

博士論文

Annihilation of local cohomology groups by
separable extensions in positive characteristic

(正標数における局所コホモロジー群の分離拡大による消滅)

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ANNIHILATION OF LOCAL COHOMOLOGY GROUPS BY SEPARABLE EXTENSIONS IN POSITIVE CHARACTERISTIC

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ABSTRACT. For any local domain (R, \mathfrak{m}) of prime characteristic that is a homomorphic image of a Gorenstein ring, Huneke and Lyubeznik proved that there exists a module-finite extension domain S such that the induced map on local cohomology modules $H_{\mathfrak{m}}^i(R) \rightarrow H_{\mathfrak{m}}^i(S)$ is zero for each $i < \dim R$. We prove that the extension S may be chosen to be generically Galois, and analyze the Galois groups between them.

1. INTRODUCTION

Let R be a commutative Noetherian integral domain. We use R^+ to denote the integral closure of R in an algebraic closure of its fraction field. Hochster and Huneke proved the following:

Theorem 1.1. [HH2, Theorem 1.1] *If R is an excellent local domain of prime characteristic, then each system of parameters for R is a regular sequence on R^+ , i.e., R^+ is a balanced big Cohen-Macaulay algebra for R .*

It follows that for a ring R as above, and $i < \dim R$, the local cohomology module $H_{\mathfrak{m}}^i(R^+)$ is zero. Hence, given an element $[\eta]$ of $H_{\mathfrak{m}}^i(R)$, there exists a module-finite extension domain S such that $[\eta]$ maps to 0 under the induced map $H_{\mathfrak{m}}^i(R) \rightarrow H_{\mathfrak{m}}^i(S)$. This was strengthened by Huneke and Lyubeznik, albeit under mildly different hypotheses:

Theorem 1.2. [HL, Theorem 2.1] *Let (R, \mathfrak{m}) be a local domain of prime characteristic that is a homomorphic image of a Gorenstein ring. Then there exists a module-finite extension domain S such that the induced map $H_{\mathfrak{m}}^i(R) \rightarrow H_{\mathfrak{m}}^i(S)$ is zero for each $i < \dim R$.*

By a *generically Galois extension* of a domain R , we mean an extension domain S that is integral over R , such that the extension of fraction fields is Galois; $\text{Gal}(S/R)$ will denote the Galois group of the corresponding extension of fraction fields. We prove the following:

Theorem 1.3. *Let R be a domain of prime characteristic.*

The first author was supported by the Japan Society for Promotion of Science (JSPS), and the second author by NSF grant DMS 0856044.

- (1) Let \mathfrak{a} be an ideal of R and $[\eta]$ an element of $H_{\mathfrak{a}}^i(R)_{\text{nil}}$ (see Section 2.3). Then there exists a module-finite generically Galois extension S , with $\text{Gal}(S/R)$ a solvable group, such that $[\eta]$ maps to 0 under the induced map $H_{\mathfrak{a}}^i(R) \rightarrow H_{\mathfrak{a}}^i(S)$.
- (2) Suppose (R, \mathfrak{m}) is a homomorphic image of a Gorenstein ring. Then there exists a module-finite generically Galois extension S such that the induced map $H_{\mathfrak{m}}^i(R) \rightarrow H_{\mathfrak{m}}^i(S)$ is zero for each $i < \dim R$.

Set $R^{+\text{sep}}$ to be the R -algebra generated by the elements of R^+ that are separable over $\text{frac}(R)$. Under the hypotheses of Theorem 1.3 (2), $R^{+\text{sep}}$ is a separable balanced big Cohen-Macaulay R -algebra; see Corollary 3.3. In contrast, the algebra R^∞ , i.e., the purely inseparable part of R^+ , is not a Cohen-Macaulay R -algebra in general: take R to be an F -pure domain that is not Cohen-Macaulay; see [HH2, page 77].

For an \mathbb{N} -graded domain R of prime characteristic, Hochster and Huneke proved the existence of a \mathbb{Q} -graded Cohen-Macaulay R -algebra $R^{+\text{GR}}$, see Theorem 5.1. In view of this and the preceding paragraph, it is natural to ask whether there exists a \mathbb{Q} -graded separable Cohen-Macaulay R -algebra; in Example 5.2 we show that the answer is negative.

In Example 5.3 we construct an \mathbb{N} -graded domain of prime characteristic for which no module-finite \mathbb{Q} -graded extension domain is Cohen-Macaulay.

We also prove the following results for closure operations; the relevant definitions may be found in Section 2.1.

Theorem 1.4. *Let R be an integral domain of prime characteristic, and let \mathfrak{a} be an ideal of R .*

- (1) *Given $z \in \mathfrak{a}^F$, there exists a module-finite generically Galois extension S , with $\text{Gal}(S/R)$ a solvable group, such that $z \in \mathfrak{a}S$.*
- (2) *Given $z \in \mathfrak{a}^+$, there exists a module-finite generically Galois extension S such that $z \in \mathfrak{a}S$.*

In Example 4.1 we present a domain R of prime characteristic where $z \in \mathfrak{a}^+$ for an element z and ideal \mathfrak{a} , and conjecture that $z \notin \mathfrak{a}S$ for each module-finite generically Galois extension S with $\text{Gal}(S/R)$ a solvable group. Similarly, in Example 4.3 we present a 3-dimensional ring R where we conjecture that $H_{\mathfrak{m}}^2(R) \rightarrow H_{\mathfrak{m}}^2(S)$ is nonzero for each module-finite generically Galois extension S with $\text{Gal}(S/R)$ a solvable group.

Remark 1.5. The assertion of Theorem 1.2 does not hold for rings of characteristic zero: Let (R, \mathfrak{m}) be a normal domain of characteristic zero, and S a module-finite extension domain. Then the field trace map $\text{tr}: \text{frac}(S) \rightarrow \text{frac}(R)$ provides an R -linear splitting of $R \subseteq S$, namely

$$\frac{1}{[\text{frac}(S) : \text{frac}(R)]} \text{tr}: S \rightarrow R.$$

It follows that the induced maps on local cohomology $H_{\mathfrak{m}}^i(R) \rightarrow H_{\mathfrak{m}}^i(S)$ are R -split. A variation is explored in [RSS], where the authors investigate

whether the image of $H_m^i(R)$ in $H_m^i(R^+)$ is killed by elements of R^+ having arbitrarily small positive valuation. This is motivated by Heitmann's proof of the direct summand conjecture for rings (R, \mathfrak{m}) of dimension 3 and mixed characteristic $p > 0$, [He], which involves showing that the image of

$$H_m^2(R) \longrightarrow H_m^2(R^+)$$

is killed by $p^{1/n}$ for each positive integer n .

Throughout this paper, a *local ring* refers to a commutative Noetherian ring with a unique maximal ideal. Standard notions from commutative algebra that are used here may be found in [BH]; for more on local cohomology, consult [ILL]. For the original proof of the existence of big Cohen-Macaulay modules for equicharacteristic local rings, see [Ho].

2. PRELIMINARY REMARKS

2.1. Closure operations. Let R be an integral domain. The *plus closure* of an ideal \mathfrak{a} is the ideal $\mathfrak{a}^+ = \mathfrak{a}R^+ \cap R$.

When R is a domain of prime characteristic $p > 0$, we set

$$R^\infty = \bigcup_{e \geq 0} R^{1/p^e},$$

which is a subring of R^+ . The *Frobenius closure* of an ideal \mathfrak{a} is the ideal $\mathfrak{a}^F = \mathfrak{a}R^\infty \cap R$. Alternatively, set

$$\mathfrak{a}^{[p^e]} = \{a^{p^e} \mid a \in \mathfrak{a}\}.$$

Then $\mathfrak{a}^F = \{r \in R \mid r^{p^e} \in \mathfrak{a}^{[p^e]} \text{ for some } e \in \mathbb{N}\}$.

2.2. Solvable extensions. A finite separable field extension L/K is *solvable* if $\text{Gal}(M/K)$ is a solvable group for some Galois extension M of K containing L . Solvable extensions form a *distinguished class*, i.e.,

- (1) for finite extensions $K \subseteq L \subseteq M$, the extension M/K is solvable if and only if each of M/L and L/K are solvable;
- (2) for finite extensions L/K and M/K contained in a common field, if L/K is solvable, then so is the extension LM/M .

A finite separable extension L/K of fields of characteristic $p > 0$ is solvable precisely if it is obtained by successively adjoining

- (1) roots of unity;
- (2) roots of polynomials $T^n - a$ for n coprime to p ;
- (3) roots of *Artin-Schreier polynomials*, $T^p - T - a$.

2.3. Frobenius-nilpotent submodules. Let R be a ring of prime characteristic p . A *Frobenius action* on an R -module M is an additive map $F: M \rightarrow M$ with $F(rm) = r^p F(m)$ for each $r \in R$ and $m \in M$. In this case, $\ker F$ is a submodule of M , and we have an ascending sequence

$$\ker F \subseteq \ker F^2 \subseteq \ker F^3 \subseteq \dots$$

The union of these is the F -nilpotent submodule of M , denoted M_{nil} . If R is local and M is Artinian, then there exists a positive integer e such that $F^e(M_{\text{nil}}) = 0$; see [Ly, Proposition 4.4] or [HS, Theorem 1.12].

3. PROOFS

We record two elementary results that will be used later:

Lemma 3.1. *Let K be a field of characteristic $p > 0$. Let a and b elements of K where a is nonzero. Then the Galois group of the polynomial*

$$T^p + aT - b$$

is a solvable group.

Proof. Form an extension of K by adjoining a primitive $p - 1$ root of unity and an element c that is a root of $T^{p-1} - a$. The polynomial $T^p + aT - b$ has the same roots as

$$\left(\frac{T}{c}\right)^p - \left(\frac{T}{c}\right) - \frac{b}{c^p},$$

which is an Artin-Schreier polynomial in T/c . \square

Lemma 3.2. *Let R be a domain, and \mathfrak{p} a prime ideal. Given a domain S that is a module-finite extension of $R_{\mathfrak{p}}$, there exists a domain T , module-finite over R , with $T_{\mathfrak{p}} = S$.*

Proof. Given $s_i \in S$, there exists $r_i \in R \setminus \mathfrak{p}$ such that $r_i s_i$ is integral over R . If s_1, \dots, s_n are generators for S as an $R_{\mathfrak{p}}$ -module, set $T = R[r_1 s_1, \dots, r_n s_n]$. \square

Proof of Theorem 1.3. Since solvable extensions form a distinguished class, (1) reduces by induction to the case where $F([\eta]) = 0$. Compute $H_{\mathfrak{a}}^i(R)$ using a Čech complex $C^\bullet(x; R)$, where $x = x_0, \dots, x_n$ are nonzero elements generating the ideal \mathfrak{a} ; recall that $C^\bullet(x; R)$ is the complex

$$0 \longrightarrow R \longrightarrow \bigoplus_{i=0}^n R_{x_i} \longrightarrow \bigoplus_{i < j} R_{x_i x_j} \longrightarrow \dots \longrightarrow R_{x_0 \dots x_n} \longrightarrow 0.$$

Consider a cycle η in $C^i(x; R)$ that maps to $[\eta]$ in $H_{\mathfrak{a}}^i(R)$. Since $F([\eta]) = 0$, the cycle $F(\eta)$ is a boundary, i.e., $F(\eta) = \partial(\alpha)$ for some $\alpha \in C^{i-1}(x; R)$.

Let μ_1, \dots, μ_m be the square-free monomials of degree $i-2$ in the elements x_1, \dots, x_n , and regard $C^{i-1}(x; R) = C^{i-1}(x_0, \dots, x_n; R)$ as

$$R_{x_0 \mu_1} \oplus \dots \oplus R_{x_0 \mu_m} \oplus C^{i-1}(x_1, \dots, x_n; R).$$

There exist a power q of the characteristic p of R , and elements b_1, \dots, b_m in R , such that α can be written in the above direct sum as

$$\alpha = \left(\frac{b_1}{(x_0 \mu_1)^q}, \dots, \frac{b_m}{(x_0 \mu_m)^q}, *, \dots, * \right).$$

Consider the polynomials

$$T^p + x_0^q T - b_i \quad \text{for } i = 1, \dots, m,$$

and let L be a finite extension field where these have roots t_1, \dots, t_m respectively. By Lemma 3.1, we may assume L is Galois over $\text{frac}(R)$ with the Galois group being solvable. Let S be a module-finite extension of R that contains t_1, \dots, t_m , and has L as its fraction field; if R is excellent, we may take S to be the integral closure of R in L .

In the module $C^{i-1}(x; S)$ one then has

$$\alpha = \left(\frac{t_1^p + x_0^q t_1}{(x_0 \mu_1)^q}, \dots, \frac{t_m^p + x_0^q t_m}{(x_0 \mu_m)^q}, *, \dots, * \right) = F(\beta) + \gamma,$$

where

$$\beta = \left(\frac{t_1}{(x_0 \mu_1)^{q/p}}, \dots, \frac{t_m}{(x_0 \mu_m)^{q/p}}, 0, \dots, 0 \right) \quad \text{and} \quad \gamma = \left(\frac{t_1}{\mu_1^q}, \dots, \frac{t_m}{\mu_m^q}, *, \dots, * \right)$$

are elements of

$$C^{i-1}(x; S) = S_{x_0 \mu_1} \oplus \dots \oplus S_{x_0 \mu_m} \oplus C^{i-1}(x_1, \dots, x_n; S).$$

Since $F(\eta) = \partial(F(\beta) + \gamma)$, we have

$$F(\eta - \partial(\beta)) = \partial(\gamma).$$

But $[\eta] = [\eta - \partial(\beta)]$ in $H_a^i(S)$, so after replacing η we may assume that

$$F(\eta) = \partial(\gamma).$$

Next, note that γ is an element of $C^{i-1}(1, x_1, \dots, x_n; S)$, viewed as a submodule of $C^{i-1}(x; S)$. There exists ζ in $C^{i-2}(1, x_1, \dots, x_n; S)$ such that

$$\partial(\zeta) = \left(\frac{t_1}{\mu_1^q}, \dots, \frac{t_m}{\mu_m^q}, *, \dots, * \right).$$

Since

$$F(\eta) = \partial(\gamma - \partial(\zeta)),$$

after replacing γ we may assume that the first m coordinate entries of γ are 0, i.e., that

$$\gamma = \left(0, \dots, 0, \frac{c_1}{\lambda_1^Q}, \dots, \frac{c_l}{\lambda_l^Q} \right),$$

where Q is a power of p , the c_i belong to S , and $\lambda_1, \dots, \lambda_l$ are the square-free monomials of degree $i-1$ in x_1, \dots, x_n .

The coordinate entries of $\partial(\gamma)$ include each c_i/λ_i^Q . Since $\partial(\gamma) = F(\eta)$, each c_i/λ_i^Q is a p -th power in $\text{frac}(S)$; it follows that each c_i has a p -th root in $\text{frac}(S)$. After enlarging S by adjoining each $c_i^{1/p}$, we see that $\gamma = F(\xi)$ for an element ξ of $C^{i-1}(x; S)$. But then

$$F(\eta) = \partial(F(\xi)) = F(\partial(\xi)).$$

Since the Frobenius action on $C^i(x; S)$ is injective, we have $\eta = \partial(\xi)$, which proves (1).

For (2), it suffices to construct a module-finite generically separable extension S such that $H_m^i(R) \rightarrow H_m^i(S)$ is zero for $i < \dim R$; to obtain a

generically Galois extension, enlarge S to a module-finite extension whose fraction field is the Galois closure of $\text{frac}(S)$ over $\text{frac}(R)$.

We use induction on $d = \dim R$, as in [HL]. If $d = 0$, there is nothing to be proved; if $d = 1$, the inductive hypothesis is again trivially satisfied since $H_m^0(R) = 0$. Fix $i < \dim R$. Let (A, \mathfrak{M}) be a Gorenstein local ring that has R as a homomorphic image, and set

$$M = \text{Ext}_A^{\dim A - i}(R, A).$$

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the elements of the set $\text{Ass}_A M \setminus \{\mathfrak{M}\}$.

Let \mathfrak{q} be a prime ideal of R that is not maximal. Since R is catenary, one has

$$\dim R = \dim R_{\mathfrak{q}} + \dim R/\mathfrak{q}.$$

Thus, the condition $i < \dim R$ may be rewritten as

$$i - \dim R/\mathfrak{q} < \dim R_{\mathfrak{q}}.$$

Using the inductive hypothesis and Lemma 3.2, there exists a module-finite extension R' of R such that $\text{frac}(R')$ is a separable field extension of $\text{frac}(R_{\mathfrak{q}}) = \text{frac}(R)$, and the induced map

$$(3.2.1) \quad H_{\mathfrak{q}R_{\mathfrak{q}}}^{i - \dim R/\mathfrak{q}}(R_{\mathfrak{q}}) \longrightarrow H_{\mathfrak{q}R_{\mathfrak{q}}}^{i - \dim R/\mathfrak{q}}(R'_{\mathfrak{q}})$$

is zero. Taking the compositum of finitely many such separable extensions inside a fixed algebraic closure of $\text{frac}(R)$, there exists a module-finite generically separable extension R' of R such that the map (3.2.1) is zero when \mathfrak{q} is any of the primes $\mathfrak{p}_1R, \dots, \mathfrak{p}_sR$. We claim that the image of the induced map $H_m^i(R) \longrightarrow H_m^i(R')$ has finite length.

Using local duality over A , it suffices to show that

$$M' = \text{Ext}_A^{\dim A - i}(R', A) \longrightarrow \text{Ext}_A^{\dim A - i}(R, A) = M$$

has finite length. This, in turn, would follow if

$$M'_{\mathfrak{p}} = \text{Ext}_{A_{\mathfrak{p}}}^{\dim A - i}(R'_{\mathfrak{p}}, A_{\mathfrak{p}}) \longrightarrow \text{Ext}_{A_{\mathfrak{p}}}^{\dim A - i}(R_{\mathfrak{p}}, A_{\mathfrak{p}}) \doteq M_{\mathfrak{p}}$$

is zero for each prime ideal \mathfrak{p} in $\text{Ass}_A M \setminus \{\mathfrak{M}\}$. Using local duality over $A_{\mathfrak{p}}$, it suffices to verify the vanishing of

$$H_{\mathfrak{p}R_{\mathfrak{p}}}^{\dim A_{\mathfrak{p}} - \dim A + i}(R_{\mathfrak{p}}) \longrightarrow H_{\mathfrak{p}R_{\mathfrak{p}}}^{\dim A_{\mathfrak{p}} - \dim A + i}(R'_{\mathfrak{p}})$$

for each \mathfrak{p} in $\text{Ass}_A M \setminus \{\mathfrak{M}\}$. This, however, follows from our choice of R' since

$$\dim A_{\mathfrak{p}} - \dim A + i = i - \dim A/\mathfrak{p} = i - \dim R/\mathfrak{p}R.$$

What we have arrived at thus far is a module-finite generically separable extension R' of R such that the image of $H_m^i(R) \longrightarrow H_m^i(R')$ has finite length; in particular, this image is finitely generated. Working with one generator at a time and taking the compositum of extensions, given $[\eta]$ in $H_m^i(R')$, it suffices to construct a module-finite generically separable extension S of R' such that $[\eta]$ maps to 0 under $H_m^i(R') \longrightarrow H_m^i(S)$.

By Theorem 1.2, there exists a module-finite extension R_1 of R' such that $[\eta]$ maps to 0 under $H_m^i(R') \rightarrow H_m^i(R_1)$. Setting R_2 to be the separable closure of R' in R_1 , the image of $[\eta]$ in $H_m^i(R_2)$ lies in $H_m^i(R_2)_{\text{nil}}$. The result now follows by (1). \square

Corollary 3.3. *Let (R, \mathfrak{m}) be a local domain of prime characteristic that is a homomorphic image of a Gorenstein ring. Then $H_m^i(R^{+\text{sep}}) = 0$ for each $i < \dim R$.*

Moreover, each system of parameters for R is a regular sequence on $R^{+\text{sep}}$, i.e., $R^{+\text{sep}}$ is a separable balanced big Cohen-Macaulay algebra for R .

Proof. Theorem 1.3 (2) implies that $H_m^i(R^{+\text{sep}}) = 0$ for each $i < \dim R$. The proof that this implies the second statement is similar to the proof of [HL, Corollary 2.3]. \square

Proof of Theorem 1.4. Let p be the characteristic of R . If $z \in \mathfrak{a}^F$, there exists a prime power $q = p^e$ with $z^q \in \mathfrak{a}^{[q]}$. In this case, $z^{q/p}$ belongs to the Frobenius closure of $\mathfrak{a}^{[q/p]}$, and

$$(z^{q/p})^p \in (\mathfrak{a}^{[q/p]})^{[p]}.$$

Since solvable extensions form a distinguished class, we reduce to the case $e = 1$, i.e., $q = p$.

There exist nonzero elements, $a_0, \dots, a_m \in \mathfrak{a}$ and $b_0, \dots, b_m \in R$ with

$$z^p = \sum_{i=0}^m b_i a_i^p.$$

Consider the polynomials

$$T^p + a_0^p T - b_i \quad \text{for } i = 1, \dots, m,$$

and let L be a finite extension field where these have roots t_1, \dots, t_m respectively. By Lemma 3.1, we may assume L is Galois over $\text{frac}(R)$ with the Galois group being solvable. Set

$$(3.3.1) \quad t_0 = \frac{1}{a_0} \left(z - \sum_{i=1}^m t_i a_i \right).$$

Taking p -th powers, we have

$$t_0^p = \frac{1}{a_0^p} \left(\sum_{i=0}^m b_i a_i^p - \sum_{i=1}^m t_i^p a_i^p \right) = b_0 + \frac{1}{a_0^p} \sum_{i=1}^m (b_i - t_i^p) a_i^p = b_0 + \sum_{i=1}^m t_i a_i^p.$$

Thus, t_0 belongs to the integral closure of $R[t_1, \dots, t_m]$ in its field of fractions. Let S be a module-finite extension of R that contains t_0, \dots, t_m , and has L as its fraction field; if R is excellent, we may take S to be the integral closure of R in L . Since (3.3.1) may be rewritten as

$$z = \sum_{i=0}^m t_i a_i,$$

it follows that $z \in \mathfrak{a}S$, completing the proof of (1).

(2) follows from [Si2, Corollary 3.4], though we include a proof using (1). There exists a module-finite extension domain T such that $z \in \mathfrak{a}T$. Decompose the field extension $\text{frac}(R) \subseteq \text{frac}(T)$ as a separable extension $\text{frac}(R) \subseteq \text{frac}(T)^{+\text{sep}}$ followed by a purely inseparable extension $\text{frac}(T)^{+\text{sep}} \subseteq \text{frac}(T)$. Let T_0 be the integral closure of R in $\text{frac}(T)^{+\text{sep}}$.

Since T is a purely inseparable extension of T_0 , and $z \in \mathfrak{a}T$, it follows that z belongs to the Frobenius closure of the ideal $\mathfrak{a}T_0$. By (2) there exists a generically separable extension S_0 of T_0 with $z \in \mathfrak{a}S_0$. Enlarge S_0 to a generically Galois extension S of R . This concludes the argument in the case R is excellent; in the event that S is not module-finite over R , one may replace it by a subring satisfying $z \in \mathfrak{a}S$ and having the same fraction field. \square

The equational construction used in the proof of Theorem 1.4(1) arose from the study of symplectic invariants in [Si1].

4. SOME GALOIS GROUPS THAT ARE NOT SOLVABLE

Let R be a domain of prime characteristic, and let \mathfrak{a} be an ideal of R . If z is an element of \mathfrak{a}^F , Theorem 1.4(1) states that there exists a solvable module-finite extension S with $z \in \mathfrak{a}S$. In the following example one has $z \in \mathfrak{a}^+$, and we conjecture $z \notin \mathfrak{a}S$ for any module-finite generically Galois extension S with $\text{Gal}(S/R)$ solvable.

Example 4.1. Let a, b, c_1, c_2 be algebraically independent over \mathbb{F}_p , and set R be the hypersurface

$$\frac{\mathbb{F}_p(a, b, c_1, c_2)[x, y, z]}{(z^{p^2} + c_1(xy)^{p^2-p}z^p + c_2(xy)^{p^2-1}z + ax^{p^2} + by^{p^2})}.$$

We claim $z \in (x, y)^+$. Let u, v be elements of R^+ that are, respectively, roots of the polynomials

$$(4.1.1) \quad T^{p^2} + c_1y^{p^2-p}T^p + c_2y^{p^2-1}T + a,$$

and

$$T^{p^2} + c_1x^{p^2-p}T^p + c_2x^{p^2-1}T + b.$$

Set S to be the integral closure of R in the Galois closure of $\text{frac}(R)(u, v)$ over $\text{frac}(R)$. Then $(z - ux - vy)/xy$ is an element of S , since it is a root of the monic polynomial

$$T^{p^2} + c_1T^p + c_2T.$$

It follows that $z \in (x, y)S$.

We next show that $\text{Gal}(S/R)$ is not solvable for the extension S constructed above. Since u is a root of (4.1.1), u/y is a root of

$$(4.1.2) \quad T^{p^2} + c_1T^p + c_2T + \frac{a}{y^{p^2}}.$$

The polynomial (4.1.2) is irreducible over $\mathbb{F}_q(c_1, c_2, a/y^{p^2})$, and hence over the purely transcendental extension $\mathbb{F}_q(c_1, c_2, a, x, y, z) = \text{frac}(R)$. Since $\text{frac}(S)$ is a Galois extension of $\text{frac}(R)$ containing a root of (4.1.2), it contains all roots of (4.1.2). As (4.1.2) is separable, its roots are distinct; taking differences of roots, it follows that $\text{frac}(S)$ contains the p^2 distinct roots of

$$(4.1.3) \quad T^{p^2} + c_1 T^p + c_2 T.$$

We next verify that the Galois group of (4.1.3) over $\text{frac}(R)$ is $\text{GL}_2(\mathbb{F}_q)$.

Quite generally, let L be a field of characteristic p . Consider the standard linear action of $\text{GL}_2(\mathbb{F}_p)$ on the polynomial ring $L[x_1, x_2]$. The ring of invariants for this action is generated over L by the *Dickson invariants* c_1, c_2 , which occur as the coefficients in the polynomial

$$\prod_{\alpha, \beta \in \mathbb{F}_p} (T - \alpha x_1 - \beta x_2) = T^{p^2} + c_1 T^p + c_2 T,$$

see [Di] or [Be, Chapter 8]. Hence the extension $L(x_1, x_2)/L(c_1, c_2)$ has Galois group $\text{GL}_2(\mathbb{F}_p)$.

It follows from the above that if c_1, c_2 are algebraically independent elements over a field L of characteristic p , then the polynomial

$$T^{p^2} + c_1 T^p + c_2 T \in L(c_1, c_2)[T]$$

has Galois group $\text{GL}_2(\mathbb{F}_p)$.

The group $\text{PSL}_2(\mathbb{F}_p)$ is a subquotient of $\text{GL}_2(\mathbb{F}_p)$, and, we conjecture, a subquotient of $\text{Gal}(S/R)$ for *any* module-finite generically Galois extension S of R with $z \in \mathfrak{a}S$. For $p \geq 5$, the group $\text{PSL}_2(\mathbb{F}_p)$ is a nonabelian simple group; thus, conjecturally, $\text{Gal}(S/R)$ is not solvable for any module-finite generically Galois extension S with $z \in \mathfrak{a}S$.

Example 4.2. Extending the previous example, let a, b, c_1, \dots, c_n be algebraically independent elements over \mathbb{F}_q , and set R to be the hypersurface

$$\frac{\mathbb{F}_q(a, b, c_1, \dots, c_n)[x, y, z]}{(z^{q^n} + c_1(xy)^{q^n - q^{n-1}}z^{q^{n-1}} + c_2(xy)^{q^n - q^{n-2}}z^{q^{n-2}} + \dots + c_n(xy)^{q^n - 1}z + ax^{q^n} + by^{q^n})}.$$

Then $z \in (x, y)^+$; imitate the previous example with u, v being roots of

$$T^{q^n} + c_1 y^{q^n - q^{n-1}} T^{q^{n-1}} + c_2 y^{q^n - q^{n-2}} T^{q^{n-2}} + \dots + c_n y^{q^n - 1} T + a,$$

and

$$T^{q^n} + c_1 x^{q^n - q^{n-1}} T^{q^{n-1}} + c_2 x^{q^n - q^{n-2}} T^{q^{n-2}} + \dots + c_n x^{q^n - 1} T + b.$$

If S is any module-finite generically Galois extension of R with $z \in \mathfrak{a}S$, we conjecture that $\text{frac}(S)$ contains the splitting field of

$$(4.2.1) \quad T^{q^n} + c_1 T^{q^{n-1}} + c_2 T^{q^{n-2}} + \dots + c_n T.$$

Using a similar argument with Dickson invariants, the Galois group of (4.2.1) over $\text{frac}(R)$ is $\text{GL}_n(\mathbb{F}_q)$. Its subquotient $\text{PSL}_n(\mathbb{F}_q)$ is a nonabelian simple group for $n \geq 3$, and for $n = 2$, $q \geq 4$.

Likewise, we record conjectural examples R where $H_m^i(R) \rightarrow H_m^i(S)$ is nonzero for each module-finite generically Galois extension S with $\text{Gal}(S/R)$ solvable:

Example 4.3. Let a, b, c_1, c_2 be algebraically independent over \mathbb{F}_p , and consider the hypersurface

$$A = \frac{\mathbb{F}_p(a, b, c_1, c_2)[x, y, z]}{(z^{2p^2} + c_1(xy)^{p^2-p}z^{2p} + c_2(xy)^{p^2-1}z^2 + ax^{p^2} + by^{p^2})}.$$

Let (R, \mathfrak{m}) be the Rees ring $A[xt, yt, zt]$ localized at the maximal ideal x, y, z, xt, yt, zt . The elements $x, yt, y + xt$ form a system of parameters for R , and the relation

$$z^2t \cdot (y + xt) = z^2t^2 \cdot x + z^2 \cdot yt$$

defines an element $[\eta]$ of $H_m^2(R)$. We conjecture that if S is any module-finite generically Galois extension such that $[\eta]$ maps to 0 under the induced map $H_m^2(R) \rightarrow H_m^2(S)$, then $\text{frac}(S)$ contains the splitting field of

$$T^{p^2} + c_1T^p + c_2T,$$

and hence that $\text{Gal}(S/R)$ is not solvable if $p \geq 5$.

5. GRADED RINGS AND EXTENSIONS

Let R be an \mathbb{N} -graded domain that is finitely generated over a field R_0 . Set $R^{+\text{GR}}$ to be the $\mathbb{Q}_{\geq 0}$ -graded ring generated by elements of R^+ that can be assigned a degree such that they then satisfy a homogeneous equation of integral dependence over R . Note that $[R^{+\text{GR}}]_0$ is the algebraic closure of the field R_0 . One has the following:

Theorem 5.1. [HH2, Theorem 6.1] *Let R be an \mathbb{N} -graded domain that is finitely generated over a field R_0 of prime characteristic. Then each homogeneous system of parameters for R is a regular sequence on $R^{+\text{GR}}$.*

Let R be as in the above theorem. Since $R^{+\text{GR}}$ and $R^{+\text{sep}}$ are Cohen-Macaulay R -algebras, it is natural to ask whether there exists a \mathbb{Q} -graded separable Cohen-Macaulay R -algebra. The answer to this is negative:

Example 5.2. Let R be the Rees ring

$$\frac{\overline{\mathbb{F}}_2[x, y, z]}{(x^3 + y^3 + z^3)}[xt, yt, zt]$$

with the \mathbb{N} -grading where the generators x, y, z, xt, yt, zt have degree 1. Set B to be the R -algebra generated by the homogeneous elements of $R^{+\text{GR}}$ that are separable over $\text{frac}(R)$. We prove that B is not a balanced Cohen-Macaulay R -module.

The elements $x, yt, y + xt$ are a system of parameters for R . Suppose, to the contrary, that they form a regular sequence on B . Since

$$z^2t \cdot (y + xt) = z^2t^2 \cdot x + z^2 \cdot yt,$$

it follows that $z^2t \in (x, yt)B$. Thus, there exist elements $u, v \in B_1$ with

$$(5.2.1) \quad z^2t = u \cdot x + v \cdot yt.$$

Since $z^3 = x^3 + y^3$, we also have $z^2 = x\sqrt{xz} + y\sqrt{yz}$ in $R^{+\text{GR}}$, and hence

$$(5.2.2) \quad z^2t = t\sqrt{xz} \cdot x + \sqrt{yz} \cdot yt.$$

Comparing (5.2.1) and (5.2.2), we see that

$$(u + t\sqrt{xz}) \cdot x = (v + \sqrt{yz}) \cdot yt$$

in $R^{+\text{GR}}$. But x, yt is a regular sequence on $R^{+\text{GR}}$, so there exists an element c in $[R^{+\text{GR}}]_0$ with $u + t\sqrt{xz} = c yt$ and $v + \sqrt{yz} = cx$. Since $[R^{+\text{GR}}]_0 = \overline{\mathbb{F}}_2$, it follows that $c \in R$, and hence that $\sqrt{yz} \in B$. This contradicts the hypothesis that elements of B are separable over $\text{frac}(R)$.

The above argument shows that any graded Cohen-Macaulay R -algebra must contain the elements \sqrt{yz} and $t\sqrt{xz}$.

We next show that no module-finite \mathbb{Q} -graded extension domain of the ring R in Example 5.2 is Cohen-Macaulay.

Example 5.3. Let R be the Rees ring from Example 5.2, and let S be a graded Cohen-Macaulay ring with $R \subseteq S \subseteq R^{+\text{GR}}$. We prove that S is not finitely generated over R .

By the previous example, S contains \sqrt{yz} and $t\sqrt{xz}$. Using the symmetry between x, y, z , it follows that $\sqrt{xy}, \sqrt{xz}, t\sqrt{xy}, t\sqrt{yz}$ are all elements of S . We prove inductively that S contains

$$(5.3.1) \quad \begin{array}{lll} x^{1-2/q}(yz)^{1/q}, & y^{1-2/q}(xz)^{1/q}, & z^{1-2/q}(xy)^{1/q}, \\ tx^{1-2/q}(yz)^{1/q}, & ty^{1-2/q}(xz)^{1/q}, & tz^{1-2/q}(xy)^{1/q}, \end{array}$$

for each $q = 2^e$ with $e \geq 1$. The case $e = 1$ has been settled.

Suppose S contains the elements (5.3.1) for some $q = 2^e$. Then, one has

$$\begin{aligned} & x^{1-2/q}(yz)^{1/q} \cdot ty^{1-2/q}(xz)^{1/q} \cdot (y + xt) \\ &= tx^{1-2/q}(yz)^{1/q} \cdot ty^{1-2/q}(xz)^{1/q} \cdot x + x^{1-2/q}(yz)^{1/q} \cdot y^{1-2/q}(xz)^{1/q} \cdot yt. \end{aligned}$$

Using as before that $x, yt, y + xt$ is a regular sequence on S , we conclude

$$x^{1-2/q}(yz)^{1/q} \cdot ty^{1-2/q}(xz)^{1/q} = u \cdot x + v \cdot yt$$

for some $u, v \in S_1$. Simplifying the left hand side, the above reads

$$(5.3.2) \quad t(xy)^{1-1/q}z^{2/q} = u \cdot x + v \cdot yt.$$

Taking q -th roots in

$$z^2 = x\sqrt{xz} + y\sqrt{yz}$$

and multiplying by $t(xy)^{1-1/q}$ yields

$$(5.3.3) \quad t(xy)^{1-1/q}z^{2/q} = ty^{1-1/q}(xz)^{1/2q} \cdot x + x^{1-1/q}(yz)^{1/2q} \cdot yt.$$

Comparing (5.3.2) and (5.3.3), we see that

$$(u + ty^{1-1/q}(xz)^{1/2q}) \cdot x = (v + x^{1-1/q}(yz)^{1/2q}) \cdot yt,$$

so there exists c in $[R^{+\text{GR}}]_0 = \overline{\mathbb{F}}_2$ with

$$u + ty^{1-1/q}(xz)^{1/2q} = cyt \quad \text{and} \quad v + x^{1-1/q}(yz)^{1/2q} = cx.$$

It follows that $ty^{1-1/q}(xz)^{1/2q}$ and $x^{1-1/q}(yz)^{1/2q}$ are elements of S . In view of the symmetry between x, y, z , this completes the inductive step. Setting

$$\theta = \frac{xy}{z^2},$$

we have proved that

$$\theta^{1/q} \in \text{frac}(S) \quad \text{for each } q = 2^e.$$

We claim $\theta^{1/2}$ does not belong to $\text{frac}(R)$. Indeed if it does, then $(xy)^{1/2}$ belongs to $\text{frac}(R)$, and hence to R , as R is normal; this is readily seen to be false. The extension

$$\text{frac}(R) \subseteq \text{frac}(R)(\theta^{1/q})$$

is purely inseparable, so the minimal polynomial of $\theta^{1/q}$ over $\text{frac}(R)$ has the form $T^Q - \theta^{Q/q}$ for some $Q = 2^E$. Since $\theta^{1/2} \notin \text{frac}(R)$, we conclude that the minimal polynomial is $T^q - \theta$. Hence

$$\left[\text{frac}(R)(\theta^{1/q}) : \text{frac}(R) \right] = q \quad \text{for each } q = 2^e.$$

It follows that $[\text{frac}(S) : \text{frac}(R)]$ is not finite.

Theorem 1.2 and Theorem 1.3 (2) discuss the vanishing of the image of $H_m^i(R)$ for $i < \dim R$. In the case of graded rings, one also has the following result for $H_m^d(R)$.

Proposition 5.4. *Let R be an \mathbb{N} -graded domain that is finitely generated over a field R_0 of prime characteristic. Set $d = \dim R$. Then the submodule $[H_m^d(R)]_{\geq 0}$ maps to zero under the induced map*

$$H_m^d(R) \longrightarrow H_m^d(R^{+\text{GR}}).$$

Hence, there exists a module-finite \mathbb{Q} -graded extension domain S of R such that the induced map $[H_m^d(R)]_{\geq 0} \longrightarrow H_m^d(S)$ is zero.

Proof. Let $F^e : H_m^d(R) \longrightarrow H_m^d(R)$ denote the e -th iteration of the Frobenius map. Suppose $[\eta] \in [H_m^d(R)]_n$ for some $n \geq 0$. Then $F^e([\eta])$ belongs to $[H_m^d(R)]_{np^e}$ for each e . As $[H_m^d(R)]_{\geq 0}$ has finite length, there exists $e \geq 1$ and homogeneous elements $r_1, \dots, r_e \in R$ such that

$$(5.4.1) \quad F^e([\eta]) + r_1 F^{e-1}([\eta]) + \dots + r_e [\eta] = 0.$$

We imitate the equational construction from [HL]: Consider a homogeneous system of parameters $x = x_1, \dots, x_d$, and compute $H_m^i(R)$ as the cohomology of the Čech complex $C^\bullet(x; R)$ below:

$$0 \longrightarrow R \longrightarrow \bigoplus_{i=1}^d R_{x_i} \longrightarrow \bigoplus_{i < j} R_{x_i x_j} \longrightarrow \dots \longrightarrow R_{x_1 \dots x_d} \longrightarrow 0.$$

This complex is \mathbb{Z} -graded; let η be a homogeneous element of $C^d(\mathfrak{x}; R)$ that maps to $[\eta]$ in $H_m^d(R)$. Equation (5.4.1) implies that

$$F^e(\eta) + r_1 F^{e-1}(\eta) + \cdots + r_e \eta$$

is a boundary in $C^d(\mathfrak{x}; R)$, say it equals $\partial(\alpha)$ for a homogeneous element α of $C^{d-1}(\mathfrak{x}; R)$. Solving integral equations in each coordinate of $C^{d-1}(\mathfrak{x}; R)$, there exists a module-finite extension domain S and β in $C^{d-1}(\mathfrak{x}; S)$ with

$$F^e(\beta) + r_1 F^{e-1}(\beta) + \cdots + r_e \beta = \alpha.$$

Moreover, we may assume S is a normal ring. Since $\eta - \partial(\beta)$ is an element on $\text{frac}(S)$ satisfying

$$T^{pe} + r_1 T^{p^{e-1}} + \cdots + r_e T = 0,$$

it belongs to S . But then $\eta - \partial(\beta)$ maps to zero in $H_m^d(S)$. Thus, each homogeneous element of $[H_m^d(R)]_{\geq 0}$ maps to 0 in $H_m^d(R^{+\text{GR}})$.

For the final statement, note that $[H_m^d(R)]_{\geq 0}$ has finite length. □

The next example illustrates why Proposition 5.4 is limited to $[H_m^d(R)]_{\geq 0}$.

Example 5.5. Let K be a field of prime characteristic, and take R to be the semigroup ring

$$R = K[x_1 \cdots x_d, x_1^d, \dots, x_d^d].$$

It is easily seen that R is normal, and that $[H_m^d(R)]_n$ is nonzero for each integer $n < 0$. We claim that the induced map

$$H_m^d(R) \longrightarrow H_m^d(S)$$

is injective for each module-finite extension ring S . For this, it suffices to check that R is a *splinter* ring, i.e., that R is a direct summand of each module-finite extension ring; the splitting of $R \subseteq S$ then induces an R -splitting of $H_m^d(R) \longrightarrow H_m^d(S)$.

To check that R is a splinter ring, note that normal affine semigroup rings are weakly F -regular by [HH1, Proposition 4.12], and that weakly F -regular rings are splinter by [HH3, Theorem 5.25]. For more on splinters, we point the reader towards [Ma, HH3, Si3].

We thank Kazuhiko Kurano for pointing out an error in an earlier version of this manuscript.

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