Weyl modules, Demazure modules and finite crystals for non-simply laced type

(ノンシンプリーレースド型のワイル加群, デマズール加群および有限クリスタルについて)

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Abstract

We show that any Weyl module for a current algebra has a filtration such that each successive quotient is isomorphic to some Demazure module. We also prove that the weight sum of the path model for a tensor product of level zero fundamental representations are equal to a sum of the characters of Demazure modules. Moreover, we show that appearing Demazure modules in these two objects coincide exactly. Though these results are previously known in the simply laced case, they are new in the non-simply laced case.

1 Introduction

In this article, we study finite dimensional representations of a current algebra and crystals for a quantum affine algebra. First, we begin with an introduction of the results concerning finite dimensional representations of a current algebra.

Let $\mathfrak{g}$ be a complex simple Lie algebra. The study of finite dimensional representations of its current algebra $\mathcal{C}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t]$, together with that of a loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, has been the subjects of many articles. For example, see [1, 3, 4, 5, 6, 8].

Among finite dimensional representations of a current algebra, Weyl modules are especially important. The notion of a Weyl module was originally introduced in [5] for a loop algebra as a module having some universal property, and defined similarly for a current algebra in [4]. Let P_+ denote the set of dominant integral weights of $\mathfrak{g}$. We denote by $W(\lambda)$ the Weyl module for $\mathcal{C}\mathfrak{g}$ associated to $\lambda \in P_+$. Here, we make one remark. It is known that $W(\lambda)$ has a $\mathbb{Z}$-graded structure. In this article, we shall study $W(\lambda)$ with this $\mathbb{Z}$-graded structure, and hence we consider $W(\lambda)$ as a $\mathcal{C}\mathfrak{g}_d (= \mathcal{C}\mathfrak{g} \oplus \mathbb{C}d)$-module, where d is the degree operator.

There are other important finite dimensional $\mathcal{C}\mathfrak{g}_d$-modules called Demazure modules. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$, and let $\widehat{\mathfrak{b}} = \mathfrak{b} + \mathbb{C}K + \mathbb{C}d + \mathfrak{g} \otimes t\mathbb{C}[t]$ be the Borel subalgebra of the affine Lie algebra $\widehat{\mathfrak{g}}$, where K is the canonical central element. A Demazure module is, by definition, a $\widehat{\mathfrak{b}}$-module generated by an extremal weight vector of an irreducible highest weight $\widehat{\mathfrak{g}}$-module. Among the Demazure modules, we are mainly interested in $\mathcal{C}\mathfrak{g}_d$-stable ones, which are denoted by $\mathcal{D}(\ell, \lambda)[m]$ for some $\ell \in \mathbb{Z}_{>0}, \lambda \in P_+, m \in \mathbb{Z}$ (for a precise definition, see the subsection 3.2).

In [8], the following remarkable result was proved: if $\mathfrak{g}$ is simply laced, the Weyl module $W(\lambda)$ is isomorphic to the Demazure module $\mathcal{D}(1, \lambda)[0]$. (In [8], the authors gave a $\mathcal{C}\mathfrak{g}$-module isomorphism between these modules, but it is easily seen that this isomorphism is in fact a $\mathcal{C}\mathfrak{g}_d$-module isomorphism). Using this result, several corollaries about the Weyl module in simply laced case were obtained in [8].

Then, it is natural to ask what happens in the non-simply laced case. As stated in [8], a similar result does not hold any longer in the non-simply laced case.

In this article, we generalize the above result for a non-simply laced $\mathfrak{g}$. Assume that $\mathfrak{g}$ is non-simply laced. To state our result, we prepare some notation. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and let Δ be the root system of $\mathfrak{g}$. We denote by Δ^{sh} the root subsystem of Δ generated by short simple roots, and let $\mathfrak{g}^{\mathrm{sh}}$ be the simple Lie subalgebra of $\mathfrak{g}$ corresponding to Δ^{sh}. Put $\mathfrak{h}^{\mathrm{sh}} = \mathfrak{h} \cap \mathfrak{g}^{\mathrm{sh}}$, which is a Cartan subalgebra of $\mathfrak{g}^{\mathrm{sh}}$, and put $C\mathfrak{g}_d^{\mathrm{sh}} = \mathfrak{g}^{\mathrm{sh}} \otimes \mathbb{C}[t] \oplus \mathbb{C}d$. Since we need to consider Demazure modules for both $\mathfrak{g}$ and $\mathfrak{g}^{\mathrm{sh}}$, we denote a Demazure module for $\mathfrak{g}^{\mathrm{sh}}$ by $\mathcal{D}^{\mathrm{sh}}(\ell, \nu)[m]$, where ν is a dominant integral weight of $\mathfrak{g}^{\mathrm{sh}}$. Let $\overline{P}_+$ denote the set of dominant integral weights of $\mathfrak{g}^{\mathrm{sh}}$. We define a positive integer r by

$$r = \begin{cases} 2 & \text{if } \mathfrak{g} \text{ is of type } B_n, C_n \text{ or } F_4, \\ 3 & \text{if } \mathfrak{g} \text{ is of type } G_2. \end{cases}$$

For $\lambda \in P_+$, we denote by $\overline{\lambda} \in \overline{P}_+$ the image of λ under the canonical projection $\mathfrak{h}^* \to (\mathfrak{h}^{\mathrm{sh}})^*$. Using a result of Joseph in [16], we can show that $\mathcal{D}^{\mathrm{sh}}(1, \overline{\lambda})[0]$ has a $C\mathfrak{g}_d^{\mathrm{sh}}$-module filtration $0 = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_k = \mathcal{D}^{\mathrm{sh}}(1, \overline{\lambda})[0]$ such that each subquotient D_i/D_{i-1} is isomorphic to $\mathcal{D}^{\mathrm{sh}}(r, \nu_i)[m_i]$ for some $\nu_i \in \overline{P}_+$ and $m_i \in \mathbb{Z}_{\geq 0}$. We define a map $i_{\mathrm{sh}} : (\mathfrak{h}^{\mathrm{sh}})^* \to \mathfrak{h}^*$ by $\alpha_i|_{\mathfrak{h}^{\mathrm{sh}}} \mapsto \alpha_i$ for $\alpha \in \Delta^{\mathrm{sh}}$. We put $\lambda' = \lambda - i_{\mathrm{sh}}(\overline{\lambda})$, and $\mu_i = i_{\mathrm{sh}}(\nu_i) + \lambda'$ for each $1 \leq i \leq k$. Then we can show the following theorem, which is proved in Theorem 9.3:

Theorem A. *The Weyl module $W(\lambda)$ has a $C\mathfrak{g}_d$-module filtration $0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_k = W(\lambda)$ such that each subquotient W_i/W_{i-1} is isomorphic to the Demazure module $\mathcal{D}(1, \mu_i)[m_i]$.*

When a Weyl module is isomorphic to a Demazure module, the successive quotient of a trivial filtration is of course isomorphic to a Demazure module. Hence, this result is a sort of generalization of the result in the simply laced case.

We should remark that we need some results on the crystal theory to prove Theorem A. In fact, we can show without using any crystal theory that the Weyl module has a filtration such that each subquotient is a quotient of the Demazure module. However, it is in the final section that we prove that they are isomorphic to the Demazure modules, since we need some results on crystals to prove this statement.

Theorem A gives a lot of information about Weyl modules for non-simply laced $\mathfrak{g}$. Let ϖ_i denote the fundamental weight of $\mathfrak{g}$. Then, as shown in [8] in the simply laced case, we can easily obtain the following corollary from Theorem A, which is proved in Corollary 9.5:

Corollary A. *Let $\lambda \in P_+$.*
(i) *If $\lambda = \sum_{i \in I} \lambda_i \varpi_i$, then*

$$\dim W(\lambda) = \prod_{i \in I} \dim W(\varpi_i)^{\lambda_i}.$$

(ii) *Let* $\lambda_1, \ldots, \lambda_k \in P_+$ *be elements satisfying* $\lambda = \lambda_1 + \cdots + \lambda_k$. *Then for arbitrary pairwise distinct complex numbers* $c_1, \ldots, c_k$, *we have*

$$W(\lambda) \cong W(\lambda_1)_{c_1} * \cdots * W(\lambda_k)_{c_k}$$

as $C\mathfrak{g}_d$*-modules.*

We should explain some notation in (ii). The notation $*$ denotes the fusion product introduced in [6], and $c_1, \ldots, c_n$ are parameters used to define the fusion product. The statement (i) implies the dimension conjecture of the Weyl module, which is conjectured in [5]. The statement (ii), which easily follows from (i), shows that the fusion product of Weyl modules is associative and independent of the parameters $c_1, \ldots, c_n$. This statement was conjectured in [6] for more general modules for $C\mathfrak{g}$. It should be remarked that it is known that the corollary (i) can be proved using the global basis theory. (The proof is not written in any literature, but a brief sketch of this proof can be found in the introduction of [8]). Our approach is quite different from this proof.

Moreover, we can obtain from Theorem A the $\mathbb{Z}$-graded characters of Weyl modules. From this result, we can find some connection between Weyl modules and the classically restricted one-dimensional sums, which are polynomials defined in the crystal theory. We state this result after giving the introduction of our results in the crystal theory.

Next, we introduce our theorem in the crystal theory. Let $U_q(\widehat{\mathfrak{g}})$ be the quantum affine algebra associated to $\widehat{\mathfrak{g}}$, and $U_q'(\widehat{\mathfrak{g}})$ be the one without the degree operator. Crystals we mainly study in this article are realized by path models, which are originally introduced by Littelmann in [23, 24]. Let $\widehat{P}$ denote the integral weight lattice of $\widehat{\mathfrak{g}}$. A path with weight in $\widehat{P}$ is, by definition, a piecewise linear, continuous map $\pi : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} \widehat{P}$ such that $\pi(0) = 0$ and $\pi(1) \in \widehat{P}$. Let $\mathbb{P}$ denote the set of paths with weight in $\widehat{P}$, which was shown to have a $U_q(\widehat{\mathfrak{g}})$-crystal structure by Littelmann. Let $\lambda \in \widehat{P}$ be a level zero weight that is dominant integral for $\mathfrak{g}$, and let $\mathbb{B}_0(\lambda)$ be the connected component of $\mathbb{P}$ containing the straight line path $\pi_\lambda : \pi_\lambda(t) = t\lambda$. Put $\widehat{P}_{\mathrm{cl}} = \widehat{P}/\mathbb{Z}\delta$ where δ is the indivisible null root. By projecting $\mathbb{B}_0(\lambda)$ to $\mathbb{R} \otimes_{\mathbb{Z}} \widehat{P}_{\mathrm{cl}}$, we can obtain a finite $U_q'(\widehat{\mathfrak{g}})$-crystal, which is denoted by $\mathbb{B}(\lambda)_{\mathrm{cl}}$. Naito and Sagaki have verified in [27, 28] that this $\mathbb{B}(\lambda)_{\mathrm{cl}}$ is isomorphic to the tensor product of the crystal bases of level zero fundamental $U_q'(\widehat{\mathfrak{g}})$-representations, which is introduced by Kashiwara in [20].

Here, we make one remark. By definition, elements in $\mathbb{B}(\lambda)_{\mathrm{cl}}$ have only $\widehat{P}_{\mathrm{cl}}$-weights, but we define $\widehat{P}$-weights on these elements using the degree function introduced in [30]. These $\widehat{P}$-weights are important for our theorem.

Demazure modules have counterparts in the crystal theory. Let $\mathcal{B}(\Lambda)$ be the crystal basis of the irreducible highest weight $U_q(\widehat{\mathfrak{g}})$-module with highest weight Λ. Similarly as the classical case, a Demazure module is defined for $U_q(\widehat{\mathfrak{g}})$ as a submodule of an irreducible highest weight module. Then for each Demazure module, Kashiwara defined in [18] a subset of $\mathcal{B}(\Lambda)$ called a Demazure crystal, and he proved that there exists strong connection between a Demazure module and the corresponding Demazure crystal. For example, the character of a Demazure module coincides with the weight sum of the corresponding Demazure crystal. We denote by $\mathcal{B}(\ell, \lambda)[m]$ the Demazure crystal corresponding to (the quantized version of) $\mathcal{D}(\ell, \lambda)[m]$.

Now, we state our second main theorem, which is proved in Theorem 9.4. Let μ_i $(1 \le i \le k)$ be the dominant integral weights defined just above Theorem A. We denote by b_Λ the highest weight element of $\mathcal{B}(\Lambda)$, and by Λ_0 the fundamental weight associated with the additional node of the extended Dynkin diagram of $\mathfrak{g}$:

Theorem B. *$\mathcal{B}(\Lambda_0) \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}$ is isomorphic as a $U_q'(\widehat{\mathfrak{g}})$-crystal to the direct sum of crystal bases of irreducible highest weight $U_q(\widehat{\mathfrak{g}})$-modules. Moreover, the restriction of the given isomorphism on $b_{\Lambda_0} \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}$ preserves the $\widehat{P}$-weights, and the image of $b_{\Lambda_0} \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}$ coincides with $\coprod_{1 \le i \le k} \mathcal{B}(1, \mu_i)[m_i]$.*

In order to prove this theorem, we show the following proposition in advance (which corresponds to Corollary 6.8 and Proposition 7.6):

Proposition. *Let Λ be an arbitrary dominant integral weight of $\widehat{\mathfrak{g}}$.*
(i) *$\mathcal{B}(\Lambda) \otimes \mathbb{B}_0(\lambda)$ is isomorphic as a $U_q(\widehat{\mathfrak{g}})$-crystal to the direct sum of the crystal bases of irreducible highest weight $U_q(\widehat{\mathfrak{g}})$-modules.*
(ii) *Under the isomorphism given in* (i), *the image of $b_\Lambda \otimes \mathbb{B}_0(\lambda)$ coincides with the disjoint union of some Demazure crystals.*

Since it is known that, at least in some cases, $\mathbb{B}_0(\lambda)$ is related to the crystal basis of some level zero extremal weight $U_q(\widehat{\mathfrak{g}})$-modules ([28, 29]), this proposition itself seems important and interesting.

It was proved in [4] that the $\mathbb{Z}$-graded multiplicity of $W(\lambda)$ is equal to the Kostka polynomial when $\mathfrak{g} = \mathfrak{sl}_n$. From Theorem A and B, we can obtain a generalization of this result. Let $\mathbf{i} = (i_1, \dots, i_\ell)$ be an arbitrary sequence of indices of simple roots of $\mathfrak{g}$, and let $\mathbb{B}_{\mathbf{i}} = \mathbb{B}(\varpi_{i_1})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_\ell})_{\mathrm{cl}}$. Then we can define a classically restricted one-dimensional sum $X(\mathbb{B}_{\mathbf{i}}, \mu; q)$ for each $\mu \in P_+$, and it is known that $X(\mathbb{B}_{\mathbf{i}}, \mu; q^{-1})$ is equal to a Kostka polynomial in the $\mathfrak{sl}_n$-case ([31]). Using Theorem A, B and the result in [30], we show the following in Corollary 9.7:

Corollary B. *We have for some constant $C \in \mathbb{Z}$ that*

$$\sum_{n \in \mathbb{Z}_{\ge 0}} (W(\lambda)_n : V_{\mathfrak{g}}(\mu)) q^n = q^C X(\mathbb{B}_{\mathbf{i}}, \mu; q^{-1}),$$

where $V_{\mathfrak{g}}(\mu)$ denotes the irreducible $\mathfrak{g}$-module with highest weight μ, and we denote by $(W(\lambda)_n : V_{\mathfrak{g}}(\mu))$ the multiplicity of $V_{\mathfrak{g}}(\mu)$ in the subspace of $W(\lambda)$ with degree n.

The plan of this article is as follows. In Section 2, we fix basic notation used in the article. In Section 3, we review some results on finite dimensional representations of a current algebra, almost of which have been already known. Section 4 is the main part in the first half of this article. We give defining relations of the Demazure module of level 1, and we show using this defining relations the existence of a filtration on the Weyl module whose subquotients are quotients of the Demazure modules.

In Section 5, we review the theory of path models. In Section 6, we show that $\mathcal{B}(\Lambda) \otimes \mathbb{B}_0(\lambda)$ is isomorphic to the direct sum of the crystal bases of highest weight modules, and in Section 7, we show that the image of $b_\Lambda \otimes \mathbb{B}_0(\lambda)$ under this isomorphism coincides with the disjoint union of Demazure crystals. In the final part of Section 7, we show that $\mathcal{B}(\Lambda) \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}$ also has similar properties. In particular, $b_{\Lambda_0} \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}$ decomposes to the disjoint union of some Demazure crystals. In Section 8, we study this decomposition of $b_{\Lambda_0} \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}$ in more detail.

Then in Section 9, we show the Theorem A and B, and also show Corollary A and B.

Index of notation

We provide for the reader's convenience a brief index of the notation which is used repeatedly in this paper:

Section 2: $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{b}$, Δ, $\Delta_{\pm}$, Π, α_i, I, θ, $\mathrm{ht}\,\alpha$, $\alpha^{\vee}$, $\mathfrak{g}_{\alpha}$, e_{α}, f_{α}, e_i, f_i, $\mathfrak{n}_{\pm}$, $(\ ,\ )$, ν, Q, Q_+, ϖ_i, P, P_+, W, s_{α}, s_i, w_0, r, $V_{\mathfrak{g}}(\lambda)$, $\widehat{\mathfrak{g}}$, K, d, $\widehat{\mathfrak{h}}$, $\widehat{\mathfrak{b}}$, $\widehat{\Delta}$, δ, $\widehat{\Pi}$, $\widehat{I}$, Λ_i, $\widehat{P}$, $\widehat{P}_+$, $\widehat{W}$, $\mathcal{C}\mathfrak{g}$, $\mathcal{C}\mathfrak{g}_d$, $\mathfrak{h}_d$, Π^{sh}, I^{sh}, Δ^{sh}, $\Delta^{\mathrm{sh}}_{\pm}$, Q^{sh}, Q^{sh}_{+}, W^{sh}, $\mathfrak{h}^{\mathrm{sh}}$, $\mathfrak{g}^{\mathrm{sh}}$, $\mathfrak{n}^{\mathrm{sh}}_{\pm}$, $\widehat{\mathfrak{g}}^{\mathrm{sh}}$, $\mathcal{C}\mathfrak{g}^{\mathrm{sh}}$, $\mathcal{C}\mathfrak{g}^{\mathrm{sh}}_d$, $\widehat{\mathfrak{h}}^{\mathrm{sh}}$, $\mathfrak{h}^{\mathrm{sh}}_d$, $\overline{\lambda}$, i_{sh}, $\overline{P}$, $\overline{P}_+$, $U_q(\widehat{\mathfrak{g}})$, $U'_q(\widehat{\mathfrak{g}})$, $U_q(\widehat{\mathfrak{g}}^{\mathrm{sh}})$, $U'_q(\widehat{\mathfrak{g}}^{\mathrm{sh}})$, M_{λ}, wt_H, ch_H, P_S.
Section 3.1: $W(\lambda)$.
Section 3.2: $V(\Lambda)$, $V_w(\Lambda)$, $\mathcal{D}(\ell,\lambda)[m]$, $\mathcal{D}(\ell,\lambda)$.
Section 3.3: $W^1_{c_1} * \cdots * W^k_{c_k}$.
Section 4.1: M^{λ}.
Section 4.3: $V_q(\Lambda)$, $V_{q,w}(\Lambda)$, $\preceq$, $(M : \mathcal{D}(\ell,\lambda)[m])$.
Section 4.4: $\mathcal{D}^{\mathrm{sh}}(\ell,\nu)[m]$, $\leq$ on $\mathbb{Z}[\mathfrak{h}^*_d]$.
Section 5.1: $[a,b]$, $\mathbb{P}$, $(\underline{\mu},\underline{\sigma})$, H^{π}_i, m^{π}_i, $\mathbb{P}_{\mathrm{int}}$, $\tilde{e}_i$, $\tilde{f}_i$, $\mathbf{0}$, wt, ε_i, φ_i, S_i, S_w, $\pi_1 * \pi_2$, $\mathbb{B}_1 * \mathbb{B}_2$.
Section 5.2: $C(b)$, π_{λ}, $\mathbb{B}_0(\lambda)$, $\mathcal{B}(\Lambda)$, b_{Λ}, $\widehat{P}_{\mathrm{cl}}$, cl, $\mathbb{P}_{\mathrm{cl}}$, η_{ξ}, $\mathrm{cl}(\pi)$, $\mathbb{B}(\lambda)_{\mathrm{cl}}$, $W_q(\varpi_i)$, $\mathcal{B}(W_q(\varpi_i))$.
Section 5.3: $\iota(\pi)$, $\iota(\eta)$, d_{λ}, i_{cl}, π_{η}, $\mathrm{Deg}(\eta)$, $\mathrm{wt}_{\widehat{P}}$.
Section 6.1: $\widehat{W}_J$.
Section 6.2: $\mathbb{B}_0(\lambda)^{\Lambda}$.
Section 7.1: $\mathcal{F}_i(\mathcal{C})$, $\mathcal{F}_{\mathrm{i}}(\mathcal{C})$, $\mathcal{B}_w(\Lambda)$.
Section 7.2: $\widehat{I}_{\Lambda}$.
Section 7.3: $\mathbb{B}(\lambda)^{\Lambda}_{\mathrm{cl}}$.
Section 8.1: $\mathcal{B}(\ell,\lambda)[m]$, κ.
Section 8.2: θ^{sh}, α^{sh}_0, s^{sh}_0, $\widehat{W}^{\mathrm{sh}}$, $\widehat{I}^{\mathrm{sh}}$, $\widehat{P}^{\mathrm{sh}}$, $\mathbb{P}^{\mathrm{sh}}$, $\mathbb{P}^{\mathrm{sh}}_{\mathrm{int}}$, $\tilde{e}^{\mathrm{sh}}_i$, $\tilde{f}^{\mathrm{sh}}_i$, $H^{\mathrm{sh},\pi}_i$, $m^{\mathrm{sh},\pi}_i$, $\mathbb{B}^{\mathrm{sh}}_0(\lambda)$.

2 Notation and elementary lemmas

In this section, we fix the notation used in this article. Let $\mathfrak{g}$ be a complex simple Lie algebra. We fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ and a Borel subalgebra $\mathfrak{b} \supseteq \mathfrak{h}$. Let $\Delta \subseteq \mathfrak{h}^*$ denote the root system of $\mathfrak{g}$, and by Δ_+ and Δ_- we denote the set of positive roots and negative roots corresponding to $\mathfrak{b}$ respectively. Denote by $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots, and by $I = \{1, \ldots, n\}$ its index set. Let θ be the highest root of Δ. For $\alpha = \sum_i n_i \alpha_i \in \Delta$, we define the height of α by $\mathrm{ht}\,\alpha = \sum_i n_i$, and we denote the coroot of α by $\alpha^{\vee} \in \mathfrak{h}$.

Denote the root space associated with $\alpha \in \Delta$ by $\mathfrak{g}_{\alpha}$. For each $\alpha \in \Delta_+$, we fix $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[e_\alpha, f_\alpha] = \alpha^\vee$, and we abbreviate $e_i = e_{\alpha_i}, f_i = f_{\alpha_i}$. Let

$$\mathfrak{n}_\pm = \bigoplus_{\alpha\in\Delta_\pm} \mathfrak{g}_\alpha.$$

Let $(\ ,\ )$ be the unique non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$ normalized by $(\theta^\vee, \theta^\vee) = 2$, and let $\nu : \mathfrak{h} \to \mathfrak{h}^*$ be the linear isomorphism defined by the restriction of $(\ ,\ )$ to $\mathfrak{h}$. We define a bilinear form on $\mathfrak{h}^*$ by $(\nu^{-1}(*), \nu^{-1}(*))$, which is also denoted by $(\ ,\ )$. Note that we have $(\theta, \theta) = 2$. In this article, we say $\alpha \in \Delta$ is a *long root* if $(\alpha, \alpha) = (\theta, \theta)(= 2)$, and a *short root* otherwise. Note that all roots are long if $\mathfrak{g}$ is simply laced.

Let $Q = \sum_i \mathbb{Z}\alpha_i$ be the root lattice of $\mathfrak{g}$, and let $Q_+ = \sum_i \mathbb{Z}_{\geq 0}\alpha_i \subseteq Q$. We denote by ϖ_i the fundamental weight corresponding to α_i. Let $P = \sum_i \mathbb{Z}\varpi_i$ be the weight lattice of $\mathfrak{g}$, and let $P_+ = \sum_i \mathbb{Z}_{\geq 0}\varpi_i$ be the set of dominant weights. Let W be the Weyl group of $\mathfrak{g}$. For $\alpha \in \Delta_+$, we denote by $s_\alpha \in W$ the reflection associated with α, and we abbreviate $s_i = s_{\alpha_i}$ for $i \in I$. We denote by w_0 the longest element of W.

When $\mathfrak{g}$ is non-simply laced, by r we denote the number $2 \cdot$ (square length of a short root)$^{-1}$, and we put $r = 1$ when $\mathfrak{g}$ is simply laced. Then we have

$$r = \begin{cases} 1 & \text{if } \mathfrak{g} \text{ is simply laced}, \\ 2 & \text{if } \mathfrak{g} \text{ is of type } B_n, C_n, F_4, \\ 3 & \text{if } \mathfrak{g} \text{ is of type } G_2. \end{cases} \tag{1}$$

For $\lambda \in P_+$, we denote by $V_\mathfrak{g}(\lambda)$ the irreducible $\mathfrak{g}$-module with highest weight λ. The following lemma is well-known:

Lemma 2.1. *Let $v \in V_\mathfrak{g}(\lambda)$ be a highest weight vector. Then $V_\mathfrak{g}(\lambda)$ is generated by v with the defining relations:*

$$\mathfrak{n}_+.v = 0, \quad h.v = \langle \lambda, h\rangle v, \quad f_i^{\langle \lambda, \alpha_i^\vee\rangle + 1}.v = 0$$

for $h \in \mathfrak{h}$ and $i \in I$.

Let $\widehat{\mathfrak{g}}$ be the non-twisted affine Lie algebra corresponding to the extended Dynkin diagram of $\mathfrak{g}$:

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

where K denotes the canonical central element and d is the degree operator. The Lie bracket of $\widehat{\mathfrak{g}}$ is given by

$$\begin{aligned}[][x \otimes t^m + a_1K + b_1d, y \otimes t^n + a_2K + b_2d] \\ = [x, y] \otimes t^{m+n} + nb_1 y \otimes t^n - mb_2 x \otimes t^m + m\delta_{m,-n}(x, y)K.\end{aligned}$$

We naturally consider $\mathfrak{g}$ as a Lie subalgebra of $\widehat{\mathfrak{g}}$. The Cartan subalgebra $\widehat{\mathfrak{h}} \subseteq \widehat{\mathfrak{g}}$ and the Borel subalgebra $\widehat{\mathfrak{b}} \subseteq \widehat{\mathfrak{g}}$ are as follows:

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d, \qquad \widehat{\mathfrak{b}} = \mathfrak{b} \oplus \mathbb{C}K \oplus \mathbb{C}d \oplus \mathfrak{g} \otimes t\mathbb{C}[t].$$

Denote by $\widehat{\Delta}$ the root system of $\widehat{\mathfrak{g}}$. Considering Δ naturally as a subset of $\widehat{\Delta}$, we can write $\widehat{\Delta} = \{\alpha + s\delta \mid \alpha \in \Delta, s \in \mathbb{Z}\} \cup \{s\delta \mid s \in \mathbb{Z} \setminus \{0\}\}$, where $\delta \in \widehat{\mathfrak{h}}^*$ is a unique element satisfying $\langle \delta, \mathfrak{h} + \mathbb{C}K \rangle = 0, \langle \delta, d \rangle = 1$. The set of simple roots of $\widehat{\Delta}$ are denoted by $\widehat{\Pi} = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$ where $\alpha_0 = \delta - \theta$. We denote by $\widehat{I} = \{0, 1, \dots, n\}$ its index set.

Let $\{\Lambda_0, \dots, \Lambda_n\} \subseteq \widehat{\mathfrak{h}}^*$ be the fundamental weights corresponding to $\widehat{\Pi}$, $\widehat{P} = \sum_{i \in \widehat{I}} \mathbb{Z}\Lambda_i + \mathbb{Z}\delta$ be the weight lattice, and $\widehat{P}_+ = \sum_{i \in \widehat{I}} \mathbb{Z}_{\geq 0}\Lambda_i + \mathbb{Z}\delta$ the set of dominant weights. For $\Lambda \in \widehat{P}$, the level of Λ is defined by the integer $\langle \Lambda, K \rangle$. Let $\widehat{W}$ be the Weyl group of $\widehat{\mathfrak{g}}$ generated by simple reflections s_i $(i \in \widehat{I})$, and we see W naturally as a subgroup of $\widehat{W}$. We denote the Bruhat order on $\widehat{W}$ by $\leq$.

We define Lie subalgebras of $\widehat{\mathfrak{g}}$ by

$$\mathcal{C}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t] \subseteq \widehat{\mathfrak{g}}, \quad \mathcal{C}\mathfrak{g}_d = \mathcal{C}\mathfrak{g} \oplus \mathbb{C}d \subseteq \widehat{\mathfrak{g}}.$$

We write $\mathfrak{h}_d = \mathfrak{h} \oplus \mathbb{C}d \subseteq \widehat{\mathfrak{h}}$. We usually consider $\mathfrak{h}^*$ and $\mathfrak{h}_d^*$ as subspaces of $\widehat{\mathfrak{h}}^*$ canonically to be compatible with the decomposition $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$. Note that under this identification ϖ_i is an element of $\widehat{\mathfrak{h}}^*$ satisfying

$$\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij} \text{ for } j \in I, \quad \langle \varpi_i, \mathbb{C}K \oplus \mathbb{C}d \rangle = 0,$$

and P is a subgroup of $\widehat{P}$.

In this article, we need to consider the subset of Π consisting of short simple roots:

$$\Pi^{\mathrm{sh}} = \{\alpha_i \in \Pi \mid \alpha_i \text{ is short}\}.$$

Note that $\Pi^{\mathrm{sh}} = \emptyset$ if $\mathfrak{g}$ is simply laced. Let $I^{\mathrm{sh}} = \{i \in I \mid \alpha_i \in \Pi^{\mathrm{sh}}\}$ be its index set, and let

$$\Delta^{\mathrm{sh}} = \Delta \cap \sum_{i \in I^{\mathrm{sh}}} \mathbb{Z}\alpha_i.$$

Let $\Delta_\pm^{\mathrm{sh}} = \Delta^{\mathrm{sh}} \cap \Delta_\pm$, and let

$$Q^{\mathrm{sh}} = \sum_{i \in I^{\mathrm{sh}}} \mathbb{Z}\alpha_i, \quad Q_+^{\mathrm{sh}} = \sum_{i \in I^{\mathrm{sh}}} \mathbb{Z}_{\geq 0}\alpha_i, \quad W^{\mathrm{sh}} = \langle s_i \mid i \in I^{\mathrm{sh}} \rangle \subseteq W.$$

Later we need the following elementary lemma:

Lemma 2.2. *If $\alpha \in \Delta_+ \setminus \Delta_+^{\mathrm{sh}}$ and $w \in W^{\mathrm{sh}}$, then $w\alpha \in \Delta_+ \setminus \Delta_+^{\mathrm{sh}}$ follows.*

Proof. If we write $\alpha = \sum_{i \in I} n_i \alpha_i$, there exists some $j \in I \setminus I^{\mathrm{sh}}$ such that $n_j > 0$. For $k \in I^{\mathrm{sh}}$, the coefficient of $s_k\alpha = \alpha - \langle \alpha, \alpha_k^\vee \rangle \alpha_k$ on α_j is $n_j > 0$, which implies $s_k\alpha \in \Delta_+ \setminus \Delta_+^{\mathrm{sh}}$. □

Put $\mathfrak{h}^{\mathrm{sh}} = \bigoplus_{i \in I^{\mathrm{sh}}} \mathbb{C}\alpha_i^\vee \subseteq \mathfrak{h}$, and denote the simple Lie subalgebra corresponding to Δ^{sh} by $\mathfrak{g}^{\mathrm{sh}}$:

$$\mathfrak{g}^{\mathrm{sh}} = \mathfrak{h}^{\mathrm{sh}} \oplus \bigoplus_{\alpha \in \Delta^{\mathrm{sh}}} \mathfrak{g}_\alpha.$$

Let $\mathfrak{n}_\pm^{\mathrm{sh}} = \bigoplus_{\alpha \in \Delta_\pm^{\mathrm{sh}}} \mathfrak{g}_\alpha$. Note that the type of $\mathfrak{g}^{\mathrm{sh}}$ is as follows:

$$\begin{cases} \{0\} & \text{if } \mathfrak{g} \text{ is simply laced}, \\ A_1 & \text{if } \mathfrak{g} \text{ is of type } B_n, G_2, \\ A_2 & \text{if } \mathfrak{g} \text{ is of type } F_4, \\ A_{n-1} & \text{if } \mathfrak{g} \text{ is of type } C_n. \end{cases}$$

Let $\widehat{\mathfrak{g}}^{\text{sh}}$ be the non-twisted affine Lie algebra corresponding to the extended Dynkin diagram of $\mathfrak{g}^{\text{sh}}$, and we consider $\widehat{\mathfrak{g}}^{\text{sh}}$ naturally as a Lie subalgebra of $\widehat{\mathfrak{g}}$. Also we denote

$$\mathcal{C}\mathfrak{g}^{\text{sh}} = \mathfrak{g}^{\text{sh}} \otimes \mathbb{C}[t] \subseteq \widehat{\mathfrak{g}}^{\text{sh}}, \quad \mathcal{C}\mathfrak{g}_d^{\text{sh}} = \mathcal{C}\mathfrak{g}^{\text{sh}} \oplus \mathbb{C}d \subseteq \widehat{\mathfrak{g}}^{\text{sh}},$$

and $\widehat{\mathfrak{h}}^{\text{sh}} = \mathfrak{h}^{\text{sh}} \oplus \mathbb{C}K \oplus \mathbb{C}d$, $\mathfrak{h}_d^{\text{sh}} = \mathfrak{h}^{\text{sh}} \oplus \mathbb{C}d$.

Throughout this article, we denote by $\overline{\lambda}$ the projection image of $\lambda \in \widehat{\mathfrak{h}}^*$ under the canonical projection $\widehat{\mathfrak{h}}^* \to (\widehat{\mathfrak{h}}^{\text{sh}})^*$, and we fix one splitting of this projection: let i_{sh} denote the linear map $(\widehat{\mathfrak{h}}^{\text{sh}})^* \to \widehat{\mathfrak{h}}^*$ defined by

$$i_{\text{sh}}(\overline{\alpha}_i) = \alpha_i \text{ for } i \in I^{\text{sh}}, \quad i_{\text{sh}}(\overline{\Lambda}_0) = \Lambda_0, \quad i_{\text{sh}}(\overline{\delta}) = \delta.$$

We write

$$\overline{P} = \{\overline{\lambda} \mid \lambda \in P\} = \sum_{i \in I^{\text{sh}}} \mathbb{Z}\overline{\varpi}_i, \quad \overline{P}_+ = \sum_{i \in I^{\text{sh}}} \mathbb{Z}_{\geq 0}\overline{\varpi}_i.$$

We denote by $U_q(\widehat{\mathfrak{g}})$ the quantum affine algebra associated with $\widehat{\mathfrak{g}}$ over $\mathbb{C}(q)$, and by $U_q'(\widehat{\mathfrak{g}})$ the quantum affine algebra without the degree operator. Let $U_q(\widehat{\mathfrak{g}}^{\text{sh}})$ and $U_q'(\widehat{\mathfrak{g}}^{\text{sh}})$ denote the ones associated with $\widehat{\mathfrak{g}}^{\text{sh}}$.

Let H be one of the vector spaces $\widehat{\mathfrak{h}}$, $\mathfrak{h}_d$, $\mathfrak{h}$, $\widehat{\mathfrak{h}}^{\text{sh}}$, $\mathfrak{h}_d^{\text{sh}}$ and $\mathfrak{h}^{\text{sh}}$. For an H-module M and $\lambda \in H^*$, we denote the weight space of M with weight λ by

$$M_\lambda = \{v \in M \mid h.v = \langle \lambda, h\rangle v \text{ for } h \in H\},$$

and if $M = \bigoplus_{\lambda \in H^*} M_\lambda$, we say M is an H-weight module. We denote the set of H-weights by $\operatorname{wt}_H(M) = \{\lambda \in H^* \mid M_\lambda \neq 0\}$. If M is an finite dimensional H-weight module, we define

$$\operatorname{ch}_H M = \sum_{\lambda \in H^*} (\dim M_\lambda) e(\lambda),$$

where $e(\lambda)$ are formal basis elements of the group algebra $\mathbb{C}[H^*]$ with multiplication defined by $e(\lambda)e(\mu) = e(\lambda + \mu)$. For a subset $S \subseteq H^*$, we denote by P_S the linear map $\mathbb{C}[H^*] \to \mathbb{C}[H^*]$ defined by

$$P_S(e(\lambda)) = \begin{cases} e(\lambda) & \text{if } \lambda \in S, \\ 0 & \text{if } \lambda \notin S. \end{cases}$$

The map $i_{\text{sh}} : (\widehat{\mathfrak{h}}^{\text{sh}})^* \to \widehat{\mathfrak{h}}^*$ induces naturally a linear map $\mathbb{C}[(\widehat{\mathfrak{h}}^{\text{sh}})^*] \to \mathbb{C}[\widehat{\mathfrak{h}}^*]$, which we denote by i_{sh}, too. The following lemma is used later:

Lemma 2.3. *Let $\lambda \in \mathfrak{h}^*$, and assume that M is a $\mathfrak{h}$-weight $\mathcal{C}\mathfrak{g}$-module generated by $v \in M_\lambda$ and v is annihilated by $\mathfrak{n}_+ \otimes \mathbb{C}[t] \oplus \mathfrak{h} \otimes t\mathbb{C}[t]$. Then $W = U(\mathcal{C}\mathfrak{g}^{\text{sh}}).v$ satisfies*

$$P_{\lambda - Q_+^{\text{sh}}} \operatorname{ch}_{\mathfrak{h}} M = \operatorname{ch}_{\mathfrak{h}} W = e(\lambda - i_{\text{sh}}(\overline{\lambda}))i_{\text{sh}}\operatorname{ch}_{\mathfrak{h}^{\text{sh}}} W.$$

Proof. Put $\mathfrak{n}_-' = \sum_{\alpha \in \Delta_- \setminus \Delta_-^{\text{sh}}} \mathfrak{g}_\alpha$. Then we have $\mathfrak{n}_- = \mathfrak{n}_-^{\text{sh}} \oplus \mathfrak{n}_-'$, and we have

$$\begin{aligned} M &= U(\mathfrak{n}_- \otimes \mathbb{C}[t]).v \\ &= U(\mathfrak{n}_-^{\text{sh}} \otimes \mathbb{C}[t]).v + U(\mathfrak{n}_-^{\text{sh}} \otimes \mathbb{C}[t])U(\mathfrak{n}_-' \otimes \mathbb{C}[t])_+.v \\ &= W + U(\mathfrak{n}_-^{\text{sh}} \otimes \mathbb{C}[t])U(\mathfrak{n}_-' \otimes \mathbb{C}[t])_+.v, \end{aligned}$$

where $U(\mathfrak{n}'_-\otimes\mathbb{C}[t])_+$ is the augmentation ideal. It is obvious that $\mathrm{wt}_{\mathfrak{h}}(W)\subseteq\lambda-Q_+^{\mathrm{sh}}$, and

$$\mathrm{wt}_{\mathfrak{h}}\Big(U(\mathfrak{n}_-^{\mathrm{sh}}\otimes\mathbb{C}[t])U(\mathfrak{n}'_-\otimes\mathbb{C}[t])_+.v\Big)\cap(\lambda-Q_+^{\mathrm{sh}})=\emptyset.$$

Hence, the first equality follows. The second equality follows from the following fact, which is easily checked: if $\mu\in\lambda-Q_+^{\mathrm{sh}}$, we have $\mu=i_{\mathrm{sh}}(\overline{\mu})+(\lambda-i_{\mathrm{sh}}(\overline{\lambda}))$. □

3 Weyl modules and Demazure modules

3.1 Weyl modules

In this article, we consider the following $\mathcal{C}\mathfrak{g}_d$-module:

Definition 3.1. For $\lambda\in P_+$, the $\mathcal{C}\mathfrak{g}_d$-module $W(\lambda)$ is the module generated by an element v with the relations:

$$\mathfrak{n}_+\otimes\mathbb{C}[t].v=0,\quad h\otimes t^s.v=\delta_{s0}\langle\lambda,h\rangle v \text{ for } h\in\mathfrak{h}, s\in\mathbb{Z}_{\geq0},\quad d.v=0,\qquad(2)$$

and

$$f_i^{\langle\lambda,\alpha_i^\vee\rangle+1}.v=0 \text{ for } i\in I.$$

We call $W(\lambda)$ the *Weyl module* for $\mathcal{C}\mathfrak{g}_d$ associated with $\lambda\in P_+$.

Remark 3.2. $W(\lambda)$ is a *$\mathbb{Z}$-graded version* of the Weyl module for $\mathcal{C}\mathfrak{g}$ introduced in [4], [8]. Indeed, it is easily seen that $W(\lambda)$ is cyclic as a $\mathcal{C}\mathfrak{g}$-module and the defining relations of $W(\lambda)$ as a $\mathcal{C}\mathfrak{g}$-module are

$$\mathfrak{n}_+\otimes\mathbb{C}[t].v=0,\quad h\otimes t^s.v=\delta_{s0}\langle\lambda,h\rangle v \text{ for } h\in\mathfrak{h}, s\in\mathbb{Z}_{\geq0},$$

and

$$f_i^{\langle\lambda,\alpha_i^\vee\rangle+1}.v=0 \text{ for } i\in I,$$

which is the ones of the Weyl module for $\mathcal{C}\mathfrak{g}$ defined in these articles.

The following theorem follows from [4, Theorem 1.2.2]:

Theorem 3.3. *For $\lambda\in P_+$, the Weyl module $W(\lambda)$ is finite dimensional. Moreover, any finite dimensional $\mathcal{C}\mathfrak{g}_d$-module generated by an element v satisfying the relations* (2) *is a quotient of $W(\lambda)$.*

3.2 Demazure modules

We denote by $V(\Lambda)$ the irreducible highest weight $\widehat{\mathfrak{g}}$-module with highest weight $\Lambda\in\widehat{P}_+$. Recall that for any $w\in\widehat{W}$ we have $\dim V(\Lambda)_{w\Lambda}=1$.

Definition 3.4. For $w\in\widehat{W}$, the $\widehat{\mathfrak{b}}$-module

$$V_w(\Lambda)=U(\widehat{\mathfrak{b}}).V(\Lambda)_{w\Lambda}=U(\widehat{\mathfrak{n}}_+).V(\Lambda)_{w\Lambda}$$

is called the *Demazure submodule* of $V(\Lambda)$ associated with w.

Remark 3.5. Note that $V_w(\Lambda)=V_{w'}(\Lambda)$ if $w\Lambda=w'\Lambda$.

Since $f_i.V(\Lambda)_{w\Lambda} = 0$ follows if and only if $\langle w\Lambda, \alpha_i^\vee \rangle \le 0$, we can see that $V_w(\Lambda)$ is f_i-stable if and only if $\langle w\Lambda, \alpha_i^\vee \rangle \le 0$. In this article, we are mainly interested in the Demazure modules which are $\mathfrak{g}$-stable. From the above observation, $V_w(\Lambda)$ is $\mathfrak{g}$-stable if and only if $\langle w\Lambda, \alpha_i^\vee \rangle \le 0$ for all $i \in I$, which is equivalent to that $w\Lambda \in -P_+ + \ell\Lambda_0 + \mathbb{Z}\delta$, where ℓ is the level of Λ. For simplicity, we use the following alternative notation: Let $\lambda \in P_+, \ell \in \mathbb{Z}_{>0}, m \in \mathbb{Z}$. There exists a unique $\Lambda \in \widehat{P}_+$ such that $w_0\lambda + \ell\Lambda_0 + m\delta \in \widehat{W}\Lambda$. For an element $w \in \widehat{W}$ such that $w\Lambda = w_0\lambda + \ell\Lambda_0 + m\delta$, we write

$$\mathcal{D}(\ell, \lambda)[m] = V_w(\Lambda),$$

which is a $\mathcal{C}\mathfrak{g}_d \oplus \mathbb{C}K$-module as stated above. We usually consider only the $\mathcal{C}\mathfrak{g}_d$-module structure of $\mathcal{D}(\ell, \lambda)[m]$. For any $\Lambda \in \widehat{P}_+$ and $m \in \mathbb{Z}$, we have $V(\Lambda) \cong V(\Lambda + m\delta)$ as $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$-modules. Therefore, the $\mathcal{C}\mathfrak{g}$-module structure of $\mathcal{D}(\ell, \lambda)[m]$ is independent of m. We denote this $\mathcal{C}\mathfrak{g}$-module isomorphism class simply by $\mathcal{D}(\ell, \lambda)$.

$\mathcal{D}(\ell, \lambda)$ and $\mathcal{D}(\ell, \lambda)[m]$ have descriptions in terms of generators and relations as follows:

Proposition 3.6. (i) *$\mathcal{D}(\ell, \lambda)$ is isomorphic as a $\mathcal{C}\mathfrak{g}$-module to the cyclic module generated by an element v with relations:*

$$\mathfrak{n}_+ \otimes \mathbb{C}[t].v = 0, \;\; h \otimes t^s.v = \delta_{s0}\langle \lambda, h \rangle v \text{ for } h \in \mathfrak{h}, s \in \mathbb{Z}_{\ge 0}, \tag{3}$$

and for $\gamma \in \Delta_+$ and $s \in \mathbb{Z}_{\ge 0}$,

$$(f_\gamma \otimes t^s)^{k_{\gamma,s}+1}.v = 0 \text{ where } k_{\gamma,s} = \begin{cases} \max\{0, \langle \lambda, \gamma^\vee \rangle - \ell s\} & \text{if } \gamma \text{ is long,} \\ \max\{0, \langle \lambda, \gamma^\vee \rangle - r\ell s\} & \text{if } \gamma \text{ is short} \end{cases} \tag{4}$$

(r is the number defined in (1)).
(ii) *$\mathcal{D}(\ell, \lambda)[m]$ is isomorphic as a $\mathcal{C}\mathfrak{g}_d$-module to the cyclic module generated by an element v with relations (3), (4) and $d.v = mv$.*

Proof. (ii) follows easily from (i), and (i) can be proved by the same way as [8, Corollary 1] using

$$\begin{aligned} -\langle \lambda + \ell\Lambda_0 + m\delta, (-\gamma + s\delta)^\vee \rangle &= \langle \lambda + \ell\Lambda_0 + m\delta, \gamma^\vee - \frac{2s}{(\gamma,\gamma)}K \rangle \\ &= \langle \lambda, \gamma^\vee \rangle - \frac{2\ell s}{(\gamma,\gamma)}. \end{aligned}$$

□

The following theorem is a reformulation of [8, Theorem 7] for our setting:

Theorem 3.7. *Assume that $\mathfrak{g}$ is simply laced. Then the Weyl module $W(\lambda)$ is isomorphic to $\mathcal{D}(1, \lambda)[0]$ as $\mathcal{C}\mathfrak{g}_d$-module.*

Proof. Note that the notation "$D(\ell, \lambda^\vee)$" in [8] coincides with $\mathcal{D}(\ell, \ell\nu(\lambda^\vee))$ in our notation. Since $\nu(\lambda^\vee) = \lambda$ in the simply laced case, [8, Theorem 7] says that $W(\lambda) \cong \mathcal{D}(1, \lambda)$ as $\mathcal{C}\mathfrak{g}$-modules. Then $W(\lambda) \cong \mathcal{D}(1, \lambda)[0]$ as $\mathcal{C}\mathfrak{g}_d$-modules obviously holds. □

Later, we need the following lemma:

Lemma 3.8. *Assume that* $\mathfrak{g}$ *is of type* A_n*. Then for any* $\ell \in \mathbb{Z}_{>0}$ *and* $i \in I$*,* $\mathcal{D}(\ell, \varpi_i)$ *is isomorphic to* $V_{\mathfrak{g}}(\varpi_i)$ *as a* $\mathfrak{g}$*-module.*

Proof. Although this lemma can be shown directly from the definition, we prove this using Proposition 3.6. Since $\mathfrak{g}$ is of type A_n, we have $\langle \varpi_i, \gamma^\vee \rangle = 0$ or 1 for all $\gamma \in \Delta_+$. From this and Proposition 3.6, the generator $v \in \mathcal{D}(\ell, \varpi_i)$ satisfies $f_\gamma \otimes t\mathbb{C}[t].v = 0$ for all $\gamma \in \Delta_+$, which implies

$$\mathcal{D}(\ell, \varpi_i) = U(\mathcal{C}\mathfrak{g}).v = U(\mathfrak{g}).v.$$

Since v satisfies the defining relations of $V_{\mathfrak{g}}(\varpi_i)$ in Lemma 2.1, $\mathcal{D}(\ell, \varpi_i)$ is a $\mathfrak{g}$-module quotient of $V_{\mathfrak{g}}(\varpi_i)$. Since $\mathcal{D}(\ell, \varpi_i)$ is non-trivial, the lemma follows. □

3.3 Fusion product

We briefly recall the definition of fusion products of $\mathcal{C}\mathfrak{g}$-modules introduced in [6] and some facts on them.

Let W be a $\mathcal{C}\mathfrak{g}$-module. For $a \in \mathbb{C}$, we define a $\mathcal{C}\mathfrak{g}$-module W_a by the pullback $\varphi_a^* W$, where φ_a is an automorphism of $\mathcal{C}\mathfrak{g}$ defined by $x \otimes t^s \mapsto x \otimes (t+a)^s$. $U(\mathcal{C}\mathfrak{g})$ has a natural grading such that

$$G^s(U(\mathcal{C}\mathfrak{g})) = \{X \in U(\mathcal{C}\mathfrak{g}) \mid [d, X] = sX\},$$

from which we define a natural filtration on $U(\mathcal{C}\mathfrak{g})$ by

$$F^s(U(\mathcal{C}\mathfrak{g})) = \bigoplus_{p \leq s} G^p(U(\mathcal{C}\mathfrak{g})).$$

Let now W be a cyclic $\mathcal{C}\mathfrak{g}$-module with a cyclic vector w. Denote by W_s the subspace $F^s(U(\mathcal{C}\mathfrak{g})).w$ of W, and denote the associated $\mathcal{C}\mathfrak{g}$-module by $\mathrm{gr}(W)$:

$$\mathrm{gr}(W) = \bigoplus_{s \geq 0} W_s / W_{s-1},$$

where we put $W_{-1} = 0$.

Now we recall the definition of fusion products. Let $W^1, \ldots, W^k$ be $\mathbb{Z}$-graded cyclic finite dimensional $\mathcal{C}\mathfrak{g}$-modules with cyclic vectors $w_1, \ldots, w_k$, and let $c_1, \ldots, c_k$ be pairwise distinct complex numbers. As shown in [6], $W^1_{c_1} \otimes \cdots \otimes W^k_{c_k}$ is a cyclic $U(\mathcal{C}\mathfrak{g})$-module generated by $w_1 \otimes \cdots \otimes w_k$.

Definition 3.9 ([6]). The $\mathcal{C}\mathfrak{g}$-module

$$W^1_{c_1} * \cdots * W^k_{c_k} = \mathrm{gr}(W^1_{c_1} \otimes \cdots \otimes W^k_{c_k})$$

is called the *fusion product.*

Remark 3.10. Put $X = W^1_{c_1} \otimes \cdots \otimes W^k_{c_k}$. By letting d act on X_s / X_{s-1} by a scalar s, we sometimes consider $W^1_{c_1} * \cdots * W^k_{c_k}$ as a $\mathcal{C}\mathfrak{g}_d$-module.

Lemma 3.11. (i)

$$\mathrm{ch}_{\mathfrak{h}} W^1_{c_1} * \cdots * W^k_{c_k} = \prod_{1\leq i\leq k} \mathrm{ch}_{\mathfrak{h}} W^i.$$

(ii) *Let $\lambda_1,\dots,\lambda_k \in P_+$, and let $\lambda = \lambda_1+\cdots+\lambda_k$. Then there exists a surjective $C\mathfrak{g}_d$-module homomorphism from $W(\lambda)$ to $\mathcal{D}(1,\lambda_1)_{c_1} * \cdots * \mathcal{D}(1,\lambda_k)_{c_k}$.*

(iii) *If $\mathfrak{g}$ is simply laced, the surjection in* (ii) *is an isomorphism.*

Proof. Since $W^1_{c_1} * \cdots * W^k_{c_k}$ is isomorphic to $W^1 \otimes \cdots \otimes W^k$ as a $\mathfrak{g}$-module, (i) follows. We can show (ii) by the same way as [8, Lemma 5]. From Theorem 3.7 and [8, Theorem 8], (iii) follows . □

4 Filtrations on the Weyl modules

4.1 The defining relations of the Demazure modules

The goal of this section is to show that the Weyl module admits a filtration whose subquotients are surjective images of Demazure modules. To do this, however, the defining relations of $\mathcal{D}(\ell,\lambda)$ given in Proposition 3.6 is insufficient, and we need to reduce the relations in the case $\ell = 1$. This and the next subsections are devoted to prove that $\mathcal{D}(1,\lambda)$ has the following refined version of the defining relations:

Proposition 4.1. *For $\lambda \in P_+$, $\mathcal{D}(1,\lambda)$ is isomorphic as a $C\mathfrak{g}$-module to the cyclic module generated by an element v with the following relations:*
(D1) $\mathfrak{n}_+ \otimes \mathbb{C}[t].v = 0$,
(D2) $h \otimes t^s.v = \delta_{s0}\langle\lambda, h\rangle v$ *for* $h \in \mathfrak{h}, s \in \mathbb{Z}_{\geq 0}$,
(D3) $f_i^{\langle\lambda,\alpha_i^\vee\rangle+1}.v = 0$ *for* $i \in I$,
(D4) $(f_\gamma \otimes t^s)^{\max\{0,\langle\lambda,\gamma^\vee\rangle - rs\}+1}.v = 0$ *for* $\gamma \in \Delta_+^{\mathrm{sh}}, s \in \mathbb{Z}_{\geq 0}$.

This proposition obviously implies the following corollary:

Corollary 4.2. *For $\lambda \in P_+$ and $m \in \mathbb{Z}$, $\mathcal{D}(1,\lambda)[m]$ is isomorphic as a $C\mathfrak{g}_d$-module to the cyclic module generated by an element v with the relations* (D1)–(D4) *and $d.v = mv$.*

If $\mathfrak{g}$ is simply laced, the relations in Proposition 4.1 are just the relations of the Weyl module $W(\lambda)$ (see Remark 3.2 for the defining relations of $W(\lambda)$ as a $C\mathfrak{g}$-module). Hence, in this case, the proposition follows from Theorem 3.7. Thus, to the end of the proof of the proposition, we assume that $\mathfrak{g}$ is non-simply laced.

We denote by M^λ the cyclic $C\mathfrak{g}$-module generated by an element v with the relations (D1)–(D4). By Proposition 3.6, to show the above proposition we need to show that $v \in M^\lambda$ satisfies

$$(f_\gamma \otimes t^s)^{k_{\gamma,s}+1}.v = 0 \text{ where } k_{\gamma,s} = \begin{cases} \max\{0, \langle\lambda,\gamma^\vee\rangle - s\} & \text{if } \gamma \text{ is long,} \\ \max\{0, \langle\lambda,\gamma^\vee\rangle - rs\} & \text{if } \gamma \text{ is short} \end{cases} \tag{5}$$

for all $\gamma \in \Delta_+$ and $s \in \mathbb{Z}_{\geq 0}$. We show the equation (5) by separating the proof into several cases. First, the following case is elementary:

Lemma 4.3. *For $\gamma \in \Delta_+$ and $s = 0$, (5) follows.*

Proof. From (D1)–(D3), we can see that $U(\mathfrak{g}).v \subseteq M^\lambda$ is a $\mathfrak{g}$-module quotient of $V_{\mathfrak{g}}(\lambda)$. Hence $\mathfrak{sl}_2$-theory implies the lemma. □

The proof of the following case uses the same argument with the one used in the proof of [8, Theorem 7]:

Lemma 4.4. *When γ is a long root, (5) follows for all $s \in \mathbb{Z}_{\geq 0}$.*

Proof. Take a Lie subalgebra

$$\mathfrak{sl}_{2,\gamma} = \mathbb{C}e_\gamma \oplus \mathbb{C}\gamma^\vee \oplus \mathbb{C}f_\gamma \subseteq \mathfrak{g}$$

which is isomorphic to $\mathfrak{sl}_2$, and let $C\mathfrak{sl}_{2,\gamma} = \mathfrak{sl}_{2,\gamma} \otimes \mathbb{C}[t]$. Let $N = U(C\mathfrak{sl}_{2,\gamma}).v$. Note that v satisfies the relations

$$e_\gamma \otimes \mathbb{C}[t].v = 0, \quad \gamma^\vee \otimes t^s.v = \delta_{s0}\langle \lambda, \gamma^\vee \rangle v, \quad f_\gamma^{\langle \lambda, \gamma^\vee \rangle + 1}.v = 0,$$

which is the defining relations of the Weyl module $W_\gamma(\langle \lambda, \gamma^\vee \rangle)$ for $C\mathfrak{sl}_{2,\gamma}$ (see Remark 3.2). Here, we identify the weight lattice of $\mathfrak{sl}_{2,\gamma}$ with $\mathbb{Z}$. Hence, N is a quotient of this Weyl module $W_\gamma(\langle \lambda, \gamma^\vee \rangle)$. By Theorem 3.7, $W_\gamma(\langle \lambda, \gamma^\vee \rangle)$ is isomorphic to $C\mathfrak{sl}_{2,\gamma}$-Demazure module $\mathcal{D}_\gamma(1, \langle \lambda, \gamma^\vee \rangle)$. In particular, v satisfies the defining relations of $\mathcal{D}_\gamma(1, \langle \lambda, \gamma^\vee \rangle)$ stated in Proposition 3.6, which contain the relations

$$(f_\gamma \otimes t^s)^{\max\{0, \langle \lambda, \gamma^\vee \rangle - s\} + 1}.v = 0$$

for all $s \in \mathbb{Z}_{\geq 0}$. □

Before starting the proof of remaining cases, we prepare an elementary lemma:

Lemma 4.5. *Assume that the rank of $\mathfrak{g}$ is 2, that is $\dim \mathfrak{h} = 2$, and let $\Pi = \{\alpha, \beta\}$. Let N be a $U(\mathfrak{n}_-)$-module, and we assume that an element $v \in N$ satisfies*

$$f_\alpha^{a+1}.v = 0, \quad f_\beta^{b+1}.v = 0$$

for some $a, b \in \mathbb{Z}_{\geq 0}$. Then for $\gamma \in \Delta_+$ such that $\gamma^\vee = n_1\alpha^\vee + n_2\beta^\vee$, we have

$$f_\gamma^{n_1 a + n_2 b + 1}.v = 0.$$

Proof. Let ϖ_α and ϖ_β denote the fundamental weights corresponding to α and β respectively, and let $\mu = a\varpi_\alpha + b\varpi_\beta$. The following isomorphism as $U(\mathfrak{n}_-)$-modules is well-known:

$$V_{\mathfrak{g}}(\mu) \cong U(\mathfrak{n}_-)/U(\mathfrak{n}_-)(\mathbb{C}f_\alpha^{a+1} + \mathbb{C}f_\beta^{b+1}).$$

Therefore, there exists a $U(\mathfrak{n}_-)$-module homomorphism from $V_{\mathfrak{g}}(\mu)$ to N which maps a highest weight vector to v. Since a highest weight vector of $V_{\mathfrak{g}}(\mu)$ satisfies the relation for γ, so does v. □

It remains to prove that the equation (5) follows for short $\gamma \in \Delta_+$. In the rest of this subsection, we prove that this statement is true if $\mathfrak{g}$ is of type B_n, C_n or F_4, and we prove the statement in G_2 case in the next subsection. Note that if $\mathfrak{g}$ is of type B_n, C_n or F_4, we have $r = 2$.

Lemma 4.6. *Assume that* $\mathfrak{g}$ *is of type* B_n, C_n *or* F_4. *If* $\gamma \in \Delta_+ \setminus \Delta_+^{\mathrm{sh}}$ *is a short root, there exists a short root* $\alpha \in \Delta_+$ *and a long root* $\beta \in \Delta_+$ *such that* $\gamma = \alpha + \beta$.

Proof. We prove by induction on $\operatorname{ht}\gamma$. Put

$$S = \{\alpha \in \Delta_+ \mid \alpha \notin \Delta_+^{\mathrm{sh}},\ \alpha \text{ is short}\},$$

and take arbitrary $\gamma \in S$. Since $\gamma \in \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$ and $(\gamma, \gamma) > 0$, there exists some $i \in I$ such that $(\gamma, \alpha_i) > 0$. Since $\gamma \notin \Pi^{\mathrm{sh}}$ and γ is short, we have $\gamma \notin \Pi$, in particular $\gamma \neq \alpha_i$. Then, we have $\langle \gamma, \alpha_i^\vee \rangle = 1$ since γ is short, and therefore we have $\gamma = s_i(\gamma) + \alpha_i$. Note that $s_i(\gamma)$ is a short root. If α_i is long, this implies the lemma. Assume that $\alpha_i \in \Pi^{\mathrm{sh}}$. Then by Lemma 2.2, we have $s_i(\gamma) \in S$. Since $\operatorname{ht} s_i(\gamma) < \operatorname{ht}\gamma$, by the induction hypothesis there exist short $\alpha \in \Delta_+$ and long $\beta \in \Delta_+$ such that $s_i(\gamma) = \alpha + \beta$. If $\alpha = \alpha_i$, we have $\gamma = \beta + 2\alpha$, which contradicts that γ is short. Hence we have $\alpha \neq \alpha_i$, which implies $s_i(\alpha) \in \Delta_+$, and then $\gamma = s_i(\alpha) + s_i(\beta)$ implies the lemma. □

Now, we complete the proof of Proposition 4.1 for $\mathfrak{g}$ of type B_n, C_n or F_4:

Proposition 4.7. *Assume that* $\mathfrak{g}$ *is of type* B_n, C_n *or* F_4. *Then* (5) *follows for all short* $\gamma \in \Delta_+$ *and* $s \in \mathbb{Z}_{\geq 0}$.

Proof. We have to show that

$$(f_\gamma \otimes t^s)^{\max\{0, \langle \lambda, \gamma^\vee \rangle - 2s\}+1}.v = 0. \tag{6}$$

We show this by induction on $\operatorname{ht}\gamma$. If $\operatorname{ht}\gamma = 1$, this trivially follows from (D4) since $\gamma \in \Pi^{\mathrm{sh}}$. Assume $\operatorname{ht}\gamma > 1$. We can also assume that $\gamma \notin \Delta_+^{\mathrm{sh}}$. By Lemma 4.6, there exist short $\alpha \in \Delta_+$ and long $\beta \in \Delta_+$ such that $\gamma = \alpha + \beta$. Put

$$a = \langle \lambda, \alpha^\vee \rangle, \quad b = \langle \lambda, \beta^\vee \rangle.$$

Now, fix arbitrary $s \in \mathbb{Z}_{\geq 0}$, and put $q = \min\{b, s\}$, $p = s - q$. By the induction hypothesis, we have

$$(f_\alpha \otimes t^p)^{\max\{0, a-2p\}+1}.v = 0, \tag{7}$$

and we have from Lemma 4.4 that

$$(f_\beta \otimes t^q)^{b-q+1}.v = 0. \tag{8}$$

It is easily checked that the root subsystem $(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Delta$ is the root system of type B_2 with a basis $\{\alpha, \beta\}$. Hence, the Lie subalgebra

$$\mathbb{C}f_\alpha \otimes t^p + \mathbb{C}f_\beta \otimes t^q + \mathbb{C}f_\gamma \otimes t^{p+q} + \mathbb{C}f_{2\alpha+\beta} \otimes t^{2p+q} \subseteq C\mathfrak{g}$$

is isomorphic to the nilradical of the Borel subalgebra of $\mathfrak{so}_5$ (simple Lie algebra of type B_2). Since $\gamma^\vee = \alpha^\vee + 2\beta^\vee$, we have from Lemma 4.5, (7) and (8) that

$$(f_\gamma \otimes t^{p+q})^{\max\{0, a-2p\}+2(b-q)+1}.v = 0.$$

Since $\langle \lambda, \gamma^\vee \rangle = a + 2b$ and $s = p + q$, it is easily checked that this equation is equivalent to (6). □

4.2 Proof for the type G_2

In this subsection, we assume $\mathfrak{g}$ is of type G_2. We denote by α the short simple root and by β the long simple root. Note that

$$\Delta_+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}.$$

Let $\varpi_\alpha, \varpi_\beta$ be the corresponding fundamental weights.

By Lemma 4.4, to complete the proof of Proposition 4.1 for $\mathfrak{g}$, we need to show the equation (5) for $\gamma = \alpha+\beta, 2\alpha+\beta$. The first one is easy:

Lemma 4.8. *For $\gamma = \alpha+\beta$ and $s \in \mathbb{Z}_{\geq 0}$, (5) follows.*

Proof. This proof is similar to the one given in Proposition 4.7, and we shall only give the sketch of it. Assume that $\lambda = a\varpi_\alpha + b\varpi_\beta$ with $a, b \in \mathbb{Z}_{\geq 0}$. Since $(\alpha+\beta)^\vee = \alpha^\vee + 3\beta^\vee$, what we have to show is that $v \in M^\lambda$ satisfies

$$(f_{\alpha+\beta} \otimes t^s)^{\max\{0, a+3b-3s\}+1}.v = 0 \tag{9}$$

for all $s \in \mathbb{Z}_{\geq 0}$. Fix arbitrary $s \in \mathbb{Z}_{\geq 0}$, and put $q = \min\{b, s\}$, $p = s - q$. Since

$$(f_\alpha \otimes t^p)^{\max\{0, a-3p\}+1}.v = 0 \text{ and } (f_\beta \otimes t^q)^{b-q+1}.v = 0$$

follow, we have from Lemma 4.5 that

$$(f_{\alpha+\beta} \otimes t^{p+q})^{\max\{0, a-3p\}+3(b-q)+1}.v = 0.$$

Then this equation is equivalent to (9). $\square$

It remains to prove (5) for $\gamma = 2\alpha+\beta$. The proof in this case is a relatively heavy task. Throughout the rest of this subsection, to simplify the notation we abbreviate $X \otimes t^k$ for $X \in \mathfrak{g}$ as Xt^k, and also abbreviate $\max\{k_1, k_2\}$ as $\{k_1, k_2\}$.

For $a, b \in \mathbb{Z}_{\geq 0}$, we define a subspace $I_{a,b}$ of $U(\mathcal{C}\mathfrak{g})$ by

$$I_{a,b} = \mathfrak{n}_+\mathbb{C}[t] + \mathfrak{h}t\mathbb{C}[t] + \mathbb{C}(\alpha^\vee - a) + \mathbb{C}(\beta^\vee - b) + \sum_{s\geq 0} \mathbb{C}(f_\alpha t^s)^{\{0, a-3s\}+1} + \mathbb{C}f_\beta^{b+1}.$$

Note that $U(\mathcal{C}\mathfrak{g})I_{a,b}$ is the left ideal generated by the relations in Proposition 4.1 with $\lambda = a\varpi_\alpha + b\varpi_\beta$. We shall prove the equation

$$(f_{2\alpha+\beta}t^s)^{\{0, 2a+3b-3s\}+1} \in U(\mathcal{C}\mathfrak{g})I_{a,b} \tag{10}$$

for all $a, b \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{Z}_{\geq 0}$, which is equivalent to (5) with $\lambda = a\varpi_\alpha + b\varpi_\beta$ and $\gamma = 2\alpha+\beta$. In the following proof, we use repeatedly the fact that $X \in U(\mathcal{C}\mathfrak{g})$ annihilates $v \in M^\lambda$ if and only if $X \in U(\mathcal{C}\mathfrak{g})I_{a,b}$ without any mention.

Before starting the proof of (10), we prepare two lemmas:

Lemma 4.9. *For $s_1, s_2 \in \mathbb{Z}_{\geq 0}$, we have*

$$(f_{2\alpha+\beta}t^{2s_1+s_2})^{2\{0, a-3s_1\}+3\{0, b-s_2\}+1} \in U(\mathcal{C}\mathfrak{g})I_{a,b}.$$

Proof. By Lemma 4.4, we have

$$(f_\beta t^{s_2})^{\{0,b-s_2\}+1} \in U(C\mathfrak{g})I_{a,b}.$$

Then since $\gamma^\vee = 2\alpha^\vee + 3\beta^\vee$, we have from Lemma 4.5 that

$$(f_{2\alpha+\beta}t^{2s_1+s_2})^{2\{0,a-3s_1\}+3\{0,b-s_2\}+1} \\ \in U(C\mathfrak{g})(\mathbb{C}(f_\alpha t^{s_1})^{\{0,a-3s_1\}+1} + \mathbb{C}(f_\beta t^{s_2})^{\{0,b-s_2\}+1}) \subseteq U(C\mathfrak{g})I_{a,b}.$$

□

Lemma 4.10. *Let $\{e,h,f\}$ be the Chevalley basis of $\mathfrak{sl}_2$, and let $a \in \mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{Z}_{>0}$. We define a Lie subalgebra $\mathfrak{a}$ of $C\mathfrak{sl}_2 = \mathfrak{sl}_2 \otimes \mathbb{C}[t]$ by*

$$\mathfrak{a} = et\mathbb{C}[t] + h\mathbb{C}[t] + f\mathbb{C}[t],$$

and let I be a subspace of $U(\mathfrak{a})$ defined by

$$I = et\mathbb{C}[t] + ht\mathbb{C}[t] + \mathbb{C}(h-a) + \sum_{s\geq 0}\mathbb{C}(ft^s)^{\{0,a-\ell s\}+1}.$$

Then we have for all $p \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{Z}_{\geq 0}$ that

$$e^p(ft^s)^{\{0,a-\ell s\}+1} \in U(\mathfrak{a})I + U(C\mathfrak{sl}_2)e.$$

Proof. Applying an involution on $C\mathfrak{sl}_2$ defined by $et^k \leftrightarrow ft^k$, $ht^k \leftrightarrow -ht^k$, we prove the following statement which is equivalent to the lemma: put $\mathfrak{a}' = ft\mathbb{C}[t] + h\mathbb{C}[t] + e\mathbb{C}[t]$ and

$$I' = ft\mathbb{C}[t] + ht\mathbb{C}[t] + \mathbb{C}(h+a) + \sum_{s\geq 0}\mathbb{C}(et^s)^{\{0,a-\ell s\}+1}.$$

Then we have for all $p \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{Z}_{\geq 0}$ that

$$f^p(et^s)^{\{0,a-\ell s\}+1} \in U(\mathfrak{a}')I' + U(C\mathfrak{sl}_2)f.$$

Fix arbitrary p and s. Since $U(C\mathfrak{sl}_2) = U(\mathfrak{a}') \oplus U(C\mathfrak{sl}_2)f$, there exists $X \in U(C\mathfrak{sl}_2)$ such that

$$f^p(et^s)^{\{0,a-\ell s\}+1} - Xf \in U(\mathfrak{a}').$$

Consider the $\widehat{\mathfrak{sl}_2}$-Demazure module $V_w(\Lambda)$ such that $w\Lambda = \ell\Lambda_0 - a\varpi_1$, and let $v_{w\Lambda}$ be a nonzero vector in $V_w(\Lambda)_{w\Lambda}$. By [8, Theorem 1], the $U(\mathfrak{a}')$-annihilator of $v_{w\Lambda}$ is $\mathrm{Ann}_{U(\mathfrak{a}')}v_{w\Lambda} = U(\mathfrak{a}')I'$. Then since

$$(f^p(et^s)^{\{0,a-\ell s\}+1} - Xf).v_{w\Lambda} = 0,$$

we have $f^p(et^s)^{\{0,a-\ell s\}+1} - Xf \in U(\mathfrak{a}')I'$. □

We fix arbitrary $b \in \mathbb{Z}_{\geq 0}$, and we prove the equation (10) for fixed b by induction on $a \in \mathbb{Z}_{\geq 0}$. To begin the induction, we first prove (10) for $a = 0,1,2$.
(i) When $a = 0$, Lemma 4.9 with $s_1 = 0, s_2 = s$ implies (10).
(ii) Assume $a = 1$. If $s \leq b$, Lemma 4.9 with $s_1 = 0, s_2 = s$ implies (10). If $s \geq b+2$, Lemma 4.9 with $s_1 = 1, s_2 = s-2$ implies (10). Let $s = b+1$. By Lemma 4.4 and $\langle \varpi_\alpha + b\varpi_\beta, (3\alpha+\beta)^\vee\rangle = b+1$, we have

$$f_{3\alpha+\beta}t^{b+1} \in U(C\mathfrak{g})I_{1,b}.$$

Then (10) follows since

$$[e_\alpha, f_{3\alpha+\beta}t^{b+1}] = e_\alpha f_{3\alpha+\beta}t^{b+1} - f_{3\alpha+\beta}t^{b+1}e_\alpha \in U(\mathcal{C}\mathfrak{g})I_{1,b}.$$

(iii) Assume $a = 2$. If $s \neq b+1$, we can show (10) by the same way as (ii). Let $s = b+1$. By Lemma 4.4 and 4.8, we have

$$(f_{3\alpha+\beta}t^{b+1})^2 \in U(\mathcal{C}\mathfrak{g})I_{2,b} \text{ and } f_{\alpha+\beta}t^{b+1} \in U(\mathcal{C}\mathfrak{g})I_{2,b}.$$

Then (10) follows from the following calculation:

$$\begin{aligned} U(\mathcal{C}\mathfrak{g})I_{2,b} \ni e_\alpha^2(f_{3\alpha+\beta}t^{b+1})^2 &= e_\alpha(f_{3\alpha+\beta}t^{b+1})^2 e_\alpha + 2[e_\alpha, f_{3\alpha+\beta}t^{b+1}]^2 \\ &+ 2f_{3\alpha+\beta}t^{b+1}[e_\alpha, f_{3\alpha+\beta}t^{b+1}]e_\alpha + 2f_{3\alpha+\beta}t^{b+1}\mathrm{ad}(e_\alpha)^2(f_{3\alpha+\beta}t^{b+1}). \end{aligned}$$

From the above results, to proceed the induction it suffices to show that the equation (10) for given $a, b \in \mathbb{Z}_{\geq 0}$ and all $s \in \mathbb{Z}_{\geq 0}$ imply the following equation:

$$(f_{2\alpha+\beta}t^s)^{\{0,2(a+3)+3b-3s\}+1} \in U(\mathcal{C}\mathfrak{g})I_{a+3,b} \text{ for all } s \in \mathbb{Z}_{\geq 0}.$$

Define a Lie subalgebra $\mathfrak{a}$ of $\mathcal{C}\mathfrak{g}$ by

$$\mathfrak{a} = \mathfrak{n}_-\mathbb{C}[t] + \mathfrak{h}\mathbb{C}[t] + \sum_{\gamma\in\Delta_+} e_\gamma t^{\langle\gamma,\varpi_\alpha^\vee\rangle}\mathbb{C}[t],$$

where $\varpi_\alpha^\vee$ is the fundamental coweight corresponding to α, that is, an element in $\mathfrak{h}$ satisfying $\langle\alpha, \varpi_\alpha^\vee\rangle = 1$, $\langle\beta, \varpi_\alpha^\vee\rangle = 0$, and define a Lie subalgebra $\mathfrak{a}_0$ of $\mathcal{C}\mathfrak{g}$ by

$$\mathfrak{a}_0 = \sum_{\substack{\gamma\in\Delta_+\setminus\{\beta\} \\ 0\leq s<\langle\gamma,\varpi_\alpha^\vee\rangle}} \mathbb{C}e_\gamma t^s.$$

Note that $\mathcal{C}\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}_0$. Let $I'_{a,b}$ be a subspace of $I_{a,b}$ defined by

$$\begin{aligned} I'_{a,b} &= I_{a,b} \cap U(\mathfrak{a}) \\ &= \sum_{\gamma\in\Delta_+} e_\gamma t^{\langle\gamma,\varpi_\alpha^\vee\rangle}\mathbb{C}[t] + \mathfrak{h}t\mathbb{C}[t] + \mathbb{C}(\alpha^\vee - a) + \mathbb{C}(\beta^\vee - b) \\ &+ \sum_{s\geq 0}\mathbb{C}(f_\alpha t^s)^{\{0,a-3s\}+1} + \mathbb{C}f_\beta^{b+1}. \end{aligned}$$

Now we show the following lemma, which is crucial to proceed the induction:

Lemma 4.11.

$$U(\mathcal{C}\mathfrak{g})I_{a,b} \subseteq U(\mathfrak{a})I'_{a,b} \oplus U(\mathcal{C}\mathfrak{g})\mathfrak{a}_0.$$

Proof. Set

$$I = I_{a,b}, \quad I' = I'_{a,b}, \quad J = U(\mathfrak{a})I' \oplus U(\mathcal{C}\mathfrak{g})\mathfrak{a}_0.$$

Since $U(\mathcal{C}\mathfrak{g}) = U(\mathfrak{a})U(\mathfrak{a}_0)$ and $U(\mathfrak{a})J = J$, it suffices to show that $U(\mathfrak{a}_0)I \subseteq J$. First, we prove that

$$I_1 = \mathfrak{n}_+\mathbb{C}[t] + \mathfrak{h}t\mathbb{C}[t] + \mathbb{C}(\alpha^\vee - a) + \mathbb{C}(\beta^\vee - b)$$

satisfies $U(\mathfrak{a}_0)I_1 \subseteq J$, which is equivalent to that for any $k \geq 0$ and a sequence $X_1, \dots, X_k \in \mathfrak{a}_0$, we have $X_1 \cdots X_k I_1 \subseteq J$. We prove a stronger result that

$$X_1 \cdots X_k I_1 \subseteq (I' \cap I_1) \oplus U(\mathcal{C}\mathfrak{g})\mathfrak{a}_0 \tag{11}$$

by induction on k. If $k = 0$, this follows since $I_1 = (I' \cap I_1) \oplus \mathfrak{a}_0$. We can easily check that

$$\mathrm{ad}(\mathfrak{a}_0)I_1 \subseteq \mathfrak{n}_+\mathbb{C}[t] \subseteq (I' \cap I_1) \oplus \mathfrak{a}_0,$$

and hence if $k > 0$, we have

$$\begin{aligned} X_1 \cdots X_k I_1 &\subseteq X_1 \cdots X_{k-1} I_1 X_k + X_1 \cdots X_{k-1}(\mathrm{ad}(X_k)I_1) \\ &\subseteq X_1 \cdots X_{k-1}(I' \cap I_1) + U(\mathcal{C}\mathfrak{g})\mathfrak{a}_0. \end{aligned}$$

This together with the induction hypothesis implies (11).

Next we prove $U(\mathfrak{a}_0)f_\beta^{b+1} \subseteq J$. Since $f_\beta^{b+1} \in I'$, it suffices to show that $U(\mathfrak{a}_0)_+ f_\beta^{b+1} \subseteq J$, where $U(\mathfrak{a}_0)_+$ denotes the augmentation ideal. The $\mathfrak{h}$-weight set of $U(\mathfrak{a}_0)_+ f_\beta^{b+1}$ with respect to the adjoint action obviously satisfies

$$\mathrm{wt}_\mathfrak{h}(U(\mathfrak{a}_0)_+ f_\beta^{b+1}) \subseteq \mathbb{Z}_{>0}\alpha + \mathbb{Z}\beta. \tag{12}$$

Since $\mathfrak{a}_0 \oplus \mathbb{C}f_\beta$ is a Lie subalgebra and $U(\mathfrak{a}_0 \oplus \mathbb{C}f_\beta) = \mathbb{C}[f_\beta] \oplus \mathbb{C}[f_\beta]U(\mathfrak{a}_0)_+$, (12) implies by weight consideration that

$$U(\mathfrak{a}_0)_+ f_\beta^{b+1} \subseteq \mathbb{C}[f_\beta]U(\mathfrak{a}_0)_+ \subseteq J.$$

Let

$$I_2 = \sum_{s \geq 0} \mathbb{C}(f_\alpha t^s)^{\{0, a-3s\}+1}.$$

Since $I = I_1 + \mathbb{C}f_\beta^{b+1} + I_2$, to complete the lemma it suffices to show that $U(\mathfrak{a}_0)I_2 \subseteq J$. To do this, we put

$$\mathfrak{a}_0' = \sum_{\substack{\gamma \in \Delta_+ \setminus \{\alpha, \beta\} \\ 0 \leq s < \langle \gamma, \varpi_\alpha^\vee \rangle}} \mathbb{C}e_\gamma t^s,$$

and we first prove that $U(\mathfrak{a}_0')_+ I_2 \subseteq U(\mathcal{C}\mathfrak{g})\mathfrak{n}_+\mathbb{C}[t]$. Note that $\mathfrak{a}_0 = \mathfrak{a}_0' \oplus \mathbb{C}e_\alpha$. The $\mathfrak{h}$-weight set of $U(\mathfrak{a}_0')_+ I_2$ with respect to the adjoint action satisfies

$$\mathrm{wt}_\mathfrak{h}(U(\mathfrak{a}_0')_+ I_2) \subseteq \mathbb{Z}\alpha + \mathbb{Z}_{>0}\beta. \tag{13}$$

Put $\mathfrak{n}_+^{(\alpha)} = \sum_{\gamma \in \Delta_+ \setminus \{\alpha\}} \mathfrak{g}_\gamma$. Since $f_\alpha\mathbb{C}[t] \oplus \mathfrak{n}_+^{(\alpha)}\mathbb{C}[t]$ is a Lie subalgebra and

$$U(f_\alpha\mathbb{C}[t] \oplus \mathfrak{n}_+^{(\alpha)}\mathbb{C}[t]) = U(f_\alpha\mathbb{C}[t]) \oplus U(f_\alpha\mathbb{C}[t])U(\mathfrak{n}_+^{(\alpha)}\mathbb{C}[t])_+,$$

(13) implies by weight consideration that

$$U(\mathfrak{a}_0')_+ I_2 \subseteq U(f_\alpha\mathbb{C}[t])U(\mathfrak{n}_+^{(\alpha)}\mathbb{C}[t])_+ \subseteq U(\mathcal{C}\mathfrak{g})\mathfrak{n}_+\mathbb{C}[t].$$

Then since Lemma 4.10 with $\ell = 3$ implies $\mathbb{C}[e_\alpha]I_2 \subseteq J$, we have

$$\begin{aligned} U(\mathfrak{a}_0)I_2 &\subseteq \mathbb{C}[e_\alpha]I_2 + \mathbb{C}[e_\alpha]U(\mathfrak{a}_0')_+ I_2 \\ &\subseteq J + U(\mathcal{C}\mathfrak{g})\mathfrak{n}_+\mathbb{C}[t] \subseteq J + U(\mathcal{C}\mathfrak{g})I_1 \subseteq J. \end{aligned}$$

□

Now, we show the following proposition, which completes the proof of Proposition 4.1:

Proposition 4.12. (10) *follows for all* $a, b \in \mathbb{Z}_{\geq 0}$ *and* $s \in \mathbb{Z}_{\geq 0}$.

Proof. As stated above, it suffices to show that if

$$(f_{2\alpha+\beta}t^s)^{\{0,2a+3b-3s\}+1} \in U(C\mathfrak{g})I_{a,b} \tag{14}$$

follows for given $a, b \in \mathbb{Z}_{\geq 0}$ and all $s \in \mathbb{Z}_{\geq 0}$, then we have

$$(f_{2\alpha+\beta}t^s)^{\{0,2(a+3)+3b-3s\}+1} \in U(C\mathfrak{g})I_{a+3,b} \tag{15}$$

for all $s \in \mathbb{Z}_{\geq 0}$.

Since $U(C\mathfrak{g}) = U(\mathfrak{a}) \oplus U(C\mathfrak{g})\mathfrak{a}_0$, (14) and Lemma 4.11 implies

$$(f_{2\alpha+\beta}t^s)^{\{0,2a+3b-3s\}+1} \in U(C\mathfrak{g})I_{a,b} \cap U(\mathfrak{a}) \subseteq U(\mathfrak{a})I'_{a,b} \tag{16}$$

for all $s \geq 0$. Define a $\mathbb{C}$-linear map $\Phi : \mathfrak{a} \to U(C\mathfrak{g})$ by

$$\Phi(e_\gamma t^k) = e_\gamma t^{k-\langle\gamma,\varpi_\alpha^\vee\rangle}, \ \Phi(f_\gamma t^k) = f_\gamma t^{k+\langle\gamma,\varpi_\alpha^\vee\rangle} \text{ for } \gamma \in \Delta_+,$$

$$\Phi(\alpha^\vee t^k) = \alpha^\vee t^k - 3\delta_{k0}, \ \Phi(\beta^\vee t^k) = \beta^\vee t^k.$$

It is easily checked that Φ satisfies $\Phi([X_1, X_2]) = [\Phi(X_1), \Phi(X_2)]$ for $X_1, X_2 \in \mathfrak{a}$. Hence Φ induces a $\mathbb{C}$-algebra homomorphism $U(\mathfrak{a}) \to U(C\mathfrak{g})$, which we also denote by Φ. Applying Φ to (16), we have

$$\begin{aligned}(f_{2\alpha+\beta}t^{s+2})^{\{0,2(a+3)+3b-3(s+2)\}+1} &\in \Phi(U(\mathfrak{a})I'_{a,b}) \subseteq U(C\mathfrak{g})\Phi(I'_{a,b}) \\ &\subseteq U(C\mathfrak{g})(\mathfrak{n}_+\mathbb{C}[t] + \mathfrak{h}t\mathbb{C}[t] + \mathbb{C}(\alpha^\vee - (a+3)) + \mathbb{C}(\beta^\vee - b) \\ &\quad + \sum_{s\geq 0}\mathbb{C}(f_\alpha t^{s+1})^{\{0,(a+3)-3(s+1)\}+1} + \mathbb{C}f_\beta^{b+1}) \subseteq U(C\mathfrak{g})I_{a+3,b},\end{aligned}$$

and hence we have the equation (15) for $s \geq 2$. Since (15) for $s = 0$ follows from Lemma 4.3, it remains only to prove (15) for $s = 1$, that is, the equation

$$(f_{2\alpha+\beta}t)^{2a+3b+4} \in U(C\mathfrak{g})I_{a+3,b}.$$

If $b \geq 1$, Lemma 4.9 with $s_1 = 0, s_2 = 1$ implies this. Assume $b = 0$, and put

$$N = U(C\mathfrak{g})/U(C\mathfrak{g})I_{a+3,0},$$

which is a quotient of the Weyl module $W((a+3)\varpi_\alpha)$ and hence a finite dimensional $C\mathfrak{g}$-module. Let $\bar{1} \in N$ be the image of 1. It suffices to show that $(f_{2\alpha+\beta}t)^{2a+4}.\bar{1} = 0$. Since the $\mathfrak{h}$-weight of $(f_{2\alpha+\beta}t)^{2a+4}.\bar{1}$ is $-(a+1)\varpi_\alpha$ and

$$\langle(2\alpha+\beta)^\vee, -(a+1)\varpi_\alpha\rangle = -2a-2,$$

if we can show that

$$e_{2\alpha+\beta}^{2a+2}(f_{2\alpha+\beta}t)^{2a+4}.\bar{1} = 0,$$

then $(f_{2\alpha+\beta}t)^{2a+4}.\bar{1} = 0$ also follows from $\mathfrak{sl}_2$-theory. By [9, Lemma 7.1], we have

$$e_{2\alpha+\beta}^{2a+2}(f_{2\alpha+\beta}t)^{2a+4}.\bar{1} \in \sum_{s_1+s_2=2a+4}\mathbb{C}f_{2\alpha+\beta}t^{s_1}f_{2\alpha+\beta}t^{s_2}.\bar{1}.$$

Using the equation (15) for $s \geq 2$, we can prove that the right hand side is 0. □

4.3 Quantized Demazure modules and Joseph's results

The quantized version of the Demazure modules also can be defined in a similar manner as the classical case. For $\Lambda \in \widehat{P}_+$, we denote by $V_q(\Lambda)$ the irreducible highest weight $U_q(\widehat{\mathfrak{g}})$-module with highest weight Λ. Similarly as the classical case, we have $\dim_{\mathbb{C}(q)} V_q(\Lambda)_{w\Lambda} = 1$ for all $w \in \widehat{W}$. We denote by $U_q(\widehat{\mathfrak{n}}_+)$ the positive part of $U_q(\widehat{\mathfrak{g}})$.

Definition 4.13. We call $V_{q,w}(\Lambda) = U_q(\widehat{\mathfrak{n}}_+).V_q(\Lambda)_{w\Lambda}$ the *quantized Demazure submodule* of $V_q(\Lambda)$ associated with w.

Joseph posed in [14, §5.8] a question which asks if the tensor product of a one-dimensional Demazure module by an arbitrary Demazure module admits a filtration whose subquotients are isomorphic to the Demazure modules. Polo [32] and Mathieu [26] gave the positive answer to this question in the case of semisimple Lie algebras, and Joseph [16] himself gave the positive answer in the case of the quantized enveloping algebras associated with simply laced Kac-Moody Lie algebras. Here, we briefly recall the Joseph's result since we use the result later. Although his result is applicable to any quantized enveloping algebras associated with simply laced Kac-Moody Lie algebras, we concentrate only on affine case here.

Let $A = \mathbb{Z}[q, q^{-1}]$, let $U_q^{\mathbb{Z}}(\widehat{\mathfrak{n}}_+)$ and $U_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})$ be the A-forms of $U_q(\widehat{\mathfrak{n}}_+)$ and $U_q(\widehat{\mathfrak{g}})$ respectively, and denote by T the Cartan part of $U_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})$. (for precise definitions, see [16, §2.2]). We denote by $U_q^{\mathbb{Z}}(\widehat{\mathfrak{b}})$ the subring of $U_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})$ generated by $U_q^{\mathbb{Z}}(\widehat{\mathfrak{n}}_+)$ and T. For $w \in \widehat{W}$, let $u_{w\Lambda}$ be a nonzero element of weight $w\Lambda$ in $V_q(\Lambda)$, and let

$$V_{q,w}^{\mathbb{Z}}(\Lambda) = U_q^{\mathbb{Z}}(\widehat{\mathfrak{n}}_+).u_{w\Lambda},$$

which is obviously a $U_q^{\mathbb{Z}}(\widehat{\mathfrak{b}})$-module. Taking the classical limit, we have

$$\mathbb{C} \otimes_A V_{q,w}^{\mathbb{Z}}(\Lambda) \cong V_w(\Lambda),$$

where A acts on $\mathbb{C}$ by letting q act by 1. Joseph has proved the following theorem:

Theorem 4.14 ([16, Theorem 5.22]). *Assume that $\mathfrak{g}$ is simply laced, and let $\Lambda, \Lambda' \in \widehat{P}_+, w \in \widehat{W}$. Then a $U_q^{\mathbb{Z}}(\widehat{\mathfrak{b}})$-submodule $u_\Lambda \otimes_A V_{q,w}^{\mathbb{Z}}(\Lambda')$ of $V_q(\Lambda) \otimes_{\mathbb{C}(q)} V_q(\Lambda')$ has a $U_q^{\mathbb{Z}}(\widehat{\mathfrak{b}})$-module filtration*

$$0 = Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_k = u_\Lambda \otimes_A V_{q,w}^{\mathbb{Z}}(\Lambda')$$

such that each subquotient Y_i/Y_{i-1} satisfies

$$Y_i/Y_{i-1} \cong V_{q,y_i}^{\mathbb{Z}}(\Lambda^i) \text{ for some } \Lambda^i \in \widehat{P}_+, y_i \in \widehat{W}.$$

Remark 4.15. (i) In [16, Theorem 5.22], a given Kac-Moody Lie algebra is assumed to be simply laced, and this condition excludes the case where $\widehat{\mathfrak{g}}$ is of type $A_1^{(1)}$. In Joseph's proof, however, this condition is only used in [16, Lemma 3.14] to apply a positivity result of Lusztig. We can check that the proof of this positivity result in [25, §22.1.7] is also applicable to $A_1^{(1)}$ without

any modification, and hence the above theorem is true for any simply laced $\mathfrak{g}$.
(ii) [16, Theorem 5.22] states only that the above filtration is a $U_q^{\mathbb{Z}}(\widehat{\mathfrak{n}}_+)$-module filtration. However, it is easily seen that this is a $U_q^{\mathbb{Z}}(\widehat{\mathfrak{b}})$-module filtration since each Y_i is defined by an A-span of some weight vectors ([16, §5.7]).

Taking the classical limit, the following result is obtained:

Corollary 4.16. *Assume that* $\mathfrak{g}$ *is simply laced. For* $\ell' > \ell$, $\mathcal{D}(\ell,\lambda)[m]$ *has a* $C\mathfrak{g}_d$*-module filtration*

$$0 = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_k = \mathcal{D}(\ell,\lambda)[m]$$

such that each subquotient D_i/D_{i-1} *satisfies*

$$D_i/D_{i-1} \cong \mathcal{D}(\ell',\mu_i)[m_i] \text{ for some } \mu_i \in P_+, m_i \in \mathbb{Z}_{\geq m}.$$

Proof. We prove this statement for $\ell' = \ell+1$. The results for general ℓ' can be obtained by applying this case repeatedly. Take $\Lambda' \in \widehat{P}_+$ and $w \in \widehat{W}$ so that $w\Lambda' = w_0\lambda + \ell\Lambda_0 + m\delta$. By Theorem 4.14, $u_{\Lambda_0} \otimes V_{q,w}^{\mathbb{Z}}(\Lambda')$ has a $U_q^{\mathbb{Z}}(\widehat{\mathfrak{b}})$-module filtration $0 = Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_k = u_{\Lambda_0} \otimes V_{q,w}^{\mathbb{Z}}(\Lambda')$ such that

$$Y_i/Y_{i-1} \cong V_{q,y_i}^{\mathbb{Z}}(\Lambda^i) \text{ for some } \Lambda^i \in \widehat{P}_+, y_i \in \widehat{W}.$$

Put $D_i = \mathbb{C} \otimes_A Y_i$. Then we have

$$D_k = \mathbb{C} \otimes_A (u_{\Lambda_0} \otimes_A V_{q,w}^{\mathbb{Z}}(\Lambda')) = \mathbb{C}_{\Lambda_0} \otimes_{\mathbb{C}} V_w(\Lambda') = \mathbb{C}_{\Lambda_0} \otimes_{\mathbb{C}} \mathcal{D}(\ell,\lambda)[m],$$

where $\mathbb{C}_{\Lambda_0}$ denotes a 1-dimensional $\widehat{\mathfrak{b}}$-module spanned by a vector of weight Λ_0 on which $\widehat{\mathfrak{n}}_+$ acts trivially. Since all Y_i and Y_i/Y_{i-1} are free A-modules ([16, §5.7]), $\mathbb{C}_{\Lambda_0} \otimes \mathcal{D}(\ell,\lambda)[m]$ has a $\widehat{\mathfrak{b}}$-module filtration

$$0 = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_k = \mathbb{C}_{\Lambda_0} \otimes \mathcal{D}(\ell,\lambda)[m],$$

and each subquotient satisfies $D_i/D_{i-1} \cong V_{y_i}(\Lambda^i)$. Obviously, each Λ^i is of level $\ell+1$. By [16, Theorem 5.9], there exists for each $1 \leq i \leq k$ a $U_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})$-submodule Z_i of $V_q(\Lambda) \otimes_{\mathbb{C}(q)} V_q(\Lambda')$ such that

$$Y_i = Z_i \cap (u_{\Lambda_0} \otimes_A V_{q,w}^{\mathbb{Z}}(\Lambda')).$$

Then since $\mathbb{C}_{\Lambda_0} \otimes_{\mathbb{C}} \mathcal{D}(\ell,\lambda)[m]$ is a $C\mathfrak{g}_d \oplus \mathbb{C}K$-module, we can see that each D_i is also a $C\mathfrak{g}_d \oplus \mathbb{C}K$-module, and so is D_i/D_{i-1}. Hence, each D_i/D_{i-1} is isomorphic to $\mathcal{D}(\ell+1,\mu_i)[m_i]$ for some $\mu_i \in P_+, m_i \in \mathbb{Z}$. Each m_i obviously satisfies $m_i \in \mathbb{Z}_{\geq m}$ since $\mathrm{wt}_{\mathfrak{h}_d}\mathcal{D}(\ell,\lambda)[m] \in \lambda - Q_+ + \mathbb{Z}_{\geq m}\delta$. Now, since $\mathbb{C}_{\Lambda_0}$ is a trivial $C\mathfrak{g}_d$-module, we obtain the corollary by restricting these results to $C\mathfrak{g}_d$. □

We define a partial order $\preceq$ on $P_+ + \mathbb{Z}\delta$ by $\lambda_1 + m_1\delta \preceq \lambda_2 + m_2\delta$ if $\lambda_2 - \lambda_1 \in Q_+$ and $m_1 \geq m_2$. Since $\mathcal{D}(\ell,\lambda)[m]$ is $U(\mathfrak{n}_- \otimes \mathbb{C}[t])$-cyclic, if $\lambda_1 + m_1\delta \in \mathrm{wt}_{\mathfrak{h}_d}\mathcal{D}(\ell,\lambda)[m]$, then $\lambda_1 + m_1\delta \preceq \lambda + m\delta$ follows. This implies that $\{\mathrm{ch}_{\mathfrak{h}_d}\mathcal{D}(\ell,\lambda)[m] \mid \lambda \in P_+, m \in \mathbb{Z}\}$ are linearly independent for each $\ell \in \mathbb{Z}_{>0}$.

When a $\mathfrak{h}_d$-weight $C\mathfrak{g}_d$-module M has a filtration $0 = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_k = M$ such that

$$D_i/D_{i-1} \cong \mathcal{D}(\ell,\mu_i)[m_i] \text{ for some } \mu_i \in P_+, m_i \in \mathbb{Z}$$

for fixed $\ell \in \mathbb{Z}_{>0}$, we define

$$(M : \mathcal{D}(\ell,\lambda)[m]) = \#\{i \mid D_i/D_{i-1} \cong \mathcal{D}(\ell,\lambda)[m]\},$$

which is independent of the choice of a filtration from the linearly independence of the characters.

4.4 Filtrations on the Weyl modules

In this subsection, we assume that $\mathfrak{g}$ is non-simply laced. Here, we need to consider the Demazure modules for both $\mathfrak{g}$ and $\mathfrak{g}^{\mathrm{sh}}$. Hence, we denote the $\mathcal{C}\mathfrak{g}^{\mathrm{sh}}$-Demazure module by $\mathcal{D}^{\mathrm{sh}}(\ell,\nu)$ and the $\mathcal{C}\mathfrak{g}_d^{\mathrm{sh}}$-Demazure module by $\mathcal{D}^{\mathrm{sh}}(\ell,\nu)[m]$, where $\nu \in \overline{P}_+$, $\ell \in \mathbb{Z}_{>0}$, and $m \in \mathbb{Z}$.

Lemma 4.17. *Let v be the generator of $W(\lambda)$ in Definition 3.1, and let $W = U(\mathcal{C}\mathfrak{g}_d^{\mathrm{sh}}).v \subseteq W(\lambda)$. Then W is isomorphic to $\mathcal{D}^{\mathrm{sh}}(1,\overline{\lambda})[0]$ as a $\mathcal{C}\mathfrak{g}_d^{\mathrm{sh}}$-module.*

Proof. By Corollary 4.2, $\mathcal{D}^{\mathrm{sh}}(1,\overline{\lambda})[0]$ is isomorphic to the cyclic $\mathcal{C}\mathfrak{g}_d^{\mathrm{sh}}$-module generated by an element v' with relations:

$$\mathfrak{n}_+^{\mathrm{sh}}\otimes\mathbb{C}[t].v' = 0,\quad h\otimes t^s.v' = \delta_{s0}\langle\overline{\lambda},h\rangle v' \text{ for } h \in \mathfrak{h}_d^{\mathrm{sh}},\quad f_i^{\langle\overline{\lambda},\alpha_i^\vee\rangle+1}.v' = 0 \text{ for } i \in I^{\mathrm{sh}}.$$

From these relations, we can check that there exists a surjective homomorphism $\varphi : \mathcal{D}^{\mathrm{sh}}(1,\overline{\lambda})[0] \to W$ of $\mathcal{C}\mathfrak{g}_d^{\mathrm{sh}}$-modules. We need to show that φ is injective. In this proof, by $V(\mu)$ for $\mu \in P_+$ we denote the $\mathcal{C}\mathfrak{g}_d$-module defined by the extension of $V_\mathfrak{g}(\mu)$ by letting $\mathfrak{g}\otimes t\mathbb{C}[t] \oplus \mathbb{C}d$ act by 0. Write λ in the form $\lambda = \sum_{i\in I}\lambda_i\varpi_i$ where $\lambda_i \in \mathbb{Z}_{\geq 0}$, and put $p = \sum_i \lambda_i$. Let $c_1,\dots,c_p$ be pairwise distinct complex numbers, and define a $\mathcal{C}\mathfrak{g}$-module W_1 by

$$W_1 = V(\varpi_1)_{c_1} * \cdots * V(\varpi_1)_{c_{\lambda_1}} * \cdots * V(\varpi_n)_{c_{p-\lambda_n+1}} * \cdots * V(\varpi_n)_{c_p},$$

where each $V(\varpi_i)$ occurs λ_i times. By the same way as [8, Lemma 5], we can show that there exists a surjective homomorphism $\psi : W(\lambda) \to W_1$ of $\mathcal{C}\mathfrak{g}$-modules. It suffices to show that the $\mathcal{C}\mathfrak{g}^{\mathrm{sh}}$-module homomorphism $\psi\circ\varphi : \mathcal{D}^{\mathrm{sh}}(1,\lambda)[0] \to W_1$ is injective. By Lemma 2.3 and Lemma 3.11 (i), we have

$$\mathrm{ch}_\mathfrak{h}\,(\mathrm{Im}\,\psi\circ\varphi) = P_{\lambda-Q_+^{\mathrm{sh}}}\mathrm{ch}_\mathfrak{h}\,W_1 = P_{\lambda-Q_+^{\mathrm{sh}}}\prod_{i\in I}\mathrm{ch}_\mathfrak{h}\,V(\varpi_i)^{\lambda_i},$$

and using $\mathrm{wt}_\mathfrak{h} V(\varpi_i) \subseteq \varpi_i - Q_+$, we have

$$P_{\lambda-Q_+^{\mathrm{sh}}}\prod_{i\in I}\mathrm{ch}_\mathfrak{h}\,V(\varpi_i)^{\lambda_i} = \prod_{i\in I}(P_{\varpi_i-Q_+^{\mathrm{sh}}}\mathrm{ch}_\mathfrak{h}\,V(\varpi_i))^{\lambda_i}.$$

We have from Lemma 2.3 that $P_{\varpi_i-Q_+^{\mathrm{sh}}}\mathrm{ch}_\mathfrak{h}\,V(\varpi_i) = \mathrm{ch}_\mathfrak{h}\,U(\mathfrak{g}^{\mathrm{sh}}).v_i$, where v_i is a nonzero highest weight vector of $V(\varpi_i)$, and we can easily see that

$$U(\mathfrak{g}^{\mathrm{sh}}).v_i \cong V_{\mathfrak{g}^{\mathrm{sh}}}(\overline{\varpi}_i) = \begin{cases} V_{\mathfrak{g}^{\mathrm{sh}}}(\overline{\varpi}_i) & \text{if } i \in I^{\mathrm{sh}}, \\ V_{\mathfrak{g}^{\mathrm{sh}}}(0) & \text{if } i \notin I^{\mathrm{sh}} \end{cases}$$

as $\mathfrak{g}^{\mathrm{sh}}$-modules. From these equations, we have

$$\dim(\mathrm{Im}\,\psi\circ\varphi) = \prod_{i\in I^{\mathrm{sh}}}\dim V_{\mathfrak{g}^{\mathrm{sh}}}(\overline{\varpi}_i)^{\lambda_i}.$$

On the other hand, we have that

$$\dim \mathcal{D}^{\mathrm{sh}}(1,\overline{\lambda}) = \prod_{i\in I^{\mathrm{sh}}} \dim \mathcal{D}^{\mathrm{sh}}(1,\overline{\varpi}_i)^{\lambda_i} = \prod_{i\in I^{\mathrm{sh}}} \dim V_{\mathfrak{g}^{\mathrm{sh}}}(\overline{\varpi}_i)^{\lambda_i},$$

where the first equality follows from [7, Theorem 1], and the second follows from Lemma 3.8. Hence we have $\dim(\mathrm{Im}\,\psi\circ\varphi) = \dim \mathcal{D}^{\mathrm{sh}}(1,\overline{\lambda})$, which implies that $\psi\circ\varphi$ is injective. □

Let $\lambda \in P_+$. Using Corollary 4.16, we can take a $\mathcal{C}\mathfrak{g}_d^{\mathrm{sh}}$-module filtration of $\mathcal{D}^{\mathrm{sh}}(1,\overline{\lambda})[0]$: $0 = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_k = \mathcal{D}^{\mathrm{sh}}(1,\overline{\lambda})[0]$ such that

$$D_i/D_{i-1} \cong \mathcal{D}^{\mathrm{sh}}(r,\nu_i)[m_i] \text{ for some } \nu_i \in \overline{P}_+, m_i \in \mathbb{Z}_{\geq 0}.$$

Now, using this filtration, we show the following proposition, which is the main result of the first half of this article:

Proposition 4.18. *Let $\mu_i = i_{\mathrm{sh}}(\nu_i) + (\lambda - i_{\mathrm{sh}}(\overline{\lambda}))$ for each $1 \leq i \leq k$. Then the Weyl module $W(\lambda)$ has a $\mathcal{C}\mathfrak{g}_d$-module filtration $0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_k = W(\lambda)$ such that each subquotient W_i/W_{i-1} is a quotient of $\mathcal{D}(1,\mu_i)[m_i]$.*

Proof. By Lemma 4.17, $W(\lambda) \supseteq W = U(\mathcal{C}\mathfrak{g}_d^{\mathrm{sh}}).v$ has a $\mathcal{C}\mathfrak{g}_d^{\mathrm{sh}}$-module filtration $0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_k = W$ such that

$$X_i/X_{i-1} \cong \mathcal{D}^{\mathrm{sh}}(r,\nu_i)[m_i]. \tag{17}$$

For each $1 \leq i \leq k$, take $x_i \in X_i$ so that the image on X_i/X_{i-1} coincides with the generator in Proposition 3.6 under the isomorphism (17). We can assume that each x_i is a $\mathfrak{h}^{\mathrm{sh}}$-weight vector of weight $\nu_i + m_i\overline{\delta}$. Then each x_i satisfies

$$\mathfrak{n}_+^{\mathrm{sh}} \otimes \mathbb{C}[t].x_i \subseteq X_{i-1},\ \mathfrak{h}^{\mathrm{sh}} \otimes t\mathbb{C}[t].x_i \subseteq X_{i-1},\ h.x_i = \langle \nu_i + m_i\overline{\delta}, h\rangle x_i \text{ for } h \in \mathfrak{h}_d^{\mathrm{sh}},$$

and for $\gamma \in \Delta^{\mathrm{sh}}$ and $s \in \mathbb{Z}_{\geq 0}$,

$$(f_\gamma \otimes t^s)^{\max\{0,\langle \nu_i,\gamma^\vee\rangle - rs\}+1}.x_i \in X_{i-1}.$$

First, we show that each x_i is a $\mathfrak{h}$-weight vector of weight μ_i. Set $(\mathfrak{h}^{\mathrm{sh}})^\perp = \{h \in \mathfrak{h} \mid (h,h_1) = 0 \text{ for } h_1 \in \mathfrak{h}^{\mathrm{sh}}\}$, which obviously satisfies $\mathfrak{h} = \mathfrak{h}^{\mathrm{sh}} \oplus (\mathfrak{h}^{\mathrm{sh}})^\perp$ and $[(\mathfrak{h}^{\mathrm{sh}})^\perp, \mathcal{C}\mathfrak{g}_d^{\mathrm{sh}}] = 0$. Then, since $x_i \in U(\mathcal{C}\mathfrak{g}_d^{\mathrm{sh}}).v$, we have

$$h.x_i = \langle \lambda, h\rangle x_i \text{ for } h \in (\mathfrak{h}^{\mathrm{sh}})^\perp.$$

On the other hand, we can easily check from the definition of i_{sh} that $\langle \mu_i, h\rangle = \langle \lambda, h\rangle$ for $h \in (\mathfrak{h}^{\mathrm{sh}})^\perp$ and $\langle \mu_i, h\rangle = \langle \nu_i, h\rangle$ for $h \in \mathfrak{h}^{\mathrm{sh}}$. Hence, we have checked that x_i is a $\mathfrak{h}$-weight vector of weight μ_i. Now, let $W_i = U(\mathcal{C}\mathfrak{g}_d).X_i$ for each i and let $W_0 = 0$. Then $W_k = W(\lambda)$ is obvious. Let $\overline{x_i}$ be the image of x_i on W_i/W_{i-1}. Since

$$U(\mathcal{C}\mathfrak{g}_d).x_i + W_{i-1} = U(\mathcal{C}\mathfrak{g}_d).(\mathbb{C}x_i + X_{i-1}) = U(\mathcal{C}\mathfrak{g}_d).X_i = W_i,$$

to show the proposition, it suffices to show that each $\overline{x_i}$ satisfies the defining relations of $\mathcal{D}(1,\mu_i)[m_i]$ in Corollary 4.2. From the relations which x_i satisfies, we can see that $\overline{x}_i$ is a weight vector of weight $\mu_i + m_i\delta$, and satisfies (D4) since we have $\langle \mu_i, \gamma \rangle = \langle \nu_i, \gamma \rangle$ for $\gamma \in \Delta^{\mathrm{sh}}$. To show (D1), it suffices to show that

$$e_\gamma \otimes t^s.x_i = 0 \quad \text{for } \gamma \in \Delta_+ \setminus \Delta_+^{\mathrm{sh}}, s \in \mathbb{Z}_{\geq 0}, \tag{18}$$

which follows since the $\mathfrak{h}$-weight of $e_\gamma \otimes t^s.x_i$ is $\mu_i + \gamma \in \lambda - Q_+^{\mathrm{sh}} + \gamma$ and hence $\mu_i + \gamma \notin \lambda - Q_+$. Then (D3) follows since W_i/W_{i-1} is finite dimensional. To show (D2), it suffices to show that

$$(\mathfrak{h}^{\mathrm{sh}})^\perp \otimes t\mathbb{C}[t].x_i = 0, \tag{19}$$

which follows since $x_i \in U(\mathcal{C}\mathfrak{g}_d^{\mathrm{sh}}).v$ and $[(\mathfrak{h}^{\mathrm{sh}})^\perp \otimes t\mathbb{C}[t], \mathcal{C}\mathfrak{g}_d^{\mathrm{sh}}] = 0$. □

Remark 4.19. In Section 9, we show that the subquotients of the given filtration on $W(\lambda)$ are actually isomorphic to the Demazure modules.

For $F_1, F_2 \in \mathbb{Z}[\mathfrak{h}_d^*]$, we write $F_1 \leq F_2$ if $F_2 - F_1 \in \mathbb{Z}_{\geq 0}[\mathfrak{h}_d^*]$. The above proposition implies the following corollary:

Corollary 4.20. *Let $\lambda \in P_+$, and we set $\lambda' = \lambda - i_{\mathrm{sh}}(\overline{\lambda})$. Then we have*

$$\mathrm{ch}_{\mathfrak{h}_d} W(\lambda) \leq \sum_{\nu \in \overline{P}_+, m \in \mathbb{Z}_{\geq 0}} (\mathcal{D}^{\mathrm{sh}}(1, \overline{\lambda})[0] : \mathcal{D}^{\mathrm{sh}}(r, \nu)[m]) \mathrm{ch}_{\mathfrak{h}_d} \mathcal{D}(1, i_{\mathrm{sh}}(\nu) + \lambda')[m].$$

Before ending this section, we show the following lemma that we need later:

Lemma 4.21. *Let $\lambda \in P_+, m \in \mathbb{Z}$, and we set $\lambda' = \lambda - i_{\mathrm{sh}}(\overline{\lambda})$. Then we have*

$$P_{\lambda - Q_+^{\mathrm{sh}} + \mathbb{Z}\delta} \mathrm{ch}_{\mathfrak{h}_d} \mathcal{D}(1, \lambda)[m] = e(\lambda') i_{\mathrm{sh}} \mathrm{ch}_{\mathfrak{h}_d^{\mathrm{sh}}} \mathcal{D}^{\mathrm{sh}}(r, \overline{\lambda})[m].$$

Proof. Let v be the generator of $\mathcal{D}(1, \lambda)[m]$ in Corollary 4.2, and let $D' = U(\mathcal{C}\mathfrak{g}_d^{\mathrm{sh}}).v$. Similarly as Lemma 2.3, we have

$$P_{\lambda - Q_+^{\mathrm{sh}} + \mathbb{Z}\delta} \mathrm{ch}_{\mathfrak{h}_d} \mathcal{D}(1, \lambda)[m] = e(\lambda') i_{\mathrm{sh}} \mathrm{ch}_{\mathfrak{h}_d^{\mathrm{sh}}} D'.$$

From proposition 3.6, we can see that D' is a quotient of $\mathcal{D}^{\mathrm{sh}}(r, \overline{\lambda})[m]$ as a $\mathcal{C}\mathfrak{g}_d^{\mathrm{sh}}$-module. Hence, we have

$$P_{\lambda - Q_+^{\mathrm{sh}} + \mathbb{Z}\delta} \mathrm{ch}_{\mathfrak{h}_d} \mathcal{D}(1, \lambda)[m] \leq e(\lambda') i_{\mathrm{sh}} \mathrm{ch}_{\mathfrak{h}_d^{\mathrm{sh}}} \mathcal{D}^{\mathrm{sh}}(r, \overline{\lambda})[m].$$

We show the opposite inequality. It is enough to show in the case $m = 0$. Here, we use the notation in the proof of Proposition 4.18. Similarly as (18) and (19), we can show for each $1 \leq i \leq k$ that

$$e_\gamma.X_i = 0 \text{ for } \gamma \in \Delta_+ \setminus \Delta_+^{\mathrm{sh}}, \quad (\mathfrak{h}^{\mathrm{sh}})^\perp \otimes t\mathbb{C}[t].X_i = 0,$$

which implies

$$W_i = U(\mathcal{C}\mathfrak{g}_d).X_i = U(\mathfrak{n}'_- \otimes \mathbb{C}[t]).X_i,$$

where we set $\mathfrak{n}'_- = \bigoplus_{\gamma \in \Delta_- \setminus \Delta_-^{\mathrm{sh}}} \mathfrak{g}_\gamma$. From this, we have that

$$X_i \cap W_{i-1} = X_i \cap (U(\mathfrak{n}'_- \otimes \mathbb{C}[t]).X_{i-1}) = X_{i-1},$$

and hence we have

$$U(\mathcal{C}\mathfrak{g}_d^{\mathrm{sh}}).\overline{x_i} = X_i / X_i \cap W_{i-1} = X_i / X_{i-1} \cong \mathcal{D}^{\mathrm{sh}}(r, \nu_i)[m_i],$$

where $\overline{x_i}$ is the image of x_i on W_i/W_{i-1}. Similarly as Lemma 2.3, we have

$$\begin{aligned} P_{\lambda-Q_+^{\mathrm{sh}}+\mathbb{Z}\delta}\mathrm{ch}_{\mathfrak{h}_d} W_i/W_{i-1} &= \mathrm{ch}_{\mathfrak{h}_d} U(\mathcal{C}\mathfrak{g}_d^{\mathrm{sh}}).\overline{x_i} \\ &= e(\mu_i - i_{\mathrm{sh}}(\overline{\mu}_i))i_{\mathrm{sh}}\mathrm{ch}_{\mathfrak{h}_d^{\mathrm{sh}}} \mathcal{D}^{\mathrm{sh}}(r,\nu_i)[m_i]. \end{aligned}$$

It is easily checked that $\mu_i - i_{\mathrm{sh}}(\overline{\mu}_i) = \lambda - i_{\mathrm{sh}}(\overline{\lambda})$. Hence, we have

$$P_{\lambda-Q_+^{\mathrm{sh}}+\mathbb{Z}\delta}\mathrm{ch}_{\mathfrak{h}_d} \mathcal{D}(1,\mu_i)[m_i] \geq e(\lambda')i_{\mathrm{sh}}\mathrm{ch}_{\mathfrak{h}_d^{\mathrm{sh}}} \mathcal{D}^{\mathrm{sh}}(r,\nu_i)[m_i], \tag{20}$$

since W_i/W_{i-1} is a quotient of $\mathcal{D}(1,\mu_i)[m_i]$. From

$$\sum_{1\leq i\leq k} \mathrm{ch}_{\mathfrak{h}_d^{\mathrm{sh}}} \mathcal{D}^{\mathrm{sh}}(r,\nu_i)[m_i] = \mathrm{ch}_{\mathfrak{h}_d^{\mathrm{sh}}} \mathcal{D}^{\mathrm{sh}}(1,\overline{\lambda})[0],$$

we can easily see that there exists some j such that $\nu_j = \overline{\lambda}$ and $m_j = 0$. Hence, we have

$$P_{\lambda-Q_+^{\mathrm{sh}}+\mathbb{Z}\delta}\mathrm{ch}_{\mathfrak{h}_d} \mathcal{D}(1,\lambda)[0] \geq e(\lambda')i_{\mathrm{sh}}\mathrm{ch}_{\mathfrak{h}_d^{\mathrm{sh}}} \mathcal{D}^{\mathrm{sh}}(r,\overline{\lambda})[0]$$

by (20). □

5 Path models

In this section, we review the theory of path models originally introduced by Littelmann [23], [24] (this theory can be applied to any Kac-Moody algebras, but we state only in affine case here). We do not review the definition of (abstract) crystals, but we refer the reader to [13, §4.5].

5.1 Definition of path models

For $a, b \in \mathbb{R}$ with $a < b$, we set $[a,b] = \{t \in \mathbb{R} \mid a \leq t \leq b\}$. A path with weight in $\widehat{P}$ is, by definition, a piecewise linear, continuous map $\pi : [0,1] \to \widehat{\mathfrak{h}}_{\mathbb{R}}^* = \mathbb{R}\otimes_{\mathbb{Z}}\widehat{P}$ such that $\pi(0) = 0, \pi(1) \in \widehat{P}$. We denote by $\mathbb{P}$ the set of all paths with weight in $\widehat{P}$. For π_1, π_2, we define $\pi_1 + \pi_2 \in \mathbb{P}$ by $(\pi_1 + \pi_2)(t) = \pi_1(t) + \pi_2(t)$.

Remark 5.1. In [23] and [24], paths are considered modulo reparametrization. In this article, however, we do not do this since there is no need to do so. Indeed, it can be checked that all results in [23] and [24] used in this article still hold in this setting.

Let $\pi \in \mathbb{P}$. A pair $(\underline{\mu}, \underline{\sigma})$ of a sequence $\underline{\mu} : \mu_1, \mu_2, \ldots, \mu_N$ of elements of $\widehat{\mathfrak{h}}_{\mathbb{R}}^*$ and a sequence $\underline{\sigma} : 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_N = 1$ of real numbers is called an expression of $\pi \in \mathbb{P}$ if the following equation holds:

$$\pi(t) = \sum_{p'=1}^{p-1}(\sigma_{p'} - \sigma_{p'-1})\mu_{p'} + (t - \sigma_{p-1})\mu_p \quad \text{for } t \in [\sigma_{p-1}, \sigma_p],\ 1 \leq p \leq N.$$

In this case, we write $\pi = (\underline{\mu}, \underline{\sigma})$.

For $\pi \in \mathbb{P}$ and $i \in \widehat{I}$, we define $H_i^\pi : [0,1] \to \mathbb{R}$ and $m_i^\pi \in \mathbb{R}$ by

$$H_i^\pi(t) = \langle \pi(t), \alpha_i^\vee \rangle, \quad m_i^\pi = \min\{H_i^\pi(t) \mid t \in [0,1]\}. \tag{21}$$

We denote by $\mathbb{P}_{\mathrm{int}}$ the subset of $\mathbb{P}$ consisting of paths π such that m_i^π is a nonpositive integer for all $i \in \widehat{I}$.

Littelmann has defined root operators $\tilde{e}_i, \tilde{f}_i$ ($i \in \widehat{I}$) on $\mathbb{P}$ in [23] and [24]. (In these articles, root operators are denoted by e_α, f_α). Here, for simplicity, we recall the actions of them only on elements of $\mathbb{P}_{\mathrm{int}}$, which are enough for this article since all the paths we consider are contained in $\mathbb{P}_{\mathrm{int}}$. For $\pi \in \mathbb{P}_{\mathrm{int}}$ and $i \in \widehat{I}$, we define $\tilde{e}_i\pi$ as follows: if $m_i^\pi = 0$, then we define $\tilde{e}_i\pi = \mathbf{0}$, where $\mathbf{0}$ is an additional element corresponding to '0' in the theory of crystals. If $m_i^\pi \leq -1$, then we define $\tilde{e}_i\pi \in \mathbb{P}$ by

$$(\tilde{e}_i\pi)(t) = \begin{cases} \pi(t) & \text{for } t \in [0, t_0], \\ \pi(t_0) + s_i(\pi(t) - \pi(t_0)) & \text{for } t \in [t_0, t_1], \\ \pi(t) + \alpha_i & \text{for } t \in [t_1, 1], \end{cases}$$

where we set

$$t_1 = \min\{t \in [0,1] \mid H_i^\pi(t) = m_i^\pi\},$$
$$t_0 = \max\{t \in [0,t_1] \mid H_i^\pi(t) = m_i^\pi + 1\}.$$

Similarly, we define $\tilde{f}_i\pi \in \mathbb{P} \cup \{\mathbf{0}\}$ as follows: if $H_i^\pi(1) = m_i^\pi$, then $\tilde{f}_i\pi = \mathbf{0}$. If $H_i^\pi(1) \geq m_i^\pi + 1$, then we define $\tilde{f}_i\pi$ by

$$(\tilde{f}_i\pi)(t) = \begin{cases} \pi(t) & \text{for } t \in [0, t_0], \\ \pi(t_0) + s_i(\pi(t) - \pi(t_0)) & \text{for } t \in [t_0, t_1], \\ \pi(t) - \alpha_i & \text{for } t \in [t_1, 1], \end{cases}$$

where we set

$$t_0 = \max\{t \in [0,1] \mid H_i^\pi(t) = m_i^\pi\},$$
$$t_1 = \min\{t \in [t_0,1] \mid H_i^\pi(t) = m_i^\pi + 1\}.$$

We set $\mathrm{wt}(\pi) = \pi(1) \in \widehat{P}$ for $\pi \in \mathbb{P}_{\mathrm{int}}$, and we define $\varepsilon_i : \mathbb{P}_{\mathrm{int}} \to \mathbb{Z}_{\geq 0}$ and $\varphi_i : \mathbb{P}_{\mathrm{int}} \to \mathbb{Z}_{\geq 0}$ for $i \in \widehat{I}$ by

$$\varepsilon_i(\pi) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^k\pi \neq \mathbf{0}\}, \quad \varphi_i(\pi) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i^k\pi \neq \mathbf{0}\}.$$

Then the following theorem follows from [24, §2]:

Theorem 5.2. *Let $\mathbb{B}$ be a subset of $\mathbb{P}_{\mathrm{int}}$ such that $\tilde{e}_i\mathbb{B} \subseteq \mathbb{B} \cup \{\mathbf{0}\}$, $\tilde{f}_i\mathbb{B} \subseteq \mathbb{B} \cup \{\mathbf{0}\}$ for all $i \in \widehat{I}$. Then $\mathbb{B}$, together with the root operators $\tilde{e}_i, \tilde{f}_i$ for $i \in \widehat{I}$ and the maps $\mathrm{wt}, \varepsilon_i, \varphi_i$ for $i \in \widehat{I}$, becomes a $U_q(\widehat{\mathfrak{g}})$-crystal* ([13, Definition 4.5.1]). *Moreover, we have*

$$\varepsilon_i(\pi) = -m_i^\pi, \quad \varphi_i(\pi) = H_i^\pi(1) - m_i^\pi. \tag{22}$$

The following lemma is easily checked from the definition of the root operators.

Lemma 5.3. *Let $\pi \in \mathbb{P}_{\mathrm{int}}, i \in \widehat{I}$, and let $0 < u \leq 1$ be a real number.*
(i) *If π satisfies $H_i^\pi(t) \geq m_i^\pi + 1$ for all $t \in [0, u]$, then we have*

$$\tilde{e}_i\pi(t) = \pi(t) \quad \textit{for all } t \in [0, u].$$

(ii) *Let* $M \in \mathbb{Z}_{\geq 0}$. *If* π *satisfies*

$$H_i^\pi(t) \geq -M \text{ for all } t \in [0,u] \text{ and } H_i^\pi(u) = -M,$$

then we have

$$\tilde{f}_i\pi(t) = \pi(t) \quad \text{for all } t \in [0,u].$$

Let $\mathbb{B} \subseteq \mathbb{P}_{\text{int}}$ be a subset such that $\tilde{e}_i\mathbb{B} \subseteq \mathbb{B} \cup \{\mathbf{0}\}$, $\tilde{f}_i\mathbb{B} \subseteq \mathbb{B} \cup \{\mathbf{0}\}$ for all $i \in \widehat{I}$. For each $i \in \widehat{I}$, we define $S_i : \mathbb{B} \to \mathbb{B}$ by

$$S_i(\pi) = \begin{cases} \tilde{f}_i^l\pi & \text{if } l = \langle \pi(1), \alpha_i^\vee \rangle \geq 0, \\ \tilde{e}_i^{-l}\pi & \text{if } l = \langle \pi(1), \alpha_i^\vee \rangle < 0. \end{cases}$$

The following theorem is verified in [24, Theorem 8.1]:

Theorem 5.4. *The map* $s_i \mapsto S_i$ *on the simple reflections in* $\widehat{W}$ *extends to a unique group action of* $\widehat{W}$ *on* $\mathbb{B} : w \mapsto S_w$.

Lemma 5.5. *Let* $\pi \in \mathbb{B}$ *and* $i \in \widehat{I}$, *and assume that* H_i^π *is non-decreasing or non-increasing. Then* $S_i(\pi)$ *satisfies*

$$S_i(\pi)(t) = s_i(\pi(t)) \text{ for all } t \in [0,1].$$

Proof. Assume that H_i^π is non-decreasing. Then we have $\varphi_i(\pi) = H_i^\pi(1) \in \mathbb{Z}_{\geq 0}$ by (22). Put $M = \varphi_i(\pi)$, and for $k \in \{0, \dots, M\}$ we define $\sigma_k \in [0,1]$ by

$$\sigma_k = \max\{u \in [0,1] \mid H_i^\pi(t) = k\}.$$

Then we have $0 \leq \sigma_0 < \sigma_1 < \cdots < \sigma_M = 1$, and we can show inductively from the definition of $\tilde{f}_i$ that

$$\tilde{f}_i^k\pi(t) = \begin{cases} s_i(\pi(t)) & \text{for } t \in [0, \sigma_k], \\ \pi(t) - k\alpha_i & \text{for } t \in [\sigma_k, 1]. \end{cases}$$

Hence, we have $\tilde{f}_i^M\pi(t) = s_i(\pi(t))$ for all $t \in [0,1]$. When H_i^π is non-increasing, we can show the lemma similarly. □

Now, we recall the definition of a concatenation of paths in $\mathbb{P}$ (cf. [24, §1]). For $\pi_1, \pi_2 \in \mathbb{P}$, we define a concatenation $\pi_1 * \pi_2 \in \mathbb{P}$ by

$$(\pi_1 * \pi_2)(t) \begin{cases} \pi_1(2t) & \text{if } t \in [0, \frac{1}{2}], \\ \pi_1(1) + \pi_2(2t-1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

It is obvious that if $\pi_1, \pi_2 \in \mathbb{P}_{\text{int}}$, then $\pi_1 * \pi_2 \in \mathbb{P}_{\text{int}}$. For notational convenience, we set $\mathbf{0} * \pi = \pi * \mathbf{0} = \mathbf{0}$ for any $\pi \in \mathbb{P}$.

Lemma 5.6 ([24, Lemma 2.7]). *For* $\pi_1, \pi_2 \in \mathbb{P}_{\text{int}}$ *and* $i \in \widehat{I}$, *we have*

$$\tilde{e}_i(\pi_1 * \pi_2) = \begin{cases} (\tilde{e}_i\pi_1) * \pi_2 & \text{if } \varphi_i(\pi_1) \geq \varepsilon_i(\pi_2), \\ \pi_1 * (\tilde{e}_i\pi_2) & \text{if } \varphi_i(\pi_1) < \varepsilon_i(\pi_2), \end{cases} \tag{23}$$

and

$$\tilde{f}_i(\pi_1 * \pi_2) = \begin{cases} (\tilde{f}_i\pi_1) * \pi_2 & \text{if } \varphi_i(\pi_1) > \varepsilon_i(\pi_2), \\ \pi_1 * (\tilde{f}_i\pi_2) & \text{if } \varphi_i(\pi_1) \leq \varepsilon_i(\pi_2). \end{cases} \tag{24}$$

This lemma implies the following:

Proposition 5.7. *Let $\mathbb{B}_1, \mathbb{B}_2$ be subsets of $\mathbb{P}_{\rm int}$ such that $\tilde{e}_i\mathbb{B}_j \subseteq \mathbb{B}_j \cup \{\mathbf{0}\}$, $\tilde{f}_i\mathbb{B}_j \subseteq \mathbb{B}_j \cup \{\mathbf{0}\}$ for all $i \in \widehat{I}$ $(j = 1, 2)$. Set*

$$\mathbb{B}_1 * \mathbb{B}_2 = \{\pi_1 * \pi_2 \mid \pi_1 \in \mathbb{B}_1, \pi_2 \in \mathbb{B}_2\}.$$

Then $\tilde{e}_i(\mathbb{B}_1 * \mathbb{B}_2) \subseteq (\mathbb{B}_1 * \mathbb{B}_2) \cup \{\mathbf{0}\}$ and $\tilde{f}_i(\mathbb{B}_1 * \mathbb{B}_2) \subseteq (\mathbb{B}_1 * \mathbb{B}_2) \cup \{\mathbf{0}\}$ follow for all $i \in \widehat{I}$, and $\mathbb{B}_1 * \mathbb{B}_2$ is isomorphic to $\mathbb{B}_1 \otimes \mathbb{B}_2$ (for a tensor product of crystals, see [13, Definition 4.5.3]) *as a $U_q(\widehat{\mathfrak{g}})$-crystal, where the isomorphism is given by $\pi_1 * \pi_2 \mapsto \pi_1 \otimes \pi_2$.*

Lemma 5.8. *Let $\mathbb{B}_1, \mathbb{B}_2$ be as in the above proposition, and let $\pi_1 \in \mathbb{B}_1, \pi_2 \in \mathbb{B}_2$. There exists some $p \in \mathbb{Z}_{\geq 0}$ such that*

$$\tilde{e}_i^{p+1}(\pi_1 * \pi_2) = (\tilde{e}_i\pi_1) * (\tilde{e}_i^p\pi_2) \text{ and } \tilde{e}_i^p\pi_2 \neq \mathbf{0}.$$

Proof. From (23), we can see that $p = \max\{0, \varepsilon_i(\pi_2) - \varphi_i(\pi_1)\}$ satisfies the lemma. □

5.2 Relations between crystal bases and path models

The crystal bases ([13, Definition 4.2.3]) of $U_q(\widehat{\mathfrak{g}})$-modules and $U_q'(\widehat{\mathfrak{g}})$-modules are typical and very important examples of crystals. In this subsection, we review some realizations of crystal bases using the path models.

We prepare some notation. For a $U_q(\widehat{\mathfrak{g}})$-crystal $\mathcal{B}$ and an element $b \in \mathcal{B}$, we denote by $C(b)$ the connected component of $\mathcal{B}$ containing b, that is, the subset of $\mathcal{B}$ consisting of elements obtained from b by applying $\tilde{e}_i$'s and $\tilde{f}_i$'s ($i \in \widehat{I}$) several times. Note that $C(b)$ is a connected $U_q(\widehat{\mathfrak{g}})$-crystal. For $\lambda \in \widehat{P}$, we denote by π_λ the straight line path $\pi_\lambda(t) = t\lambda$, and we write

$$\mathbb{B}_0(\lambda) = C(\pi_\lambda).$$

It is known that $\mathbb{B}_0(\lambda) \subseteq \mathbb{P}_{\rm int}$ for all $\lambda \in \widehat{P}$ ([24, Lemma 4.5 (d) and Corollary 2]). It is easily seen from the definition of the root operators that for any $\pi = (\mu_1, \ldots, \mu_N; \underline{\sigma}) \in \mathbb{B}_0(\lambda)$, we have $\mu_j \in \widehat{W}\lambda$ for all $1 \leq j \leq N$.

It is well-known that the irreducible highest weight $U_q(\widehat{\mathfrak{g}})$-module $V_q(\Lambda)$ with highest weight $\Lambda \in \widehat{P}_+$ has a crystal basis, which we denote by $\mathcal{B}(\Lambda)$. By b_Λ we denote the highest weight element of $\mathcal{B}(\Lambda)$. From the construction of $\mathcal{B}(\Lambda)$ ([13, Chapter 5]), we have that

$$\mathcal{B}(\Lambda) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_s} b_\Lambda \mid s \geq 0, i_k \in \widehat{I}\} \setminus \{0\}, \tag{25}$$

and

$$\{b \in \mathcal{B}(\Lambda) \mid \tilde{e}_i b = 0 \text{ for all } i \in \widehat{I}\} = \{b_\Lambda\}. \tag{26}$$

Theorem 5.9. *Let $\Lambda \in \widehat{P}_+$.*
(i) ([24, §7]) *If $\pi \in \mathbb{P}$ satisfies $m_i^\pi = 0$ for all $i \in \widehat{I}$ and $\pi(1) = \Lambda$, then we have $C(\pi) \subseteq \mathbb{P}_{\rm int}$, and there exists a unique $U_q(\widehat{\mathfrak{g}})$-crystal isomorphism from $C(\pi)$ to $\mathbb{B}_0(\Lambda)$ which maps π to π_Λ.*
(ii) ([19], [15]) *There exists a unique $U_q(\widehat{\mathfrak{g}})$-crystal isomorphism from $\mathbb{B}_0(\Lambda)$ to $\mathcal{B}(\Lambda)$ which maps π_Λ to b_Λ.*

Corollary 5.10. *If $\pi \in \mathbb{P}$ satisfies $m_i^\pi = 0$ for all $i \in \widehat{I}$ and $\pi(1) = \Lambda \in \widehat{P}_+$, then there exists a unique $U_q(\widehat{\mathfrak{g}})$-crystal isomorphism from $C(\pi)$ to $\mathcal{B}(\Lambda)$ which maps π to b_Λ.*

Remark 5.11. It is known that $\mathbb{B}_0(\Lambda)$ with $\Lambda \in \widehat{P}_+$ coincides with the set $\mathbb{B}(\Lambda)$ of all Lakshmibai-Seshadri paths (LS paths, for short) of shape Λ ([24]). We do not recall the definition of LS paths here, since we do not need it in this article.

We write $\widehat{P}_{\mathrm{cl}} = \widehat{P}/\mathbb{Z}\delta$, and we denote the canonical projection $\widehat{P} \to \widehat{P}_{\mathrm{cl}}$ by cl. Note that $\widehat{P}_{\mathrm{cl}}$ is the weight lattice of $U_q'(\widehat{\mathfrak{g}})$. A path with weight in $\widehat{P}_{\mathrm{cl}}$ is a piecewise linear, continuous map $\eta : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} \widehat{P}_{\mathrm{cl}}$ such that $\eta(0) = 0, \eta(1) \in \widehat{P}_{\mathrm{cl}}$. Let $\mathbb{P}_{\mathrm{cl}}$ denote the set consisting of paths with weight in $\widehat{P}_{\mathrm{cl}}$. For $\xi \in \widehat{P}_{\mathrm{cl}}$, we denote by $\eta_\xi \in \mathbb{P}_{\mathrm{cl}}$ the straight line path $\eta_\xi(t) = t\xi$. We define an expression of $\eta \in \mathbb{P}_{\mathrm{cl}}$ similarly as that for a path with weight in $\widehat{P}$.

For $\pi \in \mathbb{P}$, we define $\mathrm{cl}(\pi) \in \mathbb{P}_{\mathrm{cl}}$ by

$$(\mathrm{cl}(\pi))(t) = \mathrm{cl}(\pi(t)) \quad \text{for } t \in [0,1].$$

Let $\lambda \in P_+ (\subseteq \widehat{P})$, and let

$$\mathbb{B}(\lambda)_{\mathrm{cl}} = \{\mathrm{cl}(\pi) \mid \pi \in \mathbb{B}_0(\lambda)\},$$

which is known to be a finite set. Note that $\eta_{\mathrm{cl}(\lambda)} = \mathrm{cl}(\pi_\lambda) \in \mathbb{B}(\lambda)_{\mathrm{cl}}$. Now, we define a $U_q'(\widehat{\mathfrak{g}})$-crystal structure on $\mathbb{B}(\lambda)_{\mathrm{cl}}$. Define a weight map $\mathrm{wt} : \mathbb{B}(\lambda)_{\mathrm{cl}} \to \widehat{P}_{\mathrm{cl}}$ by $\mathrm{wt}(\eta) = \eta(1)$, and define root operators $\tilde{e}_i, \tilde{f}_i : \mathbb{B}(\lambda)_{\mathrm{cl}} \to \mathbb{B}(\lambda)_{\mathrm{cl}} \cup \{\mathbf{0}\}$ for $i \in \widehat{I}$ by

$$\tilde{e}_i(\mathrm{cl}(\pi)) = \mathrm{cl}(\tilde{e}_i(\pi)), \quad \tilde{f}_i(\mathrm{cl}(\pi)) = \mathrm{cl}(\tilde{f}_i(\pi)),$$

where $\mathrm{cl}(\mathbf{0})$ is understood as $\mathbf{0}$. We also define $\varepsilon_i : \mathbb{B}(\lambda)_{\mathrm{cl}} \to \mathbb{Z}_{\geq 0}$ and $\varphi_i : \mathbb{B}(\lambda)_{\mathrm{cl}} \to \mathbb{Z}_{\geq 0}$ for $i \in \widehat{I}$ by $\varepsilon_i(\mathrm{cl}(\pi)) = \varepsilon_i(\pi)$, $\varphi_i(\mathrm{cl}(\pi)) = \varphi_i(\pi)$. Then these maps are all well-defined, and $\mathbb{B}(\lambda)_{\mathrm{cl}}$ together with these maps becomes a finite $U_q'(\widehat{\mathfrak{g}})$-crystal ([28, §3.3]).

Remark 5.12. For $\lambda \in P_+$, $\mathbb{B}_0(\lambda)$ does not necessarily coincide with the set $\mathbb{B}(\lambda)$ of all LS paths of shape λ. However, it is known that the set $\{\mathrm{cl}(\pi) \mid \pi \in \mathbb{B}(\lambda)\}$ coincides with $\mathbb{B}(\lambda)_{\mathrm{cl}}$ defined above ([29, Lemma 4.5 (1)]). This is why we use the notation '$\mathbb{B}(\lambda)_{\mathrm{cl}}$' in stead of '$\mathbb{B}_0(\lambda)_{\mathrm{cl}}$'.

In [20, §5.2], Kashiwara has introduced a finite dimensional irreducible integrable $U_q'(\widehat{\mathfrak{g}})$-module $W_q(\varpi_i)$ for each $i \in I$ called a level zero fundamental representation, and has proved that it has a crystal basis. We denote this crystal basis of $W_q(\varpi_i)$ by $\mathcal{B}(W_q(\varpi_i))$.

Naito and Sagaki have verified the following facts:

Theorem 5.13. (i) ([27, Theorem 3.2]) *Let $\lambda = \sum_{i \in I} \lambda_i \varpi_i \in P_+$ with $\lambda_i \in \mathbb{Z}_{\geq 0}$. Then there exists a unique isomorphism of $U_q'(\widehat{\mathfrak{g}})$-crystals from $\mathbb{B}(\lambda)_{\mathrm{cl}}$ to $\bigotimes_{i \in I} \mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes \lambda_i}$ (which does not depend on the choice of the ordering of the tensor factors).*
(ii) ([28, Corollary of Theorem 1]) *For all $i \in I$, $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is isomorphic to $\mathcal{B}(W_q(\varpi_i))$ as a $U_q'(\widehat{\mathfrak{g}})$-crystal.*

5.3 Degree function on $\mathbb{B}(\lambda)_{\rm cl}$

Let $\pi \in \mathbb{P}$. If $\pi = (\mu_1, \ldots, \mu_N; \underline{\sigma})$, we call $\mu_1 \in \mathbb{R} \otimes_{\mathbb{Z}} \widehat{P}$ the initial direction of π (which does not depend on the choice of an expression), and we denote the initial direction of π by $\iota(\pi)$. The initial direction of $\eta \in \mathbb{P}_{\rm cl}$ is defined similarly, and denoted by $\iota(\eta)$. Note that $\iota({\rm cl}(\pi)) = {\rm cl}(\iota(\pi))$ follows.

For $\lambda \in P_+$, let d_λ be the nonnegative integer satisfying

$$\widehat{W}\lambda \cap (\lambda + \mathbb{Z}\delta) = \lambda + d_\lambda \mathbb{Z}\delta.$$

Then we have

$$\widehat{W}\lambda = W\lambda + d_\lambda \mathbb{Z}\delta. \tag{27}$$

Lemma 5.14. *Let $\lambda \in P_+$.*
(i) *For any $\pi \in \mathbb{B}_0(\lambda)$, we have ${\rm wt}(\pi) \in \lambda - Q_+ + \mathbb{Z}\delta$.*
(ii) *For any $\eta \in \mathbb{B}(\lambda)_{\rm cl}$, we have ${\rm wt}(\eta) \in {\rm cl}(\lambda - Q_+)$.*

Proof. By the definition of $\mathbb{B}_0(\lambda)$, ${\rm wt}(\pi) \in \lambda + \sum_{i \in \widehat{I}} \mathbb{Z}\alpha_i = \lambda + Q + \mathbb{Z}\delta$. Let $\pi = (\mu_1, \ldots, \mu_N; \underline{\sigma})$ be an expression of π. Since $\mu_j \in \widehat{W}\lambda \subseteq W\lambda + \mathbb{Z}\delta \subseteq \lambda - Q_+ + \mathbb{Z}\delta$ for each j, we have that

$${\rm wt}(\pi) = \sum_{p=1}^{N} (\sigma_p - \sigma_{p-1})\mu_p \in \lambda - \sum_{i \in I} \mathbb{R}_{\geq 0}\alpha_i + \mathbb{R}\delta.$$

Hence, (i) follows. Then (ii) is obvious from (i). □

Lemma 5.15. *Let $\eta \in \mathbb{B}(\lambda)_{\rm cl}$.*
(i) *For arbitrary $\pi \in \mathbb{B}_0(\lambda) \cap {\rm cl}^{-1}(\eta)$, we have*

$$\mathbb{B}_0(\lambda) \cap {\rm cl}^{-1}(\eta) = \{\pi + \pi_{kd_\lambda\delta} \mid k \in \mathbb{Z}\}.$$

(ii) *Let $\mu \in \widehat{W}\lambda$ be an element satisfying ${\rm cl}(\mu) = \iota(\eta)$. Then there exists a unique element $\pi \in \mathbb{B}_0(\lambda) \cap {\rm cl}^{-1}(\eta)$ such that $\iota(\pi) = \mu$.*

Proof. (i) has been proved in [27, Lemma 4.5]. To prove (ii), let $\pi' \in \mathbb{B}_0(\lambda) \cap {\rm cl}^{-1}(\eta)$ be an arbitrary element. From (i),

$$\mathbb{B}_0(\lambda) \cap {\rm cl}^{-1}(\eta) = \{\pi' + \pi_{kd_\lambda\delta} \mid k \in \mathbb{Z}\}.$$

Since ${\rm cl}(\iota(\pi')) = \iota(\eta) = {\rm cl}(\mu)$ and $\iota(\pi') \in \widehat{W}\lambda$, from (27) there exists some $s \in \mathbb{Z}$ such that $\iota(\pi') = \mu + sd_\lambda\delta$. Now, it is obvious that $\pi' - \pi_{sd_\lambda\delta}$ is the unique element in $\mathbb{B}_0(\lambda) \cap {\rm cl}^{-1}(\eta)$ whose initial direction is μ. □

Now, we recall the definition of the degree function on $\mathbb{B}(\lambda)_{\rm cl}$ ($\lambda \in P_+$) introduced in [30]. Let $i_{\rm cl} : \widehat{P}_{\rm cl} \to \widehat{P}$ be the map uniquely determined by conditions

$${\rm cl} \circ i_{\rm cl}(\xi) = \xi, \quad i_{\rm cl}(\xi) \in P + \mathbb{Z}\Lambda_0$$

for $\xi \in \widehat{P}_{\rm cl}$. For $\eta \in \mathbb{B}(\lambda)_{\rm cl}$, we denote by π_η the element in $\mathbb{B}_0(\lambda) \cap {\rm cl}^{-1}(\eta)$ such that $\iota(\pi_\eta) \in W\lambda$, which is unique by Lemma 5.15 (ii). Then we define ${\rm Deg}(\eta) \in \mathbb{Z}$ by an integer satisfying

$$\pi_\eta(1) = i_{\rm cl} \circ \eta(1) - \delta{\rm Deg}(\eta),$$

and call ${\rm Deg}(\eta)$ the degree of η. By [30, Lemma 3.1.1], ${\rm Deg}(\eta) \leq 0$ for all $\eta \in \mathbb{B}(\lambda)_{\rm cl}$.

Remark 5.16. The main theorem in [30] says that the degree function on $\mathbb{B}(\lambda)_{\mathrm{cl}}$ where $\lambda = \sum_{i\in I} \lambda_i \varpi_i$ can be identified with the energy function (see [11], [12]) on $\bigotimes_{i\in I} \mathcal{B}(W_q(\varpi_i))^{\otimes \lambda_i}$ up to some constant through the isomorphism given in Theorem 5.13.

For $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, we define

$$\mathrm{wt}_{\widehat{P}}(\eta) := \mathrm{wt}(\pi_\eta) = i_{\mathrm{cl}} \circ \mathrm{wt}(\eta) - \delta \mathrm{Deg}(\eta) \in \widehat{P},$$

and we call $\mathrm{wt}_{\widehat{P}}(\eta)$ the $\widehat{P}$-weight of η.

Remark 5.17. By the definition of π_η, we have

$$\{\pi_\eta \mid \eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}\} = \{\pi \in \mathbb{B}_0(\lambda) \mid \iota(\pi) \in P\}.$$

Hence, we have

$$\sum_{\eta\in\mathbb{B}(\lambda)_{\mathrm{cl}}} e(\mathrm{wt}_{\widehat{P}}(\eta)) = \sum_{\eta\in\mathbb{B}(\lambda)_{\mathrm{cl}}} e(\mathrm{wt}(\pi_\eta)) = \sum_{\substack{\pi\in\mathbb{B}_0(\lambda)\\ \iota(\pi)\in P}} e(\mathrm{wt}(\pi)).$$

Lemma 5.18. *Let $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$. In the following equations, $\pi_{\mathbf{0}}$ is understood as* $\mathbf{0}$.
(i) *If $i \in I$, we have $\tilde{e}_i \pi_\eta = \pi_{\tilde{e}_i\eta}$ and $\tilde{f}_i \pi_\eta = \pi_{\tilde{f}_i\eta}$.*
(ii) *If $\tilde{e}_0\eta \neq \mathbf{0}$, then we have $\tilde{f}_0 \pi_\eta = \pi_{\tilde{f}_0\eta}$.*

Proof. (i) By the definition of the root operator, we have

$$\iota(\tilde{e}_i\pi_\eta) = \iota(\pi_\eta) \text{ or } s_i\iota(\pi_\eta),$$

which implies $\iota(\tilde{e}_i\pi_\eta) \in W\lambda$. Since $\mathrm{cl}(\tilde{e}_i\pi_\eta) = \tilde{e}_i\mathrm{cl}(\pi_\eta) = \tilde{e}_i\eta$, we have $\tilde{e}_i\pi_\eta = \pi_{\tilde{e}_i\eta}$ by definition. The proof of $\tilde{f}_i\pi_\eta = \pi_{\tilde{f}_i\eta}$ is similar. (ii) If $\tilde{f}_0\eta = \mathbf{0}$, (ii) follows since $\tilde{f}_0\eta = \mathrm{cl}(\tilde{f}_0\pi_\eta)$. Assume $\tilde{f}_0\eta \neq \mathbf{0}$. Then we have $\tilde{f}_0\pi_\eta \neq \mathbf{0}$, and we also have $\tilde{e}_0\pi_\eta \neq \mathbf{0}$ by the assumption. Hence we have from [23, Lemma 5.3] that

$$\iota(\tilde{f}_0\pi_\eta) = \iota(\pi_\eta) \in W\lambda.$$

Since $\mathrm{cl}(\tilde{f}_0\pi_\eta) = \tilde{f}_0\eta$, we have $\tilde{f}_0\pi_\eta = \pi_{\tilde{f}_0\eta}$ by definition. □

6 Decomposition of $\mathcal{B}(\Lambda) \otimes \mathbb{B}_0(\lambda)$

Throughout this section, we assume that $\Lambda \in \widehat{P}_+ \setminus \mathbb{Z}\delta$ and $\lambda \in P_+$, and we show that $\mathcal{B}(\Lambda) \otimes \mathbb{B}_0(\lambda)$ is isomorphic to the direct sum of the crystal bases of irreducible highest weight $U_q(\widehat{\mathfrak{g}})$-modules. This result is motivated by the tensor product rule in [24] (see also [10]).

6.1 Technical lemmas

Here, we prepare several technical lemmas.

Lemma 6.1. *Let u be a real number such that $0 < u \leq 1$, and assume that $\pi \in \mathbb{B}_0(\lambda)$ satisfies*

$$H_i^\pi(t) \geq -\langle \Lambda, \alpha_i^\vee \rangle \quad \text{for all } t \in [0,u], \ i \in \widehat{I}. \tag{28}$$

(i) *For any $i \in \widehat{I}$ such that $\tilde{e}_i(\pi_\Lambda * \pi) \neq \mathbf{0}$, we have $\tilde{e}_i \pi \neq \mathbf{0}$ and*

$$\tilde{e}_i \pi(t) = \pi(t) \quad \text{for all } t \in [0,u]. \tag{29}$$

(ii) *For any sequence $i_1, \ldots, i_k$ of elements of $\widehat{I}$ such that $\tilde{e}_{i_1} \cdots \tilde{e}_{i_k}(\pi_\Lambda * \pi) \neq \mathbf{0}$, we have $\tilde{e}_{i_1} \cdots \tilde{e}_{i_k} \pi \neq \mathbf{0}$ and*

$$\tilde{e}_{i_1} \cdots \tilde{e}_{i_k} \pi(t) = \pi(t) \quad \text{for all } t \in [0,u].$$

Proof. It suffices only to show (i) since we can prove (ii) inductively from (i). The statement $\tilde{e}_i \pi \neq \mathbf{0}$ follows since otherwise we have $\tilde{e}_i(\pi_\Lambda * \pi) = \mathbf{0}$. From $\tilde{e}_i(\pi_\Lambda * \pi) \neq \mathbf{0}$, we have $m_i^{\pi_\Lambda * \pi} \leq -1$, which implies from the definition of the concatenation that $m_i^\pi \leq -\langle \Lambda, \alpha_i^\vee \rangle - 1$. Then Lemma 5.3 (i) together with the assumption (28) implies (29). □

Lemma 6.2. *There exist a positive integer N and a sequence of real numbers $\underline{\sigma} : 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_N = 1$ such that any $\pi \in \mathbb{B}_0(\lambda)$ has a unique expression in the form $(\mu_1, \ldots, \mu_N; \underline{\sigma})$ with $\mu_i \in \widehat{W}\lambda$.*

Proof. For each $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, take $N_\eta \in \mathbb{Z}_{>0}$ and a sequence $\underline{\sigma}^\eta : 0 = \sigma_0^\eta < \sigma_1^\eta < \cdots < \sigma_{N_\eta}^\eta = 1$ so that π_η has an expression in the form $(\mu_1, \ldots, \mu_{N_\eta}; \underline{\sigma}^\eta)$. Now, we define $\underline{\sigma} : 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_N = 1$ by the unique sequence such that

$$\{\sigma_0, \ldots, \sigma_N\} = \bigcup_{\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}} \{\sigma_0^\eta, \ldots, \sigma_{N_\eta}^\eta\}.$$

We prove that these N and $\underline{\sigma}$ satisfy the statement of the lemma. Let $\pi_1 \in \mathbb{B}_0(\lambda)$ be an arbitrary element, and let $\eta_1 = \mathrm{cl}(\pi_1)$. Then π_{η_1} has an expression in the form $(\mu_1, \ldots, \mu_{N_{\eta_1}}; \underline{\sigma}^{\eta_1})$. By Lemma 5.15, there exists some $s \in \mathbb{Z}$ such that $\pi_1 = \pi_{\eta_1} + \pi_{sd_\lambda \delta}$, and then we have $\pi_1 = (\mu_1 + sd_\lambda\delta, \ldots, \mu_{N_{\eta_1}} + sd_\lambda\delta; \underline{\sigma}^{\eta_1})$. Define a sequence $\mu'_1, \ldots, \mu'_N$ of elements of $\widehat{W}\lambda$ by

$$\mu'_i = \mu_j + sd_\lambda\delta \quad \text{if } \sigma_{j-1}^{\eta_1} < \sigma_i \leq \sigma_j^{\eta_1}.$$

Then we can easily see that $\pi = (\mu'_1, \ldots, \mu'_N; \underline{\sigma})$. Uniqueness of an expression is obvious. □

If a sequence $\underline{\sigma} : 0 = \sigma_0 < \cdots < \sigma_N = 1$ satisfies the condition of this lemma, we say that $\underline{\sigma}$ is *sufficiently fine* for λ.

Remark 6.3. Assume that $\underline{\sigma}$ is sufficiently fine for λ. Then for any $\pi \in \mathbb{B}_0(\lambda)$, it is obvious that the function $H_i^\pi(t)$ is strictly increasing, strictly decreasing, or constant on each $[\sigma_p, \sigma_{p+1}]$.

Lemma 6.4. *Assume that $\underline{\sigma} : 0 = \sigma_0 < \cdots < \sigma_N = 1$ is sufficiently fine for λ. Then for any $\pi \in \mathbb{B}_0(\lambda), i \in \widehat{I}$ and $M \in \mathbb{Z}_{\geq 0}$, we have*

$$\max\{u \in [0,1] \mid H_i^\pi(t) \geq -M \text{ for all } t \in [0,u]\} \in \{\sigma_0, \sigma_1, \ldots, \sigma_N\}.$$

Proof. Set

$$u_0 = \max\{u \in [0,1] \mid H_i^\pi(t) \geq -M \text{ for all } t \in [0,u]\},$$

and assume that $u_0 \notin \{\sigma_0, \sigma_1, \ldots, \sigma_N\}$. If $m_i^\pi \geq -M$, we have $u_0 = 1 = \sigma_N$, and hence $m_i^\pi \leq -M-1$ follows. Let $p \in \{0, \ldots, N-1\}$ be the number such that $\sigma_p < u_0 < \sigma_{p+1}$. By the definition of u_0 and the assumption that $\underline{\sigma}$ is sufficiently fine, we have that

$$H_i^\pi(\sigma_p) > -M, \; H_i^\pi(u_0) = -M \text{ and } H_i^\pi(\sigma_{p+1}) < -M.$$

Let $q = -M - m_i^\pi$. Then we have from (22) that

$$m_i^{\tilde{e}_i^r \pi} \leq -M-1 \text{ for } r < q \text{ and } m^{\tilde{e}_i^q \pi} = -M.$$

Then since $H_i^\pi(t) \geq -M$ for all $t \in [0, u_0]$, we can show inductively from Lemma 5.3 (i) that

$$\tilde{e}_i^q \pi(t) = \pi(t) \quad \text{for all } t \in [0, u_0].$$

Hence, we have $H_i^{\tilde{e}_i^q \pi}(\sigma_p) > -M$ and $H_i^{\tilde{e}_i^q \pi}(u_0) = -M$. On the other hand, we have $H_i^{\tilde{e}_i^q \pi}(\sigma_{p+1}) \geq -M$ since $m_i^{\tilde{e}_i^q \pi} = -M$, which contradicts the assumption that $\underline{\sigma}$ is sufficiently fine. □

For a subset J of $\widehat{I}$, we denote by $\widehat{W}_J$ the subgroup of $\widehat{W}$ generated by simple reflections $\{s_i \mid i \in J\}$. It is well-known that $\widehat{W}_J$ is a finite subgroup if J is proper.

Lemma 6.5. *Let J be a proper subset of $\widehat{I}$. Then for any $\pi \in \mathbb{B}_0(\lambda)$, the set $\{\tilde{e}_{i_1} \cdots \tilde{e}_{i_s} \pi \mid s \geq 0, \; i_k \in J\} \setminus \{\mathbf{0}\}$ is finite.*

Proof. Assume that $\underline{\sigma} : 0 = \sigma_0 < \cdots < \sigma_N = 1$ is sufficiently fine for λ, and let $(\mu_1, \ldots, \mu_N; \underline{\sigma})$ be the expression of π. By the definition of root operators and the sufficiently fineness of $\underline{\sigma}$, we can see that

$$\{\tilde{e}_{i_1} \cdots \tilde{e}_{i_s} \pi \mid s \geq 0, \; i_k \in J\} \setminus \{\mathbf{0}\} \subseteq \{(w_1\mu_1, \ldots, w_N\mu_N; \underline{\sigma}) \mid w_j \in \widehat{W}_J\},$$

and right hand side is a finite set since $\widehat{W}_J$ is finite. □

Lemma 6.6. *For any $\pi \in \mathbb{B}_0(\lambda)$ and $u \in [0,1]$, a subset $\{i \in \widehat{I} \mid H_i^\pi(u) = -\langle \Lambda, \alpha_i^\vee \rangle\}$ of $\widehat{I}$ is proper.*

Proof. Let $(\mu_1, \ldots, \mu_s; \underline{\sigma})$ be an expression of π. Recall that if $\sigma_{p-1} \leq u \leq \sigma_p$, we have

$$\pi(u) = \sum_{1 \leq p' \leq p-1} (\sigma_{p'} - \sigma_{p'-1})\mu_{p'} + (u - \sigma_{p-1})\mu_p.$$

Since $\mu_j \in \widehat{W}\lambda = W\lambda + d_\lambda \mathbb{Z}\delta$, we have that $\langle \mu_j, K \rangle = 0$ for all $1 \leq j \leq s$, and hence we have $\langle \pi(u), K \rangle = 0$. On the other hand, we have $\langle \Lambda, K \rangle > 0$ from $\Lambda \in \widehat{P}_+ \setminus \mathbb{Z}\delta$ since $K \in \sum_{i \in \widehat{I}} \mathbb{Z}_{>0} \alpha_i^\vee$ ([17, §6]). Hence $\langle \pi(u) + \Lambda, K \rangle > 0$ follows, which implies that $\langle \pi(u) + \Lambda, \alpha_i^\vee \rangle > 0$ for some $i \in \widehat{I}$ since $K \in \sum_{i \in \widehat{I}} \mathbb{Z}_{>0} \alpha_i^\vee$. □

6.2 Decomposition of $\mathcal{B}(\Lambda)\otimes\mathbb{B}_0(\lambda)$

We set

$$\mathbb{B}_0(\lambda)^\Lambda=\{\pi\in\mathbb{B}_0(\lambda)\mid m_i^\pi\geq-\langle\Lambda,\alpha_i^\vee\rangle\text{ for all }i\in\widehat{I}\}.$$

Note that $\Lambda+\pi_0(1)\in\widehat{P}_+$ for $\pi_0\in\mathbb{B}_0(\lambda)^\Lambda$.

Proposition 6.7. *We have*

$$\mathbb{B}_0(\Lambda)*\mathbb{B}_0(\lambda)=\bigoplus_{\pi_0\in\mathbb{B}_0(\lambda)^\Lambda}C(\pi_\Lambda*\pi_0).$$

Proof. If $\pi_0\in\mathbb{B}_0(\lambda)^\Lambda$, $\pi_\Lambda*\pi_0$ satisfies $m_i^{\pi_\Lambda*\pi_0}=0$ for all $i\in\widehat{I}$, which implies by Corollary 5.10 that there exists a $U_q(\widehat{\mathfrak{g}})$-crystal isomorphism from $C(\pi_\Lambda*\pi_0)$ to $\mathcal{B}(\Lambda+\pi_0(1))$ that maps $\pi_\Lambda*\pi_0$ to $b_{\Lambda+\pi_0(1)}$. From this isomorphism and (26), we have that

$$\{\pi\in C(\pi_\Lambda*\pi_0)\mid\tilde{e}_i\pi=0\text{ for all }i\in\widehat{I}\}=\{\pi_\Lambda*\pi_0\},$$

which implies that a sum $\bigcup_{\pi_0\in\mathbb{B}_0(\lambda)^\Lambda}B(\pi_\Lambda*\pi_0)$ is disjoint, and we have

$$\mathbb{B}_0(\Lambda)*\mathbb{B}_0(\lambda)\supseteq\bigoplus_{\pi_0\in\mathbb{B}_0(\lambda)^\Lambda}C(\pi_\Lambda*\pi_0).$$

We need to show the opposite containment, and this is equivalent to the following statement: for any $\pi_1\in\mathbb{B}_0(\Lambda)$ and $\pi_2\in\mathbb{B}_0(\lambda)$, there exists a sequence $i_1,\dots,i_k$ of elements of $\widehat{I}$ such that

$$\tilde{e}_{i_1}\cdots\tilde{e}_{i_k}(\pi_1*\pi_2)=\pi_\Lambda*\pi_0\quad\text{for some }\pi_0\in\mathbb{B}_0(\lambda)^\Lambda.$$

By Lemma 5.8, we can see that there exists a sequence $j_1,\dots,j_s$ of elements of $\widehat{I}$ such that

$$\tilde{e}_{j_1}\cdots\tilde{e}_{j_s}(\pi_1*\pi_2)=\pi_\Lambda*\pi_2'\quad\text{for some }\pi_2'\in\mathbb{B}_0(\lambda).$$

Hence it suffices to prove that for any $\pi_2\in\mathbb{B}_0(\lambda)$, there exists a sequence $i_1,\dots,i_k$ of elements of $\widehat{I}$ such that

$$\tilde{e}_{i_1}\cdots\tilde{e}_{i_k}(\pi_\Lambda*\pi_2)=\pi_\Lambda*\pi_0\quad\text{for some }\pi_0\in\mathbb{B}_0(\lambda)^\Lambda.$$

Let $\underline{\sigma}:0=\sigma_0<\sigma_1<\cdots<\sigma_N=1$ be a sufficiently fine sequence for λ. By Lemma 6.4, there exists some $0\leq p_0\leq N$ such that

$$\sigma_{p_0}=\max\{u\in[0,1]\mid H_i^{\pi_2}(t)\geq-\langle\Lambda,\alpha_i^\vee\rangle\text{ for }t\in[0,u],\ i\in\widehat{I}\}.$$

We show the above statement by the descending induction on p_0. If $p_0=N$, then we have $\pi_2\in\mathbb{B}_0(\lambda)^\Lambda$, and there is nothing to prove. Assume $p_0<N$, and set

$$J=\{i\in\widehat{I}\mid H_i^{\pi_2}(\sigma_{p_0})=-\langle\Lambda,\alpha_i^\vee\rangle\},$$

which is a proper subset of $\widehat{I}$ by Lemma 6.6. We have that

$$\begin{aligned}\{\tilde{e}_{i_1}\cdots\tilde{e}_{i_s}(\pi_\Lambda*\pi_2)\mid s\geq0,i_k\in J\}\setminus\{\mathbf{0}\}\\ \subseteq\pi_\Lambda*\{\tilde{e}_{i_1}\cdots\tilde{e}_{i_s}\pi_2\mid s\geq0,i_k\in J\}\setminus\{\mathbf{0}\},\end{aligned}$$

and since the right hand side is a finite set by Lemma 6.5, the left hand side is also finite. Hence, we can see by weight consideration that there exists

$$\pi_\Lambda * \pi_2' \in \{\tilde{e}_{i_1} \cdots \tilde{e}_{i_s}(\pi_\Lambda * \pi_2) \mid s \geq 0, i_k \in J\} \setminus \{\mathbf{0}\}$$

such that

$$\tilde{e}_i(\pi_\Lambda * \pi_2') = \mathbf{0} \text{ for all } i \in J. \tag{30}$$

Note that we have from Lemma 6.1 (ii) and the definition of p_0 that

$$\pi_2'(t) = \pi_2(t) \quad \text{for all } t \in [0, \sigma_{p_0}]. \tag{31}$$

Let $0 \leq p_0' \leq N$ be an integer such that

$$\sigma_{p_0'} = \max\{u \in [0,1] \mid H_i^{\pi_2'}(t) \geq -\langle \Lambda, \alpha_i^\vee \rangle \text{ for } t \in [0,u],\ i \in \widehat{I}\}.$$

For $i \in J$, we have $m_i^{\pi_\Lambda * \pi_2'} = 0$ by (30), which implies that $H_i^{\pi_2'}(t) \geq -\langle \Lambda, \alpha_i^\vee \rangle$ for all $t \in [0,1]$. On the other hand, if $i \in \widehat{I} \setminus J$, we have from (31) and the definition of J that

$$H_i^{\pi_2'}(t) \geq -\langle \Lambda, \alpha_i^\vee \rangle \text{ for all } t \in [0, \sigma_{p_0}] \text{ and } H_i^{\pi_2'}(\sigma_{p_0}) > -\langle \Lambda, \alpha_i^\vee \rangle.$$

Hence we have $p_0' > p_0$, and this together with the induction hypothesis completes the proof. □

The above proposition, together with Proposition 5.7, Theorem 5.9 and Corollary 5.10, implies the following corollary, which is some sort of generalization of [10, Theorem 1.6], in which $\mathfrak{g} = \mathfrak{sl}_{\ell+1}$ and $\lambda = m\varpi_1$.

Corollary 6.8. *We have*

$$\mathcal{B}(\Lambda) \otimes \mathbb{B}_0(\lambda) \xrightarrow{\sim} \bigoplus_{\pi_0 \in \mathbb{B}_0(\lambda)^\Lambda} \mathcal{B}(\Lambda + \pi_0(1))$$

as $U_q(\widehat{\mathfrak{g}})$-crystals, where the given isomorphism maps each $b_\Lambda \otimes \pi_0 \in b_\Lambda \otimes \mathbb{B}_0(\lambda)^\Lambda$ to $b_{\Lambda+\pi_0(1)} \in \mathcal{B}(\Lambda + \pi_0(1))$.

Note that we have shown the following fact in the proof of proposition 6.7, which is needed again in the next section:

Lemma 6.9. *Assume that $\underline{\sigma} : 0 = \sigma_0 < \cdots < \sigma_N = 1$ is sufficiently fine for λ and $\pi_2 \in \mathbb{B}_0(\lambda)$. Let $0 \leq p_0 \leq N$ be an integer such that*

$$\sigma_{p_0} = \max\{u \in [0,1] \mid H_i^{\pi_2}(t) \geq -\langle \Lambda, \alpha_i^\vee \rangle \textit{ for all } t \in [0,u], i \in \widehat{I}\},$$

and assume $p_0 < N$. Let $J = \{i \in \widehat{I} \mid H_i^{\pi_2}(\sigma_{p_0}) = -\langle \Lambda, \alpha_i^\vee \rangle\}$. Then there exists $\pi_2' \in \mathbb{B}_0(\lambda)$ such that

$$\pi_\Lambda * \pi_2' \in \{\tilde{e}_{i_1} \cdots \tilde{e}_{i_s}(\pi_\Lambda * \pi_2) \mid s \geq 0, i_k \in J\} \setminus \{\mathbf{0}\}$$

and $p_0' > p_0$, where p_0' denotes the integer such that

$$\sigma_{p_0'} = \max\{u \in [0,1] \mid H_i^{\pi_2'}(t) \geq -\langle \Lambda, \alpha_i^\vee \rangle \textit{ for all } t \in [0,u], i \in \widehat{I}\}.$$

7 Relations among the Demazure crystals, $\mathbb{B}_0(\lambda)$ and $\mathbb{B}(\lambda)_{\rm cl}$

7.1 Demazure crystals

For a $U_q(\widehat{\mathfrak{g}})$-crystal $\mathcal{B}$ and a subset $\mathcal{C} \subseteq \mathcal{B}$, we define a subset $\mathcal{F}_i\mathcal{C}$ of $\mathcal{B}$ for $i \in \widehat{I}$ by

$$\mathcal{F}_i\mathcal{C} = \{\tilde{f}_i^k b \mid b \in \mathcal{C}, k \geq 0\} \setminus \{0\},$$

and for a sequence $\mathbf{i} : i_1, i_2, \ldots, i_m$ of elements of $\widehat{I}$, we define $\mathcal{F}_{\mathbf{i}}\mathcal{C}$ by

$$\mathcal{F}_{\mathbf{i}}\mathcal{C} = \mathcal{F}_{i_1}\mathcal{F}_{i_2}\cdots\mathcal{F}_{i_m}\mathcal{C}.$$

For notational convenience, we set $\mathcal{F}_{\emptyset}\mathcal{C} = \mathcal{C}$, and $\mathcal{F}_{\mathbf{i}}b = \mathcal{F}_{\mathbf{i}}\{b\}$ for $b \in \mathcal{B}$.

Let $\Lambda \in \widehat{P}_+$, $w \in \widehat{W}$, and let $w = s_{i_1}\cdots s_{i_m}$ be a reduced expression.

Proposition 7.1 ([18]). *The subset $\mathcal{F}_{i_1,\ldots,i_m}b_\Lambda \subseteq \mathcal{B}(\Lambda)$ is independent of the choice of a reduced expression of w.*

We denote this subset of $\mathcal{B}(\Lambda)$ by $\mathcal{B}_w(\Lambda)$. It is known that $\mathcal{B}_w(\Lambda)$ has the following properties:

Proposition 7.2 ([18]). (i) *For any $i \in \widehat{I}$, we have $\tilde{e}_i\mathcal{B}_w(\Lambda) \subseteq \mathcal{B}_w(\Lambda) \cup \{0\}$.*
(ii) *We have*

$$\mathrm{ch}_{\widehat{\mathfrak{h}}}\, V_w(\Lambda) = \sum_{b\in\mathcal{B}_w(\Lambda)} e(\mathrm{wt}(b)).$$

Definition 7.3. We call the subset $\mathcal{B}_w(\Lambda) \subseteq \mathcal{B}(\Lambda)$ the *Demazure crystal* associated with Λ and w. (Note that $\mathcal{B}_w(\Lambda)$ does not have a structure of a $U_q(\widehat{\mathfrak{g}})$-crystal).

Lemma 7.4. (i) *For any $i \in \widehat{I}$ and $w \in \widehat{W}$, we have*

$$\mathcal{F}_i\mathcal{B}_w(\Lambda) = \begin{cases} \mathcal{B}_{s_iw}(\Lambda) & \text{if } s_iw > w, \\ \mathcal{B}_w(\Lambda) & \text{if } s_iw < w. \end{cases}$$

(ii) *For an arbitrary sequence $\mathbf{i} : i_1, \ldots, i_m$ of elements of $\widehat{I}$, there exists some $w \in \widehat{W}$ such that $\mathcal{F}_{\mathbf{i}}b_\Lambda = \mathcal{B}_w(\Lambda)$.*

Proof. If $s_iw > w$, (i) follows from the definition. If $s_iw < w$, then w has a reduced expression in the form $w = s_is_{j_1}\cdots s_{j_m}$ by the exchange condition ([17, Lemma 3.11]), and hence (i) follows since

$$\mathcal{F}_i\mathcal{B}_w(\Lambda) = \mathcal{F}_i\mathcal{F}_{i,j_1,\ldots,j_m}b_\Lambda = \mathcal{F}_{i,j_1,\ldots,j_m}b_\Lambda = \mathcal{B}_w(\Lambda).$$

Then (ii) can be shown inductively from (i). □

Proposition 7.5. *Let J be a proper subset of $\widehat{I}$, and let $\mathbf{i} : i_1, \ldots, i_m$ be a sequence of $\widehat{I}$. We assume that there exists some $1 \leq m' \leq m$ such that $i_k \in J$ for all $1 \leq k \leq m'$ and $s_{i_1}\cdots s_{i_{m'}}$ is a reduced expression of the longest element of $\widehat{W}_J$. Then there exists some element $w \in \widehat{W}$ satisfying $\mathcal{F}_{\mathbf{i}}b_\Lambda = \mathcal{B}_w(\Lambda)$ and $\langle w\Lambda, \alpha_i^\vee\rangle \leq 0$ for $i \in J$. Moreover, this Demazure crystal $\mathcal{B}_w(\Lambda)$ satisfies*

$$\tilde{f}_i\mathcal{B}_w(\Lambda) \subseteq \mathcal{B}_w(\Lambda) \cup \{0\} \text{ for all } i \in J. \tag{32}$$

Proof. It suffices to show that there exists $w \in \widehat{W}$ such that

$$s_i w < w \text{ for all } i \in J \text{ and } \mathcal{F}_1 b_\Lambda = \mathcal{B}_w(\Lambda).$$

Indeed if this is true, since $s_i w < w$ if and only if $w^{-1}\alpha_i$ is a negative root of $\widehat{\Delta}$ ([17, Lemma 3.11]), we have that

$$\langle w\Lambda, \alpha_i^\vee \rangle = \langle \Lambda, w^{-1}\alpha_i^\vee \rangle \leq 0 \text{ for } i \in J,$$

and (32) also follows from Lemma 7.4 (i). We show by the descending induction on k that there exists an element $w_k \in \widehat{W}$ for each $1 \leq k \leq m'+1$ satisfying the following two conditions: w_k satisfies $\mathcal{F}_{i_k,\ldots,i_m} b_\Lambda = \mathcal{B}_{w_k}(\Lambda)$, and has a reduced expression in the form

$$w_k = s_{i_k} \cdots s_{i_{m'}} s_{j_1} \cdots s_{j_l} \text{ for some } l \in \mathbb{Z}_{\geq 0},\ j_1, \ldots, j_l \in \widehat{I}.$$

Note that this statement for $k = 1$ implies the above assertion. The statement for $k = m'+1$ follows from Lemma 7.4 (ii) since the second condition is trivial in this case. Assume $k \leq m'$. By the induction hypothesis, w_{k+1} has a reduced expression in the form $w_{k+1} = s_{i_{k+1}} \cdots s_{i_{m'}} s_{j_1} \cdots s_{j_l}$. If $s_{i_k} w_{k+1} > w_{k+1}$, then we have

$$\mathcal{F}_{i_k,\ldots,i_m} b_\Lambda = \mathcal{F}_{i_k} \mathcal{B}_{w_{k+1}}(\Lambda) = \mathcal{B}_{s_{i_k} w_{k+1}}(\Lambda)$$

by Lemma 7.4 (i), and hence $w_k = s_{i_k} w_{k+1}$ satisfies the above conditions. Assume that $s_{i_k} w_{k+1} < w_{k+1}$. Then by the exchange condition, there exists some $k+1 \leq p \leq m'$ such that

$$s_{i_k} s_{i_{k+1}} \cdots s_{i_{p-1}} = s_{i_{k+1}} \cdots s_{i_p},$$

or there exists some $1 \leq q \leq l$ such that

$$s_{i_k} s_{i_{k+1}} \cdots s_{i_{m'}} s_{j_1} \cdots s_{j_{q-1}} = s_{i_{k+1}} \cdots s_{i_{m'}} s_{j_1} \cdots s_{j_q}.$$

However, since $s_{i_1} \cdots s_{i_{m'}}$ is a reduced expression, the first case cannot occur. Hence the second case occurs, and

$$w_{k+1} = s_{i_k} s_{i_{k+1}} \cdots s_{i_{m'}} s_{j_1} \cdots s_{j_{q-1}} s_{j_{q+1}} \cdots s_{j_l}$$

is a reduced expression of w_{k+1}. Moreover, we have

$$\mathcal{F}_{i_k,\ldots,i_m} b_\Lambda = \mathcal{F}_{i_k} \mathcal{B}_{w_{k+1}}(\Lambda) = \mathcal{B}_{w_{k+1}}(\Lambda)$$

by Lemma 7.4 (i). Hence $w_k = w_{k+1}$ satisfies the above conditions. □

7.2 Demazure crystal decomposition of $b_\Lambda \otimes \mathbb{B}_0(\lambda)$

Throughout the rest of this section, we assume that $\Lambda \in \widehat{P}_+ \setminus \mathbb{Z}\delta$, $\lambda \in P_+$ and a sequence of real numbers $\underline{\sigma} : 0 = \sigma_0 < \cdots < \sigma_N = 1$ is sufficiently fine for λ. For Λ, we denote by $\widehat{I}_\Lambda$ the subset of $\widehat{I}$ defined by

$$\widehat{I}_\Lambda = \{ i \in \widehat{I} \mid \langle \Lambda, \alpha_i^\vee \rangle = 0 \}.$$

This subsection is devoted to show the following proposition. The proof is carried out in the similar line as [22, Proposition 12].

Proposition 7.6. *For each $\pi_0 \in \mathbb{B}_0(\lambda)^\Lambda$, there exists some $w_{\pi_0} \in \widehat{W}$ such that the image of a subset $b_\Lambda \otimes \mathbb{B}_0(\lambda)$ in $\mathcal{B}(\Lambda) \otimes \mathbb{B}_0(\lambda)$ under the isomorphism given in Corollary 6.8 coincides with the disjoint union of the Demazure crystals*

$$\coprod_{\pi_0 \in \mathbb{B}_0(\lambda)^\Lambda} \mathcal{B}_{w_{\pi_0}}(\Lambda + \pi_0(1)).$$

Moreover, each w_{π_0} satisfies $\langle w_{\pi_0}(\Lambda + \pi_0(1)), \alpha_i^\vee \rangle \leq 0$ for all $i \in \widehat{I}_\Lambda$.

First we show the following lemma:

Lemma 7.7. *Let $0 < u \leq 1$ be a real number, J be a subset of $\widehat{I}$, and let $\pi \in \mathbb{B}_0(\lambda)$ be a path satisfying for all $i \in J$ that*

$$H_i^\pi(t) \geq -\langle \Lambda, \alpha_i^\vee \rangle \text{ for all } t \in [0, u] \text{ and } H_i^\pi(u) = -\langle \Lambda, \alpha_i^\vee \rangle. \tag{33}$$

(i) *For $i \in J$ such that $\tilde{f}_i(\pi_\Lambda * \pi) \neq \mathbf{0}$, we have*

$$\tilde{f}_i(\pi_\Lambda * \pi) = \pi_\Lambda * \tilde{f}_i\pi \text{ and } \tilde{f}_i\pi(t) = \pi(t) \quad \text{for all } t \in [0, u].$$

(ii) *For a sequence $i_1, \dots, i_s$ of elements of J such that $\tilde{f}_{i_1} \cdots \tilde{f}_{i_s}(\pi_\Lambda * \pi) \neq \mathbf{0}$, we have*

$$\tilde{f}_{i_1} \cdots \tilde{f}_{i_k}(\pi_\Lambda * \pi) = \pi_\Lambda * (\tilde{f}_{i_1} \cdots \tilde{f}_{i_k}\pi) \text{ and } \tilde{f}_{i_1} \cdots \tilde{f}_{i_k}\pi(t) = \pi(t) \text{ for all } t \in [0, u].$$

Proof. It suffices only to show (i) since (ii) can be proved inductively from this. The statement $\tilde{f}_i(\pi_\Lambda * \pi) = \pi_\Lambda * (\tilde{f}_i\pi)$ follows from (24) since we have from (22) that

$$\varphi_i(\pi_\Lambda) = \langle \Lambda, \alpha_i^\vee \rangle \text{ and } \varepsilon_i(\pi) = -m_i^\pi \geq \langle \Lambda, \alpha_i^\vee \rangle,$$

where the second inequality follows from (33), and $\tilde{f}_i\pi(t) = \pi(t)$ for all $t \in [0, u]$ follows from Lemma 5.3 (ii) and (33). □

We define a sequence $\mathbf{i}_{\pi_0}$ of elements of $\widehat{I}$ for each $\pi_0 \in \mathbb{B}_0(\lambda)^\Lambda$ as follows. Fix $\pi_0 \in \mathbb{B}_0(\lambda)^\Lambda$. For $0 \leq p \leq N-1$, we set

$$J^p = \{i \in \widehat{I} \mid H_i^{\pi_0}(\sigma_p) = -\langle \Lambda, \alpha_i^\vee \rangle\}.$$

Note that J^p is a proper subset by Lemma 6.6, and hence $\widehat{W}_{J^p}$ is finite. For each $0 \leq p \leq N-1$ such that $J^p \neq \emptyset$, we fix a sequence $\mathbf{i}^p : i_1^p, \dots, i_{m_p}^p$ of elements of J^p so that $s_{i_1^p} \dots s_{i_{m_p}^p}$ is a reduced expression of the longest element of $\widehat{W}_{J^p}$. If $J^p = \emptyset$, we set $\mathbf{i}^p = \emptyset$, and we define a sequence $\mathbf{i}_{\pi_0}$ by $\mathbf{i}^0, \mathbf{i}^1, \dots, \mathbf{i}^{N-1}$. The following lemma is essential for the proof of Proposition 7.6:

Lemma 7.8. *For $\pi_0 \in \mathbb{B}_0(\lambda)^\Lambda$, we have*

$$C(\pi_\Lambda * \pi_0) \cap (\pi_\Lambda * \mathbb{B}_0(\lambda)) = \mathcal{F}_{\mathbf{i}_{\pi_0}}(\pi_\Lambda * \pi_0).$$

Proof. For $0 \leq p \leq N-1$, we set $\mathbf{i}^{\geq p} = \mathbf{i}^p, \mathbf{i}^{p+1}, \dots, \mathbf{i}^{N-1}$, and we set $\mathbf{i}^{\geq N} = \emptyset$. First, we show the containment $\supseteq$. Let

$$\mathbb{B}_0(\lambda)^p = \{\pi \in \mathbb{B}_0(\lambda) \mid \pi(t) = \pi_0(t) \text{ for all } t \in [0, \sigma_p]\}$$

for $0 \leq p \leq N$, and we show by the descending induction on p that

$$\pi_\Lambda * \mathbb{B}_0(\lambda)^p \supseteq \mathcal{F}_{\mathrm{i}_{\geq p}}(\pi_\Lambda * \pi_0),$$

which for $p = 0$ implies the desired containment. If $p = N$, there is nothing to prove. Assume $p \leq N-1$. By the induction hypothesis, we have $\pi_\Lambda * \mathbb{B}_0(\lambda)^{p+1} \supseteq \mathcal{F}_{\mathrm{i}_{\geq p+1}}(\pi_\Lambda * \pi_0)$. For $\pi \in \mathbb{B}_0(\lambda)^{p+1}$, we have that $H_i^\pi(t) = H_i^{\pi_0}(t)$ for all $t \in [0, \sigma_{p+1}]$, $i \in \widehat{I}$. Hence, in particular, we have for all $i \in J^p$ that

$$H_i^\pi(t) \geq -\langle \Lambda, \alpha_i^\vee \rangle \text{ for all } t \in [0, \sigma_p] \text{ and } H_i^\pi(\sigma_p) = -\langle \Lambda, \alpha_i^\vee \rangle.$$

These equations and Lemma 7.7 (ii) imply that for any sequence $i_1, \ldots, i_s$ of elements of J^p satisfying $\tilde{f}_{i_1} \cdots \tilde{f}_{i_s}(\pi_\Lambda * \pi) \neq 0$, we have

$$\tilde{f}_{i_1} \cdots \tilde{f}_{i_s}(\pi_\Lambda * \pi) = \pi_\Lambda * (\tilde{f}_{i_1} \cdots \tilde{f}_{i_s}\pi)$$

and

$$\tilde{f}_{i_1} \cdots \tilde{f}_{i_s}\pi(t) = \pi(t) = \pi_0(t) \quad \text{for all } t \in [0, \sigma_p].$$

Hence, we have

$$\mathcal{F}_{\mathrm{i}_{\geq p}}(\pi_\Lambda * \pi_0) \subseteq \mathcal{F}_{\mathrm{i}^p}(\pi_\Lambda * \mathbb{B}_0(\lambda)^{p+1}) = \pi_\Lambda * \mathcal{F}_{\mathrm{i}^p}\mathbb{B}_0(\lambda)^{p+1} \subseteq \pi_\Lambda * \mathbb{B}_0(\lambda)^p.$$

Now, we show the opposite containment $\subseteq$. For arbitrary $\pi \in \mathbb{B}_0(\lambda)$, we denote by $p_0(\pi) \in \{0, \ldots, N\}$ the integer satisfying

$$\sigma_{p_0(\pi)} = \max\{u \in [0,1] \mid H_i^\pi(t) \geq -\langle \Lambda, \alpha_i^\vee \rangle \text{ for all } t \in [0,u], i \in \widehat{I}\}.$$

Let π be an arbitrary path in $\mathbb{B}_0(\lambda)$ such that $\pi_\Lambda * \pi \in C(\pi_\Lambda * \pi_0)$, and we show by the descending induction on $p_0(\pi)$ that $\pi_\Lambda * \pi \in \mathcal{F}_{\mathrm{i}_{\geq p_0(\pi)}}(\pi_\Lambda * \pi_0)$. If $p_0(\pi) = N$, then $\tilde{e}_i(\pi_\Lambda * \pi) = 0$ follows for all $i \in \widehat{I}$, which implies $\pi = \pi_0$ since $\pi_\Lambda * \pi_0$ is the unique element in $C(\pi_\Lambda * \pi_0)$ satisfying this condition. Assume that $p_0(\pi) < N$. By Lemma 6.9, there exists $\pi' \in \mathbb{B}_0(\lambda)$ such that

$$\pi_\Lambda * \pi' \in \{\tilde{e}_{i_1} \cdots \tilde{e}_{i_s}(\pi_\Lambda * \pi) \mid s \geq 0, i_k \in J^{p_0(\pi)}\} \tag{34}$$

and $p_0(\pi') > p_0(\pi)$. By the induction hypothesis, we have

$$\pi_\Lambda * \pi' \in \mathcal{F}_{\mathrm{i}_{\geq p_0(\pi')}}(\pi_\Lambda * \pi_0) \subseteq \mathcal{F}_{\mathrm{i}_{\geq p_0(\pi)}}(\pi_\Lambda * \pi_0).$$

Since $C(\pi_\Lambda * \pi_0) \cong \mathcal{B}(\Lambda + \pi_0(1))$, we have for all $i \in J^{p_0(\pi)}$ that

$$\tilde{f}_i \mathcal{F}_{\mathrm{i}_{\geq p_0(\pi)}}(\pi_\Lambda * \pi_0) \subseteq \mathcal{F}_{\mathrm{i}_{\geq p_0(\pi)}}(\pi_\Lambda * \pi_0) \cup \{\mathbf{0}\}$$

by Proposition 7.5. Hence, we have from (34) that

$$\pi_\Lambda * \pi \in \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_s}(\pi_\Lambda * \pi') \mid s \geq 0, i_k \in J^{p_0(\pi)}\} \subseteq \mathcal{F}_{\mathrm{i}_{\geq p_0(\pi)}}(\pi_\Lambda * \pi_0).$$

□

Proof of Proposition 7.6. For $\pi_0 \in \mathbb{B}_0(\lambda)^\Lambda$, Lemma 7.8 implies that the image of $C(b_\Lambda \otimes \pi_0) \cap (b_\Lambda \otimes \mathbb{B}_0(\lambda))$ under the isomorphism given in Corollary 6.8 is $\mathcal{F}_{\mathrm{i}_{\pi_0}} b_{\Lambda + \pi_0(1)}$. Then by Proposition 7.5, there exists an element $w_{\pi_0} \in \widehat{W}$ satisfying

$$\mathcal{F}_{\mathrm{i}_{\pi_0}} b_{\Lambda + \pi_0(1)} = \mathcal{B}_{w_{\pi_0}}(\Lambda + \pi_0(1)),$$

and $\langle w_{\pi_0}(\Lambda + \pi_0(1)), \alpha_i^\vee \rangle \leq 0$ for all $i \in J^0 = \widehat{I}_\Lambda$. Now, the proposition follows since

$$b_\Lambda \otimes \mathbb{B}_0(\lambda) = \coprod_{\pi_0 \in \mathbb{B}_0(\lambda)^\Lambda} C(b_\Lambda \otimes \pi_0) \cap (b_\Lambda \otimes \mathbb{B}_0(\lambda)).$$

□

7.3 Demazure crystal decomposition of $b_\Lambda \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}$

First, we remark an elementary fact about crystals:

Remark 7.9. Let $\mathcal{B}$ be a $U_q(\widehat{\mathfrak{g}})$-crystal with a weight map $\mathrm{wt} : \mathcal{B} \to \widehat{P}$. Then we can canonically consider $\mathcal{B}$ as a $U_q'(\widehat{\mathfrak{g}})$-crystal by replacing the weight map with $\mathrm{cl} \circ \mathrm{wt} : \mathcal{B} \to \widehat{P}_{\mathrm{cl}}$.

Similarly as $\mathbb{B}_0(\lambda)^\Lambda$, we define $\mathbb{B}(\lambda)_{\mathrm{cl}}^\Lambda$ by

$$\mathbb{B}(\lambda)_{\mathrm{cl}}^\Lambda = \{\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}} \mid \langle \eta(t), \alpha_i^\vee \rangle \geq -\langle \Lambda, \alpha_i^\vee \rangle \text{ for all } t \in [0,1], i \in \widehat{I}\}.$$

Then it is easily checked that

$$\mathbb{B}_0(\lambda)^\Lambda = \coprod_{\eta_0 \in \mathbb{B}(\lambda)_{\mathrm{cl}}^\Lambda} \mathrm{cl}^{-1}(\eta_0) \cap \mathbb{B}_0(\lambda). \tag{35}$$

For a $U_q'(\widehat{\mathfrak{g}})$-crystal $\mathcal{B}$ and $b \in \mathcal{B}$, in the same way as a $U_q(\widehat{\mathfrak{g}})$-crystal, we denote by $C(b)$ the connected component of $\mathcal{B}$ containing b .

Lemma 7.10. *Let $\eta_0 \in \mathbb{B}(\lambda)_{\mathrm{cl}}^\Lambda$. Then for any $\pi_0 \in \mathrm{cl}^{-1}(\eta_0) \cap \mathbb{B}_0(\lambda) \subseteq \mathbb{B}_0(\lambda)^\Lambda$, the map $\mathrm{id} \otimes \mathrm{cl} : \mathcal{B}(\Lambda) \otimes \mathbb{B}_0(\lambda) \to \mathcal{B}(\Lambda) \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}$ induces a $U_q'(\widehat{\mathfrak{g}})$-crystal isomorphism from $C(b_\Lambda \otimes \pi_0)$ to $C(b_\Lambda \otimes \eta_0)$.*

Proof. By the definition of $\mathrm{cl} : \mathbb{P} \to \mathbb{P}_{\mathrm{cl}}$ in the subsection 5.2, we can see that $\mathrm{id} \otimes \mathrm{cl}$ preserves a $\widehat{P}_{\mathrm{cl}}$-weight, ε_i, φ_i, and commutes with root operators. Hence, it suffices to show that the induced map is bijective. The surjectivity is obvious. We show the injectivity. Let $b \in \mathcal{B}(\Lambda)$ and $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$ be arbitrary elements such that $b \otimes \eta \in C(b_\Lambda \otimes \eta_0)$, and assume that $\pi_1, \pi_2 \in \mathbb{B}_0(\lambda)$ satisfy

$$b \otimes \pi_j \in C(b_\Lambda \otimes \pi_0) \text{ and } \mathrm{cl}(\pi_j) = \eta$$

for $j = 1, 2$. By Lemma 5.15 (i), there exists $k \in \mathbb{Z}$ such that $\pi_2 = \pi_1 + \pi_{kd_\lambda\delta}$. By $C(b_\Lambda \otimes \pi_0) \cong \mathcal{B}(\Lambda + \pi_0(1))$, there exists a sequence $i_1, \ldots, i_s$ of elements of $\widehat{I}$ such that $\tilde{e}_{i_1} \cdots \tilde{e}_{i_s}(b \otimes \pi_1) = b_\Lambda \otimes \pi_0$, and then it is easily seen (cf. [28, Lemma 3.3.1]) that

$$\tilde{e}_{i_1} \cdots \tilde{e}_{i_s}(b \otimes \pi_2) = \tilde{e}_{i_1} \cdots \tilde{e}_{i_s}(b \otimes (\pi_1 + \pi_{kd_\lambda\delta})) = b_\Lambda \otimes (\pi_0 + \pi_{kd_\lambda\delta}).$$

Hence $b_\Lambda \otimes (\pi_0 + \pi_{kd_\lambda\delta}) \in C(b_\Lambda \otimes \pi_0)$, which together with (26) implies $k = 0$. Therefore, we have $\pi_1 = \pi_2$, and the injectivity follows. □

Recall that we have

$$\mathcal{B}(\Lambda) \otimes \mathbb{B}_0(\lambda) = \bigoplus_{\pi_0 \in \mathbb{B}_0(\lambda)^\Lambda} C(b_\Lambda \otimes \pi_0)$$

by Proposition 6.7. Applying $\mathrm{id} \otimes \mathrm{cl}$ to this equation, we have from (35) that

$$\mathcal{B}(\Lambda) \otimes \mathbb{B}(\lambda)_{\mathrm{cl}} = \bigoplus_{\eta_0 \in \mathbb{B}(\lambda)_{\mathrm{cl}}^\Lambda} C(b_\Lambda \otimes \eta_0). \tag{36}$$

Now, we can obtain some results about a decomposition of $\mathcal{B}(\Lambda) \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}$ and $b_\Lambda \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}$. Take an arbitrary $\pi^{\eta_0} \in \mathrm{cl}^{-1}(\eta_0) \cap \mathbb{B}_0(\Lambda)$ for each $\eta_0 \in \mathbb{B}(\lambda)_{\mathrm{cl}}^\Lambda$. Then the following proposition follows:

Proposition 7.11. (i) *We have*

$$\mathcal{B}(\Lambda) \otimes \mathbb{B}(\lambda)_{\mathrm{cl}} \xrightarrow{\sim} \bigoplus_{\eta_0 \in \mathbb{B}(\lambda)_{\mathrm{cl}}^{\Lambda}} \mathcal{B}(\Lambda + \pi^{\eta_0}(1))$$

as $U_q'(\widehat{\mathfrak{g}})$-crystals, where the given isomorphism maps each $b_\Lambda \otimes \eta_0 \in b_\Lambda \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}^{\Lambda}$ to $b_{\Lambda+\pi^{\eta_0}(1)} \in \mathcal{B}(\Lambda + \pi^{\eta_0}(1))$.
(ii) *Under the isomorphism given in* (i), *the image of the subset $b_\Lambda \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}$ coincides with the disjoint union of the Demazure crystals*

$$\coprod_{\eta_0 \in \mathbb{B}(\lambda)_{\mathrm{cl}}^{\Lambda}} \mathcal{B}_{w_{\eta_0}}(\Lambda + \pi^{\eta_0}(1)) \textit{ for some } w_{\eta_0} \in \widehat{W}.$$

Moreover, each w_{η_0} satisfies $\langle w_{\eta_0}(\Lambda + \pi^{\eta_0}(1)), \alpha_i^\vee \rangle \leq 0$ for all $i \in \widehat{I}_\Lambda$.

Proof. (i) follows from (36) since for each $\eta_0 \in \mathbb{B}(\lambda)_{\mathrm{cl}}^{\Lambda}$, we have from Lemma 7.10 and Corollary 6.8 that

$$C(b_\Lambda \otimes \eta_0) \cong C(b_\Lambda \otimes \pi^{\eta_0}) \cong \mathcal{B}(\Lambda + \pi^{\eta_0}(1))$$

as $U_q'(\widehat{\mathfrak{g}})$-crystals, and (ii) also follows from these isomorphisms and Proposition 7.6. □

8 Study on the decomposition of $b_{\Lambda_0} \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}$

8.1 Preliminaries about the weight sum of $\mathbb{B}(\lambda)_{\mathrm{cl}}$

In the previous section, we have seen that $b_\Lambda \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}$ coincides with the disjoint union of some Demazure crystals. In this section, we study in more detail this result with $\Lambda = \Lambda_0$.

First, we prepare some notation. Let $\Lambda \in \widehat{P}_+$ and $w \in \widehat{W}$ be elements satisfying $w\Lambda = w_0\lambda + \ell\Lambda_0 + m\delta$ for some $\lambda \in P_+, \ell \in \mathbb{Z}_{>0}, m \in \mathbb{Z}$. Then we use the following notation which is compatible with that of modules:

$$\mathcal{B}(\ell, \lambda)[m] = \mathcal{B}_w(\Lambda).$$

Note that we have from Proposition 7.2 (ii) that

$$\mathrm{ch}_{\widehat{\mathfrak{h}}} \mathcal{D}(\ell, \lambda)[m] = \sum_{b \in \mathcal{B}(\ell,\lambda)[m]} e(\mathrm{wt}(b)). \tag{37}$$

Let $\lambda \in P_+$ and $\eta_0 \in \mathbb{B}(\lambda)_{\mathrm{cl}}^{\Lambda_0}$. Then we have $\pi_{\eta_0} \in \mathrm{cl}^{-1}(\eta_0) \cap \mathbb{B}_0(\lambda)$, where π_{η_0} is defined in the subsection 5.3. Hence from Proposition 7.11, we obtain a $U_q'(\widehat{\mathfrak{g}})$-crystal isomorphism

$$\kappa : \mathcal{B}(\Lambda_0) \otimes \mathbb{B}(\lambda)_{\mathrm{cl}} \xrightarrow{\sim} \bigoplus_{\eta_0 \in \mathbb{B}(\lambda)_{\mathrm{cl}}^{\Lambda_0}} \mathcal{B}(\Lambda_0 + \pi_{\eta_0}(1))$$

which is uniquely determined by $\kappa(b_{\Lambda_0} \otimes \eta_0) = b_{\Lambda_0 + \pi_{\eta_0}(1)}$ for all $\eta_0 \in \mathbb{B}(\lambda)_{\mathrm{cl}}^{\Lambda_0}$.

Lemma 8.1. *For each $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, we have*

$$\mathrm{wt} \circ \kappa(b_{\Lambda_0} \otimes \eta) = \mathrm{wt}(\pi_\eta) + \Lambda_0 = \mathrm{wt}_{\widehat{P}}(\eta) + \Lambda_0.$$

Proof. The second equality follows from the definition of the $\widehat{P}$-weight of η. We show the first one. By (36), there exists some $\eta_0 \in \mathbb{B}(\lambda)_{\mathrm{cl}}^{\Lambda_0}$ such that $b_{\Lambda_0} \otimes \eta \in C(b_{\Lambda_0} \otimes \eta_0)$. Recall that κ is defined by the composition of a $U_q'(\widehat{\mathfrak{g}})$-crystal isomorphism from $C(b_{\Lambda_0} \otimes \eta_0)$ to $C(b_{\Lambda_0} \otimes \pi_{\eta_0})$, which we denote by κ' here, with a $U_q(\widehat{\mathfrak{g}})$-crystal isomorphism from $C(b_{\Lambda_0} \otimes \pi_{\eta_0})$ to $\mathcal{B}(\Lambda_0 + \pi_{\eta_0}(1))$. Hence, it suffices to show that

$$\kappa'(b_{\Lambda_0} \otimes \eta) = b_{\Lambda_0} \otimes \pi_\eta.$$

Recall that $\kappa'(b_{\Lambda_0} \otimes \eta_0) = b_{\Lambda_0} \otimes \pi_{\eta_0}$. Since there exists a sequence $i_1, \dots, i_k$ such that $b_{\Lambda_0} \otimes \eta = \tilde{f}_{i_1} \cdots \tilde{f}_{i_k}(b_{\Lambda_0} \otimes \eta_0)$, the above equation can be proved inductively from the following assertion: if η and $i \in \widehat{I}$ satisfies $\tilde{f}_i(b_{\Lambda_0} \otimes \eta) = b_{\Lambda_0} \otimes \tilde{f}_i\eta$, then we have $\tilde{f}_i(b_{\Lambda_0} \otimes \pi_\eta) = b_{\Lambda_0} \otimes \pi_{\tilde{f}_i\eta}$. This assertion easily follows from Lemma 5.18. □

By Proposition 7.11 (ii), $\kappa(b_{\Lambda_0} \otimes \mathbb{B}(\lambda)_{\mathrm{cl}})$ is the disjoint union of some Demazure crystals in the form $\mathcal{B}_w(\Lambda')$ such that the level of Λ' is 1 and $\langle w\Lambda', \alpha_i^\vee \rangle \leq 0$ for all $i \in I$, which can be written as $\mathcal{B}(1, \mu)[n]$ for some $\mu \in P_+$ and $n \in \mathbb{Z}$.

In conclusion, we have the following:

Proposition 8.2. *Let $\lambda \in P_+$. Then there exists sequences $\mu_1, \dots, \mu_\ell \in P_+$ and $n_1, \dots, n_\ell \in \mathbb{Z}$ such that*

$$\kappa(b_{\Lambda_0} \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}) = \coprod_{1 \leq j \leq \ell} \mathcal{B}(1, \mu_j)[n_j].$$

Moreover, we have $\mathrm{wt} \circ \kappa(b_{\Lambda_0} \otimes \eta) = \mathrm{wt}_{\widehat{P}}(\eta) + \Lambda_0$.

Corollary 8.3.

$$\sum_{\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}} e(\mathrm{wt}_{\widehat{P}}(\eta)) = \sum_{1 \leq j \leq \ell} \mathrm{ch}_{\mathfrak{h}_d} \mathcal{D}(1, \mu_j)[n_j].$$

If $\mathfrak{g}$ is simply laced, we can easily determine μ_j's and n_j's in Corollary 8.3 from a result in [8]:

Proposition 8.4. *If $\mathfrak{g}$ is simply laced, then we have*

$$\sum_{\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}} e(\mathrm{wt}_{\widehat{P}}(\eta)) = \mathrm{ch}_{\mathfrak{h}_d} \mathcal{D}(1, \lambda)[0].$$

Proof. From [8, Proposition 3], we have $\kappa(b_{\Lambda_0} \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}) = \mathcal{B}(1, \lambda)[m]$ for some $m \in \mathbb{Z}$. By definition, we have $\pi_{\eta_{\mathrm{cl}(\lambda)}} = \pi_\lambda$, which implies $\mathrm{wt} \circ \kappa(b_{\Lambda_0} \otimes \eta_{\mathrm{cl}(\lambda)}) = \lambda + \Lambda_0$ by Proposition 8.2. Since $\eta_{\mathrm{cl}(\lambda)} \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, this equation means that $\mathcal{B}(1, \lambda)[m]$ contains an element with weight $\lambda + \Lambda_0$, which forces $m = 0$. □

If $\mathfrak{g}$ is non-simply laced, the decomposition can be more complicated, and it seems hard to determine μ_j's and n_j's by straightforward calculations. In the fundamental weight case, however, we can obtain the following proposition using the result in [21] and Theorem 5.13.

Proposition 8.5. *For general $\mathfrak{g}$, we have for each $i \in I$ that*

$$\sum_{\eta \in \mathbb{B}(\varpi_i)_{\mathrm{cl}}} e(\mathrm{wt}_{\widehat{P}}(\eta)) = \mathrm{ch}_{\mathfrak{h}_d} \mathcal{D}(1, \varpi_i)[0].$$

Proof. Since $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is isomorphic to $\mathcal{B}(W_q(\varpi_i))$ by Theorem 5.13 (ii), we can see from [21, Corollary 4.8] that $\kappa(b_{\Lambda_0} \otimes \mathbb{B}(\varpi_i)_{\mathrm{cl}}) = \mathcal{B}(1, \varpi_i)[m]$ for some $m \in \mathbb{Z}$. The rest of the proof is same as Proposition 8.4. $\square$

Corollary 8.6. *If $\lambda = \sum_{i \in I} \lambda_i \varpi_i \in P_+$, then we have*

$$\sum_{\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}} e(p \circ \mathrm{wt}(\eta)) = \prod_{i \in I} \mathrm{ch}_{\mathfrak{h}} \mathcal{D}(1, \varpi_i)^{\lambda_i},$$

where p denotes the canonical projection $\widehat{P}_{\mathrm{cl}} \to P$.

Proof. From the above proposition, we have

$$\sum_{\eta \in \mathbb{B}(\varpi_i)_{\mathrm{cl}}} e(p \circ \mathrm{wt}(\eta)) = \mathrm{ch}_{\mathfrak{h}} \mathcal{D}(1, \varpi_i)$$

for each $i \in I$. Then the corollary follows since $\mathbb{B}(\lambda)_{\mathrm{cl}} \cong \bigotimes_{i \in I} \mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes \lambda_i}$ by Theorem 5.13 (i). $\square$

From the next subsection, we begin to determine μ_j's and n_j's in the non-simply laced case using the Demazure crystals for $U_q(\widehat{\mathfrak{g}}^{\mathrm{sh}})$.

8.2 Path models for $U_q(\widehat{\mathfrak{g}}^{\mathrm{sh}})$

In the rest of this section, we assume that $\mathfrak{g}$ is non-simply laced, and we apply the theory of path models for $U_q(\widehat{\mathfrak{g}}^{\mathrm{sh}})$. Here, we fix some notation, which are used throughout the rest of this section.

Let $\theta^{\mathrm{sh}} \in \Delta_+$ denote the highest root in Δ^{sh}, and let $\alpha_0^{\mathrm{sh}} = \delta - \theta^{\mathrm{sh}} \in \widehat{\Delta}$, which corresponds to a simple root of $\widehat{\mathfrak{g}}^{\mathrm{sh}}$. Note that $(\alpha_0^{\mathrm{sh}})^\vee = rK - (\theta^{\mathrm{sh}})^\vee$. Let $s_0^{\mathrm{sh}} \in \widehat{W}$ denote the reflection associated with α_0^{sh}, and we denote by $\widehat{W}^{\mathrm{sh}}$ the subgroup of $\widehat{W}$ generated by $\{s_0^{\mathrm{sh}}\} \cup \{s_i \mid i \in I^{\mathrm{sh}}\}$. Set $\widehat{I}^{\mathrm{sh}} = \{0\} \cup I^{\mathrm{sh}}$, and

$$\widehat{P}^{\mathrm{sh}} = \sum_{i \in I^{\mathrm{sh}}} \mathbb{Z}\overline{\varpi_i} + r^{-1}\mathbb{Z}\overline{\Lambda}_0 + \mathbb{Z}\overline{\delta} \subseteq (\widehat{\mathfrak{h}}^{\mathrm{sh}})^*.$$

Note that s_0^{sh} acts on $\widehat{P}^{\mathrm{sh}}$ by $s_0^{\mathrm{sh}}(\nu) = \nu - \langle \nu, (\alpha_0^{\mathrm{sh}})^\vee \rangle \overline{\alpha}_0^{\mathrm{sh}}$ for $\nu \in \widehat{P}^{\mathrm{sh}}$, and s_i for $i \in I^{\mathrm{sh}}$ also acts similarly.

Let $\mathbb{P}^{\mathrm{sh}}$ be the set of paths with weight in $\widehat{P}^{\mathrm{sh}}$, and define $\mathbb{P}^{\mathrm{sh}}_{\mathrm{int}}$ similarly as $\mathbb{P}_{\mathrm{int}}$. As described in the subsection 5.1, we can define root operators associated to $i \in \widehat{I}^{\mathrm{sh}}$ on $\mathbb{P}^{\mathrm{sh}}_{\mathrm{int}}$ using the above actions of simple reflections. To distinguish them from $\tilde{e}_i$ and $\tilde{f}_i$, we denote them by $\tilde{e}_i^{\mathrm{sh}}$ and $\tilde{f}_i^{\mathrm{sh}}$ ($i \in \widehat{I}^{\mathrm{sh}}$). The maps $\mathrm{wt} : \mathbb{P}^{\mathrm{sh}}_{\mathrm{int}} \to \widehat{P}^{\mathrm{sh}}$, and $\varepsilon_i, \varphi_i : \mathbb{P}^{\mathrm{sh}} \to \mathbb{Z}_{\geq 0}$ for $i \in \widehat{I}^{\mathrm{sh}}$ are defined similarly. Then Theorem 5.2 implies the following:

Proposition 8.7. *Let $\mathbb{B} \subseteq \mathbb{P}^{\mathrm{sh}}_{\mathrm{int}}$ be a subset such that $\tilde{e}_i^{\mathrm{sh}}\mathbb{B} \subseteq \mathbb{B} \cup \{0\}$ and $\tilde{f}_i^{\mathrm{sh}}\mathbb{B} \subseteq \mathbb{B} \cup \{0\}$ for all $i \in \widehat{I}^{\mathrm{sh}}$. Then $\mathbb{B}$, together with the root operators $\tilde{e}_i^{\mathrm{sh}}, \tilde{f}_i^{\mathrm{sh}}$ for $i \in \widehat{I}^{\mathrm{sh}}$ and the maps $\mathrm{wt}, \varepsilon_i, \varphi_i$, becomes a $U_q(\widehat{\mathfrak{g}}^{\mathrm{sh}})$-crystal.*

We denote by $H_i^{\mathrm{sh},\pi}$ and $m_i^{\mathrm{sh},\pi}$ for $\pi \in \mathbb{P}^{\mathrm{sh}}_{\mathrm{int}}$ and $i \in \widehat{I}^{\mathrm{sh}}$ the counterparts of $H_i^{\pi'}$ and $m_i^{\pi'}$ respectively. For $\nu \in \widehat{P}^{\mathrm{sh}}$, let π_ν denote the straight line path:

$\pi_\nu(t) = t\nu$, and we denote by $\mathbb{B}_0^{\mathrm{sh}}(\nu)$ the connected component containing π_ν, which is a $U_q(\widehat{\mathfrak{g}}^{\mathrm{sh}})$-crystal.

Since $\mathfrak{g}^{\mathrm{sh}}$ is simply laced, the following lemma follows from Proposition 8.4 and Remark 5.17:

Lemma 8.8. *Let $\nu \in \overline{P}_+$. Then we have*

$$\sum_{\substack{\pi \in \mathbb{B}_0^{\mathrm{sh}}(\nu) \\ \iota(\pi) \in \overline{P}}} e(\mathrm{wt}(\pi)) = \mathrm{ch}_{\mathfrak{h}_d^{\mathrm{sh}}} \mathcal{D}^{\mathrm{sh}}(1,\nu)[0].$$

8.3 Identity on the weight sum of $\mathbb{B}(\lambda)_{\mathrm{cl}}$

This subsection is devoted to the proof of the following proposition:

Proposition 8.9. *Let $\lambda \in P_+$, and set $\lambda' = \lambda - i_{\mathrm{sh}}(\overline{\lambda})$. Then we have*

$$\sum_{\substack{\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}} \\ \mathrm{wt}_{\widehat{P}}(\eta) \in \lambda - Q_+^{\mathrm{sh}} + \mathbb{Z}\delta}} e(\mathrm{wt}_{\widehat{P}}(\eta)) = e(\lambda') i_{\mathrm{sh}} \mathrm{ch}_{\mathfrak{h}_d^{\mathrm{sh}}} \mathcal{D}^{\mathrm{sh}}(1,\overline{\lambda})[0].$$

Remark 8.10. In the final part of this article, we show the equation

$$\mathrm{ch}_{\mathfrak{h}_d} W(\lambda) = \sum_{\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}} e(\mathrm{wt}_{\widehat{P}}(\eta)).$$

Note that the above proposition is compatible with this result and Proposition 4.17.

First, we prepare a technical lemma:

Lemma 8.11. *There exist an element $\tau \in \widehat{W}$ and an element $j \in I^{\mathrm{sh}}$ satisfying the following two conditions:*
(i) $\tau(\alpha_j) = \alpha_0^{\mathrm{sh}}$,
(ii) *τ has an expression $\tau = s_{i_1} \dots s_{i_M}$ such that*

$$s_{i_1} \dots s_{i_{L-1}}(\alpha_{i_L}) \notin \{\alpha + k\delta \mid \alpha \in \Delta^{\mathrm{sh}}, k \in \mathbb{Z}\} \tag{38}$$

for all $1 \leq L \leq M$.

Proof. Here, we give $\tau \in \widehat{W}$, an expression of τ, and $j \in I^{\mathrm{sh}}$ for each type of $\mathfrak{g}$ using the numbering of index for simple roots in [17, §4]:

- Type B_ℓ: Let $\tau = s_{\ell-1}s_{\ell-2}\cdots s_2 s_0 s_1 s_2 \cdots s_{\ell-1}$ and $j = \ell$.
  In this case, $\alpha_0^{\mathrm{sh}} = \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$.
- Type C_ℓ: Let $\tau = s_\ell s_{\ell-1} \cdots s_1 s_0$ and $j = 1$.
  In this case, $\alpha_0^{\mathrm{sh}} = \alpha_0 + \alpha_1 + \cdots + \alpha_\ell$.
- Type F_4: Let $\tau = s_2 s_3 s_1 s_2 s_3 s_4 s_0 s_1 s_2$ and $j = 3$.
  In this case, $\alpha_0^{\mathrm{sh}} = \alpha_0 + 2\alpha_1 + 3\alpha_2 + 3\alpha_3 + \alpha_4$.
- Type G_2: Let $\tau = s_1 s_2 s_0 s_1$ and $j = 2$.
  In this case, $\alpha_0^{\mathrm{sh}} = \alpha_0 + 2\alpha_1 + 2\alpha_2$.

Though it is a little troublesome work, we can check directly that these elements actually satisfy the conditions (for informations of the root systems, for example see [2, Ch. VI. §4]). Note that if α_{i_L} is a long root, the condition (38) is trivial since the right hand side consists of short roots. Using this fact, we can reduce a bit the amount of calculations. □

Now, we begin the proof of the proposition. Write $\lambda = \sum_{i \in I} \lambda_i \varpi_i \in P_+$ with $\lambda_i \in \mathbb{Z}_{\geq 0}$. For $\pi \in \mathbb{B}_0^{\mathrm{sh}}(\overline{\lambda})$, we define a piecewise linear, continuous map $i_{\mathrm{sh}}(\pi) : [0,1] \to \widehat{P} \otimes_{\mathbb{Z}} \mathbb{R}$ by $i_{\mathrm{sh}}(\pi)(t) = i_{\mathrm{sh}}(\pi(t))$ for $t \in [0,1]$, and let $\pi_{\lambda'}$ be a straight line path: $\pi_{\lambda'}(t) = t\lambda'$. Note that $\langle \alpha_i, \lambda' \rangle = 0$ for all $i \in I^{\mathrm{sh}}$. Now, we define $\varphi(\pi)$ for $\pi \in \mathbb{B}_0^{\mathrm{sh}}(\overline{\lambda})$ by

$$\varphi(\pi) = i_{\mathrm{sh}}(\pi) + \pi_{\lambda'}.$$

The following lemma is essential for the proof of the above proposition.

Lemma 8.12. *We have $\varphi(\pi) \in \mathbb{B}_0(\lambda)$ for all $\pi \in \mathbb{B}_0^{\mathrm{sh}}(\overline{\lambda})$.*

It is easily seen that $\varphi(\pi_{\overline{\lambda}}) = \pi_\lambda \in \mathbb{B}_0(\lambda)$. Hence, by the definition of $\mathbb{B}_0^{\mathrm{sh}}(\overline{\lambda})$, it suffices to show the following lemma:

Lemma 8.13. *Assume that $\pi \in \mathbb{B}_0^{\mathrm{sh}}(\overline{\lambda})$ satisfies $\varphi(\pi) \in \mathbb{B}_0(\lambda)$. Then for each $i \in \widehat{I}^{\mathrm{sh}}$, the following statements hold.*
(i) If $\tilde{e}_i^{\mathrm{sh}}\pi \neq \mathbf{0}$, then $\varphi(\tilde{e}_i^{\mathrm{sh}}\pi) \in \mathbb{B}_0(\lambda)$.
(ii) If $\tilde{f}_i^{\mathrm{sh}}\pi \neq \mathbf{0}$, then $\varphi(\tilde{f}_i^{\mathrm{sh}}\pi) \in \mathbb{B}_0(\lambda)$.

Proof. We show only (i). The proof of (ii) is similar.

claim 1. For $w \in \widehat{W}^{\mathrm{sh}}$ and $\nu \in \widehat{P}^{\mathrm{sh}}$, we have $i_{\mathrm{sh}}(w\nu) = w i_{\mathrm{sh}}(\nu)$.

It suffices to show that $i_{\mathrm{sh}}(s_i\nu) = s_i i_{\mathrm{sh}}(\nu)$ for $i \in I^{\mathrm{sh}}$ and $i_{\mathrm{sh}}(s_0^{\mathrm{sh}}\nu) = s_0^{\mathrm{sh}} i_{\mathrm{sh}}(\nu)$. Let $i \in I^{\mathrm{sh}}$. Since $\alpha_i^\vee \in \widehat{\mathfrak{h}}^{\mathrm{sh}}$, we have $\langle \nu, \alpha_i^\vee \rangle = \langle i_{\mathrm{sh}}(\nu), \alpha_i^\vee \rangle$. Then this together with the fact $i_{\mathrm{sh}}(\overline{\alpha}_i) = \alpha_i$ implies $i_{\mathrm{sh}}(s_i\nu) = s_i i_{\mathrm{sh}}(\nu)$. We can also prove $i_{\mathrm{sh}}(s_0^{\mathrm{sh}}\nu) = s_0^{\mathrm{sh}} i_{\mathrm{sh}}(\nu)$ in the same way.

claim 2. If $i \in I^{\mathrm{sh}}$, then we have $i_{\mathrm{sh}}(\tilde{e}_i^{\mathrm{sh}}(\pi)) = \tilde{e}_i(i_{\mathrm{sh}}(\pi))$. (Though $i_{\mathrm{sh}}(\pi)$ may not be contained in $\mathbb{P}_{\mathrm{int}}$, we define $\tilde{e}_i(i_{\mathrm{sh}}(\pi))$ by the same way described in the subsection 5.1).

We have

$$H_i^{i_{\mathrm{sh}}(\pi)}(t) = \langle i_{\mathrm{sh}}(\pi(t)), \alpha_i^\vee \rangle = \langle \pi(t), \alpha_i^\vee \rangle = H_i^{\mathrm{sh},\pi}(t) \text{ for } t \in [0,1].$$

Then claim 2 follows from the definition of the root operators and claim 1.

claim 3. If $i \in I^{\mathrm{sh}}$, then (i) follows.

We have $H_i^{i_{\mathrm{sh}}(\pi)}(t) = H_i^{i_{\mathrm{sh}}(\pi)+\pi_{\lambda'}}$ since $\langle \lambda', \alpha_i^\vee \rangle = 0$. Then by the definition of $\tilde{e}_i$, we can check that $\tilde{e}_i i_{\mathrm{sh}}(\pi) + \pi_{\lambda'} = \tilde{e}_i(i_{\mathrm{sh}}(\pi) + \pi_{\lambda'})$. Now, we have from claim 2 that

$$\varphi(\tilde{e}_i^{\mathrm{sh}}\pi) = i_{\mathrm{sh}}(\tilde{e}_i^{\mathrm{sh}}\pi) + \pi_{\lambda'} = \tilde{e}_i i_{\mathrm{sh}}(\pi) + \pi_{\lambda'} = \tilde{e}_i(i_{\mathrm{sh}}(\pi) + \pi_{\lambda'}) = \tilde{e}_i\varphi(\pi) \in \mathbb{B}_0(\lambda).$$

It remains to show the case $i = 0$. Let $(\nu_1, \dots, \nu_N; \underline{\sigma})$ be an expression of π, where $\nu_i \in \widehat{W}^{\mathrm{sh}}\overline{\lambda}$. For each $1 \le k \le N$, we can take $w_k \in W^{\mathrm{sh}}$ and $p_k \in \mathbb{Z}$ such that $\nu_k = w_k\overline{\lambda} + p_k\overline{\delta}$ by (27).

claim 4. We have $\varphi(\pi) = (w_1\lambda + p_1\delta, \dots, w_N\lambda + p_N\delta; \underline{\sigma})$.

For each $1 \le k \le N$, we have

$$i_{\mathrm{sh}}(w_k\overline{\lambda} + p_k\overline{\delta}) + \lambda' = w_k i_{\mathrm{sh}}(\overline{\lambda}) + p_k\delta + \lambda' = w_k(i_{\mathrm{sh}}(\overline{\lambda}) + \lambda') + p_k\delta = w_k\lambda + p_k\delta,$$

which implies claim 4.

We set $\mu_k = w_k\lambda + p_k\delta$ for $1 \le k \le N$. We put

$$t_1 = \min\{t \in [0,1] \mid H_0^{\mathrm{sh},\pi}(t) = m_0^{\mathrm{sh},\pi}\},$$
$$t_0 = \max\{t \in [0,t_1] \mid H_0^{\mathrm{sh},\pi}(t) = m_0^{\mathrm{sh},\pi} + 1\}.$$

By replacing $\underline{\sigma}$ if need, we can assume that $t_0, t_1 \in \{\sigma_0, \dots, \sigma_N\}$. Take $q_0, q_1 \in \{0, \dots, N\}$ so that $\sigma_{q_0} = t_0$ and $\sigma_{q_1} = t_1$. Then by the definition of $\tilde{e}_0^{\mathrm{sh}}$ and an expression, we have

$$\tilde{e}_0^{\mathrm{sh}}\pi = (\nu_1, \dots, \nu_{q_0}, s_0^{\mathrm{sh}}\nu_{q_0+1}, \dots, s_0^{\mathrm{sh}}\nu_{q_1}, \nu_{q_1+1}, \dots, \nu_N; \underline{\sigma}).$$

Since $i_{\mathrm{sh}}(s_0^{\mathrm{sh}}\nu_k) + \lambda' = s_0^{\mathrm{sh}}(i_{\mathrm{sh}}(\nu_k) + \lambda')$, we have that

$$\varphi(\tilde{e}_0^{\mathrm{sh}}\pi) = (\mu_1, \dots, \mu_{q_0}, s_0^{\mathrm{sh}}\mu_{q_0+1}, \dots, s_0^{\mathrm{sh}}\mu_{q_1}, \mu_{q_1+1}, \dots, \mu_N; \underline{\sigma}). \tag{39}$$

Let $\tau \in \widehat{W}$ and $j \in I^{\mathrm{sh}}$ be elements satisfying Lemma 8.11, and let $\tau = s_{i_1} \cdots s_{i_M}$ be an expression satisfying the condition (ii). In the following proof, by $w\pi'$ for $w \in \widehat{W}$ and $\pi' \in \mathbb{P}$ we denote the path defined by $w\pi'(t) = w(\pi'(t))$.

claim 5. We have

$$S_{\tau^{-1}}\varphi(\pi) = \tau^{-1}\varphi(\pi) = (\tau^{-1}\mu_1, \dots, \tau^{-1}\mu_N; \underline{\sigma}),$$

where the action $S_{\tau^{-1}}$ is defined in Theorem 5.4.

Set $\tau_L = s_{i_1} \cdots s_{i_L}$ for $1 \le L \le M$, and $\tau_0 = \mathrm{id}$. We show by induction on L that

$$S_{\tau_L^{-1}}\varphi(\pi) = \tau_L^{-1}\varphi(\pi).$$

The case $L = 0$ is trivial. Assume that $L \ge 1$, and assume that $S_{\tau_{L-1}^{-1}}\varphi(\pi) = \tau_{L-1}^{-1}\varphi(\pi)$. By the condition (ii), we have for some $\beta \in \Delta \setminus \Delta^{\mathrm{sh}}$ and $\ell \in \mathbb{Z}$ that $\tau_{L-1}\alpha_{i_L} = \beta + \ell\delta$. Then, we have for each $1 \le k \le N$ that

$$\langle \tau_{L-1}^{-1}\mu_k, \alpha_{i_L}^\vee \rangle = \langle \lambda, w_k^{-1}\tau_{L-1}\alpha_{i_L}^\vee \rangle = \langle \lambda, (w_k^{-1}\beta)^\vee \rangle.$$

By Lemma 2.2, if $\beta \in \Delta_+ \setminus \Delta_+^{\mathrm{sh}}$, then we have $w_k^{-1}\beta \in \Delta_+$ for all $1 \le k \le N$, and hence the last term is nonnegative for all k. On the other hand, if $\beta \in \Delta_- \setminus \Delta_-^{\mathrm{sh}}$, then we have $w_k^{-1}\beta \in \Delta_-$ for all k, and hence the last term is nonpositive for all k. Therefore, we have that the function $H_{i_L}^{\tau_{L-1}^{-1}\varphi(\pi)}$ is non-decreasing or non-increasing, and then Lemma 5.5 completes the proof of the claim.

Now, the following claim completes the proof of lemma 8.13 (i):

claim 6. We have $\varphi(\tilde{e}_0^{\rm sh}\pi) = S_\tau \tilde{e}_j S_{\tau^{-1}} \varphi(\pi) \in \mathbb{B}_0(\lambda)$.

Since $\varphi(\pi) = i_{\rm sh}(\pi) + \pi_{\lambda'}$ and $\langle \lambda', (\alpha_0^{\rm sh})^\vee \rangle = 0$, we have $\langle \varphi(\pi)(t), (\alpha_0^{\rm sh})^\vee \rangle = H_0^{{\rm sh},\pi}(t)$ for all $t \in [0,1]$. Hence we have

$$\min\{\langle \varphi(\pi)(t), (\alpha_0^{\rm sh})^\vee \rangle \mid t \in [0,1]\} = m_0^{{\rm sh},\pi}, \tag{40}$$

and

$$\begin{aligned} &\min\{t \in [0,1] \mid \langle \varphi(\pi)(t), (\alpha_0^{\rm sh})^\vee \rangle = m_0^{{\rm sh},\pi}\} = t_1 = \sigma_{q_1}, \\ &\max\{t \in [0,t_1] \mid \langle \varphi(\pi)(t), (\alpha_0^{\rm sh})^\vee \rangle = m_0^{{\rm sh},\pi} + 1\} = t_0 = \sigma_{q_0}. \end{aligned} \tag{41}$$

By the condition (i), we have

$$\langle \tau^{-1}\varphi(\pi)(t), \alpha_j^\vee \rangle = \langle \varphi(\pi)(t), (\alpha_0^{\rm sh})^\vee \rangle \text{ for all } t \in [0,1].$$

Then we have $m_j^{\tau^{-1}\varphi(\pi)} = m_0^{{\rm sh},\pi}$ from this equation and (40), and we also have from (41) that

$$\begin{aligned} &\min\{t \in [0,1] \mid H_j^{\tau^{-1}\varphi(\pi)} = m_j^{\tau^{-1}\varphi(\pi)}\} = t_1 = \sigma_{q_1}, \\ &\max\{t \in [0,t_1] \mid H_j^{\tau^{-1}\varphi(\pi)} = m_j^{\tau^{-1}\varphi(\pi)} + 1\} = t_0 = \sigma_{q_0}. \end{aligned}$$

Hence, we have from claim 5 that

$$\begin{aligned} &\tilde{e}_j S_{\tau^{-1}} \varphi(\pi) = \tilde{e}_j(\tau^{-1}\mu_1, \dots, \tau^{-1}\mu_N; \underline{\sigma}) \\ &= (\tau^{-1}\mu_1, \dots, \tau^{-1}\mu_{q_0}, s_j\tau^{-1}\mu_{q_0+1}, \dots, s_j\tau^{-1}\mu_{q_1}, \tau^{-1}\mu_{q_1+1}, \dots, \tau^{-1}\mu_N; \underline{\sigma}). \\ &= (\tau^{-1}\mu_1, \dots, \tau^{-1}\mu_{q_0}, \tau^{-1}s_0^{\rm sh}\mu_{q_0+1}, \dots, \tau^{-1}s_0^{\rm sh}\mu_{q_1}, \tau^{-1}\mu_{q_1+1}, \dots, \tau^{-1}\mu_N; \underline{\sigma}), \end{aligned}$$

where the last equality follows from the condition (i). Then similarly as claim 5, we can show that

$$S_\tau \tilde{e}_j S_{\tau^{-1}} \varphi(\pi) = (\mu_1, \dots, \mu_{q_0}, s_0^{\rm sh}\mu_{q_0+1}, \dots, s_0^{\rm sh}\mu_{q_1}, \mu_{q_1+1}, \dots, \mu_N; \underline{\sigma}),$$

which is equal to $\varphi(\tilde{e}_0^{\rm sh}\pi)$ by (39). □

As stated above, Lemma 8.12 follows from Lemma 8.13. By Lemma 8.8, we have

$$\sum_{\substack{\pi \in \mathbb{B}_0^{\rm sh}(\overline{\lambda}) \\ \iota(\pi) \in \overline{P}}} e(\mathrm{wt}(\pi)) = \mathrm{ch}_{\mathfrak{h}_d^{\rm sh}} \mathcal{D}^{\rm sh}(1, \overline{\lambda})[0].$$

Since $\mathrm{wt}(\varphi(\pi)) = i_{\rm sh}(\pi)(1) + \lambda' = i_{\rm sh}(\mathrm{wt}(\pi)) + \lambda'$, this equation implies that

$$\sum_{\substack{\pi \in \mathbb{B}_0^{\rm sh}(\overline{\lambda}) \\ \iota(\pi) \in \overline{P}}} e(\mathrm{wt}(\varphi(\pi))) = e(\lambda') i_{\rm sh} \mathrm{ch}_{\mathfrak{h}_d^{\rm sh}} \mathcal{D}^{\rm sh}(1, \overline{\lambda})[0].$$

On the other hand, as stated in Remark 5.17, we have

$$\sum_{\eta \in \mathbb{B}(\lambda)_{\rm cl}} e(\mathrm{wt}_{\overline{P}}(\eta)) = \sum_{\substack{\pi' \in \mathbb{B}_0(\lambda) \\ \iota(\pi') \in P}} e(\mathrm{wt}(\pi')).$$

Hence, the following lemma completes the proof of Proposition 8.9.

Lemma 8.14. *The map φ induces a bijection*

$$\{\pi \in \mathbb{B}_0^{\mathrm{sh}}(\overline{\lambda}) \mid \iota(\pi) \in \overline{P}\} \xrightarrow{\sim} \{\pi' \in \mathbb{B}_0(\lambda) \mid \iota(\pi') \in P, \mathrm{wt}(\pi') \in \lambda - Q_+^{\mathrm{sh}} + \mathbb{Z}\delta\}.$$

Proof. Using Lemma 8.12, we can easily check that the image of the left hand side is contained in the right hand side, and this map is obviously injective. Hence, it suffices to show that the number of elements of these sets are same. As stated in Remark 5.17,

$$\{\pi' \in \mathbb{B}_0(\lambda) \mid \iota(\pi') \in P\} = \{\pi_\eta \mid \eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}\}$$

holds. Hence the number of elements of the right hand side of the lemma is equal to that of elements of the set $\{\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}} \mid \mathrm{wt}(\eta) \in \mathrm{cl}(\lambda - Q_+^{\mathrm{sh}})\}$. By Corollary 8.6, we have

$$\sum_{\substack{\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}} \\ \mathrm{wt}(\eta) \in \mathrm{cl}(\lambda - Q_+^{\mathrm{sh}})}} e(p \circ \mathrm{wt}(\eta)) = P_{\lambda - Q_+^{\mathrm{sh}}} \prod_{i \in I} \mathrm{ch}_{\mathfrak{h}}\, \mathcal{D}(1, \varpi_i)^{\lambda_i},$$

where p denotes the canonical projection $\widehat{P}_{\mathrm{cl}} \to P$, and since $\mathrm{wt}_{\mathfrak{h}}\mathcal{D}(1, \varpi_i) \subseteq \varpi_i - Q_+$, the right hand side of the above equation is equal to

$$\prod_{i \in I} \left(P_{\varpi_i - Q_+^{\mathrm{sh}}} \mathrm{ch}_{\mathfrak{h}}\, \mathcal{D}(1, \varpi_i)\right)^{\lambda_i}.$$

By Lemma 4.21, we have

$$P_{\varpi_i - Q_+^{\mathrm{sh}}} \mathrm{ch}_{\mathfrak{h}}\, \mathcal{D}(1, \varpi_i) = e(\varpi_i - i_{\mathrm{sh}}(\overline{\varpi}_i)) i_{\mathrm{sh}} \mathrm{ch}_{\mathfrak{h}^{\mathrm{sh}}}\, \mathcal{D}^{\mathrm{sh}}(r, \overline{\varpi}_i),$$

and from Lemma 3.8, we have that

$$\dim \mathcal{D}^{\mathrm{sh}}(r, \overline{\varpi}_i) = \begin{cases} \dim V_{\mathfrak{g}^{\mathrm{sh}}}(\overline{\varpi}_i) & \text{if } i \in I^{\mathrm{sh}}, \\ 1 & \text{if } i \notin I^{\mathrm{sh}}. \end{cases}$$

In conclusion, we can see from these equations that

$$\#\{\pi' \in \mathbb{B}_0(\lambda) \mid \iota(\pi') \in P, \mathrm{wt}(\pi') \in \lambda - Q_+^{\mathrm{sh}} + \mathbb{Z}\delta\} = \prod_{i \in I^{\mathrm{sh}}} \dim V_{\mathfrak{g}^{\mathrm{sh}}}(\overline{\varpi}_i)^{\lambda_i}.$$

On the other hand, we have

$$\begin{aligned} \#\{\pi \in \mathbb{B}_0^{\mathrm{sh}}(\overline{\lambda}) \mid \iota(\pi) \in \overline{P}\} &= \dim \mathcal{D}^{\mathrm{sh}}(1, \overline{\lambda}) \\ &= \prod_{i \in I^{\mathrm{sh}}} \dim \mathcal{D}^{\mathrm{sh}}(1, \overline{\varpi}_i)^{\lambda_i} = \prod_{i \in I^{\mathrm{sh}}} \dim V_{\mathfrak{g}^{\mathrm{sh}}}(\overline{\varpi}_i)^{\lambda_i}, \end{aligned}$$

where the first equality follows from Lemma 8.8, the second follows from [7, Theorem 1], and the last one follows from Lemma 3.8. Hence, we completes the proof. □

8.4 Lower bound for the weight sum of $\mathbb{B}(\lambda)_{\mathrm{cl}}$

Recall that, by Corollary 4.16, $\mathcal{D}^{\mathrm{sh}}(1,\overline{\lambda})[0]$ has a $C\mathfrak{g}_d^{\mathrm{sh}}$-module filtration $0 = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_k = \mathcal{D}^{\mathrm{sh}}(1,\overline{\lambda})[0]$ such that

$$D_i/D_{i-1} \cong \mathcal{D}^{\mathrm{sh}}(r,\nu_i)[m_i] \text{ for some } \nu_i \in \overline{P}_+, m_i \in \mathbb{Z}_{\geq 0}.$$

On the other hand, by Corollary 8.3, there exist a sequence $\mu_1,\dots,\mu_\ell$ of elements of P_+ and a sequence $n_1,\dots,n_\ell$ of integers such that

$$\sum_{\eta\in\mathbb{B}(\lambda)_{\mathrm{cl}}} e(\mathrm{wt}_{\widetilde{P}}(\eta)) = \sum_{1\leq j\leq \ell} \mathrm{ch}_{\mathfrak{h}_d}\mathcal{D}(1,\mu_j)[n_j].$$

Now, we state the following proposition, which is the main result in this section:

Proposition 8.15. *Let $\lambda' = \lambda - i_{\mathrm{sh}}(\overline{\lambda})$. There exists a sequence $j_1,\dots,j_k$ of pairwise distinct elements of $\{1,\dots,\ell\}$ such that $\mu_{j_i} = i_{\mathrm{sh}}(\nu_i)+\lambda'$ and $n_{j_i} = m_i$ for $1\leq i\leq k$.*

Proof. Let $\overline{Q}_+^{\mathrm{sh}} = \sum_{i\in I^{\mathrm{sh}}} \mathbb{Z}_{\geq 0}\overline{\alpha}_i$, and we define a partial order $\preceq_{\mathrm{sh}}$ on $\overline{P}_+ + \mathbb{Z}\overline{\delta}$ by $\nu_1'+p_1\overline{\delta} \preceq_{\mathrm{sh}} \nu_2'+p_2\overline{\delta}$ if $\nu_2'-\nu_1' \in \overline{Q}_+^{\mathrm{sh}}$ and $p_1\geq p_2$. This partial order satisfies that, if $\nu_1'+p_1\overline{\delta} \in \mathrm{wt}_{\mathfrak{h}_d^{\mathrm{sh}}}\mathcal{D}^{\mathrm{sh}}(\ell,\nu_2')[p_2]$, then we have $\nu_1'+p_1\overline{\delta} \preceq_{\mathrm{sh}} \nu_2'+p_2\overline{\delta}$. Let $\zeta_i = \nu_i + m_i\overline{\delta}$ for each $1\leq i\leq k$. By changing the numbering, we can assume that the sequence $\zeta_1,\dots,\zeta_k$ satisfies that, if $\zeta_i \succeq_{\mathrm{sh}} \zeta_j$, then $i\leq j$. We show by induction on k' that there exists a sequence $j_1,\dots,j_{k'}$ of pairwise distinct elements of $\{1,\dots,\ell\}$ such that $\mu_{j_i} = i_{\mathrm{sh}}(\nu_i)+\lambda'$ and $n_{j_i}=m_i$. The case $k'=0$ is trivial. Assume $0<k'\leq k$. By the induction hypothesis, we have

$$\begin{aligned}\sum_{\eta\in\mathbb{B}(\lambda)_{\mathrm{cl}}} e(\mathrm{wt}_{\widetilde{P}}(\eta)) - \sum_{i=1}^{k'-1}\mathrm{ch}_{\mathfrak{h}_d}\mathcal{D}(1, i_{\mathrm{sh}}(\nu_i)+\lambda')[m_i] \\ = \sum_{j\notin\{j_1,\dots,j_{k'-1}\}} \mathrm{ch}_{\mathfrak{h}_d}\mathcal{D}(1,\mu_j)[n_j].\end{aligned}$$

Denote by F the element of $\mathbb{Z}[\mathfrak{h}_d^*]$ in the both sides of the above equation. We have from Proposition 8.9 and Lemma 4.21 that

$$\begin{aligned}P_{\lambda-Q_+^{\mathrm{sh}}+\mathbb{Z}\delta}F &= e(\lambda')i_{\mathrm{sh}}\left(\mathrm{ch}_{\mathfrak{h}_d^{\mathrm{sh}}}\mathcal{D}^{\mathrm{sh}}(1,\overline{\lambda})[0] - \sum_{i=1}^{k'-1}\mathrm{ch}_{\mathfrak{h}_d^{\mathrm{sh}}}\mathcal{D}^{\mathrm{sh}}(r,\nu_i)[m_i]\right) \\ &= e(\lambda')i_{\mathrm{sh}}\left(\sum_{i=k'}^{k}\mathrm{ch}_{\mathfrak{h}_d^{\mathrm{sh}}}\mathcal{D}^{\mathrm{sh}}(r,\nu_i)[m_i]\right).\end{aligned} \tag{42}$$

Define a subset $S\subseteq(\mathfrak{h}_d^{\mathrm{sh}})^*$ by

$$S = \{\zeta\in(\mathfrak{h}_d^{\mathrm{sh}})^* \mid \text{the coefficient of } e(\zeta) \text{ in } \sum_{i=k'}^{k}\mathrm{ch}_{\mathfrak{h}_d^{\mathrm{sh}}}\mathcal{D}^{\mathrm{sh}}(r,\nu_i)[m_i] \text{ is nonzero}\}.$$

From the assumption on $\zeta_1,\dots,\zeta_k$, we can see that $\zeta_{k'} = \nu_{k'}+m_{k'}\overline{\delta}$ is maximal in S with respect to the partial order $\preceq_{\mathrm{sh}}$. Then, since $\nu_2'-\nu_1' \in \overline{Q}_+^{\mathrm{sh}}$ if and only if $(i_{\rm sh}(\nu_2') + \lambda') - (i_{\rm sh}(\nu_1') + \lambda') \in Q_+^{\rm sh}$ for $\nu_1', \nu_2' \in \overline{P}$, we have from (42) that $i_{\rm sh}(\zeta_{k'}) + \lambda'$ is maximal with respect to $\preceq$ in the set

$$\{\xi \in \lambda - Q_+^{\rm sh} + \mathbb{Z}\delta \mid \text{the coefficient of } e(\xi) \text{ in } F \text{ is nonzero}\}.$$

Then, since $\{\xi \in \mathfrak{h}_d^* \mid \text{the coefficient of } e(\xi) \text{ in } F \text{ is nonzero}\} \subseteq \lambda - Q_+ + \mathbb{Z}\delta$ by Lemma 5.14 (ii), we can see that $i_{\rm sh}(\zeta_{k'}) + \lambda'$ is a maximal element in this set. Now, from the weight consideration, there exists some $j' \in \{1, \dots, \ell\} \setminus \{j_1, \dots, j_{k'-1}\}$ such that $\mu_{j'} + n_{j'}\delta = i_{\rm sh}(\zeta_{k'}) + \lambda'$, which completes the proof. □

Corollary 8.16. *Let $\lambda \in P_+$, and we set $\lambda' = \lambda - i_{\rm sh}(\overline{\lambda})$. Then we have*

$$\sum_{\eta \in \mathbb{B}(\lambda)_{\rm cl}} e(\mathrm{wt}_{\overline{P}}(\eta)) \geq \sum_{\nu \in \overline{P}_+, m \in \mathbb{Z}_{\geq 0}} (\mathcal{D}^{\rm sh}(1, \overline{\lambda})[0] : \mathcal{D}^{\rm sh}(r, \nu)[m]) \mathcal{D}(1, i_{\rm sh}(\nu) + \lambda')[m].$$

Remark 8.17. In the next section, we show that the left hand side and the right hand side in the above inequality are equal.

9 Main theorem and corollaries

To prove the main theorem, we need the following lemma:

Lemma 9.1. *For $\lambda \in P_+$, we have*

$$\dim W(\lambda) \geq \#\mathbb{B}(\lambda)_{\rm cl}.$$

Proof. Write $\lambda = \sum_{i \in I} \lambda_i \varpi_i$ with $\lambda_i \in \mathbb{Z}_{\geq 0}$. The inequality follows since

$$\dim W(\lambda) \geq \prod_{i \in I} \dim \mathcal{D}(1, \varpi_i)^{\lambda_i} = \prod_{i \in I} (\#\mathbb{B}(\varpi_i)_{\rm cl})^{\lambda_i} = \#\mathbb{B}(\lambda)_{\rm cl},$$

where the first inequality follows from Lemma 3.11 (ii), the second equality follows from Proposition 8.5, and the third follows from Theorem 5.13 (i). □

Now, we state the main theorems in this article. The following theorem is easily follows from Corollary 4.20, Corollary 8.16, and the above lemma:

Theorem 9.2. *Assume that $\mathfrak{g}$ is non-simply laced. Let $\lambda \in P_+$, and set $\lambda' = \lambda - i_{\rm sh}(\overline{\lambda})$. Then we have*

$$\begin{aligned} \mathrm{ch}_{\mathfrak{h}_d} W(\lambda) &= \sum_{\eta \in \mathbb{B}(\lambda)_{\rm cl}} e(\mathrm{wt}_{\overline{P}}(\eta)) \\ &= \sum_{\nu \in \overline{P}_+, m \in \mathbb{Z}_{\geq 0}} (\mathcal{D}^{\rm sh}(1, \overline{\lambda})[0] : \mathcal{D}^{\rm sh}(r, \nu)[m]) \mathrm{ch}_{\mathfrak{h}_d} \mathcal{D}(1, i_{\rm sh}(\nu) + \lambda')[m]. \end{aligned}$$

When $\mathfrak{g}$ is non-simply laced, the Demazure module $\mathcal{D}^{\rm sh}(1, \overline{\lambda})[0]$ has a $C\mathfrak{g}_d^{\rm sh}$-module filtration $0 = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_k$ such that

$$D_i / D_{i-1} \cong \mathcal{D}^{\rm sh}(r, \nu_i)[m_i] \text{ for some } \nu_i \in \overline{P}_+, m_i \in \mathbb{Z}_{\geq 0}$$

by Corollary 4.16. Then we set $\mu_i = i_{\rm sh}(\nu_i) + (\lambda - i_{\rm sh}(\overline{\lambda}))$ for each $1 \leq i \leq k$. When $\mathfrak{g}$ is simply laced, we set $k = 1$, $\mu_1 = \lambda$ and $m_1 = 0$. Now, we obtain the following result on the structure of the Weyl module for the current algebra:

Theorem 9.3. *The Weyl module $W(\lambda)$ has a $C\mathfrak{g}_d$-module filtration $0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_k = W(\lambda)$ such that each subquotient W_i/W_{i-1} is isomorphic to the Demazure module $\mathcal{D}(1, \mu_i)[m_i]$.*

Proof. The theorem in the simply laced case is just Theorem 3.7. Proposition 4.18 together with Theorem 9.2 implies this theorem in the non-simply laced case. □

On the other hand, in the crystal theory, we have the following theorem:

Theorem 9.4. *There exists a $U'_q(\widehat{\mathfrak{g}})$-crystal isomorphism*

$$\kappa : \mathcal{B}(\Lambda_0) \otimes \mathbb{B}(\lambda)_{\mathrm{cl}} \to \bigoplus_{\eta_0 \in \mathbb{B}(\lambda)_{\mathrm{cl}}^{\Lambda_0}} \mathcal{B}(\Lambda_0 + \pi_{\eta_0}(1))$$

such that the restriction of κ on $b_{\Lambda_0} \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}$ preserves the $\widehat{P}$-weights, and we have

$$\kappa(b_{\Lambda_0} \otimes \mathbb{B}(\lambda)_{\mathrm{cl}}) = \coprod_{1 \le i \le k} \mathcal{B}(1, \mu_i)[m_i].$$

Proof. The theorem in the simply laced case follows from Proposition 8.2 and Proposition 8.4. On the other hand, the theorem in the non-simply laced case follows from Proposition 8.2 and Theorem 9.2. □

Next, we introduce two corollaries which easily follow from our main theorems. The following one has been verified in the simply laced case in [8, Proposition 1, Corollary 4].

Corollary 9.5. *Let $\lambda \in P_+$.*
(i) *If $\lambda = \sum_{i \in I} \lambda_i \varpi_i$, then*

$$\dim W(\lambda) = \prod_{i \in I} \dim W(\varpi_i)^{\lambda_i}.$$

(ii) *Let $\lambda_1, \ldots, \lambda_k \in P_+$ be elements satisfying $\lambda = \lambda_1 + \cdots + \lambda_k$. Then for arbitrary pairwise distinct complex numbers $c_1, \ldots, c_k$, we have*

$$W(\lambda) \cong W(\lambda_1)_{c_1} * \cdots * W(\lambda_k)_{c_k}$$

as $C\mathfrak{g}_d$-modules.

Proof. We need only to show in the non-simply laced case. By Theorem 9.2 and Theorem 5.13 (i), we have

$$\dim W(\lambda) = \#\mathbb{B}(\lambda)_{\mathrm{cl}} = \prod_{i \in I} (\#\mathbb{B}(\varpi_i)_{\mathrm{cl}})^{\lambda_i} = \prod_{i \in I} \dim W(\varpi_i)^{\lambda_i},$$

and (i) follows. By the same way as [8, Lemma 5], we can check that $W(\lambda_1)_{c_1} * \cdots * W(\lambda_k)_{c_k}$ satisfies the defining relations of $W(\lambda)$, and hence there exists a surjective homomorphism from $W(\lambda)$ to $W(\lambda_1)_{c_1} * \cdots * W(\lambda_k)_{c_k}$. From (i), this surjection is an isomorphism. □

Before stating the next corollary, we prepare some notation. For a $C\mathfrak{g}_d$-module M and $c \in \mathbb{C}$, we set $M_c = \{v \in M \mid d.v = cv\}$, which is obviously a $\mathfrak{g}$-module. For a finite dimensional $\mathfrak{g}$-module N and $\mu \in P_+$, we denote by $(N : V_{\mathfrak{g}}(\mu))$ the multiplicity of $V_{\mathfrak{g}}(\mu)$ in N.

In [30], an expression of the classically restricted one-dimensional sums (1dsums for short) in terms of paths has been given. Here, we briefly recall the result. For the precise definitions of energy functions and classically restricted 1dsums, we refer the reader to [12, 11]. Let $\mathbf{i} = (i_1, i_2, \ldots, i_\ell)$ be a sequence of elements of I, and put

$$\mathbb{B}_{\mathbf{i}} = \mathbb{B}(\varpi_{i_1})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_2})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_\ell})_{\mathrm{cl}}.$$

We denote by $D_{\mathbf{i}}$ the energy function defined on $\mathbb{B}_{\mathbf{i}}$ (for definition, see [30, §4.1]), which is a function from $\mathbb{B}_{\mathbf{i}}$ to $\mathbb{Z}$. Then, for an element $\mu \in P_+$, the classically restricted 1dsum $X(\mathbb{B}_{\mathbf{i}}, \mu; q)$ is defined by

$$X(\mathbb{B}_{\mathbf{i}}, \mu; q) = \sum_{\substack{b \in \mathbb{B}_{\mathbf{i}} \\ \tilde{e}_j b = 0 \ (j \in I) \\ \mathrm{wt}(b) = \mathrm{cl}(\mu)}} q^{D_{\mathbf{i}}(b)}.$$

Remark 9.6. In [12] and [11], classically restricted 1dsums are defined on the tensor products of Kirillov-Reshetikhin crystals $\mathcal{B}^{i,s}$ indexed by $i \in I$ and $s \in \mathbb{Z}_{\geq 1}$. As noted in [30, §5.1], $\mathbb{B}(\varpi_i)_{\mathrm{cl}} \cong \mathcal{B}(W_q(\varpi_i))$ is the Kirillov-Reshetikhin crystal $\mathcal{B}^{i,s}$ with $s = 1$.

By [30, Corollary 5.1.1], we have for some constant $C \in \mathbb{Z}$ that

$$\sum_{\substack{\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}} \\ \tilde{e}_j \eta = 0 \ (j \in I) \\ \eta(1) = \mathrm{cl}(\mu)}} q^{\mathrm{Deg}(\eta)} = q^C X(\mathbb{B}_{\mathbf{i}}, \mu; q). \tag{43}$$

The second corollary is as follows:

Corollary 9.7. *Let $\lambda \in P_+$, and let $\mu_i \in P_+$ and $m_i \in \mathbb{Z}$ for $1 \leq i \leq k$ be the elements given just below Theorem 9.2. Then for any sequence $\mathbf{i} = (i_1, \ldots, i_\ell)$ such that $\lambda = \sum_{1 \leq j \leq \ell} \varpi_{i_j}$ and $\mu \in P_+$, we have for some constant $C \in \mathbb{Z}$ that*

$$\begin{aligned}\sum_{n \in \mathbb{Z}_{\geq 0}} (W(\lambda)_n : V_{\mathfrak{g}}(\mu)) q^n &= \sum_{1 \leq i \leq k} \sum_{n \in \mathbb{Z}_{\geq 0}} (\mathcal{D}(1, \mu_i)[m_i]_n : V_{\mathfrak{g}}(\mu)) q^n \\ &= q^C X(\mathbb{B}_{\mathbf{i}}, \mu; q^{-1}).\end{aligned}$$

Proof. The first equality is obvious from Theorem 9.3. Let $\mathcal{D}_q(1, \mu_i)[m_i]$ be the quantized Demazure module whose classical limit is $\mathcal{D}(1, \mu_i)[m_i]$, and let $V_q(\mu)$ be the irreducible $U_q(\mathfrak{g})$-module whose classical limit is $V_{\mathfrak{g}}(\mu)$, where $U_q(\mathfrak{g})$ is the quantized enveloping algebra associated with $\mathfrak{g}$. Then we have

$$(\mathcal{D}(1, \mu_i)[m_i]_n : V_{\mathfrak{g}}(\mu)) = (\mathcal{D}_q(1, \mu_i)[m_i]_n : V_q(\mu)),$$

where $\mathcal{D}_q(1, \mu_i)[m_i]_n$ is defined similarly. Since $\mathcal{B}(1, \mu_i)[m_i]$ is a $U_q(\mathfrak{g})$-crystal basis of $\mathcal{D}_q(1, \mu_i)[m_i]$, we have

$$\begin{aligned}&(\mathcal{D}_q(1, \mu_i)[m_i]_n : V_q(\mu)) \\ &\quad = \#\{b \in \mathcal{B}(1, \mu_i)[m_i] \mid \tilde{e}_j b = 0 \text{ for } j \in I, \ \mathrm{wt}(b) = \mu + \Lambda_0 + n\delta\}.\end{aligned}$$

Hence the second term in the corollary is equal to

$$\sum_{1\le i\le k} \sum_{\substack{b\in\mathcal{B}(1,\mu_i)[m_i] \\ \tilde{e}_j b=0\ (j\in I) \\ \mathrm{wt}(b)\in\mu+\Lambda_0+\mathbb{Z}\delta}} q^{\langle \mathrm{wt}(b),d\rangle}.$$

We can easily see that

$$\eta' \in \{\eta\in\mathbb{B}(\lambda)_{\mathrm{cl}} \mid \tilde{e}_j\eta = \mathbf{0} \text{ for } j\in I,\ \eta(1)=\mathrm{cl}(\mu),\ \mathrm{Deg}(\eta)=n\}$$

if and only if

$$\kappa(b_{\Lambda_0}\otimes\eta') \in \coprod_{1\le i\le k} \{b\in\mathcal{B}(1,\mu_i)[m_i] \mid \tilde{e}_j b = 0 \text{ for } j\in I,\ \mathrm{wt}(b)=\mu+\Lambda_0-n\delta\},$$

where κ is the $U_q'(\widehat{\mathfrak{g}})$-crystal isomorphism given in Theorem 9.4. Hence, we have that

$$\sum_{1\le i\le k} \sum_{\substack{b\in\mathcal{B}(1,\mu_i)[m_i] \\ \tilde{e}_j b=0\ (j\in I) \\ \mathrm{wt}(b)\in\mu+\Lambda_0+\mathbb{Z}\delta}} q^{\langle \mathrm{wt}(b),d\rangle} = \sum_{\substack{\eta\in\mathbb{B}(\lambda)_{\mathrm{cl}} \\ \tilde{e}_j\eta=0\ (j\in I) \\ \eta(1)=\mathrm{cl}(\mu)}} q^{-\mathrm{Deg}(\eta)},$$

which together with (43) implies the second equality. □

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