_f$ satisfies that $a_\gamma = 0$ if $\deg f_*(\gamma) \le 0$. We denote by W the subgroup of the unit group of Λ_f consisting of elements of the form $1+\lambda$ with $\lambda \in \Lambda_f^+$. For $x, y \in W_{ab} := W/[W,W]$, we write

$$x \sim y$$

if for any $k \in \mathbb{Z}$, there exist representatives $\sum_{\gamma\in\pi_1 X} a_\gamma\gamma, \sum_{\gamma\in\pi_1 X} b_\gamma\gamma \in W$ of x, y respectively such that for any $\gamma \in \pi_1 X$ with $\deg f_*(\gamma) \le k$, $a_\gamma = b_\gamma$.

Lemma 2.2.1. *The relation $\sim$ is an equivalence relation in W_{ab}.*

Proof. We only need to show the transitivity. We assume that $x \sim y$ and $y \sim z$ for $x, y, z \in W_{ab}$ and for any $k \in \mathbb{Z}$ take representatives $\sum_\gamma a_\gamma\gamma$, $\sum_\gamma b_\gamma\gamma$ and $\sum_\gamma b'_\gamma\gamma$, $\sum_\gamma c'_\gamma\gamma$ of x, y and y, z respectively such that for any γ with $\deg f_*(\gamma) \le k$, $a_\gamma = b_\gamma$ and $b'_\gamma = c'_\gamma$. There exists $\lambda = \sum_\gamma d_\gamma\gamma \in [W,W]$ such that $\sum_\gamma b_\gamma\gamma = (\sum_\gamma b'_\gamma\gamma)\lambda$. We set c_γ so that $\sum_\gamma c_\gamma\gamma = (\sum_\gamma c'_\gamma\gamma)\lambda$. Then $\sum_\gamma c_\gamma\gamma$ is also a representative of z. Since for any γ with $\deg f_*(\gamma) < 0$, $d_\gamma = 0$, for any γ with $\deg f_*(\gamma) \le k$, $b_\gamma = c_\gamma$ and so $a_\gamma = c_\gamma$. We thus get $x \sim z$. □

We define $\overline{W}_{ab}$ to be the quotient set by the equivalence relation. The group structure of W_{ab} naturally induces that of $\overline{W}_{ab}$.

A *closed orbit* is a non-constant map $o\colon S^1 \to X$ with $\frac{do}{ds} = -\beta\nabla f$ for some $\beta > 0$. Two closed orbits are called equivalent if they differ by linear parameterization. We denote by O the set of the equivalence classes of closed orbits. The *period* $p(o)$ is the largest integer p such that o factors through a p-fold covering $S^1 \to S^1$. We assume that all the closed orbits are *nondegenerate*, namely the determinant of $id - d\phi\colon T_xX/T_xo(S^1) \to T_xX/T_xo(S^1)$ does not vanish for any $[o] \in O$, where ϕ is a $p(o)$th return map around a point $x \in o(S^1)$. The *Lefschetz sign* $\epsilon(o)$ is the sign of the determinant. We denote by $i_-(o)$ and $i_0(o)$ the numbers of real eigenvalues of $d\phi\colon T_xX/T_xo(S^1) \to T_xX/T_xo(S^1)$ for a return map ϕ which are in $(-\infty,-1)$ and in $(-1,1)$ respectively.

Definition 2.2.2. We number $[o] \in O$ with $p(o) = 1$ as $\{[o_i]\}_{i=1}^\infty$ and choose a path σ_{o_i} from the base point of X to a point of $o_i(S^1)$ for each $[o_i]$. Then we have $[\sigma_{o_i} o_i \bar{\sigma}_{o_i}] \in \pi_1 X$, where $\bar{\sigma}_{o_i}$ is the inverse path of σ_{o_i}. We define

$$\zeta_f := \left[\prod_{i=1}^{\infty}(1-(-1)^{i_-(o_i)}[\sigma_{o_i} o_i \bar{\sigma}_{o_i}])^{(-1)^{i_0(o_i)+1}}\right] \in \overline{W}_{ab}.$$

By the completeness of Λ_f we can easily check that the infinite product $\prod_{i=1}^\infty(1 - (-1)^{i_-(o_i)}[\sigma_{o_i} o_i \bar{\sigma}_{o_i}])^{(-1)^{i_+(o_i)+i_-(o_i)+1}} \in W$ makes sense.

Lemma 2.2.3. *The zeta function ζ_f does not depend on the choices of $\{[o_i]\}_{i=1}^\infty$ and σ_{o_i}.*

Proof. We take another sequence $\{[o'_i]\}_{i=1}^\infty$ and another path σ'_{o_i} for each $[o_i]$. For any $k \in \mathbb{Z}$, since $\{[o_i] \in O\,;\ \deg f_*([o_i]) \le k\}$, which equals $\{[o'_i] \in O\,;\ \deg f_*([o'_i]) \le k\}$, is a finite set and

$$[1 \pm [\sigma_{o_i} o_i \bar{\sigma}_{o_i}]] = [1 \pm [\sigma'_{o_i} o_i \bar{\sigma}'_{o_i}]]$$

in W_{ab},

$$\begin{aligned}
&\left[\prod_{i=1}^{\infty}(1 \pm [\sigma_{o_i} o_i \bar{\sigma}_{o_i}])^{\pm 1}\right] \\
&= \left[\prod_{1\le i\le\infty, \deg f_*([o_i])\le k} (1 \pm [\sigma_{o_i} o_i \bar{\sigma}_{o_i}])^{\pm 1}\right] \left[\prod_{1\le i\le\infty, \deg f_*([o_i])> k} (1 \pm [\sigma_{o_i} o_i \bar{\sigma}_{o_i}])^{\pm 1}\right] \\
&= \left[\prod_{1\le i\le\infty, \deg f_*([o'_i])\le k} (1 \pm [\sigma_{o'_i} o'_i \bar{\sigma}_{o'_i}])^{\pm 1}\right] \left[\prod_{1\le i\le\infty, \deg f_*([o_i])> k} (1 \pm [\sigma_{o_i} o_i \bar{\sigma}_{o_i}])^{\pm 1}\right] \\
&= \left[\prod_{1\le i\le\infty, \deg f_*([o'_i])\le k} (1 \pm [\sigma'_{o'_i} o'_i \bar{\sigma}'_{o'_i}])^{\pm 1}\right] \left[\prod_{1\le i\le\infty, \deg f_*([o_i])> k} (1 \pm [\sigma_{o_i} o_i \bar{\sigma}_{o_i}])^{\pm 1}\right].
\end{aligned}$$

in W_{ab}. Therefore for any $k \in \mathbb{Z}$, the products $\left[\prod_{i=1}^{\infty}(1 \pm [\sigma_{o_i} o_i \bar{\sigma}_{o_i}])^{\pm 1}\right]$ and $\left[\prod_{i=1}^{\infty}(1 \pm [\sigma'_{o'_i} o'_i \bar{\sigma}'_{o'_i}])^{\pm 1}\right]$ have representatives in W such that for any $\gamma \in \pi_1 X$ with $\deg f_*(\gamma) \le k$, the coefficients of γ are same, and the lemma follows. □

By the above lemma we can write

$$\zeta_f = \prod_{[o]\in O, p(o)=1} [1 - (-1)^{i_-(o)}[\sigma_o o \bar{\sigma}_o]]^{(-1)^{i_0(o)+1}}.$$

There is a formal exponential $\exp\colon \Lambda_f^+ \to W$ given by $\exp(\lambda) := \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}$. Since

$$\epsilon(o^j) = (-1)^{ji_-(o)+i_0(o)}$$

and

$$\exp\left(\sum_{j=1}^{\infty} \frac{(\pm\gamma)^j}{j}\right) = (1 \mp \gamma)^{-1}$$

for $\gamma \in \pi_1 X$ with $\deg f_*(\gamma) > 0$,

$$\zeta_f = \prod_{[o]\in O, p(o)=1} \left[\exp\left(\sum_{j=1}^{\infty} \frac{\epsilon(o^j)}{j}[\sigma_o o^j \bar{\sigma}_o]\right)\right],$$

where o^j is the composition of a j-fold covering $S^1 \to S^1$ and o.

2.2.2. *Novikov-type skew fields.* We proceed to construct Novikov-type non-commutative coefficients for torsion and formulate the main theorem.

A group G is called *poly-torsion-free-abelian (PTFA)* if there exists a filtration

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{n-1} \triangleleft G_n = G$$

such that G_i/G_{i-1} is torsion-free abelian.

Proposition 2.2.4 ([35]). *If G is a PTFA group, then $\mathbb{Q}[G]$ is a right (and left) Ore domain; namely $\mathbb{Q}[G]$ embeds in its classical right ring of quotients $\mathbb{Q}(G) := \mathbb{Q}[G](\mathbb{Q}[G] \setminus 0)^{-1}$.*

Let G be a PTFA group and $\rho\colon \pi_1X \to G$, $\alpha\colon G \to \langle t\rangle$ be group homomorphisms such that $\alpha\circ\rho = f_*$. Then $\operatorname{Ker}\alpha$ is also PTFA, and so we have the classical ring of quotients $\mathcal{K}$ of $\mathbb{Q}[\operatorname{Ker}\alpha]$. We denote by l the nonnegative integer such that t^l generates $\operatorname{Im}\alpha$. We pick $\mu\in G$ such that $\alpha(\mu) = t^l$ and let $\theta\colon \mathcal{K}\to\mathcal{K}$ be the automorphism given by $\theta(k) = \mu k\mu^{-1}$ for $k\in\mathcal{K}$. Now we have a Novikov type skew field $\mathcal{K}_\theta((t^l))$. More precisely, the elements of $\mathcal{K}_\theta((t^l))$ are formal sums $\sum_{i=n}^{\infty} a_i t^{li}$ with $n\in\mathbb{Z}$ and $a_i\in\mathcal{K}$, and the multiplication is defined by using the rule $t^l k = \theta(k)t^l$. Note that the isomorphism type of the ring $\mathcal{K}_\theta((t^l))$ dose not depend on the choice of μ, and we can regard $\mathbb{Z}[G]$ as a subring of $\mathcal{K}_\theta((t^l))$.

For $x, y\in \mathcal{K}_\theta((t))^\times_{ab}$, we write

$$x\sim y$$

if for any $k\in\mathbb{Z}$, there exist representatives $\sum_{i\in\mathbb{Z}} a_i t^{li}, \sum_{i\in\mathbb{Z}} b_i t^{li} \in \mathcal{K}_\theta((t^l))^\times$ of x, y respectively such that for any $i\le k$, $a_i = b_i$.

The following lemma can be similarly proved as Lemma 2.2.1 and so we omit the proof.

Lemma 2.2.5. *The relation $\sim$ is an equivalence relation in $\mathcal{K}_\theta((t^l))^\times_{ab}$.*

We define $\overline{\mathcal{K}_\theta((t^l))}^\times_{ab}$ to be the quotient set by the equivalence relation, which is also an abelian group. Note that if $\mathcal{K}_\theta((t^l))$ is commutative, then $\overline{\mathcal{K}_\theta((t^l))}^\times_{ab} = \mathcal{K}_\theta((t^l))^\times_{ab}$.

The group homomorphism ρ naturally extends to a ring homomorphism $\Lambda_f \to \mathcal{K}_\theta((t^l))$. By abuse of notation, we also denote it by ρ. By virtue of Theorem 2.1.7 $H^\rho_*(X;\mathcal{K}_\theta((t^l)))$ is isomorphic to $H_*(C^{Nov}_*(f)\otimes_\Lambda \mathcal{K}_\theta((t^l)))$.

Definition 2.2.6. If $H^\rho_*(X;\mathcal{K}_\theta((t^l))) = 0$, then we define the *Novikov torsion* associated to ρ as

$$\tau^{Nov}_\rho(f) := [\tau(C^{Nov}_*(f)\otimes_{\Lambda_f}\mathcal{K}_\theta((t^l)), \langle\tilde p\otimes 1\rangle_p)] \in \mathcal{K}_\theta((t^l))^\times_{ab}/\pm\rho(\pi_1X).$$

The ring homomorphism $\rho\colon \Lambda_f \to \mathcal{K}_\theta((t^l))$ naturally induces a group homomorphism $\rho_*\colon \overline{W}_{ab} \to \overline{\mathcal{K}_\theta((t^l))}^\times_{ab}$ and $\overline{\mathcal{K}_\theta((t^l))}^\times_{ab}/\pm\rho(\pi_1X)$ is a quotient group of $\mathcal{K}_\theta((t^l))^\times_{ab}/\pm\rho(\pi_1X)$.

Theorem 2.2.7 (Main theorem). *For a given pair (ρ,α) as above, if $H^\rho_*(X;\mathcal{K}_\theta((t^l))) = 0$, then*

$$\tau_\rho(X) = \rho_*(\zeta_f)\tau^{Nov}_\rho(f) \in \overline{\mathcal{K}_\theta((t^l))}^\times_{ab}/\pm\rho(\pi_1X).$$

Remark 2.2.8. More generally, the same construction makes sense and the theorem also holds under the assumption that $\mathbb{Q}[\operatorname{Ker}\alpha]$ is a right Ore domain instead of that G is PTFA. Moreover, it is expected that we can eliminate the ambiguity of multiplication by an element of $\rho(\pi_1X)$, carefully considering *Euler structures* by Turaev [49], [50] as in [22].

An important example of a pair (ρ,α) is provided by Harvey's *rational derived series* in [19].

Definition 2.2.9. For a group Π, let $\Pi^{(0)}_r = \Pi$ and we inductively define

$$\Pi^{(i)}_r = \{\gamma\in\Pi^{(i-1)}_r\ ;\ \gamma^k\in[\Pi^{(i-1)}_r,\Pi^{(i-1)}_r] \text{ for some } k\in\mathbb{Z}\setminus 0\}.$$

Lemma 2.2.10 ([19]). *For any group Π and any n,*

$$\Pi^{(n-1)}_r/\Pi^{(n)}_r = (\Pi^{(n-1)}_r/[\Pi^{(n-1)}_r,\Pi^{(n-1)}_r])/\ \textit{torsion}$$

and $\Pi/\Pi^{(n)}_r$ is a PTFA group.

For any n, we have the natural surjection $\rho^{(n)}\colon \pi_1X\to\pi_1X/(\pi_1X)^{(n+1)}_r$ and the induced homomorphism $\alpha^{(n)}\colon \pi_1X/(\pi_1X)^{(n+1)}_r \to\langle t\rangle$ by f_*. By the above construction we obtain the group homomorphism $\rho^{(n)}\colon \mathbb{Z}[\pi_1X]\to\mathcal{K}^{(n)}_\theta(t^l)$ and the extended one $\tilde\rho^{(n)}\colon \Lambda_f\to\mathcal{K}^{(n)}_\theta((t^l))$, where $\mathcal{K}^{(n)}_\theta(t^l)$ is the subfield of $\mathcal{K}^{(n)}_\theta((t^l))$ consisting of rational elements.

Definition 2.2.11. If $H_*^{\rho^{(n)}}(X;\mathcal{K}_\theta^{(n)}(t^l))=0$, then $\tau_{\rho^{(n)}}(X)\in\mathcal{K}_\theta^{(n)}(t^l)_{ab}^\times/\pm\rho^{(n)}(\pi_1X)$ is defined. We call it the *higher-order Reidemeister torsion* of order n.

Remark 2.2.12. It is known by Friedl that $\tau_{\rho^{(n)}}(X)$ equals an alternating product of non-commutative Alexander polynomials. See [9] for the details.

If $H_*^{\rho^{(n)}}(X;\mathcal{K}_\theta^{(n)}(t^l))=0$, then we have $H_*^{\tilde\rho^{(n)}}(X;\mathcal{K}_\theta^{(n)}((t^l)))=0$, and we can also define $\tau_{\tilde\rho^{(n)}}(X)\in\mathcal{K}_\theta^{(n)}((t^l))_{ab}^\times/\pm\rho^{(n)}(\pi_1X)$. By the functoriality of Reidemeister torsion the image of $\tau_{\rho^{(n)}}(X)$ by the natural map $\mathcal{K}_\theta^{(n)}(t^l)_{ab}^\times/\pm\rho^{(n)}(\pi_1X)\to\mathcal{K}_\theta^{(n)}((t^l))_{ab}^\times/\pm\rho^{(n)}(\pi_1X)$ equals $\tau_{\tilde\rho^{(n)}}(X)$. Thus for the pair $(\rho^{(n)},\alpha^{(n)})$, the main theorem gives a Morse theoretical and dynamical presentation of $\tau_{\rho^{(n)}}(X)$ in $\overline{\mathcal{K}_\theta^{(n)}((t^l))_{ab}}^\times/\pm\rho^{(n)}(\pi_1X)$ as a corollary.

2.3. **Proof.** The proof of the main theorem is divided into two parts. In the first part we construct an "approximate" CW complex X' which is adapted to ∇f, and we show that the Reidemeister torsion of X' equals that of X. The second part is devoted to computation of the torsion of X', and we see that it has the desired form.

2.3.1. *An approximate CW-complex.* Let Σ be a level set of a regular value of f and let Y be the compact Riemannian manifold obtained by cutting X along Σ. We can pick a Morse function $f_0\colon Y\to\mathbb{R}$ induced by f. We write $\partial Y=\Sigma_0\sqcup\Sigma_1$, where Σ_0,Σ_1 are the cutting hypersurfaces and $-\nabla f_0$ points outward along Σ_0. We denote by $\mathcal{A}_0(p),\mathcal{D}_0(p)$ the stable and unstable manifolds of a critical point p of f_0.

We take a smooth triangulation T_1 of Σ_1 such that each simplex is transverse to $\mathcal{A}_0(p)$ for each critical point p of f_0. For $\sigma\in T_1$, let us denote by $\mathcal{F}(\sigma)$ the set of all $y\in Y$ such that the flow of ∇f_0 starting at y hits σ. It is well-known that the submanifolds $\mathcal{D}_0(p)$ and $\mathcal{F}(\sigma)$ have natural compactifications $\overline{\mathcal{D}_0(p)}$ and $\overline{\mathcal{F}(\sigma)}$ respectively by adding broken flow lines of $-\nabla f_0$. (See for instance [21].) We choose a cell decomposition T_0 of Σ_0 such that $\overline{\mathcal{D}_0(p)}\cap\Sigma_0$ and $\overline{\mathcal{F}(\sigma)}\cap\Sigma_0$ are subcomplexes for each critical point p and each simplex σ. Then we can check that the cells in $T_0,T_1,\overline{\mathcal{D}_0(p)}$ and $\overline{\mathcal{F}(\sigma)}$ give a cell decomposition T_Y of Y.

Let $h\colon(\Sigma_0,T_0)\to(\Sigma_1,T_1)$ be a cellular approximation to the canonical identification $\Sigma_0\to\Sigma_1$. We consider the mapping cylinder M_h of h:

$$M_h:=((\Sigma_0\times[0,1])\sqcup\Sigma_1)/(x,1)\sim h(x).$$

It has a natural cell decomposition induced by T_0 and T_1.

Definition 2.3.1. Let X' be the space obtained by gluing Y and M_h along $\Sigma_0\sqcup\Sigma_1$.

For a cell Δ in T_Y of the form $\overline{\mathcal{D}_0(p)}$ and $\overline{\mathcal{F}(\sigma)}$, we define $\widehat{\Delta}$ to be the set obtained by gluing Δ and

$$(((\Delta\cap\Sigma_0)\times[0,1])\sqcup h(\Delta\cap\Sigma_0))/(x,1)\sim h(x)$$

along $\Delta\cap\Sigma_0$. Cells of the form $\widehat{\mathcal{D}_0(p)}$, σ and $\widehat{\mathcal{F}(\sigma)}$ for a critical point p of f_0 and $\sigma\in T_1$ give a cell decomposition of X'.

We pick a homotopy equivalence map $X'\to X$ and identify π_1X' with π_1X.

Lemma 2.3.2. *Under the assumptions of Theorem 2.2.7 we have*

$$\tau_\rho(X)=\tau_\rho(X').$$

Proof. In all of the calculations below, we implicitly tensor the chain complexes with the skew field $\mathcal{K}_\theta((t^l))$, and brackets mean that they are in $\mathcal{K}_\theta((t^l))^\times_{ab}/\pm\rho(\pi_1 X)$.

We regard X as the union of Y and $\Sigma\times[0,1]$ along $\Sigma_0\sqcup\Sigma_1$, then we have short exact sequences

$$0\to C_*(\widetilde{\Sigma}_0)\oplus C_*(\widetilde{\Sigma}_1)\to C_*(\widetilde{\Sigma}\times[0,1])\oplus C_*(\widetilde{Y})\to C_*(\widetilde{X})\to 0,$$
$$0\to C_*(\widetilde{\Sigma}_0)\oplus C_*(\widetilde{\Sigma}_1)\to C_*(\widetilde{M}_h)\oplus C_*(\widetilde{Y})\to C_*(\widetilde{X})\to 0.$$

The natural map $\Sigma\times[0,1]\to M_h$ induces an isomorphism between $H^\rho_*(\Sigma\times[0,1];\mathcal{K}_\theta((t^l)))$ and $H^\rho_*(M_h;\mathcal{K}_\theta((t^l)))$, and there is an isomorphism between the long exact sequences in homology for the above sequences. Let $\boldsymbol{c}$ and $\boldsymbol{c}'$ be bases of $C_*(\widetilde{\Sigma}\times[0,1])$ and $C_*(\widetilde{M}_h)$ consisting of lifts of cells. We pick bases $\boldsymbol{h}$ and $\boldsymbol{h}'$ of $H^\rho_*(\Sigma\times[0,1];\mathcal{K}_\theta((t^l)))$ and $H^\rho_*(M_h;\mathcal{K}_\theta((t^l)))$ such that the isomorphism maps $\boldsymbol{h}$ to $\boldsymbol{h}'$. Then from Lemma 2.1.2 we obtain

$$\frac{\tau_\rho(X)}{\tau_\rho(X')}=\frac{[\tau(C_*(\widetilde{\Sigma}\times[0,1]),\boldsymbol{c},\boldsymbol{h})]}{[\tau(C_*(\widetilde{M}_h),\boldsymbol{c}',\boldsymbol{h}')]}. \tag{2.1}$$

We have short exact sequences

$$0\to C_*(\widetilde{\Sigma}\times 1)\to C_*(\widetilde{\Sigma}\times[0,1])\to C_*(\widetilde{\Sigma}\times[0,1],\widetilde{\Sigma}\times 1)\to 0,$$
$$0\to C_*(\widetilde{\Sigma}_1)\to C_*(\widetilde{M}_h)\to C_*(\widetilde{M}_h,\widetilde{\Sigma}_1)\to 0.$$

The map $\Sigma\times[0,1]\to M_h$ also induces an isomorphism between the long exact sequences in homology for the above sequences. Let $\boldsymbol{d}$ and $\boldsymbol{d}'$ be bases of $C_*(\widetilde{\Sigma}\times[0,1],\widetilde{\Sigma}\times 1)$ and $C_*(\widetilde{M}_h,\widetilde{\Sigma}_1)$ induced by $\boldsymbol{c}$ and $\boldsymbol{c}'$. Then again from Lemma 2.1.2 we obtain

$$\frac{[\tau(C_*(\widetilde{\Sigma}\times[0,1]),\boldsymbol{c},\boldsymbol{h})]}{[\tau(C_*(\widetilde{M}_h),\boldsymbol{c}',\boldsymbol{h}')]}=\frac{[\tau(C_*(\widetilde{\Sigma}\times[0,1],\widetilde{\Sigma}\times 1),\boldsymbol{d})]}{[\tau(C_*(\widetilde{M}_h,\widetilde{\Sigma}_1),\boldsymbol{d}')]}. \tag{2.2}$$

By direct computations we have

$$[\tau(C_*(\widetilde{\Sigma}\times[0,1],\widetilde{\Sigma}\times 1),\boldsymbol{d})]=[\tau(C_*(\widetilde{M}_h,\widetilde{\Sigma}_1),\boldsymbol{d}')]=[1].$$

Now the lemma follows from (2.1), (2.2) and these equalities. □

2.3.2. *Computation of the torsion.* We decompose

$$C_i(\widetilde{X}')\otimes_{\mathbb{Z}[\pi_1 X']}\mathcal{K}_\theta((t^l))=D_i\oplus E_i\oplus F_i,$$

where D_i, E_i and F_i are generated by elements of the form $\widehat{\mathcal{D}_0(p)}$, σ and $\widehat{\mathcal{F}(\sigma)}$ for a critical point p of f_0 and $\sigma\in T_1$ respectively. There are natural identifications $D_i\cong C_i^{Nov}(f)\otimes_{\Lambda_f}\mathcal{K}_\theta((t^l))$ and $F_i\cong E_{i-1}$. Then the matrix for the differential ∂_i can be written as

$$\partial_i \;=\; \begin{array}{c} \\ D_{i-1} \\ E_{i-1} \\ F_{i-1} \end{array}\!\!\begin{array}{c} \begin{array}{ccc} D_i & E_i & F_i \end{array} \\ \begin{pmatrix} \boldsymbol{N}_i & 0 & \boldsymbol{W}_i \\ -\boldsymbol{M}_i & \partial_i^\Sigma & I-\phi_{i-1} \\ 0 & 0 & -\partial_i^\Sigma \end{pmatrix} \end{array},$$

where ∂_i^Σ is the differential on $C_*(\widetilde{\Sigma})\otimes_{\mathbb{Z}[\pi_1\Sigma]}\mathcal{K}_\theta((t^l))$ and ϕ_{i-1} can be interpreted as the return map of the gradient flow in $\widetilde{X}$ after perturbation by h. We set

$$\boldsymbol{K}_i:=\boldsymbol{N}_i+\boldsymbol{W}_i(I-\phi_{i-1})^{-1}\boldsymbol{M}_i\colon D_i\to D_{i-1}.$$

Since $C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1 X]} \mathcal{K}_\theta((t^l))$ is acyclic, the Novikov complex (D_*, ∂^f_*) is also acyclic by Theorem 2.1.7, and we can choose a decomposition $D_i = D'_i \oplus D''_i$ such that D'_i and D''_i are spanned by lifts of the critical points of f and ∂_f induces an isomorphism $D'_i \to D''_{i-1}$. We denote by $K_i\colon D'_i \to D''_{i-1}$ the induced map by $\boldsymbol{K}_i$.

Lemma 2.3.3. *Under the assumptions of Theorem 2.2.7, if K_i is non-singular for each i, then*

$$\tau_\rho(X') = \prod_{i=1}^{d} [\det(I - \phi_{i-1}) \det K_i]^{(-1)^i}.$$

Proof. We consider the matrix

$$\Omega_i := \begin{array}{c c} & \begin{array}{cc} D'_i & F_i \end{array} \\ \begin{array}{c} D''_{i-1} \\ E_{i-1} \end{array} & \begin{pmatrix} N_i & W_i \\ -M_i & I - \phi_{i-1} \end{pmatrix} \end{array},$$

where $M_i\colon D'_i \to E_{i-1}$, $N_i\colon D'_i \to D''_{i-1}$ and $W_i\colon F_i \to D''_{i-1}$ be the induced maps by $\boldsymbol{M}_i$, $\boldsymbol{N}_i$ and $\boldsymbol{W}_i$ respectively. After elementary row operations we can turn Ω_i into the matrix

$$\begin{pmatrix} K_i & 0 \\ -M_i & I - \phi_{i-1} \end{pmatrix}.$$

Since K_i is nonsingular, Ω_i is also nonsingular and

$$\det \Omega_i = \det(I - \phi_{i-1}) \det K_i.$$

By Lemma 2.1.3 we have

$$\tau_\rho(X') = \prod_{i=1}^{d} [\det \Omega_i]^{(-1)^i},$$

which proves the lemma. □

For a positive integer k and $x, y \in \overline{\mathcal{K}_\theta((t^l))_{ab}^{\times}} / \pm \rho(\pi_1 X)$, we write

$$x \sim_k y$$

if there exist representatives $\sum_{i=0}^{\infty} a_i t^{li}, \sum_{i=0}^{\infty} b_i t^{li} \in \mathcal{K}_\theta((t))^\times$ of x, y respectively such that $a_0 b_0 \neq 0$ and $a_i = b_i$ for $i = 0, 1, \dots, k$. Note that $x = y$ if and only if for any positive integer k, $x \sim_k y$.

Lemma 2.3.4. *For any positive integer k, if we choose T_1 sufficiently fine and h sufficiently close to the identity, then*

$$\prod_{i=1}^{d} [\det(I - \phi_{i-1})]^{(-1)^i} \sim_k [\rho_*(\zeta_f)].$$

We prepare some notation and a lemma for the proof.

Let $\varphi\colon \Sigma \setminus \sqcup_p \mathcal{A}_0(p) \to \Sigma \setminus \sqcup_p \mathcal{D}_0(p)$ be the diffeomorphism defined by the downward gradient flows and let $H\colon \Sigma \times [0,1] \to \Sigma$ be the homotopy from id to h. We can consider the i times iterate maps φ^i and $(H(\cdot, t) \circ \varphi)^i$ for $t \in [0,1]$ which are partially defined. A natural compactification $\overline{\Gamma}^i_t$ of the graph $\Gamma^i_t \in \Sigma \times \Sigma$ of $(H(\cdot, t) \circ \varphi)^i$ is defined by attaching pairs $(x, H(y, t))$, where x is the starting point and y is the end point of a broken flow line of $-\nabla f_0$. (See [21], [22] for more details.)

It is known that there exists a positive integer N such that if the simplexes in T_1 are all contained in balls of radius ϵ, then H can be chosen so that the distance between x and $H(x, t)$ is

$< N\epsilon$ for all $x \in \Sigma$ and $t \in [0,1]$. (See for instance [48].) Since the set of fixed points of φ^i lies in the interior of $\overline{\Gamma}^i_0$ under the diagonal map $\Sigma \to \Sigma \times \Sigma$ and is compact from the nondegenerate assumption, it follows for any positive integer k that we can choose ϵ so that $\overline{\Gamma}^i_t$ does not cross the diagonal in $\Sigma \times \Sigma$ for all $i \le k$.

Lemma 2.3.5. *Let k be a positive integer and suppose that $l = 1$. Let $(a_{i,j})$ be the n-dimensional matrix over $\mathcal{K}_\theta((t))$ such that $a_{i,j} = c_{i,j}t$, where $c_{i,j} \in \mathcal{K}$. If $a_{1,i_1}a_{i_1,i_2}\cdots a_{i_{j-1},1} = 0$ for any sequence $i_1, i_2, \dots, i_{j-1}$ with $j \le k$, then*

$$[\det(\delta_{i,j} - a_{i,j})_{1\le i,j\le n}] \sim_k [\det(\delta_{i,j} - a_{i,j})_{2\le i,j\le n}],$$

where $\delta_{i,j}$ is Kronecker's delta.

Proof. We set

$$b^{(0)}_{i,j} = \delta_{i,j} - a_{i,j}$$

and inductively define $b^{(m)}_{i,j}$ for $m = 1, \dots, n$ as follows:

$$b^{(m)}_{i,j} = b^{(m-1)}_{i,j} - b^{(m-1)}_{i,m}(b^{(m-1)}_{m,m})^{-1}b^{(m-1)}_{m,j}.$$

This is an elementary row operation with respect to the ith row and $b^{(m)}_{i,j} = 0$ if $i \neq j \le m$. We have

$$[\det(b^{(0)}_{i,j})_{1\le i,j\le n}] = [\det(b^{(n)}_{i,j})_{1\le i,j\le n}] = \left[\prod_{i=1}^{n} b^{(n)}_{i,i}\right].$$

By induction on m we first show the following observation concerning any nonzero term in $b^{(m)}_{i,j} - \delta_{i,j}$:
(i) The term has positive degree.
(ii) The term has elements $a_{i,i_1}, a_{i_1,i_2}, \dots, a_{i_{j'-1},j}$ as factors for a sequence $i_1, i_2, \dots, i_{j'-1}$.
(iii) If the term has $a_{i',1}$ as a factor for some i', then the degree of the term is $> k$ or we can make such a sequence in (ii) contains 1.

It is easy to check them for $m = 0$. We assume them for $m = m' - 1$. Since

$$(b^{(m'-1)}_{m',m'})^{-1} = 1 + \sum_{i=1}^{\infty}(-1)^i(b^{(m'-1)}_{m',m'} - 1)^i,$$

(i) for $m = m'$ follows from (i) for $m = m' - 1$. By (ii) for $m = m' - 1$ we see at once (ii) for $m = m'$. We take any nonzero term c in $(b^{(m'-1)}_{m',m'})^{-1}$ which has $a_{i',1}$ as a factor. Then there is a nonzero term c' in $b^{(m'-1)}_{m',m'}$ which has $a_{i',1}$ as a factor such that c has c' as a factor. If c' has elements $a_{m',i_1}, a_{i_1,i_2}, \dots, a_{i_{j'-1},m'}$ as factors for a sequence $i_1, i_2, \dots, i_{j'-1}$ containing 1, then $j' \ge k$ by the assumption, and $\deg c' > k$. Hence by (ii) and (iii) for $m = m' - 1$, $\deg c \ge \deg c > k$. Now we can immediately check (iii) for $m = m'$.

As a consequence of the above argument the degree of any nonzero term in $b^{(n)}_{i,i}$ which has $a_{i',1}$ as a factor is $> k$, and the part of $\left[\prod_{i=1}^{n} b^{(n)}_{i,i}\right]$ up to degree k is invariant even if we erase such terms. Therefore in considering the equivalence class we can regard $a_{i,1}$ as 0 for all i, which deduce the lemma. □

Proof of Lemma 2.3.4. We only consider the case where $l = 1$. If $l = 0$, then $\phi_i = 0$ for all i and there is no closed orbit, and so there is nothing to prove. If $l > 1$, then we can prove it by a similar argument.

We set

$$\mathcal{I}_k := \{[o] \in O\ ;\ p(o) = 1, \deg f_*([o]) \le k\}.$$

Then we see that

$$[\rho_*(\zeta_f)] \sim_k \prod_{[o]\in\mathcal{I}_k} [1 - (-1)^{i_-(o)} \rho([\sigma_o o \bar{\sigma}_o])]^{(-1)^{i_0(o)+1}}. \tag{2.3}$$

For $[o] \in \mathcal{I}_k$, there is a sequence $x_0, x_1, \dots, x_{\deg f([o])-1}$ of fixed points of $\varphi^{\deg f_*([o])}$ such that $\varphi(x_{j-1}) = x_j$ for $j = 1, 2, \dots, \deg f([o])-1$. Choosing T_1 sufficiently fine and H as above, we can pick mutually disjoint contractible subcomplexes N_{x_j} of T_1 satisfying the following conditions:
(i) Each N_{x_j} contains all the fixed points of $(H(\cdot, t) \circ \varphi)^{\deg f_*([o])}$ which are close to x_j for all $t \in [0, 1]$.
(ii) There is a one-to-one correspondence between cells of $N_{x_{j-1}}$ and ones of N_{x_j} by $h \circ \varphi$ and $h \circ \varphi(N_{\deg f_*([o])-1}) \cap N_j = \emptyset$ for $j = 1, 2, \dots, \deg f_*([o]) - 1$.
(iii) If there is a sequence $\sigma = \sigma_0, \sigma_1, \dots, \sigma_i = \sigma$ for a simplex σ not contained in any N_{x_j} such that $\sigma_j \subset h \circ \varphi(\sigma_{j-1})$ for $j = 1, \dots, i$, then $i > k$.

We denote by $N_{[o]}$ and $N'_{[o]}$ for $[o] \in \mathcal{I}_k$ the union of all N_{x_j} for a fixed point $x_j \in o(S^1)$ and that of all N_{x_0} for such a sequence of $[o]$. Then we define

$$\phi_{[o],i} \colon C_i(\widetilde{N}_{[o]}) \otimes \mathcal{K}_\theta((t)) \to C_i(\widetilde{N}_{[o]}) \otimes \mathcal{K}_\theta((t))$$
$$\phi'_{[o],i} \colon C_i(\widetilde{N}'_{[o]}) \otimes \mathcal{K}_\theta((t)) \to C_i(\widetilde{N}'_{[o]}) \otimes \mathcal{K}_\theta((t))$$

to be the maps induced by $h \circ \varphi$ and $(h \circ \varphi)^{\deg f_*([o])}$ respectively.

Note that all the entries of ϕ_i are monomials. By condition (iii) we can apply Lemma 2.3.5 repeatedly and obtain

$$[\det(I - \phi_i)] \sim_k \prod_{[o]\in\mathcal{I}_k} [\det(I - \phi_{[o],i})]. \tag{2.4}$$

By condition (ii) we can take simplexes $\sigma_j \subset N_{x_j}$ such that $h \circ \varphi(\sigma_{j-1}) = \sigma_j$ for $j = 1, 2, \dots, \deg f([o]) - 1$. The matrix of the restriction of $I - \phi_{[o],i}$ on these simplexes has the form

$$\begin{pmatrix} 1 & 0 & \dots & 0 & m\rho(\gamma_{\deg f_*([o])}) \\ \pm\rho(\gamma_1) & 1 & \dots & 0 & 0 \\ 0 & \pm\rho(\gamma_2) & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \dots & \pm\rho(\gamma_{\deg f_*([o])-1}) & 1 \end{pmatrix},$$

where $m \in \mathbb{Z}$ and $\gamma_j \in \pi_1 X$. Since $\prod_{j=0}^{\deg f_*([o])-1} \gamma_{\deg f_*([o])-j} = [\sigma_o o \bar{\sigma}_o]$ for a path σ_o from the base point of X to x_0, the determinant of the matrix is $(1 - (\pm m\rho([\sigma_o o \bar{\sigma}_o]))$ and the coefficient of σ_0 of $\phi_{[o],i}(\sigma_0 \otimes 1)$ is $\pm m\rho([\sigma_o o \bar{\sigma}_o])$. According to the above argument, we have

$$[\det(I - \phi_{[o],i})] = [\det(I - \phi'_{[o],i})]. \tag{2.5}$$

Since the entries of the matrix $\phi'_{[o],i}$ are all in an abelian ring $\rho(\mathbb{Z}[[\sigma_o o \bar{\sigma}_o]])$,

$$\begin{aligned}\prod_{i=0}^{d-1}[\det(I-\phi'_{[o],i})] &= \left[\exp\sum_{j=0}^{\infty}\sum_{i=0}^{d-1}\frac{(-1)^i}{j}\operatorname{tr}(\phi'_{[o],i})^j\right]\\ &\sim_k \left[\exp\sum_{j=1}^{\infty}\frac{\operatorname{Fix}((\varphi|_{N'_{[o]}})^{j\deg f_*([o])})}{j}\rho([\sigma_o o\bar{\sigma}_o])^j\right]\\ &= \left[\exp\sum_{j=1}^{\infty}\frac{\epsilon(o^j)}{j}\rho([\sigma_o o\bar{\sigma}_o])^j\right]\\ &= [1-(-1)^{i_-(o)}\rho([\sigma_o o\bar{\sigma}_o])]^{(-1)^{i_0(o)+1}},\end{aligned}$$

where $\operatorname{Fix}((\varphi|_{N'_{[o]}})^{j\deg f_*([o])})$ counts fixed points of $(\varphi|_{N'_{[o]}})^{j\deg f_*([o])}$ with sign. The second equivalence follows from the machinery used to prove the Lefschetz fixed point theorem. From (2.3), (2.4), (2.5) and this the lemma is proved. □

Lemma 2.3.6. *For any positive integer k, if we choose T_1 sufficiently fine and h sufficiently close to the identity, then K_i is non-singular and*

$$\prod_{i=1}^{d}[\det K_i]^{(-1)^i}\sim_k[\tau_\rho^{Nov}(f)].$$

Proof. Suppose to begin that $h=id$. The unstable manifold $\mathcal{D}(p)$ of a critical point p of f has a natural compactification $\overline{\mathcal{D}(p)}$ such as $\mathcal{D}_0(p)$. The compactification $\overline{\mathcal{D}(p)}$ can be represented as

$$\overline{\mathcal{D}_0(p)}+\sum_{j=0}^{\infty}\overline{\mathcal{F}(\phi^j_{i-1}M_i(\overline{\mathcal{D}_0(p)}))},$$

where by abuse of notation we also denote by $\mathcal{F}$ the linear extension of $\mathcal{F}$. So if we identify D_i with $C_i^{Nov}(f)\otimes_{\Lambda_f}\mathcal{K}_\theta((t^l))$, then

$$\begin{aligned}\partial_i^f(\overline{\mathcal{D}_0(p)}) &= pr_{D_{i-1}}\circ\partial_i\left(\overline{\mathcal{D}_0(p)}+\sum_{j=0}^{\infty}\overline{\mathcal{F}(\phi^j_{i-1}M_i(\overline{\mathcal{D}_0(p)}))}\right)\\ &= \boldsymbol{K}_i(\overline{D_0(p)})\end{aligned}$$

for a critical point p with index i. Hence ∂_i^f induces $K_i\colon D'_i\to D''_{i-1}$, and K_i is nonsingular. From Lemma 2.1.3 we have

$$\begin{aligned}\prod_{i=1}^{d}[\det K_i]^{(-1)^i} &= \prod_{i=1}^{d}[\det pr_{D''_i}\circ\partial_i^f|_{D'_i}]^{(-1)^i}\\ &= [\tau_\rho^{Nov}(f)].\end{aligned}$$

Next we consider the case where $h\neq id$.

Let $pr_1, pr_2\colon\overline{\Gamma}_t^j\to\Sigma$ be the restriction of the first and second projections of $\Sigma\times\Sigma$. We define

$$B_t^j(p):=pr_2(pr_1^{-1}(H(\cdot,t)(\overline{\mathcal{D}_0(p)}\cap\Sigma_0)))$$

for $j=0,1,\ldots,k-1$, $t\in[0,1]$ and a critical point p of f. Since the set of the intersection points of $B_0^j(p)$ and $\mathcal{A}_0(q)\cap\Sigma_1$ for any critical point q lies the interior of $B_0^j(p)$ and is compact from the transverse condition, it follows that we can choose T_1 sufficiently fine and H as above so that $B_t^j(p)$ does not cross $\mathcal{A}(q) \cap \Sigma_1$ for all $j < k$, where we naturally identify Σ_1 with Σ.

By a similar argument to that of Lemma 2.3.4 we can check that the image of the hat of $\mathcal{F}(\phi_{i-1}^j \boldsymbol{M}_i(\widehat{\mathcal{D}_0(p)}))$ by $pr_{D_{i-1}} \circ \partial_i$ can be computed from the local intersection numbers of $B_1^j(p)$ and $\mathcal{A}_0(q) \cap \Sigma_1$ and the elements of $\pi_1 X$ determined by the perturbed flows by h from p to q, which are invariant on t for $j < k$. Hence from the computation of the case where $h = id$, we obtain

$$\partial_i^f(\widehat{\mathcal{D}_0(p)}) \sim_k \boldsymbol{K}_i(\widehat{\mathcal{D}_0(p)})$$

for all p with index i, and K_i is non-singular. Again from Lemma 2.1.3 we analogously see the desired relation. □

From the proofs of Lemmas 2.3.4 and 2.3.6, if we choose appropriate T_1 and H, then the conclusions of the lemmas simultaneously hold for any positive integer k, and so

$$\prod_{i=1}^{d} [\det(I - \phi_{i-1}) \det K_i]^{(-1)^i} \sim_k [\rho_*(\zeta_f)][\tau_\rho^{Nov}(f)].$$

Now we can establish Theorem 2.2.7 at once from Lemma 2.3.2 and 2.3.3.

3. Homology cylinders of higher-order

3.1. Definitions.

3.1.1. *The monoids of homology cylinders of higher-order.* We begin with introducing the monoids of homology cylinders of higher-order, which give a filtration of the monoid of ordinary homology cylinders introduced in [16], [17]. See [18], [45] for more details on homology cylinders.

To simplify notation we often write Σ, π instead of $\Sigma_{g,n}, \pi_1\Sigma_{g,n}$, respectively. We set $\Gamma_m := \pi/\pi^{(m+1)}$ for each $m \geq 0$.

Definition 3.1.1. For an integer $m \geq 0$, a *homology cylinder* $(M, i_\pm)$ of order m over Σ is defined to be a compact oriented 3-manifold M together with embeddings $i_+, i_- \colon \Sigma \to \partial M$ satisfying the following:

(i) i_+ is orientation preserving and i_- is orientation reversing,

(ii) $\partial M = i_+(\Sigma) \cup i_-(\Sigma)$ and $i_+(\Sigma) \cap i_-(\Sigma) = i_+(\partial M) = i_-(\partial M)$,

(iii) $i_+|_{\partial\Sigma} = i_-|_{\partial\Sigma}$,

(iv) $(i_+)_*, (i_-)_* \colon \Gamma_m \to \pi_1 M/(\pi_1 M)^{(m+1)}$ are isomorphisms.

Two homology cylinders $(M, i_\pm), (N, j_\pm)$ are called isomorphic if there exists an orientation preserving homeomorphism $f \colon M \to N$ satisfying $j_\pm = f \circ i_\pm$. We denote by $C_{g,n}^{(m)}$ the set of all isomorphism classes of homology cylinders of order m over $\Sigma_{g,n}$.

Remark 3.1.2. By the Hurewicz theorem and a standard argument on homology groups it follows from the condition (iv) that $(i_+)_*, (i_-)_* \colon H_*(\Sigma) \to H_*(M)$ are isomorphism. In particular, a homology cylinder of order 0 is nothing but an ordinary homology cylinder, i.e., $C_{g,n}^{(0)} = C_{g,n}$.

For $\psi \in \mathcal{M}_{g,n}$, we define $M(\psi)$ to be the homology cylinder $\Sigma \times [0,1]/\sim$ (of order m for all m) equipped with $(i_+ = id \times 1, i_- = \psi \times 0)$, where $(x, s) \sim (x, t)$ for $x \in \partial\Sigma$ and $s, t \in [0,1]$. A product operation on $C_{g,n}^{(m)}$ is given by stacking:

$$(M, i_\pm) \cdot (N, j_\pm) := (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-),$$

which turns $\mathcal{C}_{g,n}^{(m)}$ into a monoid from the following lemma. The unit is given by $M(id)$. For all m, $\mathcal{C}_{g,n}^{(m+1)}$ is a submonoid of $\mathcal{C}_{g,n}^{(m)}$. The correspondence $\psi \mapsto M(\psi)$ gives a monoid homomorphism $\mathcal{M}_{g,n} \to \mathcal{C}_{g,n}^{(m)}$ for each m.

Lemma 3.1.3. *For $(M, i_\pm), (N, j_\pm) \in \mathcal{C}_{g,n}^{(m)}$, $(M, i_\pm) \cdot (N, j_\pm) \in \mathcal{C}_{g,n}^{(m)}$.*

Proof. We only need to check that $(M, i_\pm) \cdot (N, j_\pm)$ satisfies the condition (iv).

By the van Kampen theorem $\pi_1(M \cup_{i_- \circ (j_+)^{-1}} N) = \pi_1 M *_\pi \pi_1 N$. Let Π be the subgroup normally generated by elements of $(\pi_1 M)^{(m+1)} *_{\pi^{(m+1)}} (\pi_1 N)^{(m+1)}$. Then

$$(\pi_1 M *_\pi \pi_1 N)/\Pi \cong (\pi_1 M/(\pi_1 M)^{(m+1)}) *_{\Gamma_m} (\pi_1 N/(\pi_1 N)^{(m+1)}) \cong \Gamma_m.$$

Since Π is a subgroup of $(\pi_1 M *_\pi \pi_1 N)^{(m+1)}$, $(i_+)_*, (j_-)_* \colon \Gamma_m \to (\pi_1 M *_\pi \pi_1 N)/(\pi_1 M *_\pi \pi_1 N)^{(m+1)}$ are isomorphisms. □

The following lemma can be seen at once from the definition and the observation that $\mathcal{C}_{0,0}^{(0)}$ and $\mathcal{C}_{0,1}^{(0)}$ are naturally isomorphic to the monoid θ_3 of integral homology 3-spheres with the connected sum operation. See Theorem 3.3.4 for the other cases.

Lemma 3.1.4. *(i) $\mathcal{C}_{0,0}^{(m)} = \mathcal{C}_{0,1}^{(m)} = \mathcal{C}_{0,0}^{(0)} = \mathcal{C}_{0,1}^{(0)} = \theta_3$ for $m \geq 0$.*
(ii) $\mathcal{C}_{0,2}^{(m)} = \mathcal{C}_{0,2}^{(1)}$ for $m \geq 1$.
(iii) $\mathcal{C}_{1,0}^{(m)} = \mathcal{C}_{1,0}^{(1)}$ for $m \geq 1$.

The same argument as the proof of [15, Proposition 2. 4] gives the following proposition.

Proposition 3.1.5. *The monoid $\mathcal{C}_{g,n}^{(m)}$ is not finitely generated.*

In fact, we have an epimorphism $F\colon \mathcal{C}_{g,n}^{(m)} \to \theta_3$ as follows. For $(M, i_\pm) \in \mathcal{C}_{g,n}^{(m)}$, we can write $M = M' \sharp M''$, where M' is the prime factor of M containing ∂M. Then $F(M, i_\pm) := M''$. Therefore as pointed out in [15] it is reasonable to consider the submonoid $\overline{\mathcal{C}}_{g,n}^{(m)}$ consisting of all $(M, i_\pm) \in \mathcal{C}_{g,n}^{(m)}$ with irreducible M. Note that $\overline{\mathcal{C}}_{0,0}^{(m)} = \emptyset$ and $\overline{\mathcal{C}}_{0,1}^{(m)} = 1$ for all m.

Let $\zeta_i \in \pi$ be a representative of each boundary circle. We denote by $\mathrm{Out}^*(\Gamma_m)$ the group of outer automorphisms of Γ_m which preserve the conjugacy class of $[\zeta_i] \in \Gamma_m$ for all i. A homomorphism $\varphi_m \colon \mathcal{C}_{g,n}^{(m)} \to \mathrm{Out}^*(\Gamma_m)$ is given by

$$\varphi_m(M, i_\pm) := [(i_+)_*^{-1} \circ (i_-)_*].$$

It is easily seen that φ_m does not depend on the choices of a base point and ζ_i. We define

$$\mathcal{IC}_{g,n}^{(m)} := \operatorname{Ker} \varphi_m,$$
$$\mathcal{I}\overline{\mathcal{C}}_{g,n}^{(m)} := \operatorname{Ker} \varphi_m|_{\overline{\mathcal{C}}_{g,n}^{(m)}}.$$

Remark 3.1.6. In [14, Proposition 2. 3] Goda and Sakasai showed that $\phi_0(M, i_\pm)$ preserves the intersection form of Σ for all homology cylinders $(M, i_\pm)$.

3.1.2. *The homology cobordism groups of homology cylinders of higher-order.* Next we consider homology cobordisms for homology cylinders of higher-order. See [11], [27] for the case of ordinary homology cylinders.

Definition 3.1.7. For $(M, i_\pm), (N, j_\pm) \in C_{g,n}^{(m)}$, we write $(M, i_\pm) \sim_m (N, j_\pm)$ if there exists a compact oriented smooth 4-manifold satisfying the following:

(i) $\partial W = M \cup_{i_+\circ j_+^{-1}, i_-\circ j_-^{-1}} (-N)$,

(ii) $H_*(M) \to H_*(W), H_*(N) \to H_*(W)$ are isomorphisms,

(iii) $\pi_1 M/(\pi_1 M)^{(m+1)} \to \pi_1 W/(\pi_1 W)^{(m+1)}, \pi_1 N/(\pi_1 N)^{(m+1)} \to \pi_1 W/(\pi_1 W)^{(m+1)}$ are isomorphisms.

Lemma 3.1.8. *The relation $\sim_m$ is an equivalence relation on $C_{g,n}^{(m)}$ which is compatible with the product operation.*

For the proof in the case where $m = 0$, we refer to [27, p. 246]. Using an almost same technique as in the proof of Lemma 3.1.3, we can also prove the lemma in the general case, and so we omit the proof.

We define $\mathcal{H}_{g,n}^{(m)} := C_{g,n}^{(m)}/\sim_m$, which has a natural group structure induced by the monoid structure of $C_{g,n}^{(m)}$. The inverse of $[M, i_\pm] \in \mathcal{H}_{g,n}^{(m)}$ is given by $[-M, i_\mp]$. There is a natural homomorphism $\mathcal{H}_{g,n}^{(m+1)} \to \mathcal{H}_{g,n}^{(m)}$ for each m.

Remark 3.1.9. The group $\mathcal{H}_{g,n}^{(0)}$ is nothing but the smooth homology cobordism group $\mathcal{H}_{g,n}$ of ordinary homology cylinders. We can see that $\mathcal{H}_{0,0}^{(0)}$ and $\mathcal{H}_{0,1}^{(0)}$ are isomorphic to the smooth homology cobordism group Θ_3 of integral homology 3-spheres, and that $\mathcal{H}_{0,2}^{(0)}$ is isomorphic to $\mathbb{Z} \oplus C_\mathbb{Z}$, where $C_\mathbb{Z}$ is the smooth concordance group of knots in integral homology 3-spheres. The $\mathbb{Z}$ factor comes from framings of knots.

We can also consider topological homology cobordisms instead of smooth ones in Definition 3.1.7. Let $\mathcal{H}_{g,n}^{(m)\ \mathrm{top}}$ be the topological homology cobordism group of homology cylinders of order m. Since we consider only topological methods, in fact, we can obtain analogous results on $\mathcal{H}_{g,n}^{(m)\ \mathrm{top}}$ to all the theorems on $\mathcal{H}_{g,n}^{(m)}$ in this paper.

Using the results of Freedman [8], Furuta [10] and Fintushel and Stern [7], Cha-Friedl-Kim [3, Theorem 1. 1] showed that the kernel of the epimorphism $\mathcal{H}_{g,n}^{(0)} \to \mathcal{H}_{g,n}^{(0)\ \mathrm{top}}$ contains an abelian group of infinite rank, which is given by the image of a homomorphism $\Theta_3 \to \mathcal{H}_{g,n}^{(0)}$. Since the homomorphism $\Theta_3 \to \mathcal{H}_{g,n}^{(0)}$ factors through $\mathcal{H}_{g,n}^{(m)}$, the kernel of the epimorphism $\mathcal{H}_{g,n}^{(m)} \to \mathcal{H}_{g,n}^{(m)\ \mathrm{top}}$ also contains an abelian group of infinite rank.

The proof of the following lemma is straightforward from the definition and Lemma 3.1.4. See Theorem 3.2.11 for general cases.

Lemma 3.1.10. *(i) $\mathcal{H}_{0,0}^{(m)} = \mathcal{H}_{0,1}^{(m)} = \mathcal{H}_{0,0}^{(0)} = \mathcal{H}_{0,1}^{(0)} = \Theta_3$ for $m \geq 0$.*

(ii) $\mathcal{H}_{0,2}^{(m)} = \mathcal{H}_{0,2}^{(1)}$ for $m \geq 1$.

(iii) $\mathcal{H}_{1,0}^{(m)} = \mathcal{H}_{1,0}^{(1)}$ for $m \geq 1$.

Proposition 3.1.11 ([3], [11], [27]). *The homomorphism $\mathcal{M}_{g,n} \to C_{g,n}^{(0)} \to \mathcal{H}_{g,n}^{(0)}$ is injective.*

Since this injection factors through $\mathcal{H}_{g,n}^{(m)}$, we obtain the following corollary.

Corollary 3.1.12. *The homomorphism $\mathcal{M}_{g,n} \to C_{g,n}^{(m)} \to \mathcal{H}_{g,n}^{(m)}$ is injective for all m.*

The homomorphism $\varphi_m \colon C_{g,n}^{(m)} \to \mathrm{Out}^*(\Gamma_m)$ induces a homomorphism $\mathcal{H}_{g,n}^{(m)} \to \mathrm{Out}^*(\Gamma_m)$. By abuse of notation we also write φ_m for the homomorphism. We define

$$\mathcal{IH}_{g,n}^{(m)} := \mathrm{Ker}(\varphi_m \colon \mathcal{H}_{g,n}^{(m)} \to \mathrm{Out}^*(\Gamma_m)).$$

3.2. **Reidemeister torsion homomorphisms.**

3.2.1. *Torsion of homology cylinders of higher-order.* First we define non-commutative torsion invariants of homology cylinders of higher-order. Torsion invariants of homology cylinders were first studied by Sakasai [43, 44].

Lemma 3.2.1. *For a CW-pair (X, Y) and a homomorphism $\rho\colon \pi_1 M \to \Gamma$ to a PTFA group Γ, if $H_*(X, Y; \mathbb{Q}) = 0$, then $H_*^{\rho}(X, Y; \mathbb{Q}(\Gamma)) = 0$.*

See [5, Proposition 2. 10] for a proof.

For $(M, i_\pm) \in C_{g,n}^{(m)}$, we denote by ρ_m the pullback $\pi_1 M \to \Gamma_m$ of $(i_+)_*^{-1}\colon \pi_1 M/(\pi M)^{(m+1)} \to \Gamma_m$. It is well-known that Γ_m is torsion-free for all m, and hence Γ_m is PTFA for all m. It follows from the above lemma and Remark 3.1.2 that $H_*^{\rho_m}(M, i_+(\Sigma); \mathbb{Q}(\Gamma_m)) = 0$.

Definition 3.2.2. We define a map $\tau_m\colon C_{g,n}^{(m)} \to \mathbb{Q}(\Gamma_m)_{ab}^{\times}/\pm\Gamma_m$ by $\tau_m(M, i_\pm) := \tau_{\rho_m}(M, i_+(\Sigma))$.

It can be easily computed that for all $\psi \in \mathcal{M}_{g,n}$ and all m, $\tau_m(M(\psi)) = 1$.

We denote by $\mathrm{Wh}(\Gamma) := K_1(\mathbb{Z}[\Gamma])/\pm\Gamma$ the Whitehead group of a group Γ. The Dieudonné determinant induces a homomorphism $\mathrm{Wh}(\Gamma_k) \to \mathbb{Q}(\Gamma)_{ab}^{\times}/\pm\Gamma_m$.

Proposition 3.2.3. *For all $(M, i_\pm) \in C_{g,n}^{(m)}$ and all $k < m$, $\tau_k(M, i_\pm)$ is in the image of* $\mathrm{Wh}(\Gamma_k) \to \mathbb{Q}(\Gamma)_{ab}^{\times}/\pm\Gamma_m$. *Furthermore, for all $(M, i_\pm) \in C_{g,n}^{(0)}$, $(M, i_\pm) \in C_{g,n}^{(1)}$ if and only if $\tau_0(M, i_\pm) = 1$.*

Proof. First we suppose that $(M, i_\pm) \in C_{g,n}^{(m)}$. Since M can be obtained by attaching 1-handles and 2-handles of same numbers to $\Sigma \times [0, 1]$, $C_*^{\rho_k}(M, i_+(\Sigma); \mathbb{Z}[\Gamma_k])$ is simple homotopy equivalent to a chain complex

$$0 \to C_2 \xrightarrow{\partial} C_1 \to 0$$

with $\operatorname{rank} C_1 = \operatorname{rank} C_2$ for $k \le m$. Hence $\tau_k(M, i_\pm) = [\det\partial]$ for $k \le m$.

Since $(i_+)*\colon \pi^{(k+1)}/\pi^{(k+2)} \to (\pi_1 M)^{(k+1)}/(\pi_1 M)^{(k+2)}$ is an isomorphism for $k < m$, $H_1^{\rho_k}(i_+(\Sigma); \mathbb{Z}[\Gamma_k]) \to H_1^{\rho_k}(M; \mathbb{Z}[\Gamma_k])$ is also an isomorphism for $k < m$. From the long exact homology sequence for $(M, i_+(\Sigma))$, $H_1^{\rho_k}(M, i_+(\Sigma); \mathbb{Z}[\Gamma_k]) = 0$ for $k < m$. Therefore ∂ is a surjection, and so an isomorphism for $k < m$, which proves the first statement.

Now the necessary condition in the second statement follows from the result by [1] that for any free abelian group Γ, $\mathrm{Wh}(\Gamma) = 1$.

Next we suppose that $\tau_0(M, i_\pm) = 1$ for $(M, i_\pm) \in C_{g,n}^{(0)}$. Since $\det\partial \in \pm H_1(\Sigma)$, ∂ is an isomorphism for $k = 0$. Hence $H_*^{\rho_0}(M, i_+(\Sigma); \mathbb{Z}[H_1(\Sigma)]) = 0$. From the Poincaré duality and the universal coefficient theorem, we have $H_*^{\rho_0}(M, i_-(\Sigma); \mathbb{Z}[H_1(\Sigma)]) = 0$. From the long exact sequence for $(M, i_\pm(\Sigma))$, $H_1^{\rho_0}(i_\pm(\Sigma); \mathbb{Z}[H_1(\Sigma)]) \to H_1^{\rho_0}(M; \mathbb{Z}[H_1(\Sigma)])$ are isomorphisms, and so $(i_\pm)_*\colon \pi^{(1)}/\pi^{(2)} \to (\pi_1 M)^{(1)}/(\pi_1 M)^{(2)}$ are also isomorphisms. Therefore $(i_\pm)_*\colon \pi/\pi^{(2)} \to \pi_1 M/(\pi_1 M)^{(2)}$ are isomorphisms, which gives the sufficient condition in the second statement. □

Remark 3.2.4. Though it is a well-known conjecture that for any finitely generated torsion-free group Γ, $\mathrm{Wh}(\Gamma) = 1$, to author's knowledge there seems to be no appropriate reference on whether $\mathrm{Wh}(\Gamma_m) = 1$ for $m > 0$.

Each $\varphi \in \mathrm{Out}^*(\Gamma_m)$ induces an automorphism of $\mathbb{Q}(\Gamma_m)_{ab}^{\times}/\pm\Gamma_m$, which we also denote by φ. The following proposition is an extension of [3, Proposition 3. 5]. See also [44, Proposition 6. 6] for a related result.

Proposition 3.2.5. *For* $(M,i_\pm),(N,j_\pm)\in C_{g,n}^{(m)}$,

$$\tau_m((M,i_\pm)\cdot(N,j_\pm))=\tau_m(M,i_\pm)\cdot\varphi_m(M,i_\pm)(\tau_m(N,j_\pm)).$$

Proof. In all of the calculations below, we implicitly tensor the chain complexes with $\mathbb{Q}(\Gamma_m)$. We write $W:=M\cup_{i_-\circ(j_+)^{-1}}N$. We have the following short exact sequences:

$$0\to C_*(\widetilde{j_+(\Sigma)})\to C_*(\widetilde{M},\widetilde{i_+(\Sigma)})\oplus C_*(\widetilde{N})\to C_*(\widetilde{W},\widetilde{i_+(\Sigma)})\to 0,$$
$$0\to C_*(\widetilde{j_+(\Sigma)})\to C_*(\widetilde{N})\to C_*(\widetilde{N},\widetilde{j_+(\Sigma)})\to 0,$$

where we consider the homomorphisms

$$\rho\colon \pi_1W\to\pi_1W/(\pi_1W)^{(m+1)}\xrightarrow{(i_+)_*^{-1}}\Gamma_m,$$
$$\rho'\colon \pi_1N\to\pi_1N/(\pi_1N)^{(m+1)}\xrightarrow{\sim}\pi_1W/(\pi_1W)^{(m+1)}\xrightarrow{(i_+)_*^{-1}}\Gamma_m$$

respectively. It follows from the long exact homology sequence that the inclusion map $j_+(\Sigma)\to N$ induces an isomorphism $H_*^{\rho'}(j_+(\Sigma);\mathbb{Q}(\Gamma_m))\to H_*^{\rho'}(N;\mathbb{Q}(\Gamma_m))$. Let $\boldsymbol{c},\boldsymbol{c}'$ be bases of $C_*(\widetilde{N}),C_*(\widetilde{j_+(\Sigma)})$ consisting of cells and let $\boldsymbol{h},\boldsymbol{h}'$ be bases of $H_*^{\rho'}(N;\mathbb{Q}(\Gamma_m)),H_*^{\rho'}(j_+(\Sigma);\mathbb{Q}(\Gamma_m))$ such that $\boldsymbol{h}$ is the image of $\boldsymbol{h}'$ by the isomorphism. By Lemma 2.1.2 we obtain the following equations:

$$\tau_\rho(M,i_+(\Sigma))\cdot[\tau(C_*(\widetilde{N}),\boldsymbol{c},\boldsymbol{h})]=[\tau(C_*(\widetilde{j_+(\Sigma)}),\boldsymbol{c}',\boldsymbol{h}')]\cdot\tau_\rho(W,i_+(\Sigma)),$$
$$[\tau(C_*(\widetilde{N}),\boldsymbol{c},\boldsymbol{h})]=[\tau(C_*(\widetilde{j_+(\Sigma)}),\boldsymbol{c}',\boldsymbol{h}')]\cdot\tau_{\rho'}(N,j_+(\Sigma)).$$

Hence

$$\tau_\rho(W,i_+(\Sigma))=\tau_\rho(M,i_+(\Sigma))\cdot\tau_{\rho'}(N,j_+(\Sigma)).$$

By the functoriality of Reidemeister torsion $\tau_{\rho'}(N,j_+(\Sigma))=\varphi_m(M,i_\pm)(\tau_m(N,j_\pm))$, which establishes the formula. □

Corollary 3.2.6. *The map* $\tau_m\rtimes\varphi_m\colon C_{g,n}^{(m)}\to(\mathbb{Q}(\Gamma_m)_{ab}^\times/\pm\Gamma_m)\rtimes\mathrm{Out}^*(\Gamma_m)$ *is a homomorphism.*

Corollary 3.2.7. *The map* $\tau_m\colon \mathcal{IC}_{g,n}^{(m)}\to\mathbb{Q}(\Gamma_m)_{ab}^\times/\pm\Gamma_m$ *is a homomorphism.*

3.2.2. *Torsion and homology cobordisms.* We denote by $\bar{\cdot}\colon\mathbb{Z}[\Gamma_m]\to\mathbb{Z}[\Gamma_m]$ the involution defined by $\bar{\gamma}=\gamma^{-1}$ for $\gamma\in\Gamma_m$ and naturally extend it to $\mathbb{Q}(\Gamma_m)$ for each m.

The following theorem is an extension of [3, Theorem 3. 10].

Theorem 3.2.8. *Let* $(M,i_\pm),(N,j_\pm)\in C_{g,n}^{(m)}$. *If* $(M,i_\pm)\sim_m(N,j_\pm)$, *then*

$$\tau_m(M,i_\pm)=\tau_m(N,j_\pm)\cdot q\cdot\bar{q}$$

for some $q\in\mathbb{Q}(\Gamma)_{ab}^\times/\pm\Gamma_m$.

Proof. We pick a homology cobordism W between M and N as in Definition 3.1.7. Let ρ be the homomorphism $\pi_1W\to\pi_1W/(\pi_1W)^{(m+1)}\xrightarrow{(i_+)_*^{-1}}\Gamma_m$. The long exact homology sequences for $(W,M),(W,N),(W,i_+(\Sigma))$ give $H_*(W,M)=H_*(W,N)=H_*(W,i_+(\Sigma))=0$. By Lemma 3.2.1 we obtain $H_*^\rho(W,M;\mathbb{Q}(\Gamma_m))=H_*^\rho(W,N;\mathbb{Q}(\Gamma_m))=H_*^\rho(W,i_+(\Sigma);\mathbb{Q}(\Gamma_m))=0$. By applying Lemma 2.1.2 to the following exact sequence

$$0\to C_*(\widetilde{M},\widetilde{i_+(\Sigma)})\otimes\mathbb{Q}(\Gamma_m)\to C_*(\widetilde{W},\widetilde{i_+(\Sigma)})\otimes\mathbb{Q}(\Gamma_m)\to C_*(\widetilde{W},\widetilde{M})\otimes\mathbb{Q}(\Gamma_m)\to 0,$$

we get

$$\tau_\rho(W,i_+(\Sigma))=\tau_\rho(M,i_+(\Sigma))\cdot\tau_\rho(W,M).$$

Similarly,

$$\tau_\rho(W, i_+(\Sigma)) = \tau_\rho(N, j_+(\Sigma)) \cdot \tau_\rho(W, N).$$

By the duality of Reidemeister torsion

$$\tau_\rho(W, M) = \overline{\tau_\rho(W, N)}^{-1}$$

(e.g., see [2, 25, 29]). Hence

$$\tau_\rho(M, i_+(\Sigma)) = \tau_\rho(N, j_+(\Sigma)) \cdot \tau_\rho(W, N) \cdot \overline{\tau_\rho(W, N)},$$

which proves the theorem. □

We set

$$N_m := \{\pm\gamma \cdot q \cdot \bar{q} \in \mathbb{Q}(\Gamma_m)^\times_{ab}\ ;\ \gamma \in \Gamma_m, q \in \mathbb{Q}(\Gamma_m)^\times_{ab}\}.$$

Corollary 3.2.9. *The map $\tau_m \rtimes \varphi_m : \mathcal{H}^{(m)}_{g,n} \to (\mathbb{Q}(\Gamma_m)^\times_{ab}/N_m) \rtimes \mathrm{Out}^*(\Gamma_m)$ is a homomorphism.*

Corollary 3.2.10. *The map $\tau_m : \mathcal{IH}^{(m)}_{g,n} \to \mathbb{Q}(\Gamma_m)^\times_{ab}/N_m$ is a homomorphism.*

The following theorem showed that if $(g, n) \neq (0, 0), (0, 1)$ and $m > 0$, then $\mathcal{H}^{(m)}_{g,n}$ is another enlargement of $\mathcal{M}_{g,n}$.

Theorem 3.2.11. *If $(g, n) \neq (0, 0), (0, 1)$, then the homomorphisms $\mathcal{H}^{(m)}_{g,n} \to \mathcal{H}^{(0)}_{g,n}, \mathcal{IH}^{(m)}_{g,n} \to \mathcal{IH}^{(0)}_{g,n}$ are not surjective for $m > 0$.*

Proof. By Proposition 3.2.3 the image of the composition of $\mathcal{H}^{(m)}_{g,n} \to \mathcal{H}^{(0)}_{g,n}$ and $\tau_0 \rtimes \varphi_0 : \mathcal{H}^{(0)}_{g,n} \to (\mathbb{Q}(\Gamma_0)^\times_{ab}/N_0) \rtimes \mathrm{Out}^*(\Gamma_0)$ is contained in $1 \times \mathrm{Out}^*(\Gamma_0)$ and that of $\mathcal{IH}^{(m)}_{g,n} \to \mathcal{IH}^{(0)}_{g,n}$ and $\tau_0 : \mathcal{IH}^{(0)}_{g,n} \to \mathbb{Q}(\Gamma_0)^\times_{ab}/N_0$ is trivial. On the other hand, in [3] Cha, Friedl and Kim detected elements of the image of $\tau_0 \rtimes \varphi_0 : \mathcal{H}^{(0)}_{g,n} \to (\mathbb{Q}(\Gamma_0)^\times_{ab}/N_0) \rtimes \mathrm{Out}^*(\Gamma_0)$ not contained in $1 \times \mathrm{Out}^*(\Gamma_0)$ and nontrivial ones of $\tau_0 : \mathcal{IH}^{(0)}_{g,n} \to \mathbb{Q}(\Gamma_0)^\times_{ab}/N_0$ when $(g, n) \neq (0, 0), (0, 1)$. These prove the theorem. □

Remark 3.2.12. It is an important question whether the homomorphisms $\mathcal{H}^{(m)}_{g,n} \to \mathcal{H}^{(0)}_{g,n}, \mathcal{IH}^{(m)}_{g,n} \to \mathcal{IH}^{(0)}_{g,n}$ are in general injective or not.

3.3. **Construction and computation.** For $\gamma \in \pi$ and a tame knot $K \subset S^3$, we construct a homology cylinder $M(\gamma, K)$ as follows. See [3, Section 4] for various constructions of homology cylinders.

Let $* \in \Sigma$ be the base point for π. We choose a smooth path $f : [0, 1] \to \Sigma$ representing γ such that $f^{-1}(*) = \{0, 1\}$, and define $\tilde{f} : [0, 1] \to \Sigma \times [0, 1]$, $c : [0, 1] \to \Sigma \times [0, 1]$ by $\tilde{f}(t) = (f(t), t)$ and $c(t) = (*, 1 - t)$. After pushed into the interior, $\tilde{f} \cdot c$ determines a tame knot $J \subset \mathrm{Int}\, M(id)$. We take a framing of J representing the conjugacy class of the image of γ by $\pi \to \pi_1(M(id) \setminus J)$. Let E_J, E_K be the exteriors of J, K. Now $M(\gamma, K)$ is the result of attaching $-E_K$ to E_J along the boundaries so that a longitude and a meridian of K correspond to a meridian and a longitude of J respectively. Note that if $(g, n) = (0, 0)$ or $(0, 1)$, then $M(1, K) = M(id)$ for all K.

Proposition 3.3.1. *If $(g, n) \neq (0, 0), (0, 1)$ and $\gamma \in \pi^{(m)}$, then $M(\gamma, K) \in \overline{C}^{(m)}_{g,n}$ for all K.*

Proof. It follows from irreducibility of E_J and E_K that $M(\gamma, K)$ is also irreducible.

Extending a degree 1 map $(E_K, \partial E_K) \to (D^2 \times S^1, \partial D^2 \times S^1)$ by the identity map on E_J, we have $f\colon M(\gamma, K) \to M(id)$. We show that $f_*\colon \pi_1 M(\gamma, K)/(\pi_1 M(\gamma, K))^{(m+1)} \to \pi_1 M(id)/(\pi_1 M(id))^{(m+1)}$ is an isomorphism, which immediately gives the desired conclusion.

Let $\lambda_J, \mu_J \in \pi_1 E_J$ and $\lambda_K, \mu_K \in \pi_1 E_K$ be longitude-meridian pairs. By the van Kampen theorem $\pi_1 M(\gamma, K)$ is the amalgamated product of $\pi_1 E_J$ and $\pi_1 E_K$ with $\lambda_J = \mu_K$ and $\mu_J = \lambda_K$, and $\pi_1 M(id)$ is that of $\pi_1 E_J$ and $\langle t \rangle$ with $\lambda_J = t$ and $\mu_J = 1$. Here $f_*\colon \pi_1 M(\gamma, K) \to \pi_1 M(id)$ is the identity map on $\pi_1 E_J$ and is the Hurewicz map on $\pi_1 E_K$. Hence the kernel is the normal closure of $(\pi_1 E_K)^{(1)}$ in $\pi_1 M(\gamma, K)$. Thus it is suffice to show that $\pi_1 E_K \subset (\pi_1 M(\gamma, K))^{(m)}$. Since $\pi_1 E_K$ is normally generated by μ_K, it suffice to show that $\mu_K \in (\pi_1 M(\gamma, K))^{(m)}$.

Suppose that $\mu_K \in (\pi_1 M(\gamma, K))^{(k)}$ for an integer $k < m$. Since $\gamma \in \pi^{(m)}$, a longitude of J bounds a map from a symmetric m-stage grope in $M(id)$ such that the grope stages meet J transversely (e.g., see [6]). Hence it bounds a map from a punctured symmetric m-stage grope in E_J, where the boundaries of these punctures are meridians of E_J. Therefore for some $\xi_i \in \pi_1 E_J$,

$$\lambda_J = \prod_i \xi_i \mu_J^{\pm 1} \xi_i^{-1} \prod_j [a_j, b_j],$$

where representatives of $a_j, b_j \in \pi_1 E_J$ bounds a map from punctured $(m-1)$-stage gropes in E_J. Hence a_j, b_j have similar expressions as λ_J. Continuing in this fashion, we see from $\mu_J = \lambda_K \in (\pi_1 M(\gamma, K))^{(k+1)}$ that $\mu_K = \lambda_J \in (\pi_1 M(\gamma, K))^{(k+1)}$. It follows by induction that $\mu_K \in (\pi_1 M(\gamma, K))^{(m)}$. □

Proposition 3.3.2. *Let $\gamma \in \pi^{(m)}$. Then $\tau_m(M(\gamma, K)) = [\Delta_K(\gamma)]$ for all K.*

Proof. In all of the calculations below, we implicitly tensor the chain complexes with $\mathbb{Q}(\Gamma_m)$.

First we suppose that $[\gamma] = 1 \in \Gamma_m$. We have the following short exact sequences:

$$0 \to C_*(\widetilde{\partial E_K}) \to C_*(\widetilde{E_J}, \widetilde{i_+(\Sigma)}) \oplus C_*(\widetilde{E_K}) \to C_*(\widetilde{M(\gamma, K)}, \widetilde{i_+(\Sigma)}) \to 0,$$

$$0 \to C_*(\widetilde{\partial D^2 \times S^1}) \to C_*(\widetilde{E_J}, \widetilde{i_+(\Sigma)}) \oplus C_*(\widetilde{D^2 \times S^1}) \to C_*(\widetilde{M(id)}, \widetilde{i_+(\Sigma)}) \to 0,$$

where we consider $\rho_m\colon \pi_1 M(\gamma, K) \to \Gamma_m$, $\rho'_m\colon \pi_1 M(id) \to \Gamma_m$. Let $f\colon M(\gamma, K) \to M(id)$ be the map taken in the proof of Proposition 3.3.1. It is easily seen that the induced maps $H_*^{\rho_m}(\partial E_K; \mathbb{Q}(\Gamma_m)) \to H_*^{\rho'_m}(\partial D^2 \times S^1; \mathbb{Q}(\Gamma_m))$, $H_*^{\rho_m}(E_K; \mathbb{Q}(\Gamma_m)) \to H_*^{\rho'_m}(D^2 \times S^1; \mathbb{Q}(\Gamma_m))$ are isomorphisms. We pick bases $\boldsymbol{h}, \boldsymbol{h}', \boldsymbol{h}''$ of $H_*^{\rho_m}(\partial E_K; \mathbb{Q}(\Gamma_m))$, $H_*^{\rho_m}(E_J, i_+(\Sigma); \mathbb{Q}(\Gamma_m))$, $H_*^{\rho_m}(E_K; \mathbb{Q}(\Gamma_m))$ respectively such that the isomorphism $H_*^{\rho_m}(\partial E_K; \mathbb{Q}(\Gamma_m)) \to H_*^{\rho_m}(E_J, i_+(\Sigma); \mathbb{Q}(\Gamma_m)) \oplus H_*^{\rho_m}(E_K; \mathbb{Q}(\Gamma_m))$ maps $\boldsymbol{h}$ to $\boldsymbol{h}' \oplus \boldsymbol{h}''$. By Lemma 2.1.2 we obtain

$$[\tau(C_*(\widetilde{E_J}, \widetilde{i_+(\Sigma)}), \boldsymbol{h}')] \cdot [\tau(C_*(\widetilde{E_K}), \boldsymbol{h}'')] = [\tau(C_*(\widetilde{\partial E_K}), \boldsymbol{h})] \cdot \tau_{\rho_m}(M(\gamma, K), i_+(\Sigma)),$$

$$[\tau(C_*(\widetilde{E_J}, \widetilde{i_+(\Sigma)}), \boldsymbol{h}')] \cdot [\tau(C_*(\widetilde{D^2 \times S^1}), f_*(\boldsymbol{h}''))] = [\tau(C_*(\widetilde{\partial D^2 \times S^1}), f_*(\boldsymbol{h}))] \cdot \tau_{\rho'_m}(M(id), i_+(\Sigma)),$$

where we consider bases of chain complexes consisting of cells and the notation of these bases is omitted. Since

$$[\tau(C_*(\widetilde{\partial E_K}), \boldsymbol{h})] = [\tau(C_*(\widetilde{\partial D^2 \times S^1}), f_*(\boldsymbol{h}))],$$

$$[\tau(C_*(\widetilde{E_K}), \boldsymbol{h}'')] = [\tau(C_*(\widetilde{D^2 \times S^1}), f_*(\boldsymbol{h}''))],$$

$$\tau_{\rho'_m}(M(id), i_+(\Sigma)) = 1,$$

we have

$$\tau_{\rho_m}(M(\gamma, K), i_+(\Sigma)) = 1 = [\Delta_K(\gamma)].$$

Next we suppose that $[\gamma] \neq 1 \in \Gamma_m$. In this case $H_*^{\rho_m}(\partial E_K; \mathbb{Q}(\Gamma_m))$, $H_*^{\rho_m}(E_K; \mathbb{Q}(\Gamma_m))$, $H_*^{\rho'_m}(\partial D^2 \times S^1; \mathbb{Q}(\Gamma_m))$, $H_*^{\rho'_m}(D^2 \times S^1; \mathbb{Q}(\Gamma_m))$ vanish. Therefore $H_*^{\rho_m}(E_J, i_+(\Sigma); \mathbb{Q}(\Gamma_m))$ also vanishes. By Lemma 2.1.2 we obtain

$$\tau_{\rho_m}(E_J, i_+(\Sigma)) \cdot \tau_{\rho'_m}(E_K) = \tau_{\rho_m}(\partial E_K) \cdot \tau_{\rho_m}(M(\gamma, K), i_+(\Sigma)),$$
$$\tau_{\rho'_m}(E_J, i_+(\Sigma)) \cdot \tau_{\rho'_m}(D^2 \times S^1) = \tau_{\rho'_m}(\partial D^2 \times S^1) \cdot \tau_{\rho'_m}(M(id), i_+(\Sigma)).$$

Here

$$\tau_{\rho_m}(E_K) = [\Delta_K(\gamma)(\gamma - 1)^{-1}],$$
$$\tau_{\rho'_m}(D^2 \times S^1) = [(\gamma - 1)^{-1}],$$
$$\tau_{\rho_m}(\partial E_K) = \tau_{\rho'_m}(\partial D^2 \times S^1) = \tau_{\rho'_m}(M(id), i_+(\Sigma)) = 1,$$

which are easy to check. Now these equations give the desired formula. □

Considering the homology long exact sequences of the chain complexes with $\mathbb{Z}[\Gamma]$ coefficients instead of Lemma 2.1.2 in the proof, we obtain the following lemma.

Lemma 3.3.3. *Let $\gamma \in \pi^{(m)}$. Then*

$$H_1^{\rho_m}(M(\gamma, K), i_+(\Sigma); \mathbb{Z}[\Gamma]) \cong \mathcal{A}_K \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}[\Gamma_m],$$

where $\mathcal{A}_K$ is the Alexander module of K and we consider the homomorphism $\mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[\Gamma_m]$ defined by $t \mapsto \gamma$.

Now we are in position to show the difference between $C_{g,n}^{(m)}$ and $C_{g,n}^{(m+1)}$.

Theorem 3.3.4. *(i) $\overline{\mathcal{IC}}_{0,2}^{(1)} \neq \overline{\mathcal{IC}}_{0,2}^{(0)}$.*

(ii) $\overline{\mathcal{IC}}_{1,0}^{(1)} \neq \overline{\mathcal{IC}}_{1,0}^{(0)}$.

(iii) If $(g, n) \neq (0,0), (0,1), (0,2), (1,0)$, then $\overline{\mathcal{IC}}_{g,n}^{(m+1)} \neq \overline{\mathcal{IC}}_{g,n}^{(m)}$ for all m.

Proof. Suppose that $\pi^{(m+1)} \neq \pi^{(m)}$. Let $\gamma \in \pi^{(m)} \setminus \pi^{(m+1)}$ and let $K \subset S^3$ be a tame knot with nontrivial $\mathcal{A}_K$. By Proposition 3.3.1 we see at once $M(\gamma, K) \cdot M(\phi_m(M(\gamma, K))^{-1}) \in \overline{\mathcal{IC}}_{g,n}^{(m)}$. By Lemma 3.3.3 we have

$$H_1^{\rho_m}(M(\gamma, K) \cdot M(\phi_0(M(\gamma, K))^{-1}), i_+(\Sigma); \mathbb{Z}[\Gamma_m]) = \mathcal{A}_K \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}[\Gamma_m] \neq 0.$$

On the other hand, $H_1^{\rho_m}(M, i_+(\Sigma); \mathbb{Z}[\Gamma_m]) = 0$ for every $(M, i_\pm) \in C_{g,n}^{(m+1)}$ (e.g., see the proof of Proposition 3.2.3). Therefore $M(\gamma, K) \cdot M(\phi_m(M(\gamma, K))^{-1}) \notin C_{g,n}^{(m+1)}$, which gives the theorem. □

3.4. **Reduction of the torsion group.** A *bi-order* $\leq$ of a group Γ is a total order of Γ satisfying that if $x \leq y$, then $axb \leq ayb$ for all $a, b, x, y \in \Gamma$. A group Γ is called *bi-orderable* if Γ admits a bi-order. It is well-known that an abelian group is bi-orderable if and only if it is torsion-free. The following lemma is an immeadiate consequence of [33, Corollary 2. 4. 2, Corollary 2. 4. 3].

Lemma 3.4.1. *For a free group F, $F/F^{(m)}$ is bi-orderable for all m.*

Remark 3.4.2. It is well-known that every finitely generated torsion-free nilpotent group is residually p for any prime p. Rhemtulla [42] showed that a group which is residually p for infinitely many p is bi-orderable. To author's knowledge there seems to be no appropriate reference on whether Γ_m is residually nilpotent, residually p for infinitely many p or bi-orderable in the case where $m > 0$ and $n = 0$.

In the following we assume $n > 0$ and fix a bi-order of Γ_{m-1}. Let $A_m := \pi^{(m)}/\pi^{(m+1)}$ and let

$$C_m := \{\pm a \cdot p \cdot \gamma p^{-1} \gamma^{-1} \in \mathbb{Q}(A_m)^\times \ ; \ a \in A_m, \gamma \in \Gamma_m, p \in \mathbb{Q}(A_m)^\times\}.$$

We define a map $d\colon \mathbb{Z}[\Gamma_m] \setminus 0 \to \mathbb{Q}(A_m)^\times / C_m$ by

$$d\left(\sum_{\delta\in\Gamma_{m-1}} \sum_{\gamma\in\Gamma_m, [\gamma]=\delta} a_\gamma \gamma\right) = \left[\left(\sum_{\gamma\in\Gamma_m, [\gamma]=\delta_{max}} a_\gamma\gamma\right)\gamma_0^{-1}\right],$$

where $\delta_{\max} \in \Gamma_{m-1}$ is the maximum with respect to the fixed bi-order such that for some $\gamma \in \Gamma_m$ with $[\gamma] = \delta_{max}$, $a_\gamma \neq 0$, and $\gamma_0 \in \Gamma_m$ is an element with $[\gamma_0] = \delta_{max}$. The proof of the following lemma is straightforward.

Lemma 3.4.3. *The map $d\colon \mathbb{Z}[\Gamma_m] \setminus 0 \to \mathbb{Q}(A_m)^\times / C_m$ does not depend on the choice of γ_0 and is a monoid homomorphism.*

By the lemma we have a group homomorphism $\mathbb{Q}(\Gamma_m)^\times_{ab} \to \mathbb{Q}(A_m)^\times/C_m$ which maps $f \cdot g^{-1}$ to $d(f) \cdot d(g)^{-1}$ for $f, g \in \mathbb{Z}[\Gamma_m] \setminus 0$. By abuse of notation, we use the same letter d for the homomorphism. Since there is a natural section $\mathbb{Q}(A_m)^\times/C_m \to \mathbb{Q}(\Gamma_m)^\times_{ab}$ of d, $\mathbb{Q}(A_m)^\times/C_m$ can be seen as a direct summand of $\mathbb{Q}(\Gamma_m)^\times_{ab}$.

For irreducible $p, q \in \mathbb{Z}[A_m] \setminus 0$, we write $p \sim q$ if there exist $a \in A_m$ and $\gamma \in \Gamma_m$ such that $p = \pm a \cdot \gamma q \gamma^{-1}$. Since $\mathbb{Z}[A_m]$ is a unique factorization domain, every $x \in \mathbb{Q}(A_m)^\times/C_m$ can be written as $x = \prod_{[p]} [p]^{e_{[p]}}$, where $e_{[p]}$ is a uniquely determined integer. Let $e\colon \mathbb{Q}(A_m)^\times/C_m \to \oplus_{[p]}\mathbb{Z}$ be the isomorphism given by $x \mapsto \sum_{[p]} e_{[p]}$.

Recall that for every monoid S, there exists a monoid homomorphism $g\colon S \to \mathcal{U}(S)$ to a group $\mathcal{U}(S)$ satisfying the following: For every monoid homomorphism $f\colon S \to G$ to a group G, there exists a unique group homomorphism $f'\colon \mathcal{U}(S) \to G$ such that $f = f' \circ g$. By the universality $\mathcal{U}(S)$ is uniquely determined up to isomorphisms. The following theorem is an analogous result of Goda and Sakasai in [15].

Theorem 3.4.4. *If $n > 0$ and $(g, n) \neq (0, 1), (0, 2)$, then the abelianization of $\mathcal{U}(\mathcal{I}\overline{C}^{(m)}_{g,n})$ has infinite rank for all m.*

Proof. Let $\gamma \in \pi^{(m)} \setminus \pi^{(m+1)}$ and let $K \subset S^3$ be a tame knot. By Proposition 3.3.1 we see at once $M(\gamma, K) \cdot M(\phi_m(M(\gamma, K))^{-1}) \in \mathcal{I}\overline{C}^{(m)}_{g,n}$. By Propositions 3.2.5, 3.3.2 we have

$$d \circ \tau_m(M(\gamma, K) \cdot M(\phi_m(M(\gamma, K))^{-1})) = [\Delta_K(\gamma)].$$

Since it is well-known that for any $p \in \mathbb{Z}[t, t^{-1}]$ with $p(t^{-1}) = p(t)$ and $p(1) = 1$, there exists a knot $K \subset S^3$ such that $\Delta_K = p$, the image of $e \circ d \circ \tau_m\colon \mathcal{I}\overline{C}^{(m)}_{g,n} \to \oplus_{[p]}\mathbb{Z}$ contains a submonoid isomorphic to $\mathbb{Z}^\infty_{\geq 0}$. Therefore the image of the induced map $\mathcal{U}(\mathcal{I}\overline{C}^{(m)}_{g,n}) \to \oplus_{[p]}\mathbb{Z}$ is a free abelian group of infinite rank, which proves the theorem. □

Remark 3.4.5. If we know that $\mathrm{Wh}(\Gamma_m) = 1$, we could conclude by Proposition 3.2.3 that under the same assumption $\mathcal{U}(\mathcal{I}\overline{C}^{(m)}_{g,n})/\mathcal{U}(\mathcal{I}\overline{C}^{(m+1)}_{g,n})$ should have infinite rank for all m.

Corollary 3.4.6. *If $n > 0$ and $(g, n) \neq (0,1), (0,2)$, then $\overline{\mathcal{IC}}_{g,n}^{(m)}$ is not finitely generated for all m.*

We conclude with an observation concerning the abelianization of $\mathcal{IH}_{g,n}^{(m)}$. We set

$$N'_m := \{\pm a \cdot q \cdot \overline{q} \in \mathbb{Q}(A_m)^\times \,;\, a \in A_m, q \in \mathbb{Q}(A_m)^\times\}.$$

There is a natural map $\iota\colon \mathbb{Q}(A_m)^\times/(C_m \cdot N'_m) \to \mathbb{Q}(\Gamma_m)^\times_{ab}/N_m$. From the unique factorization property of $\mathbb{Z}[A_m]$ we have the isomorphism $e'\colon \mathbb{Q}(A_m)^\times/(C_m \cdot N'_m) \to (\oplus_{[p]=[\overline{p}]}\mathbb{Z}/2) \oplus (\oplus_{[p]\neq[\overline{p}]}\mathbb{Z})$ induced by $e\colon \mathbb{Q}(A_m)^\times/C_m \to \oplus_{[p]}\mathbb{Z}$. If $n > 0$ and $(g, n) \neq (0,1), (0,2)$, then from the argument in the proof of Theorem 3.4.4 the image of $\tau_m\colon \mathcal{IH}_{g,n}^{(m)} \to \mathbb{Q}(\Gamma_m)^\times_{ab}/N_m$ contains the image of a direct summand isomorphic to $(\mathbb{Z}/2)^\infty$ by ι. Thus to investigate $\operatorname{Ker}\iota$ is essential to detect size of the image of τ_m.

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