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INDUCTIVE CONSTRUCTION OF THE p-ADIC ZETA FUNCTIONS FOR NON-COMMUTATIVE p-EXTENSIONS OF EXPONENT p OF TOTALLY REAL FIELDS

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ABSTRACT. We construct the p-adic zeta function for a one-dimensional (as a p-adic Lie extension) non-commutative p-extension F_∞ of a totally real number field F such that the finite part of its Galois group G is a p-group of exponent p. We first calculate the Whitehead groups of the Iwasawa algebra $\Lambda(G)$ and its canonical Ore localisation $\Lambda(G)_S$ by using Oliver-Taylor's theory of integral logarithms. This calculation reduces the existence of the non-commutative p-adic zeta function to certain congruences between abelian p-adic zeta pseudomeasures. Then we finally verify these congruences by using Deligne-Ribet's theory and a certain inductive technique. As an application we shall prove a special case of (the p-part of) the non-commutative equivariant Tamagawa number conjecture for critical Tate motives.

0. INTRODUCTION

One of the most important topics in non-commutative Iwasawa theory is to construct the p-adic zeta function and to verify the main conjecture, as well as in classical theory. Up to the present, there have been several successful examples upon this topic for p-adic Lie extensions of totally real number fields: the results of Jürgen Ritter and Alfred Weiss [RW7], Kazuya Kato [Kato], Mahesh Kakde [Kakde1] and so on. In this article, we shall construct different type of example for certain non-commutative p-extensions of totally real number fields.

Let p be a positive odd prime number and F a totally real number field. Let F_∞ be a totally real p-adic Lie extension of F which contains the cyclotomic $\mathbb{Z}_p$-extension F_{cyc} of F, and assume that only finitely many primes of F ramify in F_∞. For a moment we admit Iwasawa's $\mu = 0$ conjecture to simplify conditions (see Section 1 (F_∞-3) for a general $\mu = 0$ condition). The aim of this article is to prove the following theorem under these conditions:

Theorem 0.1 (=Theorem 3.1). *Let G denote the Galois group of F_∞/F. Then for F_∞/F the p-adic zeta function $\xi_{F_\infty/F}$ exists and the Iwasawa main conjecture is true if G is isomorphic to the direct product of a finite p-group*

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G^f of exponent p and the Galois group Γ of the cyclotomic $\mathbb{Z}_p$-extension F_{cyc}/F.

We shall review the characterisation of the p-adic zeta function and the precise statement of the non-commutative Iwasawa main conjecture in Section 1. In the preceding paper [H], we constructed the p-adic zeta function and verified the main conjecture when the Galois group $\mathrm{Gal}(F_\infty/F)$ is isomorphic to the pro-p group

$$\begin{pmatrix} 1 & \mathbb{F}_p & \mathbb{F}_p & \mathbb{F}_p \\ 0 & 1 & \mathbb{F}_p & \mathbb{F}_p \\ 0 & 0 & 1 & \mathbb{F}_p \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \Gamma$$

and p is not equal to either 2 or 3. Theorem 0.1 generalises this result.

Philosophically the Iwasawa main conjecture is closely related to the special values of L-functions (as implied by many people including Kazuya Kato, Annette Huber-Klawitter, Guido Kings, David Burns, Matthias Flach,......); hence the non-commutative Iwasawa main conjecture should also suggest validity of conjectures concerning these values even in non-commutative coefficient cases (see also [HubKin1] and [FukKat]). In fact we may verify a special case of (the p-part of) the equivariant Tamagawa number conjecture for critical Tate motives with non-commutative coefficient combining Theorem 0.1 with descent theory established by David Burns and Otmar Venjakob [BurVen].

Corollary 0.2 (=Corollary 3.6). *Let F_∞ be a p-adic Lie extension of a totally real number field F as in Theorem 0.1 and F' an arbitrary finite Galois subextension of F_∞/F. Then for an arbitrary natural number r divisible by $p-1$, the p-part of the equivariant Tamagawa number conjecture for $\mathbb{Q}(1-r)_{F'/F}$ is true (here $\mathbb{Q}(1-r)_{F'/F} = h^0(\mathrm{Spec}\, F')(1-r)$ denotes the $(1-r)$-fold Tate motive regarded as a motive over F).*

This may also be regarded as an analogue of the cohomological Lichtenbaum conjecture (in special cases), which was proven by Barry Mazur and Andrew Wiles [MazWil, Wiles] when F' is the same field as F—the Bloch-Kato conjecture case— as the direct consequence of the main conjecture (for commutative cases) which they verified.

Now let us summarise the main idea to prove Theorem 0.1. Consider the family $\mathfrak{F}_B$ of all pairs (U, V) such that U is an open subgroup of G containing Γ and V is the commutator subgroup of U. By classical induction theorem of Richard Brauer [Serre1, Théorèm 22], an arbitrary Artin representation of G is isomorphic to a $\mathbb{Z}$-linear combination of representations induced by characters of abelian groups U/V (as a virtual representation) where each (U, V) is in $\mathfrak{F}_B$. Let $\theta_{U,V}$ and $\theta_{S,U,V}$ denote the composite maps

$$K_1(\Lambda(G)) \xrightarrow{\text{norm}} K_1(\Lambda(U)) \xrightarrow{\text{canonical}} \Lambda(U/V)^\times,$$

$$K_1(\Lambda(G)_S) \xrightarrow{\text{norm}} K_1(\Lambda(U)_S) \xrightarrow{\text{canonical}} \Lambda(U/V)_S^\times$$

for each (U, V) in $\mathfrak{F}_B$ where $\Lambda(G)_S$ (resp. $\Lambda(U)_S$, $\Lambda(U/V)_S$) is *the canonical Ore localisation* of the Iwasawa algebra $\Lambda(G)$ (resp. $\Lambda(U)$, $\Lambda(U/V)$)

introduced in [CFKSV, Section 2] (see also Section 1 in this article). Set $\theta = (\theta_{U,V})_{(U,V)\in\mathfrak{F}_B}$ and $\theta_S = (\theta_{S,U,V})_{(U,V)\in\mathfrak{F}_B}$.

Let F_U (resp. F_V) be the maximal subfield of F_∞ fixed by U (resp. V). Then *the p-adic zeta function* exists for each *abelian* extension F_V/F_U; Pierre Deligne and Kenneth Alan Ribet first constructed it [DR], and by using their results Jean-Pierre Serre reconstructed it as a unique element $\xi_{U,V}$ in the total quotient ring of $\Lambda(U/V)$ which satisfies certain *interpolation formulae* [Serre2]. Now suppose that there exists an element ξ in $K_1(\Lambda(G)_S)$ which satisfies the equation

$$\theta_S(\xi) = (\xi_{U,V})_{(U,V)\in\mathfrak{F}_B}. \tag{0.1}$$

Then we may verify by Brauer induction that ξ satisfies an interpolation formula which characterises ξ as *the p-adic zeta function for F_∞/F*. This observation motivates us to prove that $(\xi_{U,V})_{(U,V)\in\mathfrak{F}_B}$ is contained in the image of θ_S. It seems, however, to be difficult in general to characterise the image of the theta map θ_S completely for the localised Iwasawa algebra $\Lambda(G)_S$. Therefore we shall first determine the image of the theta map θ for the (integral) Iwasawa algebra $\Lambda(G)$, and then construct an element ξ satisfying (0.1) by using this calculation and certain diagram chasing. The strategy which we introduced here was first proposed by David Burns (and hence we call this method *Burns' technique* in this article). We shall discuss its details in Section 2.

Let $(U, \{e\})$ be an element in $\mathfrak{F}_B$ such that the cardinality of the finite part of U is at most p^2 (and U is hence abelian). Let $I_{S,U}$ denote the image of $\theta_{S,U,\{e\}}$ for each of such $(U, \{e\})$'s. By virtue of Burns' technique, we may reduce the condition for $(\xi_{U,V})_{(U,V)\in\mathfrak{F}_B}$ to be contained in the image of θ_S to the following type of congruence:

$$\xi_{U,\{e\}} \equiv \varphi(\xi_{G,[G,G]})^{(G:U)/p} \qquad \mod I_{S,U}$$

where φ is the Frobenius endomorphism $\varphi\colon \Lambda(G^{\mathrm{ab}})_S \to \Lambda(\Gamma)_{(p)}$ induced by the group homomorphism $G^{\mathrm{ab}} \to \Gamma; g \mapsto g^p$. Kato, Ritter, Weiss and Kakde verified such type of congruence when the index $(G : U)$ exactly equals p [Kato, RW6, Kakde1] by using the theory of Deligne and Ribet concerning Hilbert modular forms [DR]. It seems, however, to be almost impossible to deduce such congruences only from Deligne-Ribet's theory when the index $(G : U)$ is strictly greater than p. Nevertheless in Sections 8 and 9 we shall verify these congruences by combining Deligne-Ribet's theory with a certain *inductive technique* which was first introduced in [H].

In computation of the images of θ and θ_S we use theory of p-adic logarithmic homomorphisms. This causes ambiguity of p-power torsion elements in the whole calculation, and hence we have to eliminate this ambiguity as the final step of the proof. We shall complete this step by utilising the existence of the p-adic zeta functions for Ritter-Weiss-type extensions [RW7] and certain inductive arguments.

The detailed content of this article is as follows. We shall briefly review the basic formulations of the non-commutative Iwasawa main conjecture in Section 1. Then we discuss David Burns' outstanding strategy for construction of the p-adic zeta function in Section 2. The precise statement of our main theorem and its application will be dealt with in Section 3. Sections 4, 5 and 6 are devoted to computation of the image of (a certain variant $\tilde{\theta}$ of) the theta map; we first construct 'the additive theta isomorphism' θ^+ in Section 4, and then translate it into the multiplicative morphism $\tilde{\theta}$ by utilising logarithmic homomorphisms in Section 6. Section 5 is the preliminary section for Section 6. We study the image of the theta map $\tilde{\theta}_S$ for the localised Iwasawa algebra $\Lambda(G)_S$ in Section 7, and derive certain 'weak congruences' between abelian p-adic zeta pseudomeasures in Section 8 by applying Deligne-Ribet's q-expansion principle [DR] and Ritter-Weiss' approximation technique [RW6]. In Section 9 we refine the congruences obtained in the previous section by using induction, and construct the p-adic zeta function 'modulo p-torsion.' We shall finally eliminate ambiguity of the p-power torsion part.

Note added in proof.[1] After this article was submitted, there have been outstanding developments of the proof of the non-commutative Iwasawa main conjecture for totally real number fields, which should be mentioned here.

Jürgen Ritter and Alfred Weiss proved the 'main conjecture' of their equivariant Iwasawa theory for p-adic Lie extensions of totally real number fields of rank one under mild conditions on the cyclotomic μ-invariant [RW8].

David Burns extended the results of Ritter and Weiss for p-adic Lie extensions of totally real number fields of arbitrary rank [Burns]. In [Burns], he also developed his patching arguments (that we call Burns' technique in this article) in somewhat axiomatic way, and described various applications of the main conjecture to other conjectures concerning special values of L-functions including the equivariant Tamagawa number conjecture.

Mahesh Kakde also proved the main conjecture for arbitrary p-adic Lie extensions of totally real number fields [Kakde3] by adopting the strategy of Burns and Kato which is the same one as this article is based upon.

CONTENTS

Notation. In this article, p always denotes a positive prime number. We denote by $\mathbb{N}$ the set of natural numbers (the set of *strictly* positive integers).

[1]This article is the submitted version to appear in Duke Mathematical Journal.

We also denote by $\mathbb{Z}$ (resp. $\mathbb{Z}_p$) the ring of integers (resp. p-adic integers). The symbol $\mathbb{Q}$ (resp. $\mathbb{Q}_p$) denotes the rational number field (resp. the p-adic number field). For an arbitrary group G let $\mathrm{Conj}(G)$ denote the set of all conjugacy classes of G. For an arbitrary pro-finite group P, we always denote by $\Lambda(P)$ its Iwasawa algebra over $\mathbb{Z}_p$ and by $\Omega(P)$ its Iwasawa algebra over $\mathbb{F}_p$. More specifically, $\Lambda(P)$ and $\Omega(P)$ are defined by

$$\Lambda(P) = \varprojlim_U \mathbb{Z}_p[P/U], \qquad \Omega(P) = \varprojlim_U \mathbb{F}_p[P/U]$$

where U runs over all open normal subgroups of P. Let Γ denote the commutative p-adic Lie group isomorphic to $\mathbb{Z}_p$ (corresponding to the Galois group of the cyclotomic $\mathbb{Z}_p$-extension). Throughout this paper, we fix a topological generator γ of Γ. In other words, we fix Iwasawa-Serre isomorphisms

$$\Lambda(\Gamma) \xrightarrow{\sim} \mathbb{Z}_p[[T]], \qquad \Omega(\Gamma) \xrightarrow{\sim} \mathbb{F}_p[[T]]$$
$$\gamma \mapsto 1+T \qquad\qquad \gamma \mapsto 1+T$$

where $\mathbb{Z}_p[[T]]$ (resp. $\mathbb{F}_p[[T]]$) is the formal power series ring over $\mathbb{Z}_p$ (resp. $\mathbb{F}_p$). For an arbitrary p-adic Lie group W isomorphic to the direct product of a finite p-group and Γ, W^f denotes the finite part of W. We always assume that every associative ring has a unit. The centre of an associative ring R is denoted by $Z(R)$. For an associative ring R, we denote by $\mathrm{M}_n(R)$ the ring of $n \times n$-matrices with entries in R and by $\mathrm{GL}_n(R)$ the multiplicative group of $\mathrm{M}_n(R)$. We always consider that all Grothendieck groups are *additive* abelian groups, whereas all Whitehead groups are *multiplicative* abelian groups. For an arbitrary multiplicative abelian group A, let $A_{p\text{-tors}}$ (resp. A_{tors}) denote the p-power torsion part (resp. the torsion part) of A. We set $\widetilde{K}_1(R) = K_1(R)/K_1(R)_{p\text{-tors}}$ for an arbitrary associative ring R. Similarly we set $\widetilde{\Lambda}(P)^\times = \Lambda(P)^\times/\Lambda(P)^\times_{p\text{-tors}}$ for an arbitrary pro-finite group P.

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1. Review of non-commutative Iwasawa theory

In this section we review the formulation of the non-commutative Iwasawa main conjecture for totally real number fields following Coates, Fukaya, Kato, Sujatha and Venjakob [CFKSV, FukKat]. We remark that Ritter and Weiss also formulated the non-commutative Iwasawa main conjecture —*the 'main conjecture' of equivariant Iwasawa theory* in their terminology— for p-adic Lie extensions of rank one in somewhat different manner [RW1, RW2,

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RW3, RW4]. Refer to [Bass, Swan] for basic results upon (low-dimensional) algebraic K-theory used in this section.

Let F be a totally real number field and p a positive odd prime number. Let F_∞ be a totally real p-adic Lie extension of F satisfying the following three conditions:

(F_∞-1) the cyclotomic $\mathbb{Z}_p$-extension F_{cyc} of F is contained in F_∞;
(F_∞-2) only finitely many primes of F ramify in F_∞;
(F_∞-3) there exists a finite subextension F' of F_∞/F such that F_∞/F' is pro-p and the Iwasawa μ-invariant for its cyclotomic $\mathbb{Z}_p$-extension F'_{cyc}/F' equals zero.

Fix a finite set Σ_F of finite places of F which contains all of those ramifying in F_∞. For an arbitrary algebraic extension E of F, we shall denote by Σ_E the set of all finite places of E above Σ_F and by $\Sigma_E^\vee$ the union of Σ_E and the set of all infinite places of E. Set $G = \mathrm{Gal}(F_\infty/F)$, $H = \mathrm{Gal}(F_\infty/F_{\mathrm{cyc}})$ and $\Gamma = \mathrm{Gal}(F_{\mathrm{cyc}}/F)$. The pro-$p$ group Γ is isomorphic to $\mathbb{Z}_p$ by definition.

Let S be the subset of $\Lambda(G)$ consisting of all elements f such that the quotient module $\Lambda(G)/\Lambda(G)f$ is finitely generated as a left $\Lambda(H)$-module. The set S is a left and right Ore set of $\Lambda(G)$ with no zero divisors [CFKSV, Theorem 2.4], which is called *the canonical Ore set for* F_∞/F (refer to [McRob, Stenström] for general theory of Ore localisation). The Ore localisation $\Lambda(G) \to \Lambda(G)_S$ induces the following localisation exact sequence in algebraic K-theory (due to Weibel, Yao, Berrick and Keating [WeibYao, BerKeat]):

$$K_1(\Lambda(G)) \to K_1(\Lambda(G)_S) \xrightarrow{\partial} K_0(\Lambda(G), \Lambda(G)_S) \to 0.$$

Surjectivity of the connecting homomorphism ∂ was proven in [CFKSV, Proposition 3.4].

Let $\mathscr{C}^{\mathrm{Perf}}(\Lambda(G))$ denote the category of perfect complexes of finitely generated left $\Lambda(G)$-modules (that is, the category of complexes of finitely generated left $\Lambda(G)$-modules which are quasi-isomorphic to bounded complexes of finitely generated *projective* left $\Lambda(G)$-modules), and let $\mathscr{C}_S^{\mathrm{Perf}}(\Lambda(G))$ denote the full subcategory of $\mathscr{C}^{\mathrm{Perf}}(\Lambda(G))$ generated by all objects whose cohomology groups are S-torsion left $\Lambda(G)$-modules. Note that the relative algebraic K-group $K_0(\Lambda(G), \Lambda(G)_S)$ is identified with the abelian group $K_0(\mathscr{C}_S^{\mathrm{Perf}}(\Lambda(G)))$ which we may define by modifying the construction of Fukaya and Kato in [FukKat, Definition 1.3.14].[3] Let us consider the complex $C_{F_\infty/F}$ defined by

$$C_{F_\infty/F} = R\mathrm{Hom}(R\Gamma_{\text{ét}}(\operatorname{Spec} \mathcal{O}_{F_\infty, \Sigma^\vee_{F_\infty}}, \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p)$$

where $\Gamma_{\text{ét}}$ denotes the global section functor for the étale topology and $\mathcal{O}_{F_\infty, \Sigma^\vee_{F_\infty}}$ denotes the $\Sigma^\vee_{F_\infty}$-integer ring of F_∞. Its cohomology groups are calculated as follows:

$$H^i(C_{F_\infty/F}) = \begin{cases} \mathbb{Z}_p & \text{if } i = 0, \\ X_{\Sigma_{F_\infty}} & \text{if } i = -1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

[3] It might be more natural to consider such identification in Waldhausen K-theory.

Here $X_{\Sigma_{F_\infty}} = \mathrm{Gal}(M_{\Sigma_{F_\infty}}/F_\infty)$ is the Galois group of the maximal abelian pro-p extension $M_{\Sigma_{F_\infty}}$ of F_∞ unramified outside Σ_{F_∞}. Note that $\mathbb{Z}_p$ is an S-torsion module since it is finitely generated as a left $\Lambda(H)$-module (see [CFKSV, Proposition 2.3] for details). The Galois group $X_{\Sigma_{F_\infty}}$ is also an S-torsion module by condition (F_∞-3) (use the universal coefficient spectral sequence and Nakayama-Azumaya-Krull's Lemma). Therefore we may regard $C_{F_\infty/F}$ as an object of $\mathscr{C}_S^{\mathrm{Perf}}(\Lambda(G))$, and by surjectivity of ∂ there exists an element $f_{F_\infty/F}$ in $K_1(\Lambda(G)_S)$ satisfying

$$\partial(f_{F_\infty/F}) = -[C_{F_\infty/F}], \tag{1.2}$$

which is called *a characteristic element for F_∞/F*. Characteristic elements are determined uniquely up to multiplication by elements in the image of the canonical homomorphism $K_1(\Lambda(G)) \to K_1(\Lambda(G)_S)$ (due to the localisation exact sequence).

We next consider the p-adic zeta function for F_∞/F. From now on we fix an algebraic closure of the p-adic number field $\overline{\mathbb{Q}}_p$, and we also fix embeddings of the algebraic closure $\overline{\mathbb{Q}}$ of the rational number field $\mathbb{Q}$ into $\mathbb{C}$ —the complex number field— and $\overline{\mathbb{Q}}_p$ till the end of this section. By the argument in [CFKSV, p.p. 172–173], we may define *the evaluation map*

$$K_1(\Lambda(G)_S) \to \overline{\mathbb{Q}}_p \cup \{\infty\}; f \mapsto f(\varrho)$$

for an arbitrary continuous representation $\varrho\colon G \to \mathrm{GL}_d(\mathcal{O})$ (where $\mathcal{O}$ is the ring of integers of a certain finite extension of $\mathbb{Q}_p$). Now let $L_{\Sigma_F}(s; F_\infty/F, \rho)$ be the complex Artin L-function associated to an Artin representation ρ for G (recall that ρ is an Artin representation if its image is finite) whose local factors at places belonging to Σ_F are removed. If there exists an element $\xi_{F_\infty/F}$ in $K_1(\Lambda(G)_S)$ which satisfies *the interpolation formula*

$$\xi_{F_\infty/F}(\rho\kappa^r) = L_{\Sigma_F}(1-r; F_\infty/F, \rho) \tag{1.3}$$

for an arbitrary Artin representation ρ of G and an arbitrary natural number r divisible by $p-1$, we call $\xi_{F_\infty/F}$ *the p-adic zeta function for F_∞/F*. The Iwasawa main conjecture for totally real number fields is then formulated as follows:

Conjecture 1.1. *Let p, F and F_∞/F be as above. Then*

(1) (*existence of the p-adic zeta function*)
the p-adic zeta function $\xi_{F_\infty/F}$ for F_∞/F exists;

(2) (*the Iwasawa main conjecture*)
the equation $\partial(\xi_{F_\infty/F}) = -[C_{F_\infty/F}]$ holds.

2. Burns' technique

There exists a standard strategy to construct the p-adic zeta functions for non-commutative extensions by 'patching' Serre's p-adic zeta pseudomeasures for abelian extensions. It was first observed by Burns and applied by Kato to his pioneering work [Kato] (see also [Burns]). Here we shall introduce this outstanding technique in a little generalised way.

Throughout this section we fix embeddings of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$. Let p be a positive odd prime number, F a totally real number field and F_∞/F a totally real p-adic Lie extension satisfying conditions (F_∞-1), (F_∞-2) and

(F_∞-3) in the previous section. Let G, H and Γ be p-adic Lie groups defined as in Section 1.

Definition 2.1 (Brauer families). Let $\mathfrak{F}_B$ be a family consisting of pairs (U, V) where U is an open subgroup of G and V is that of H such that V is normal in U and the quotient group U/V is abelian. We call $\mathfrak{F}_B$ *a Brauer family for the group* G if it satisfies the following condition $(\sharp)_B$:

> $(\sharp)_B$ an arbitrary Artin representation of G is isomorphic to *a $\mathbb{Z}$-linear combination* (as a virtual representation) of induced representations $\mathrm{Ind}_U^G(\chi_{U/V})$, where each (U, V) is an element in $\mathfrak{F}_B$ and $\chi_{U/V}$ is a character of finite order of the abelian group U/V.

Suppose that there exists a Brauer family $\mathfrak{F}_B$ for G. Let $\theta_{U,V}$ be the composition

$$K_1(\Lambda(G)) \xrightarrow{\mathrm{Nr}_{\Lambda(G)/\Lambda(U)}} K_1(\Lambda(U)) \xrightarrow{\text{canonical}} K_1(\Lambda(U/V)) \xrightarrow{\sim} \Lambda(U/V)^\times$$

for each (U, V) in $\mathfrak{F}_B$ where $\mathrm{Nr}_{\Lambda(G)/\Lambda(U)}$ is the norm map in algebraic K-theory. Set

$$\theta = (\theta_{U,V})_{(U,V)\in\mathfrak{F}_B} : K_1(\Lambda(G)) \to \prod_{(U,V)\in\mathfrak{F}_B} \Lambda(U/V)^\times.$$

Similarly we may construct the map[4]

$$\theta_S = (\theta_{S,U,V})_{(U,V)\in\mathfrak{F}_B} : K_1(\Lambda(G)_S) \to \prod_{(U,V)\in\mathfrak{F}_B} \Lambda(U/V)_S^\times$$

for the localised Iwasawa algebra $\Lambda(G)_S$. Then we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
K_1(\Lambda(G)) & \longrightarrow & K_1(\Lambda(G)_S) & \xrightarrow{\partial} & K_0(\Lambda(G), \Lambda(G)_S) & \longrightarrow & 0 \\
\downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta_S} & & \downarrow{\scriptstyle\text{norm}} & & \\
\prod_{\mathfrak{F}_B} \Lambda(U/V)^\times & \hookrightarrow & \prod_{\mathfrak{F}_B} \Lambda(U/V)_S^\times & \xrightarrow[\partial]{} & \prod_{\mathfrak{F}_B} K_0(\Lambda(U/V), \Lambda(U/V)_S) & \longrightarrow & 0.
\end{array}$$

Let f be an arbitrary characteristic element for F_∞/F (that is, an element in $K_1(\Lambda(G)_S)$ satisfying the relation (1.2)) and $(f_{U,V})_{(U,V)\in\mathfrak{F}_B}$ its image under the map θ_S. Then each $f_{U,V}$ satisfies the relation $\partial(f_{U,V}) = -[C_{U,V}]$ by the functoriality of the connecting homomorphism ∂. Now recall that for each pair (U, V) in $\mathfrak{F}_B$, the p-adic zeta pseudomeasure $\xi_{U,V}$ for the abelian extension F_V/F_U exists as an element in $\Lambda(U/V)_S^\times$ (see [Serre2] for details) and the Iwasawa main conjecture $\partial(\xi_{U,V}) = -[C_{U,V}]$ holds (due to the wonderful results of Wiles [Wiles]). Each p-adic zeta pseudomeasure $\xi_{U,V}$ is characterised by the interpolation formula

$$\xi_{U,V}(\chi\kappa^r) = L_{\Sigma_{F_U}}(1-r; F_V/F_U, \chi) \tag{2.1}$$

for an arbitrary character of finite order of the abelian group U/V and an arbitrary natural number r divisible by $p-1$. Let $w_{U,V}$ be the element defined as $\xi_{U,V} f_{U,V}^{-1}$ which is in fact an element in $\Lambda(U/V)^\times$ by the localisation exact sequence. Here we further consider the following assumption:

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[4] We use the same symbol S for the canonical Ore set for F_V/F_U by abuse of notation.

Assumption ($\mathfrak{b}$) the element $(w_{U,V})_{(U,V)\in\mathfrak{F}_B}$ is contained in the image of θ.

Under Assumption ($\mathfrak{b}$) there exists an element w in $K_1(\Lambda(G))$ which satisfies $\theta(w) = (w_{U,V})_{(U,V)\in\mathfrak{F}_B}$. Let ξ be the element in $K_1(\Lambda(G)_S)$ defined as fw. Then ξ readily satisfies the following two conditions by easy diagram chasing:

(ξ-1) the equation $\partial(\xi) = -[C_{F_\infty/F}]$ holds;

(ξ-2) the equation $\theta_S(\xi) = (\xi_{U,V})_{(U,V)\in\mathfrak{F}_B}$ holds.

By using condition $(\sharp)_B$, condition (ξ-2) and the interpolation formulae (2.1), we may verify that ξ satisfies the interpolation formula (1.3) as follows:

$$\begin{aligned}
\xi(\rho\kappa^r) &= \xi\left(\kappa^r \sum_{(U,V)\in\mathfrak{F}_B} a_{U,V}\mathrm{Ind}_U^G(\chi_{U/V})\right) \qquad \text{(by } (\sharp)_B)\\
&= \prod_{(U,V)\in\mathfrak{F}_B} \mathrm{Nr}_{\Lambda(G)_S/\Lambda(U)_S}(\xi)(\chi_{U/V}\kappa^r)^{a_{U,V}}\\
&= \prod_{(U,V)\in\mathfrak{F}_B} \xi_{U,V}(\chi_{U/V}\kappa^r)^{a_{U,V}} \qquad \text{(by } (\xi\text{-}2))\\
&= \prod_{(U,V)\in\mathfrak{F}_B} L_{\Sigma_{F_U}}(1-r; F_V/F_U, \chi_{U/V})^{a_{U,V}} \qquad \text{(by (2.1))}\\
&= L_{\Sigma_F}(1-r; F_\infty/F, \sum_{(U,V)\in\mathfrak{F}_B} a_{U,V}\mathrm{Ind}_U^G(\chi_{U/V}))\\
&= L_{\Sigma_F}(1-r; F_\infty/F, \rho)
\end{aligned}$$

where ρ is an arbitrary Artin representation of G and r is an arbitrary natural number divisible by $p-1$. Therefore ξ is the desired p-adic zeta function for F_∞/F. Furthermore (ξ-1) implies that ξ is also a characteristic element for F_∞/F; in other words Conjecture 1.1 holds for F_∞/F.

By virtue of Burns' technique, both construction of the p-adic zeta function and verification of the Iwasawa main conjecture are reduced to the following two tasks:

- characterisation of the images of θ and θ_S;
- verification of Assumption ($\mathfrak{b}$).

In general, there are so many pairs in a Brauer family $\mathfrak{F}_B$ that it is hard to compute and characterise the images of the norm maps θ and θ_S. Therefore we shall use not only Brauer families but also *Artinian families* in arguments of the rest of this article;

Definition 2.2 (Artinian families). If a family $\mathfrak{F}_A$ consisting of abelian open subgroups of G satisfies the following condition $(\sharp)_A$, we call $\mathfrak{F}_A$ *an Artinian family for the group G*:

$(\sharp)_A$ an arbitrary Artin representation of G is isomorphic to *a $\mathbb{Z}[1/p]$-linear combination* (as a virtual representation) of induced representations $\mathrm{Ind}_U^G(\chi_U)$, where each U is an element in $\mathfrak{F}_A$ and χ_U is a character of finite order of the abelian group U.

Artinian families tend to contain much fewer and simpler elements than Brauer families, by virtue of which we may often avoid hard computation.

3. The main theorem and its application

3.1. The main theorem. The precise statement of the main result in this article is as follows. Here we continuously fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

Theorem 3.1 (Main theorem). *Let p be a positive odd prime number, F a totally real number field and F_∞ a totally real p-adic Lie extension of F satisfying conditions (F_∞-1), (F_∞-2) and (F_∞-3) in Section 1. Suppose that the Galois group of F_∞/F is isomorphic to the direct product of a finite p-group G^f of exponent p and the commutative p-adic Lie group Γ. Then the p-adic zeta function $\xi_{F_\infty/F}$ for F_∞/F exists uniquely up to multiplication by an element in $SK_1(\mathbb{Z}_p[G^f])$. Moreover, the Iwasawa main conjecture (Conjecture 1.1 (2)) is true for F_∞/F.*

Remark 3.2. If the group $SK_1(\mathbb{Z}_p[G^f])$ is a trivial group, we may establish the uniqueness assertion of the p-adic zeta function $\xi_{F_\infty/F}$ in Theorem 3.1. However there exists finite groups G^f for which the SK_1-group $SK_1(\mathbb{Z}_p[G^f])$ does not vanish (see [Oliver, Example 8.11] for instance), and it is easy to show that the image of $SK_1(\mathbb{Z}_p[G^f])$ in $K_1(\Lambda(G)_S)$ never vanishes unless $SK_1(\mathbb{Z}_p[G^f])$ itself is trivial. Therefore as an element in $K_1(\Lambda(G)_S)$ the uniqueness assertion of the p-adic zeta function $\xi_{F_\infty/F}$ does not hold in general. Nevertheless the image of $\xi_{F_\infty/F}$ in $K_1(\Lambda(G)_S[1/p])$ is unique in our case since $SK_1(\mathbb{Q}_p[G^f])$ is trivial (see for instance [Oliver, Theorem 2.3]) and the natural map $SK_1(\mathbb{Z}_p[G^f]) \to K_1(\Lambda(G)_S[1/p])$ factors through $SK_1(\mathbb{Q}_p[G^f])$.

Now let us consider an easy but interesting application of our main theorem. Let $B^N(\mathbb{F}_p)$ be a multiplicative p-group defined as the subgroup of the general linear group $\mathrm{GL}_{N+1}(\mathbb{F}_p)$ of degree $N+1$ generated by all strictly upper-triangular matrices; that is,

$$B^N(\mathbb{F}_p) = \begin{array}{cc} & \begin{matrix} 1 & 2 & \dots & \dots & N & N+1 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ \vdots \\ \vdots \\ N \\ N+1 \end{matrix} & \begin{pmatrix} 1 & \mathbb{F}_p & \mathbb{F}_p & \dots & \mathbb{F}_p & \mathbb{F}_p \\ 0 & 1 & \mathbb{F}_p & \dots & & \mathbb{F}_p \\ 0 & 0 & 1 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & & & & 1 & \mathbb{F}_p \\ 0 & 0 & \dots & \dots & 0 & 1 \end{pmatrix} \end{array}.$$

Corollary 3.3. *Let p be a positive odd prime number, F a totally real number field and F_∞/F a totally real p-adic Lie extension satisfying conditions (F_∞-1), (F_∞-2) and (F_∞-3) in Section 1. Assume also that*

(i) *there exists a certain non-negative integer N such that the Galois group of F_∞/F is isomorphic to the direct product of $B^N(\mathbb{F}_p)$ and the commutative p-adic Lie group Γ;*
(ii) *the prime number p is strictly greater than N.*

Then the p-adic zeta function $\xi_{F_\infty/F}$ for F_∞/F exists uniquely and the Iwasawa main conjecture (Conjecture 1.1 (2)) is true for F_∞/F.

Proof. For each p strictly greater than N, the exponent of $B^N(\mathbb{F}_p)$ equals p. Therefore the claim is directly deduced from Theorem 3.1. Note that in this case $SK_1(\mathbb{Z}_p[B^N(\mathbb{F}_p)])$ is trivial by the results of Schneider and Venjakob [SchVen, Proposition 4.1].[5] □

Remark 3.4. The case where N is equal to 2 in Corollary 3.3 is a special case of Kato's Heisenberg-type extensions [Kato]. The case where N is equal to 3 is nothing but the main results of the preceding paper of the author [H]. The original motivation for this study was to generalise these results to the cases where N is greater than or equal to 4. It became, however, clear that it was convenient to consider the problem under more general conditions as in Theorem 3.1.

3.2. **Application to the equivariant Tamagawa number conjecture for critical Tate motives.** In this subsection, we shall show that the p-part of the (non-commutative) equivariant Tamagawa number conjecture for critical Tate motives follows from the Iwasawa main conjecture (Conjecture 1.1) by applying standard descent arguments of Burns and Venjakob [BurVen]. Refer to [BurFl3] for terminologies regarding the equivariant Tamagawa number conjecture. We often abbreviate an affine scheme Spec $\mathcal{O}$ to just $\mathcal{O}$ if there is no risk of confusion in the following arguments.

Proposition 3.5. *Let p be a positive odd prime number and F a totally real number field. Let F_∞ be a totally real p-adic Lie extension of F satisfying conditions (F_∞-1), (F_∞-2) and (F_∞-3) in Section 1. Assume that Conjectures 1.1 (1), (2) are true for F^∞/F. Then the p-part of the equivariant Tamagawa number conjecture for $\mathbb{Q}(1-r)_{F'/F}$ is true for an arbitrary finite Galois subextension F' of F_∞/F and an arbitrary natural number r divisible by $p-1$.*

Note that the Tate motive $\mathbb{Q}(1-r)_{F'/F}$ is *critical* in the sense of Deligne [Deligne1, Definition 1.3] since both F and F' are totally real and r is even. Combining this proposition with Theorem 3.1, we obtain:

Corollary 3.6. *Let p, F and F_∞/F be as in Proposition 3.5. Suppose that the Galois group of F_∞/F is isomorphic to the direct product of a finite p-group G^f of exponent p and the commutative p-adic Lie group Γ. Then the p-part of the equivariant Tamagawa number conjecture for $\mathbb{Q}(1-r)_{F'/F}$ is true for an arbitrary finite Galois subextension F' of F_∞/F and an arbitrary natural number r divisible by $p-1$.*

This corollary gives a simple but non-trivial example strongly suggesting validity of the equivariant Tamagawa number conjecture for motives *with non-commutative coefficient.* Proposition 3.5 is just the direct consequence of the Iwasawa main conjecture and descent theory established by Burns and Venjakob [BurVen], and all materials used in the proof should be essentially contained in [BurVen]. There, however, does not seem to exist explicit

[5]We also remark that Oliver had already proved the triviality of $SK_1(\mathbb{Z}_p[B^2(F_p)])$. Refer to [Oliver, Proposition 12.7].

suggestion upon critical Tate motives there, and thus we shall give the proof of Proposition 3.5 in the rest of this subsection. Set $G_{F'/F} = \mathrm{Gal}(F'/F)$.

Remark 3.7. If we take $F' = F$, Proposition 3.5 is equivalent to the p-part of the *cohomological Lichtenbaum conjecture*

$$|\zeta_F(1-r)|_p^{-1} = \frac{|\sharp H^2_{c,\text{ét}}(\mathcal{O}_{F,\Sigma_F^\vee}, \mathbb{Z}_p(1-r))|_p^{-1}}{|\sharp H^1_{c,\text{ét}}(\mathcal{O}_{F,\Sigma_F^\vee}, \mathbb{Z}_p(1-r))|_p^{-1}}$$

via certain specialisation (here $\zeta_F(s)$ is the complex Dedekind zeta function for F and $|\cdot|_p$ is the p-adic valuation normalised by $|p|_p = 1/p$). This is directly deduced from the (classical) Iwasawa main conjecture for totally real number fields verified by Wiles [Wiles]. Proposition 3.5 gives its certain generalisation for cases with non-commutative coefficient.

Remark 3.8. If $F'/\mathbb{Q}$ is a finite abelian Galois extension and F is a subfield of F', the equivariant Tamagawa number conjecture for the Tate motives $\mathbb{Q}(m)_{F'/F}$ has been proven for an arbitrary prime number p and an arbitrary integer m (not necessarily negative) by Burns, Greither and Flach. Refer to [BurGr1] (for negative m and odd p), [Flach] (for negative m and arbitrary p) and [BurFl4] (for arbitrary m and p). Independently Huber, Kings and Itakura have also proven the Bloch-Kato conjecture [BlKat] for Dirichlet motives —a somewhat weaker conjecture than the equivariant Tamagawa number conjecture— by using rather different technique. See [HubKin2] (for $p \neq 2$) and [Itakura] (for $p = 2$) for details.

Remark 3.9. If F is a totally real number field and K/F is a finite abelian CM-extension, Burns and Greither proved the p-part of the equivariant Tamagawa number conjecture for the pair $(\mathbb{Q}(1-r)_{K/F}, \mathbb{Z}_p[G_{K/F}]e_r)$ under some mild conditions on p [BurGr2, Theorem 5.2, (8)] where r is an arbitrary natural number and e_r is an idempotent of $\mathbb{Z}_p[G_{K/F}]$ defined as $(1+(-1)^r c)/2$ (here c denotes the complex conjugation). If r is even, we may deduce the p-part of the equivariant Tamagawa number conjecture for $\mathbb{Q}(1-r)_{F'/F}$ (here $F' = K^+$ denotes the maximal totally real subfield of K)

$$\det\nolimits^{-1}_{\mathbb{Z}_p[G_{F'/F}]}(R\Gamma_{c,\text{ét}}(\mathcal{O}_{F',\Sigma_{F'}^\vee}, \mathbb{Z}_p(1-r))) = L_{\Sigma_F}(1-r) \cdot \mathbb{Z}_p[G_{F'/F}] \tag{3.1}$$

by standard Galois descent arguments (refer to [BurGr2, p. 173] for the definition of $L_{\Sigma_F}(1-r)$). Proposition 3.5 is a natural non-commutative generalisation of their result; in particular the diagram (3.6) may be regarded as a non-commutative generalisation of (3.1).

We now begin the proof of Proposition 3.5. First note that the fundamental line [BurFl3, (23)]

$$\Xi(\mathbb{Q}(1-r)_{F'/F}) = \det\nolimits_{\mathbb{Q}[G_{F'/F}]}(K_{2r-1}(F')^*_{\mathbb{Q}}) \cdot \det\nolimits^{-1}_{\mathbb{Q}[G_{F'/F}]}(\mathbb{Q}(1-r)^+_{F'/F,B})$$

is trivial (here 'det' denotes Deligne's determinant functor [Deligne2]) since both the $+$-part of Betti realization $\mathbb{Q}(1-r)^+_{F'/F,B}$ and the rational K-group $K_{2r-1}(F')_{\mathbb{Q}}$ are trivial (and hence the $\mathbb{Q}$-dual $K_{2r-1}(F')^*_{\mathbb{Q}}$ is also trivial): this follows from the fact that F and F' are totally real and r is even (for the triviality of $K_{2r-1}(F')_{\mathbb{Q}}$ we use Borel's computation [Borel1, Borel2]). Therefore the period-regulator map [BurFl3, p. 529]

$$\vartheta_\infty \colon \Xi(\mathbb{Q}(1-r)_{F'/F})_{\mathbb{R}} \xrightarrow{\sim} \mathbf{1}_{\mathbb{R}[G_{F'/F}]}$$

degenerates to the identity map on the unit object $\mathbf{1}_{\mathbb{R}[G_{F'/F}]}$ of Deligne's category of virtual objects $V(\mathbb{R}[G_{F'/F}])$ [Deligne2], which reduces the rationality conjecture [BurFl3, Conjecture 5] to the rationality of the leading term of the equivariant Artin L-function $L^*(\mathbb{Q}(1-r)_{F'/F})$ (which is denoted by $L^*({}_{\mathbb{Q}[G_{F'/F}]}\mathbb{Q}(1-r)_{F'/F}, 0)$ in [BurFl3, p. 533]): this can be easily verified by the same type of arguments as used in the proof of [Deligne1, Proposition 6.7] (use Klingen-Siegel's theorem [Klingen, Siegel], see also [CL, Section 1.1]).

Under the isomorphism [BrBur, Lemma 5.1]

$$K_0(\mathbb{Z}_p[G_{F'/F}], \overline{\mathbb{Q}}_p[G_{F'/F}]) \xrightarrow{\sim} \pi_0(V(\mathbb{Z}_p[G_{F'/F}], \overline{\mathbb{Q}}_p[G_{F'/F}])), \tag{3.2}$$

we regard (the isomorphism class of) a pair $[E, a]$ as an element of the relative K-group $K_0(\mathbb{Z}_p[G_{F'/F}], \overline{\mathbb{Q}}_p[G_{F'/F}])$ where E is a virtual object of $V(\mathbb{Z}_p[G_{F'/F}])$ and a is a trivialization

$$a \colon E_{\overline{\mathbb{Q}}_p} = E \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbf{1}_{\overline{\mathbb{Q}}_p[G_{F'/F}]}$$

in $V(\overline{\mathbb{Q}}_p[G_{F'/F}])$. In order to verify the integrality conjecture [BurFl3, Conjecture 6], it suffices to prove that

(3.3)
$$\begin{aligned}\bar{\partial}_p(&\mathrm{nrd}^{-1}_{\overline{\mathbb{Q}}_p[G_{F'/F}]}(L^*(\mathbb{Q}(1-r)_{F'/F}))\\ &+ [\det_{\mathbb{Z}_p[G_{F'/F}]}(R\Gamma_{c,\text{ét}}(\mathcal{O}_{F',\Sigma^\vee_{F'}}), \mathbb{Z}_p(1-r)), \vartheta^{-1}_{p,\overline{\mathbb{Q}}_p}] = 0\end{aligned}$$

holds as an equation in $K_0(\mathbb{Z}_p[G_{F'/F}], \overline{\mathbb{Q}}_p[G_{F'/F}])$; here

$$\mathrm{nrd}_{\overline{\mathbb{Q}}_p[G_{F'/F}]} \colon K_0(\overline{\mathbb{Q}}_p[G_{F'/F}]) \to Z(\overline{\mathbb{Q}}_p[G_{F'/F}])^\times$$

denotes the reduced norm map which is in fact bijective (see [CR, Theorem (45.3)]),

$$\bar{\partial}_p \colon K_1(\overline{\mathbb{Q}}_p[G_{F'/F}]) \to K_0(\mathbb{Z}_p[G_{F'/F}], \overline{\mathbb{Q}}_p[G_{F'/F}])$$

denotes the connecting homomorphism and $\vartheta_{p,\overline{\mathbb{Q}}_p}$ denotes the scalar extension from $\mathbb{Q}_p$ to $\overline{\mathbb{Q}}_p$ of the p-adic period-regulator map [BurFl3, p. 526]

$$\vartheta_p \colon \Xi(\mathbb{Q}(1-r)_{F'/F})_{\mathbb{Q}_p} \xrightarrow{\sim} \det_{\mathbb{Q}_p[G_{F'/F}]}(R\Gamma_{c,et}(\mathcal{O}_{F',\Sigma^\vee_{F'}}, \mathbb{Q}_p(1-r)))$$

(recall that $\Xi(\mathbb{Q}(1-r)_{F'/F})$ is now trivial). Indeed we may easily check that the left hand side of (3.3) is the image of $T\Omega(\mathbb{Q}(1-r)_{F'/F}, \mathbb{Z}_p[G_{F'/F}])$ defined in [BurFl3, Conjecture 6] under the canonical map induced by the embedding $\mathbb{Q}_p \hookrightarrow \overline{\mathbb{Q}}_p$. Note that the leading term of the equivariant Artin L-function $L^*(\mathbb{Q}(1-r)_{F'/F})$ as an element in $Z(\overline{\mathbb{Q}}_p[G_{F'/F}])^\times$ is identified with a set of special values of ($\overline{\mathbb{Q}}_p$-valued) Artin L-functions $(L(1-r, j_p\rho))_{\rho\in\mathrm{Irr}(G_{F'/F})}$ for an embedding $j_p\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ via the Wedderburn decomposition

$$Z(\overline{\mathbb{Q}}_p[G_{F'/F}])^\times \xrightarrow{\sim} \prod_{\rho\in \mathrm{Irr}(G_{F'/F})} \overline{\mathbb{Q}}_p^\times \tag{3.4}$$

where $\mathrm{Irr}(G_{F'/F})$ denotes the set of all isomorphism classes of ($\overline{\mathbb{Q}}$-valued) irreducible Artin representations of $G_{F'/F}$.

Proof of Proposition 3.5. Let $\xi_{F_\infty/F}$ be the p-adic zeta function for F_∞/F and assume that the Iwasawa main conjecture $\partial(\xi_{F_\infty/F}) = -[C_{F_\infty/F}]$ is valid (for an arbitrary embedding $j_p\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$). Since $R\Gamma_{\text{ét}}(\mathcal{O}_{F_\infty,\Sigma^\vee_{F_\infty}}, \mathbb{Q}_p/\mathbb{Z}_p)$ is identified with the injective limit of complexes $R\Gamma_{\text{ét}}(\mathcal{O}_{L,\Sigma^\vee_L}, \mathbb{Q}_p/\mathbb{Z}_p)$ for all finite Galois subextensions L/F of F^∞/F, we may easily see that $C_{F_\infty/F}$ is isomorphic to the complex $\varprojlim_L R\Gamma_{\mathrm{c,ét}}(\operatorname{Spec}\mathcal{O}_{L,\Sigma^\vee_L}, \mathbb{Z}_p(1))[3]$ by virtue of Poitou-Tate/Artin-Verdier duality theorem. Furthermore for each L the complex $R\Gamma_{\mathrm{c,ét}}(\mathcal{O}_{L,\Sigma^\vee_L}, \mathbb{Z}_p(1))$ is isomorphic to $R\Gamma_{\mathrm{c,ét}}(\mathcal{O}_{F,\Sigma^\vee_F}, \mathbb{Z}_p[G_{L/F}]^\sharp(1))$ by Shapiro's lemma (here $\mathbb{Z}_p[G_{L/F}]^\sharp$ denotes $\mathbb{Z}_p[G_{L/F}]$ regarded as a left G_F-module upon which an element σ in G_F acts by the right multiplication of its inverse σ^{-1}). Hence the following equation holds in $K_0(\Lambda(G), \Lambda(G)_S)$:

$$\begin{aligned}\partial(\xi_{F_\infty/F}) &= [\varprojlim_L R\Gamma_{\mathrm{c,ét}}(\operatorname{Spec}\mathcal{O}_{F,\Sigma^\vee_F}, \mathbb{Z}_p[G_{L/F}]^\sharp(1))] \\ &= [R\Gamma_{\mathrm{c,ét}}(\operatorname{Spec}\mathcal{O}_{F,\Sigma^\vee_F}, \Lambda(G)^\sharp(1))].\end{aligned}$$

Now for each natural number r divisible by $p-1$ consider the $\mathbb{Z}_p$-linear map $\mathrm{tw}^r_\kappa\colon \Lambda(G) \to \Lambda(G)$ induced by $\sigma \mapsto \kappa^r(\sigma)\sigma$ for σ in G, which is in fact a ring automorphism because $\kappa^r(\sigma)$ is an element in the centre of $\Lambda(G)$. This map also induces a ring automorphism $\mathrm{tw}^r_{S,\kappa}$ on the canonical Ore localisation $\Lambda(G)_S$ of $\Lambda(G)$. Moreover the composition of $\mathrm{tw}^r_{S,\kappa}$ with the homomorphism $\Lambda(G)_S \to \mathrm{Frac}(\Lambda(\Gamma))$ induced by the projection $G \to \Gamma$ coincides with the morphism Φ_{κ^r} introduced in [CFKSV, Lemma 3.3]. Hence the definition of the evaluation map asserts that the interpolation formula

$$\mathrm{tw}^r_{S,\kappa}(\xi_{F_\infty/F})(\rho) = L_{\Sigma_F}(1-r; F_\infty/F, \rho)$$

holds for an arbitrary Artin representation ρ of G. On the other hand the Tate twist $\Lambda(G)^\sharp(1) \to \Lambda(G)^\sharp(1-r)$ defines a $\mathrm{tw}^r_{S,\kappa}$-semilinear isomorphism (due to the $\Lambda(G)$-module structure of $\Lambda(G)^\sharp$), and hence there is a canonical isomorphism

$$\Lambda(G)\otimes_{\Lambda(G),\mathrm{tw}^r_\kappa} R\Gamma_{\mathrm{c,ét}}(\mathcal{O}_{F,\Sigma^\vee_F}, \Lambda(G)^\sharp(1)) \xrightarrow{\sim} R\Gamma_{\mathrm{c,ét}}(\mathcal{O}_{F,\Sigma^\vee_F}, \Lambda(G)^\sharp(1-r))$$

in the derived category of $\mathscr{C}^{\mathrm{Perf}}(\Lambda(G))$. Then we obtain

$$\partial(\mathrm{tw}^r_{S,\kappa}(\xi_{F_\infty/F})) = [R\Gamma_{\mathrm{c,ét}}(\mathcal{O}_{F,\Sigma^\vee_F}, \Lambda(G)^\sharp(1-r))] \tag{3.5}$$

by the functoriality of the connecting homomorphism. The descent theory of Burns and Venjakob [BurVen, Theorem 2.2] asserts that the equation (3.5) descends to

(3.6)

$$\begin{array}{ccc} K_1(\overline{\mathbb{Q}}_p[G_{F'/F}]) & \xrightarrow{\partial_p} & K_0(\mathbb{Z}_p[G_{F'/F}], \overline{\mathbb{Q}}_p[G_{F'/F}]) \\ \mathrm{nrd}^{-1}_{\overline{\mathbb{Q}}_p[G_{F'/F}]}(L^*_{\Sigma_F}(\mathbb{Q}(1-r)_{F'/F})) & \mapsto & [R\Gamma_{c,\text{ét}}(\mathcal{O}_{F',\Sigma^\vee_{F'}}, \mathbb{Z}_p(1-r))] \end{array}$$

where $L^*_\Sigma(\mathbb{Q}(1-r)_{F'/F})$ denotes an element corresponding to special values of Σ_F-truncated Artin L-functions $(L_{\Sigma_F}(1-r, j_p\rho))_{\rho\in\mathrm{Irr}(G_{F'/F})}$ via the Wedderburn decomposition (3.4); this is because

$$\begin{aligned}&\mathbb{Z}_p[G_{F'/F}]\otimes^{\mathbb{L}}_{\Lambda(G)} R\Gamma_{c,\text{ét}}(\mathcal{O}_{F,\Sigma_F^\vee}, \Lambda(G)^\sharp(1-r))\\ &= R\Gamma_{c,\text{ét}}(\mathcal{O}_{F,\Sigma_F^\vee}, \mathbb{Z}_p[G_{F'/F}]^\sharp(1-r)) = R\Gamma_{c,\text{ét}}(\mathcal{O}_{F',\Sigma_{F'}^\vee}, \mathbb{Z}_p(1-r))\end{aligned}$$

holds for an arbitrary finite Galois subextension F'/F of F_∞/F and the localisation $R\Gamma_{c,\text{ét}}(\mathcal{O}_{F',\Sigma_{F'}^\vee}, \mathbb{Z}_p(1-r))_{\mathbb{Q}_p}$ is acyclic (essentially due to the criticalness of $\mathbb{Q}(1-r)_{F'/F}$; refer to [BurFl2, (9),(10)]).

The element $[R\Gamma_{c,\text{ét}}(\mathcal{O}_{F',\Sigma_{F'}^\vee}, \mathbb{Z}_p(1-r))]$ in $K_0(\mathbb{Z}_p[G_{F'/F}], \overline{\mathbb{Q}}_p[G_{F'/F}])$ corresponds to $-[\det_{\mathbb{Z}_p[G_{F'/F}]}(R\Gamma_{c,\text{ét}}(\mathcal{O}_{F',\Sigma_{F'}^\vee}, \mathbb{Z}_p(1-r))), \mathrm{acyc}]$ via (3.2) where 'acyc' denotes the natural trivialisation induced by acyclicity of the complex $R\Gamma_{c,\text{ét}}(\mathcal{O}_{F',\Sigma_{F'}^\vee}, \overline{\mathbb{Q}}_p(1-r))$ (see Remark 3.10 for sign convention concerning the normalisation of the relative K-group). The difference between two trivialisations $\vartheta^{-1}_{p,\overline{\mathbb{Q}}_p}$ and 'acyc' was calculated in [BurFl2] as

$$\mathrm{acyc} = \vartheta^{-1}_{p,\overline{\mathbb{Q}}_p}\cdot \prod_{v\in\Sigma_F}\phi_v^{-1}$$

where each $\overline{\mathbb{Q}}_p$-isomorphism $\phi_v\colon V\to V$ is defined as in [BurFl1, Section 1.2] or [FukKat, Section 2.4.2] which we regard as an element in $K_1(\overline{\mathbb{Q}}_p[G_{F'/F}])$. Then, by definition, the image of $\mathrm{nrd}_{\overline{\mathbb{Q}}_p[G_{F'/F}]}(\phi_v^{-1})$ under the Wedderburn decomposition (3.4) coincides with $(L_v(1-r, j_p\rho))_{\rho\in\mathrm{Irr}(G_{F'/F})}$, the local factors of Artin L-functions at v. Combining this fact with the relation (3.6), we can easily obtain the desired result (3.3). □

Remark 3.10 (sign convention). Let R be an associative ring and S a left Ore subset of R. We let $S^{-1}R$ denote the left Ore localisation of R with respect to S. In [Swan], the relative algebraic K-group $K_0(R, S^{-1}R)$ is defined as a certain quotient of the free abelian group generated by all triples $[P,\lambda,Q]$ where each P and Q are finitely generated projective left R-modules and λ is an $S^{-1}R$-isomorphism $\lambda\colon S^{-1}R\otimes_R P \xrightarrow{\sim} S^{-1}R\otimes_R Q$. Then we may identify the homomorphism $P\to Q$ induced by λ with a cochain complex concentrated in terms of degree 0 and 1, and we use this identification as a normalisation of the isomorphism between $K_0(R,S^{-1}R)$ and $K_0(\mathscr{C}_S^{\mathrm{Perf}}(R))$ (this normalisation is the same one as used in [FukKat]). In [BurVen], however, they identify $K_0(R,S^{-1}R)$ with $\pi_0(V(R,S^{-1}R))$ in the following manner: when both $\mathrm{Ker}(\lambda)$ and $\mathrm{Coker}(\lambda)$ are projective, the element $[P,\lambda,Q]$ in $K_0(R,S^{-1}R)$ is identified with an element in $\pi_0(V(R,S^{-1}R))$ defined as $[\det_R^{-1}(P)\cdot\det_R(Q), \det_R^{-1}(\lambda)\cdot \mathrm{id}_{\det_R(Q)}]$; in other words they implicitly regard $P\to Q$ as a complex concentrated in terms of degree -1 and 0. Hence there appears difference in sign convention

$$\begin{array}{ccc} K_0(\mathscr{C}_S^{\mathrm{Perf}}(R)) & \xrightarrow{\sim} & \pi_0(V(R,S^{-1}R)) \\ [C] & \leftrightarrow & -[\det_R(C), \mathrm{acyc}] \end{array}$$

(on the other hand they adopted, in [BrBur], a different normalisation

$$[P,\lambda,Q] \quad\leftrightarrow\quad [\det_R(P)\cdot\det_R^{-1}(Q), \det_R(\lambda)\cdot\mathrm{id}_{\det_R^{-1}(Q)}],$$

and therefore each element $[C]$ in $K_0(\mathscr{C}_S^{\mathrm{Perf}}(R))$ corresponds to an element $[\det_R(C), \mathrm{acyc}]$ in $\pi_0(V(R, S^{-1}R)))$.

4. Construction of the theta isomorphism I —additive theory—

In the rest of this article we prove our main theorem (Theorem 3.1). In this section we first define Artinian families $\mathfrak{F}_A$, $\mathfrak{F}_{A,c}$ and a Brauer family $\mathfrak{F}_B$ (see Section 4.1 for definition), which will play important roles in the following arguments. We then construct an additive version of the theta isomorphism (see Section 4.3). Later we shall translate it into a multiplicative morphism in Section 6. We remark that Mahesh Kakde has recently established more general construction of the additive theta isomorphism [Kakde2] (his construction can be applied to cases in which G^f is an arbitrary finite p-group not necessarily of exponent p).

4.1. **Artinian families $\mathfrak{F}_A$, $\mathfrak{F}_{A,c}$ and Brauer family $\mathfrak{F}_B$.** Let p, F and F_∞/F be as in Theorem 3.1. Let G be the Galois group of F_∞/F and p^N the order of the finite part G^f of G (and N is hence a non-negative integer). The finite p-group G^f acts upon the set of all its cyclic subgroups by conjugation. Choose a set of representatives of the orbital decomposition under this action, and choose also a generator for each representative cyclic group. Let $\mathfrak{H}$ denote the set of these fixed generators, and for each h in $\mathfrak{H}$ let U_h^f be the cyclic subgroup of G^f generated by h. Since the exponent of G^f is equal to p, the degree of each U_h^f exactly equals p except for $U_e^f = \{e\}$ (here we denote the unit of G^f by e). Let U_h be the open subgroup of G isomorphic to the direct product of U_h^f and Γ for each h in $\mathfrak{H}$, and consider the family of open subgroups of G consisting of all such U_h which we denote by $\mathfrak{F}_A$ (we always identify U_e with Γ).

Proposition 4.1. *The family $\mathfrak{F}_A$ satisfies condition $(\sharp)_A$. In other words, the family $\mathfrak{F}_A$ is an Artinian family for the group G.*

Proof. The claim is directly deduced from the classical Artin induction theorem (see, for example, [Serre1, Corollaire de Théorème 15]). □

For the usage of induction in Section 9, we now introduce another Artinian family $\mathfrak{F}_{A,c}$. When N equals zero, we set $\mathfrak{F}_{A,c} = \mathfrak{F}_A = \{(\Gamma, \{e\})\}$. When N is greater than or equal to 1, choose a central element $c \neq e$ in $\mathfrak{H}$ and fix it (there exists such an element c because G^f is a p-group). For each h in $\mathfrak{H}$, let $U_{h,c}^f$ be the abelian subgroup of G^f generated by h and c, and let $U_{h,c}$ be the open subgroup of G isomorphic to the direct product of $U_{h,c}^f$ and Γ. Let $\mathfrak{F}_{A,c}$ denote the family of open subgroups of G consisting of all elements in $\mathfrak{F}_A$ and all $U_{h,c}$ (we identify both $U_{e,c}$ and $U_{c,c}$ with U_c). Then the family $\mathfrak{F}_{A,c}$ is also an Artinian family for G because $\mathfrak{F}_{A,c}$ contains the Artinian family $\mathfrak{F}_A$.

We finally define $\mathfrak{F}_B$ as the family consisting of all pairs (U, V) such that U is an open subgroup of G containing Γ and V is the commutator subgroup of U. Then the family $\mathfrak{F}_B$ satisfies condition $(\sharp)_B$ by Brauer's induction theorem [Serre1, Théorèm 22] (note that for an arbitrary finite p-group, the family of all its Brauer elementary subgroups coincides with that of all its subgroups by definition); hence $\mathfrak{F}_B$ is a Brauer family.

4.2. **Calculation of the images of trace homomorphisms.** First recall the definition of *trace homomorphisms*; for an arbitrary finite group Δ, let $\mathbb{Z}_p[\mathrm{Conj}(\Delta)]$ be the free $\mathbb{Z}_p$-module of finite rank with basis $\mathrm{Conj}(\Delta)$, and for an arbitrary pro-finite group P, let $\mathbb{Z}_p[[\mathrm{Conj}(P)]]$ be the projective limit of free $\mathbb{Z}_p$-modules $\mathbb{Z}_p[\mathrm{Conj}(P_\lambda)]$ over all finite quotients P_λ of P.

Definition 4.2 (trace homomorphisms). Let P be an arbitrary pro-finite group and U an arbitrary open subgroup of P. Let $\{a_1, a_2, \ldots, a_s\}$ be a set of representatives of the left coset decomposition P/U. For an arbitrary conjugacy class $[g]$ of P and for each integer $1 \leq j \leq s$, define $\tau_j([g])$ as

$$\tau_j([g]) = \begin{cases} [a_j^{-1} g a_j] & \text{if } a_j^{-1} g a_j \text{ is contained in } U, \\ 0 & \text{otherwise.} \end{cases}$$

Then the element $\mathrm{Tr}_{\mathbb{Z}_p[[\mathrm{Conj}(P)]]/\mathbb{Z}_p[[\mathrm{Conj}(U)]]}([g]) = \sum_{j=1}^{s} \tau_j([g])$ is determined independently of the choice of representatives $\{a_j\}_{j=1}^{s}$. We call the induced $\mathbb{Z}_p$-module homomorphism

$$\mathrm{Tr}_{\mathbb{Z}_p[[\mathrm{Conj}(P)]]/\mathbb{Z}_p[[\mathrm{Conj}(U)]]} \colon \mathbb{Z}_p[[\mathrm{Conj}(P)]] \to \mathbb{Z}_p[[\mathrm{Conj}(U)]]$$

the trace homomorphism from $\mathbb{Z}_p[[\mathrm{Conj}(P)]]$ *to* $\mathbb{Z}_p[[\mathrm{Conj}(U)]]$.

Let c be the fixed central element in G^f as in the previous subsection and let θ_U^+ denote the trace homomorphism $\mathrm{Tr}_{\mathbb{Z}_p[[\mathrm{Conj}(G)]]/\mathbb{Z}_p[[U]]}$ for each U in $\mathfrak{F}_{A,c}$. We now calculate each image I_U of θ_U^+. Let NU^f denote the normaliser of U^f in G^f for each U in $\mathfrak{F}_{A,c}$. We denote by p^{n_h} the cardinality of NU_h^f for each h in $\mathfrak{H}$.

Calculation of $I_\Gamma (= I_{U_e})$. When N is equal to zero, the $\mathbb{Z}_p$-module I_Γ obviously coincides with $\Lambda(G) = \Lambda(\Gamma)$. Now suppose that N is greater than or equal to 1. In this case, $\theta_\Gamma^+([g])$ does not vanish if and only if g is contained in Γ. We may regard the finite part G^f as a set of representatives of the left coset decomposition G/Γ, and for each γ in Γ, its conjugate $a^{-1}\gamma a$ equals γ (note that γ is central). Therefore we have

$$I_\Gamma = p^N \mathbb{Z}_p[[\Gamma]]$$

(this equality is also valid for the case in which N equals zero).

Calculation of I_{U_h} for h in $\mathfrak{H}$ except for e ($N \geq 1$). When N is equal to 1, the $\mathbb{Z}_p$-module I_{U_h} obviously coincides with $\Lambda(G) = \Lambda(U_h)$. Hence suppose that N is greater than or equal to 2. In this case $\theta_{U_h}^+([g])$ does not vanish if and only if g is contained in one of the conjugates of U_h, and we may therefore assume that g itself is contained in U_h without loss of generality. The normaliser NU_h^f acts upon U_h^f by conjugation, which induces a group antihomomorphism $\mathrm{inn}\colon (NU_h^f)^{\mathrm{op}} \to \mathrm{Aut}(U_h^f) \cong \mathbb{F}_p^\times$. Note that it is trivial since NU_h^f is a p-group. Hence for every g in U_h not contained in Γ, its conjugate $a^{-1}ga$ is equal to g if a is contained in NU_h^f and is not contained in U_h otherwise. For each γ in Γ, its conjugate $a^{-1}\gamma a$ always equals γ as in the previous case. Consequently we have

$$I_{U_h} = p^{N-1}\mathbb{Z}_p[[\Gamma]] \oplus \bigoplus_{i=1}^{p-1} p^{n_h-1}h^i\mathbb{Z}_p[[\Gamma]]$$

(this equality is also valid when N equals 1).

Calculation of $I_{U_{h,c}}$ for h in $\mathfrak{H}$ except for e and c ($N \geq 2$). We obtain a group antihomomorphism

$$\mathrm{inn}\colon (NU^f_{h,c})^{\mathrm{op}} \to \mathrm{Aut}(U^f_{h,c})$$

in the same argument as in the previous case. Since the automorphism group $\mathrm{Aut}(U^f_{h,c})$ is isomorphic to the general linear group $\mathrm{GL}_2(\mathbb{F}_p)$ and its cardinality is equal to $p(p-1)^2(p+1)/2$, we have to consider the following two cases:

Case (a) the image of 'inn' is trivial;
Case (b) the image of 'inn' is a cyclic group of degree p.

In Case (a) it is easy to see that $NU^f_{h,c}$ coincides with NU^f_h (in particular the cardinality of $NU^f_{h,c}$ is equal to p^{n_h}). Therefore we may calculate $I_{U_{h,c}}$ in the same way as I_{U_h}, and we obtain

$$I_{U_{h,c}} = p^{N-2}\mathbb{Z}_p[[U_c]] \oplus \bigoplus_{i=1}^{p-1} p^{n_h-2}h^i\mathbb{Z}_p[[U_c]].$$

In Case (b) we may readily show by easy computation that the image of the map 'inn' is generated by automorphisms induced by $h^ic^j \mapsto h^ic^{ki+j}$ for each $0 \leq k \leq p-1$. The kernel of 'inn' obviously coincides with NU^f_h, and the cardinality of $NU^f_{h,c}$ is thus equal to p^{n_h+1}. This enables us to calculate $I_{U_{h,c}}$ as

$$I_{U_{h,c}} = p^{N-2}\mathbb{Z}_p[[U_c]] \oplus \bigoplus_{i=1}^{p-1} p^{n_h-2}h^i(1+c+\ldots+c^{p-1})\mathbb{Z}_p[[\Gamma]].$$

4.3. **Additive theta isomorphisms.** Now set

$$\theta^+_A = (\theta^+_U)_{U\in\mathfrak{F}_A}\colon \mathbb{Z}_p[[\mathrm{Conj}(G)]] \to \prod_{U\in\mathfrak{F}_A} \mathbb{Z}_p[[U]]$$

and let Φ be the $\mathbb{Z}_p$-submodule of $\prod_{U\in\mathfrak{F}_A}\mathbb{Z}_p[[U]]$ consisting of all elements $y_\bullet$ satisfying the following two conditions:

- (trace relation) the equation $\mathrm{Tr}_{\mathbb{Z}_p[[U_h]]/\mathbb{Z}_p[[\Gamma]]}y_h = y_e$ holds for each $\mathbb{Z}_p[[U_h]]$-component y_h of $y_\bullet$ (see Figure 1);
- each $\mathbb{Z}_p[[U]]$-component y_U of $y_\bullet$ is contained in I_U.

Proposition 4.3. *The map θ^+_A induces an isomorphism of $\mathbb{Z}_p$-modules*

$$\theta^+_A\colon \mathbb{Z}_p[[\mathrm{Conj}(G)]] \xrightarrow{\sim} \Phi.$$

We call the induced isomorphism θ^+_A the additive theta isomorphism for $\mathfrak{F}_A$.

$$\begin{array}{c} \mathbb{Z}_p[[U_h]] \\ \Big\downarrow {\scriptstyle \mathrm{Tr}_{\mathbb{Z}_p[[U_h]]/\mathbb{Z}_p[[\Gamma]]}} \\ \mathbb{Z}_p[[\Gamma]] \\ (= \mathbb{Z}_p[[U_e]]) \end{array}$$

FIGURE 1. Trace and norm relation for $\mathfrak{F}_A$.

Proof. It is easy to see that Φ contains the image of θ_A^+ by construction.

Injectivity. Take an element y from the kernel of θ_A^+ and let ρ be an arbitrary Artin representation of G. Note that ρ is isomorphic to a $\mathbb{Z}[1/p]$-linear combination $\sum_{U\in\mathfrak{F}_A} a_U \mathrm{Ind}_U^G \chi_U$ by condition $(\sharp)_A$ where each χ_U is a character of finite order of the abelian group U. If we let χ_ρ denote the character associated to the Artin representation ρ, we obtain an equation

$$\chi_\rho(y) = \sum_{U\in\mathfrak{F}_A} a_U\chi_U \circ \mathrm{Tr}_{\mathbb{Z}_p[[\mathrm{Conj}(G)]]/\mathbb{Z}_p[[U]]}(y)$$

by the explicit formula for induced characters [Serre1, Section 7.2]. This implies that $\chi_\rho(y)$ vanishes (recall that y is an element in the kernel of θ_A^+); in other words, the evaluation at y of an arbitrary class function on G is equal to zero. Therefore y itself is trivial.

Surjectivity. For an arbitrary element $y_\bullet$ in Φ, let y be the element in $\mathbb{Z}_p[[\mathrm{Conj}(G)]]$ defined by

$$y = p^{-N}[y_e] + \sum_{h\in\mathfrak{H}\setminus\{e\}} p^{-n_h+1}([y_h] - p^{-1}[y_e])$$

(we use the bracket $[\cdot]$ for the corresponding element in $\mathbb{Z}_p[[\mathrm{Conj}(G)]]$). Note that the definition of Φ guarantees that y has coefficients in $\mathbb{Z}_p$. Then it is not difficult at all to check that the image of y under the map θ_A^+ coincides with $y_\bullet$. □

Corollary 4.4. *Every element y in $\mathbb{Z}_p[[\mathrm{Conj}(G)]]$ is completely determined by its trace images $\{\theta_U^+(y)\}_{U\in\mathfrak{F}_A}$.*

Next we extend the notion of the additive theta isomorphism to the Brauer family $\mathfrak{F}_B$; for each (U,V) in $\mathfrak{F}_B$ let $\theta_{U,V}^+$ be the composite map

$$\mathbb{Z}_p[[\mathrm{Conj}(G)]] \xrightarrow{\mathrm{Tr}_{\mathbb{Z}_p[[\mathrm{Conj}(G)]]/\mathbb{Z}_p[[\mathrm{Conj}(U)]]}} \mathbb{Z}_p[[\mathrm{Conj}(U)]] \xrightarrow{\text{canonical}} \mathbb{Z}_p[[U/V]]$$

and set $\theta_B^+ = (\theta_{U,V}^+)_{(U,V)\in\mathfrak{F}_B}$. We define the $\mathbb{Z}_p$-submodule Φ_B of the direct product $\prod_{(U,V)\in\mathfrak{F}_B} \mathbb{Z}_p[[U/V]]$ as the submodule consisting of all elements $(y_{U,V})_{(U,V)\in\mathfrak{F}_B}$ satisfying the following *trace compatibility condition* (TCC, see Figure 2):

> the equation $\mathrm{Tr}_{\mathbb{Z}_p[[U/V]]/\mathbb{Z}_p[[U'/V]]}(y_{U,V}) = \mathrm{can}_V^{V'}(y_{U',V'})$ holds for arbitrary pairs (U,V) and (U',V') in $\mathfrak{F}_B$ such that U contains U' and U' contains V respectively (we denote by $\mathrm{can}_V^{V'}$ the canonical surjection $\mathbb{Z}_p[[U'/V']] \to \mathbb{Z}_p[[U'/V]]$);

$$\begin{tikzcd}[column sep=small] \mathbb{Z}_p[[U/V]] \arrow[dr, "\mathrm{Tr}_{\mathbb{Z}_p[[U/V]]/\mathbb{Z}_p[[U'/V]]}"'] & & \mathbb{Z}_p[[U'/V']] \arrow[dl, "\mathrm{can}_V^{V'}"] \\ & \mathbb{Z}_p[[U'/V]] & \end{tikzcd}$$

FIGURE 2. Trace compatibility condition for $\mathfrak{F}_B$.

and the following *conjugacy compatibility condition* (CCC+):

> the equation $y_{U',V'} = \psi_a(y_{U,V})$ holds if (U, V) and (U', V') are elements in $\mathfrak{F}_B$ such that $U' = a^{-1}Ua$ and $V' = a^{-1}Va$ hold for a certain element a in G (we denote by ψ_a the isomorphism $\mathbb{Z}_p[[U/V]] \xrightarrow{\sim} \mathbb{Z}_p[[U'/V']]$ induced by the conjugation $U/V \to U'/V'; u \mapsto a^{-1}ua$).

Note that we may naturally regard $\mathfrak{F}_A$ as a subfamily of $\mathfrak{F}_B$ (by identifying U in $\mathfrak{F}_A$ with the pair $(U, \{e\})$ in $\mathfrak{F}_B$).

Proposition 4.5. *Let $(y_{U,V})_{(U,V)\in\mathfrak{F}_B}$ be an element in Φ_B and assume that $(y_{U,\{e\}})_{U\in\mathfrak{F}_A}$ is contained in Φ. Then there exists a unique element y in $\mathbb{Z}_p[[\mathrm{Conj}(G)]]$ which satisfies $\theta_B^+(y) = (y_{U,V})_{(U,V)\in\mathfrak{F}_B}$.*

Proof. Consider the following commutative diagram (we denote the canonical projection by 'proj'):

$$\begin{tikzcd} \mathbb{Z}_p[[\mathrm{Conj}(G)]] \arrow[rr, "\theta_B^+"] \arrow[d, equal] & & \prod_{(U,V)\in\mathfrak{F}_B} \mathbb{Z}_p[[U/V]] \arrow[d, "\mathrm{proj}"] \\ \mathbb{Z}_p[[\mathrm{Conj}(G)]] \arrow[r, "\sim", "\theta_A^+"'] & \Phi \arrow[r, hook] & \prod_{U\in\mathfrak{F}_A} \mathbb{Z}_p[[U]]. \end{tikzcd}$$

Then $\mathrm{proj}((y_{U,V})_{(U,V)\in\mathfrak{F}_B})$ is contained in Φ by assumption, and thus there exits a unique element y in $\mathbb{Z}_p[[\mathrm{Conj}(G)]]$ which corresponds to the element $\mathrm{proj}((y_{U,V})_{(U,V)\in\mathfrak{F}_B})$ via the additive theta isomorphism θ_A^+ for $\mathfrak{F}_A$ (Proposition 4.3). We have to show that $\theta_B^+(y)$ coincides with $(y_{U,V})_{(U,V)\in\mathfrak{F}_B}$, and for this purpose it suffices to show that 'proj' induces an injection on Φ_B (note that the element $(\theta_{U,V}^+(y))_{(U,V)\in\mathfrak{F}_B}$ obviously satisfies both (TCC) and (CCC+) by construction; hence $\theta_B^+(y)$ is also an element in Φ_B). Let $(z_{U,V})_{(U,V)\in\mathfrak{F}_B}$ be an element in Φ_B satisfying the following equation:

$$\mathrm{proj}((z_{U,V})_{(U,V)\in\mathfrak{F}_B}) = (z_{U,\{e\}})_{U\in\mathfrak{F}_A} = 0. \tag{4.1}$$

We shall prove that $z_{U,V} = 0$ holds for each (U, V) in $\mathfrak{F}_B$ by induction on the cardinality of U^f. First note that $z_{U,\{e\}} = 0$ holds for $(U, \{e\})$ if the cardinality of U^f is less than or equal to p (use (4.1) and the conjugacy compatibility condition (CCC+)). Now let (U, V) be an element in $\mathfrak{F}_B$ such that the degree of U^f is equal to p^k for certain k greater than or equal to 2 and set $W = U^f/V^f$. Then the abelian group W is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{\oplus d}$ for a certain natural number d less than or equal to k (due to the structure theorem of finite abelian groups). Moreover we may assume that d is greater than or equal to 2.[6] Since the element $z_{U,V}$ is represented as a $\Lambda(\Gamma)$-linear combination $\sum_{w\in W} a_w w$, it suffices to prove that a_w equals zero for every w in W. Take an arbitrary element x of degree p in W, and let U^f_x denote the inverse image of $\langle x\rangle$—the cyclic subgroup of W generated by x— under the canonical surjection $U^f \to W$. Obviously the cardinality of U^f_x is strictly less than p^k. If we set $U_x = U^f_x \times \Gamma$, we may explicitly calculate the image of $z_{U,V}$ under the trace map from $\mathbb{Z}_p[[U/V]]$ to $\mathbb{Z}_p[[U_x/V]]$ as $\sum_{i=0}^{p-1} p^{d-1} a_{x^i} x^i$. On the other hand the element z_{U_x,V_x} is equal to zero by induction hypothesis (here V_x denotes the commutator subgroup of U_x). Therefore $a_{x^i} = 0$ holds for each i by (TCC). Replacing x appropriately, we may verify that $a_w = 0$ holds for every w in W. □

5. Preliminaries for logarithmic translation

This section is devoted to technical preliminaries for arguments in Section 6.

5.1. **Augmentation theory.** For each (U, V) in $\mathfrak{F}_B$, let $\mathrm{aug}_{U,V}$ denote the augmentation map from $\Lambda(U/V)$ to $\Lambda(\Gamma)$ (namely it is a ring homomorphism induced by the projection $U/V \to \Gamma$), and let $\overline{\mathrm{aug}}_{U,V}\colon \Omega(U/V) \to \Omega(\Gamma)$ be its reduction modulo p. Let $\varphi\colon \Lambda(G) \to \Lambda(\Gamma)$ denote 'the Frobenius endomorphism' on $\Lambda(G)$ defined as the composition

$$\Lambda(G) \xrightarrow{\mathrm{aug}_G} \Lambda(\Gamma) \xrightarrow{\varphi_\Gamma} \Lambda(\Gamma)$$

where aug_G is the canonical augmentation map and φ_Γ is the Frobenius endomorphism on $\Lambda(\Gamma)$ induced by $\gamma \mapsto \gamma^p$. Let $\theta_{U,V}$ denote the composition of the norm map $\mathrm{Nr}_{\Lambda(G)/\Lambda(U)}$ with the canonical map $K_1(\Lambda(U)) \to \Lambda(U/V)^\times$.

The author is grateful to Takeshi Tsuji for presenting the following useful proposition to him.

Proposition 5.1. *Let (U, V) be an element in $\mathfrak{F}_B$ and $J_{U,V}$ the kernel of the composite map*

$$\Lambda(U/V) \xrightarrow{\mathrm{aug}_{U,V}} \Lambda(\Gamma) \xrightarrow{\mathrm{mod}\, p} \Omega(\Gamma).$$

Then the element $\varphi(x)^{-(G:U)/p}\theta_{U,V}(x)$ is contained in $1 + J_{U,V}$ for each x in $K_1(\Lambda(G))$ if U is a proper *subgroup of G. In other words, the congruence $\theta_{U,V}(x) \equiv \varphi(x)^{(G:U)/p} \bmod J_{U,V}$ holds unless U coincides with G.*

Before the proof we remark that the image of an element x in $K_1(\Lambda(G))$ under the map $\theta_{U,V}$ can be calculated as follows: since the Iwasawa algebra $\Lambda(G)$ is regarded as a left free $\Lambda(U)$-module of rank $r = (G : U)$, the 'right multiplication by x' map is represented by an invertible matrix A_x with entries in $\Lambda(U)$.[7] The element $\theta_{U,V}(x)$ then coincides with the determinant of the image of A_x under the canonical map $\mathrm{GL}_r(\Lambda(U)) \to \mathrm{GL}_r(\Lambda(U/V))$.

Proof. The claim is equivalent to the following Proposition 5.2 since both $\Lambda(G)$ and $\Lambda(U/V)$ are p-adically complete. □

[6] We may easily verify that the cardinality of V^f is always less than or equal to p^{k-2} by induction on the cardinality of U^f.

[7] By abuse of notation, we use the same symbol x for an arbitrary lift of x to $\Lambda(G)^\times$.

Let $\bar{\varphi}\colon \Omega(G) \to \Omega(\Gamma)$ denote the Frobenius endomorphism on $\Omega(G)$ defined as $\varphi \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ and let $\bar{\theta}_{U,V}$ denote the composition of the norm map $\mathrm{Nr}_{\Omega(G)/\Omega(U)}$ with the canonical map $K_1(\Omega(U)) \to \Omega(U/V)^\times$.

Proposition 5.2. *Let $\bar{J}_{U,V}$ be the kernel of the augmentation map*

$$\Omega(U/V) \xrightarrow{\overline{\mathrm{aug}}_{U,V}} \Omega(\Gamma).$$

Then the element defined as $\bar{\varphi}(x)^{-(G:U)/p}\bar{\theta}_{U,V}(x)$ is contained in $1+\bar{J}_{U,V}$ for each x in $K_1(\Omega(G))$.

Remark 5.3. Proposition 5.2 is valid even if U coincides with G (indeed $\bar{\varphi}(x)$ can be described as a p-th power of a certain element, see the proof of Proposition 5.2). However there exists an obstruction for taking the projective limit if the exponent $(G : U)/p$ of $\varphi(x)$ is *not* integral. Therefore the case where U coincides with G remains as an exception to Proposition 5.1.

Proposition 5.2 is deduced from the following elementary lemma in modular representation theory.

Lemma 5.4. *Let K be a field of positive characteristic p, Δ a finite p-group and V a finite dimensional representation space of Δ over K. Let 'aug' denote the canonical augmentation map $K[\Delta] \to K$. Take a natural number n such that p^n is greater than the K-dimension of V. Then for each x in $K[\Delta]$, the action of x^{p^n} upon V coincides with the multiplication by $\mathrm{aug}(x)^{p^n}$. In particular the equation $x^{\sharp\Delta} = \mathrm{aug}(x)^{\sharp\Delta}$ holds.*

Proof. Let d be the K-dimension of V. The group ring $K[\Delta]$ is a local ring whose maximal ideal is the augmentation ideal since K is of characteristic p and Δ is a p-group. Therefore the only simple $K[\Delta]$-module (up to isomorphisms) is K endowed with trivial Δ-action, and moreover there exists a Jordan-Schreier composition series

$$V = V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_d \supsetneq V_{d+1} = \{0\}$$

such that each quotient space V_i/V_{i+1} is isomorphic to K. Take an arbitrary element e_i in V_i not contained in V_{i+1} for each $1 \le i \le d$. Then $\{e_1, e_2, \dots, e_d\}$ forms a basis of V over K, with respect to which the action of x is represented by an upper triangular matrix all of whose diagonal components are equal to $\mathrm{aug}(x)$. This implies the first claim. The second claim is directly deduced from the first one (take n as the p-order of Δ and apply the claim to the regular representation $V = K[\Delta]$). □

Proof of Proposition 5.2. Identify the modulo p Iwasawa algebra $\Omega(U/V)$ with the group ring $\Omega(\Gamma)[U^f/V^f]$, and let K be the fractional field of $\Omega(\Gamma)$. Then we may naturally regard each t in $\Omega(U/V)$ as an element in $K[U^f/V^f]$, and therefore the equation

$$t^{\sharp(U^f/V^f)} = \overline{\mathrm{aug}}_{U,V}(t)^{\sharp(U^f/V^f)} \tag{5.1}$$

holds by Lemma 5.4. Now let x be an arbitrary element in $\Omega(G)^\times$, and set $z = \overline{\mathrm{aug}}_G(x)$ and $y = xz^{-1}$. Then we obtain $\overline{\mathrm{aug}}_G(y) = 1$ and

$$\bar{\theta}(x) = \bar{\theta}(y)\bar{\theta}(z) \tag{5.2}$$

by definition (here we denote the map $\bar{\theta}_{U,V}$ by $\bar{\theta}$ to simplify the notation). Since z is an element in $\Omega(\Gamma)$ (and hence z is contained in the centre of $\Omega(G)$), the image of z under the norm map $\bar{\theta}$ coincides with $z^{(G:U)}$ by direct calculation. On the other hand we may calculate $\bar{\varphi}(x)$ as follows:

$$(5.3) \qquad \bar{\varphi}(x) = \bar{\varphi}(y)\bar{\varphi}(z) = \bar{\varphi}(\overline{\mathrm{aug}}_G(y))\bar{\varphi}(z) = z^p \qquad (\text{use } \overline{\mathrm{aug}}_G(y) = 1).$$

Hence the equation $\bar{\varphi}(x)^{-(G:U)/p}\bar{\theta}(x) = \bar{\theta}(y)$ holds by (5.2) and (5.3). Moreover (5.1) implies that y^{p^N} is equal to $\overline{\mathrm{aug}}_G(y)^{p^N} = 1$, and therefore $\bar{\theta}(y)^{p^N}$ is also trivial. The same argument as above derives a similar equation $\bar{\theta}(y)^{\sharp(U^f/V^f)} = \overline{\mathrm{aug}}_{U,V}(\bar{\theta}(y))^{\sharp(U^f/V^f)}$, and consequently the equation

$$\overline{\mathrm{aug}}_{U,V}(\bar{\theta}(y))^{p^N} = \bar{\theta}(y)^{p^N} = 1$$

holds. Since $\Omega(\Gamma)$ is a domain (recall that $\Omega(\Gamma)$ is isomorphic to the formal power series ring $\mathbb{F}_p[[T]]$), the last equation implies that $\overline{\mathrm{aug}}_{U,V}(\bar{\theta}(y))$ itself is trivial; in other words $\bar{\theta}(y)$ is contained in $1 + \bar{J}_{U,V}$. □

The last paragraph of the proof implies that $\theta_{U,V}(y)$ is contained in $1 + J_{U,V}$ if y is an element in $\Lambda(G)$ satisfying $\mathrm{aug}_G(y) \equiv 1 \bmod p$. By replacing G and U appropriately, we obtain the following useful corollary:

Corollary 5.5. *Let (U, V) be an element in $\mathfrak{F}_B$ such that U does not contain a non-trivial central element c. Then the norm map $\mathrm{Nr}_{\Lambda(U\times\langle c\rangle/V)/\Lambda(U/V)}$ induces a group homomorphism from $1 + J_{U\times\langle c\rangle,V}$ to $1 + J_{U,V}$.*

Remark 5.6. Both $1 + J_{U\times\langle c\rangle,V}$ and $1 + J_{U,V}$ are actually multiplicative groups; see Proposition 5.7 for details.

5.2. **Logarithmic theory.** Let us study the p-adic logarithm on $1 + J_{U,V}$ for each (U, V) in $\mathfrak{F}_B$, as well as those on $1 + I_U$ for each U in $\mathfrak{F}_{A,c}$.

Proposition 5.7. *For each (U, V) in $\mathfrak{F}_B$, let $J_{U,V}$ be as in Proposition* 5.1. *Then*

(1) *the subset $1 + J_{U,V}$ of $\Lambda(U/V)$ is a multiplicative subgroup of $\Lambda(U/V)^\times$;*
(2) *for each y in $J_{U,V}$, the logarithm $\log(1 + y) = \sum_{m=1}^{\infty}(-1)^{m-1}y^m/m$ converges p-adically in $\Lambda(U/V) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$;*
(3) *the kernel (resp. image) of the induced homomorphism*

$$\log\colon 1 + J_{U,V} \to \Lambda(U/V) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is $\mu_p(\Lambda(U/V))$ (resp. is contained in $\Lambda(U/V)$) where $\mu_p(\Lambda(U/V))$ denotes the multiplicative subgroup of $\Lambda(U/V)^\times$ consisting of all p-power roots of unity.

Proof. Define $\overline{\mathrm{aug}}_{U,V}\colon \Omega(U/V) \to \Omega(\Gamma)$ similarly to the previous subsection, and let $\bar{J}_{U,V}$ be the kernel of $\overline{\mathrm{aug}}_{U,V}$. Since $\Omega(U/V)$ is commutative, we have

$$\bar{y}^p = \sum_{u\in U^f/V^f} \bar{y}_u^p u^p = \sum_{u\in U^f/V^f} \bar{y}_u^p = \left(\sum_{u\in U^f/V^f} \bar{y}_u\right)^p = (\overline{\mathrm{aug}}_{U,V}(\bar{y}))^p = 0$$

for an element $\bar{y} = \sum_{u\in U^f/V^f} \bar{y}_u u$ in $\bar{J}_{U,V}$ (each $\bar{y}_u$ is an element in $\Omega(\Gamma)$). Therefore y^p is contained in $p\Lambda(U/V)$ for each y in $J_{U,V}$.

(1) By the above remark, $(1+y)^{-1} = \sum_{m=0}^{\infty}(-y)^m$ converges p-adically in $1 + J_{U,V}$ for each $1+y$ in $1+J_{U,V}$.
(2) In a similar way the element y^m is contained in $p^{[m/p]}\Lambda(U/V)$ for each y in $J_{U,V}$ (for a real number x, we denote by $[x]$ the largest integer not greater than x), and hence the claim holds.
(3) If we take an element $x = 1+y$ from $1+J_{U,V}$, we may calculate as $\bar{x}^p = 1 + \bar{y}^p = 1$ where $\bar{x} = 1+\bar{y}$ is the image of x in $1 + \bar{J}_{U,V}$. This implies that x^p is an element in $1+p\Lambda(U/V)$ since the p-adic exponential map and the p-adic logarithmic map define an isomorphism between $p\Lambda(U/V)$ and $1+p\Lambda(U/V)$ in general (recall that p is odd). Therefore $p\log x (= \log x^p)$ is contained in $p\Lambda(U/V)$, or equivalently $\log(1+J_{U,V})$ is contained in $\Lambda(U/V)$. Furthermore if we assume that $\log x = 0$ holds for x in $1+J_{U,V}$, we obtain $x^p = 1$ by the above calculation, which implies that x is an element in $\mu_p(\Lambda(U/V))$. Conversely $\log x$ vanishes for an arbitrary element x in $\mu_p(\Lambda(U/V))$ since $\Lambda(U/V)$ is free of p-torsion.

$\square$

Lemma 5.8. *For each U in $\mathfrak{F}_{A,c}$, let I_U be the $\mathbb{Z}_p$-submodule of $\Lambda(U)$ defined as in Section 3. Then I_U^2 is contained in I_U. Moreover,*
(1) *when N is greater than or equal to 1, the $\mathbb{Z}_p$-module I_Γ is contained in $J_\Gamma = p\Lambda(\Gamma)$;*
(2) *when N is greater than or equal to 2, the $\mathbb{Z}_p$-module I_{U_h} is contained in $p\Lambda(U_h)$ (hence also in J_{U_h}) for each h in $\mathfrak{H} \setminus \{e\}$, and there exist canonical inclusions $p^{k(N-1)}I_{U_h} \subseteq I_{U_h}^{k+1} \subseteq p^{k(n_h-1)}I_{U_h}$ for an arbitrary natural number k;*
(3) *when N is greater than or equal to 3, the $\mathbb{Z}_p$-module $I_{U_{h,c}}$ is contained in $p\Lambda(U_{h,c})$ (hence also in $J_{U_{h,c}}$) for each h in $\mathfrak{H} \setminus \{e,c\}$ satisfying the condition of Case (a), and $p^{k(N-2)}I_{U_{h,c}} \subseteq I_{U_{h,c}}^{k+1} \subseteq p^{k(n_h-2)}I_{U_{h,c}}$ holds for an arbitrary natural number k;*
(4) *when N is greater than or equal to 3, the $\mathbb{Z}_p$-module $I_{U_{h,c}}$ is contained in $J_{U_{h,c}}$ for each h in $\mathfrak{H} \setminus \{e,c\}$ satisfying the condition of Case (b), and there exist canonical inclusions $p^{k(N-2)}I_{U_{h,c}}^2 \subseteq I_{U_{h,c}}^{k+2} \subseteq p^{k(n_h-1)}I_{U_{h,c}}^2$ for an arbitrary natural number k.*

Proof. The first claim is easily checked by direct calculation.
(1) Obvious from the exact description of I_Γ (see Section 4.3).
(2) If h is contained in the centre of G^f, the equation $n_h = N$ clearly holds. Otherwise NU_h^f has to contain the centre of G^f, and therefore n_h is at least 2. In both cases I_{U_h} is contained in $p\Lambda(U_h)$. The last claim is obvious from the explicit description of I_{U_h}.
(3) Recall that $\sharp NU_{h,c}^f = p^{n_h}$ holds in Case (a). Let $\bar{U}_h^f$ denote the quotient group $U_{h,c}^f/U_c^f$. If $\bar{h}$ (the image of h in $\bar{U}_h^f$) is contained in the centre of $\bar{G}^f = G^f/U_c^f$, the normaliser of $\bar{U}_h^f$ obviously coincides with $\bar{G}^f$, which implies that n_h is equal to N. Otherwise there exists a non-trivial element $\bar{a}$ in the centre of $\bar{G}^f$. Let a be its lift to G^f, then the finite subgroup of G^f generated by c, h and a is contained in $NU_{h,c}^f$ by construction. This implies that n_h is at least 3. In both cases we may

conclude that $I_{U_{h,c}}$ is contained in $p\Lambda(U_{h,c})$. The last claim is obvious from the explicit description of $I_{U_{h,c}}$.

(4) First note that $2 \leq n_h \leq N-1$ holds since the cardinality of $NU^f_{h,c}$ (which is less than or equal to p^N) is equal to p^{n_h+1}. By using this fact, we may exactly calculate as

$$I^k_{U_{h,c}} = (p^{k(N-2)}\mathbb{Z}_p[[U_c]] + p^{k(n_h-1)-1}(1+c+\cdots+c^{p-1})\mathbb{Z}_p[[\Gamma]]) \oplus \bigoplus_{i=1}^{p-1} p^{k(n_h-1)-1}h^i(1+c+\cdots+c^{p-1})\mathbb{Z}_p[[\Gamma]]$$

for each k greater than or equal to 2. The claim holds by this calculation. □

Proposition 5.9. *Let U be an element in $\mathfrak{F}_{A,c}$ and assume that U does not coincide with G. Then*

(1) *the subset $1+I_U$ of $\Lambda(U)$ is a multiplicative subgroup of $\Lambda(U)^\times$;*
(2) *for each y in I_U, the logarithm $\log(1+y) = \sum_{m=1}^{\infty}(-1)^{m-1}y^m/m$ converges p-adically in I_U;*
(3) *the p-adic logarithmic homomorphism induces an isomorphism between $1+I_U$ and I_U.*

Proof. The claims of (1) and (2) follow from Lemma 5.8 (use the fact that y^{p^m}/p^m is contained in I_U for each y in I_U if p is odd). For (3), first note that $1+I_U^k$ is a multiplicative subgroup of $1+I_U$ and the p-adic logarithm induces a homomorphism from $1+I_U^k$ to I_U^k for each natural number k (similarly to (1) and (2)). Moreover the I_U-adic topology on I_U coincides with the p-adic topology by Lemma 5.8. Therefore it suffices to show that the p-adic logarithm induces an isomorphism

$$\log\colon 1+I_U^k/1+I_U^{k+1} \to I_U^k/I_U^{k+1}; 1+y \mapsto y$$

for each natural number k. Let y be an element in I_U^k. We have only to show that y^{p^m}/p^m is contained in I_U^{k+1} for each $m \geq 1$, or equivalently, $p^{-m}I_U^{kp^m}$ is contained in I_U^{k+1} for every k and m. We may verify it by direct calculation.[8] □

Remark 5.10. Suppose that N equals either $0, 1$ or 2. Then the Artinian family $\mathfrak{F}_{A,c}$ contains the whole group G by definition. The $\mathbb{Z}_p$-module I_G obviously coincides with $\Lambda(G)$, and thus the p-adic logarithm never converges on $1+I_G$. We remark that the $\mathbb{Z}_p$-module $I_G = \Lambda(G)$ is the only exception to our logarithmic theory discussed in this subsection.

6. Construction of the theta isomorphism II —translation—

In this section we shall construct the multiplicative theta isomorphism by using the facts studied in Section 5.

[8]In this calculation we use the fact that p is greater than 2.

6.1. The multiplicative theta isomorphism. Let (U, V) be an element in $\mathfrak{F}_B$. We use the notion '$x \equiv y \,\widetilde{\mathrm{mod}}\, \mathfrak{I}$' for elements x and y in $\widetilde{\Lambda}(U/V)^\times$ such that xy^{-1} is contained in $1+\mathfrak{I}^{\sim}$ —the image of $1+\mathfrak{I}$ under the canonical surjection $\Lambda(U/V)^\times \to \widetilde{\Lambda}(U/V)^\times$— if $\mathfrak{I}$ is a $\mathbb{Z}_p$-submodule of $\Lambda(U/V)$ such that $1+\mathfrak{I}$ is a multiplicative subgroup of $\Lambda(U/V)^\times$. Let $\widetilde{\Psi}'$ denote the subgroup of $\prod_{(U,V)\in\mathfrak{F}_B} \widetilde{\Lambda}(U/V)^\times$ consisting of all elements $\eta_\bullet = (\eta_{U,V})_{(U,V)\in\mathfrak{F}_B}$ satisfying the following three conditions:

- (norm compatibility condition, NCC)
  the equation $\mathrm{Nr}_{\Lambda(U/V)/\Lambda(U'/V)}(\eta_{U,V}) = \mathrm{can}_V^{V'}(\eta_{U',V'})$ holds for (U, V) and (U', V') in $\mathfrak{F}_B$ such that U contains U' and U' contains V respectively (here $\mathrm{can}_V^{V'}$ is the canonical map $\Lambda(U'/V') \to \Lambda(U'/V)$);
- (conjugacy compatibility condition, CCC)
  the equation $\eta_{U',V'} = \psi_a(\eta_{U,V})$ holds for (U, V) and (U', V') in $\mathfrak{F}_B$ such that $U' = a^{-1}Ua$ and $V' = a^{-1}Va$ hold for a certain element a in G (we denote by ψ_a the isomorphism $\Lambda(U/V)^\times \xrightarrow{\sim} \Lambda(U'/V')^\times$ induced by the conjugation $U/V \to U'/V'; u \mapsto a^{-1}ua$);
- (congruence condition)
  the congruence $\eta_{U,V} \equiv \varphi(\eta_{\mathrm{ab}})^{(G:U)/p} \,\widetilde{\mathrm{mod}}\, J_{U,V}$ holds for (U, V) in $\mathfrak{F}_B$ except for $(G, [G, G])$ where η_{ab} denotes the $\widetilde{\Lambda}(G^{\mathrm{ab}})^\times$-component of $\eta_\bullet$ (see the previous section for the definition of $J_{U,V}$).

Let $\widetilde{\Psi}$ (resp. $\widetilde{\Psi}_c$) be the subgroup of $\widetilde{\Psi}'$ consisting of all elements $\eta_\bullet$ in $\widetilde{\Psi}'$ satisfying the following *additional congruence condition* (see Section 4 for the definition of I_U):

- (additional congruence condition)
  the congruence $\eta_U \equiv \varphi(\eta_{\mathrm{ab}})^{(G:U)/p} \,\widetilde{\mathrm{mod}}\, I_U$ holds for each U in $\mathfrak{F}_A$ (resp. $\mathfrak{F}_{A,c}$).

Remark 6.1. When N equals either 0, 1 or 2, we regard the additional congruence condition for the total group G as *the trivial condition* (in other words, we do not impose any congruence condition upon G). Therefore we have only to consider an element (U, V) in $\mathfrak{F}_B$ (resp. U in $\mathfrak{F}_{A,c}$) such that U is a *proper* subgroup of G in arguments concerning congruence conditions.

Remark 6.2. For each U in $\mathfrak{F}_{A,c}$, we may easily check that the ideal J_U contains I_U unless U coincides with G by using the explicit description of I_U given in Section 4.2; in particular $\widetilde{\Psi}_c$ is a subgroup of $\widetilde{\Psi}$.

Let $\theta_{U,V}$ be as in Section 5.1 and set $\theta = (\theta_{U,V})_{(U,V)\in\mathfrak{F}_B}$, then the map θ induces a group homomorphism $\tilde{\theta}\colon \widetilde{K}_1(\Lambda(G)) \to \prod_{(U,V)\in\mathfrak{F}_B} \widetilde{\Lambda}(U/V)^\times$.

Proposition 6.3. *The multiplicative group $\widetilde{\Psi}$ coincides with $\widetilde{\Psi}_c$. Moreover the map $\tilde{\theta}$ induces an isomorphism*

$$\tilde{\theta}\colon \widetilde{K}_1(\Lambda(G)) \xrightarrow{\sim} \widetilde{\Psi} \quad (= \widetilde{\Psi}_c).$$

In order to prove Proposition 6.3, it suffices to verify surjectivity of $\widetilde{K}(\Lambda(G)) \to \widetilde{\Psi}$ and injectivity of $\widetilde{K}(\Lambda(G)) \to \widetilde{\Psi}_c$ (see Remark 6.2). The arguments to verify these two claims will occupy the rest of this section.

6.2. **Integral logarithmic homomorphism.** We now introduce *the integral logarithmic homomorphisms*; for an arbitrary finite p-group Δ, Oliver and Taylor defined a homomorphism of abelian groups (called the integral logarithm)

$$\Gamma_\Delta \colon K_1(\mathbb{Z}_p[\Delta]) \to \mathbb{Z}_p[\mathrm{Conj}(\Delta)];\ x \mapsto \log(x) - p^{-1}\varphi(\log(x))$$

where φ is 'the Frobenius correspondence' on $\mathbb{Z}_p[\mathrm{Conj}(\Delta)]$ characterised by

$$\varphi\left(\sum_{[d]\in\mathrm{Conj}(\Delta)} a_{[d]}[d]\right) = \sum_{[d]\in\mathrm{Conj}(\Delta)} a_{[d]}[d^p].$$

The integral logarithmic homomorphisms are compatible with group homomorphisms; that is, the diagram

$$\begin{array}{ccc} K_1(\mathbb{Z}_p[\Delta]) & \xrightarrow{\Gamma_\Delta} & \mathbb{Z}_p[\mathrm{Conj}(\Delta)] \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ K_1(\mathbb{Z}_p[\Delta']) & \xrightarrow[\Gamma_{\Delta'}]{} & \mathbb{Z}_p[\mathrm{Conj}(\Delta')] \end{array} \tag{6.1}$$

commutes for an arbitrary homomorphism $f\colon \Delta \to \Delta'$ of finite p-groups (the symbol f_* denotes the homomorphism of K-groups induced by f). It is known that the sequence

$$1 \to K_1(\mathbb{Z}_p[\Delta])/K_1(\mathbb{Z}_p[\Delta])_{\mathrm{tors}} \xrightarrow{\Gamma_\Delta} \mathbb{Z}_p[\mathrm{Conj}(\Delta)] \xrightarrow{\omega_\Delta} \Delta^{\mathrm{ab}} \to 1 \tag{6.2}$$

is exact where ω_Δ is the homomorphism of abelian groups defined by

$$\omega_\Delta\left(\sum_{[d]\in\mathrm{Conj}(\Delta)} a_{[d]}[d]\right) = \prod_{[d]\in\mathrm{Conj}(\Delta)} \bar{d}^{a_{[d]}}$$

(here we denote by $\bar{d}$ the image of $[d]$ in Δ^{ab}). Refer to [Oliver, OT] for details of properties of integral logarithms.

Now consider the case $G = G^f \times \Gamma$: let us apply the exact sequence (6.2) to the finite p-group $G^{(n)} = G^f \times \Gamma/\Gamma^{p^n}$ for each natural number n. The structure of the torsion part of $K_1(\mathbb{Z}_p[G^{(n)}])$ has been well studied in [H, Section 4.4]; in fact, it is described as

$$K_1(\mathbb{Z}_p[G^{(n)}])_{\mathrm{tors}} \cong \mu_{p-1}(\mathbb{Z}_p) \times G^{(n),\mathrm{ab}} \times SK_1(\mathbb{Z}_p[G^f]) \tag{6.3}$$

by Wall's theorem [Wall, Theorem 4.1] where $\mu_{p-1}(\mathbb{Z}_p)$ denotes the subgroup of $\mathbb{Z}_p^\times$ consisting of all $(p-1)$-th roots of unity. By taking the projective limit,[9] we obtain the following exact sequence (note that the projective limit $\varprojlim_n K_1(\mathbb{Z}_p[G^{(n)}])$ actually coincides with $K_1(\Lambda(G))$; see [FukKat, Proposition 1.5.1]):

(6.4)

$$1 \to K_1(\Lambda(G))/\varprojlim_n K_1(\mathbb{Z}_p[G^{(n)}])_{\mathrm{tors}} \xrightarrow{\Gamma_G} \mathbb{Z}_p[[\mathrm{Conj}(G)]] \xrightarrow{\omega_G} G^{\mathrm{ab}} \to 1.$$

[9] Since $K_1(\mathbb{Z}_p[G^{(n+1)}])/K_1(\mathbb{Z}_p[G^{(n+1)}])_{\mathrm{tors}} \to K_1(\mathbb{Z}_p[G^{(n)}])/K_1(\mathbb{Z}_p[G^{(n)}])_{\mathrm{tors}}$ is surjective, the exact sequence (6.2) for projective systems with respect to $\{G^{(n)}\}_{n\in\mathbb{N}}$ satisfies so-called Mittag-Leffler condition. Therefore we may take the projective limit.

Moreover (6.3) implies that the projective limit $\varprojlim_n K_1(\mathbb{Z}_p[G^{(n)}])_{\mathrm{tors}}$ is isomorphic to the direct product $\mu_{p-1}(\mathbb{Z}_p) \times G^{\mathrm{ab}} \times SK_1(\mathbb{Z}_p[G^f])$. We may, therefore, identify the p-torsion part $K_1(\Lambda(G))_{p\text{-tors}}$ of the Whitehead group $K_1(\Lambda(G))$ with $G^{f,\mathrm{ab}} \times SK_1(\mathbb{Z}_p[G^f])$ (recall that $SK_1(\mathbb{Z}_p[G^f])$ is a finite p-group [Wall, Theorem 2.5]).

We remark that the p-th power Frobenius endomorphism $g \mapsto g^p$ is well defined on G in our case since the exponent of G^f equals p. We use the same symbol φ for the Frobenius endomorphism on G, and then it obviously induces the Frobenius correspondence on $\mathbb{Z}_p[[\mathrm{Conj}(G)]]$. Note that the notion φ introduced here is compatible with the one defined in Section 5.

6.3. **The group $\widetilde{\Psi}_c$ contains the image of $\tilde{\theta}$.** In this subsection we prove that $\widetilde{\Psi}_c$ contains the image of $\tilde{\theta}$ (and hence $\widetilde{\Psi}$ also does by Remark 6.1).

Lemma 6.4. *The multiplicative group $\widetilde{\Psi}'$ contains the image of $\tilde{\theta}$.*

Proof. The element $(\tilde{\theta}_{U,V}(\eta))_{(U,V)\in\mathfrak{F}_B}$ satisfies both (NCC) and (CCC) for each η in $\widetilde{K}_1(\Lambda(G))$ by the basic properties of norm maps in algebraic K-theory. Moreover the congruence $\tilde{\theta}_{U,V}(\eta) \equiv \varphi(\tilde{\theta}_{\mathrm{ab}}(\eta))^{(G:U)/p} \mathbin{\widetilde{\mathrm{mod}}} J_{U,V}$ holds unless U coincides with G by Proposition 5.1 (we denote by $\tilde{\theta}_{\mathrm{ab}}$ the homomorphism $\widetilde{K}_1(\Lambda(G)) \to \widetilde{\Lambda}(G^{\mathrm{ab}})^\times$ induced by the abelianisation map; note that $\varphi(\tilde{\theta}_{\mathrm{ab}}(\eta))$ obviously coincides with $\varphi(\eta)$ by definition). $\square$

By virtue of Lemma 6.4 we have only to verify the following proposition to show that $\widetilde{\Psi}_c$ contains the image of $\tilde{\theta}$.

Proposition 6.5. *Let η be an element in $\widetilde{K}_1(\Lambda(G))$. Then the congruence $\tilde{\theta}_U(\eta) \equiv \varphi(\tilde{\theta}_{\mathrm{ab}}(\eta))^{(G:U)/p} \mathbin{\widetilde{\mathrm{mod}}} I_U$ holds for each U in $\mathfrak{F}_{A,c}$.*

The following lemma relates norm maps in algebraic K-theory to trace homomorphisms defined in Section 4.2 via p-adic logarithms.

Lemma 6.6 (compatibility lemma). *Let (U,V) and (U',V') be elements in $\mathfrak{F}_B$ such that U contains U'. Then the following diagram commutes:*

$$\begin{array}{ccc}
K_1(\Lambda(U)) & \xrightarrow{\log} & \mathbb{Q}_p[[\mathrm{Conj}(U)]] \\
{\scriptstyle \mathrm{Nr}_{\Lambda(U)/\Lambda(U')}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_{\mathbb{Q}_p[[\mathrm{Conj}(U)]]/\mathbb{Q}_p[[\mathrm{Conj}(U')]]}} \\
K_1(\Lambda(U')) & \xrightarrow[\log]{} & \mathbb{Q}_p[[\mathrm{Conj}(U')]].
\end{array}$$

Proof. We may prove that the diagram commutes for each finite quotient $U^{(n)} = U^f \times \Gamma/\Gamma^{p^n}$ and $U'^{(n)} = U'^f \times \Gamma/\Gamma^{p^n}$ by the same argument as that in [H, Lemma 4.7]. Hence the claim holds if we take the projective limit. $\square$

Proof of Proposition 6.5. We may assume that U does not coincide with G without loss of generality (see Remark 6.1). Let θ_{ab} (resp. θ^+_{ab}) be the homomorphism $K_1(\Lambda(G)) \to \Lambda(G^{\mathrm{ab}})^\times$ (resp. $\mathbb{Z}_p[[\mathrm{Conj}(G)]] \to \mathbb{Z}_p[[G^{\mathrm{ab}}]]$) induced by the abelianisation map $G \to G^{\mathrm{ab}}$. Then we may easily check that the following diagram commutes for each (U,V) in $\mathfrak{F}_B$:

$$\begin{tikzcd}
\mathbb{Q}_p[[\mathrm{Conj}(G)]] \arrow[r, "\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\theta^+_{\mathrm{ab}}"] \arrow[d, "\frac{1}{p}\varphi"'] & \mathbb{Q}_p[[G^{\mathrm{ab}}]] \arrow[d, "\frac{(G:U)}{p}\varphi"] \\
\mathbb{Q}_p[[\mathrm{Conj}(G)]] \arrow[r, "\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\theta^+_{U,V}"'] & \mathbb{Q}_p[[U/V]].
\end{tikzcd} \tag{6.5}$$

Note that $\varphi(\tilde{\theta}_{\mathrm{ab}}(\eta))^{-(G:U)/p}\tilde{\theta}_U(\eta)$ is contained in $1+J_U^{\sim}$ for each U in $\mathfrak{F}_{A,c}$ because $(\tilde{\theta}_{U,V}(\eta))_{(U,V)\in\mathfrak{F}_B}$ is an element in $\widetilde{\Psi}'$ (Lemma 6.4). Then Proposition 5.7 (3) asserts that the element $\log(\varphi(\tilde{\theta}_{\mathrm{ab}}(\eta))^{-(G:U)/p}\tilde{\theta}_U(\eta))$ is contained in $\Lambda(U)$. On the other hand, we may calculate as

(6.6)

$$\begin{aligned}
\theta^+_{U,V}\circ\Gamma_G(\eta) &= (\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\theta^+_{U,V})(\log(\eta)) - (\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\theta^+_{U,V})(p^{-1}\varphi(\log(\eta))) \\
&= \log(\theta_{U,V}(\eta)) - \frac{(G:U)}{p}\varphi(\log(\theta_{\mathrm{ab}}(\eta))) \\
&= \log\frac{\theta_{U,V}(\eta)}{\varphi(\theta_{\mathrm{ab}}(\eta))^{(G:U)/p}}
\end{aligned}$$

for each (U,V) in $\mathfrak{F}_B$ (the first equality is nothing but the definition of the integral logarithm and the second follows from Lemma 6.6 and (6.5)).[10] In particular $\log(\varphi(\theta_{\mathrm{ab}}(\eta))^{-(G:U)/p}\theta_U(\eta))$ is contained in I_U for each U in $\mathfrak{F}_{A,c}$ by definition. Recall that for each U in $\mathfrak{F}_{A,c}$ the p-adic logarithm is injective on $1+J_U^{\sim}$ (Proposition 5.7) and it induces an isomorphism between $1+I_U^{\sim}$ and I_U (Proposition 5.9 (3)) unless U coincides with G. Therefore we may conclude that $\varphi(\tilde{\theta}_{\mathrm{ab}}(\eta))^{-(G:U)/p}\tilde{\theta}_U(\eta)$ is contained in $1+I_U^{\sim}$, which implies the desired additional congruence for U. □

By Lemma 6.4 and Proposition 6.5, we may conclude that $\widetilde{\Psi}$ (resp. $\widetilde{\Psi}_c$) contains the image of $\tilde{\theta}$; in other words, $\tilde{\theta}$ induces a homomorphism

$$\tilde{\theta}\colon \widetilde{K}_1(\Lambda(G)) \to \widetilde{\Psi} \quad (\text{resp. } \widetilde{\Psi}_c).$$

6.4. **Proof of the bijectivity of $\tilde{\theta}$.** We shall verify the bijectivity of the above induced map $\tilde{\theta}$ in this subsection.

Proposition 6.7. *The homomorphism $\tilde{\theta}\colon \widetilde{K}_1(\Lambda(G)) \to \widetilde{\Psi}_c$ is injective.*

Proof. Take an arbitrary element from the kernel of $\tilde{\theta}$ and let η denote its lift to $K_1(\Lambda(G))$. Then $\theta^+_{U,V}\circ\Gamma_G(\eta)$ vanishes for each (U,V) in $\mathfrak{F}_B$ by (6.6). Hence $\Gamma_G(\eta)$ coincides with zero since θ^+_B is injective (Proposition 4.5); equivalently the element η is contained in the kernel of the integral logarithm Γ_G. Combining this fact with Wall's theorem (see [Wall, Theorem 4.1] and (6.3)), we may regard η as an element in $\mu_{p-1}(\mathbb{Z}_p)\times G^{\mathrm{ab}}\times SK_1(\mathbb{Z}_p[G^f])$. Furthermore the abelianisation map θ_{ab} induces the canonical projection

[10]For the abelianisation $\theta_{\mathrm{ab}} = \theta_{G,[G,G]}$, we use the notation $\log(\varphi(\eta)^{-1/p}\theta_{\mathrm{ab}}(\eta))$ for an element defined as $\Gamma_{G_{\mathrm{ab}}}(\theta_{\mathrm{ab}}(\eta)) = \log(\theta_{\mathrm{ab}}(\eta)) - p^{-1}\log(\varphi(\theta_{\mathrm{ab}}(\eta)))$ by abuse of notation.

from $\mu_{p-1}(\mathbb{Z}_p) \times G^{\mathrm{ab}} \times SK_1(\mathbb{Z}_p[G^f])$ onto $\mu_{p-1}(\mathbb{Z}_p) \times G^{\mathrm{ab}}$ when it is restricted to the kernel of Γ_G. Since $\tilde{\theta}_{\mathrm{ab}}(\eta)$ vanishes by assumption, the element η is contained in $G^{\mathrm{ab},f} \times SK_1(\mathbb{Z}_p[G^f])$, and in particular η is a p-torsion element. This implies that the image of η in $\widetilde{K}_1(\Lambda(G))$ is trivial. □

Proposition 6.8. *The homomorphism $\tilde{\theta}\colon \widetilde{K}_1(\Lambda(G)) \to \widetilde{\Psi}$ is surjective.*

Let $\eta_\bullet$ be an element in $\widetilde{\Psi}$. Since $\eta_\bullet$ is in particular contained in $\widetilde{\Psi}'$, the element $\log(\varphi(\eta_{\mathrm{ab}})^{-(G:U)/p}\eta_{U,V})$ can be defined as an element in $\Lambda(U/V)$ for each (U,V) in $\mathfrak{F}_B$ (Proposition 5.7 (2) and the definition of the integral logarithm for G^{ab}).

Lemma 6.9. *The element $(\log(\varphi(\eta_{\mathrm{ab}})^{-(G:U)/p}\eta_{U,V}))_{(U,V)\in\mathfrak{F}_B}$ is contained in Φ_B. Moreover $(\log(\varphi(\eta_{\mathrm{ab}})^{-(G:U)/p}\eta_U))_{U\in\mathfrak{F}_A}$ is contained in Φ.*

Proof. Set $y_{U,V} = \log(\varphi(\eta_{\mathrm{ab}})^{-(G:U)/p}\eta_{U,V})$ for each (U,V) in $\mathfrak{F}_B$. Then we may easily verify that $(y_{U,V})_{(U,V)\in\mathfrak{F}_B}$ satisfies both (TCC) and (CCC+) (due to (NCC), (CCC) and Lemma 6.6). Hence $(y_{U,V})_{(U,V)\in\mathfrak{F}_B}$ is contained in Φ_B. Moreover $\varphi(\eta_{\mathrm{ab}})^{-(G:U)/p}\eta_U$ is contained in $1+\widetilde{I_U}$ for each U in $\mathfrak{F}_A$ by additional congruence condition, and thus $y_U = \log(\varphi(\eta_{\mathrm{ab}})^{-(G:U)/p}\eta_U)$ is contained in I_U by Proposition 5.9. This implies that $(y_U)_{U\in\mathfrak{F}_A}$ is an element in Φ. □

Proof of Proposition 6.8. First note that there exists a unique element y in $\mathbb{Z}_p[[\mathrm{Conj}(G)]]$ which satisfies $\theta_B^+(y) = (\log(\varphi(\eta_{\mathrm{ab}})^{-(G:U)/p}\eta_{U,V}))_{(U,V)\in\mathfrak{F}_B}$ by Proposition 4.5 and Lemma 6.9. In particular the equation

$$\theta_{\mathrm{ab}}^+(y) = \log \eta_{\mathrm{ab}} - \frac{1}{p}\varphi(\log \eta_{\mathrm{ab}}) = \Gamma_{G_{\mathrm{ab}}}(\eta_{\mathrm{ab}}) \tag{6.7}$$

holds. Then we may calculate as

$$\omega_G(y) = \omega_{G^{\mathrm{ab}}} \circ \theta_{\mathrm{ab}}^+(y) = \omega_{G^{\mathrm{ab}}} \circ \Gamma_{G^{\mathrm{ab}}}(\eta_{\mathrm{ab}}) = 1$$

where the first equality directly follows from the definition of ω_G and $\omega_{G^{\mathrm{ab}}}$ (see Section 6.2), the second follows from (6.7) and the last follows from (6.4). The exact sequence (6.4) also asserts that there exists an element η' in $K_1(\Lambda(G))$ which satisfies $\Gamma_G(\eta') = y$. Furthermore we obtain

$$\Gamma_{G^{\mathrm{ab}}}(\tilde{\theta}_{\mathrm{ab}}(\eta')) = \theta_{\mathrm{ab}}^+ \circ \Gamma_G(\eta') = \theta_{\mathrm{ab}}^+(y) = \Gamma_{G^{\mathrm{ab}}}(y_{\mathrm{ab}})$$

by using (6.7). Since the kernel of $\Gamma_{G^{\mathrm{ab}}}$ is identified with $\mu_{p-1}(\mathbb{Z}_p) \times G^{\mathrm{ab}}$ by the theorem of Higman [Higman], there exists an element τ in $\mu_{p-1}(\mathbb{Z}_p) \times G^{\mathrm{ab}}$ such that the equation $\tilde{\theta}_{\mathrm{ab}}(\eta')\tau = \eta_{\mathrm{ab}}$ holds. Set $\eta = \eta'\tau$. By construction, the abelianisation $\tilde{\theta}_{\mathrm{ab}}(\eta)$ of η coincides with η_{ab} and

$$\log \frac{\eta_{U,V}}{\varphi(\eta_{\mathrm{ab}})^{(G:U)/p}} = \theta_{U,V}^+(y) = \theta_{U,V}^+ \circ \Gamma_G(\eta) = \log \frac{\tilde{\theta}_{U,V}(\eta)}{\varphi(\tilde{\theta}_{\mathrm{ab}}(\eta))^{(G:U)/p}}$$

holds for each (U,V) in $\mathfrak{F}_B$ except for $(G,[G,G])$ (the first equality is due to the construction of y and the last is due to (6.6)). Then $\tilde{\theta}_{U,V}(\eta)$ coincides with $\eta_{U,V}$ because the p-adic logarithm induces an injection on $1+\widetilde{J_{U,V}}$ (Proposition 5.7); in other words the image of η under the map $\tilde{\theta}$ coincides with $\eta_\bullet$, which asserts that $\tilde{\theta}\colon \widetilde{K}_1(\Lambda(G)) \to \widetilde{\Psi}$ is surjective. □

7. Localised version

In this section we study 'the localised theta map;' more precisely, let $\theta_{S,U,V}$ be the composition of the norm map $\mathrm{Nr}_{\Lambda(G)_S/\Lambda(U)_S}$ with the canonical homomorphism $K_1(\Lambda(U)_S) \to \Lambda(U/V)_S^\times$ for each (U,V) in $\mathfrak{F}_B$ and set $\theta_S = (\theta_{S,U,V})_{(U,V)\in\mathfrak{F}_B}$. It is obvious that θ_S induces a group homomorphism $\tilde{\theta}_S \colon \widetilde{K}_1(\Lambda(G)_S) \to \prod_{(U,V)\in\mathfrak{F}_B} \widetilde{\Lambda}(U/V)_S^\times$. We shall study the image of $\tilde{\theta}_S$.

Let $\Lambda(\Gamma)_{(p)}$ denote the localisation of the Iwasawa algebra $\Lambda(\Gamma)$ with respect to the prime ideal $p\Lambda(\Gamma)$, and let R denote its p-adic completion $\Lambda(\Gamma)^{\widehat{}}_{(p)}$ for simplicity. We remark that for each finite p-group Δ, the localised Iwasawa algebra $\Lambda(\Delta\times\Gamma)_S$ is identified with a group ring $\Lambda(\Gamma)_{(p)}[\Delta]$ under the identification $\Lambda(\Delta\times\Gamma) \cong \Lambda(\Gamma)[\Delta]$ (see [CFKSV, Lemma 2.1]). Now for each (U,V) in $\mathfrak{F}_B$, let $J_{S,U,V}$ (resp. $\widehat{J_{U,V}}$) be the kernel of the composition

$$\Lambda(U/V)_S \xrightarrow{\text{augmentation}} \Lambda(\Gamma)_{(p)} \to \Lambda(\Gamma)_{(p)}/p\Lambda(\Gamma)_{(p)}$$
$$(\text{resp. } R[U^f/V^f] \xrightarrow{\text{augmentation}} R \longrightarrow R/pR).$$

Then we may easily verify that the intersection of $\widehat{J_{U,V}}$ and $\Lambda(U/V)_S$ (resp. $J_{S,U,V}$ and $\Lambda(U/V)$) coincides with $J_{S,U,V}$ (resp. $J_{U,V}$) under the identification $\Lambda(U/V)_S \cong \Lambda(\Gamma)_{(p)}[U^f/V^f]$. Since the group ring $R[U^f/V^f]$ is p-adically complete, the p-adic logarithm converges on $1+\widehat{J_{U,V}}$ and induces an injection $\log\colon (1+\widehat{J_{U,V}})^{\sim} \to R[U^f/V^f]$ unless U coincides with G (can be verified similarly to Proposition 5.7). Let $\widetilde{\Psi}'_S$ be the subgroup of the direct product $\prod_{(U,V)\in\mathfrak{F}_B} \widetilde{\Lambda}(U/V)_S^\times$ consisting of all elements $\eta_{S,\bullet}$ satisfying norm compatibility condition $(\mathrm{NCC})_S$, conjugacy compatibility condition $(\mathrm{CCC})_S$ and the following congruence for each (U,V) in $\mathfrak{F}_B$ except for $(G,[G,G])$:[11]

$$\eta_{S,U,V} \equiv \varphi(\eta_{S,\mathrm{ab}})^{(G:U)/p} \mathbin{\widetilde{\bmod}} J_{S,U,V}.$$

Let $\widetilde{\Psi}_S$ (resp. $\widetilde{\Psi}_{S,c}$) be the subgroup of $\widetilde{\Psi}'_S$ consisting of all elements $\eta_{S,\bullet}$ in $\widetilde{\Psi}'_S$ satisfying the following additional congruence condition:

- (additional congruence condition)
  the congruence $\eta_{S,U} \equiv \varphi(\eta_{S,\mathrm{ab}})^{(G:U)/p} \mathbin{\widetilde{\bmod}} I_{S,U}$ holds for each U in $\mathfrak{F}_A$ (resp. $\mathfrak{F}_{A,c}$) where $I_{S,U}$ is the $\Lambda(\Gamma)_{(p)}$-module defined as $I_U \otimes_{\Lambda(\Gamma)} \Lambda(\Gamma)_{(p)}$.

The group $\widetilde{\Psi}_{S,c}$ is a subgroup of $\widetilde{\Psi}_S$ (as $\widetilde{\Psi}_c$ is that of $\widetilde{\Psi}$; see also Remark 6.1).

Lemma 7.1. *The intersection of $\widehat{I_U}$ and $\Lambda(U)_S$ (resp. $I_{S,U}$ and $\Lambda(U)$) coincides with $I_{S,U}$ (resp. I_U).*

Proof. We shall only prove the claim $I_{S,U} \cap \Lambda(U) = I_U$ (the other one is verified by much simpler calculation). The $\mathbb{Z}_p$-module I_U is obviously contained in the intersection $I_{S,U} \cap \Lambda(U)$ by construction. Note that $I_{S,U}$ is a free $\Lambda(\Gamma)_{(p)}$-submodule of $\Lambda(U)_S$ each of whose generators is obtained as a finite sum of $\{p^j u\}_{0\leq j\leq N, u\in U^f}$ (see the explicit description of I_U given in Section 4.2). Hence an arbitrary element in $I_{S,U} \cap \Lambda(U)$ is described

[11] We may naturally extend both (NCC) and (CCC) to the localised versions $(\mathrm{NCC})_S$ and $(\mathrm{CCC})_S$ in an obvious manner.

as a $\Lambda(\Gamma)_{(p)} \cap \Lambda(\Gamma)[p^{-1}]$-linear combination of generators of $I_{S,U}$, which implies that the intersection $I_{S,U} \cap \Lambda(U)$ is contained in I_U (observe that $\Lambda(\Gamma)_{(p)} \cap \Lambda(\Gamma)[p^{-1}]$ coincides with $\Lambda(\Gamma)$ and generators of $I_{S,U}$ over $\Lambda(\Gamma)_{(p)}$ coincide with those of I_U over $\Lambda(\Gamma)$). □

Proposition 7.2. *Both $\widetilde{\Psi}_S$ and $\widetilde{\Psi}_{S,c}$ contain the image of $\tilde{\theta}_S$.*

Sketch of the proof. Let η_S be an arbitrary element in $\widetilde{K}_1(\Lambda(G)_S)$. By the same argument as that in the proof of Lemma 6.4, we may verify that $\widetilde{\Psi}'_S$ contains the image of $\tilde{\theta}_S$. Then the element $\varphi(\theta_{S,\mathrm{ab}}(\eta_S))^{-(G:U)/p}\theta_{S,U,V}(\eta_S)$ (which we denote by $\eta'_{S,U,V}$ in the following) is contained in $1+J_{S,U,V}$ for (U,V) in $\mathfrak{F}_B$ except for $(G,[G,G])$ by congruence condition, and it is regarded as an element in $1+\widehat{J_{U,V}}$ in a natural way. Hence we may define $\log \eta'_{S,U,V}$ as an element in $R[U^f/V^f]$. On the other hand we may easily show that for each U in $\mathfrak{F}_{A,c}$ the image of the trace map $\mathrm{Tr}_{R[\mathrm{Conj}(G^f)]/R[U^f]}$ coincides with the R-module $\widehat{I_U}$ defined as $I_U \otimes_{\Lambda(\Gamma)} R$ (here we assume that U does not coincide with G; see Remark 6.1). Moreover the image of η_S under the composite map $\mathrm{Tr}_{R[\mathrm{Conj}(G^f)]/R[U^f]} \circ \Gamma_{R,G^f}$ is calculated as $\log \eta'_{S,U}$ by simple calculation similar to (6.6) where Γ_{R,G^f} is the integral logarithm $K_1(R[G^f]) \to R[\mathrm{Conj}(G^f)]$ with coefficient in R (see [H, Section 1.1 and Remark 5.2]). This implies that $\log \eta'_{S,U}$ is contained in $\widehat{I_U}$ for each U in $\mathfrak{F}_{A,c}$, and therefore we obtain $\tilde{\theta}_{S,U}(\eta_S) \equiv \varphi(\tilde{\theta}_{S,\mathrm{ab}}(\eta_S))^{(G:U)/p} \mathbin{\widetilde{\mathrm{mod}}} I_{S,U}$ by the logarithmic isomorphism $1+\widehat{I_U} \xrightarrow{\sim} \widehat{I_U}$ (readily verified in the same manner as Proposition 5.9) and the relation $\widehat{I_U} \cap \Lambda(U)_S = I_{S,U}$ (Lemma 7.1). Consequently $(\tilde{\theta}_{S,U,V}(\eta_S))_{(U,V)\in\mathfrak{F}_B}$ is contained in $\widetilde{\Psi}_{S,c}$ (and hence in $\widetilde{\Psi}_S$). □

Proposition 7.3. *The intersection of $\widetilde{\Psi}_S$ (resp. $\widetilde{\Psi}_{S,c}$) and the direct product $\prod_{(U,V)\in\mathfrak{F}_B} \widetilde{\Lambda}(U/V)^\times$ coincides with $\widetilde{\Psi}$ $(=\widetilde{\Psi}_c)$.*

Proof. Use relations $I_{S,U} \cap \Lambda(U) = I_U$ for each U in $\mathfrak{F}_{A,c}$ (Lemma 7.1) and $J_{S,U,V} \cap \Lambda(U/V) = J_{U,V}$ for each (U,V) in $\mathfrak{F}_B$. □

8. Weak congruences between abelian p-adic zeta functions

In this section we study properties of the p-adic zeta pseudomeasures for extensions corresponding to certain abelian subquotients of G —especially *congruences* which they satisfy. In the rest of this article, *we fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.*

8.1. Weak Congruences.

For each (U,V) in $\mathfrak{F}_B$, let $\xi_{U,V}$ denote Serre's p-adic zeta pseudomeasure for the abelian extension F_V/F_U (which is an element in $\Lambda(U/V)^\times_S$).

Lemma 8.1. *The element $(\xi_{U,V})_{(U,V)\in\mathfrak{F}_B}$ in $\prod_{(U,V)\in\mathfrak{F}_B} \widetilde{\Lambda}(U/V)^\times_S$ satisfies both norm compatibility condition* (NCC)$_S$ *and conjugacy compatibility condition* (CCC)$_S$ *in Section 7.*

Proof. Let (U,V) and (U',V') be elements in $\mathfrak{F}_B$ such that U contains U' and U' contains V respectively. Then we may easily verify that

$$\mathrm{Nr}_{\Lambda(U/V)_S/\Lambda(U'/V)_S}(f)(\rho) = f(\mathrm{Ind}_{U'}^{U}(\rho))$$

holds for an arbitrary element f in $\Lambda(U/V)_S^\times$ and an arbitrary continuous p-adic character ρ of the abelian group U'/V (due to the definition of the evaluation map). Hence for an arbitrary character χ of finite order of U'/V and an arbitrary natural number r divisible by $p-1$, the following equation holds by the interpolation property (2.1) of $\xi_{U,V}$:

$$\begin{aligned}\mathrm{Nr}_{\Lambda(U/V)_S/\Lambda(U'/V)_S}(\xi_{U,V})(\chi\kappa^r) &= \xi_{U,V}(\mathrm{Ind}_{U'}^{U}(\chi\kappa^r)) \\ &= L_{\Sigma_{F_U}}(1-r; F_V/F_U, \mathrm{Ind}_{U'}^{U}(\chi)) \\ &= L_{\Sigma_{F_{U'}}}(1-r; F_V/F_{U'}, \chi) = \xi_{U',V}(\chi\kappa^r).\end{aligned}$$

Then uniqueness of the abelian p-adic zeta pseudomeasures for $F_V/F_{U'}$ asserts that the norm image $\mathrm{Nr}_{\Lambda(U/V)_S/\Lambda(U'/V)_S}(\xi_{U,V})$ of $\xi_{U,V}$ coincides with $\xi_{U',V}$. The equation $\mathrm{can}_V^{V'}(\xi_{U',V'}) = \xi_{U',V}$ is also straightforward to verify, and therefore $(\xi_{U,V})_{(U,V)\in\mathfrak{F}_B}$ satisfies $(\mathrm{NCC})_S$. By a similar formal argument we may also prove that $(\xi_{U,V})_{(U,V)\in\mathfrak{F}_B}$ satisfies $(\mathrm{CCC})_S$, but we omit the details. □

Therefore if $(\xi_{U,V})_{(U,V)\in\mathfrak{F}_B}$ satisfies both congruence condition and additional congruence condition, we may conclude that $(\xi_{U,V})_{(U,V)\in\mathfrak{F}_B}$ is contained in $\widetilde{\Psi}_{S,c}$ (hence also in $\widetilde{\Psi}_S$). It seems, however, to be difficult to prove the desired congruences for $\{\xi_{U,V}\}_{(U,V)\in\mathfrak{F}_B}$ directly. In the rest of this section we shall prove the following *weak congruences* by using Deligne-Ribet's theory concerning Hilbert modular forms [DR] (especially using *the q-expansion principle*).

Proposition 8.2 (weak congruences). *Let (U,V) be an element in $\mathfrak{F}_B$ such that U does not coincide with G, then there exists an element $c_{U,V}$ in $\widetilde{\Lambda}(\Gamma)_{(p)}^\times$ and the congruence*

$$\xi_{U,V} \equiv c_{U,V} \quad \widetilde{\mathrm{mod}}\, J_{S,U,V} \tag{8.1}$$

holds. If U is an element in $\mathfrak{F}_{A,c}$, the congruence

$$\xi_U \equiv c_U \quad \widetilde{\mathrm{mod}}\, I'_{S,U} \tag{8.2}$$

also holds where I'_U is the image of the trace map from $\mathbb{Z}_p[[\mathrm{Conj}(NU)]]$ to $\mathbb{Z}_p[[U]]$ and $I'_{S,U}$ is its scalar extension $I'_U \otimes_{\Lambda(\Gamma)} \Lambda(\Gamma)_{(p)}$.

Remark 8.3. We may obtain an explicit description of each I'_U by easy calculation similar to that in Section 4.2 as follows:

$$\begin{aligned}
I'_\Gamma &= p^N \mathbb{Z}_p[[\Gamma]]; \\
I'_{U_h} &= p^{n_h-1}\mathbb{Z}_p[[U_h]] && \text{for } h \text{ in } \mathfrak{H}\setminus\{e\}, \\
I'_{U_{h,c}} &= p^{n_h-2}\mathbb{Z}_p[[U_{h,c}]] && \text{for } h \text{ in } \mathfrak{H}\setminus\{e,c\} \text{ satisfying (Case-1)}, \\
I'_{U_{h,c}} &= p^{n_h-1}\mathbb{Z}_p[[U_c]] \oplus \bigoplus_{i=1}^{p-1} p^{n_h-2}h^i(1+c^2+\cdots+c^{p-1})\mathbb{Z}_p[[\Gamma]] \\
& && \text{for } h \text{ in } \mathfrak{H}\setminus\{e,c\} \text{ satisfying (Case-2)}.
\end{aligned}$$

Each I'_U (resp. $I'_{S,U}$) obviously contains I_U (resp. $I_{S,U}$). Moreover the p-adic logarithm induces an isomorphism between $1+I'_U$ and I'_U (resp. between $1+(I'_U)^\frown$ and $(I'_U)^\frown$ where $(I'_U)^\frown$ is defined as $I'_U \otimes_{\Lambda(\Gamma)} R$) by the same argument as that in the proof of Proposition 5.9.

8.2. **Ritter-Weiss' approximation technique.** In their work concerning the 'main conjecture' of equivariant Iwasawa theory, Ritter and Weiss approximated p-adic zeta pseudomeasures by using special values of partial zeta functions, and derived certain congruences between p-adic zeta pseudomeasures [RW6]. In this section we shall derive sufficient condition for Proposition 8.2 to hold by applying their approximation technique. Fix an element (U,V) in $\mathfrak{F}_B$ such that U does not coincide with G, and let W denote the quotient group U/V (which is abelian by definition). For an arbitrary open subgroup $\mathcal{U}$ of W, we define the natural number $m(\mathcal{U})$ by $\kappa^{p-1}(\mathcal{U}) = 1+p^{m(\mathcal{U})}\mathbb{Z}_p$ where κ is the p-adic cyclotomic character. Then we obtain an isomorphism

$$\mathbb{Z}_p[[W]] \xrightarrow{\sim} \varprojlim_{\mathcal{U}\trianglelefteq W:\text{ open}} \mathbb{Z}_p[W/\mathcal{U}]/p^{m(\mathcal{U})}\mathbb{Z}_p[W/\mathcal{U}] \tag{8.3}$$

(see [RW6, Lemma 1] for details).

Definition 8.4 (partial zeta function). Let ε be a $\mathbb{C}$-valued locally constant function on W. If ε is constant on an open subgroup $\mathcal{U}$ of W, we may identify ε with a $\mathbb{C}$-linear combination $\sum_{x\in W/\mathcal{U}} \varepsilon(x)\delta^{(x)}$ where $\delta^{(x)}$ is 'the Dirac delta function at x' (that is, $\delta^{(x)}(w)$ equals 1 if w is in the coset x and 0 otherwise). Then we define *the (Σ_{F_U}-truncated) partial zeta function* $\zeta^{\Sigma_{F_U}}_{F_V/F_U}(s,\varepsilon)$ *for F_V/F_U with respect to the locally constant function ε* as $\sum_{x\in W/\mathcal{U}} \varepsilon(x)\zeta^{\Sigma_{F_U}}_{F_V/F_U}(s,\delta^{(x)})$ where $\zeta^{\Sigma_{F_U}}_{F_V/F_U}(s,\delta^{(x)})$ is defined as the Dirichlet series

$$\zeta^{\Sigma_{F_U}}_{F_V/F_U}(s,\delta^{(x)}) = \sum_{0\neq\mathfrak{a}\subseteq\mathcal{O}_{F_U}:\text{ integral ideal prime to }\Sigma_{F_U}} \frac{\delta^{(x)}((F_V/F_U,\mathfrak{a}))}{(\mathcal{N}\mathfrak{a})^s}$$

(the symbol $(F_V/F_U,-)$ denotes the Artin symbol for the abelian extension F_V/F_U and $\mathcal{N}\mathfrak{a}$ denotes the absolute norm of the ideal $\mathfrak{a}$). It is meromorphically continued to the whole complex plane $\mathbb{C}$.

For an arbitrary natural number k divisible by $p-1$ and an arbitrary element w in W, set

$$\Delta^{w}_{F_V/F_U}(1-k,\varepsilon) = \zeta^{\Sigma_{F_U}}_{F_V/F_U}(1-k,\varepsilon) - \kappa(w)^k \zeta^{\Sigma_{F_U}}_{F_V/F_U}(1-k,\varepsilon_w)$$

which is a p-adic rational number due to the results of Klingen and Siegel [Klingen, Siegel] (we denote by ε_w the translation of ε by w; in other words ε_w is defined by $\varepsilon_w(w') = \varepsilon(ww')$).

Proposition 8.5 (approximation lemma). *Let $\mathcal{U}$ be an arbitrary open normal subgroup of W. Then for each k divisible by $p-1$ and each w in W, the image of the element $(1-w)\xi_{U,V}$ under the canonical surjection $\mathbb{Z}_p[[W]] \to \mathbb{Z}_p[W/\mathcal{U}]/p^{m(\mathcal{U})}\mathbb{Z}_p[W/\mathcal{U}]$ is described as*

$$\sum_{x\in W/\mathcal{U}} \Delta^{w}_{F_V/F_U}(1-k,\delta^{(x)})\kappa(x)^{-k}x \qquad \mathrm{mod}\, p^{m(\mathcal{U})}.$$

Proof. See [RW6, Proposition 2]. □

Let j be a natural number and NU the normaliser of U. Then the quotient group NU/U acts upon W/Γ^{p^j} by conjugation (recall that Γ^{p^j} is abelian). For each coset y of W/Γ^{p^j}, let $(NU/U)_y$ denote the isotropy subgroup of NU/U at y under this action.

Proposition 8.6 (sufficient condition). *Let (U,V) be an element in $\mathfrak{F}_B$ except for $(G,[G,G])$. Then the congruence* (8.1) *holds if the congruence*

$$(8.4) \qquad \Delta^{w}_{F_V/F_U}(1-k,\delta^{(y)}) \equiv 0 \qquad \mathrm{mod}\, \sharp(NU/U)_y\mathbb{Z}_p$$

holds for an arbitrary element w in Γ, an arbitrary natural number k divisible by $p-1$ and an arbitrary coset y of W/Γ^{p^j} not contained in Γ. If $U=(U,\{e\})$ is an element in $\mathfrak{F}_{A,c}$ and does not coincide with G, the congruence (8.4) *also gives sufficient condition for the congruence* (8.2) *to hold.*

Proof. Apply the approximation lemma (Proposition 8.5) to the element $(1-w)\xi_{U,V}$. Then its image under the canonical surjection from $\mathbb{Z}_p[[W]]$ onto $\mathbb{Z}_p[W/\Gamma^{p^j}]/p^{m(\Gamma^{p^j})}\mathbb{Z}_p[W/\Gamma^{p^j}]$ is described as

$$(8.5) \qquad \sum_{y\in W/\Gamma^{p^j}} \Delta^{w}_{F_V/F_U}(1-k,\delta^{(y)})\kappa(y)^{-k}y \qquad \mathrm{mod}\, p^{m(\Gamma^{p^j})}$$

for an arbitrary natural number k divisible by $p-1$. Let y be a coset of W/Γ^{p^j} not contained in Γ, and consider the NU/U-orbital sum in (8.5) containing the term associated to y. We may calculate it by applying (8.4) as follows:[12]

$$\sum_{\sigma\in(NU/U)/(NU/U)_y} \Delta^w_{F_V/F_U}(1-k,\delta^{(\sigma^{-1}y\sigma)})\kappa(\sigma^{-1}y\sigma)^{-k}\sigma^{-1}y\sigma$$
$$=\Delta^w_{F_V/F_U}(1-k,\delta^{(y)})\kappa(y)^{-k}\sum_{\sigma\in(NU/U)/(NU/U)_y}\sigma^{-1}y\sigma$$
$$\equiv \sharp(NU/U)_y \sum_{\sigma\in(NU/U)/(NU/U)_y}\sigma^{-1}y\sigma \qquad \bmod \sharp(NU/U)_y.$$

If we set

$$P_y=\sharp(NU/U)_y\sum_{\sigma\in(NU/U)/(NU/U)_y}\sigma^{-1}y\sigma,$$

the element $\mathrm{aug}_{U,V}(P_y)=\sharp(NU/U)\mathrm{aug}_{U,V}(y)$ is obviously divisible by p (note that the normaliser of U is strictly greater than U since U is a proper open subgroup of the pro-p group G). This calculation implies that P_y is an element in $J_{U,V}\otimes_{\Lambda(\Gamma)}\mathbb{Z}_p[\Gamma/\Gamma^{p^j}]/p^{m(\Gamma^{p^j})}$. If U is an element in $\mathfrak{F}_{A,c}$, it is also obvious that P_y is no other than the image of y under the trace map from $\mathbb{Z}_p[[\mathrm{Conj}(NU)]]$ to $\mathbb{Z}_p[[U]]$. Therefore P_y is contained in $I'_U\otimes_{\Lambda(\Gamma)}\mathbb{Z}_p[\Gamma/\Gamma^{p^j}]/p^{m(\Gamma^{p^j})}$. Clearly $\Delta^w_{F_V/F_U}(1-k,\delta^{(y)})\kappa(y)^{-k}y$ is an element in $\mathbb{Z}_p[\Gamma/\Gamma^{p^j}]/p^{m(\Gamma^{p^j})}$ if y is a coset contained in Γ, and hence we may show by taking the projective limit that the element $(1-w)\xi_{U,V}$ (resp. $(1-w)\xi_U$ for U in $\mathfrak{F}_{A,c}$) is contained in $\Lambda(\Gamma)+J_{U,V}$ (resp. $\Lambda(\Gamma)+I'_U$). Since $1-w$ is an invertible element in $\Lambda(U/V)_S$, we obtain the desired congruences (8.1) and (8.2). □

8.3. **Deligne-Ribet's theory of Hilbert modular forms.** We briefly summarise the theory of Deligne and Ribet concerning Hilbert modular forms [DR] in this subsection, which we shall use in verification of sufficient condition (8.4).

Let K be a totally real number field of degree r and K_∞/K an *abelian* totally real p-adic Lie extension. Let $\mathfrak{D}$ be the different of K and Σ a finite set of prime ideals of K. Assume that Σ contains all primes which ramify in K_∞ (we fix such a finite set Σ throughout the following argument). We denote by $\mathfrak{h}_K$ the Hilbert upper-half space associated to K defined as $\{\tau\in K\otimes\mathbb{C}\mid \mathrm{Im}(\tau)\gg 0\}$. For an even natural number k, we define the action of $\mathrm{GL}_2(K)^+$—subgroup of $\mathrm{GL}_2(K)$ consisting of all matrices with totally positive determinants— upon the set of $\mathbb{C}$-valued functions on $\mathfrak{h}_K$ by

$$(F|_k\begin{pmatrix}a&b\\c&d\end{pmatrix})(\tau)=\mathcal{N}(ad-bc)^{k/2}\mathcal{N}(c\tau+d)^{-k}F(\frac{a\tau+b}{c\tau+d})$$

where $\mathcal{N}\colon K\otimes\mathbb{C}\to\mathbb{C}$ denotes the usual norm map.

[12] We need the fact that w commutes with each σ in NU/U to guarantee the equality $\Delta^w_{F_V/F_U}(1-k,\delta^{(\sigma^{-1}y\sigma)})=\Delta^w_{F_V/F_U}(1-k,\delta^{(y)})$.

Definition 8.7 (Hilbert modular forms). Let $\mathfrak{f}$ be an integral ideal of $\mathcal{O}_K$ all of whose prime factors are contained in Σ and set

$$\Gamma_{00}(\mathfrak{f}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K) \mid a, d \in 1+\mathfrak{f}, b \in \mathfrak{D}^{-1}, c \in \mathfrak{f}\mathfrak{D}\}.$$

Then *a Hilbert modular form F of (parallel) weight k on $\Gamma_{00}(\mathfrak{f})$* is defined as a holomorphic function $F\colon \mathfrak{h}_K \to \mathbb{C}$ which is fixed by the action of $\Gamma_{00}(\mathfrak{f})$ (namely $F|_k M = F$ holds for an arbitrary element M in $\Gamma_{00}(\mathfrak{f})$).[13]

Let $\mathbb{A}_K^{\mathrm{fin}}$ denote the finite adèle ring of K. Then $\mathrm{SL}_2(\mathbb{A}_K^{\mathrm{fin}})$ is decomposed as $\hat{\Gamma}_{00}(\mathfrak{f})\cdot\mathrm{SL}_2(K)$ by the strong approximation theorem (we denote by $\hat{\Gamma}_{00}(\mathfrak{f})$ the topological closure of $\Gamma_{00}(\mathfrak{f})$ in $\mathrm{SL}_2(\mathbb{A}_K^{\mathrm{fin}})$). We define the action of $\mathrm{SL}_2(\mathbb{A}_K^{\mathrm{fin}})$ upon the set of all $\mathbb{C}$-valued functions on $\mathfrak{h}_K$ by $F|_k M = F|_k M_{\mathrm{SL}_2(K)}$ where $M_{\mathrm{SL}_2(K)}$ is the $\mathrm{SL}_2(K)$-factor of M in $\mathrm{SL}_2(\mathbb{A}_K^{\mathrm{fin}})$. For a finite idèle α of K and a Hilbert modular form F of weight k on $\Gamma_{00}(\mathfrak{f})$, set

$$F_\alpha = F|_k \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}.$$

Then F_α has a Fourier series expansion

$$F_\alpha = c(0,\alpha) + \sum_{\mu \in \mathcal{O}_K, \mu \gg 0} c(\mu,\alpha) q_K^\mu, \quad q_K^\mu = \exp(2\pi\sqrt{-1}\mathrm{Tr}_{K/\mathbb{Q}}(\mu\tau))$$

which we call *the q-expansion of F at the cusp determined by α*. Especially, the q-expansion of F at the cusp ∞ (determined by 1) is called *the standard q-expansion of F*. Deligne and Ribet proved the following deep theorem concerning the integrality of coefficients appearing in the q-expansion of a Hilbert modular form.

Theorem 8.8 ([DR, Theorem (0.2)]). *Let F_k be a Hilbert modular form of weight k on $\Gamma_{00}(\mathfrak{f})$. Assume that all coefficients of the q-expansion of F_k at an arbitrary cusp are rational numbers, and assume also that F_k is equal to zero for all but finitely many k. Set $\mathscr{F}(\alpha) = \sum_{k\geq 0} \mathcal{N}\alpha_p^{-k} F_{k,\alpha}$ for a finite idèle α of K whose p-th component we denote by α_p. Then if the q-expansion of $\mathscr{F}(\gamma)$ has all its coefficients in $p^j\mathbb{Z}_p$ for a certain finite idèle γ and a certain integer j, the q-expansion of $\mathscr{F}(\alpha)$ for an arbitrary finite idèle α also has all its coefficients in $p^j\mathbb{Z}_p$.*

The following corollary—so-called *the q-expansion principle*— plays the most important role in verification of sufficient condition (8.4).

Corollary 8.9 (q-expansion principle). *Let F_k and $\mathscr{F}(\alpha)$ be as in Theorem 8.8 and j an integer. Suppose that the q-expansion of $\mathscr{F}(\gamma)$ has all its non-constant coefficients in $p^j\mathbb{Z}_{(p)}$ for a certain finite idèle γ. Then for arbitrary two distinct finite idèles α and β, the difference between the constant terms of the q-expansions of $\mathscr{F}(\alpha)$ and $\mathscr{F}(\beta)$ is also contained in $p^j\mathbb{Z}_{(p)}$.*

Proof. Just apply Theorem 8.8 to $\mathscr{F}(\alpha) - c(0,\gamma)$ and $\mathscr{F}(\beta) - c(0,\gamma)$ where $c(0,\gamma)$ is the constant term of the q-expansion of $\mathscr{F}(\gamma)$. See also [DR, Corollary (0.3)]. □

[13] If K is the rational number field $\mathbb{Q}$, we assume that F is holomorphic at the cusp ∞.

Finally we introduce the Hilbert-Eisenstein series attached to a locally constant $\mathbb{C}$-valued function ε on $\mathrm{Gal}(K_\infty/K)$.

Theorem 8.10 (Hilbert-Eisenstein series). *Let ε be a locally constant function on $\mathrm{Gal}(K_\infty/K)$ and k an even natural number. Then there exists an integral ideal $\mathfrak{f}$ of $\mathcal{O}_K$ all of whose prime factors are contained in Σ, and there exists a Hilbert modular form $G_{k,\varepsilon}$ of weight k on $\Gamma_{00}(\mathfrak{f})$ (which is called* the Hilbert-Eisenstein series of weight k attached to ε) *whose standard q-expansion is given by*

$$2^{-r}\zeta^{\Sigma}_{K_\infty/K}(1-k,\varepsilon)+\sum_{\mu\in\mathcal{O}_K,\mu\gg 0}\left(\sum_{\mu\in\mathfrak{a}\subseteq\mathcal{O}_K,\text{prime to }\Sigma}\varepsilon(\mathfrak{a})\kappa(\mathfrak{a})^{k-1}\right)q_K^{\mu}$$

(we use the notation $\varepsilon(\mathfrak{a})$ and $\kappa(\mathfrak{a})$ for elements defined as $\varepsilon((K_\infty/K,\mathfrak{a}))$ and $\kappa((K_\infty/K,\mathfrak{a}))$ respectively where $(K_\infty/K,-)$ denotes the Artin symbol for the abelian extension K_∞/K). The q-expansion of $G_{k,\varepsilon}$ at the cusp determined by a finite idèle α is given by

(8.6)
$$\mathcal{N}((\alpha))^k\left\{2^{-r}\zeta^{\Sigma}_{K_\infty/K}(1-k,\varepsilon_a)+\sum_{\substack{\mu\in\mathcal{O}_K\\ \mu\gg 0}}\left(\sum_{\substack{\mu\in\mathfrak{a}\subseteq\mathcal{O}_K\\ \text{prime to }\Sigma}}\varepsilon_a(\mathfrak{a})\kappa(\mathfrak{a})^{k-1}\right)q_K^{\mu}\right\}$$

where (α) is the ideal generated by α and a is an element in $\mathrm{Gal}(K_\infty/K)$ defined as $(K_\infty/K,(\alpha)\alpha^{-1})$.

For details, see [DR, Theorem (6.1)].

8.4. **Proof of sufficient conditions.** In the rest of this section we shall verify sufficient condition (8.4). This part is a subtle generalisation of the argument in [H, Section 6.6]. Let j be a sufficiently large integer and y a coset of W/Γ^{p^j} not contained in Γ. Choose an integral ideal $\mathfrak{f}$ of $\mathcal{O}_{F_{NU}}$ such that the Hilbert-Eisenstein series $G_{k,\delta^{(y)}}$ over $\mathfrak{h}_{F_U}$ is defined on $\Gamma_{00}(\mathfrak{f}\mathcal{O}_{F_U})$. Then it is easy to see that the restriction $\mathscr{G}=G_{k,\delta^{(y)}}|_{\mathfrak{h}_{F_{NU}}}$ of $G_{k,\delta^{(y)}}$ to $\mathfrak{h}_{F_{NU}}$ is also a Hilbert modular form of weight $p^{n_U}k$ on $\Gamma_{00}(\mathfrak{f})$ where p^{n_U} is the cardinality of the quotient group NU/U. The q-expansion of $\mathscr{G}$ is directly calculated as

$$2^{-[F_U:\mathbb{Q}]}\zeta^{\Sigma_{F_U}}_{F_V/F_U}(1-k,\delta^{(y)})+\sum_{\substack{\nu\in\mathcal{O}_{F_U}\\ \nu\gg 0}}\left(\sum_{\substack{\nu\in\mathfrak{b}\subseteq\mathcal{O}_{F_U}\\ \text{prime to }\Sigma_{F_U}}}\delta^{(y)}(\mathfrak{b})\kappa(\mathfrak{b})^{k-1}\right)q_{F_{NU}}^{\mathrm{tr}(\nu)}$$

where $q_{F_{NU}}^{\mathrm{tr}(\nu)}$ denotes $\exp(2\pi\sqrt{-1}\mathrm{Tr}_{F_{NU}/\mathbb{Q}}(\mathrm{Tr}_{F_U/F_{NU}}(\nu)\tau))$. Note that the quotient group NU/U naturally acts upon the set of all pairs $(\mathfrak{b},\nu)$ such that $\mathfrak{b}$ is a non-zero integral ideal of $\mathcal{O}_{F_U}$ prime to Σ_{F_U} and ν is a totally positive element in $\mathfrak{b}$. First suppose that the isotropy subgroup $(NU/U)_{(\mathfrak{b},\nu)}$ is trivial. Then we can easily calculate the NU/U-orbital sum in the q-expansion of $\mathscr{G}$ containing the term associated to $(\mathfrak{b},\nu)$ as follows:

$$\sum_{\sigma\in NU/U}\delta^{(y)}(\mathfrak{b}^\sigma)\kappa(\mathfrak{b}^\sigma)^{k-1}q_{F_{NU}}^{\mathrm{tr}(\nu^\sigma)}=\sharp(NU/U)_y\sum_{\sigma\in(NU/U)/(NU/U)_y}\delta^{(\sigma y\sigma^{-1})}(\mathfrak{b})\kappa(\mathfrak{b})^{k-1}q_{F_{NU}}^{\mathrm{tr}(\nu)}$$

(use the obvious formula $\mathrm{Tr}_{F_U/F_{NU}}(\nu^\sigma) = \mathrm{Tr}_{F_U/F_{NU}}(\nu)$).

Next suppose that the isotropy subgroup $(NU/U)_{(\mathfrak{b},\nu)}$ is not trivial. Let $F_{(\mathfrak{b},\nu)}$ be the fixed subfield of F_U by $(NU/U)_{(\mathfrak{b},\nu)}$ and $F^{\mathrm{comm}}_{(\mathfrak{b},\nu)}$ the fixed subfield of F_∞ by the commutator subgroup of $NU_{(\mathfrak{b},\nu)}$. Then $(\mathfrak{b},\nu)$ is fixed by the action of $\mathrm{Gal}(F_U/F_{(\mathfrak{b},\nu)})$, and hence ν is an element in $F_{(\mathfrak{b},\nu)}$ and there exists a non-zero integral ideal $\mathfrak{a}$ of $\mathcal{O}_{F_{(\mathfrak{b},\nu)}}$ such that $\mathfrak{a}\mathcal{O}_{F_U}$ coincides with $\mathfrak{b}$. For such $(\mathfrak{a},\nu)$, the equation

$$\delta^{(y)}(\mathfrak{b}) = \delta^{(y)}((F_V/F_U, \mathfrak{a}\mathcal{O}_{F_U})) = \delta^{(y)} \circ \mathrm{Ver}((F^{\mathrm{comm}}_{(\mathfrak{b},\nu)}/F_{(\mathfrak{b},\nu)}, \mathfrak{a})) = 0$$

holds because the image of the Verlagerung homomorphism is contained in Γ (indeed the Verlagerung coincides with the n_U-th power of the Frobenius endomorphism φ^{n_U} if the finite part of the Galois group is of exponent p; see [H, Lemma 4.3] for details) but y is not contained in Γ.

The above calculation implies that $\mathscr{G}$ has all its non-constant coefficients in $\sharp(NU/U)_y\mathbb{Z}_{(p)}$. Take a finite idèle γ such that $(F_V/F_U, (\gamma)\gamma^{-1})$ coincides with w. Then by Deligne-Ribet's q-expansion principle (Corollary 8.9) the constant term of $\mathscr{G} - \mathscr{G}(\gamma)$ is also contained in $\sharp(NU/U)_y\mathbb{Z}_{(p)}$, which we may calculate as $2^{-[F_U:\mathbb{Q}]}\Delta^w_{F_V/F_U}(1-k, \delta^{(y)})$ (use the explicit formula (8.6) for the q-expansion of $\mathscr{G}(\gamma)$). Therefore sufficient condition (8.4) holds (recall that 2 is invertible in $\mathbb{Z}_{(p)}$ since p is odd).

9. Inductive construction of the p-adic zeta functions

We shall complete the proof of our main theorem (Theorem 3.1). We first construct the p-adic zeta function 'modulo p-torsion' for F_∞/F, and then eliminate ambiguity of the p-torsion part.

9.1. Choice of the central element c. In order to let induction work effectively, we have to choose a 'good' central element c which is used in the construction of the Artinian family $\mathfrak{F}_{A,c}$ (see Section 4.1). The following elementary lemma implies how to choose such a 'good' central element.

Lemma 9.1. *Let Δ be a finite p-group of exponent p and c a non-trivial central element in Δ. If c is not contained in the commutator subgroup of Δ, the p-group Δ is isomorphic to the direct product of the cyclic group $\langle c\rangle$ generated by c and the quotient group $\Delta/\langle c\rangle$.*

Proof. By the structure theorem of finite abelian groups, the abelianisation Δ^{ab} of Δ is decomposed as the direct product of the image of the cyclic group $\langle c\rangle$ and a certain finite abelian p-group $\bar{H}$ of exponent p. Let H denote the inverse image of $\bar{H}$ under the abelianisation map $\Delta \twoheadrightarrow \Delta^{\mathrm{ab}}$. Then one may easily verify that H and $\langle c\rangle$ generate Δ. The intersection of H and $\langle c\rangle$ is obviously trivial, and H commutes with elements in $\langle c\rangle$ since c is central. □

This lemma implies that there exists a non-trivial central element c which is contained in the commutator subgroup of G^f if G is not abelian. We may assume that G is non-commutative without loss of generality (abelian cases are just the results of Deligne, Ribet and Wiles), and thus we may always find a non-trivial central element contained in $[G,G]$. In the following argument we take a non-trivial central element c from the commutator subgroup of G and fix it. Let $F_{\langle c\rangle}$ denote the maximal subfield of F_∞ fixed by $\langle c\rangle$.

9.2. Construction of the p-adic zeta function 'modulo p-torsion'. In this subsection we construct the p-adic zeta function 'modulo p-torsion' for F_∞/F, by mimicking Burns' technique (see Section 2). First consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
K_1(\Lambda(G)) \arrow[r] \arrow[d, "\theta"'] & K_1(\Lambda(G)_S) \arrow[r, "\partial"] \arrow[d, "\theta_S"] & K_0(\Lambda(G), \Lambda(G)_S) \arrow[r] \arrow[d, "\mathrm{norm}"] & 0 \\
\prod_{\mathfrak{F}_B} \Lambda(U/V)^\times \arrow[r, hook] & \prod_{\mathfrak{F}_B} \Lambda(U/V)_S^\times \arrow[r, "\partial"'] & \prod_{\mathfrak{F}_B} K_0(\Lambda(U/V), \Lambda(U/V)_S) \arrow[r] & 0.
\end{tikzcd}$$

Let f be an arbitrary characteristic element for F_∞/F (see Section 1) and set $\theta_S(f) = (f_{U,V})_{(U,V)\in\mathfrak{F}_B}$. For each (U,V) in $\mathfrak{F}_B$ let $w_{U,V}$ be the element defined as $\xi_{U,V} f_{U,V}^{-1}$, which is contained in $\Lambda(U/V)^\times$ by an argument similar to Burns' technique. Let $\tilde{w}_{U,V}$ denote the image of $w_{U,V}$ in $\widetilde{\Lambda}(U/V)^\times$. Since both $(f_{U,V})_{(U,V)\in\mathfrak{F}_B}$ and $(\xi_{U,V})_{(U,V)\in\mathfrak{F}_B}$ satisfy conditions (NCC)$_S$ and (CCC)$_S$ (see Proposition 7.2 and Lemma 8.1), the element $(\tilde{w}_{U,V})_{(U,V)\in\mathfrak{F}_B}$ also satisfies (NCC) and (CCC). Moreover there exists an element $\tilde{d}_{U,V}$ (resp. $\tilde{d}_U$) in $\widetilde{\Lambda}(\Gamma)_{(p)}^\times$ such that the congruence

$$\tag{9.1} \tilde{w}_{U,V} \equiv \tilde{d}_{U,V} \quad \widetilde{\bmod}\, J_{S,U,V} \qquad (\text{resp. } \tilde{w}_U \equiv \tilde{d}_U \quad \widetilde{\bmod}\, I'_{S,U})$$

holds for each (U,V) in $\mathfrak{F}_B$ except for $(G,[G,G])$ (resp. for each U in $\mathfrak{F}_{A,c}$ except for G) by Proposition 7.2 and Proposition 8.2. We remark that these congruences are *not* sufficient to prove that $(\tilde{w}_{U,V})_{(U,V)\in\mathfrak{F}_B}$ is contained in $\widetilde{\Psi}$ (or equivalently in $\widetilde{\Psi}_c$).

Remark 9.2. Unfortunately the congruences (9.1) hold not in $\Lambda(U/V)$ (resp. $\Lambda(U)$) but in $\Lambda(U/V)_S$ (resp. $\Lambda(U)_S$). Nevertheless we may obtain the *integral* congruences (9.5) later by 'eliminating $\tilde{d}_{U,V}$ and $\tilde{d}_U$.' The author would like to thank Mahesh Kakde for pointing out wrong arguments around these phenomena in the preliminary version of this article.

Theorem 9.3 (strong congruences modulo p-torsion). *The congruence*

$$\tilde{w}_{U,V} \equiv \varphi(\tilde{w}_{\mathrm{ab}})^{(G:U)/p} \quad \widetilde{\bmod}\, J_{U,V} \quad (\textit{resp.} \quad \tilde{w}_U \equiv \varphi(\tilde{w}_{\mathrm{ab}})^{(G:U)/p} \quad \widetilde{\bmod}\, I_U)$$

holds for each (U,V) in $\mathfrak{F}_B$ except for $(G,[G,G])$ (resp. for each U in $\mathfrak{F}_{A,c}$ except for G).

Proof. Recall that the non-negative integer N is defined by $\sharp G^f = p^N$. We shall prove the claim by induction on N. We first assume that G is abelian. Then the element $(\xi_{U,\{e\}})_{(U,\{e\})\in\mathfrak{F}_B}$ is in fact contained in the image of $\tilde{\theta}_S$ (use the existence of the p-adic zeta pseudomeasure for F_∞/F), and hence $(\xi_{U,\{e\}})_{(U,\{e\})\in\mathfrak{F}_B}$ satisfies desired congruence condition and additional congruence condition. This implies that $(\tilde{w}_{U,\{e\}})_{(U,\{e\})\in\mathfrak{F}_B}$ also satisfies them. In particular the cases where N equals either 0, 1 or 2 are done. Therefore we assume that N is greater than or equal to 3 and G is non-commutative in the following argument.

Now let (U,V) (resp. U) be an element in $\mathfrak{F}_B$ (resp. $\mathfrak{F}_{A,c}$) such that U contains the fixed central element c chosen as in Section 9.1. Set $\bar{G} = G/\langle c\rangle$, $\bar{U} = U/\langle c\rangle$ and $\bar{V} = V\langle c\rangle/\langle c\rangle$ respectively. Clearly the set of all such $(\bar{U},\bar{V})$ is a Brauer family $\bar{\mathfrak{F}}_B$ for $\bar{G}$, and the set of $\bar{U}_h = U_{h,c}/\langle c\rangle$ for all h in $\mathfrak{H}$ is an Artinian family $\bar{\mathfrak{F}}_A$ for $\bar{G}$. We may easily see that the following diagram commutes because U contains c:

$$\begin{tikzcd} K_1(\Lambda(G)_S) \arrow[r] \arrow[d, "\mathrm{Nr}_{\Lambda(G)_S/\Lambda(U)_S}"'] & K_1(\Lambda(\bar{G})_S) \arrow[d, "\mathrm{Nr}_{\Lambda(\bar{G})_S/\Lambda(\bar{U})_S}"] \\ \Lambda(U)_S^\times \arrow[r] & \Lambda(\bar{U})_S^\times. \end{tikzcd}$$

Note that the image of $\xi_{U,V}$ under the quotient map $\Lambda(U/V)_S^\times \to \Lambda(\bar{U}/\bar{V})_S^\times$ coincides with the p-adic zeta pseudomeasure $\xi_{\bar{U},\bar{V}}$ for $F_{\bar{V}}/F_{\bar{U}}$ (easily follows from its interpolation property). Hence we may apply the induction hypothesis to the image $\bar{\tilde{w}}_{\bar{U},\bar{V}}$ of $\tilde{w}_{U,V}$ in $\widetilde{\Lambda}(\bar{U}/\bar{V})^\times$; in other words, we may assume that the congruences

$$(9.2)\quad \bar{\tilde{w}}_{\bar{U},\bar{V}} \equiv \varphi(\bar{\tilde{w}}_{\mathrm{ab}})^{(\bar{G}:\bar{U})/p} \quad \widetilde{\mathrm{mod}}\, J_{\bar{U},\bar{V}}, \quad \bar{\tilde{w}}_{\bar{U}} \equiv \varphi(\bar{\tilde{w}}_{\mathrm{ab}})^{(\bar{G}:\bar{U})/p} \quad \widetilde{\mathrm{mod}}\, I_{\bar{U}}$$

hold for each $(\bar{U},\bar{V})$ in $\bar{\mathfrak{F}}_B$ except for $(\bar{G},[\bar{G},\bar{G}])$ and for each $\bar{U}$ in $\bar{\mathfrak{F}}_A$ if we define $J_{\bar{U},\bar{V}}$ and $I_{\bar{U}}$ analogously to $J_{U,V}$ and I_U. On the other hand we may readily verify that the natural surjection $\Lambda(U/V) \to \Lambda(\bar{U}/\bar{V})$ maps $J_{U,V}$ to $J_{\bar{U},\bar{V}}$ and I_U to $I_{\bar{U}}$ respectively (use the definition of $J_{U,V}$ and the explicit description of I_U). Let $I'_{\bar{U}}$ denote the image of the trace map $\mathrm{Tr}_{\mathbb{Z}_p[[\mathrm{Conj}(N\bar{U})]]/\mathbb{Z}_p[[\bar{U}]]}$ for each $\bar{U}$ in $\bar{\mathfrak{F}}_A$, and let $J_{S,\bar{U},\bar{V}}$ (resp. $I'_{S,\bar{U}}$) denote the scalar extension $J_{\bar{U},\bar{V}} \otimes_{\Lambda(\Gamma)} \Lambda(\Gamma)_{(p)}$ (resp. $I'_{\bar{U}} \otimes_{\Lambda(\Gamma)} \Lambda(\Gamma)_{(p)}$). Then we obtain the congruences

$$(9.3)\qquad \bar{\tilde{w}}_{\bar{U},\bar{V}} \equiv \tilde{d}_{U,V} \qquad \widetilde{\mathrm{mod}}\, J_{S,\bar{U},\bar{V}}, \qquad \bar{\tilde{w}}_{\bar{U}} \equiv \tilde{d}_U \qquad \widetilde{\mathrm{mod}}\, I'_{S,\bar{U}}$$

by applying the canonical surjection $\Lambda(U/V) \to \Lambda(\bar{U}/\bar{V})$ to (9.1) (recall that $I'_{S,\bar{U}}$ contains $I_{S,\bar{U}}$). The congruences (9.2) and (9.3) imply that for (U,V) in $\mathfrak{F}_B$ except for $(G,[G,G])$ the element $\varphi(\bar{\tilde{w}}_{\mathrm{ab}})^{-(\bar{G}:\bar{U})/p}\tilde{d}_{U,V}$ is contained in $1 + \tilde{J}_{S,\bar{U},\bar{V}} \cap \widetilde{\Lambda}(\Gamma)^\times_{(p)}$, which coincides with $1 + p\Lambda(\Gamma)^{\sim}_{(p)}$ by definition. Furthermore for U in $\mathfrak{F}_{A,c}$ the element $\varphi(\bar{\tilde{w}}_{\mathrm{ab}})^{-(\bar{G}:\bar{U})/p}\tilde{d}_U$ is contained in $(1 + I'_{S,\bar{U}})^{\sim} \cap \widetilde{\Lambda}(\Gamma)^\times_{(p)}$, which coincides with $1 + p^{n_h-\epsilon}\Lambda(\Gamma)^{\sim}_{(p)}$ by the explicit description of $I'_{S,\bar{U}}$ (the integer ϵ is defined to be 2 for (Case-1) and 1 for (Case-2)). Obviously $\varphi(\bar{\tilde{w}}_{\mathrm{ab}})$ coincides with $\varphi(\tilde{w}_{\mathrm{ab}})$ and $(\bar{G}:\bar{U})$ equals $(G:U)$ by construction, and therefore the congruences

$$(9.4)\qquad \begin{aligned} \tilde{d}_{U,V} &\equiv \varphi(\tilde{w}_{\mathrm{ab}})^{(G:U)/p} \quad \widetilde{\mathrm{mod}} \quad p\Lambda(\Gamma)_{(p)}, \\ \tilde{d}_U &\equiv \varphi(\tilde{w}_{\mathrm{ab}})^{(G:U)/p} \quad \widetilde{\mathrm{mod}}\, p^{n_h-\epsilon}\Lambda(\Gamma)_{(p)}. \end{aligned}$$

hold. Combining (9.4) with (9.1), we obtain the following congruences:[14]

$$(9.5)\quad \tilde{w}_{U,V} \equiv \varphi(\tilde{w}_{\mathrm{ab}})^{(G:U)/p} \quad \widetilde{\mathrm{mod}}\, J_{U,V}, \quad \tilde{w}_U \equiv \varphi(\tilde{w}_{\mathrm{ab}})^{(G:U)/p} \quad \widetilde{\mathrm{mod}}\, I'_U.$$

The former congruence is no other than the desired one. The latter one for U_c is also the desired one because I'_{U_c} coincides with I_{U_c} by definition. Now consider the latter congruence for $U_{h,c}$. Note that $U_{h,c}$ is a proper subgroup of G since we now assume that N is greater than or equal to 3.

[14]Since both $\tilde{w}_{U,V}$ (resp. $\tilde{w}_U$) and $\varphi(\tilde{w}_{\mathrm{ab}})^{(G:U)/p}$ are contained in $\widetilde{\Lambda}(U/V)^\times$, the congruence (9.5) actually holds in $\Lambda(U/V)$ (resp. in $\Lambda(U)$) and we may remove the sub-index S from the congruence. This is the 'eliminating $\tilde{d}$' procedure mentioned in Remark 9.2.

Since $\log(\varphi(\tilde{w}_{\rm ab})^{-p^{N-3}}\tilde{w}_{U_{h,c}})$ is contained in $I'_{U_{h,c}}$ by (9.5), it is explicitly described as

$$\log\frac{\tilde{w}_{U_{h,c}}}{\varphi(\tilde{w}_{\rm ab})^{p^{N-3}}}=\sum_{i=0}^{p-1}p^{n_h-\epsilon}a_ic^i+(\text{terms containing } h)$$

where each a_i is an element in $\Lambda(\Gamma)$. Furthermore the equation

$$\log\frac{\tilde{w}_{U_c}}{\varphi(\tilde{w}_{\rm ab})^{p^{N-2}}}=\mathrm{Tr}_{\mathbb{Z}_p[[U_{h,c}]]/\mathbb{Z}_p[[U_c]]}(\log\frac{\tilde{w}_{U_{h,c}}}{\varphi(\tilde{w}_{\rm ab})^{p^{N-3}}})=\sum_{i=0}^{p-1}p^{n_h-\epsilon+1}a_ic^i$$

holds by (TCC) and Lemma 6.6. The first expression of the above equation is contained in $I_{U_c}=p^{N-1}\mathbb{Z}_p[[U_c]]$ as we have already remarked, and hence there exists an element b_i in $\Lambda(\Gamma)$ such that $p^{n_h-\epsilon+1}a_i$ coincides with $p^{N-1}b_i$ for each i. We may thus conclude that $\log(\varphi(\tilde{w}_{\rm ab})^{-p^{N-3}}\tilde{w}_{U_{h,c}})$ is contained in $I_{U_{h,c}}$. This implies the desired congruence for $U_{h,c}$ because the logarithm induces an injection on $1+\widetilde{J_{U_{h,c}}}$ (Proposition 5.7) and an isomorphism between $1+\widetilde{I_{U_{h,c}}}$ and $I_{U_{h,c}}$ (Proposition 5.9).

Next let (U,V) be an element in $\mathfrak{F}_B$ such that U does not contain the fixed central element c. We claim that $U\times\langle c\rangle$ does not coincide with G; indeed if it does, the commutator subgroup of G automatically coincides with V which does not contain c. This is contradiction since we choose such c as contained in $[G,G]$ in Section 9.1. Now we apply the above argument to the pair $(U\times\langle c\rangle,V)$ and obtain the congruence

$$\tilde{w}_{U\times\langle c\rangle,V}\equiv\varphi(\tilde{w}_{\rm ab})^{(G:U)/p^2}\quad\widetilde{\mathrm{mod}}\,J_{U\times\langle c\rangle,V}$$

(use the obvious relation $(G:U\times\langle c\rangle)=(G:U)/p$). By using (NCC) and the fact that $\varphi(\tilde{w}_{\rm ab})$ is contained in the centre of $\widetilde{\Lambda}(U\times\langle c\rangle)^\times$, we have

$$\mathrm{Nr}_{\Lambda(U\times\langle c\rangle/V)/\Lambda(U/V)}(\varphi(\tilde{w}_{\rm ab})^{-(G:U)/p^2}\tilde{w}_{U\times\langle c\rangle,V})=\varphi(\tilde{w}_{\rm ab})^{-(G:U)/p}\tilde{w}_{U,V}.$$

On the other hand the left hand side of the above equation is contained in $1+\widetilde{J_{U,V}}$ by Corollary 5.5. The desired congruence thus holds for (U,V).

Finally let U_h be an element in $\mathfrak{F}_A$ and assume that h does not coincide with c. By the same argument as above, we may conclude that $\varphi(\tilde{w}_{\rm ab})^{-p^{N-2}}\tilde{w}_{U_h}$ is contained in $1+\widetilde{J_{U_h}}$. On the other hand the element $\varphi(\tilde{w}_{\rm ab})^{-p^{N-3}}\tilde{w}_{U_{h,c}}$ is contained in $1+\widetilde{I_{U_{h,c}}}$ by the above argument. Now the compatibility lemma (Lemma 6.6) enables us to calculate as follows:

$$\begin{aligned}\mathrm{Tr}_{\mathbb{Z}_p[[U_{h,c}]]/\mathbb{Z}_p[[U_h]]}(\log\frac{\tilde{w}_{U_{h,c}}}{\varphi(\tilde{w}_{\rm ab})^{p^{N-3}}})&=\log(\mathrm{Nr}_{\Lambda(U_{h,c})/\Lambda(U_h)}(\frac{\tilde{w}_{U_{h,c}}}{\varphi(\tilde{w}_{\rm ab})^{p^{N-3}}}))\\&=\log\frac{\tilde{w}_{U_h}}{\varphi(\tilde{w}_{\rm ab})^{p^{N-2}}}.\end{aligned}$$

The $\mathbb{Z}_p$-module $\mathrm{Tr}_{\mathbb{Z}_p[[U_{h,c}]]/\mathbb{Z}_p[[U_h]]}(I_{U_{h,c}})$ is contained in I_{U_h} by definition, and thus $\log(\varphi(\tilde{w}_{\rm ab})^{-p^{N-2}}\tilde{w}_{U_h})$ is also contained in I_{U_h}. The desired congruence now holds for U_h because the p-adic logarithm induces an injection on $1+\widetilde{J_{U_h}}$ (Proposition 5.7) and an isomorphism between $1+\widetilde{I_{U_h}}$ and I_{U_h} (Proposition 5.9). □

By virtue of Theorem 9.3, we may conclude that $(\tilde{w}_{U,V})_{(U,V)\in\mathfrak{F}_B}$ is an element in $\widetilde{\Psi}_c$. Hence there exists a unique element $\tilde{w}$ in $\widetilde{K}_1(\Lambda(G))$ such that $\tilde{\theta}(\tilde{w}) = (\tilde{w}_{U,V})_{(U,V)\in\mathfrak{F}_B}$ holds (Proposition 6.7 and Proposition 6.8). Take an arbitrary lift of $\tilde{w}$ to $K_1(\Lambda(G))$ and set $\tilde{\xi} = f\tilde{w}$. Then by construction, we may easily check that $\tilde{\xi}$ satisfies the following two properties:

($\tilde{\xi}$-1) the equation $\partial(\tilde{\xi}) = -[C_{F_\infty/F}]$ holds;

($\tilde{\xi}$-2) there exists an element $\tau_{U,V}$ in $\Lambda(U/V)^\times_{p\text{-tors}}$ for each (U,V) in $\mathfrak{F}_B$ such that the equation $\theta_S(\tilde{\xi}) = (\xi_{U,V}\tau_{U,V})_{(U,V)\in\mathfrak{F}_B}$ holds.

By using $(\sharp)_A$ and ($\tilde{\xi}$-2), we may show that there exists a p-power root of unity $\zeta_{\rho,r}$ such that the equation $\tilde{\xi}(\rho\kappa^r) = \zeta_{\rho,r}L_{\Sigma_F}(1-r; F_\infty/F, \rho)$ holds for an arbitrary Artin representation ρ of G and an arbitrary natural number r divisible by $p-1$. Roughly speaking, the element $\tilde{\xi}$ is the p-adic zeta function 'modulo p-torsion' for F_∞/F which interpolates special values of complex Artin L-functions *up to multiplication by a p-power root of unity.*

9.3. **Refinement of the p-torsion part.** We shall finally modify $\tilde{\xi}$ and reconstruct the p-adic zeta function ξ for F_∞/F without any ambiguity of p-torsion elements. The author strongly believes that our argument to remove ambiguity of the p-torsion part is based upon essentially the same spirits as 'the torsion congruence method' used by Jürgen Ritter and Alfred Weiss [RW5]. We shall, however, adopt somewhat different formalism from theirs.

Let $\tilde{\xi}$ be the p-adic zeta function 'modulo p-torsion' for F_∞/F and set $\tau_{U,V} = \xi_{U,V}\theta_{S,U,V}(\tilde{\xi})^{-1}$ for each (U,V) in $\mathfrak{F}_B$. Then $\tau_{U,V}$ is a p-torsion element by definition. Moreover $\tau_{U,V}$ is an element in $\Lambda(U/V)^\times$ by the same argument as Burns' technique. Since the p-torsion part of $K_1(\Lambda(G))$ is identified with $G^{f,\mathrm{ab}} \times SK_1(\mathbb{Z}_p[G^f])$ and that of $\Lambda(G^{\mathrm{ab}})^\times$ is identified with $G^{f,\mathrm{ab}}$ respectively (see Section 6.2), we may naturally regard the p-torsion element $\tau_{\mathrm{ab}} = \xi_{\mathrm{ab}}\theta_{S,\mathrm{ab}}(\tilde{\xi})^{-1}$ as an element in $K_1(\Lambda(G))_{p\text{-tors}}$. Set $\xi = \tau_{\mathrm{ab}}\tilde{\xi}$. Then $\theta_{S,\mathrm{ab}}(\xi) = \xi_{\mathrm{ab}}$ obviously holds by construction.

Theorem 9.4. *The equation $\theta_{S,U,V}(\xi) = \xi_{U,V}$ holds for each (U,V) in $\mathfrak{F}_B$.*

If the claim is verified, we may conclude that ξ satisfies the interpolation formula (1.3) without any ambiguity by Brauer induction (see Section 2). Therefore ξ is no other than the 'true' p-adic zeta function for F_∞/F.

We shall prove Theorem 9.4 by induction on N. First assume that G is abelian. Then the obvious equation $\xi = \theta_{S,\mathrm{ab}}(\xi) = \xi_{\mathrm{ab}}$ implies that ξ is actually the p-adic zeta function for F_∞/F. In particular the cases in which N equals either $0, 1$ or 2 are done.

Now suppose that N is strictly greater than 2 and G is non-commutative. Let c be a non-trivial central element in G chosen as in Section 9.1 and set $\bar{G} = G/\langle c\rangle$. We denote by $\bar{\xi}$ the image of ξ under the canonical map $K_1(\Lambda(G)_S) \to K_1(\Lambda(\bar{G})_S)$. Then the element $\bar{\xi}$ is the p-adic zeta function 'modulo p-torsion' for $F_{\langle c\rangle}/F$ by construction. Furthermore the following diagram commutes since c is contained in the commutator subgroup of G

(here $\bar{\theta}_{S,\mathrm{ab}}$ denotes the abelianisation map for $\Lambda(\bar{G})_S$):

$$\begin{tikzcd} K_1(\Lambda(G)_S) \arrow[r, "\theta_{S,\mathrm{ab}}"] \arrow[d, "\text{canonical}"'] & \Lambda(G^{\mathrm{ab}})_S^\times \arrow[d, equal] \\ K_1(\Lambda(\bar{G})_S) \arrow[r, "\bar{\theta}_{S,\mathrm{ab}}"'] & \Lambda(\bar{G}^{\mathrm{ab}})_S^\times . \end{tikzcd}$$

This asserts that $\bar{\theta}_{S,\mathrm{ab}}(\bar{\xi}) = \xi_{\mathrm{ab}}$ holds, and we may thus apply the induction hypothesis to $\bar{\xi}$; in other words we may assume that $\bar{\xi}$ is the 'true' p-adic zeta function for $F_{\langle c\rangle}/F$. Now take an arbitrary pair (U, V) in $\mathfrak{F}_B$.

(Case-1). Suppose that c is contained in V. Let $\bar{U}$ and $\bar{V}$ denote the quotient groups $U/\langle c\rangle$ and $V/\langle c\rangle$ respectively. Let $\bar{\theta}_{S,\bar{U},\bar{V}}$ be the composition of the norm map $\mathrm{Nr}_{\Lambda(\bar{G})_S/\Lambda(\bar{U})_S}$ with the canonical homomorphism $K_1(\Lambda(\bar{U})_S) \to \Lambda(\bar{U}/\bar{V})_S^\times$. Then it is clear that U/V coincides with $\bar{U}/\bar{V}$ and the theta maps $\theta_{S,U,V}$ and $\bar{\theta}_{S,\bar{U},\bar{V}}$ are compatible. Hence we obtain

$$\theta_{S,U,V}(\xi) = \bar{\theta}_{S,\bar{U},\bar{V}}(\bar{\xi}) = \xi_{\bar{U},\bar{V}} = \xi_{U,V}$$

which is the desired result (the last equality follows from the fact that $F_{\bar{V}}/F_{\bar{U}}$ is the completely same extension as F_V/F_U).

(Case-2). Suppose that c is contained in U but not contained in V. Let U' be an open subgroup of G which contains U as a subgroup of index p (and U is hence normal in U'). Let V' denote the commutator subgroup of U'. We claim that we may assume without loss of generality that $\theta_{S,U',V'}(\xi) = \xi_{U',V'}$ holds; indeed the desired equation holds if V' contains c by (Case-1). Assume that V' does not contain c. Then the pair $(U_1, V_1) = (U', V')$ also satisfies the condition of (Case-2), and recursively we may obtain a sequence of pairs $\{(U_i, V_i)\}_{i\in\mathbb{Z}_{\geq 0}}$ such that (U_0, V_0) is equal to (U, V) and U_{i+1} contains U_i as its normal subgroup of index p (here each V_i is the commutator subgroup of U_i). Hence there exists a natural number n such that V_n does not contain c but V_{n+1} contains c (recall the assumption that the commutator subgroup of G contains c). Now it suffices to replace (U, V) with (U_n, V_n) and (U', V') with (U_{n+1}, V_{n+1}) respectively.

The key to remove ambiguity of the p-torsion part is the fact that *the p-adic zeta function $\xi_{U',V}$ exists uniquely for the p-adic Lie extension $F_V/F_{U'}$* by applying the following result of Ritter and Weiss [RW7] to $F_V/F_{U'}$;[15]

Theorem 9.5. *Let p be a positive odd prime number and F a totally real number field. Let F_∞ be a totally real p-adic Lie extension of F satisfying conditions (F_∞-1), (F_∞-2) and (F_∞-3) in Section* 1. *Suppose that the Galois group G of F_∞/F is a one-dimensional pro-p p-adic Lie group and has an* abelian *open normal subgroup A of index p. Then the p-adic zeta function $\xi_{F_\infty/F}$ for F_∞/F exists uniquely as an element in $K_1(\Lambda(G)_S)$.*

Let $\mathrm{can}_{U',V}$ denote the canonical map $K_1(\Lambda(U')_S) \to K_1(\Lambda(U'/V)_S)$. Note that the element $\mathrm{can}_{U',V} \circ \mathrm{Nr}_{\Lambda(G)_S/\Lambda(U')_S}(\xi)$ is the p-adic zeta function 'modulo p-torsion' for $F_V/F_{U'}$ by the interpolation property, and hence there

[15] We remark that we may also prove Theorem 9.5 by arguments based upon Burns' technique similar to those of this article.

exists an element τ in U'^f/V'^f such that $\mathrm{can}_{U',V} \circ \mathrm{Nr}_{\Lambda(G)_S/\Lambda(U')_S}(\xi)$ coincides with $\tau\xi_{U',V}$ (here we remark that the p-torsion part of $K_1(\Lambda(U'/V))$ coincides with $(U'^f/V^f)^{\mathrm{ab}} = U'^f/V'^f$ because $SK_1(\mathbb{Z}_p[U'^f/V^f])$ is trivial by [Oliver, Theorem 8.10]). Then easy calculation verifies that the equation

$$\begin{aligned}\xi_{U',V'} = \theta_{S,U',V'}(\xi) &= \mathrm{can}^V_{V'} \circ \mathrm{can}_{U',V} \circ \mathrm{Nr}_{\Lambda(G)_S/\Lambda(U')_S}(\xi)\\ &= \mathrm{can}^V_{V'}(\tau\xi_{U',V}) = \tau\xi_{U',V'}\end{aligned}$$

holds where $\mathrm{can}^V_{V'} \colon K_1(\Lambda(U'/V)_S) \to \Lambda(U'/V')^\times_S$ denotes the canonical homomorphism. This implies that τ is trivial. On the other hand, the norm relation $\mathrm{Nr}_{\Lambda(U'/V)_S/\Lambda(U/V)_S}(\xi_{U',V}) = \xi_{U,V}$ holds since $\xi_{U',V}$ is the p-adic zeta function for $F_V/F_{U'}$. Therefore we obtain the desired equation

$$\begin{aligned}\theta_{S,U,V}(\xi) &= \mathrm{Nr}_{\Lambda(U'/V)_S/\Lambda(U/V)_S} \circ \mathrm{can}_{U',V} \circ \mathrm{Nr}_{\Lambda(G)_S/\Lambda(U')_S}(\xi)\\ &= \mathrm{Nr}_{\Lambda(U'/V)_S/\Lambda(U/V)_S}(\xi_{U',V}) = \xi_{U,V}.\end{aligned}$$

(Case-3). Suppose that c is contained in neither U nor V. In this case the pair $(U \times \langle c \rangle, V)$ satisfies the condition of (Case-2), and thus the equation $\theta_{S,U\times\langle c\rangle,V}(\xi) = \xi_{U\times\langle c\rangle,V}$ holds. Then by using the commutative diagram

$$\begin{tikzcd} K_1(\Lambda(G)_S) \arrow[r, "\theta_{S,U\times\langle c\rangle,V}"] \arrow[dr, "\theta_{S,U,V}"'] & \Lambda(U\times\langle c\rangle/V)^\times_S \arrow[d, "\mathrm{Nr}_{\Lambda(U\times\langle c\rangle/V)_S/\Lambda(U/V)_S}"] \\ & \Lambda(U/V)^\times_S, \end{tikzcd}$$

we obtain

$$\begin{aligned}\theta_{S,U,V}(\xi) &= \mathrm{Nr}_{\Lambda(U\times\langle c\rangle/V)_S/\Lambda(U/V)_S} \circ \theta_{S,U\times\langle c\rangle,V}(\xi)\\ &= \mathrm{Nr}_{\Lambda(U\times\langle c\rangle/V)_S/\Lambda(U/V)_S}(\xi_{U\times\langle c\rangle,V}) = \xi_{U,V},\end{aligned}$$

which is the desired result.[16]

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[16] We may derive the desired result for (Case-3) even if we only assume that $\theta_{S,U\times\langle c\rangle,V}(\xi) = c^j\xi_{U\times\langle c\rangle,V}$ holds for certain j (which we may verify by the arguments similar to (Case-1)); hence the essentially difficult part in the proof of Theorem 9.4 is just (Case-2). Note that $\mathrm{Nr}_{\Lambda(U\times\langle c\rangle/V)_S/\Lambda(U/V)_S}(c^j)$ coincides with $(c^j)^p = 1$ because c^j is contained in the centre of $\Lambda(U \times \langle c\rangle/V)^\times_S$.

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