Dispersive and Strichartz estimates for Schrödinger equations
(シュレディンガー方程式に対する 分散型及びストリッカーツ評価)

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Abstract

The present paper is concerned with L^p-smoothing properties of solutions to time dependent Schrödinger equations. We study two kinds of L^p-smoothing properties, namely the *Strichartz estimates* and the *dispersive estimates.*

The first part is concerned with Strichartz estimates for Schrödinger equations on scattering manifolds which are non-compact manifolds with asymptotically conic ends. It is shown that time-local Strichartz estimates, outside a large compact set centered at origin, hold for any admissible pair (including the endpoint). Moreover, we prove global in space Strichartz estimates under the nontrapping condition of the geodesic flow. We also study a relationship between Strichartz estimates and microlocal properties of the solution. More precisely, it is shown that (local in time) Strichartz estimates follow from the dispersive estimates for spatial and frequency localized solutions. This part is based on a paper " Strichartz estimates for Schrödinger equations on scattering manifolds " .

The second part is concerned with dispersive estimates for scattering solutions to one-dimensional Schrödinger equations with potentials. We prove a weighted dispersive estimate for the non-resonant state with stronger time decay than the case of the usual unweighted estimate. Furthermore asymptotic expansions in time of scattering solutions are given. This part is based on a paper "Dispersive estimates and asymptotic expansions for Schrödinger equations in dimension one" to appear in *Journal of the Mathematical Society of Japan.*

keywords: Strichartz estimate; Dispersive estimate; Asymptotic expansion; Schrödinger equation; Scattering manifold.

Contents

Chapter 1

Introduction

The present paper is concerned with smoothing properties of solutions to time-dependent Schödinger equations. In particular, we study the following two estimates

- **the Strichartz estimate,**
- **the dispersive estimate.**

To motivate our results, let us first recall the case of the free Schödinger equation on the Euclidean space $\mathbb{R}^d$:

$$\begin{cases} i\partial_t u(t) = \frac{1}{2}\Delta u(t), & t \in \mathbb{R}, \\ u(0) = u_0 \in L^2(\mathbb{R}^d). \end{cases}$$

The solution is uniquely determined by the propagator $e^{it\frac{1}{2}\Delta}$, which is unitary on $L^2(\mathbb{R}^d)$. We hence have the L^2-conservation, namely

$$||e^{it\frac{1}{2}\Delta}u_0||_{L^2(\mathbb{R}^d)} = ||u_0||_{L^2(\mathbb{R}^d)}, \quad t \in \mathbb{R}.$$

Moreover, the solution is given explicitly by

$$e^{it\frac{1}{2}\Delta}u_0(x) = \frac{1}{(2\pi it)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{2t}} u_0(y)dy.$$

By this formula, we obtain the *dispersive estimate*:

$$||e^{it\frac{1}{2}\Delta}u_0||_{L^\infty(\mathbb{R}^d)} \leq (2\pi|t|)^{-d/2}||u_0||_{L^1(\mathbb{R}^d)}, \quad u_0 \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d), \tag{1.0.1}$$

for $t \neq 0$. This property says that the L^∞-norm of the solution decays to 0 as $t \to \infty$. By the dispersive estimate, the L^2-conservation and the interpolation theorem, we have the L^p-L^q estimates:

$$||e^{it\frac{1}{2}\Delta}u_0||_{L^p(\mathbb{R}^d)} \leq (2\pi|t|)^{-d(1/2-1/q)}||u_0||_{L^q(\mathbb{R}^d)}, \quad u_0 \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d),$$

for $t \neq 0$, where $2 \leq p \leq \infty$ and $1/p + 1/q = 1$. Moreover, these estimates imply so called the *Strichartz estimates.*

Theorem 1.0.1 (The Strichartz estimates). *The solution $e^{it\frac{1}{2}\Delta}u_0$ satisfies*

$$||e^{it\frac{1}{2}\Delta}u_0||_{L^p(\mathbb{R};L^q(\mathbb{R}^d))} \le C_{pq}||u_0||_{L^2(\mathbb{R}^d)}, \quad u_0 \in L^2(\mathbb{R}^d), \tag{1.0.2}$$

provided that (p,q) satisfies the following admissible condition:

$$2 \le p, q \le \infty, \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \quad (d,p,q) \ne (2,2,\infty) \tag{1.0.3}$$

Proof. We only prove the non-endpoint case $p > 2$. For the proof of the endpoint case $p = 2$, we refer to Keel-Tao [26]. The non-endpoint estimates can be proved by the TT^*-argument due to Ginibre-Velo [16] as follows. We define a bounded operator from $L^1(\mathbb{R}; L^2(\mathbb{R}^d))$ to $L^2(\mathbb{R}^d)$ by

$$Vf = \int_{\mathbb{R}} e^{-is\frac{1}{2}\Delta} f(s)ds.$$

Its adjoint is then given by $U = e^{it\frac{1}{2}\Delta}$, and the composition of these operators

$$UVf(t) = \int_{\mathbb{R}} e^{i(t-s)\frac{1}{2}\Delta} f(s)ds$$

is bounded from $L^1(\mathbb{R}; L^2(\mathbb{R}^d))$ to $L^\infty(\mathbb{R}; L^2(\mathbb{R}^d))$. By the TT^*-argument, the estimate (1.0.2) is equivalent to

$$||UVf||_{L^p(\mathbb{R};L^q(\mathbb{R}^d)} \le C||f||_{L^{p'}(\mathbb{R};L^{q'}(\mathbb{R}^d)}, \tag{1.0.4}$$

where (p,q) satisfies the admissible condition, $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. Since the L^q-$L^{q'}$ estimates imply

$$||UVf(t,\cdot)||_{L^q(\mathbb{R}^d)} \le (2\pi|t|)^{-d(1/2-1/q)} * ||f(t,\cdot)||_{L^{q'}(\mathbb{R}^d)}.$$

applying the Hardy-Littlewood-Sobolev inequality, we obtain (1.0.4) when $p > p'$, that is $p > 2$. □

The estimate (1.0.2) with $p > 2$ has been proved by Strichartz [40], Ginibre-Velo [16] and Yajima [46]; the endpoint case ($p = 2$) has been proved by Keel-Tao [26]. It is well known that these estimates (1.0.1) and (1.0.2) are fundamental in studying nonlinear Schrödinger equations. For example, these estimates can be applied to study the well-posedness of the Cauchy problem and scattering theory.

It is a natural question if such estimates hold for Schrödinger equations on manifolds or Schrödinger equations with potentials. In other words, the question is: how do the geometry of manifolds or the behavior of potentials influence the behavior of solutions ? In this thesis, we will prove

- local-in-time Strichartz estimates on scattering manifolds,
- Asymptotic expansions in time on the line.

Our results are the followings.

1.1 Strichartz estimates on scattering manifolds

Let M be a non-compact Riemannian manifold of dimension $d \geq 2$ such that M can be decomposed as $M = M_c \cup M_\infty$, where $M_c \Subset M$ is relatively compact and M_∞ is diffeomorphic to $(0, \infty) \times \partial M$ with a $(d-1)$-dimensional closed manifold ∂M. Suppose that the Riemannian metric g on M is a long-range perturbation of the perfectly conic metric. Namely, g takes the form

$$g = dr^2 + r^2(h_{jk}(\theta) + a_{jk}(r, \theta))d\theta^j d\theta^k \quad \text{for} \quad (r, \theta) \in (1, \infty) \times \partial M,$$

where (h_{jk}) is the metric on ∂M and (a_{jk}) satisfies the following long-range type condition:

$$|\partial_r^l \partial_\theta^\alpha a_{jk}(r, \theta)| \leq C_{l\alpha} r^{-\mu - l} \quad \text{for} \quad (r, \theta) \in (1, \infty) \times \partial M,$$

with some $\mu > 0$. Such a manifold (M, g) is called a (long-range) scattering manifold following Melrose [30]. For the precise definition of the scattering manifold, see Section 2.1.

In Chapter 2, we study the Strichartz estimates for the Schrödinger equation on the scattering manifold M:

$$\begin{cases} i\partial_t u(t) = Pu(t), & t \in \mathbb{R}, \\ u(0) = u_0 \in L^2(M), \end{cases}$$

where $P = -\frac{1}{2}\Delta_g$ and Δ_g is the Laplace-Beltrami operator associated to the metric g. To state the result, we recall the nontrapping condition. Let $p(z, \xi)$ be the principal symbol of P. We say that M is nontrapping if for any $(z_0, \xi^0) \in T^*M$ with $\xi^0 \neq 0$, the geodesic flow $(z(t, z_0, \xi^0), \xi(t, z_0, \xi^0))$, generated by $p(x, \xi)$, satisfies

$$|z(t, z_0, \xi^0)| \to +\infty \text{ as } t \to \pm\infty.$$

The main result in this chapter is the following local-in-time Strichartz estimates:

Theorem 1.1.1 (Theorem 2.1.1 and 2.1.2 in Chapter 2). *There exist a large compact subset $K \subset M$ and $\chi_K \in C_0^\infty(M)$ with $\chi_K \equiv 1$ on K such that*

$$||(1 - \chi_K)e^{-itP}u_0||_{L^p([0,1];L^q(M))} \leq C||u_0||_{L^2(M)}, \quad u_0 \in C_0^\infty(M),$$

provided that (p, q) satisfies the admissible condition (1.0.3). *Moreover, if in addition we assume that M is a nontrapping, then*

$$||e^{-itP}u_0||_{L^p([0,1];L^q(M))} \leq C||u_0||_{L^2(M)}, \quad u_0 \in C_0^\infty(M),$$

for any admissible pair (p, q).

Remark 1.1.2. (i) Theorem 1.1.1 tells us that the existence of trapped trajectories does not affect local in time Strichartz estimates if we restrict the solution outside a large compact set.

(ii) In this chapter, we also study a relationship between Strichartz estimates and microlocal properties of the solution. More precisely, we will show that local in time Strichartz estimates follow from dispersive estimates for spatial and energy localized solutions (cf. Section 2.3).

1.2 Asymptotic expansions on the line

In Chapter 3, we study the one dimensional Schrödinger equation with a potential

$$i\partial_t u(t) = Hu(t), \quad t \in \mathbb{R},$$

where $H = -\frac{d^2}{dx^2} + V$, $V(x)$ is a real-valued potential such that $\langle x\rangle V \in L^1(\mathbb{R})$ at least and $\langle x\rangle := \sqrt{1+|x|^2}$. In order to state the result in this chapter, we recall the definition of the resonance of H. We denote by $f_{\pm}(\lambda, x)$ the solutions to the equation

$$-f''(\lambda, x) + V(x)f(\lambda, x) = \lambda^2 f(\lambda, x), \quad \lambda, x \in \mathbb{R},$$

satisfying following asymptotic conditions

$$|f_{\pm}(\lambda, x) - e^{\pm i\lambda x}| \to 0 \text{ as } x \to \pm\infty.$$

We denote by $W(\lambda)$ their Wronskian

$$W(\lambda) := f_+(\lambda, x) \cdot \partial_x f_-(\lambda, x) - \partial_x f_+(\lambda, x) \cdot f_-(\lambda, x).$$

$W(\lambda)$ is independent of x and does not vanish for $\lambda \neq 0$.

Definition 1.2.1. We say that the potential V is of generic type if $W(0) \neq 0$ and is of exceptional type if $W(0) = 0$. We also say that *zero is a resonance of H* if the potential V is of exceptional type.

We note that $V \equiv 0$ is of exceptional type. The main result in Chapter 3 is the following asymptotic expansion in time of the scattering solution:

Theorem 1.2.2 (Theorem 3.1.2 in Chapter 3). *Let m be a positive integer. Suppose that $\langle x\rangle^{2m} V \in L^1(\mathbb{R})$ and V is of generic type, or $\langle x\rangle^{2m+2} V \in L^1(\mathbb{R})$ and V is of exceptional type. Let*

$$s = \begin{cases} 2m-1 & \text{if } V \text{ is of generic type,} \\ 2m & \text{if } V \text{ is of exceptional type.} \end{cases}$$

Then,

$$||\langle x\rangle^{-s}(e^{-itH}P_{ac} - P_{m-1})u||_{L^\infty(\mathbb{R})} \leq C|t|^{-\frac{1}{2}-m}||\langle x\rangle^s u||_{L^1(\mathbb{R})}$$

for all $t \neq 0$, where P_{ac} is the projection onto the absolutely continuous subspace for H and P_{m-1} is given by

$$P_{m-1} = \sum_{j=0}^{m-1} t^{-\frac{1}{2}-j} C_{j-1}.$$

Furthermore, the coefficients C_{j-1} satisfy the following:
(1) *If V is of generic type, then $C_{-1} \equiv 0$,* $\operatorname{rank} C_{j-1} \leq 2j$ *and*

$$||\langle x\rangle^{-2j+1} C_{j-1} u||_{L^\infty(\mathbb{R})} \leq C||\langle x\rangle^{2j-1} u||_{L^1(\mathbb{R})}, \ j = 1, 2, ...m-1.$$

(2) *If V is of exceptional type, then* $\operatorname{rank} C_{j-1} \leq 2j+1$ *and*

$$||\langle x\rangle^{-2j} C_{j-1} u||_{L^\infty(\mathbb{R})} \leq C||\langle x\rangle^{2j} u||_{L^1(\mathbb{R})}, \ j = 0, 1, ...m-1.$$

Remark 1.2.3. (1) If V is of generic type and $\langle x\rangle^2 V \in L^1(\mathbb{R})$, then Theorem 1.2.2 implies a weighted dispersive estimate with stronger time decay $|t|^{-3/2}$ than that of the free case $|t|^{-1/2}$:

$$||\langle x\rangle^{-1}e^{-itH}P_{ac}u||_{L^\infty(\mathbb{R})} \le C|t|^{-\frac{3}{2}}||\langle x\rangle u||_{L^1(\mathbb{R})}, \quad t \neq 0.$$

(2) The coefficients C_{j-1} can be written explicitly in terms of the Jost functions and their Wronskian. Indeed, the distribution kernel of C_{j-1} is given by

$$\frac{1}{\sqrt{4\pi i}j!(4i)^j}\left(\frac{\partial}{\partial\lambda}\right)^{2j}(T(\lambda)f_-(\lambda,x)f_+(\lambda,y))\bigg|_{\lambda=0},$$

where $T(\lambda) := \frac{-2i\lambda}{W(\lambda)}$ is the transmission coefficient (cf. Section 3.3).

Notations: Throughout the paper, we use the following notations: We denote the set of multi-indices by $\mathbb{Z}_+^d$. For Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the Banach space of bounded operators from X to Y, and we write $\mathcal{L}(X) := \mathcal{L}(X,X)$. For $a \in \mathbb{R}$, we use the notation $a_+ = \max(a,0)$. For a vector x, $\langle x\rangle$ stands for $\sqrt{1+|x|^2}$. For $A, B \ge 0$, $A \lesssim B$ means that there exists some universal constant $C > 0$ such that $A \le CB$.

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Chapter 2

Strichartz estimates on scattering manifolds

2.1 Introduction

The purpose of this chapter is to prove local in time Strichartz estimates on scattering manifolds for any admissible pair (including the endpoint $(2, \frac{2d}{d-2})$), where the scattering manifolds are non-compact manifolds which have asymptotically conic ends. Moreover, we study a relationship between Strichartz estimates and microlocal properties of the solution. More precisely, we show that (local in time) Strichartz estimates follow from the dispersive estimates for spatial and frequency localized solutions which we call *microlocal dispersive estimates.*

We consider the following model. We mean by the scattering manifold a non-compact manifold with an asymptotically conic structure. Let M be a non-compact Riemannian manifold of dimension $d \geq 2$ such that M can be decomposed as

$$M = M_c \cup M_\infty,$$

where $M_c \Subset M$ is relatively compact, and there exists a $(d-1)$-dimensional closed Riemannian manifold ∂M such that M_∞ is diffeomorphic to $(0, \infty) \times \partial M$. Let

$$\iota : M_\infty \ni z \mapsto (r(z), \theta(z)) \in (0, \infty) \times \partial M$$

be an identification mapping which is called a boundary decomposition. Suppose that $M_c \cap M_\infty \subset (0, 1) \times \partial M$ under this identification. We fix a boundary decomposition ι and do not write it explicitly, and denote local coordinates on $(0, \infty) \times \partial M$ by (r, θ).

We next recall the definition of the scattering metric on M. Let g be a Riemannian metric on M such that, for sufficiently large $R_M > 0$, g takes the form

$$g = dr^2 + r^2(h_{jk} + a_{jk})d\theta^j d\theta^k \quad \text{for} \quad (r, \theta) \in (R_M, \infty) \times \partial M, \tag{2.1.1}$$

where (h_{jk}) is the Riemannian metric on ∂M and (a_{jk}) is a smooth and real-valued tensor. Here we used the Einstein summation convention. We may assume that $R_M = 1$ without loss of generality, and define the *scattering region* by $\widetilde{M}_\infty := (1, \infty) \times \partial M$. We also assume that there exists $\mu > 0$ such that for any $(l, \alpha) \in \mathbb{Z}_+^d$,

$$|\partial_r^l \partial_\theta^\alpha a_{jk}(r, \theta)| \le C_{l\alpha} r^{-\mu - l}, \quad (r, \theta) \in \widetilde{M}_\infty. \tag{2.1.2}$$

Such a g is said to be a long-range *scattering metric* (in normal form).

Let Δ_g be the Laplace-Beltrami operator associated to g on $L^2(M)$:

$$\Delta_g = \frac{1}{G(z)} \partial_{z_l} G(z) g^{lm}(z) \partial_{z_l}, \ (g^{lm}(z)) = (g_{lm}(z))^{-1}, \ G(z) = \sqrt{\det g_{lm}(z)},$$

and set $P = -\frac{1}{2}\Delta_g$, where $L^p(M) := L^p(M, G(z)dz)$, $1 \le p \le \infty$. Under these settings we consider the Schrödinger equation:

$$\begin{cases} i\partial_t u(t) = Pu(t), & t \in \mathbb{R}, \\ u(0) = u_0 \in L^2(M). \end{cases} \tag{2.1.3}$$

Since P is essentially self-adjoint on $C_0^\infty(M)$ under the above condition, (2.1.3) has a unique solution $u(t) = e^{-itP} u_0$ by the Stone theorem. The main result is the following.

Theorem 2.1.1 (Strichartz estimates near infinity). *There exist a large compact subset $M_c \subset K \subset M$ and $\chi_K \in C_0^\infty(M)$ with $\chi_K \equiv 1$ on K such that*

$$||(1 - \chi_K) e^{-itP} u_0||_{L^p([0,1]; L^q(M))} \le C ||u_0||_{L^2(M)}, \quad u_0 \in C_0^\infty(M), \tag{2.1.4}$$

provided that (p, q) satisfies the admissible condition (1.0.3).

Let $p(z, \xi)$ be the principal symbol of P. We say that M is nontrapping if for any $(z_0, \xi^0) \in T^*M$ with $\xi^0 \ne 0$, the geodesic flow $(z(t, z_0, \xi^0), \xi(t, z_0, \xi^0))$ generated by H_p satisfies

$$|z(t, z_0, \xi^0)| \to +\infty \text{ as } t \to \pm\infty,$$

where

$$H_p = \sum_{j=1}^d \left(\frac{\partial p}{\partial \xi_j} \frac{\partial}{\partial z^j} - \frac{\partial p}{\partial z^j} \frac{\partial}{\partial \xi_j} \right)$$

is the Hamilton vector field associated to $p(z, \xi)$. If M is nontrapping, then using local smoothing effects which follow from resolvent estimates proved by [8] (see also [11]), we obtain local in time Strichartz estimates (without loss of derivatives) for $\chi_K e^{-itP} u_0$ (see Section 2.9 for more details). Combining with Theorem 2.1.1, we have the following:

Theorem 2.1.2 (Global in space estimates). *Suppose that M is a nontrapping scattering manifold. Then, for any (p, q) satisfying* (1.0.3), *we have*

$$||e^{-itP} u_0||_{L^p([0,1]; L^q(M))} \le C ||u_0||_{L^2(M)}, \quad u_0 \in C_0^\infty(M). \tag{2.1.5}$$

Remark 2.1.3. (i) Let $T > 0$. Since e^{-itP} is unitary on $L^2(M)$, the time interval $[0, 1]$ in (2.1.4) and (2.1.5) can be replaced by $[-T, T]$ provided that we replace the constant $C > 0$ by some $C_T > 0$ depending on T.

(ii) (Metrics in not normal form) Let g be a more general Riemannian metric than in normal form such that g takes the form

$$g = (1 + a^1)dr^2 + ra_j^2(drd\theta^j + d\theta^j dr) + r^2(h_{jk} + a_{jk}^3)d\theta^j d\theta^k$$

on $(1, \infty) \times \partial M$, where $a^N(r, \theta)$ are smooth and real-valued tensors satisfying (2.1.2) with some $0 < \mu_N < 1$. Then, there exists a change of coordinates $(r', \theta') = (R(r, \theta), \Theta(r, \theta))$ such that, for some $0 < \mu < \mu_N$,

$$R(r, \theta) = r + O(r^{1-\mu}), \ \Theta(r, \theta) = \theta + O(r^{-\mu}) \text{ as } r \to +\infty,$$

and g can be bought to a long-range metric in normal form as above. Hence the statements of Theorem 2.1.1 and Theorem 2.1.2 also hold for such a metric in not normal form. We refer to [21, Section 10.6] for more details.

(iii) (Potential perturbations) Let V be a smooth and real-valued potential on M of short-range type *i.e.*,

$$|\partial_r^l \partial_\theta^\alpha V(r, \theta)| \le C_{k\alpha} r^{-1-\nu-l} \quad \text{for} \quad (r, \theta) \in (1, \infty) \times \partial M,$$

with some $\nu > 0$. Our proof still work well if we replace P with $P + V$. Hence the statements of Theorem 2.1.1 and Theorem 2.1.2 are still hold for $P + V$.

The local in time Strichartz estimates on manifolds recently have been studied by many authors. Staffilani-Tataru [39], Robbiano-Zuily [34] and Bouclet-Tzvetkov [5] studied the case of Schrödinger equations on the Euclidean space with the asymptotically flat metric under several settings. In [7], Burq-Gérard-Tzvetkov proved Strichartz estimates with a loss of derivative $1/p$ on any compact manifolds without boundaries. They also proved that the loss $1/p$ is optimal in the case of $M = \mathbb{S}^d$. Hassell-Tao-Wunsch [21] considered the case of nontrapping scattering manifolds except the endpoint estimate. Our result thus is regarded as a generalization of their result to the critical exponent case, however the method of the proof is considerably different. More recently, Bouclet [2, 3, 4] studied the case of the asymptotically hyperbolic manifold which is a non-compact Riemannian manifold with the metric of the form

$$dr^2 + e^{2r}(h_{jk} + a_{jk})d\theta^j d\theta^k,$$

in the scattering region, and he proved Strichartz estimates localized near infinity without the nontrapping condition. The present paper is motivated by his works. Global in time Strichartz estimates has been studied by [6, 41, 31] in the case of Euclidean space with an asymptotically flat metric.

On the other hand, short time dispersive estimates for Schrödinger equations with potentials on the flat Euclidean space $(\mathbb{R}^d, \delta_{jk})$ also have been studied by many authors. In particular, it was shown by Fujiwara [15] that if $V(x)$ is smooth and real-valued and increases at most quadratically at infinity, namely $|\partial_x^\alpha V(x)| \le C_\alpha$ if $|\alpha| \ge 2$, then the fundamental solution $E(t, x, y)$ of the propagator e^{-itP} for $P = -\frac{1}{2}\Delta + V(x)$ satisfies the dispersive estimate

$$|E(t, x, y)| \le C|t|^{-d/2} \quad \text{on} \quad \mathbb{R}^d,$$

provided that $t \neq 0$ is small enough. Local in time Strichartz estimates are immediate consequences of the above estimate and the TT^*-argument due to Ginibre-Velo [16].

The rest of this chapter is devoted to the proof of Theorem 2.1.1 and 2.1.2. Though the proof is based on Bouclet's argument in [4], the behavior of classical trajectories at infinity $r \to +\infty$ is different from the case of the asymptotically hyperbolic manifold, and the class of the phase function of the parametrix becomes even worse. We thus cannot apply straightforward his method to the case of the scattering manifold. To overcome this difficulty, we introduce a localization of the r-variable by using the dyadic decomposition. The proof of Theorem 2.1.1 is then reduced to that of microlocal dispersive estimates.

This chapter is organized as follows. In Section 2.2, we fix notations and the pseudodifferential setup, and collect results of the functional calculus recently proved by Bouclet [2, 3]. Section 2.3 discusses a localization of both the space and energy, and we show that Theorem 2.1.1 follows from microlocal dispersive estimates. We study some properties of the geodesic flow in Section 2.4. In Section 2.5, we construct the semiclassical Isozaki-Kitada parametrix and prove microlocal dispersive estimates on the strongly outgoing and incoming regions (cf. Definition 2.2.5). In Section 2.6, we prove microlocal smoothing properties of the propagator on intermediate regions, which implies microlocal dispersive estimates. In Section 2.7, we construct the semiclassical WKB parametrix and prove short time microlocal dispersive estimates on the outgoing and incoming regions. We complete the proof of Theorem 2.1.1 in Section 2.8. We give the sketch of the proof of Theorem 2.1.2 in Section 2.9.

2.2 Preliminaries

In this section we set up some standard notations on scattering manifolds. Notice that a boundary decomposition is always fixed. The first step is to choose a suitable atlas and a partition of unity on the scattering region $\widetilde{M}_\infty$. Let

$$\{\kappa : \widetilde{M}_\infty \supset (1,\infty) \times V_\kappa \to (1,\infty) \times U_\kappa \subset \mathbb{R}^d\}_\kappa$$

be a finite atlas on $\widetilde{M}_\infty$ such that $\kappa = \mathrm{Id} \otimes \kappa_b$, where

$$\{\kappa_b : \partial M \Supset V_\kappa \to U_\kappa \Subset \mathbb{R}^{d-1}\}_\kappa$$

is a finite atlas on ∂M. We denote the associated pull-back and push-forward by κ^* and $\kappa_* = (\kappa^{-1})^*$, respectively. We also denote the induced chart diffeomorphism

$$T^*((1,\infty) \times V_\kappa) \to T^*((1,\infty) \times U_\kappa) \cong (1,\infty) \times U_\kappa \times \mathbb{R}^d .$$

by the same symbol κ_* if there is no confusion. Let $\{\psi_{\kappa_b}\}_\kappa \subset C_0^\infty(V_\kappa)$ be a partition of unity subordinate to $\{V_\kappa\}$ and $\psi \in C^\infty((2,\infty))$ a cut-off function such that $\psi \equiv 1$ for $r \geq 3$. We set $\psi_\kappa := \psi \times \psi_{\kappa_b}$. Then,

$$\{\psi_\kappa\}_\kappa \subset C^\infty((2,\infty) \times V_\kappa)$$

is a partition of unity subordinate to $\{(1,\infty) \times V_\kappa\}_\kappa$. Let $\tilde{\psi}_\kappa \in C^\infty(V_\kappa)$ be a cut-off function such that $\tilde{\psi}_\kappa$ takes the form

$$\tilde{\psi}_\kappa = \tilde{\psi} \times \tilde{\psi}_{\kappa_b},$$

where $\tilde{\psi} \in C^\infty((3/2,\infty))$, $\tilde{\psi}_{\kappa_b} \in C_0^\infty(V_\kappa)$, $\tilde{\psi} \equiv 1$ on $(2,\infty)$ and $\tilde{\psi}_{\kappa_b} \equiv 1$ close to $\operatorname{supp}\psi_{\kappa_b}$. Define smooth functions Ψ_κ and $\tilde{\Psi}_\kappa$ on $(1,\infty)\times U_\kappa$ by $\Psi_\kappa = \kappa_*\psi_\kappa$ and $\tilde{\Psi}_\kappa = \kappa_*\tilde{\psi}_\kappa$, respectively. Then, $\{\Psi_\kappa\}_\kappa$ is a partition of unity subordinate to $\{(1,\infty)\times U_\kappa\}_\kappa$ and $\tilde{\Psi}_\kappa$ satisfies $\tilde{\Psi}_\kappa = 1$ on $\operatorname{supp}\Psi_\kappa$.

2.2.1 Manifolds with asymptotically cylindrical ends

Noting that $r(z)$ can be extended to a positive smooth function on M and is bounded on M_c from above and below, we define a new density

$$\widehat{G}(z)dz := r(z)^{1-d}G(z)dz,\ z\in M,$$

and set

$$L^p(\widehat{M}) := L^p(M,\widehat{G}(z)dz) \quad \text{for} \quad 1\le p<\infty$$

and $L^\infty(\widehat{M}) := L^\infty(M)$. (2.1.2) implies that

$$|\partial_r^l\partial_\theta^\alpha(\widehat{G}(r,\theta) - (\det h_{jk}(\theta))^{1/2})| \le C_{l\alpha}r^{-\mu-l} \quad \text{for} \quad r>1,$$

and that $\widehat{G}(r,\theta)drd\theta$ is comparable with $drd\theta$ for $r>1$. This fact implies that, for every function u supported in $(1,\infty)\times V$ with $V \Subset V_\kappa$, $||u||_{L^p(\widehat{M})}$ is equivalent to $||\kappa_*u||_{L^p(\mathbb{R}^d)}$ for all $1\le p\le\infty$.

We next define an operator $\widehat{P}$ on $L^2(\widehat{M})$ by

$$\widehat{P} = r(z)^{\frac{d-1}{2}}Pr(z)^{-\frac{d-1}{2}}.$$

It easily see that $\widehat{P}$ is unitarily equivalent to P under the unitary diffeomorphism

$$L^2(\widehat{M}) \ni u \mapsto r(z)^{-\frac{d-1}{2}}u \in L^2(M),$$

and $\widehat{P}$ is essentially self-adjoint on $C_0^\infty(M)$. We denote the unique self-adjoint extension by the same symbol $\widehat{P}$. By definition, $(M,\widehat{G})$ can be regarded as a manifold with an asymptotically cylindrical end. We shall work with $\widehat{P}$ instead of P. By the formula (2.1.1) of the scattering metric g, $\widehat{P}_\kappa := \kappa_*\widehat{P}\kappa^*$ takes the form

$$\widehat{P}_\kappa = -\frac{1}{2}\widehat{G}^{-1}(\partial_r,\partial_\theta/r)\widehat{G}\begin{pmatrix}1 & 0\\ 0 & h+a\end{pmatrix}\begin{pmatrix}\partial_r\\ \partial_\theta/r\end{pmatrix} + W, \quad (r,\theta)\in(1,\infty)\times U_\kappa,$$

where $h=(h^{jk}) := (h_{jk})^{-1}$ and $a=(a^{jk})$ are smooth and real-valued tensors, and W is a smooth and real-valued potential such that

$$|\partial_r^l\partial_\theta^\alpha a^{jk}(r,\theta)| \le C_{k\alpha}r^{-\mu-l}, \quad |\partial_r^l\partial_\theta^\alpha W(r,\theta)| \le C_{k\alpha}r^{-1-\mu-l} \quad \text{for} \quad r>1. \tag{2.2.1}$$

Denoting by (ρ,ω) the dual coordinate to the vector field $(\partial_r,\partial_\theta)$, the principal symbol of $\widehat{P}_\kappa$ is written in the form

$$p_\kappa(r,\theta,\rho,\omega) = \frac{1}{2}\rho^2 + \frac{1}{2r^2}(h^{jk}(\theta)+a^{jk}(r,\theta))\omega_j\omega_k \quad \text{on} \quad T^*((1,\infty)\times U_\kappa). \tag{2.2.2}$$

Moreover, the full symbol of $\widehat{P}_\kappa$ takes the form $p_\kappa + p_{\kappa 1} + p_{\kappa 2}$, where $p_{\kappa j}$ can be written in the form

$$p_{\kappa j}(r,\theta,\rho,\omega) = \sum_{k+|\beta|=2-j} b_{\kappa j}^{k\beta}(r,\theta)\rho^k(\omega/r)^\beta,$$

with some smooth functions $b_{\kappa j}^{k\beta}(r,\theta)$ on $(1,\infty)\times U_\kappa$ satisfying

$$|\partial_r^l \partial_\theta^\alpha b_{\kappa j}^{k\beta}(r,\theta)| \le C_{l\alpha k\beta} r^{-1-\mu-l}. \tag{2.2.3}$$

2.2.2 Pseudodifferential calculus on scattering manifolds

In this subsection we define pseudodifferential operators and study their properties. Moreover we collect known results of the functional calculus on scattering manifolds which was proved by [2] in more general setting. We begin with the definition of our symbol class.

Definition 2.2.1. Let X be an open subset of $T^*((1,\infty)\times\mathbb{R}^{d-1}) = (1,\infty)_r \times \mathbb{R}_\theta^{d-1} \times \mathbb{R}_\rho \times \mathbb{R}_\omega^{d-1}$ such that $\pi_\theta \circ X$ is relatively compact, and that

$$|\omega| \lesssim r \quad \text{in} \quad X, \tag{2.2.4}$$

where π_θ is the projection onto the θ-space. We define the symbol class $S_{\mathrm{sc}}(X)$ as the set of all $a \in C^\infty(\mathbb{R}^{2d})$ such that $\operatorname{supp} a \subset X$, and that

$$|\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta a(r,\theta,\rho,\omega)| \le C_{j\alpha k\beta} r^{-j-|\beta|} \quad \text{on} \quad X. \tag{2.2.5}$$

Example 2.2.2. Let $\tilde{U}_\kappa \Subset U_\kappa$ be an open subset with $\tilde{\Psi}_\kappa = 1$ on $(2,\infty)\times\tilde{U}_\kappa$, p_κ the principal symbol of P_κ and $\varphi \in C_0^\infty((0,\infty))$. For $R \ge 2$ and an open interval $J \Subset (0,\infty)$ with $\operatorname{supp}\varphi \Subset J$, we set $X_\kappa = (R,\infty)\times\tilde{U}_\kappa\times\mathbb{R}^d \cap p_\kappa^{-1}(J)$. Since

$$|\rho|^2 + |\omega/r|^2 \le C p_\kappa(r,\theta,\rho,\omega) \le C \sup J,$$

X_κ satisfies (2.2.4). Moreover, by (2.2.1) and (2.2.2), we have

$$|\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta p_\kappa(r,\theta,\rho,\omega)| \le C_{j\alpha k\beta} r^{-j-|\beta|}(1+|\rho|^2+|\omega/r|^2).$$

Therefore, $\varphi\circ p_\kappa \in S_{\mathrm{sc}}(X_\kappa)$. $\varphi\circ p_\kappa$ is the principal symbol of the semiclassical pseudodifferential approximation of $\varphi(h^2\widehat{P})$ in the coordinated neighborhood $(1,\infty)\times U_\kappa$ (cf. Lemma 2.2.4).

Suppose that $X_\kappa \subset (1,\infty)\times U_\kappa\times\mathbb{R}^d$ satisfies (2.2.4) and $\tilde{\Psi}_\kappa = 1$ near $\pi\circ X_\kappa$, where $\pi : T^*\mathbb{R}^d \to \mathbb{R}^d$ is the projection onto the base space. For all $h\in(0,1]$ and $a \in S_{\mathrm{sc}}(X_\kappa)$, we define the semiclassical pseudodifferential operator (h-PDO for short) by

$$\mathrm{Op}_\kappa(a)u := \kappa^*\left(a(r,\theta,hD_r,hD_\theta)\kappa_*(\tilde{\psi}_\kappa u)\right) : C_0^\infty(M) \to C^\infty(M),$$

where $a(r,\theta,hD_r,hD_\theta)$ is the standard h-PDO which has the kernel

$$(2\pi h)^{-d}\int_{\mathbb{R}^d} e^{\frac{i}{h}[(r-r')\rho+(\theta-\theta')\omega]} a(r,\theta,\rho,\omega)\,d\rho\, d\omega.$$

Note that since $\pi \circ \operatorname{supp} a \subset (2,\infty) \times \tilde{U}_\kappa$, for any $f_\kappa \in C^\infty(M)$ with $\kappa_* f_\kappa \equiv 1$ on $\pi \circ \operatorname{supp} a$, we see that $\operatorname{Op}_\kappa(a)u = f_\kappa \operatorname{Op}_\kappa(a)u$. $\operatorname{Op}_\kappa(a)$ is thus well-defined on M. Moreover, the Calderón-Vaillancourt theorem shows that $\operatorname{Op}_\kappa(a)$ extends to a bounded operator on $L^2(\widehat{M})$ and satisfies

$$\|\operatorname{Op}_\kappa(a)\|_{\mathcal{L}(L^2(\widehat{M}))} \le C_d \sum_{|\gamma| \le M_d} \|\partial^\gamma_{(r,\theta,\rho,\omega)} a\|_{L^\infty(\mathbb{R}^{2d})} < \infty, \tag{2.2.6}$$

uniformly with respect to $h \in (0,1]$, where $C_d, M_d \ge 0$ depend only on d.

We next describe basic symbolic calculus for $S_{\rm sc}(X_\kappa)$. We first note that (2.2.4) and (2.2.5) are invariant under coordinate transformations since any chart diffeomorphism κ takes the form $\kappa = \operatorname{Id} \otimes \kappa_b$. Let $a \in S_{\rm sc}(X_\kappa)$ and $b \in S_{\rm sc}(X_{\kappa'})$. The above fact allow us to define the composition $\operatorname{Op}_\kappa(a) \circ \operatorname{Op}_{\kappa'}(b)$, though $\operatorname{Op}_{\kappa'}(b)$ is not a proper map in general. A standard symbolic calculus implies that the symbol of composition has the following semiclassical expansion:

$$\sum_{l+|\alpha|=0}^{N} \frac{h^{l+|\alpha|}}{l!\alpha!} \partial^l_\rho \partial^\alpha_\omega a(r,\theta,\rho,\omega) D^l_r D^\alpha_\theta \nu_{b*}(\tilde{\Psi}_{\kappa'} b)(r,\theta,\rho,\omega) + h^{N+1} r_N(r,\theta,\rho,\omega),$$

where $r_N \in S_{\rm sc}(X_\kappa)$ and ν_{b*} is the induced chart diffeomorphism with respect to $\nu_b = \kappa'_b \circ \kappa_b^{-1}$. Note that if $V_\kappa \cap V_{\kappa'} = \emptyset$, then $\operatorname{Op}_\kappa(a_\kappa) \circ \operatorname{Op}_{\kappa'}(b_{\kappa'}) = O(h^N)$ for any $N \ge 0$.

Following [2], we also define the *properly supported* h-PDO.

Definition 2.2.3. Let $\Psi \in C_0^\infty(\mathbb{R}^d)$ be a cut-off function such that $\operatorname{supp} \Psi \subset \{|z| < \delta\}$ and $\Psi \equiv 1$ on $\{|z| < \delta/2\}$ with some small $\delta > 0$. For $a \in S_{\rm sc}(X_\kappa)$, we then define the properly supported h-PDO

$$\operatorname{Op}^{pr}_\kappa(a) : C_0^\infty(M) \to C_0^\infty(M)$$

as the operator so that $\kappa_* \operatorname{Op}^{pr}_\kappa(a) \kappa^*$ has the kernel

$$(2\pi h)^{-d} \int e^{\frac{i}{h}[(r-r')\rho + (\theta-\theta')\omega]} a(r,\theta,\rho,\omega) \Psi(r-r', \theta-\theta') d\rho d\omega.$$

When δ is small, $\Psi \equiv 1$ sufficiently near $\pi \circ X_\kappa$ since $\operatorname{supp} a \subset X_\kappa$. We hence removed the factor $\tilde{\Psi}_\kappa$ of the amplitude. We also note that $\operatorname{Op}^{pr}_h(a)$ is uniquely determined on $L^2(\widehat{M})$ up to $O(h^\infty)$. More precisely, we have for any $N \ge 0$,

$$\|\operatorname{Op}_\kappa(a) - \operatorname{Op}^{pr}_\kappa(a)\|_{\mathcal{L}(L^2(\widehat{M}))} \le C_N h^N, \tag{2.2.7}$$

uniformly with respect to $h \in (0,1]$ since $\operatorname{supp}(1-\Psi)$ away from the diagonal. We here describe a simple property of the properly supported h-PDO. Choose arbitrarily $\chi_0, \chi_1 \in C_0^\infty((2,\infty))$ so that $\chi_1 \equiv 1$ on $\{r | \operatorname{dist}(\operatorname{supp} \chi_0, r) < 2\delta\}$. We then have

$$\chi_0(r)\Psi(r-r', \theta-\theta') = \chi_0(r)\Psi(r-r', \theta-\theta')\chi_1(r').$$

In particular, we have

$$\chi_0(r) \operatorname{Op}^{pr}_\kappa(a) = \chi_0(r) \operatorname{Op}^{pr}_\kappa(a)\chi_1(r'), \quad \operatorname{Op}^{pr}_\kappa(a)\chi_0(r') = \chi_1(r) \operatorname{Op}^{pr}_\kappa(a)\chi_0(r').$$

This property plays an important role in the spatial localization.

Fix $\varphi \in C_0^\infty((0,\infty))$ and a relatively compact open interval $J \Subset (0,\infty)$ so that $\operatorname{supp}\varphi \Subset J$. Let $\chi \in C_0^\infty(M)$ be a smooth cut-off function such that $\chi(z) = 1$ for $z \in M_c \cup \iota^{-1}((0,R_0) \times \partial M)$, $\chi(z) = 0$ for $z \in \iota^{-1}((R_0+1,\infty) \times \partial M)$ with some $R_0 > 1$. By using above h-PDO's, we have two kinds of the semiclassical approximation of $(1-\chi)\varphi(h^2\widehat{P})$.

Lemma 2.2.4 ([2]). *Let $\delta > 0$ be small enough. Then, for each κ and all $N \geq 0$, there exist semiclassical symbols*

$$a_{\kappa,h} = \sum_{j=0}^{N} h^j a_{\kappa,j} \quad \textit{with} \quad a_{\kappa,j} \in S_{\mathrm{sc}}((R_0,\infty) \times \tilde{U}_\kappa \times \mathbb{R}^d \cap p_\kappa^{-1}(J)),$$

such that

$$(1-\chi)\varphi(h^2\widehat{P}) = \sum_\kappa \mathrm{Op}_\kappa(a_{\kappa,h}) + h^{N+1}R_N(h) = \sum_\kappa \mathrm{Op}_\kappa^{pr}(a_{\kappa,h}) + h^{N+1}R_N^{pr}(h)$$

on $L^2(\widehat{M})$. Moreover, there exists $C_N > 0$ such that the followings hold true uniformly with respect to $h \in (0,1]$:
(i) *($L^2(\widehat{M})$-boundedness)*

$$\|\mathrm{Op}_\kappa(a_{\kappa,h})\|_{\mathcal{L}(L^2(\widehat{M}))} + \|R_N(h)\|_{\mathcal{L}(L^2(\widehat{M}))} \leq C_N; \tag{2.2.8}$$

(ii) *(semiclassical Sobolev embedding) for $2 \leq q \leq \infty$,*

$$\|r^{-\frac{d-1}{2}} \mathrm{Op}_\kappa^{pr}(a_{\kappa,h})\|_{\mathcal{L}(L^2(\widehat{M}),L^q(M))} \leq C_N h^{-d(1/2-1/q)}, \tag{2.2.9}$$

$$\|r^{-\frac{d-1}{2}} R_N^{pr}(h)\|_{\mathcal{L}(L^2(\widehat{M}),L^q(M))} \leq C_N h^{-d(1/2-1/q)}; \tag{2.2.10}$$

(iii) *(weighted $L^q(\widehat{M})$-boundedness) for $1 \leq q \leq \infty$ and all $s \in \mathbb{R}$,*

$$\|r^{-s} \mathrm{Op}_\kappa^{pr}(a_{\kappa,h}) r^s\|_{\mathcal{L}(L^q(\widehat{M}))} \leq C_N. \tag{2.2.11}$$

Proof. The proof was essentially given by [2]. We hence only check that

$$a_{\kappa,j} \in S_{\mathrm{sc}}((R_0,\infty) \times \tilde{U}_\kappa \times \mathbb{R}^d \cap p_\kappa^{-1}(J)).$$

$a_{\kappa,0}$ is explicitly given by $a_{\kappa,0} := \kappa_*(1-\chi)\cdot\varphi\circ p_\kappa$. Moreover, for each j, $a_{\kappa,j}$ is of the form

$$\sum_{k\leq N_j} d_{jk}\cdot(\partial^k\varphi)\circ p_\kappa \quad \text{for some} \quad 0 < N_j < \infty.$$

For each k, d_{jk} is a polynomial of degree $2k-j \geq 0$ with respect to $(\rho,\omega/r)$, and its coefficients are linear combinations of products of derivatives of $\tilde{\Psi}_\kappa$, $\kappa_*(1-\chi)$ and the full symbol of P_κ. Therefore, $a_{\kappa,j}$ takes the form

$$a_{\kappa,j}(r,\theta,\rho,\omega) = b_{\kappa,j}(r,\theta,\rho,\omega/r),$$

where $b_{\kappa,j}$ is compactly supported with respect to ρ and ω, and satisfies

$$|\partial_r^l\partial_\theta^\alpha\partial_\rho^m\partial_\omega^\beta b_{\kappa,j}(r,\theta,\rho,\omega)| \leq C_{l\alpha k\beta R_0} r^{-l}.$$

We hence obtain $a_{\kappa,j} \in S_{\mathrm{sc}}((R_0,\infty) \times \tilde{U}_\kappa \times \mathbb{R}^d \cap p_\kappa^{-1}(J))$. □

2.2.3 Outgoing and incoming regions

In this subsection we recall the definition of the outgoing and incoming regions and study some basic properties of these regions needed later. Let $R \geq 1$, $\tilde{U}_\kappa \Subset U_\kappa$ an open subset, $J \Subset (0, \infty)$ an open interval and $\sigma \in (-1, 1)$.

Definition 2.2.5. (i) We set

$$\Gamma^{\pm}(R, \tilde{U}_\kappa, J, \sigma) = \{(r, \theta, \rho, \omega) \in \mathbb{R}^{2d} \mid r > R,\ \theta \in \tilde{U}_\kappa,\ p_\kappa \in J,\ \pm\rho > -\sigma\sqrt{2p_\kappa}\},$$

where $p_\kappa = p_\kappa(r, \theta, \rho, \omega)$. $\Gamma^+(R, \tilde{U}_\kappa, J, \sigma)$ (resp. $\Gamma^-(R, \tilde{U}_\kappa, J, \sigma)$) is said to be the outgoing (resp. incoming) region.
(ii) Let $\tilde{U}_{\kappa,\sqrt{\varepsilon}}$ and J_{ε^2} be an $\sqrt{\varepsilon}$-neighborhood of $\tilde{U}_\kappa$ and an ε^2-neighborhood of J, respectively:

$$\tilde{U}_{\kappa,\sqrt{\varepsilon}} := \{\theta \in \mathbb{R}^{d-1} | \operatorname{dist}(\tilde{U}_\kappa, \theta) < \sqrt{\varepsilon}\}, \quad J_{\varepsilon^2} := \{\rho \pm \varepsilon^2 \in (0, \infty) | \rho \in J\}.$$

For sufficiently small $\varepsilon > 0$ such that $\tilde{U}_{\kappa,\sqrt{\varepsilon}} \Subset U_\kappa$, we define the strongly outgoing and incoming regions as follows:

$$\Gamma_s^{\pm}(R, \tilde{U}_\kappa, J, \varepsilon) := \Gamma^{\pm}(R, \tilde{U}_{\kappa,\sqrt{\varepsilon}}, J_{\varepsilon^2}, -\sqrt{1-\varepsilon^2}).$$

(iii) For sufficiently small $\varepsilon, \delta > 0$, and for any $L > 0$ and $\sigma_l \in (-1, 1/2]$, $l = 0, 2, ..., L$, satisfying

$$\begin{cases} (-\sqrt{1-\varepsilon^2/4}, 1/2) = \bigcup_{l=1}^{L-1} (\sigma_{l-1}, \sigma_{l+1}), \\ -\sqrt{1-\varepsilon^2/4} = \sigma_0 < \sigma_1 < ... < \sigma_L = 1/2, \quad |\sigma_{l+1} - \sigma_{l-1}| \leq \delta, \end{cases} \tag{2.2.12}$$

the intermediate outgoing region and incoming region are defined by

$$\begin{aligned} &\Gamma_i^{\pm}(R, \tilde{U}_\kappa, J, \varepsilon, \delta, l) \\ &:= \Gamma^{\pm}(R, \tilde{U}_{\kappa,\sqrt{\varepsilon}}, J_{\varepsilon^2}, 1/2) \cap \{-\sigma_{l+1}\sqrt{2p_\kappa} < \pm\rho < -\sigma_{l-1}\sqrt{2p_\kappa}\}. \end{aligned}$$

We describe basic properties of these regions. $\Gamma^{\pm}(R, \tilde{U}_\kappa, J, \sigma)$ are monotone decreasing with respect to R, and monotone increasing with respect to $\tilde{U}_\kappa$, J and σ. By definition, we have

$$(R, \infty) \times \tilde{U}_\kappa \times \mathbb{R}^d \cap p_\kappa^{-1}(J) \subset \bigcup_{\pm} \Gamma^{\pm}(R, \tilde{U}_\kappa, J, 1/2).$$

We also obtain

$$\Gamma^{\pm}(R, \tilde{U}_\kappa, J, 1/2) \subset \Gamma_s^{\pm}(R, \tilde{U}_\kappa, J, \varepsilon) \cup \bigcup_{l=1}^{L-1} \Gamma_i^{\pm}(R, \tilde{U}_\kappa, J, \varepsilon, \delta, l),$$

respectively. Moreover for sufficiently large $R_0, C > 0$, all $0 < \varepsilon < 1/2$ and $R \geq R_0$,

$$|\omega/r| \leq C\varepsilon \quad \text{on} \quad \Gamma_s^{\pm}(R, \tilde{U}_\kappa, J, \varepsilon).$$

Indeed, since $\rho^2 \geq (1-\varepsilon^2)\left(\rho^2 + \frac{1}{r^2}(h^{jk} + a^{jk})\omega_j\omega_k\right)$ on $\Gamma_s^{\pm}(R, \tilde{U}_\kappa, J, \varepsilon)$, taking $R_0 > 0$ large enough so that $(h^{jk} + a^{jk})_{j,k} \geq C_0^{-1}\,\mathrm{Id}$ for some fixed $C_0 > 0$, we obtain $|\omega/r|^2 \leq C_0\rho^2\varepsilon^2/(1-\varepsilon^2)$.

We also define spatial localized regions as follows.

Definition 2.2.6. Let $\varepsilon, \delta, L, \sigma_l$ be as above. For $R_2 > R_1 > 1$, we define the spatial localized outgoing and incoming regions $\Omega^{\pm}(R_1, R_2, \tilde{U}_\kappa, J, \sigma)$ by

$$\Omega^{\pm}(R_1, R_2, \tilde{U}_\kappa, J, \sigma) := \Gamma^{\pm}(R_1, \tilde{U}_\kappa, J, \sigma) \cap \{R_1 < r < 4R_2\}.$$

We shall use the notation $\Omega^{\pm}(R, \tilde{U}_\kappa, J, \sigma) = \Omega^{\pm}(R, R, \tilde{U}_\kappa, J, \sigma)$. We also define the corresponding spatial localized strongly outgoing (incoming) and intermediate regions

$$\Omega_s^{\pm}(R_1, R_2, \tilde{U}_\kappa, J, \varepsilon),\ \Omega_s^{\pm}(R, \tilde{U}_\kappa, J, \varepsilon),\ \Omega_i^{\pm}(R_1, R_2, \tilde{U}_\kappa, J, \varepsilon, \delta, l),\ \Omega_i^{\pm}(R, \tilde{U}_\kappa, J, \varepsilon, \delta, l)$$

in the same manner, respectively.

Remark 2.2.7. Since the principal symbol of P is invariant under coordinate transformations, these regions define invariant subsets in $T^*\widetilde{M_\infty}$ except the choice of the boundary decomposition. Moreover we will prove in Section 2.4 that these regions are also invariant under the geodesic flow generated by p. This property will be used to prove microlocal smoothing properties of the propagator (see Section 2.6).

2.3 Reduction to microlocal dispersive estimates

In this section we shall show that (2.1.4) follows from microlocal dispersive estimates. We first recall the frequency localization using the Littlewood-Paley decomposition. The following theorem was proved by Bouclet [3] for a large class of non-compact manifolds with asymptotically ends (including scattering manifolds).

Proposition 2.3.1 ([3]). *Let $\psi \in C_0^\infty((0,\infty))$ be a smooth cut-off function such that*

$$\operatorname{supp}\psi \subset [1/4, 4], \quad 0 \le \psi \le 1, \quad \sum_{j=0}^{\infty} \psi(2^{-2j}\lambda) = 1 \quad \text{for } \lambda \in [1,\infty).$$

Then, for all $\chi \in C_0^\infty(M)$ with $\operatorname{supp}(1-\chi) \subset M_\infty$ and $2 \le q \le \infty$ with $0 \le d\,(1/2 - 1/q) \le 1$, there exists $C > 0$ such that

$$||(1-\chi)u||_{L^q(M)} \le C||u||_{L^2(M)} + C\left(\sum_{j=0}^{\infty} ||(1-\chi)\psi(2^{-2j}P)u||^2_{L^q(M)}\right)^{\frac{1}{2}}.$$

Using this proposition, we see that (2.1.4) follows from semiclassical Strichartz estimates. More precisely, it suffices to prove that for $\chi \in C_0^\infty(M)$ as above and every $\varphi \in C_0^\infty((0,\infty))$,

$$||(1-\chi)\varphi(h^2P)e^{-itP}u_0||_{L^p([0,1];L^q(M))} \le C||u_0||_{L^2(M)} \tag{2.3.1}$$

uniformly with respect to $h \in (0,1]$, where (p,q) satisfies the admissible condition (1.0.3). Indeed, applying Proposition 2.3.1 to $u = e^{-itP}u_0$, taking the $L^p([0,1])$ norm for t and using Minkowski's inequality, we have

$$\begin{aligned}&||\chi_0 e^{-itP}u_0||_{L^p([0,1];L^q(M))}\\ &\le C||u_0||_{L^2(M)} + C\left(\sum_{j=0}^{\infty} ||\chi_0\psi(2^{-2j}P)e^{-itP}u_0||^2_{L^p([0,1];L^q(M))}\right)^{\frac{1}{2}}.\end{aligned}$$

Choosing $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi \equiv 1$ on $\operatorname{supp}\psi$, we obtain

$$\psi(2^{-2j}P)e^{-itP}u_0 = \varphi(2^{-2j}P)e^{-itP}\psi(2^{-2j}P)u_0,$$

since $\psi(h^2P)$ commutes with e^{-itP}. Applying (2.3.1) with $h = 2^{-j}$ to $\psi(2^{-2j}P)u_0$, the above inequality reads

$$\begin{aligned}||\chi_0 e^{-itP}u_0||_{L^p([0,1];L^q(M))} &\le C||u_0||_{L^2(M)} + C\left(\sum_{j=0}^{\infty}||\psi(2^{-2j}P)u_0||^2_{L^2(M)}\right)^{\frac{1}{2}} \\ &\le C||u_0||_{L^2(M)}.\end{aligned}$$

The last inequality we used the support property of ψ. By the definition of $\widehat{P}$, (2.3.1) is equivalent to

$$||r^{-\frac{d-1}{2}}(1-\chi)\varphi(h^2\widehat{P})e^{-it\widehat{P}}u_0||_{L^p([0,1];L^q(M))} \le C||u_0||_{L^2(\widehat{M})},\ h \in (0,1]. \tag{2.3.2}$$

For $R > 2$, we take a cut-off $\chi \in C_0^\infty(M)$ so that $\chi(z) = 1$ for $r(z) \le R$, $\chi(z) = 0$ for $r(z) \ge R+1$. Let $J \Subset (0,\infty)$ be an open interval so that $\operatorname{supp}\varphi \Subset J$ and $N \ge d/2$ an integer. By Lemma 2.2.4, we then can find a semiclassical symbol

$$a_{\kappa,h} \in S_{\mathrm{sc}}((R,\infty) \times \tilde{U}_\kappa \times \mathbb{R}^d \cap p_\kappa^{-1}(J))$$

such that $(1-\chi)\varphi(h^2\widehat{P})$ is well approximated by $A_h := \sum_\kappa \mathrm{Op}_\kappa^{pr}(a_{\kappa,h})$. Moreover, the following holds.

Proposition 2.3.2. *To prove* (2.3.2), *it is suffices to show that for sufficiently large $R > 1$ and all $a \in S_{\mathrm{sc}}((R,\infty) \times \tilde{U}_\kappa \times \mathbb{R}^d \cap p_\kappa^{-1}(J))$,*

$$||r^{-\frac{d-1}{2}}A_h \,\mathrm{Op}_\kappa(a)e^{-it\widehat{P}}u_0||_{L^p([0,1];L^q(M))} \le C||u_0||_{L^2(\widehat{M})}, \quad u_0 \in C_0^\infty(M), \tag{2.3.3}$$

uniformly with respect to $h \in (0,1]$, where (p,q) satisfies (1.0.3).

Proof. By Lemma 2.2.4, we have

$$\chi_0\varphi(h^2\widehat{P}) = \chi_1\chi_0\tilde{\varphi}(h^2P)\varphi(h^2\widehat{P}) = \chi_1\big(A_h + h^{N+1}R_N^{pr}(h)\big)\varphi(h^2\widehat{P}),$$

where $\chi_1 \in C^\infty(M)$ such that $\chi_1 = 1$ on $\operatorname{supp}\chi_0$ and $\chi_1(z) = 0$ for $r(z) \le 2R_0$. Since $N+1 \ge d/2$ and $e^{-it\widehat{P}}$ is unitary on $L^2(\widehat{M})$, (2.2.10) implies

$$\begin{aligned}&||r^{-\frac{d-1}{2}}\chi_1 h^{N+1}R_N^{pr}(h)\varphi(h^2\widehat{P})e^{-it\widehat{P}}u_0||_{L^p([0,1];L^q(M))} \\ &\le C_N h^{N+1-d(1/2-1/q)}||\varphi(h^2\widehat{P})e^{-it\widehat{P}}u_0||_{L^p([0,1];L^2(\widehat{M}))} \\ &\le C_N||u_0||_{L^2(\widehat{M})}.\end{aligned}$$

Take $\chi_2 \in C^\infty(M)$ such that $\chi_2 = 1$ near $\operatorname{supp}\chi_1$ and $\chi_2(z) = 0$ for $r(z) \le R_0$. Since A_h is properly supported, we can choose $R_0 > 0$ large enough such that

$$\chi_1 A_h\varphi(h^2\widehat{P}) = \chi_1 A_h\chi_2\varphi(h^2\widehat{P}).$$

Applying Lemma 2.2.4 to $\chi_2\varphi(h^2\widehat{P})$, we can find

$$\tilde{a}_{\kappa,h} = \sum_{j=0}^{N} h^j \tilde{a}_{\kappa,j} \quad \text{with} \quad \tilde{a}_{\kappa,j} \in S_{\text{sc}}((R,\infty) \times \tilde{U}_\kappa \times \mathbb{R}^d \cap p_\kappa^{-1}(J)), \; j = 0, 1, ..., N,$$

such that

$$\chi_2\varphi(h^2\widehat{P}) = \sum_\kappa \operatorname{Op}_\kappa(\tilde{a}_{\kappa,h}) + h^{N+1}R_N(h).$$

Then, (2.2.8) and (2.2.9) show that

$$\begin{aligned}
&||r^{-\frac{d-1}{2}}\chi_1 A_h h^{N+1} R_N(h) e^{-it\widehat{P}} u_0||_{L^p([0,1];L^q(M))} \\
&\le C_N h^{N+1-d(1/2-1/q)} ||R_N(h) e^{-it\widehat{P}} u_0||_{L^p([0,1];L^2(\widehat{M}))} \\
&\le C_N ||u_0||_{L^2(\widehat{M})}.
\end{aligned}$$

By the above two estimates and (2.3.3) with $a = \tilde{a}_{\kappa,h}$, we have for $h \in (0,1]$,

$$\begin{aligned}
&||r^{-\frac{d-1}{2}}\chi_0\varphi(h^2\widehat{P}) e^{-it\widehat{P}} u_0||_{L^p([0,1];L^q(M))} \\
&\le \sum_\kappa ||r^{-\frac{d-1}{2}} A_h \operatorname{Op}_\kappa(\tilde{a}_{\kappa,h}) e^{-it\widehat{P}} u_0||_{L^p([0,1];L^q(M))} + C_N ||u_0||_{L^2(\widehat{M})} \\
&\le C_N ||u_0||_{L^2(\widehat{M})},
\end{aligned}$$

and we conclude the proof. □

We next describe the spatial localization and give the main step of the proof of Theorem 2.1.1. The following theorem is the main result of this section. We here fix $\tilde{U}_\kappa$ and J, and hence do not write explicitly. We also use a notation such as $\Gamma^\pm(R, \tilde{U}_\kappa, J, \sigma) = \Gamma^\pm(R, \sigma)$ for short.

Theorem 2.3.3 (Microlocal dispersive estimates). *There exist $R_0 \ge 0$ and $\varepsilon_0 > 0$ such that the followings hold for all $R_2 \ge R_1 \ge R_0$, $0 < \varepsilon < \varepsilon_0$ and $h \in (0,1]$.*
(i) *There exists $t_0 > 0$, independent of R_2, such that for all symbols*

$$a^\pm \in S_{\text{sc}}(\Gamma^\pm(R_1, 1/2)), \quad b^\pm \in S_{\text{sc}}(\Omega^\pm(R_2, 1/2)),$$

and $0 < \pm t \le \min(R_2 t_0, h^{-1})$, we have

$$||r^{-\frac{d-1}{2}} A_h \operatorname{Op}_\kappa(a^\pm) e^{-ith\widehat{P}} \operatorname{Op}_\kappa(b^\pm)^* A_h^* r^{-\frac{d-1}{2}}||_{\mathcal{L}(L^1(\widehat{M}), L^\infty(\widehat{M}))} \le C_0 |th|^{-\frac{d}{2}}. \tag{2.3.4}$$

(ii) *For all symbols*

$$a_s^\pm \in S_{\text{sc}}(\Gamma_s^\pm(R_1, \varepsilon)), \quad b_s^\pm \in S_{\text{sc}}(\Omega_s^\pm(R_2, \varepsilon)),$$

and $0 < \pm t \le h^{-1}$, we have

$$||r^{-\frac{d-1}{2}} A_h \operatorname{Op}_\kappa(a_s^\pm) e^{-ith\widehat{P}} \operatorname{Op}_\kappa(b_s^\pm)^* A_h^* r^{-\frac{d-1}{2}}||_{\mathcal{L}(L^1(\widehat{M}), L^\infty(\widehat{M}))} \le C_1 |th|^{-\frac{d}{2}}. \tag{2.3.5}$$

(iii) *For all* $t_1 > 0$, *we can find* $\delta_{\varepsilon,t_1} > 0$ *and* $L_{\varepsilon,t_1} > 0$ *such that for all* $\sigma_l \in (-1, 1/2]$ *satisfying* (2.2.12), *all symbols*

$$a_l^\pm \in S_{\mathrm{sc}}(\Gamma_i^\pm(R_1, \varepsilon, \delta_{\varepsilon,t_0}, l)), \quad b_l^\pm \in S_{\mathrm{sc}}(\Omega_i^\pm(R_2, \varepsilon, \delta_{\varepsilon,t_0}, l)),$$

and $R_2 t_1 \leq \pm t \leq h^{-1}$, *we have*

$$||r^{-\frac{d-1}{2}} A_h \operatorname{Op}_\kappa(a_l^\pm) e^{-ith\widehat{P}} \operatorname{Op}_\kappa(b_l^\pm)^* A_h^* r^{-\frac{d-1}{2}}||_{\mathcal{L}(L^1(\widehat{M}), L^\infty(\widehat{M}))} \leq C_2 |th|^{-\frac{d}{2}}. \qquad (2.3.6)$$

Moreover constants $C_0, C_1, C_2 > 0$ *may be taken uniformly with respect to* h, t *and* R_2.

We give the proof of Theorem 2.3.3 in Section 2.8. Before proving Theorem 2.1.1, we prepare the following lemma.

Lemma 2.3.4. *Let* $R, \delta, L > 0$, $0 < \varepsilon < 1$ *and* $(\sigma_l)_{1 \leq l \leq L} \subset (-1, 1/2]$ *so that* (2.2.12) *is satisfied. Then for any* $a \in S_{\mathrm{sc}}((R, \infty) \times \tilde{U}_\kappa \times \mathbb{R}^d \cap p_\kappa^{-1}(J))$, *there exist symbols* $a^\pm \in S_{\mathrm{sc}}(\Gamma^\pm(R, 1/2))$, $a_s^\pm \in S_{\mathrm{sc}}(\Gamma_s^\pm(R, \varepsilon))$ *and* $a_l^\pm \in S_{\mathrm{sc}}(\Gamma_i^\pm(R, \varepsilon, \delta, l))$, $l = 1, 2, ..., L-1$, *such that*

$$a = a^+ + a^- = a_s^+ + a_s^- + \sum_{l=1}^{L-1} (a_l^+ + a_l^-).$$

Proof. Define

$$v(r, \theta, \rho, \omega) := \rho / \sqrt{2 p_\kappa(r, \theta, \rho, \omega)}.$$

v is then smooth on $\Gamma^\pm(R, 1/2)$, and satisfies

$$|\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta v(r, \theta, \rho, \omega)| \leq C_{j\alpha k\beta} r^{-j-|\beta|} \quad \text{on} \quad \Gamma^\pm(R, 1/2),$$

since p_κ is of the form (2.2.2). Take a cut-off function $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi(t) + \chi(-t) = 1, \quad \operatorname{supp} \chi \subset (-1/2, \infty),$$

and define $a^\pm = \chi(\pm v) \cdot a$. Then $a^\pm \in S_{\mathrm{sc}}(\Gamma^\pm(R, 1/2))$ and $a = a^+ + a^-$. Next, choose $\chi_s \in C^\infty(\mathbb{R})$ and $\chi_l \in C_0^\infty(\mathbb{R})$, $l = 1, 2, ..., L-1$, so that

$$\operatorname{supp} \chi_s \subset (\sqrt{1-\varepsilon^2}, \infty), \quad \operatorname{supp} \chi_l \subset (-\sigma_{l+1}, -\sigma_{l-1}), \quad \operatorname{supp} \chi_L \subset (-1, -\sigma_{L-1}),$$

and that $\chi_s + \sum_{l=1}^L \chi_l = 1$. We define $a_s^\pm := \chi_s(\pm v) \cdot a^\pm$ and $a_l^\pm := \chi_l(\pm v) \cdot a^\pm$, respectively. It is easy to see that they satisfy the assertion. ☐

Proof of Theorem 2.1.1. Choose $R_0, \varepsilon_0, t_0 > 0$ so that (i) and (ii) in Theorem 2.3.3 hold for all $R_2 \geq R_1 \geq R_0$ and $0 < \varepsilon \leq \varepsilon_0$. Next, fix $\varepsilon, \delta_{\varepsilon,t_0}, L_{\varepsilon,t_0} > 0$ so that (iii) holds with $t_1 = t_0$. By virtue of Proposition 2.3.2, it suffices to show (2.3.3) for any $a \in S_{\mathrm{sc}}((R_1, \infty) \times \tilde{U}_\kappa \times \mathbb{R}^d \cap p_\kappa^{-1}(J))$. Using Lemma 2.3.4 with $\delta = \delta_{\varepsilon,t_0}$ and $L = L_{\varepsilon,t_0}$, we split $\operatorname{Op}_\kappa(a)$ as follows:

$$\operatorname{Op}_\kappa(a) = \operatorname{Op}_\kappa(a_s^+) + \operatorname{Op}_\kappa(a_s^-) + \sum_{l=1}^{L_{\varepsilon,t_0}-1} \left(\operatorname{Op}_\kappa(a_l^+) + \operatorname{Op}_\kappa(a_l^-) \right),$$

where $a_s^\pm \in S_{\rm sc}(\Gamma_s^\pm(R_1,\varepsilon))$ and $a_l^\pm \in S_{\rm sc}(\Gamma_i^\pm(R_1,\varepsilon,\delta_{\varepsilon,t_0},l))$. By virtue of the TT^*-argument [26], it suffices to show the $\mathcal{L}(L^2(\widehat{M}),L^2(M))$-boundedness and $L^1(M)-L^\infty(M)$ estimates for corresponding operators uniformly with respect to $h\in(0,1]$.

By (2.2.6), (2.2.9) and the fact that $e^{-it\widehat{P}}$ is unitary on $L^2(\widehat{M})$, we obtain

$$||r^{-\frac{d-1}{2}}A_h\operatorname{Op}_\kappa(a_s^\pm)e^{-it\widehat{P}}u_0||_{L^2(M)}\le C||u_0||_{L^2(\widehat{M})},$$
$$||r^{-\frac{d-1}{2}}A_h\operatorname{Op}_\kappa(a_l^\pm)e^{-it\widehat{P}}u_0||_{L^2(M)}\le C||u_0||_{L^2(\widehat{M})},$$

uniformly with respect to $h\in(0,1]$, $t\in\mathbb{R}$ and $l=1,2,...,L_{\varepsilon,t_0}-1$. After the time rescaling $t\to th$, we set

$$U_s^\pm(t):=r^{-\frac{d-1}{2}}A_h\operatorname{Op}_\kappa(a_s^\pm)e^{-ith\widehat{P}},$$
$$U_l^\pm(t):=r^{-\frac{d-1}{2}}A_h\operatorname{Op}_\kappa(a_l^\pm)e^{-ith\widehat{P}}.$$

It remains to show that $U_s^\pm(t)U_s^\pm(s)^*$ and $U_l^\pm(t)U_l^\pm(s)^*$ satisfy dispersive estimates for $0<|t-s|\le h^{-1}$. We here use a trick by [5, Lemma 4.3]. We denote by $K^\pm(t-s,r,\theta,r',\theta')$ the kernel of $U_s^\pm(t)U_s^\pm(s)^*$, respectively. Since $U_s^\pm(t)U_s^\pm(s)^*=(U_s^\pm(s)U_s^\pm(t)^*)^*$, we see that

$$K^\pm(t-s,r,\theta,r',\theta')=\overline{K^\pm(s-t,r',\theta',r,\theta)}.$$

A same property holds for the kernel of $U_l^\pm(t)U_l^\pm(s)^*$. We hence can restrict the sign of $t-s$ so that $0\le\pm(t-s)\le h^{-1}$, and it is enough to prove the following:

$$\begin{aligned}&||U_s^\pm(t)U_s^\pm(s)^*u_0||_{L^\infty(M)}+||U_l^\pm(t)U_l^\pm(s)^*u_0||_{L^\infty(M)}\\&\le C|(t-s)h|^{-\frac{d}{2}}||u_0||_{L^1(M)}\end{aligned}\tag{2.3.7}$$

uniformly with respect to $h\in(0,1]$ and $0<\pm(t-s)\le h^{-1}$, $l=1,2,...,L_{\varepsilon,t_0}-1$, respectively. Combining with the facts $L^1(M)=r^{-(d-1)}L^1(\widehat{M})$, $L^\infty(M)=L^\infty(\widehat{M})$ and

$$(r^{-\frac{d-1}{2}})^*=r^{\frac{d-1}{2}}:L^2(M)\to L^2(\widehat{M}),$$

(2.3.7) is follows from

$$\begin{aligned}&||U_s^\pm(t)V_s^\pm(s)u_0||_{L^\infty(\widehat{M})}+||U_l^\pm(t)V_l^\pm(s)u_0||_{L^\infty(\widehat{M})}\\&\le C|(t-s)h|^{-\frac{d}{2}}||u_0||_{L^1(\widehat{M})},\end{aligned}\tag{2.3.8}$$

where $V_s^\pm(s)$ and $V_l^\pm(s)$ are given by

$$V_s^\pm(s):=e^{ish\widehat{P}}\operatorname{Op}_\kappa(a_s^\pm)^*A_h^*r^{-\frac{d-1}{2}},$$
$$V_l^\pm(s):=e^{ish\widehat{P}}\operatorname{Op}_\kappa(a_l^\pm)^*A_h^*r^{-\frac{d-1}{2}}.$$

We now introduce a spatial localization. Let $\chi\in C_0^\infty((0,\infty))$ be a smooth cut-off function such that

$$\operatorname{supp}\chi\subset[1,4],\quad 0\le\chi\le1,\quad \sum_{j=0}^\infty\chi(2^{-j}r)=1\quad\text{for } r\in[2,\infty).$$

Choose $\chi' \in C_0^\infty((0,\infty))$ so that $\operatorname{supp}\chi' \subset [1/2, 8]$, $\chi' \equiv 1$ on $\operatorname{supp}\chi$, and set $\chi_j(r) := \chi(2^{-j}r)$, $\chi_j'(r) := \chi'(2^{-j}r)$. Since Ψ is supported in a small ball centered at origin (see Definition 2.2.3), we obtain that

$$\chi_j(r)\Psi(r-r', \theta-\theta') = \chi_j(r)\Psi(r-r', \theta-\theta')\chi_j'(r'), \quad j \ge j_0,$$

with some large $j_0 \ge 0$. In particular, we have

$$\mathrm{Op}_\kappa(\chi_j a)^* A_h^* = \mathrm{Op}_\kappa(\chi_j a)^* A_h^* \chi_j' \quad \text{for} \quad j \ge j_0.$$

If we choose j_0 as $2^{j_0} > R_0$ and set $R_1 = 2^{j_0}$, then

$$a_s^\pm = \sum_{j \ge j_0} \chi_j a_s^\pm, \quad a_l^\pm = \sum_{j \ge j_0} \chi_j a_l^\pm.$$

Since $\chi_j a_s^\pm \in S_{\mathrm{sc}}(\Omega_s^\pm(2^j, \varepsilon))$ and $\chi_j a_l^\pm \in S_{\mathrm{sc}}(\Omega_l^\pm(2^j, \varepsilon, \delta_{\varepsilon,t_0}, l))$, applying Theorem 2.3.3 with $R_2 = 2^j$ to $U_s^\pm(t)V_s^\pm(s)$ and $U_l^\pm(t)V_l^\pm(s)$, we have

$$\begin{aligned}
&||U_s^\pm(t)V_s^\pm(s)u_0||_{L^\infty(\widehat{M})} + ||U_l^\pm(t)V_l^\pm(s)u_0||_{L^\infty(\widehat{M})} \\
&\le C|(t-s)h|^{-\frac{d}{2}} \sum_{j \ge j_0} ||\chi_j' u_0||_{L^1(\widehat{M})} \\
&\le 4C|(t-s)h|^{-\frac{d}{2}} ||u_0||_{L^1(\widehat{M})},
\end{aligned}$$

uniformly with respect to $h \in (0,1]$ and $0 < \pm(t-s) \le h^{-1}$. We hence obtain (2.3.8) and conclude the proof of Theorem 2.1.1. □

Remark 2.3.5. Since $b^\pm, b_s^\pm$ and $b_l^\pm$ are compactly supported with respect to both the space and the frequency, the above argument tells us that Strichartz estimates follows from microlocal dispersive estimates.

2.4 Classical Trajectories

In this section we study the behavior of the geodesic flow which we denote by

$$\exp tH_p : T^*M \to T^*M.$$

Recall that the principal symbol p_κ of $\widehat{P}_\kappa$ is of the form

$$p_\kappa(r, \theta, \rho, \omega) = \frac{1}{2}\rho^2 + \frac{1}{2r^2}(h^{jk}(\theta) + a^{jk}(r,\theta))\omega_j\omega_k, \ (r, \theta, \rho, \omega) \in (1,\infty) \times U_\kappa \times \mathbb{R}^d,$$

where $a^{jk}(r,\theta)$ satisfies (2.2.1). We put

$$(r(t), \theta(t), \rho(t), \omega(t)) = \exp tH_{p_\kappa}(r, \theta, \rho, \omega),$$

which solves the following Hamilton equation:

$$\begin{cases} \dot{r}(t) = \rho(t), \\ \dot{\theta}^j(t) = \dfrac{1}{r(t)^2} h^{jk}(\theta(t))\omega_k(t) + \dfrac{1}{r(t)^2} a^{jk}(r(t),\theta(t))\omega_k, \\ \dot{\rho}(t) = \dfrac{1}{r(t)^3} h^{jk}(\theta(t))\omega_j(t)\omega_k(t) + \dfrac{1}{r(t)^3} a^{jk}(r(t),\theta(t))\omega_j(t)\omega_k(t) \\ \qquad - \dfrac{1}{2r^2}\dfrac{\partial a^{jk}}{\partial r}(r(t),\theta(t))\omega_j(t)\omega_k(t), \\ \dot{\omega}_j(t) = -\dfrac{1}{2r(t)^2}\dfrac{\partial h^{kl}}{\partial \theta^j}(\theta(t))\omega_k(t)\omega_l(t) - \dfrac{1}{2r(t)^2}\dfrac{\partial a^{kl}}{\partial \theta^j}(r(t),\theta(t))\omega_k(t)\omega_l(t). \end{cases} \tag{2.4.1}$$

We first prepare an a priori estimate for $\exp tH_{p_\kappa}$.

Lemma 2.4.1. *Let $J \Subset (0,\infty)$ and $-1 < \sigma < 1$. Then, there exist $R_0 > 0$ such that for all $R \geq R_0$,*

$$C^{-1}(r + |t|) \leq r(t) \leq C(r + |t|), \quad |\rho(t)| + |\omega(t)/r| \leq C, \tag{2.4.2}$$

uniformly with respect to $(r,\theta,\rho,\omega) \in \Gamma^{\pm}(R,\tilde{U}_\kappa,J,\sigma)$ and $\pm t \geq 0$, where the constant C may be taken uniformly with respect to R and t.

Using Lemma 2.4.1, we obtain the behavior of the geodesic flow near infinity.

Proposition 2.4.2. *Let $R_0 > 0$ be as in Lemma 2.4.1 and $R \geq R_0$. Then the following estimates hold for all $(r,\theta,\rho,\omega) \in \Gamma^{\pm}(R,\tilde{U}_\kappa,J,\sigma)$, $\pm t \geq 0$ and $(j,\alpha,k,\beta) \in \mathbb{Z}_+^{2d}$ as long as the trajectory belongs to the same coordinate neighborhood $(1,\infty) \times U_\kappa$:*

$$\begin{cases} |\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta(r(t) - r \mp t\rho)| \leq Cr^{-j-|\beta|}|\omega/r|^{(2-|\beta|)_+}\langle r/t\rangle^{-1}|t|, \\ |\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta(\theta(t) - \theta)| \leq Cr^{-j-|\beta|}|\omega/r|^{(1-|\beta|)_+}\langle r/t\rangle^{-1}, \\ |\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta(\rho(t) - \rho)| \leq Cr^{-j-|\beta|}|\omega/r|^{(2-|\beta|)_+}\langle r/t\rangle^{-1}, \\ |\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta(\omega(t) - \omega)| \leq Cr^{1-j-|\beta|}|\omega/r|^{(2-|\beta|)_+}\langle r/t\rangle^{-1}, \end{cases} \tag{2.4.3}$$

$$\begin{cases} |\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta(r(t) - r)| + |\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta(\omega(t) - \omega)| \leq Cr^{-j-|\beta|}|t|, \\ |\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta(\theta(t) - \theta)| + |\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta(\rho(t) - \rho)| \leq Cr^{-1-j-|\beta|}|t|, \end{cases} \tag{2.4.4}$$

Moreover, for all $(r,\theta,\rho,\omega) \in \Gamma^{\pm}(R,\tilde{U}_\kappa,J,\sigma)$ and $\pm t \geq 0$, we have

$$|\sqrt{2E_0} \mp \rho(t)| \leq C|\omega/r|^2\langle t/r\rangle^{-1}, \tag{2.4.5}$$

where $E_0 := p_\kappa(r,\theta,\rho,\omega)$ is the initial energy.

Proof of Lemma 2.4.1. We prove the lemma for $t \geq 0$, and the proof for $t \leq 0$ is analogous. We first note that the energy conservation low, namely

$$E_0 = p_\kappa(r(t),\theta(t),\rho(t),\omega(t)), \quad t \in \mathbb{R},$$

implies $|\rho(t)|+|\omega(t)/r(t)|\le C_0$ as long as $r(t)$ large enough, where $C_0>0$ depends only on $\sqrt{E_0}$. In particular, we can find $R_1, C_1>0$ so large that

$$\dot r(0)=O(1),\ \dot\rho(0)=\frac{1}{r^3}h^{jk}(\theta)\omega_j\omega_k+O(r^{-1-\mu}),$$

for $r>R_1$, and hence

$$\left.\frac{d^2}{dt^2}r(t)^2\right|_{t=0}=2(\dot r(t)\rho(t)+r(t)\dot\rho(t))|_{t=0}\ge 4E_0-C_1r^{-\mu}.$$

Since $\sigma\in(-1,1)$, we can choose $0<\delta<1$ and $C_2>0$ so that

$$r^2-2\sigma r\sqrt{2E_0}t+2(1-\delta)^2E_0t^2\ge\frac{1}{C_2^2}(r+t\sqrt{2E_0})^2,\ r\ge0,\ t\ge0.$$

We now suppose that there exist $R>0$ and $t_0>0$ such that

$$4E_0-C_1r(t)^{-\mu}\ge 4(1-\delta)^2E_0 \tag{2.4.6}$$

holds true for $r>R$ and $0\le t\le t_0$. We then have

$$\begin{aligned}r(t)^2&\ge r^2+2r\rho t+2(1-\delta)^2E_0t^2\\&\ge r^2-2\sigma r\sqrt{2E_0}t+2(1-\delta)^2E_0t^2\\&\ge\frac{1}{C_2^2}(r+t\sqrt{2E_0})^2,\end{aligned}$$

for $0\le t\le t_0$ and $(r,\theta,\rho,\omega)\in\Gamma^+(R,\tilde U_\kappa,J,\sigma)$. Moreover, if we put

$$q_0(t)=h^{jk}(\theta(t))\omega_j(t)\omega_k(t),$$

then a direct computation yields

$$\begin{aligned}\dot q_0(t)&=\dot\theta^l\frac{\partial h^{jk}}{\partial\theta^l}\omega_j\omega_k+2h^{jk}\dot\omega_j\omega_k\\&=\frac{1}{r^2}(h^{lm}+a^{lm})\omega_m\frac{\partial h^{jk}}{\partial\theta^l}\omega_j\omega_k-h^{jk}\omega_k\frac{1}{r^2}\left(\frac{\partial h^{lm}}{\partial\theta^j}+\frac{\partial a^{lm}}{\partial\theta^j}\omega_l\omega_m\right)\\&=O(r(t)^{-1-\mu}q_0(t)).\end{aligned}$$

Integrating with respect to $s\in[0,t]$, we have

$$q_0(t)\le q_0(0)+C\int_0^t(r+|s|\sqrt{2E_0})^{-1-\mu}q_0(s)ds,\quad 0\le t\le t_0.$$

By Gronwall's inequality and the ellipticity of q_0, we obtain

$$|\omega(t)|\le C_3\sqrt{q_0}\le C_3r\sqrt{2E_0}\quad\text{for}\quad 0\le t\le t_0. \tag{2.4.7}$$

Applying (2.4.7) to (2.4.1), we have

$$\begin{cases}\dot\theta(t)=O(r(t)^{-2}r),\\ \dot\rho(t)=O(r(t)^{-3}r^2),\\ \dot\omega(t)=O(r(t)^{-2}r^2).\end{cases} \tag{2.4.8}$$

In particular, we see that

$$|\rho(t)| \leq C_4\sqrt{2E_0} \quad \text{for} \quad 0 \leq t \leq t_0,$$

with some large $C_4 > 0$. Therefore, it is enough to check that (2.4.6) holds with $t_0 = \infty$. Define

$$S := \{t \geq 0 \mid (2.4.6) \text{ holds for all } s \in [0,t]\}.$$

For sufficiently large $r > R_1$, the above argument shows that $0 \in S$ and $S \neq \emptyset$. Set $t_0 = \sup S$. The above argument then implies

$$r(t) \geq C_2^{-1}(r + t\sqrt{2E_0}) \quad \text{for} \quad r > R_2,\ 0 \leq t \leq t_0,$$

with some $R_2 > R_1$ large enough. Taking $R_3 > R_2$ so that

$$4\delta E_0 - \delta^2 E_0 \geq C_1 C_2^{\mu} r^{-\mu} \quad \text{for} \quad r > R_3,$$

we have

$$\begin{aligned} 4E_0 - C_1 r(t)^{-\mu} &\geq 4(1-\frac{\delta}{2})^2 E_0 + 4\delta E_0 - \delta^2 E_0 - C_1 C_2^{\mu}(r + t\sqrt{2E_0})^{-\mu} \\ &\geq 4(1-\frac{\delta}{2})^2 E_0 \end{aligned}$$

for $r > R_3$ and $0 \leq t \leq t_0$. Therefore, $t_0 + \varepsilon \in S$ for some $\varepsilon > 0$ which implies $t_0 = \infty$ by the definition of t_0. The estimate of $r(t)$ from above is obvious. □

Proof of Proposition 2.4.2. Let $t \geq 0$. The proof for $t < 0$ is similar. Take $R_0 > 0$ as in Lemma 2.4.1. (2.4.3) with $j + |\alpha| + k + |\beta| = 0$ is a direct consequence of Lemma 2.4.1 and (2.4.8) since

$$\int_0^t (r + |s|)^{-1-a} ds \leq C r^{-a} \langle r/t \rangle^{-1} \text{ for any } a > 0.$$

We next consider the derivatives. Put $z(t) := r(t) - t\rho(t)$. It is easy to see that

$$W(t) := (z(t), \theta(t), \rho(t), \omega(t)) = \exp(-tH_{\frac{1}{2}\rho^2}) \circ \exp tH_{p_\kappa}(r, \theta, \rho, \omega)$$

solves the following Hamilton equation:

$$\dot{z} = \frac{\partial K}{\partial \rho},\ \dot{\theta} = \frac{\partial K}{\partial \omega},\ \dot{\rho} = -\frac{\partial K}{\partial z},\ \dot{\omega} = -\frac{\partial K}{\partial \theta},$$

with a time dependent Hamiltonian:

$$K(t, z, \theta, \rho, \omega) := \frac{1}{2}\rho^2 - p_\kappa(z + t\rho, \theta, \rho, \omega) = -\frac{h^{jk}(\theta) + a_3^{jk}(z + t\rho, \theta)}{2(z + t\rho)^2}\omega_j\omega_k.$$

Lemma 2.4.1 shows that $K(t, W(t))$ satisfies

$$|(\partial_z^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta K)(t, W(t))| \leq C r(t)^{-2-j} |\omega|^{(2-|\beta|)_+} \quad \text{if } |\beta| \leq 2, \tag{2.4.9}$$

and $(\partial_z^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta K)(t,W(t)) = 0$ if $|\beta| \geq 3$. Let $\gamma = (j,\alpha,k,\beta)$, $|\gamma| = 1$, and denote $\partial^\gamma = \partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta$ for short. By differentiating the Hamilton equation with respect to ∂^γ, we have

$$\partial_t\partial^\gamma(W(t)-W(0)) = A(t)\partial^\gamma(W(t)-W(0)) + A(t)\partial^\gamma W(0), \tag{2.4.10}$$

where $A(t) = (A_j(t))_{1\leq j\leq 4} := dH_K(W(t))$ satisfies

$$\begin{cases} |A_1(t)| \leq Cr(t)^{-2}|\omega|^2, \\ |A_2(t)| \leq Cr(t)^{-2}|\omega|, \\ |A_3(t)| \leq Cr(t)^{-3}|\omega|^2, \\ |A_4(t)| \leq Cr(t)^{-2}|\omega|^2. \end{cases} \tag{2.4.11}$$

If we put

$$\begin{aligned} f(t) &= (f_1(t), f_2(t), f_3(t), f_4(t)) \\ &:= \left(\frac{\partial^\gamma(z(t)-r)}{r}, \partial^\gamma(\theta(t)-\theta), \partial^\gamma(\rho(t)-\rho), \frac{\partial^\gamma(\omega(t)-\omega)}{r}\right), \end{aligned}$$

then, by using (2.4.2), (2.4.9), (2.4.10) and (2.4.11), we have

$$|f_1(t)| \leq C\int_0^t \left(r^{-1}r(s)^{-2-j}|\omega|^{(2-|\beta|)_+} + r(s)^{-2}r|f(s)|\right) ds. \tag{2.4.12}$$

We similarly obtain

$$|f_2(t)| \leq C\int_0^t \left(r(s)^{-2-j}|\omega|^{(1-|\beta|)_+} + r(s)^{-2}r|f(s)|\right) ds, \tag{2.4.13}$$

$$|f_3(t)| \leq C\int_0^t \left(r^{-1}r(s)^{-2-j}|\omega|^{(2-|\beta|)_+} + r(s)^{-2}r|f(s)|\right) ds, \tag{2.4.14}$$

$$|f_4(t)| \leq C\int_0^t \left(r^{-1}r(s)^{-2-j}|\omega|^{(2-|\beta|)_+} + r(s)^{-2}r|f(s)|\right) ds. \tag{2.4.15}$$

Gronwall's inequality then implies

$$|f(t)| \leq C\int_0^t r(s)^{-2-j}|\omega|^{(1-|\beta|)_+} ds \lesssim r^{-1-j}|\omega|^{(1-|\beta|)_+}\langle r/t\rangle^{-1}, \tag{2.4.16}$$

and we obtain the estimate for $\partial^\gamma(\theta(t)-\theta)$. For the proof of other variables, we set

$$g(t) := (r^{-1}\partial^\gamma(z(t)-r), \partial^\gamma(\rho(t)-\rho), r^{-1}\partial^\gamma(\omega(t)-\omega)).$$

Combining the second estimate of (2.4.3) with (2.4.9), (2.4.12), (2.4.14) and (2.4.15), we have

$$|g(t)| \lesssim \int_0^t \left(r^{-1-j}r(s)^{-2}|\omega|^{(2-|\beta|)_+} + r(s)^{-2}r|g(s)|\right) ds.$$

Again Gronwall's inequality implies

$$|g(t)| \leq Cr^{-2-j}|\omega|^{(2-|\beta|)_+}\langle r/t\rangle^{-1},$$

and we obtain the estimates for $\partial^\gamma \rho(t)$, $\partial^\gamma \omega(t)$. Moreover the first estimate of (2.4.3) follows from

$$\partial^\gamma(\dot{r}(t)-\rho)=\partial^\gamma(\rho(t)-\rho)=O(r^{-2-j}|\omega|^{(2-|\beta|)_+}\langle r/t\rangle^{-1}),$$

since $\langle r/t\rangle^{-1}$ is monotone increasing with respect to t. Next, let l be a non-negative integer and suppose that (2.4.13) holds for any γ with $|\gamma|\le l$. Let $\gamma=(j,\alpha,k,\beta)$, $|\gamma|=l+1$. A direct computation yields

$$\partial_t\partial^\gamma(W(t)-W(0))=A(t)\partial^\gamma(W(t)-W(0))+A(t)\partial^\gamma W(0)+B(t),$$

where $B(t)$ is a linear combination of products of

$$(\partial_z^{\alpha_1}\partial_\theta^{\alpha'}\partial_\rho^{\alpha_{d+1}}\partial_\omega^{\alpha''}dH_K)(W(t)),$$

$$\partial^{\gamma_1^1}z(t)\cdots\partial^{\gamma_{\alpha_1}^1}z(t)\times\cdots\times\partial^{\gamma_1^{2d}}\omega_{d-1}(t)\cdots\partial^{\gamma_{\alpha_{2d}}^{2d}}\omega_{d-1}(t),$$

with $\alpha,\gamma_1^1,\gamma_2^1,...,\gamma_{\alpha_{2d}}^{2d}\in\mathbb{Z}_+^{2d}$ such that

$$\alpha=(\alpha_1,...,\alpha_{2d})=(\alpha_1,\alpha',\alpha_{d+1},\alpha'')\in\mathbb{Z}_+^{1+(d-1)+1+(d-1)},\quad 1\le|\alpha|\le|\gamma|,$$

$$\gamma_1^1+\gamma_2^1+\cdots+\gamma_{\alpha_{2d}}^{2d}=\gamma,\quad 1\le\gamma_l^k\le|\gamma|-1.$$

The induction hypothesis implies that each entry of $B(t)=(B_j(t))$ satisfies

$$\begin{aligned}&|B_1(t)|+|B_4(t)|\\ &\le C\sum_{\substack{1\le|\alpha|\le|\gamma|,\\ |\alpha''|\le 2}} r(t)^{-2-\alpha_1}r^{2-|\alpha''|}|\nu|^{2-|\alpha''|}r^{\alpha_1+|\alpha''|}r^{-j-|\beta|}|\nu|^{(2|\alpha|-|\alpha'|-|\alpha''|-|\beta|)_+}\\ &\le Cr(t)^{-2}r^{-j-|\beta|}|\nu|^{(2-|\beta|)_+},\end{aligned}$$

and we similarly obtain

$$|B_2(t)|\le Cr(t)^{-2}r^{-j-|\beta|}|\nu|^{(1-|\beta|)_+},\quad |B_3(t)|\le Cr(t)^{-3}r^{-j-|\beta|}|\nu|^{(2-|\beta|)_+},$$

where $\nu=\omega/r$. By a similar argument as that in the case for $|\gamma|=1$, we obtain the assertion. The proof of (2.4.4) is more simpler than (2.4.3), and we hence omit it.

Finally, we prove (2.4.5). Note that if $R>0$ is large enough, then

$$\dot{\rho}(t)=r(t)^{-3}\left(h^{jk}(\theta(t))+O(r(t)^{-\mu})\right)\omega_j(t)\omega_k(t)\ge 0,\quad t\in\mathbb{R}.$$

Therefore, integrating $\dot{\rho}(t)$ with respect to t, we have (2.4.5). □

Proposition 2.4.2 implies that the trajectory belongs to a fixed coordinate neighborhood as long as either $(r,\theta,\rho,\omega)\in\Gamma^\pm(R,\tilde{U}_\kappa,J,\sigma)$ and $0\ge\pm t\ge rt_0$ or $(r,\theta,\rho,\omega)\in\Gamma_s^\pm(R,\tilde{U}_\kappa,J,\varepsilon_0)$ and $\pm t\ge 0$, respectively, provided $t_0,\varepsilon_0>0$ are small enough. It also follows from Proposition 2.4.2 that the outgoing and incoming regions are invariant under the geodesic flow (except the choice of the boundary decomposition). More precisely we have the following.

Corollary 2.4.3. *Let $R \geq R_0$, $\tilde{U}_\kappa \Subset U_\kappa$, $J \Subset (0,\infty)$ and $-1 < \sigma < 1$ be as above. Suppose that $T_0 > 0$ and $(r,\theta,\rho,\omega) \in \Gamma^+(R,\tilde{U}_\kappa,J,\sigma)$, and choose a chart $\kappa' : V_{\kappa'} \to U_{\kappa'}$ and $\tilde{U}_{\kappa'} \Subset U_{\kappa'}$ so that*

$$\pi_\theta \circ \kappa'_* \exp T_0 H_p \kappa_*^{-1}(r,\theta,\rho,\omega) \in \tilde{U}_{\kappa'}.$$

Then, there exists $C > 0$, independent of R and T_0, such that

$$\kappa'_* \exp T_0 H_p \kappa_*^{-1}(r,\theta,\rho,\omega) \in \Gamma^+((R+T_0)/C, \tilde{U}_{\kappa'}, J, \sigma). \tag{2.4.17}$$

In particular, we can find $t_0 > 0$ such that for all $(r,\theta,\rho,\omega) \in \Gamma^+(R,\tilde{U}_\kappa,J,\varepsilon)$ and $0 \leq t \leq rt_0$,

$$\exp tH_{p_\kappa}(r,\theta,\rho,\omega) \in \Gamma^+((R+t)/C, \tilde{U}_{\kappa,\sqrt{t_0}}, J, \sigma), \tag{2.4.18}$$

where $\tilde{U}_{\kappa,\sqrt{t_0}}$ is a $\sqrt{t_0}$-neighborhood of $\tilde{U}_\kappa$. Moreover, there exists a small constant $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$, $(r,\theta,\rho,\omega) \in \Gamma_s^+(R,\tilde{U}_\kappa,J,\varepsilon)$ and $t \geq 0$, then

$$\exp tH_{p_\kappa}(r,\theta,\rho,\omega) \in \Gamma_s^+(R/C, \tilde{U}_\kappa, J, \varepsilon). \tag{2.4.19}$$

When $t < 0$, analogous results hold in the incoming region.

Proof. We first prove (2.4.18) and (2.4.19). By Lemma 2.4.1, we can find $R > 0$ large enough such that

$$\dot{\rho}(t) = \frac{1}{r(t)^3}\left(h^{jk}(\theta(t)) - O(r(t)^{-\mu})\right)\omega_j(t)\omega_k(t) \geq 0, \quad r > R,\ t \geq 0,$$

and hence $\rho(t) \geq \rho$ for $t \geq 0$. Therefore, (2.4.18) follows from (2.4.2), (2.4.4) and the energy conservation

$$p_\kappa(r(t),\theta(t),\rho(t),\omega(t)) = p_\kappa(r,\theta,\rho,\omega).$$

Next, let $(r,\theta,\rho,\omega) \in \Gamma_s^+(R,\tilde{U}_\kappa,J,\varepsilon)$. Since $\langle r/t\rangle^{-1} \leq 1$ and

$$|\omega/r| \leq C\varepsilon \quad \text{on} \quad \Gamma_s^\pm(R,\tilde{U}_\kappa,J,\varepsilon),$$

(2.4.19) follows from (2.4.3). To prove (2.4.17), divide the time interval $[0,T_0]$ as

$$[0,T_0] \subset [0,Rt_0] \cup [Rt_0, 2Rt_0] \cup \cdots \cup [T_0 - Rt_0, T_0].$$

In each interval $[jRt_0,(j+1)Rt_0]$, the flow is contained some fixed coordinate neighborhood. Since the outgoing region is invariant under coordinate transformations, applying (2.4.18) on each chart, we have the assertion. □

2.5 The Isozaki-Kitada parametrix

In this section we construct the Isozaki-Kitada parametrix of $e^{-ithP}\operatorname{Op}_\kappa(a_s^\pm)$, where symbols a_s^+ and a_s^- are supported in the strongly outgoing incoming regions, respectively. Though the method of construction is similar to the case of asymptotically hyperbolic manifold [4], the class of the phase function of the parametrix becomes even worse (see Remark 2.5.2). We thus give the full details of the proof.

By Corollary 2.4.3, we can always work on one fixed coordinate chart (U_κ, κ), and hence drop the subscript κ if there is no confusion. Fix an open set $\tilde{U}_\kappa \Subset U_\kappa$ with $\tilde{\Psi}_\kappa = 1$ on $(2, \infty) \times \tilde{U}_\kappa$ and an open interval $J \Subset (0, \infty)$ arbitrarily. We denote

$$\Gamma^\pm(R, \varepsilon) = \Gamma_s^\pm(R, \tilde{U}_\kappa, J, \varepsilon), \quad \Omega_s^\pm(R, \varepsilon) = \Omega_s^\pm(R, R, \tilde{U}_\kappa, J, \varepsilon)$$

for short. For a large parameter $\lambda \geq 1$, we also denote

$$\Gamma_s^\pm(\lambda) = \Gamma_s^\pm(R/\lambda, U, J, \lambda\varepsilon), \quad \Omega_s^\pm(\lambda) = \Omega_s^\pm(R/\lambda, \lambda R, U, J, \lambda\varepsilon).$$

Notice that $\Gamma_s^\pm(\lambda)$ and $\Omega_s^\pm(\lambda)$ are monotone increasing with respect to λ:

$$\Gamma_s^\pm(R, \varepsilon) \subset \Gamma_s^\pm(\lambda_1) \subset \Gamma_s^\pm(\lambda_2), \quad \Omega_s^\pm(R, \varepsilon) \subset \Omega_s^\pm(\lambda_1) \subset \Omega_s^\pm(\lambda_2), \quad 1 \leq \lambda_1 < \lambda_2.$$

2.5.1 Fourier integral operators for the Isozaki-Kitada parametrix

We here study Fourier integral operators (FIO's for short) on $\mathbb{R}^d$ which will be used to construct the Isozaki-Kitada parametrix. The first step is to construct the corresponding phase function.

Theorem 2.5.1. *There exist $R_0, \lambda_0 > 0$ large enough and $\varepsilon_0 > 0$ small enough such that for all $R, \varepsilon, \lambda > 0$ satisfying*

$$\lambda \geq \lambda_0, \quad R \geq \lambda R_0, \quad 0 < \varepsilon \leq \frac{\varepsilon_0}{\lambda},$$

we can find smooth and real-valued functions $S^\pm \in C^\infty(\mathbb{R}^{2d}, \mathbb{R})$ satisfying the Eikonal equation:

$$p(r, \theta, \partial_r S^\pm(r, \theta, \rho, \omega), \partial_\theta S^\pm(r, \theta, \rho, \omega)) = \frac{1}{2}\rho^2, \quad (r, \theta, \rho, \omega) \in \Gamma_s^\pm(R, \varepsilon).$$

If we put $\varphi^\pm(r, \theta, \rho, \omega) := S^\pm(r, \theta, \rho, \omega) - r\rho - \theta \cdot \omega$, then

$$\operatorname{supp} \varphi^\pm \subset \Gamma_s^\pm(\lambda),$$

and

$$|\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta \varphi^\pm(r, \theta, \rho, \omega)| \leq C r^{1-j-|\beta|} |\omega/r|^{(2-|\beta|)_+} \quad \text{on} \quad \Gamma_s^\pm(\lambda). \tag{2.5.1}$$

Furthermore, we can write

$$\varphi^\pm(r, \theta, \rho, \omega) = \frac{1}{2r\rho} q_0(\theta, \omega) + R^\pm(r, \theta, \rho, \omega) \quad \text{on} \quad \Gamma_s^\pm(R, \varepsilon), \tag{2.5.2}$$

where $q_0(\theta, \omega) := h^{jk}(\theta)\omega_j\omega_k$ and $R^\pm(r, \theta, \rho, \omega)$ satisfy

$$|\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta R^\pm(r, \theta, \rho, \omega)| \leq C r^{1-j-|\beta|} (|\omega/r|^{(3-|\beta|)_+} + r^{-\mu} |\omega/r|^{(2-|\beta|)_+}).$$

Here the constant $C > 0$ can be taken uniformly with respect to R, ε and λ.

Remark 2.5.2. We remark that $\varphi^{\pm}$ and its derivatives with respect to (θ, ρ) are not bounded with respect to r even for the perfectly conic ($a_{jk} \equiv 0$) case. This condition is even worse than that of the asymptotically flat case or asymptotically hyperbolic case. Indeed, we see that $\partial_x^\alpha \partial_\xi^\beta \varphi^{\pm} = O(\langle x \rangle^{-\varepsilon-|\alpha|})$ with some $\varepsilon > 0$ in the asymptotically flat case, and $\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta \varphi^{\pm} = O(e^{-r|\beta|})$ in the asymptotically hyperbolic case. We refer to [5, 4] for more details. This is one of the reasons why we introduce the spatial localization. We also note that (2.5.1) implies

$$|\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta \varphi^{\pm}(r, \theta, \rho, \omega)| \le C \langle r \rangle^{1-j-|\beta|} \varepsilon_0^{(2-|\beta|)+} \quad \text{on} \quad \mathbb{R}^{2d}, \tag{2.5.3}$$

since $|\omega/r| \le C\lambda\varepsilon \le C\varepsilon_0$ on $\operatorname{supp}\varphi^{\pm}$ ($\subset \Gamma_s^{\pm}(\lambda)$). For sufficiently large $R_0 > 0$ and sufficiently small $\varepsilon_0 > 0$, we hence have

$$1/2 < |\det {}^t\partial_{\rho,\omega}\partial_{r,\theta} S^{\pm}(r, \theta, \rho, \omega)| < 3/2 \quad \text{on} \quad \mathbb{R}^{2d}, \tag{2.5.4}$$

though $|{}^t\partial_{\rho,\omega}\partial_{r,\theta} S^{\pm} - \mathrm{Id}\,|$ is not bounded with respect to r in general. This estimate is crucial to obtain L^2-boundedness of FIO's.

To prove Theorem 2.5.1, we prepare several lemmas.

Lemma 2.5.3. *There exist $R_0 > 0$ large enough and $\varepsilon_0 > 0$ small enough such that, for all $R, \varepsilon > 0$, $\lambda_0 \ge 1$ satisfying $R \ge \lambda_0 R_0$, $\varepsilon \le \varepsilon_0/\lambda_0$ and all $\pm t \ge 0$, the maps*

$$f^{\pm}(t) : (r, \theta, \rho, \omega) \mapsto (r, \theta, \rho(t, r, \theta, \rho, \omega), \omega(t, r, \theta, \rho, \omega))$$

are diffeomorphisms from $\Gamma_s^{\pm}(\lambda_0)$ onto its range, respectively. Moreover, for sufficiently large $\lambda_0 > 0$, we have

$$\Gamma_s^{\pm}(R, \varepsilon) \subset f^{\pm}(t)\left(\Gamma_s^{\pm}(\lambda_0)\right), \quad \pm t \ge 0. \tag{2.5.5}$$

Proof. We prove the lemma for $t \ge 0$ only, and the proof for $t \le 0$ is similar. Let $F : (r, \theta, \rho, \omega) \mapsto (r, \theta, \rho, \omega/r)$ be a global diffeomorphism from $(0, \infty) \times \mathbb{R}^{d-1}$ onto itself, and we define for $(r, \theta, \rho, \nu) \in F\Gamma_s^{+}(\lambda_0)$,

$$\tilde{f}^{+}(t)(r, \theta, \rho, \nu) = (r, \theta, \tilde{\rho}(t), \tilde{\omega}(t)) := (F \circ f^{+}(t) \circ F^{-1})(r, \theta, \rho, \nu),$$

where $\nu = \omega/r$. By (2.4.3), we can choose $R_0, \varepsilon_0 > 0$ and $C_0 > 0$ such that

$$\begin{aligned} |\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\nu^\beta (\tilde{\rho}(t) - \rho)| &\le C_0 r^{-j} |\nu|^{(2-|\beta|)+} \le C_0 R_0^{-j} \varepsilon_0^{(2-|\beta|)+}, \\ |\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\nu^\beta (\tilde{\omega}(t) - \nu)| &\le C_0 r^{-j} |\nu|^{(2-|\beta|)+} \le C_0 R_0^{-j} \varepsilon_0^{(2-|\beta|)+}, \end{aligned} \tag{2.5.6}$$

and hence

$$|\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\nu^\beta (\partial \tilde{f}^{+}(t) - \mathrm{Id})| \le C\varepsilon_0 < 1/2, \tag{2.5.7}$$

uniformly with respect to $(r, \theta, \rho, \nu) \in F\Gamma_s^{+}(\lambda_0)$, where $\partial \tilde{f}^{+}(t)$ is the differential with respect to (r, θ, ρ, ν). Choose $\chi^{+} \in C^\infty(\mathbb{R}^{2d})$ so that

$$0 \le \chi^{+} \le 1, \quad \chi^{+} \equiv 1 \text{ on } F\Gamma_s^{+}(\lambda_0), \quad \operatorname{supp}\chi^{+} \subset F\Gamma_s^{+}(2\lambda_0),$$

and that

$$|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\nu^\beta\chi^+(r,\theta,\rho,\nu)| \le C_1\langle r\rangle^{-j} \quad \text{on} \quad \mathbb{R}^{2d}, \tag{2.5.8}$$

and define $\tilde{f}_\chi^+(t)(r,\theta,\rho,\nu) = (r,\theta,\tilde{\rho}_\chi(t),\tilde{\omega}_\chi(t))$ by

$$\tilde{f}_\chi^+(t)(r,\theta,\rho,\nu) := (r,\theta,(1-\chi^+)\rho+\chi^+\tilde{\rho}(t),(1-\chi^+)\nu+\chi^+\tilde{\omega}(t)).$$

Since

$$\tilde{f}_\chi^+(t) = \begin{cases} \tilde{f}^+(t) & \text{on} \quad F\Gamma_s^+(\lambda_0), \\ \tilde{f}_\chi^+(t) = \mathrm{Id} & \text{outside} \quad F\Gamma_s^+(2\lambda_0), \end{cases}$$

we have

$$|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\nu^\beta\tilde{f}_\chi^+(t)| \le C, \quad (r,\theta,\rho,\omega)\in\mathbb{R}^{2d},\ j+|\alpha|+k+|\beta|\ge 1.$$

Moreover, (2.5.6) and (2.5.8) imply

$$\begin{aligned} |\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\nu^\beta(\tilde{\rho}_\chi(t)-\rho)| &\le C_0C_1\langle r\rangle^{-j}|\nu|^{(2-|\beta|)_+} \le C_2R_0^{-j}\varepsilon_0^{(2-|\beta|)_+}, \\ |\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\nu^\beta(\tilde{\omega}_\chi(t)-\nu)| &\le C_0C_1\langle r\rangle^{-j}|\nu|^{(2-|\beta|)_+} \le C_2R_0^{-j}\varepsilon_0^{(2-|\beta|)_+} \end{aligned} \tag{2.5.9}$$

on $\mathbb{R}^{2d}$ with some $C_2 > 4^{j+2}C_0C_1$. $\tilde{f}_\chi^+(t)$ hence satisfies the same estimate as (2.5.7) on $\mathbb{R}^{2d}$ provided $R_0 > 0$ large enough and $\varepsilon_0 > 0$ small enough. By the Hadamard global inverse mapping theorem, we see that $\tilde{f}_\chi^+(t)$ is a diffeomorphism from $\mathbb{R}^{2d}$ onto its range. Since $f^+(t) = F^{-1}\circ\tilde{f}^+(t)\circ F$, $f^+(t)$ is a diffeomorphism from $\Gamma_s^+(\lambda_0)$ onto its range.

Next, we define $f_\chi^+(t) := F^{-1}\circ\tilde{f}_\chi^+\circ F$, and shall prove

$$\Gamma_s^+(R,\varepsilon) \subset f_\chi^+(t)\left(\Gamma_s^+(\lambda_0)\right), \quad t\ge 0, \tag{2.5.10}$$

for sufficiently large $\lambda_0 > 0$. Since $f_\chi^+(t)$ is bijective, it suffices to show that

$$\mathbb{R}^{2d}\setminus\Gamma_s^+(R,\varepsilon) \supset f_\chi^+(t)\left(\mathbb{R}^{2d}\setminus\Gamma_s^+(\lambda_0)\right), \quad t\ge 0.$$

Suppose that $Z := (r,\theta,\rho,\omega)\in\mathbb{R}^{2d}\setminus\Gamma_s^+(\lambda_0)$. If $Z\in\mathbb{R}^{2d}\setminus\Gamma_s^+(2\lambda_0)$, then

$$f_\chi^+(t)(Z) = Z \in \mathbb{R}^{2d}\setminus\Gamma_s^+(2\lambda_0) \subset \mathbb{R}^{2d}\setminus\Gamma_s^+(R,\varepsilon).$$

If $Z\in\Gamma_s^+(2\lambda_0)\setminus\Gamma_s^+(\lambda_0)$, then we have

$$p(Z)\in J_{4\lambda_0^2\varepsilon^2}\setminus J_{\lambda_0^2\varepsilon^2}, \quad \sqrt{1-4\lambda_1^2\varepsilon^2} \le \frac{\rho}{\sqrt{2p(Z)}} \le \sqrt{1-\lambda_0^2\varepsilon^2}.$$

Since

$$|p(f_\chi^+(t)(Z)) - p(Z)| \le C|\omega/r|^2 \le C\varepsilon_0^2,$$

Proposition 2.4.2 and the above argument imply that for sufficiently small ε_0 and sufficiently large λ_0,

$$p(f_\chi^+(t)(Z)) \notin J_{\varepsilon^2}, \quad \frac{\rho_\chi(t,Z)}{\sqrt{2p(f_\chi^+(t)(Z))}} < \sqrt{1-\varepsilon^2} \quad \text{for} \quad t\ge 0.$$

where $\rho_\chi(t,Z) = (1-\chi^+)\rho + \chi^+\rho(t,Z)$. Since $f_\chi^+(t) = f^+(t)$ on $\Gamma_s^+(\lambda_0)$, (2.5.5) follows from (2.5.10). □

Let $\pm t \geq 0$ and $\Gamma_s^{\pm}(R,\varepsilon) \ni (r,\theta,\rho,\omega) \mapsto (r,\theta,\hat{\rho}^{\pm}(t),\hat{\omega}^{\pm}(t)) \in \Gamma_s^{\pm}(\lambda_0)$ the inverse mappings of $f^{\pm}(t)$, respectively.

Lemma 2.5.4. *For $0 \leq \pm s \leq \pm t$ and $(r,\theta,\rho,\omega) \in \Gamma_s^{\pm}(R,\varepsilon)$, we define*

$$(r_t^{\pm}(s), \theta_t^{\pm}(s), \rho_t^{\pm}(s), \omega_t^{\pm}(s)) := (r,\theta,\rho,\omega)(s,r,\theta,(\hat{\rho}^{\pm},\hat{\omega}^{\pm})(t,r,\theta,\rho,\omega)).$$

We then have, for all $(j,\alpha,k,\beta) \in \mathbb{Z}_+^{2d}$,

$$|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta(\hat{\rho}^{\pm}(t)-\rho)| \leq Cr^{-j-|\beta|}|\omega/r|^{(2-|\beta|)_+},$$
$$|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta(\hat{\omega}^{\pm}(t)-\omega)| \leq Cr^{1-j-|\beta|}|\omega/r|^{(2-|\beta|)_+},$$

uniformly with respect to $(r,\theta,\rho,\omega) \in \Gamma_s^{\pm}(R,\varepsilon)$ and $\pm t \geq 0$.

Proof. Since $(r,\theta,\hat{\rho}^{\pm}(t),\hat{\omega}^{\pm}(t)) \in \Gamma_s^{\pm}(\lambda_0)$, we have

$$\begin{aligned}|\hat{\rho}^{\pm}(t)-\rho| + r^{-1}|\hat{\omega}^{\pm}(t)-\omega| &= |\rho_t(0)-\rho_t^{\pm}(t)| + r^{-1}|\omega_t(0)-\omega_t^{\pm}(t)| \\ &\leq C\sup_{\Gamma_s^{\pm}(\lambda_0)}(|\rho(t)-\rho| + r^{-1}|\omega(t)-\omega|) \\ &\leq C|\omega/r|^2,\end{aligned} \tag{2.5.11}$$

where C is independent of R_0, ε_0 and λ_0. We next consider the derivatives. Let $\gamma = (j,\alpha,k,\beta)$, $|\gamma| = 1$. Applying $\partial^\gamma = \partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta$ to the equality

$$(\rho,\omega) = (\rho,\omega)(t,r,\theta,\hat{\rho}^{\pm}(t),\hat{\omega}^{\pm}(t)),$$

we have

$$A(Z^{\pm}(t))\begin{pmatrix}\partial^\gamma(\hat{\rho}^{\pm}(t)-\rho)\\ r^{-1}\partial^\gamma(\hat{\omega}^{\pm}(t)-\omega)\end{pmatrix} = \begin{pmatrix}\partial^\gamma(\rho-\rho(t))\\ r^{-1}\partial^\gamma(\omega-\omega(t))\end{pmatrix}\Bigg|_{Z^{\pm}(t)}, \tag{2.5.12}$$

where $Z^{\pm}(t) = (r,\theta,\hat{\rho}^{\pm}(t),\hat{\omega}^{\pm}(t))$ and

$$A(Z^{\pm}(t)) = \begin{pmatrix}(\partial_\rho\rho)(t,Z^{\pm}(t)) & r(\partial_\omega\rho)(t,Z^{\pm}(t))\\ r^{-1}(\partial_\rho\omega)(t,Z^{\pm}(t)) & (\partial_\omega\omega)(t,Z^{\pm}(t))\end{pmatrix}.$$

(2.4.3) and (2.5.11) show that $A(Z^{\pm}(t))$ are invertible, $A(Z^{\pm}(t))$ and $A(Z^{\pm}(t))^{-1}$ are bounded on $\Gamma_s^{\pm}(R,\varepsilon)$ and the right hand side of (2.5.12) is bounded by

$$r^{-j-|\beta|}|\omega/r|^{(2-|\beta|)_+}.$$

The proof for higher derivatives is obtained by a simple induction with respect to $|\gamma|$, and we omit the details. □

The following easily follows from Lemma 2.4.1, Proposition 2.4.2 and Lemma 2.5.4.

Corollary 2.5.5. *For all $\pm t \geq 0$, $(r,\theta,\rho,\omega) \in \Gamma_s^{\pm}(R,\varepsilon)$, we have*

$$r_t^{\pm}(t) \geq C^{-1}(r+|t|).$$

Moreover, for all $(j,\alpha,k,\beta) \in \mathbb{Z}_+^{2d}$,

$$|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta(r_t^{\pm}(t)-r\mp t\rho)| \leq Cr^{1-j-|\beta|}|\omega/r|^{(2-|\beta|)_+},$$
$$|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta(\theta_t^{\pm}(t)-\theta)| \leq Cr^{-j-|\beta|}|\omega/r|^{(1-|\beta|)_+}.$$

Proof of Theorem 2.5.1. We give the proof for the case $t \geq 0$, and the proof for the case $t \leq 0$ is analogous. Define $\Lambda^+(t) \in C^\infty(\Gamma_s^+(R,\varepsilon))$ for $t \geq 0$ by

$$\Lambda^+(t,r,\theta,\rho,\omega) = r_t^+(t)\rho + \theta_t^+(t) \cdot \omega + \int_0^t L(r_t^+(s), \theta_t^+(s), \rho_t^+(s), \omega_t^+(s))ds, \qquad (2.5.13)$$

where

$$L(r,\theta,\rho,\omega) = \partial_\rho p(r,\theta,\rho,\omega)\rho + \partial_\omega p(r,\theta,\rho,\omega) \cdot \omega - p(r,\theta,\rho,\omega)$$

is the Lagrangian associated with p. Note that the smoothness of $\Lambda^+(t)$ follows from the smoothness of $(r_t^+, \theta_t^+, \rho_t^+, \omega_t^+)$. By the standard Hamilton-Jacobi theory, $\Lambda^+(t)$ solves

$$\begin{cases} \partial_t \Lambda^+(t) = p(r,\theta,\partial_r\Lambda^+(t), \partial_\theta\Lambda^+(t)), \\ \Lambda^+(0) = r\rho + \theta\cdot\omega, \\ \partial_{r,\theta,\rho,\omega}\Lambda^+(t) = (\hat{\rho}^+(t), \hat{\omega}^+(t), r_t^+(t), \theta_t^+(t)). \end{cases}$$

Put $F^+(t) = \Lambda^+(t) - \frac{1}{2}t\rho^2$. The energy conservation, namely

$$p(r,\theta,\hat{\rho}^+(t),\hat{\omega}^+(t)) = p(r_t^+(t), \theta_t^+(t), \rho, \omega),$$

implies

$$\partial_t F^+(t) = \frac{1}{2r_t^+(t)^2}\left(h^{jk}(\theta_t^+(t)) + a^{jk}(r_t^+(t), \theta_t^+(t)\right)\omega_j\omega_k^0.$$

By Lemma 2.5.4 and Corollary 2.5.5, we have

$$|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta\partial_t F^+(t)| \leq C(r+|t|)^{-2-j}r^{2-|\beta|}|\omega/r|^{(2-|\beta|)_+},$$

and hence

$$\left|\int_0^\infty \partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta\partial_t F^+(t)dt\right| \leq Cr^{1-j-|\beta|}|\omega/r|^{(2-|\beta|)_+}$$

for all $t \geq 0$, $(r,\theta,\rho,\omega) \in \Gamma_s^+(R,\varepsilon)$ and $(j,\alpha,k,\beta) \in \mathbb{Z}_+^{2d}$. If we put

$$q_0(\theta,\omega) = h^{jk}(\theta)\omega_j\omega_k,$$

then the mean value theorem and Corollary 2.5.5 imply

$$\left|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta\left(\frac{1}{2r_t^+(t)^2}q_0(\theta_t^+(t),\omega) - \frac{1}{2(r+t\rho)^2}q_0(\theta,\omega)\right)\right|$$
$$\leq C(r+|t|)^{-2}r^{2-j-|\beta|}|\omega/r|^{(3-|\beta|)_+}.$$

Therefore, $\partial_t F^+(t)$ can be written in the form

$$\frac{1}{2(r+t\rho)^2}q_0(\theta,\omega) + \tilde{R}^+(t)$$

with $\tilde{R}^+(t) \in C^\infty(\Gamma^+(R,\varepsilon))$ satisfying

$$|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta\tilde{R}^+(t,r,\theta,\rho,\omega)|$$
$$\leq C(r+|t|)^{-2}r^{2-j-|\beta|}\left(|\omega/r|^{(3-|\beta|)_+} + r^{-\mu}|\omega/r|^{(2-|\beta|)_+}\right).$$

Define $\tilde{\varphi}^+, \tilde{S}^+$ on $\Gamma_s^+(R,\varepsilon)$ by

$$\tilde{\varphi}^+(r,\theta,\rho,\omega) := \int_0^\infty \partial_t F^+(t,r,\theta,\rho,\omega)dt,$$
$$\tilde{S}^+(r,\theta,\rho,\omega) := r\rho + \theta\cdot\omega + \tilde{\varphi}^+(r,\theta,\rho,\omega).$$

The above argument shows that $\tilde{\varphi}^+, \tilde{S}^+$ are smooth and $\tilde{\varphi}^+$ satisfies (2.5.1) and (2.5.2) on $\Gamma_s^+(R,\varepsilon)$. Moreover, we have

$$\partial_{r,\theta}\tilde{S}^+ = \lim_{t\to+\infty}\partial_{r,\theta}\Lambda^+(t) \quad \text{on} \quad \Gamma_s^+(R,\varepsilon).$$

By using the energy conservation and Corollary 2.5.5, we see that $\tilde{S}^+$ satisfies the Eikonal equation on $\Gamma_s^+(R,\varepsilon)$:

$$\begin{aligned} p(r,\theta,\partial_r\tilde{S}^+,\partial_\theta\tilde{S}^+) &= \lim_{t\to+\infty} p(r,\theta,\partial_r\Lambda^+(t),\partial_\theta\Lambda^+(t)) \\ &= \lim_{t\to+\infty} p(r_t^+(t),\theta_t^+(t),\rho,\omega) \\ &= \rho^2/2. \end{aligned}$$

Choosing a smooth cut-off function χ^+ on $\mathbb{R}^{2d}$ so that $0\le\chi^+\le 1$, $\chi^+\equiv 1$ on $\Gamma_s^+(R,\varepsilon)$, $\operatorname{supp}\chi^+\subset\Gamma_s^+(\lambda)$, $\lambda\ge\lambda_0$ and that

$$|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta\chi(r,\theta,\rho,\omega)| \le C\langle r\rangle^{-j-|\beta|} \quad \text{on} \quad \mathbb{R}^{2d},$$

we define $\varphi^+, S^+ \in C^\infty(\mathbb{R}^{2d})$ by

$$\varphi^+ = \chi^+\tilde{\varphi}^+, \quad S^+ := r\rho + \theta\cdot\omega + \varphi^+.$$

Clearly, φ^+ and S^+ satisfy the statements of Theorem 2.5.1. □

Definition 2.5.6. For $a^\pm \in S_{\mathrm{sc}}(\Omega_s^\pm(R,\varepsilon))$ and $h\in(0,1]$, we define the FIO's for the Isozaki-Kitada parametrix $I_{\mathrm{IK}}^\pm(a^\pm): C_0^\infty(\mathbb{R}^d)\to C_0^\infty(\mathbb{R}^d)$ by

$$I_{\mathrm{IK}}^\pm(a^\pm)u(r,\theta) = (2\pi h)^{-d}\int e^{\frac{i}{h}(S^\pm(r,\theta,\rho,\omega)-r'\rho-\theta'\cdot\omega)}a^\pm(r,\theta,\rho,\omega)u(r',\theta')dr'd\theta'd\rho d\omega.$$

Since $|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta S^\pm| \lesssim R$, $|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta a^\pm| \lesssim 1$ on $\operatorname{supp} a^\pm$, $I_{\mathrm{IK}}^\pm(a^\pm)$ can be extended to operators from $\mathcal{S}(\mathbb{R}^d)$ to itself for each $R>0$. Moreover the following theorem shows that $I_{\mathrm{IK}}^\pm(a^\pm)$ are bounded on $L^2(\mathbb{R}^d)$ uniformly with respect to $R>0$ and $h\in(0,1]$.

Proposition 2.5.7. *Let $R_0,\varepsilon_0,\lambda_0>0$ be as in Theorem 2.5.1, $\lambda\ge\lambda_0$, $R\ge\lambda^4R_0$ and $0<\varepsilon\le\lambda^{-4}\varepsilon_0$. Then, for all $N\ge 0$ and $a^\pm,b^\pm\in S_{\mathrm{sc}}(\Omega_s^\pm(\lambda^3))$, there exist symbols $c_j^\pm\in S_{\mathrm{sc}}(\Omega_s^\pm(\lambda^4))$, $j=0,1,\dots,N$, such that*

$$\|I_{\mathrm{IK}}^\pm(a^\pm)I_{\mathrm{IK}}^\pm(b^\pm)^* - c_h^\pm(r,\theta,hD_r,hD_\theta)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \le C_N h^{N+1}, \quad h\in(0,1],$$

where

$$c_h^\pm = \sum_{j=0}^N h^j c_j^\pm.$$

In particular, we have

$$\|I_{\mathrm{IK}}^\pm(a^\pm)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \le C, \quad h\in(0,1].$$

Here $C_N, C>0$ may be taken uniformly with respect to R and h.

The following shows that any elliptic FIO (in the semiclassical sense) has a parametrix.

Proposition 2.5.8. *Let $R_0, \varepsilon_0, \lambda_0 > 0$ be as in Theorem 2.5.1, $\lambda \ge \lambda_0$, $R \ge \lambda^4 R_0$, $0 < \varepsilon \le \lambda^{-4}\varepsilon_0$ and $N \ge 0$ a non-negative integer. Choose arbitrarily sequences of symbols $(a_j^\pm)_{0\le j\le N} \subset S_{\rm sc}(\Omega_s^\pm(\lambda^3))$ satisfying*

$$a_0^\pm > C_0^{-1} \quad on \quad \Omega_s^\pm(\lambda)$$

with some fixed $C_0 > 0$, respectively. Then for all $c^\pm \in S_{\rm sc}(\Omega_s^\pm(R,\varepsilon))$, there exist sequences of symbols $(b_j^\pm)_{0\le j\le N} \subset S_{\rm sc}(\Omega_s^\pm(\lambda))$ such that

$$||I_{\rm IK}^\pm(a^\pm)I_{\rm IK}^\pm(b^\pm)^* - c^\pm(r,\theta,hD_r,hD_\theta)||_{\mathcal{L}(L^2(\mathbb{R}^d))} \le C_N h^{N+1},\ h \in (0,1],$$

where $a_h^\pm = \sum_{j=0}^N h^j a_j^\pm$ and $b_h^\pm = \sum_{j=0}^N h^j b_j^\pm$. Moreover $C_N > 0$ can be taken uniformly with respect to R and h.

To prove the above two propositions, we need the following lemma.

Lemma 2.5.9. *Define $(\rho_1^\pm, \omega_1^\pm)(r,\theta,\rho,\omega,r',\theta') : \mathbb{R}^{3d} \to \mathbb{R}^d$ by*

$$(\rho_1^\pm, \omega_1^\pm) = \int_0^1 (\partial_{r,\theta} S^\pm)(r' + \sigma(r-r'), \theta' + \sigma(\theta-\theta'), \rho, \omega)d\sigma. \tag{2.5.14}$$

Then the followings hold for all $\lambda \ge \lambda_0$, $R \ge \lambda R_0$ and $0 < \varepsilon \le \varepsilon_0/\lambda$.
(i) *If $(r,\theta,\rho,\omega) \in \Gamma_s^\pm(R,\varepsilon)$, then*

$$(r,\theta,\rho_1^\pm,\omega_1^\pm)|_{r'=r,\theta'=\theta} \in \Gamma_s^\pm(\lambda). \tag{2.5.15}$$

Conversely, if $(r,\theta,\rho_1^\pm,\omega_1^\pm)|_{r'=r,\theta'=\theta} \in \Gamma_s^\pm(R,\varepsilon)$, then

$$(r,\theta,\rho,\omega) \in \Gamma_s^\pm(\lambda). \tag{2.5.16}$$

(ii) *For all $(r,\theta,r',\theta') \in \mathbb{R}^{2d}$ satisfying $R/\lambda < r, r' < 4\lambda R$, $(\rho,\omega) \mapsto (\rho_1^\pm,\omega_1^\pm)$ are diffeomorphisms from $\mathbb{R}^d$ onto itself, respectively. Denoting by $(\rho_2^\pm,\omega_2^\pm)$ the corresponding inverses, the same properties as in* (i) *hold with $(\rho_1^\pm,\omega_1^\pm)$, $\Gamma_s^\pm(R,\varepsilon)$ and $\Gamma_s^\pm(\lambda)$ replaced by $(\rho_2^\pm,\omega_2^\pm)$, $\Omega_s^\pm(R,\varepsilon)$ and $\Omega_s^\pm(\lambda)$, respectively.*
(iii) *If $\lambda,\lambda' \ge \lambda_0$, $R \ge \max(\lambda,\lambda')R_0$ and $0 < \varepsilon \le \min(\lambda^{-1},\lambda'^{-1})\varepsilon_0$, then we have*

$$|\partial^\gamma(\rho_2^\pm - \rho)| + r^{-1}|\partial^\gamma(\omega_2^\pm - \omega)| \le Cr^{-j-j'-|\beta|}|\omega_2^\pm/r|^{(2-|\beta|)_+}, \tag{2.5.17}$$

for all $(r,\theta,\rho_2^\pm,\omega_2^\pm) \in \Omega_s^\pm(\lambda)$ and $(r',\theta',\rho_2^\pm,\omega_2^\pm) \in \Omega_s^\pm(\lambda')$, where we use the notation

$$\partial^\gamma := \partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta \partial_{r'}^{j'} \partial_{\theta'}^{\alpha'}$$

for $\gamma = (j,\alpha,k,\beta,j',\alpha') \in \mathbb{Z}_+^{3d}$.

Proof. We only consider the outgoing case. Remark that (2.5.14) is equivalent to

$$(\rho_1^+,\omega_1^+) = (\rho,\omega) + \int_0^1 (\partial_{r,\theta}\varphi^+)(r' + \sigma(r-r'), \theta' + \sigma(\theta-\theta'), \rho, \omega)d\sigma.$$

Suppose that $(r,\theta,\rho,\omega) \in \Gamma_s^+(R,\varepsilon)$. Since

$$(\rho_1^+ - \rho, \omega_1^+ - \omega)|_{r'=r,\theta'=\theta} = (\partial_{r,\theta}\varphi^+)(r,\theta,\rho,\omega),$$

(2.5.1) implies

$$(|\rho_1^+ - \rho| + r^{-1}|\omega_1^+ - \omega|)|_{r'=r,\theta'=\theta} \le C\varepsilon^2,$$

and

$$|p_1 - p| \le Cp\varepsilon^2, \quad \rho_1^+|_{r'=r,\theta'=\theta} \ge \sqrt{(1-C\varepsilon^2)2p_1},$$

with some $C>0$ which is independent of R, ε and λ_0, where we denote

$$p = p(r,\theta,\rho,\omega), \quad p_1 = p(r,\theta,\rho_1^+,\omega_1^+)|_{r'=r,\theta'=\theta},$$

for short. Choosing $\lambda > 0$ large enough such that $\lambda^2 > C$, we have

$$(r,\theta,\rho_1^+,\omega_1^+)|_{r'=r,\theta'=\theta} \in \Gamma^+(\lambda).$$

Next, consider the mapping

$$f^+ : (r,\theta,\rho,\omega) \mapsto (r,\theta,\rho_1^+,\omega_1^+)|_{r'=r,\theta'=\theta}.$$

By (2.5.1), we have

$$\begin{aligned} \left|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta(\rho_1^+|_{r'=r,\theta'=\theta} - \rho)\right| &\le C_{j\alpha k\beta} r^{-j-|\beta|}|\omega/r|^{(2-|\beta|)_+}, \\ \left|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta(\omega_1^+|_{r'=r,\theta'=\theta} - \omega)\right| &\le C_{j\alpha k\beta} r^{1-j-|\beta|}|\omega/r|^{(2-|\beta|)_+}. \end{aligned} \tag{2.5.18}$$

By same argument as that in the proof of Lemma 2.5.3, we obtain that f^+ is injective and

$$\Gamma_s^+(R,\varepsilon) \subset f^+(\Gamma_s^+(\lambda)),$$

provided that $\lambda > 0$ is large enough. We note that this λ can be taken uniformly with respect to R and ε. This fact implies (2.5.16).

We next prove (ii). Let $F_R : (\rho,\omega) \mapsto (\rho,\nu) = (\rho,\omega/R)$ be a global diffeomorphism from $\mathbb{R}^d$ onto itself. For $(r,\theta,r',\theta') \in \mathbb{R}^{2d}$ with $R/\lambda < r, r' < 4\lambda R$, we define $(\tilde\rho_1^+,\tilde\omega_1^+)$ by

$$(\tilde\rho_1^+,\tilde\omega_1^+) := F_R \circ (\rho_1^+,\omega_1^+) \circ F_R^{-1} = (\rho_1^+, R^{-1}\omega_1^+)(\rho.R\nu).$$

For sufficiently large $R_0 > 0$ and sufficiently small $\varepsilon_0 > 0$, (2.5.3) implies

$$\partial_{\rho,\nu}(\tilde\rho_1^+,\tilde\omega_1^+)(r,\theta,\rho,\omega,r',\theta') = \mathrm{Id} + O(\varepsilon_0),$$

and $(\tilde\rho_1^+,\tilde\omega_1^+)$ is diffeomorphic from $\mathbb{R}^d$ onto itself and so is (ρ_1^+,ω_1^+). Applying (2.5.16) and (2.5.15) with $(\rho,\omega) = (\rho_2^+,\omega_2^+)|_{r'=r,\theta'=\theta}$, we obtain

$$\begin{aligned} (r,\theta,\rho_2^+,\omega_2^+)|_{r'=r,\theta'=\theta} \in \Omega_s^+(\lambda) \quad &\text{if} \quad (r,\theta,\rho,\omega) \in \Omega_s^+(R,\varepsilon), \\ (r,\theta,\rho,\omega) \in \Omega_s^+(\lambda) \quad &\text{if} \quad (r,\theta,\rho_2^+,\omega_2^+)|_{r'=r,\theta'=\theta} \in \Omega_s^+(R,\varepsilon). \end{aligned}$$

Finally we shall prove (iii). Since $(r, \theta, \rho_2^+, \omega_2^+)$ satisfy

$$(\rho_2^+, \omega_2^+) = (\rho, \omega) - \int_0^1 (\partial_{r,\theta}\varphi^+)(r' + \sigma(r - r'); \theta' + \sigma(\theta - \theta'), \rho_2^+, \omega_2^+) d\sigma, \tag{2.5.19}$$

(2.5.1) implies

$$(\rho_2^+ - \rho)| + r^{-1}|\omega_2^+ - \omega| \le C|\omega_2^+/r|^{(2-|\beta|)+} \le C\varepsilon_0^{(2-|\beta|)+}$$

for $(r, \theta, \rho_2^+, \omega_2^+) \in \Gamma_s^+(\lambda)$ and $(r', \theta', \rho_2^+, \omega_2^+) \in \Gamma_s^+(\lambda')$. For the derivatives, differentiating (2.5.19) with respect to $\partial_r^j \partial_\theta^\alpha \partial_{r'}^{j'} \partial_{\theta'}^{\alpha'} \partial_\rho^k \partial_\omega^\beta$ and using (2.5.1), we obtain (2.5.17) by an induction with respect to $j + |\alpha| + k + |\beta| + j' + |\alpha'|$. □

Proof of Proposition 2.5.7. We only prove the outgoing case. Note that since $R \ge \lambda^4 R_0$ and $\varepsilon \le \lambda^{-4}\varepsilon_0$, (ρ_2^+, ω_2^+) is well-defined for $\lambda^{-3}R < r, r' < 4\lambda^3 R$. The Schwartz kernel of $I_{\mathrm{IK}}^+(a^+) I_{\mathrm{IK}}^+(b^+)^*$ can be written in the form

$$(2\pi h)^{-d} \int e^{\frac{i}{h}(S^+(r,\theta,\rho,\omega) - S^+(r',\theta',\rho,\omega))} a^+(r, \theta, \rho, \omega) \overline{b^+(r', \theta', \rho, \omega)} d\rho d\omega.$$

By (2.5.4), we have

$$1/2 < |\det \partial_{\rho,\omega}(\rho_2^+, \omega_2^+)| < 3/2. \tag{2.5.20}$$

We thus can make the change of variables $(\rho, \omega) \mapsto (\rho_2^+, \omega_2^+)(r, \theta, \rho, \omega, r', \theta')$, and the above integral can be bought to the form

$$(2\pi h)^{-d} \int e^{\frac{i}{h}[(r-r')\rho - (\theta-\theta')\cdot\omega]} A^+(r, \theta, \rho, \omega, r', \theta') d\rho d\omega,$$

where

$$A^+ = a^+(r, \theta, \rho_2^+, \omega_2^+) \overline{b^+(r', \theta', \rho_2^+, \omega_2^+)} |\det \partial_{\rho,\omega}(\rho_2^+, \omega_2^+)|.$$

By using (2.5.17), (2.5.20) and the support properties of a^+ and b^+, $\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta \partial_{r'}^{j'} \partial_{\theta'}^{\alpha'} A^+$ are uniformly bounded on $\mathbb{R}^{3d}$ for all $(j, \alpha, k, \beta, j', \alpha') \in \mathbb{Z}_+^{3d}$, and the Calderón-Vaillancourt theorem implies

$$||I_{\mathrm{IK}}^+(a^+) I_{\mathrm{IK}}^+(b^+)^*||_{\mathcal{L}(L^2(\mathbb{R}^d))} \le C_d \sum_{|\gamma| \le M_d} ||\partial^\gamma A^+||_{L^\infty(\mathbb{R}^{3d})} \le C,$$

uniformly with respect to $R > 0$ and $h \in (0, 1]$, with some $C_d, M_d > 0$ depending only on d. In particular $I_{\mathrm{IK}}^+(a^+)_{\mathcal{L}(L^2(\mathbb{R}^d))}$ is bounded on $L^2(\mathbb{R}^d)$ uniformly with respect to $R > 0$ and $h \in (0, 1]$. Furthermore, by the standard symbolic calculus (*e.g.*, see the textbook [29]), the simplified symbol of the above operator has the following asymptotic expansion

$$\sum_{l+|\alpha|=j} \frac{h^j}{l!\alpha!} \left(\partial_\rho^l \partial_\omega^\alpha D_{r'}^l D_{\theta'}^\alpha A^+\right)(r, \theta, \rho, \omega, r, \theta).$$

By Lemma 2.5.9 (ii) and (iii), we see that $(\partial_\rho^l \partial_\omega^\alpha D_{r'}^l D_{\theta'}^\alpha A^+)(r, \theta, \rho, \omega, r, \theta)$ is supported in $\Omega_s^+(\lambda^4)$ and belongs to $S_{\mathrm{sc}}(\Omega_s^\pm(\lambda^4))$. □

Proof of Proposition 2.5.8. Let $c^+ \in S_{\mathrm{sc}}(\Omega_s^+(R,\varepsilon))$. By Proposition 2.5.7, it suffices to show that there exist $b_j^+ \in S_{\mathrm{sc}}(\Omega_s^+(\lambda))$, $j = 0,1,...,N$, such that $c_0^+ = c^+$ and $c_j^+ \equiv 0$ for $j = 1,2,...,N$. We set $J_k = |\det \partial_{\rho,\omega}(\rho_k^+, \omega_k^+)|$. b_j^+ can be defined inductively as follows. We first note that, by the construction, b_h^+ should satisfy

$$c^+(r,\theta,\rho,\omega)$$
$$= \sum_{j=1}^{N} \sum_{l+|\alpha|=j} \frac{h^j}{l!\alpha!} \left(\partial_\rho^l \partial_\omega^\alpha D_{r'}^l D_{\theta'}^\alpha a_h^+(r,\theta,\rho_2^+,\omega_2^+) \overline{b_h^+(r',\theta',\rho_2^+,\omega_2^+)} J_2 \right) \Big|_{r'=r,\theta'=\theta}.$$

Define the principal symbol b_0^+ by

$$\overline{b_0^+(r,\theta,\rho,\omega)} := (a_0^+(r,\theta,\rho,\omega))^{-1} c^+(r,\theta,\rho_1^+,\omega_1^+) J_1|_{r'=r,\theta'=\theta}.$$

Since $\operatorname{supp} c^+ \subset \Omega_s^+(R,\varepsilon)$, Lemma 2.5.9 (i) implies that $c^+(r,\theta,\rho_1^+,\omega_1^+)|_{r'=r,\theta'=\theta}$ is supported in $\Omega_s^+(\lambda)$. Since a_0^+ is elliptic on $\Omega_s^+(\lambda)$, b_0^+ is well-defined and supported in $\Omega_s^+(\lambda)$. Moreover, (2.5.18) implies that $b_0^+ \in S_{\mathrm{sc}}(\Omega_s^+(\lambda))$. Next, let $j \geq 1$ and assume that $b_k^+ \in S_{\mathrm{sc}}(\Omega_s^+(\lambda))$ for all $k < j$. We then define b_j^+ by

$$a_0^+(r,\theta,\rho_2^+,\omega_2^+) \overline{b_j^+(r',\theta',\rho_2^+,\omega_2^+)} J_2|_{r'=r,\theta'=\theta} = r_j^+(r,\theta,\rho,\omega), \tag{2.5.21}$$

where r_j^+ takes the form

$$\sum_{k=0}^{j} \sum_{\substack{l+|\alpha|=k,\\ k_0+k_1=j-k,\\ k_1 \leq j-1}} \frac{1}{l!\alpha!} \left(\partial_\rho^l \partial_\omega^\alpha D_{r'}^l D_{\theta'}^\alpha a_{k_0}^+(r,\theta,\rho_2^+,\omega_2^+) \overline{b_{k_1}^+(r',\theta',\rho_2^+,\omega_2^+)} J_2 \right) \Big|_{r'=r,\theta'=\theta}.$$

Substituting $(\rho,\omega) = (\rho_1^+,\omega_1^+)$ for (2.5.21) and dividing by a_0^+, we have

$$\overline{b_j^+(r,\theta,\rho,\omega)} = (a_0^+(r',\theta',\rho,\omega))^{-1} r_j^+(r,\theta,\rho_1^+,\omega_1^+) J_1|_{r'=r,\theta'=\theta}.$$

By induction hypothesis, we conclude that $b_j^+ \in S_{\mathrm{sc}}(\Omega_s^+(\lambda))$. The proof for the incoming case is similar. □

2.5.2 Construction of the parametrix

By using the FIO defined in the previous subsection, we construct the semiclassical Isozaki-Kitada parametrix.

Theorem 2.5.10. *For any $N \geq 0$, there exist $R_{\mathrm{IK}}, \lambda_{\mathrm{IK}} > 0$ large enough and $\varepsilon_{\mathrm{IK}} > 0$ small enough such that if $R \geq R_{\mathrm{IK}}$, $0 < \varepsilon \leq \varepsilon_{\mathrm{IK}}$ and $\lambda \geq \lambda_{\mathrm{IK}}$, then we can find*

$$b_h^\pm = \sum_{j=0}^{N} b_j^\pm \quad \textit{with} \quad b_j^\pm \in S_{\mathrm{sc}}(\Omega_s^\pm(\lambda^3)),\ j = 0,1,...,N,$$

such that, for any $a_s^\pm \in S_{\mathrm{sc}}(\Omega_s^\pm(R,\varepsilon))$, there exist

$$c_h^\pm = \sum_{j=0}^{N} c_j^\pm \quad \textit{with} \quad c_j^\pm \in S_{\mathrm{sc}}(\Omega_s^\pm(\lambda)),\ j = 0,1,...,N,$$

such that, for all $T > 0$, $h \in (0,1]$ and $0 \le \pm t \le Th^{-1}$,

$$||e^{-ith\widehat{P}}\operatorname{Op}_\kappa(a_s^\pm) - \kappa^* I_{\mathrm{IK}}^\pm(b_h^\pm)e^{-ith\frac{1}{2}D_r^2}I_{\mathrm{IK}}^\pm(c_h^\pm)^*\kappa_*||_{\mathcal{L}(L^2(\widehat{M}))} \le C_{N,T}h^{N-1}, \tag{2.5.22}$$

where $C_{N,T}$ may be taken uniformly with respect to h, t and R.

We shall prove Theorem 2.5.10 for the outgoing case, and the proof for the incoming case is completely analogous. Set $B_+ := I_{\mathrm{IK}}^+(b_h^+)$, $C_+ := I_{\mathrm{IK}}^+(c_h^+)$. By the Duhamel formula, we have

$$\begin{aligned}
&e^{-ith\widehat{P}}\kappa^* B_+ C_+^*\kappa_* \\
&= \kappa^* B_+ e^{-ith\frac{1}{2}D_r^2}C_+^*\kappa_* - \frac{i}{h}\int_0^t e^{-i(t-s)h\widehat{P}}\kappa^*(h^2\widehat{P}_\kappa B_+ - B_+h^2\frac{1}{2}D_r^2)e^{-ish\frac{1}{2}D_r^2}C_+^* ds.
\end{aligned}$$

To prove (2.5.22), it suffices to show that

$$||(h^2P_\kappa B_+ - B_+h^2\frac{1}{2}D_r^2)e^{-ish\frac{1}{2}D_r^2}C_+^*||_{\mathcal{L}(L^2(\mathbb{R}^d))} \lesssim h^{N+1}, \tag{2.5.23}$$

$$||a_s^+(r,\theta,hD_r,hD_\theta) - B_+C_+^*||_{\mathcal{L}(L^2(\mathbb{R}^d))} \lesssim h^N, \tag{2.5.24}$$

uniformly with respect to $h \in (0,1]$, $0 \le s \le Th^{-1}$ and $R > 0$. To prove above two estimates, we prepare several lemmas.

Let $p + p_1 + p_2$ be the full symbol of $\widehat{P}_\kappa$:

$$\widehat{P}_\kappa = p(r,\theta,D_r,D_\theta) + p_1(r,\theta,D_r,D_\theta) + p_2(r,\theta).$$

Choosing $R_0 > 0$ and $\varepsilon_0 > 0$ so that S^+ is well-defined and solves the Hamilton-Jacobi equation on $\Gamma_s^+(R_0,\varepsilon_0)$, we define smooth tensors X^+ and Y^+ by

$$X^+ := \partial_{\rho,\omega}p(r,\theta,\partial_r S^+,\partial_\theta S^+), \quad Y^+ := (p+p_1)(r,\theta,\partial_r,\partial_\theta)S^+,$$

and define symbols d_j^+, $j = 1,2,...,N+2$, by

$$h^2\widehat{P}_\kappa B_+ - B_+h^2\frac{1}{2}D_r^2 = \sum_{j=1}^{N+2} h^j I_{\mathrm{IK}}^+(d_j^+).$$

Then d_j^+ should satisfy

$$\begin{cases} id_1^+ = X^+\cdot\partial_{r,\theta}b_0^+ + Y^+b_0^+, \\ id_j^+ = X^+\cdot\partial_{r,\theta}b_{j-1}^+ + Y^+b_{j-1}^+ + i\widehat{P}_\kappa b_{j-2}, \quad j = 2,...,N+2, \end{cases} \tag{2.5.25}$$

where $b_{N+1}^+ \equiv 0$. To construct b_j^+, we solve transport equations.

Lemma 2.5.11. *There exist $R_1 \ge R_0$, $\lambda_1 > 1$ large enough and $\varepsilon_1 \le \varepsilon_0$ small enough such that, for all $N \ge 0$, $R \ge \lambda_1^{N+5}R_1$ and $0 < \varepsilon \le \lambda_1^{-N-5}\varepsilon_1$, we can find $b_j^+ \in S_{\mathrm{sc}}(\Omega_s^+(\lambda_1^3))$, $j = 0,1,...,N$ such that b_0^+ is elliptic on $\Omega_s^+(\lambda_1)$, and that b_j^+ solve transport equations on $\Omega_s^+(\lambda_1^2)$:*

$$\begin{cases} X^+\cdot\partial_{r,\theta}b_0^+ + Y^+b_0^+ = 0, \\ X^+\cdot\partial_{r,\theta}b_j^+ + Y^+b_j^+ + i\widehat{P}_\kappa b_{j-1}^+ = 0, \quad j = 1,...,N. \end{cases} \tag{2.5.26}$$

For $(r, \theta, \rho, \omega) \in \Gamma_s^+(\lambda_1^{N+4})$, we consider the flow $(r^+(t), \theta^+(t))$ generated by X^+, *i.e.*, $(r^+(t), \theta^+(t)) = (r^+, \theta^+)(t, r, \theta, \rho, \omega)$ is the solution to

$$\begin{cases} (\dot{r}^+(t), \dot{\theta}^+(t)) = X^+(r^+(t), \theta^+(t), \rho, \omega), \\ (r^+(0), \theta^+(0)) = (r, \theta). \end{cases}$$

Then $(r^+(t), \theta^+(t))$ is defined on $[0, \infty) \times \Gamma_s^+(\lambda_1^{N+4})$, and satisfies the following:

Lemma 2.5.12. *For all $t \geq 0$, $(r, \theta, \rho, \omega) \in \Gamma_s^+(\lambda_1^{N+4})$ and $(j, \alpha, k, \beta) \in \mathbb{Z}_+^{2d}$,*

$$(r^+(t), \theta^+(t), \rho, \omega) \in \Gamma_s^+(\lambda_1^{N+5}),$$

$$|\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta (r^+(t) - r - t\rho)| \leq C_{j\alpha k\beta} r^{1-j-|\beta|} |\omega/r|^{(2-|\beta|)_+}, \tag{2.5.27}$$

$$|\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta (\theta^+(t) - \theta)| \leq C_{j\alpha k\beta} r^{-j-|\beta|} |\omega/r|^{(1-|\beta|)_+}, \tag{2.5.28}$$

Proof. Let $(r, \theta, \rho, \omega) \in \Gamma_s^+(\lambda_1^{N+4})$. Since $X^+ = (\partial_r S^+, r^{-2}(h^{jk} + a^{jk})\partial_{\theta^k} S^+)$, it follows from (2.5.1) that

$$|\dot{r}^+(t) - \rho| \leq C_0 |\omega/r^+(t)|^2,$$
$$|\dot{\theta}^+(t)| \leq C_0 r^+(t)^{-1} |\omega/r^+(t)|,$$

with some $C_0 > 0$. In particular, we have

$$|\dot{r}^+(0) - \rho| \leq C_0 |\omega/r|^2 \leq C_1 \varepsilon_1, \quad |\dot{\theta}^+(0)| \leq C_0 r^{-1} |\omega/r| \leq C_1 r^{-1} \varepsilon_1 \tag{2.5.29}$$

with some $C_1 > 0$. We set $C_2 = \inf J$ and

$$F := \{t \geq 0 \mid r^+(s) \geq r + s\rho/2, \ |\theta^+(s) - \theta| \leq 4C_0 C_2^{-1} |\omega/r| \text{ for } 0 \leq s \leq t\}.$$

By (2.5.29), it is easy to see that $0 \in F \neq \emptyset$. Let $t_0 = \sup F$. We then have

$$|\dot{r}^+(t) - \rho| \leq C_0 (r + s\rho/2)^{-2} |\omega|^2 \leq C_0 C_1 \varepsilon_1^2, \quad |\dot{\theta}^+(t)| \leq C_0 (r + s\rho/2)^{-2} |\omega|,$$

for $0 \leq t \leq t_0$, and hence

$$|r^+(t) - r - t\rho| \leq C_0 C_1 \varepsilon_1^2 t,$$
$$|\theta^+(t) - \theta| \leq 2C_0 C_2^{-1} |\omega/r|,$$

for $0 \leq t \leq t_0$. Choosing $\varepsilon_1 > 0$ such that $\rho - C_0 C_1 \varepsilon_1^2 > \rho/2$ and $\delta > 0$ small enough, we see that $t_0 + \delta \in F$ which implies $t_0 = \infty$. Therefore, $(r^+(t), \theta^+(t))$ is well-defined on $[0, \infty)$, and satisfies (2.5.27) and (2.5.28) with $(j, \alpha, k, \beta) = 0$ for all $t \geq 0$. In particular, we have

$$(r^+(t), \theta^+(t), \rho, \omega) \in \Gamma_s^+(\lambda_1^{N+5})$$

for $t \geq 0$, provided that $\lambda_1 > 0$ large enough. The proof for higher derivatives is obtained by (2.5.1) and an induction with respect to $j + |\alpha| + k + |\beta|$. □

Proof of Lemma 2.5.11. Define a smooth real-valued function Z^+ by

$$Z^+(t) := Y^+(r^+(t), \theta^+(t), \rho, \omega) \in C^\infty([0,\infty) \times \Gamma_s^+(\lambda_1^{N+4})).$$

By (2.2.3), (2.5.1) and Lemma (2.5.12), we have $Z^+(t) \in L_t^1([0,\infty))$ and

$$\left|\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta \int_0^\infty Z^+(t, r, \theta, \rho, \omega) dt\right| \le C_{j\alpha k\beta} r^{-j-|\beta|} \quad \text{on} \quad \Gamma_s^+(\lambda_1^{N+4}). \tag{2.5.30}$$

We now define smooth functions $\tilde{b}_j$ on $\Gamma_s^+(\lambda_1^{N+4-j})$ by

$$\begin{cases} \tilde{b}_0(r, \theta, \rho, \omega) = e^{\int_0^\infty Z^+(t)dt}, \\ \tilde{b}_j(r, \theta, \rho, \omega) = \displaystyle\int_0^\infty (i\widehat{P}_\kappa \tilde{b}_{j-1})(r^+(t), \theta^+(t), \rho, \omega) e^{\int_0^t Z^+(s)ds} dt, \quad j = 1, 2, ..., N. \end{cases}$$

By a standard Hamilton-Jacobi theory, $\tilde{b}_j$ solve (2.5.26). Moreover, by (2.2.3), Lemma 2.5.12 and (2.5.30), we have

$$|\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta \tilde{b}_j(r, \theta, \rho, \omega)| \le C_{j\alpha k\beta} r^{-j-|\beta|} \quad \text{on} \quad \Gamma_s^+(\lambda_1^{N+4-j}).$$

Take $\chi^+ \in C_0^\infty(\mathbb{R}^{2d})$ so that $0 \le \chi^+ \le 1$, $\chi^+ \equiv 1$ on $\Omega_s^+(\lambda_1^2)$, $\operatorname{supp}\chi^+ \subset \Omega_s^+(\lambda_1^3)$, and that

$$|\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta \chi^+(r, \theta, \rho, \omega)| \le C_{j\alpha k\beta} r^{-j-|\beta|} \quad \text{on} \quad \Gamma_s^+(\lambda_1^3),$$

and define $b_j^+ := \chi^+ \tilde{b}_j$. By the construction, b_j^+ solve (2.5.26) on $\Omega_s^+(\lambda_1^2)$, b_0^+ is elliptic on $\Omega_s^+(\lambda_1)$ and $b_j^+ \in S_{\rm sc}(\Omega_s^+(\lambda_1^3))$. ☐

Proof of Theorem 2.5.10. Let $R \ge \lambda_1^{N+5} R_1$ and $0 < \varepsilon \le \lambda_1^{-N-5} \varepsilon_1$. We first prove (2.5.24). By Proposition 2.5.8, there exists a symbol

$$c_h^+ = \sum_{j=0}^N h^j c_j^+ \quad \text{with} \quad c_j^+ \in S_{\rm sc}(\Omega_s^+(\lambda_1))$$

such that

$$||a_s^+(r, \theta, hD_r, hD_\theta) - B_+ C_+^*||_{\mathcal{L}(L^2(\mathbb{R}^d))} \le C_N h^{N+1}, \ h \in (0,1],$$

where C_N may be taken uniformly with respect to h and R.

We next prove (2.5.23). Let d_j^+, $j = 1, 2, ..., N+2$, be defined by (2.5.25). Then $d_j^+ \in S_{\rm sc}(\Omega_s^+(\lambda_1^3))$ and $d_j^+ \equiv 0$ on $\Omega_s^+(\lambda_1^2)$. For each j and k, $I_{\rm IK}^+(d_j^+) e^{-ish\frac{1}{2}D_r^2} I_{\rm IK}^+(c_k^+)^*$ has the distribution kernel

$$I(s,h) = (2\pi h)^{-d} \int e^{\frac{i}{h}((r-r')\rho_1^+ + (\theta-\theta')\cdot\omega_1^+ - \frac{1}{2}s\rho^2)} d_j^+(r, \theta, \rho, \omega) \overline{c_k^+(r', \theta', \rho, \omega)} d\rho d\omega,$$

where $(\rho_1^+, \omega_1^+)(r, \theta, \rho, \omega, r', \theta')$ is defined in Lemma 2.5.9. (2.5.1) implies that

$$\begin{aligned} |\partial_\omega \rho_1^+| &\lesssim r^{-1}|\omega/r| \\ &\lesssim r^{-1}\lambda_1^3 \varepsilon \\ &\lesssim r^{-1}\sqrt{\lambda_1 \varepsilon_1 \varepsilon}, \\ |\partial_\omega(\omega_1^+ - \omega)| &\lesssim |\omega/r| \\ &\lesssim \lambda_1^3 \varepsilon \lesssim \varepsilon_1, \end{aligned}$$

for $(r,\theta,\rho,\omega)\in\Omega_s^+(\lambda_1^3)$ and $(r',\theta',\rho,\omega)\in\Omega_s^+(\lambda_1)$. Moreover, since $d_j^+\equiv 0$ on $\Omega_s^+(\lambda_1^2)$, we have

$$\begin{aligned}|\theta-\theta'| &\geq \inf_{\theta\in U}|\theta-\theta|-\sup_{\theta\in U}|\theta-\theta'|\\ &> (\lambda_1-\sqrt{\lambda_1})\sqrt{\varepsilon}\\ &\geq \lambda_1\sqrt{\varepsilon}/2.\end{aligned}\tag{2.5.31}$$

Therefore,

$$\begin{aligned}&\left|\partial_\omega\left((r-r')\rho_1^+ +(\theta-\theta')\cdot\omega_1^+ - s\rho^2/2\right)\right|\\ &\geq |\theta-\theta'|(1-C\varepsilon_1)-C|r-r'|r^{-1}\sqrt{\lambda\varepsilon_1\varepsilon}\\ &\gtrsim (\lambda_1-C\varepsilon_1-\sqrt{\lambda_1\varepsilon_1})\sqrt{\varepsilon}\\ &> \sqrt{\varepsilon},\end{aligned}$$

for sufficiently large $\lambda_1>0$ and small $\varepsilon_1>0$. We now fix the constants $R_{\mathrm{IK}},\varepsilon_{\mathrm{IK}},\lambda_{\mathrm{IK}}$ so that $\lambda_{\mathrm{IK}}=\lambda_1$, $R_{\mathrm{IK}}=\lambda_{\mathrm{IK}}^{N+5}R_1$, $\varepsilon_{\mathrm{IK}}=\lambda_{\mathrm{IK}}^{-N-5}\varepsilon_1$. Put

$$L(r,\theta,\rho,\omega,r',\theta')=\frac{h((r-r')\partial_\omega\rho_1^+ +(\theta-\theta')\cdot\partial_\omega\omega_1^+)}{i|(r-r')\partial_\omega\rho_1^+ +(\theta-\theta')\cdot\partial_\omega\omega_1^+|^2}\cdot\partial_\omega,$$

and integrate by parts $I(s,h)$ with respect to L. For any $n\geq 0$, $I(s,h)$ then reads

$$I(s,h)=h^n(2\pi h)^{-d}\int e^{\frac{i}{h}((r-r')\rho_1^+ +(\theta-\theta')\cdot\omega_1^+ -\frac{1}{2}s\rho^2)}G^+(r,\theta,\rho,\omega,r',\theta')d\rho d\omega,$$

where

$$G^+=(L^*)^n(d_j^+(r,\theta,\rho,\omega)\overline{c_k^+(r',\theta',\rho,\omega)}).$$

Making the change of the variables $(\rho,\omega)\mapsto(\rho_2^+,\omega_2^+)$, we have

$$I(s,h)=h^n(2\pi h)^{-d}\int e^{\frac{i}{h}\Phi^+(s,r,\theta,\rho,\omega,r',\theta')}G^+(r,\theta,\rho_+^2,\omega_+^2,r',\theta')d\rho d\omega,$$

where

$$\Phi^+(s,r,\theta,\rho,\omega,r',\theta'):=(r-r')\rho+(\theta-\theta')\cdot\omega-\frac{1}{2}s\rho_2^+(r,\theta,\rho,\omega,r',\theta')^2,$$

and (ρ_2^+,ω_2^+) is the inverse of (ρ_1^+,ω_1^+). By Lemma 2.5.9 and (2.5.31), $\rho_2^+(r,\theta,\rho,\omega,r',\theta')$ and

$$G^+(r,\theta,(\rho_+^2,\omega_+^2)(r,\theta,\rho,\omega,r',\theta'),r',\theta')$$

are smooth and uniformly bounded functions on $\mathbb{R}^{3d}$. Applying the Calderón-Vaillancourt theorem, we hence have

$$||I_{\mathrm{IK}}^+(d_j^+)e^{-ish\frac{1}{2}D_r^2}I_{\mathrm{IK}}^+(c_k^+)^*||_{\mathcal{L}(L^2(\mathbb{R}^d))}\leq C_{N,d,\varepsilon}h^{n-n_d}\langle s\rangle^{n_d}\leq C_{N,d,\varepsilon,T}h^{n-2n_d}$$

for all $n\geq 0$, $h\in(0,1]$ and $0\leq s\leq Th^{-1}$, where $n_d>0$ depends only on d, and $C_{N,d,\varepsilon,T}$ is independent of h and R. Choosing $n>0$ with $n-2n_d>N+1$, we complete the proof $\square$

2.5.3 Dispersive estimates

We here prove dispersive estimates for the Isozaki-Kitada parametrix. Let $R_{\mathrm{IK}}, \varepsilon_{\mathrm{IK}}, \lambda_{\mathrm{IK}} > 0$ be as in Theorem 2.5.10. In this subsection we use the notation $\partial^\gamma := \partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta \partial_{r'}^{j'} \partial_{\theta'}^{\alpha'}$ for $\gamma = (j, \alpha, k, \beta, j', \alpha') \in \mathbb{Z}_+^{3d}$.

Theorem 2.5.13. *For sufficiently large $R > R_{\mathrm{IK}}$, small $0 < \varepsilon < \varepsilon_{\mathrm{IK}}$, all symbols $b^\pm \in S_{\mathrm{sc}}(\Omega_s^\pm(\lambda_{\mathrm{IK}}^3))$ and $c^\pm \in S_{\mathrm{sc}}(\Omega_s^\pm(\lambda_{\mathrm{IK}}))$, we can write*

$$I_{\mathrm{IK}}^\pm(b^\pm) e^{-ith\frac{1}{2}D_r^2} I_{\mathrm{IK}}^\pm(c^\pm)^* = U_{\mathrm{IK}}^\pm(t,h) + R_{\mathrm{IK}}^\pm(t,h),$$

where $U_{\mathrm{IK}}^\pm(t,h)$ satisfy dispersive estimates

$$||r^{-\frac{d-1}{2}} U_{\mathrm{IK}}^\pm(t,h) r^{-\frac{d-1}{2}}||_{\mathcal{L}(L^1(\mathbb{R}^d), L^\infty(\mathbb{R}^d))} \le C|th|^{-d/2}, \quad 0 < \pm t \le h^{-1},\ h \in (0.1],$$

and $R_{\mathrm{IK}}^\pm(t,h)$ are rapidly decaying with respect to h: for any $N \ge 0$,

$$||R_{\mathrm{IK}}^\pm(t,h)||_{\mathcal{L}(L^2(\mathbb{R}^d))} \le C_N h^N, \quad 0 < \pm t \le h^{-1},\ h \in (0.1].$$

Moreover $C, C_N > 0$ can taken uniformly with respect to h, t and R.

We prove the theorem for the case $t \ge 0$, and the proof for the case $t \le 0$ is similar. The distribution kernel of $I_{\mathrm{IK}}^+(b^+) e^{-ith\frac{1}{2}D_r^2} I_{\mathrm{IK}}^+(c^+)^*$ takes the form

$$I_{A^+}(t,h) = (2\pi h)^{-d} \int e^{\frac{i}{h}\Phi^+(t,r,\theta,\rho,\omega,r',\theta')} A^+(r,\theta,\rho,\omega,r',\theta') d\rho d\omega, \tag{2.5.32}$$

where

$$\Phi^+(t,r,\theta,\rho,\omega,r',\theta') := (r-r')\rho + (\theta-\theta')\cdot\omega - \frac{1}{2}t(\rho_2^+)^2,$$
$$A^+(r,\theta,\rho,\omega,r',\theta') := b^+(r,\theta,\rho_2^+,\omega_2^+)\overline{b^+(r',\theta',\rho_2^+,\omega_2^+)}|\det\partial_{\rho,\omega}(\rho_2^+,\omega_2^+)|,$$

and $(\rho_2^+,\omega_2^+) = (\rho_2^+,\omega_2^+)(r,\theta,\rho,\omega,r',\theta')$ is given by Lemma 2.5.9. Since $(r,\theta,\rho_2^+,\omega_2^+) \in \Omega_s^+(\lambda_{\mathrm{IK}}^3)$ and $(r',\theta',\rho_2^+,\omega_2^+) \in \Omega_s^+(\lambda_{\mathrm{IK}})$, (2.5.17) implies

$$C^{-1} \le \rho \le C, \quad |\omega/\sqrt{rr'}| \le C\varepsilon_1 \quad \text{on} \quad \operatorname{supp} A^+,$$

and $\partial^\gamma A^+$ and $\partial^\gamma \rho_2^+$ are uniformly bounded on $\mathbb{R}^{3d}$.

We first remove a smoothing term from $I_{A^+}(t,h)$. Let $\chi_\rho \in C_0^\infty(\mathbb{R})$, $\chi_\omega \in C_0^\infty(\mathbb{R}^{d-1})$ be smooth cut-off functions such that

$$\operatorname{supp}\chi_\rho \subset (-1,1), \quad \chi_\rho \equiv 1 \text{ on } (-1/2,1/2),$$
$$\operatorname{supp}\chi_\omega \subset \{|\theta| \le 1\}, \quad \chi_\omega \equiv 1 \text{ on } \{|\theta| \le 1/2\},$$

and define

$$A_\delta^+ := \chi_\rho\left(\partial_\rho \Phi^+\right)\chi_\omega(\partial_\omega\Phi^+/\delta)A^+,$$

where $\delta > 0$ is a small parameter. We denote the operator having the Schwartz kernel $I_{A_\delta^+}(t,h)$ by $U_{A_\delta^+}(t,h)$.

Lemma 2.5.14. *For all $N > 0$ and $\delta > 0$, we have*

$$||I_{\mathrm{IK}}^+(b^+)e^{-ith\frac{1}{2}D_r^2}I_{\mathrm{IK}}^+(c^+)^* - U_{A_\delta^+}(t,h)||_{\mathcal{L}(L^2(\mathbb{R}^d))} \le C_N h^N,$$

uniformly with respect to $h \in (0,1]$, $0 \le \pm t \le h^{-1}$ and $R > 0$.

Proof. We split $A^+ - A_\delta^+$ as follows:

$$A^+ - A_\delta^+ = (1-\chi_\rho)\chi_\omega A^+ - (1-\chi_\omega)A^+ =: A_1^+ + A_2^+.$$

We denote by $I_{A_j^+}$ the oscillatory integral of the form (2.5.32) with the phase Φ^+ and the amplitude A_j^+, respectively. We set

$$L_1 := (h/i\partial_\rho\Phi^+)\partial_\rho, \quad L_2 := (h/i|\partial_\omega\Phi^+|^2)(\partial_\omega\Phi^+)\cdot\partial_\omega.$$

Since

$$\left|\partial^\gamma\left(1/\partial_\rho\Phi^+\right)\right| \lesssim 1 \quad \text{on} \quad \operatorname{supp} A_1^+, \quad \left|\partial^\gamma\left(\partial_\omega\Phi^+/|\partial_\omega\Phi^+|^2\right)\right| \lesssim 1/\delta \quad \text{on} \quad \operatorname{supp} A_2^+,$$

n-times integration by parts $I_{A_j^+}$ with L_j implies that $I_{A_j^+}$ reads

$$I_{A_j^+} = (2\pi h)^{-d}\int e^{\frac{i}{h}((r-r')\rho+(\theta-\theta')\cdot\omega)}\tilde{A}_j^+(r,\theta,\rho,\omega,r',\theta')d\rho d\omega,$$

where $\tilde{A}_j^+ = e^{-\frac{i}{h}\frac{t}{2}(\rho_2^+)^2}(L_j^*)^n A_j$ satisfies

$$|\partial^\gamma\tilde{A}_j^+(r,\theta,\rho,\omega,r',\theta')| \le C_n h^{n-|\gamma|}\langle t\rangle^{|\gamma|} \le C_n h^{n-2|\gamma|} \quad \text{on} \quad \mathbb{R}^{3d},$$

for all $\gamma \in \mathbb{Z}_+^{3d}$, $h \in (0,1]$, and $0 \le \pm t \le h^{-1}$. By the Calderón-Vaillancourt theorem, there exists an integer $N_d > 0$ depending only on d such that

$$||I_{\mathrm{IK}}^+(b^+)e^{-ith\frac{1}{2}D_r^2}I_{\mathrm{IK}}^+(c^+)^* - U_{A_\delta^+}(t,h)||_{\mathcal{L}(L^2(\mathbb{R}^d))} \le C_n h^{n-2N_d}.$$

Choosing $n \ge 0$ with $n - 2N_d \ge N$, we obtain the assertion. □

To prove dispersive estimates for $U_{A_\delta^+}(t,h)$, we next study the phase function more precisely .

Lemma 2.5.15. *On* $\operatorname{supp} A^+$, $\frac{1}{2}\rho_2^+(r,\theta,\rho,\omega,r',\theta')^2$ *takes the form*

$$\frac{1}{2}\rho_2^+(r,\theta,\rho,\omega,r',\theta')^2 = \frac{1}{2}\rho^2 + \frac{1}{2rr'}q_0(\theta,\omega) + Q^+(r,\theta,\rho,\omega,r',\theta'),$$

where $q_0(\theta,\omega) = h^{jk}(\theta)\omega_j\omega_k$. Moreover we can write

$$Q^+ = Q_1^+ + (\theta-\theta')\cdot Q_2^+$$

such that, for all $(r,\theta,\rho,\omega,r',\theta) \in \operatorname{supp} A^+$ and $\gamma = (j,\alpha,k,\beta,j',\alpha') \in \mathbb{Z}_+^{3d}$,

$$\begin{cases} |\partial^\gamma Q_1^+(r,\theta,\rho,\omega,r',\theta')| \le CR_1^{-j-j'-|\beta|}\left(\varepsilon_1^{(3-|\beta|)_+} + R_1^{-\mu}\varepsilon_1^{(2-|\beta|)_+}\right), \\ |\partial^\gamma Q_2^+(r,\theta,\rho,\omega,r',\theta')| \le CR_1^{-j-j'-|\beta|}\varepsilon_1^{(2-|\beta|)_+}, \end{cases} \tag{2.5.33}$$

where R_1, ε_1 are given by Lemma 2.5.11.

Proof. We start from the formula

$$\rho_2^+ = \rho - \int_0^1 (\partial_r \varphi^+)(r_s, \theta_s, \rho_2^+, \omega_2^+) ds,$$

where $(r_s, \theta_s) = (r', \theta') + s(r - r', \theta - \theta')$. By using the mean value theorem, we have

$$(\partial_r \varphi^+)(r_s, \theta_s, \rho_2^+, \omega_2^+) = (\partial_r \varphi^+)(r_s, \theta, \rho_2^+, \omega_2^+) - (\theta - \theta') \cdot F^+(s),$$

where $F^+(s)$ is defined by

$$F^+(s) := (1-s) \int_0^1 (\partial_\theta \partial_r \varphi^+)(r_s, \theta_{s\sigma}, \rho_2^+, \omega_2^+) d\sigma, \quad \theta_{s\sigma} = \theta + \sigma(\theta_s - \theta).$$

By (2.5.1) and (2.5.17), we obtain

$$\sup_{s \in [0,1]} |\partial^\gamma F^+(s)| \lesssim r^{-j-j'-|\beta|} |\omega_2^+ / r|^{(2-|\beta|)_+} \lesssim R_1^{-j-j'-|\beta|} \varepsilon_1^{(2-|\beta|)_+}.$$

Since $R_{\mathrm{IK}} \geq \lambda_{\mathrm{IK}}^4 R_1$ and $\varepsilon_{\mathrm{IK}} \leq \lambda_{\mathrm{IK}}^{-4} \varepsilon_1$, by the mean value theorem, we can write

$$(\partial_r \varphi^+)(r_s, \theta, \rho_2^+, \omega_2^+) = -\frac{1}{2 r_s^2 \rho} q_0(\theta, \omega) - G^+(s),$$

where, by (2.5.2) and (2.5.17), $G^+(s)$ satisfies

$$\begin{aligned} \sup_{s \in [0,1]} |\partial^\gamma G^+(s)| &\lesssim r^{-j-j'-|\beta|} \left(|\omega_2^+ / r|^{(3-|\beta|)_+} + r^{-\mu} |\omega_2^+ / r|^{(2-|\beta|)_+} \right) \\ &\lesssim R_1^{-j-j'-|\beta|} \left(\varepsilon_1^{(3-|\beta|)_+} + R_1^{-\mu} \varepsilon_1^{(2-|\beta|)_+} \right). \end{aligned}$$

Set

$$\tilde{F}^+ = \int_0^1 F^+(s) ds, \quad \tilde{G}^+ = \int_0^1 G^+(s) ds.$$

Since $\int_0^1 r_s^{-2} ds = 1/\sqrt{rr'}$, we have

$$(\rho_2^+)^2 = \left(\rho + \frac{1}{2rr'\rho} q_0(\theta, \omega) + (\theta - \theta') \cdot \tilde{F}^+ + \tilde{G}^+ \right)^2.$$

If we set

$$Q_1^+ = 4\rho \tilde{G}^+ + 2 \left(\frac{1}{2rr'\rho} q_0(\theta, \omega) + (\theta - \theta') \cdot \tilde{F}^+ + \tilde{G}^+ \right)^2, \quad Q_2^+ = 4\rho \tilde{F}^+,$$

then $Q^+ = Q_1^+ + Q_2^+$ satisfy (2.5.33). □

Making the change of the variable $\omega \mapsto \sqrt{rr'}\nu$, $I_{A_\delta^+}(t,h)$ reads

$$I_{A_\delta^+}(t,h) = \frac{(rr')^{\frac{d-1}{2}}}{(2\pi h)^d} \int e^{\frac{i}{h} \Phi^+(t, r, \theta, \rho, \sqrt{rr'}\nu, r', \theta')} A_\delta^+(r, \theta, \rho, \sqrt{rr'}\nu, r', \theta') d\rho d\nu,$$

where $\tilde{A}_\delta^+(r, \theta, \rho, \nu, r', \theta') := A_\delta^+(r, \theta, \rho, \sqrt{rr'}\nu, r', \theta')$ is bounded on $\mathbb{R}^{3d}$ and compactly supported with respect to (ρ, ν). Moreover, $\pi_{\rho,\nu} \circ \operatorname{supp} \tilde{A}_\delta^+$ is bounded uniformly with respect to $R > 0$, where $\pi_{\rho,\nu} : \mathbb{R}^{3d} \to \mathbb{R}^d$ is a canonical projection onto the (ρ, ν)-space.

Proposition 2.5.16. *There exists $\delta > 0$ small enough such that, for all $h \in (0,1]$ and $0 < t \leq h^{-1}$, we have*

$$\|r^{-\frac{d-1}{2}} U_{A_\delta^+}(t,h) r^{-\frac{d-1}{2}}\|_{\mathcal{L}(L^1(\mathbb{R}^d), L^\infty(\mathbb{R}^d))} \leq C|th|^{-d/2},$$

where C may be taken uniformly with respect to R, h and t.

Proof. For $0 < t \leq h$, it follows from

$$r^{-\frac{d-1}{2}} |I_{A_\delta^+}(t,h)| r'^{-\frac{d-1}{2}} \leq Ch^{-d} \leq C|th|^{-d/2}.$$

Suppose that $h \leq t \leq h^{-1}$ and assume $2h \leq h^{-1}$ without loss of generality. Define

$$\begin{aligned}\tilde{\Phi}^+(t,r,\theta,\rho,\nu,r',\theta') &:= \frac{1}{t}\Phi^+(t,r,\theta,\rho,\sqrt{rr'}\nu,r',\theta') \\ &= \frac{r-r'}{t}\rho + \frac{\theta-\theta'}{t}\cdot\sqrt{rr'}\nu - \frac{1}{2}\rho_2^+(r,\theta,\rho,\sqrt{rr'}\nu,r',\theta')^2.\end{aligned}$$

By Lemma 2.5.15, we obtain

$$\partial_\rho \tilde{\Phi}^+(t) = \frac{r-r'}{t} - \rho + O(\varepsilon_1^2), \quad \partial_\nu \tilde{\Phi}^+(t) = \frac{\sqrt{rr'}(\theta-\theta')}{t} + O(\varepsilon_1)$$

on the support of $\tilde{A}_\delta^+$. Thus, if we restrict the support of $\tilde{A}_\delta^+$ to one of the regions

$$\left|\frac{r-r'}{t} - \rho\right| \geq \varepsilon_1 \quad \text{or} \quad \left|\frac{\sqrt{rr'}(\theta-\theta')}{t}\right| \geq 1,$$

then we have

$$|\partial_\rho \tilde{\Phi}(t)| + |\partial_\nu \tilde{\Phi}(t)| \geq \varepsilon_1,$$

with sufficiently small $\varepsilon_1 > 0$. Take $\chi_1 \in C_0^\infty(\mathbb{R})$ and $\chi_2 \in C_0^\infty(\mathbb{R}^{d-1})$ satisfying

$$\chi_1 \equiv 1 \quad \text{on} \quad (-1/2, 1/2), \quad \operatorname{supp}\chi_1 \subset (-1,1),$$

$$\chi_2 \equiv 1 \quad \text{on} \quad \{|\theta| \leq 1/2\}, \quad \operatorname{supp}\chi_2 \subset \{|\theta| \leq 1\},$$

and put

$$\tilde{\chi}_1(t,r,\theta,\rho,r',\theta') = \chi_1\left(\frac{r-r'}{\varepsilon_1 t} - \frac{\rho}{\varepsilon_1}\right)\chi_2\left(\frac{\sqrt{rr'}(\theta-\theta')}{t}\right).$$

Integrating by parts, we then obtain the non stationary estimates

$$\begin{aligned}\left|(2\pi h)^{-d}\int e^{\frac{i}{h}t\tilde{\Phi}^+(t)}(1-\tilde{\chi}_1)\tilde{A}_\delta^+ d\rho d\nu\right| &\leq C_{\varepsilon_1,\lambda_{\mathrm{IK}}} h^{-d}|t/h|^{-n} \\ &\leq C_{\varepsilon_1,\lambda_{\mathrm{IK}}}|th|^{-d/2},\end{aligned}$$

for all $h \in (0,1]$, $h \leq t \leq 2h^{-1}$, $\delta > 0$ and $n \geq d/2$, where $C_{\varepsilon_1,\lambda_{\mathrm{IK}}}$ may be taken uniformly in R. Since $C^{-1} \leq \rho \leq C$ for some $C > 0$, if $\varepsilon_1 > 0$ is sufficiently small, then

$$C_0^{-1}|t| \leq r - r' \leq C_0|t| \quad \text{with some } C_0 > 0,$$

on the support of $\tilde{\chi}_1 \tilde{A}_\delta^+$, and this estimate implies $|t/\sqrt{rr'}| \lesssim 1$. We now fix δ with $0 < \delta \le \varepsilon_1$. Since

$$\partial_\nu \tilde{\Phi}^+(t) = (\sqrt{rr'}/t)\partial_\omega \Phi_+(t),$$

we have

$$\begin{aligned} |\theta - \theta'| &\le C|t/\sqrt{rr'}|(|\partial_\nu \tilde{\Phi}^+(t)| + \varepsilon_1) \\ &\le C_1 \varepsilon_1, \end{aligned} \tag{2.5.34}$$

with some $C_1 > 0$ on the support of $\tilde{\chi}_1 \tilde{A}_\delta^+$. We fix $\lambda'_{\mathrm{IK}} > \lambda_{\mathrm{IK}}$ with $2\lambda_{\mathrm{IK}} > \lambda'_{\mathrm{IK}}$, and choose $\chi_3, \chi_4 \in C_0^\infty(\mathbb{R}^{2d})$ so that $\chi_3 \equiv 1$ on $\Omega^+(\lambda_{\mathrm{IK}}^3)$, $\operatorname{supp}\chi_3 \subset \Omega^+(\lambda_{\mathrm{IK}}'^3)$, $\chi_4 \equiv 1$ on $\Omega^+(\lambda_{\mathrm{IK}})$, $\operatorname{supp}\chi_4 \subset \Omega^+(\lambda'_{\mathrm{IK}})$ and

$$|\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta \chi_j(r,\theta,\rho,\omega)| \le C_{j\alpha k\beta\lambda_1} \langle r \rangle^{-j-|\beta|} \quad \text{on} \quad \mathbb{R}^{3d}, \quad j = 3,4. \tag{2.5.35}$$

We also choose $\chi_5 \in C_0^\infty(\mathbb{R}^{d-1})$ with $\chi_5 \equiv 1$ on $\{|\theta| \le 1/2\}$ and $\operatorname{supp}\chi_5 \subset \{|\theta| \le 1\}$. We now define

$$\begin{aligned} &\Phi_0^+(t,r,\theta,\rho,\nu,r',\theta') \\ &:= \frac{r-r'}{t}\rho + \frac{\theta-\theta'}{t} \cdot \sqrt{rr'}\nu - \frac{1}{2}\rho^2 - \frac{1}{2}q_0(\theta,\nu) - \tilde{\chi}_3\tilde{\chi}_4\tilde{\chi}_5 Q^+(r,\theta,\rho,\sqrt{rr'}\nu,r',\theta'), \end{aligned}$$

where $\tilde{\chi}_3$, $\tilde{\chi}_4$ and $\tilde{\chi}_5$ are defined by

$$\begin{aligned} \tilde{\chi}_3(r,\theta,\rho,\nu,r',\theta') &:= \chi_3(r,\theta,(\rho_2^+,\omega_2^+)(r,\theta,\rho,\sqrt{rr'}\nu,r',\theta')) \\ \tilde{\chi}_4(r,\theta,\rho,\nu,r',\theta') &:= \chi_3(r',\theta',(\rho_2^+,\omega_2^+)(r,\theta,\rho,\sqrt{rr'}\nu,r',\theta')) \\ \tilde{\chi}_5(\theta,\theta') &:= \chi_5\left(\frac{\theta-\theta'}{2C_1\varepsilon_1}\right). \end{aligned}$$

It easily see that Φ_0^+ is smooth on $[h,\infty) \times \mathbb{R}^{3d}$ and $\Phi_0^+ \equiv \tilde{\Phi}^+$ on $\operatorname{supp}\tilde{\chi}_1 \tilde{A}_\delta^+$. Moreover, by Lemma 2.5.15, (2.5.34) and (2.5.35), we have

$$|\partial_\rho^k \partial_\nu^\beta \Phi_0^+(t,r,\theta,\rho,\nu,r',\theta')| \le C_{k\beta}$$

for all $(r,\theta,\rho,\nu,r',\theta') \in \mathbb{R}^{3d}$, $h \le t \le h^{-1}$ and $|k+\beta| \ge 2$, where $C_{k\beta} > 0$ may be taken uniformly in h, t and R. We also obtain

$$\partial_{\rho,\nu}^2 \Phi_0^+ = -\begin{pmatrix} 1 & 0 \\ 0 & h^{jk} \end{pmatrix} + O(\varepsilon_1 + R_1^{-\mu}) \quad \text{on} \quad [h,h^{-1}] \times \mathbb{R}^{3d}.$$

Since $|\det(h^{jk})| \gtrsim 1$, if R_1 is large enough and ε_1 is small enough,

$$|\det \partial_{\rho,\nu}^2 \Phi_0^+| \gtrsim 1$$

on $[h,h^{-1}] \times \mathbb{R}^{3d}$ uniformly with respect to R, h and t. Therefore, the mapping

$$(\rho,\nu) \mapsto \partial_{\rho,\nu}\Phi_0^+(t,r,\theta,\rho,\nu,r',\theta')$$

is a diffeomorphism from $\mathbb{R}^d$ to $\mathbb{R}^d$, and Φ_0^+ has a unique non-degenerate critical point $(\rho_c,\nu_c) = (\rho_c,\nu_c)(t,r,r',\theta,\theta')$. Moreover, $\partial_\rho^k \partial_\nu^\beta \Phi_0^+(t,r,\theta,\rho_c,\nu_c,r',\theta')$ are bounded on $\mathbb{R}^{2d}$

uniformly with respect to R and t if $|k+\beta| \geq 2$. We hence can apply the stationary phase theorem and obtain

$$\left|(2\pi h)^{-d} \int e^{\frac{i}{h}t\tilde{\Phi}^+(t)} \tilde{\chi}_1 \tilde{A}_\delta^+ d\rho d\nu\right| = \left|(2\pi h)^{-d} \int e^{\frac{i}{h}t\Phi_0^+(t)} \tilde{\chi}_1 \tilde{A}_\delta^+ d\rho d\nu\right|$$
$$\leq C'_{\varepsilon_1,\lambda_{\mathrm{IK}}} h^{-d} |t/h|^{-d/2}$$
$$\leq C'_{\varepsilon_1,\lambda_{\mathrm{IK}}} |th|^{-d/2},$$

for $h \in (0,1]$ and $h \leq t \leq h^{-1}$, where $C'_{\varepsilon_1,\lambda_{\mathrm{IK}}} > 0$ does not depend on $R > 0$. We complete the proof □

Proof of Theorem 2.5.13. We set

$$U_{\mathrm{IK}}^+(t,h) = U_{A_\delta^+}(t,h), \quad R_{\mathrm{IK}}^+(t,h) = I_{\mathrm{IK}}^+(b^+) e^{-ith\frac{1}{2}D_r^2} I_{\mathrm{IK}}^+(c^+)^* - U_{A_\delta^+}(t,h).$$

Clearly, they satisfy the assertion. When $-h^{-1} \leq t < 0$, the proof is analogous. □

2.6 Microlocal smoothing properties

Fix arbitrarily a coordinate chart $\kappa_0 : V_{\kappa_0} \to U_{\kappa_0}$. Let $\tilde{U}_{\kappa_0} \Subset U_{\kappa_0}$, $J \Subset (0,\infty)$ and $-1 < \sigma < 1$ be an in Definition 2.2.5. In this section we prove the following:

Theorem 2.6.1. *Fix arbitrarily $t_1 > 0$ and let $\varepsilon > 0$ be small enough. Then there exist $\delta_{\varepsilon,t_1} > 0$ and $L_{\varepsilon,t_1} > 0$ such that for all $(\sigma_l)_{0 \leq l \leq L_{\varepsilon,t_1}} \subset (-1, 1/2]$ satisfying (2.2.12), sufficiently large $R_0 > 0$, all $R_2 \geq R_1 \geq R_0$, all symbols*

$$a_l^\pm \in S_{\mathrm{sc}}(\Gamma_i^\pm(R_1, \tilde{U}_{\kappa_0}, J, \varepsilon, \delta_{\varepsilon,t_1}, l)), \quad b_l^\pm \in S_{\mathrm{sc}}(\Omega_i^\pm(R_2, \tilde{U}_{\kappa_0}, J, \varepsilon, \delta_{\varepsilon,t_1}, l)),$$

and $N \geq 0$, we have

$$\| \operatorname{Op}_{\kappa_0,h}(a_l^\pm) e^{-ith\widehat{P}} \operatorname{Op}_{\kappa_0,h}(b_l^\pm) \|_{\mathcal{L}(L^2(\widehat{M}))} \leq C_{N,l} h^N, \tag{2.6.1}$$

uniformly with respect to $h \in (0,1]$, $R_2 t_1 \leq \pm t \leq h^{-1}$ and R_2.

We prove Theorem 2.6.1 for the case $t \geq 0$, and the proof for case $t \leq 0$ is analogous. We need the following Egorov theorem.

Theorem 2.6.2 (The Egorov theorem). *For any $N \geq 0$ and $b^+ \in S_{\mathrm{sc}}(\Omega^+(R_2, \tilde{U}_{\kappa_0}, J, \sigma))$, there exist symbols*

$$b_{\kappa,h}^+(t) = \sum_{j=0}^N h^j b_{\kappa,j}^+(t) \quad \text{with} \quad b_{\kappa,j}^+(t) \in S_{\mathrm{sc}}(\kappa_* \exp tH_p(\kappa_{0*}^{-1} \operatorname{supp} b^+)),$$

such that for all $T > 0$ there exists a constant $C_{N,T} > 0$, independent of R_2, such that

$$\left\| e^{-ith\widehat{P}} \operatorname{Op}_{\kappa_0}(b^+) e^{ith\widehat{P}} - \sum_\kappa \operatorname{Op}_\kappa(b_{\kappa,h}^+(t)) \right\|_{\mathcal{L}(L^2(\widehat{M}))} \leq C_{N,T} h^{N+1} |t|, \tag{2.6.2}$$

uniformly with respect to $0 \leq t \leq R_2 T$ and $h \in (0,1]$. Moreover, $b_{\kappa,j}^+(t)$ is uniformly bounded in $S_{\mathrm{sc}}(\kappa_* \exp tH_p(\kappa_{0*}^{-1} \operatorname{supp} b^+))$ with respect to $0 \leq t \leq R_2 T$.

Proof. This theorem is basically well known, and we hence give the sketch of the proof. By (2.4.19), we can choose $t_1 > 0$, independent of R_2, such that the geodesic is contained one fixed coordinate neighborhood if $0 \leq t \leq R_2 t_1$. Define the map $\tilde{\varphi}(t) = (\tilde{r}(t), \tilde{\theta}(t), \tilde{\rho}(t), \tilde{\omega}(t))$ by

$$(\tilde{r}(t), \tilde{\theta}(t), \tilde{\rho}(t), \tilde{\omega}(t)) = F_{R_2} \circ \exp tH_{p_{\kappa 0}} \circ F_{R_2}^{-1}(r, \theta, \rho, \omega),$$

where $F_{R_2}(r, \theta, \rho, \omega) = (r/R_2, \theta, \rho, \omega/R_2)$. By (2.4.4), we have

$$|\partial^\gamma(\tilde{r}(t) - x)| + |\partial^\gamma(\tilde{\theta}(t) - \theta)| + |\partial^\gamma(\tilde{\rho}(t) - \rho)| + |\partial^\gamma(\tilde{\omega}(t) - \nu)| \leq C_\gamma R_2^{-j-|\beta|} t_1,$$

and hence

$$|\partial^\gamma(\partial\tilde{\varphi}(t) - \mathrm{Id})| \leq Ct_1 < 1/2,$$

for all $(x, \theta, \rho, \nu) \in F_{R_2}\Omega^+(R_2, \tilde{U}_{\kappa 0}, J, \sigma)$, $0 \leq t \leq R_2 t_1$ and $\gamma = (j, \alpha, k, \beta)$, where $x = r/R_2$, $\nu = \omega/R_2$ and $\partial^\gamma = \partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta$. $\tilde{\varphi}(t)$ thus has the inverse

$$\tilde{\varphi}(t)^{-1} = (\tilde{r}(t)^{-1}, \tilde{\theta}(t)^{-1}, \tilde{\rho}(t)^{-1}, \tilde{\omega}(t)^{-1}),$$

for all $0 \leq t \leq R_2 t_1$ and $(r, \theta, \rho, \omega) \in \tilde{\varphi}(t, F_{R_2}\Omega^+(R_2, \tilde{U}_{\kappa 0}, J, \sigma))$, and $\tilde{\varphi}(t)^{-1}$ satisfies

$$\begin{aligned} |\partial^\gamma(\tilde{r}(t)^{-1} - x)| + |\partial^\gamma(\tilde{\theta}(t)^{-1} - \theta)| &\leq C_\gamma R_2^{-j-|\beta|} t_1, \\ |\partial^\gamma(\tilde{\rho}(t)^{-1} - \rho)| + |\partial^\gamma(\tilde{\omega}(t)^{-1} - \nu)| &\leq C_\gamma R_2^{-j-|\beta|} t_1. \end{aligned}$$

After the rescaling $(x, \nu) \mapsto (R_2 x, R_2 \nu)$, we see that

$$(r(t)^{-1}, \theta(t)^{-1}, \rho(t)^{-1}, \omega(t)^{-1}) = (\exp tH_{p_{\kappa 0}})^{-1}(r, \theta, \rho, \omega)$$

exists for all $(r, \theta, \rho, \omega) \in \exp tH_{p_{\kappa 0}}(\Omega^+(R_2, \tilde{U}_{\kappa 0}, J, \sigma))$ and $0 \leq t \leq R_2 t_1$, and satisfies

$$\begin{aligned} |\partial^\gamma(r(t)^{-1} - r)| + |\partial^\gamma(\omega(t)^{-1} - \omega)| &\leq C_{j\alpha k\beta} R_2^{1-j-|\beta|} t_1, \\ |\partial^\gamma(\theta(t)^{-1} - \theta)| + |\partial^\gamma(\rho(t)^{-1} - \rho)| &\leq C_{j\alpha k\beta} R_2^{-j-|\beta|} t_1. \end{aligned} \tag{2.6.3}$$

We now define $b_j^+(t)$ inductively as follows. Put

$$b_0^+(t, r, \theta, \rho, \omega) = b^+ \circ (\exp tH_{p_{\kappa 0}})^{-1}(r, \theta, \rho, \omega) \quad \text{on} \quad \exp tH_{p_{\kappa 0}}(\operatorname{supp} b^+),$$

and $b_0^+(t, r, \theta, \rho, \omega) = 0$ outside $\exp tH_{p_{\kappa 0}}(\operatorname{supp} b^+)$. By virtue of (2.6.3), it easily see that $(b_0^+(t))_{0 \leq t \leq R_2 t_1}$ is bounded in $S_{\mathrm{sc}}(\exp tH_{p_{\kappa 0}}(\operatorname{supp} b^+))$. It is well known that $b_0^+(t)$ solves the first transport equation

$$\frac{\partial b_0^+}{\partial t} + \{p_{\kappa 0}, b_0^+\} = 0, \quad b_0^+(0) = b^+,$$

where $\{\cdot, \cdot\}$ is the Poisson bracket. Moreover, a standard semiclassical symbolic calculus yields that

$$\frac{\partial}{\partial t} \mathrm{Op}_{\kappa 0, h}(b_0^+(t)) + \frac{i}{h}[h^2 \widehat{P}_{\kappa 0}, \mathrm{Op}_{\kappa 0, h}(b_0^+(t))] = h\, \mathrm{Op}_{\kappa 0, h}(r_0^+(t)),$$

where $r_0^+(t)$ is supported in $\operatorname{supp} b_0^+(t)$ modulo $O(h^\infty)$ on $L^2(\widehat{M})$, *i.e.*, for all $n \geq 0$, there exists a symbol $\tilde{r}_0^+(t) \in S_{\rm sc}(\operatorname{supp} b_0^+(t))$ such that

$$\| \operatorname{Op}_{\kappa_0,h}(r_0^+(t)) - \operatorname{Op}_{\kappa_0,h}(\tilde{r}_0^+(t))\|_{\mathcal{L}(L^2(\widehat{M}))} \leq C_n h^n, \quad h \in (0,1],$$

for $0 \leq t \leq R_2 t_1$. Next, put

$$b_1^+(t,r,\theta,\rho,\omega)) = \int_0^t \tilde{r}_0^+(s, \exp sH_{p_{\kappa_0}} \circ (\exp tH_{p_{\kappa_0}})^{-1}(r,\theta,\rho,\omega))ds$$

on $\exp tH_{p_{\kappa_0}}(\operatorname{supp} b^+)$, and $b_1^+(t,r,\theta,\rho,\omega)) = 0$ otherwise. Again (2.6.3) implies that $(b_1^+(t))_{0\leq t\leq R_2t_1}$ is bounded in $S_{\rm sc}(\exp tH_{p_{\kappa_0}}(\operatorname{supp} b^+))$. $b_1^+(t)$ is a solution to the second transport equation

$$\frac{\partial b_1^+}{\partial t} + \{p_{\kappa_0}, b_1^+\} = \tilde{r}_0^+, \quad b_0^+(0) = 0,$$

which implies

$$\frac{\partial}{\partial t} \operatorname{Op}_{\kappa_0,h}(b_0^+(t) + hb_1^+(t)) + \frac{i}{h}[h^2 \widehat{P}_{\kappa_0}, \operatorname{Op}_{\kappa_0,h}(b_0^+(t) + hb_1^+(t))] = h^2 \operatorname{Op}_{\kappa_0,h}(r_1^+(t)),$$

where $r_1^+(t)$ is supported in $\operatorname{supp} b_0^+(t)$ modulo $O(h^\infty)$ on $L^2(\widehat{M})$. Iterating this procedure and putting $b_h^+(t) = \sum_{j=0}^N h^j b_j^+(t)$, we have

$$\frac{\partial}{\partial t} \operatorname{Op}_{\kappa_0,h}(b_h^+(t)) + \frac{i}{h}[h^2 \widehat{P}_{\kappa_0}, \operatorname{Op}_{\kappa_0,h}(b_h^+(t))] = O(h^{N+1}),$$

and $\operatorname{Op}_{\kappa_0,h}(b_h^+(0)) = \operatorname{Op}_{\kappa_0,h}(b^+)$. Integrating the above equation with respect to $t \in [0, R_2t_1]$, we obtain the assertion for $t \in [0, R_2t_1]$. For general $T > 0$, we divide the geodesics into a finite number of small curves, as well as the proof of Corollary 2.4.3, so that each curve is contained some fixed coordinate neighborhood. Applying the above argument on each chart, we have the assertion by a partition of unity argument. □

The following tells us that the support of $e^{-ith\widehat{P}} \operatorname{Op}_{\kappa_0}(b_l^+)$ is essentially away from the support of $\operatorname{Op}_{\kappa_0}(a_l^+)$ which is crucial to prove Theorem 2.6.1.

Proposition 2.6.3. *There exists $c_0 > 0$ that for all $0 < \varepsilon < 1/2$ and $t_1 > 0$, if we choose $\delta_{\varepsilon,t_1} < c_0\varepsilon^2\langle 1/t_1\rangle^{-1}$ and $L_{\varepsilon,t_1} = \delta_{\varepsilon,t_1}^{-1}$, then for all $(\sigma_l)_{0\leq l\leq L_{\varepsilon,t_1}} \subset (-1, 1/2]$ satisfying* (2.2.12) *with $\delta = \delta_{\varepsilon,t_1}$, $L = L_{\varepsilon,t_1}$ and all $t \geq R_2t_1$,*

$$\exp tH_p\kappa_{0*}^{-1}\Omega_i^+(R_2, \tilde{U}_{\kappa_0}, J, \varepsilon, \delta_{\varepsilon,t_1}, l) \cap \kappa_{0*}^{-1}\Gamma_i^+(R_1, \tilde{U}_{\kappa_0}, J, \varepsilon, \delta_{\varepsilon,t_1}, l) = \emptyset. \tag{2.6.4}$$

Proof. It suffices to show that (2.6.4) holds with $\Omega_i^+(R_2, \tilde{U}_{\kappa_0}, J, \varepsilon, \delta_{\varepsilon,t_1}, l)$ replaced by

$$\Omega_i^+(R_1, R_2, \tilde{U}_{\kappa_0}, J, \varepsilon, \delta_{\varepsilon,t_1}, l),$$

since $\Omega_i^+(R_2, \tilde{U}_{\kappa_0}, J, \varepsilon, \delta_{\varepsilon,t_1}, l) \subset \Omega_i^+(R_1, R_2, \tilde{U}_{\kappa_0}, J, \varepsilon, \delta_{\varepsilon,t_1}, l)$. We also note that, by Corollary 2.4.3, $\Omega_i^+(R_1, R_2, \tilde{U}_{\kappa_0}, J, \varepsilon, \delta_{\varepsilon,t_1}, l)$ is invariant with respect to ρ under the geodesic flow and coordinate transformations. Therefore, by the definition of the intermediate regions and the energy conservation law, it suffices to check that

$$\frac{\rho(t)}{\sqrt{2E_0}} > \frac{\rho}{\sqrt{2E_0}} + 2\delta_{\varepsilon,t_1}, \quad t \geq R_2t_1,$$

where $E_0 = p_{\kappa_0}(r,\theta,\rho,\omega)$ and (r,θ,ρ,ω) belongs to

$$\{R_1 < r < 4R_2,\ \theta \in \tilde{U}_{\kappa_0}, E_0 \in J,\ \rho/\sqrt{2E_0} \in [-1/2, \sqrt{1-\varepsilon^2/4}]\}.$$

Note that all $\Omega_i^+(R_1, R_2, \tilde{U}_{\kappa_0}, J, \varepsilon, \delta_{\varepsilon,t_1}, l)$ are contained in the above region. By (2.4.1) and (2.4.3), for sufficiently large $r > R_1$ and all $t \geq 0$,

$$\begin{aligned}\dot{\rho}(t) &\geq \frac{1}{r(t)^3}(h^{jk}(\theta(t)) - O(r(t)^{-\mu}))\omega_j(t)\omega_k(t)\\ &\gtrsim (r+|t|)^{-3}r^2\varepsilon^2\sqrt{2E_0}.\end{aligned}$$

Since

$$\begin{aligned}\int_0^{R_2t_1}(r+|t|)^{-3}r^2dt &\gtrsim \frac{R_2t_1}{r+R_2t_1}\\ &\geq \frac{t_1}{4+t_1},\end{aligned}$$

we can find $c_0 > 0$ small enough such that

$$\begin{aligned}\frac{\rho(R_2t_1)}{\sqrt{2E_0}} &\geq \frac{\rho}{\sqrt{2E_0}} + 2c_0\varepsilon^2\frac{t_1}{1+t_1}\\ &> \frac{\rho}{\sqrt{2E_0}} + 2\delta_{\varepsilon,t_1},\end{aligned}$$

which implies the assertion since $\dot{\rho}(t)$ is non-negative. □

Proof of Theorem 2.6.1. Since (2.4.5) implies

$$|\sqrt{2E_0} - \rho(t)| \lesssim R_2/(R_2+|t|), \quad t \geq 0,$$

it follows from (2.4.17) and (2.4.18) that there exists a constant $C_0 > 0$ such that for any $0 < \varepsilon_0 < 1/2$, we can find $T_{\varepsilon_0} > 0$ such that

$$\begin{aligned}&\exp R_2T_{\varepsilon_0}H_p(\kappa_{0*}^{-1}\Omega_i^+(R_2, \tilde{U}_{\kappa_0}, J, \varepsilon, \delta_{\varepsilon,t_1}, l))\\ &\subset \bigcup_\kappa \kappa_*^{-1}\Omega_s^+(R_2T_{\varepsilon_0}/C_0, C_0R_2T_{\varepsilon_0}, \tilde{U}_\kappa, J, \varepsilon_0),\end{aligned} \tag{2.6.5}$$

where $\tilde{U}_\kappa \Subset U_\kappa$. Note that C_0 and T_{ε_0} may be taken uniformly with respect to R_2. We fix such a T_{ε_0} with $T_{\varepsilon_0} \geq C_0$. By Theorem 2.6.2, we obtain that for all $N \geq 0$,

$$\mathrm{Op}_{\kappa_0}(a_l^+)e^{-ith\widehat{P}}\,\mathrm{Op}_{\kappa_0}(b_l^+) = \mathrm{Op}_{\kappa_0}(a_l^+)\sum_\kappa \mathrm{Op}_\kappa(b_{\kappa,h}^+(t))e^{-ith\widehat{P}} + O(h^{N+1}|t|)$$

on $L^2(\widehat{M})$ uniformly with respect to R_2 and $0 \leq t \leq R_2T_{\varepsilon_0}$, where

$$b_{\kappa,h}^+(t) \in S_{\mathrm{sc}}(\kappa_*\exp tH_p(\kappa_{0*}^{-1}\Omega_i^+(R_2, \tilde{U}_{\kappa_0}, J, \varepsilon, \delta_{\varepsilon,t_1}, l))).$$

Suppose that $R_2t_1 \leq t \leq R_2T_{\varepsilon_0}$. Since the support of a_l^+ does not intersect with the support of $(\kappa\circ\kappa_0^{-1})_*b_{\kappa,h}^+(t)$ for any κ with $V_{\kappa_0}\cap V_\kappa \neq \emptyset$ by Proposition 2.6.3, The semiclassical symbolic calculus (see subsection 2.2.2) implies that the above operator is bounded on $L^2(\widehat{M})$ with the norm dominated by $C_N h^{N+1}\langle t\rangle$, where the constant $C_N > 0$ may be taken uniformly with respect to R_2, t and h. If $R_2 T_{\varepsilon_0} \geq h^{-1}$, then we obtain (2.6.1). We thus assume $R_2 T_{\varepsilon_0} \leq t \leq h^{-1}$. By (2.6.5) and Theorem 2.6.2, we can find

$$b^+_{\kappa,h}(R_2 T_{\varepsilon_0}) \in S_{\mathrm{sc}}(\Omega^+_s(R_2 T_{\varepsilon_0}/C_0, C_0 R_2 T_{\varepsilon_0}, \tilde{U}_\kappa, J, \varepsilon_0))$$

such that

$$\begin{aligned} e^{-ith\widehat{P}} \operatorname{Op}_{\kappa_0}(b^+_l) &= e^{-i(t-R_2T_{\varepsilon_0})h\widehat{P}} e^{-iR_2T_{\varepsilon_0}h\widehat{P}} \operatorname{Op}_{\kappa_0}(b^+_l) \\ &= e^{-i(t-R_2T_{\varepsilon_0})h\widehat{P}} \sum_\kappa \operatorname{Op}_\kappa(b^+_{\kappa,h}(R_2T_{\varepsilon_0}))e^{-iR_2T_{\varepsilon_0}h\widehat{P}} + O(h^N) \end{aligned}$$

on $L^2(\widehat{M})$. Put $B = \sum_\kappa \operatorname{Op}_\kappa(b^+_{\kappa,h}(R_2T_{\varepsilon_0}))$ and divide B as follows

$$B = \varphi_{\kappa_0} B \tilde{\varphi}_{\kappa_0} + (1-\varphi_{\kappa_0}) B \tilde{\varphi}_{\kappa_0} + B(1-\tilde{\varphi}_{\kappa_0}),$$

where $\varphi_{\kappa_0}, \tilde{\varphi}_{\kappa_0} \in C^\infty(\widetilde{M}_\infty)$ such that

$$\operatorname{supp}\tilde{\varphi}_{\kappa_0} \subset \operatorname{supp}\varphi_{\kappa_0} \subset (2,\infty)\times V_{\kappa_0}, \quad \varphi_{\kappa_0} \equiv 1 \text{ close to } \operatorname{supp}\tilde{\varphi}_{\kappa_0}.$$

Since

$$\operatorname{supp}(1-\varphi_{\kappa_0}) \cap \operatorname{supp}\tilde{\varphi}_{\kappa_0} = \emptyset,$$

the second term is $O(h^\infty)$ on $L^2(\widehat{M})$. The third term is also $O(h^\infty)$ on $L^2(\widehat{M})$, since

$$\operatorname{supp} b^+_{\kappa,h}(R_2T_{\varepsilon_0}) \cap \operatorname{supp}(1-\tilde{\varphi}_{\kappa_0}) = \emptyset.$$

This support property follows from the fact

$$r > R_2 > 2 \quad \text{on } \operatorname{supp} b^+_{\kappa,h}(R_2T_{\varepsilon_0}), \quad 0 < r < 2 \quad \text{on } \operatorname{supp}(1-\tilde{\varphi}_{\kappa_0}).$$

By invariance properties of the strongly outgoing region and h-PDO under coordinates transformations, we can write

$$\varphi_{\kappa_0} B \tilde{\varphi}_{\kappa_0} = \operatorname{Op}_{\kappa_0}(\tilde{b}^+_{\kappa_0,h}(R_2T_{\varepsilon_0})) + O(h^\infty)$$

on $L^2(\widehat{M})$ with some $\tilde{b}^+_{\kappa_0,h}(R_2T_{\varepsilon_0}) \in S_{\mathrm{sc}}(\Omega^+_s(R_2T_{\varepsilon_0}/C_0, C_0R_2T_{\varepsilon_0}, \tilde{U}'_{\kappa_0}, J, \varepsilon_0))$, where $\tilde{U}'_{\kappa_0} \Subset U_{\kappa_0}$. Consider a splitting of the interval $[R_2T_{\varepsilon_0}/C_0, C_0R_2T_{\varepsilon_0}]$:

$$R_2T_{\varepsilon_0}/C_0 = \tilde{R}_0 < \tilde{R}_1 < \tilde{R}_2 < \cdots < \tilde{R}_k, \ \tilde{R}_j = 2^j\tilde{R}_0, \ \tilde{R}_{k-1} < C_0R_2T_{\varepsilon_0} < \tilde{R}_k.$$

Clearly $2^k \leq 2C_0^2$. Using a method of the partition of unity, we split $\tilde{b}^+_{\kappa_0,h}(R_2T_{\varepsilon_0})$ as follows:

$$\tilde{b}^+_{\kappa_0,h}(R_2T_{\varepsilon_0}) = \sum_{j=0}^{k-2} \tilde{b}^+_{\kappa_0,h,j}$$

with $\tilde{b}^+_{\kappa_0,h,j} \in S_{\mathrm{sc}}(\Omega^+_s(\tilde{R}_j, \tilde{U}'_{\kappa_0}, J, \varepsilon_0))$. By Theorem 2.5.10, we can construct the Isozaki-Kitada parametrix of $e^{-i(t-R_2T_{\varepsilon_0})h\widehat{P}} \operatorname{Op}_{\kappa_0}(\tilde{b}^+_{\kappa_0,h,j})$ for sufficiently large $R_2 \geq R_{\mathrm{IK}}, \lambda \geq \lambda_{\mathrm{IK}}$ and small $0 < \varepsilon_0 \leq \varepsilon_{\mathrm{IK}}$, and we obtain

$$\begin{aligned} & e^{-ith\widehat{P}} \operatorname{Op}_{\kappa_0}(b^+_l) \\ &= \sum_{j=0}^{k-2} \kappa_0^* I^+_{\mathrm{IK}}(c^+_{h,j}) e^{-i(t-R_2T_{\varepsilon_0})h\frac{1}{2}D_r^2} I^+_{\mathrm{IK}}(d^+_{h,j})^* \kappa_{0*} e^{-iR_2T_{\varepsilon_0}h\widehat{P}} + Q(t,h,N,R_2), \end{aligned}$$

where, for each j, $I_{\mathrm{IK}}^{+}(c_{h,j}^{+})$ and $I_{\mathrm{IK}}^{+}(d_{h,j}^{+})$ are FIO's defined in Definition 2.5.6 with some phase function $S_{\kappa_0,j}^{+} \in C^\infty(\mathbb{R}^d;\mathbb{R})$, which satisfies the statement of Theorem 2.5.1 with $R = \tilde{R}_j$, and some amplitudes

$$c_{h,j}^{+} \in S_{\mathrm{sc}}(\Omega_s^{+}(\lambda^{-3}\tilde{R}_j, \lambda^3\tilde{R}_j, \tilde{U}'_{\kappa_0}, J, \lambda^3\varepsilon_0)),$$
$$d_{h,j}^{+} \in S_{\mathrm{sc}}(\Omega_s^{+}(\lambda^{-1}\tilde{R}_j, \lambda\tilde{R}_j, \tilde{U}'_{\kappa_0}, J, \lambda\varepsilon_0)).$$

The remainder term $Q(t,h,N,R_2)$ is uniformly bounded on $L^2(\widehat{M})$ with the norm of order h^N with respect to R_2 and $R_2T_{\varepsilon_0} \le t \le h^{-1}$. By the composition rule of FIO's with PDO's, $a_l^{+}(r,\theta,hD_r,hD_\theta)I_{\mathrm{IK}}^{+}(c_{h,j}^{+})$ is also FIO's (up to the smoothing term $O(h^\infty)$ on $L^2(\widehat{M})$) with the phase $S_{\kappa_0,j}^{+}$ and the amplitude supported in

$$X = \{(r,\theta,\rho,\omega) \mid a_l^{+}(r,\theta,\partial_r S_{\kappa_0,j}^{+}, \partial_\theta S_{\kappa_0,j}^{+})c_{h,j}^{+}(r,\theta,\rho,\omega) \neq 0\}.$$

By the support property of a_l^{+}, we see that

$$\partial_r S_{\kappa_0,j}^{+} \le \sqrt{2(1-\varepsilon^2/4)p_{\kappa_0}(r,\theta,\partial_r S_{\kappa_0,j}^{+}, \partial_\theta S_{\kappa_0,j}^{+})}.$$

Since $(r,\theta,\rho,\omega) \in \operatorname{supp} c_{h,j}^{+} \subset \Omega_s^{+}(\lambda^{-3}\tilde{R}_j, \lambda^3\tilde{R}_j, \tilde{U}_{\kappa_0}, J, \lambda^3\varepsilon_0)$, by the above estimate and (2.5.1), we have

$$\rho/\sqrt{2p_{\kappa_0}(r,\theta,\rho,\omega)} \le \sqrt{1-\varepsilon^2/4} - C|\lambda^3\varepsilon_0|^2 \le \sqrt{1-\varepsilon^2/8}.$$

On the other hand, choosing $\varepsilon_0 > 0$ small enough so that $\lambda^3\varepsilon_0 < \varepsilon^2/8$ (note that $\lambda > \lambda_{\mathrm{IK}}$ is fixed), the support property of $c_{h,j}^{+}$ implies

$$\rho/\sqrt{2p_{\kappa_0}(r,\theta,\rho,\omega)} \ge \sqrt{1-\lambda^3\varepsilon_0} > \sqrt{1-\varepsilon^2/8}.$$

The above two inequalities show that $X = \emptyset$ and hence $a_l^{+}(r,\theta,hD_r,hD_\theta)I_{\mathrm{IK}}^{+}(c_{h,j}^{+})$ is $O(h^\infty)$ in $L^2(\widehat{M})$. Since $\kappa_0^{*}e^{-i(t-R_2T_{\varepsilon_0})h\frac{1}{2}D_r^2}I_{\mathrm{IK}}^{+}(d_{h,j}^{+})^{*}\kappa_{0*}e^{-iR_2T_{\varepsilon_0}h\widehat{P}}$ is uniformly bounded on $L^2(\widehat{M})$ with respect to R_2, h and t, we obtain the assertion and conclude the proof. □

2.7 The WKB parametrix

In the previous section we proved that $\mathrm{Op}_{\kappa_0,h}(a_l^{\pm})e^{-ith\widehat{P}}\mathrm{Op}_{\kappa_0,h}(b_l^{\pm})$ are rapidly decaying with respect to $h \in (0,1]$ if $Rt_1 \le \pm t \le h^{-1}$ with any $t_1 > 0$ and large $R > 0$. Therefore, it remains to control the above operators for $0 \le \pm t \le Rt_0$ with sufficiently small t_0. This section discuss construction of the WKB parametrix of propagator $e^{-it\widehat{P}}\mathrm{Op}_\kappa(a^{\pm})$ for $0 \le \pm t \le Rt_0$, where a^{+} (resp. a^{-}) is supported in an outgoing (resp. incoming) region. By (2.4.18), we can always work on a fixed coordinate neighborhood U_κ and hence do not write the subscript κ explicitly. Let $\tilde{U}_\kappa \Subset U_\kappa$ be as in Example 2.2.2 and fix open subsets $U \Subset U_0 \Subset U_1 \Subset U_2 \Subset \tilde{U}_\kappa$, open intervals $J \Subset J_0 \Subset J_1 \Subset J_2 \Subset (0,\infty)$ and constants $-1 < \sigma < \sigma_0 < \sigma_1 < \sigma_2 < 1$ arbitrarily.

2.7.1 Fourier integral operators for the WKB parametrix

We here study time dependent FIO's which will be used to construct the WKB parametrix. We first construct the phase function.

Theorem 2.7.1. *We can choose $t_0 > 0$ small enough such that, for sufficiently large $R_2 > 0$ and all $R_1 > R_2$, there exist smooth and real-valued functions*

$$\Psi^+ \in C^\infty((0, R_1 t_0) \times \mathbb{R}^{2d}), \quad \Psi^- \in C^\infty((-R_1 t_0, 0) \times \mathbb{R}^{2d}),$$

satisfying the following Hamilton Jacobi equation on $\Gamma^\pm(R_1, U_1, J_1, \sigma_1)$:

$$\begin{cases} \partial_t \Psi^\pm + p(r, \theta, \partial_r \Psi^\pm, \partial_\theta \Psi^\pm) = 0, & 0 \le \pm t \le R_1 t_0, \\ \Psi^\pm|_{t=0} = r\rho + \theta \cdot \omega, \end{cases} \tag{2.7.1}$$

such that we have the followings all $0 \le \pm t \le R_1 t_0$:

$$\mathrm{supp}\left(\Psi^\pm(t, r, \theta, \rho, \omega) - r\rho - \theta \cdot \omega\right) \subset \Gamma^\pm(R_2, U_2, J_2, \sigma_2),$$

$$\left|\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta \left(\Psi^\pm(t, r, \theta, \rho, \omega) - r\rho - \theta \cdot \omega\right)\right| \le C_{j\alpha k\beta} \langle r\rangle^{-j-|\beta|}|t| \quad on \quad \mathbb{R}^{2d}. \tag{2.7.2}$$

Moreover, for all $(r, \theta, \rho, \omega) \in \Gamma^\pm(R_1, U_1, J_1, \sigma_1)$, we have

$$|\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta \left(\Psi^\pm(t, r, \theta, \rho, \omega) - r\rho - \theta \cdot \omega + tp(r, \theta, \rho, \omega)\right)| \le C_{j\alpha k\beta} r^{-j-|\beta|}|t| t_0. \tag{2.7.3}$$

Proof. This theorem can be proved similarly to Theorem 2.5.1. We only prove the theorem for the case $t \ge 0$, and the proof for the case $t \le 0$ is similar. Let $R_1 > R_1' > R_1/2$, $U_1 \Subset U_1' \Subset \tilde{U}_\kappa$, $J_1 \Subset J_1' \Subset (0, \infty)$ and $\sigma_1 < \sigma_1' < 1$. Let $F_{R_1} : (r, \theta, \rho, \omega) \mapsto (r/R_1, \theta, \rho, \omega)$, and define $g^+(t)$ and $\tilde{g}^+(t)$ by

$$\begin{aligned} g^+(t) &:= (r(t, r, \theta, \rho, \omega), \theta(t, r, \theta, \rho, \omega), \rho, \omega), \\ \tilde{g}^+(t) &= (\tilde{r}(t, x, \theta, \rho, \omega), \tilde{\theta}(t, x, \theta, \rho, \omega), \rho, \omega) \\ &:= F_{R_1} \circ g^+(t) \circ F_{R_1}^{-1}(x, \theta, \rho, \omega), \end{aligned}$$

where $(r(t, r, \theta, \rho, \omega), \theta(t, r, \theta, \rho, \omega))$ is the Hamilton flow generated by p, and $x = r/R_1$. By Proposition 2.4.2, we have

$$\begin{aligned} |\partial_x^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta (\tilde{r}(t) - x)| &\le C R_1^{-1}|t| \le C t_0, \\ |\partial_x^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta (\tilde{\theta}(t) - \theta)| &\le C R_1'^{-1}|t| \le C t_0, \\ |\partial_x^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta (\partial \tilde{g}^+(t) - \mathrm{Id})| &\le C t_0 < 1/2, \end{aligned}$$

for all $(x, \theta, \rho, \omega) \in F_{R_1}\Gamma^+(R_1', U_1', J_1', \sigma_1')$ and $0 \le t \le R_1 t_0$ as long as $t_0 > 0$ is small enough. Applying a same argument as that in the proof of Lemma 2.5.3 to $\tilde{g}^+(t)$, we see that $g^+(t)$ is diffeomorphic from $\Gamma^+(R_1', U_1', J_1', \sigma_1')$ onto its range for all $0 \le t \le R_1 t_0$, and satisfies

$$\Gamma^+(R_1, U_1, J_1, \sigma_1) \subset g^\pm(t)(\Gamma^+(R_1', U_1', J_1', \sigma_1')), \quad 0 \le \pm t \le R_1 t_0.$$

Let $\Gamma^+(R_1, U_1, J_1, \sigma_1) \ni (r, \theta, \rho, \omega) \mapsto (\hat{r}^+(t), \hat{\theta}^+(t), \rho, \omega)$ be the inverse of $g^+(t)$, and put

$$(r^+(t,s), \theta^+(t,s), \rho^+(t,s), \omega^+(t,s)) := (r, \theta, \rho, \omega)(s, \hat{r}^+(t), \hat{\theta}^+(t), \rho, \omega)$$

for $0 \leq s \leq t \leq R_1 t_0$. Then, by a same argument as that in the proof of Lemma 2.5.4 and Lemma 2.5.5, we have

$$\begin{cases} |\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta (\hat{r}^+(t) - r)| + |\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta (\omega^+(t,t) - \omega)| \leq C r^{-j-|\beta|} |t|, \\ |\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta (\hat{\theta}^+(t) - \theta)| + |\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta (\rho^+(t,t) - \rho)| \leq C r^{-j-|\beta|} t_0, \end{cases} \tag{2.7.4}$$

Define $\tilde{\Psi}^+ \in C^\infty((0, R_1 t_0) \times \Gamma^+(R_1, U_1, J_1, \sigma_1))$ by

$$\tilde{\Psi}^+(t) := r\rho + \theta \cdot \omega + \int_0^t L(r^+(t,s), \theta^+(t,s), \rho^+(t,s), \omega^+(t,s)) ds,$$

where $L = (\rho \partial_\rho p + \omega \cdot \partial_\omega p - p)(r, \theta, \rho, \omega)$. By a standard Hamilton-Jacobi theory, it is easy to see that $\tilde{\Psi}^+(t)$ solves (2.7.1) and satisfies

$$\partial \tilde{\Psi}(t) = (\rho^+(t,t), \omega^+(t,t), \hat{r}^+(t), \hat{\theta}^+(t)).$$

By (2.7.4) and the energy conservation law

$$p(r, \theta, \partial_r \tilde{\Psi}^+(t), \partial_\theta \tilde{\Psi}^+(t)) = p(\hat{r}^+(t), \hat{\theta}^+(t), \rho, \omega),$$

we have

$$\left| \partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta \left(p(r, \theta, \partial_r \tilde{\Psi}^+(t), \partial_\theta \tilde{\Psi}^+(t)) - p(r, \theta, \rho, \omega) \right) \right| \leq C r^{-j-|\beta|} t_0.$$

Therefore,

$$\left| \partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta \left(\tilde{\Psi}^+(t) - r\rho - \theta \cdot \omega + tp(r, \theta, \rho, \omega) \right) \right| \leq C r^{-j-|\beta|} |t| t_0.$$

Choose $\chi_+ \in C^\infty(\mathbb{R}^{2d})$ so that

$$0 \leq \chi_+ \leq 1, \quad \chi_+ \equiv 1 \quad \text{on} \quad \Gamma^+(R_1, U_1, J_1, \sigma_1), \quad \operatorname{supp} \chi_+ \subset \Gamma^+(R_2, U_2, J_2, \sigma_2),$$

and that

$$\left| \partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta \chi_+(r, \theta, \rho, \omega) \right| \leq C \langle r \rangle^{-j-|\beta|} \quad \text{on} \quad \mathbb{R}^{2d}.$$

We now define $\Psi^+(t) := r\rho + \theta \cdot \omega + \chi_+(\tilde{\Psi}^+(t) - r\rho + \theta \cdot \omega)$. Clearly, $\Psi(t)^+$ satisfies the statement of theorem 2.7.1. □

Suppose $(a^\pm(t))_{0 \leq \pm t \leq R_1 t_0}$ are bounded in $S_{\mathrm{sc}}(\Gamma^\pm(R_1, U_1, J_1, \sigma_1))$, respectively. We define the FIO's for the WKB parametrix $I^\pm_{\mathrm{WKB}}(a^\pm(t)) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ by

$$\begin{aligned} &I^\pm_{\mathrm{WKB}}(a^\pm(t)) u(r, \theta) \\ &:= \frac{1}{(2\pi h)^d} \int e^{\frac{i}{h}(\Psi^\pm(t, r, \theta, \rho, \omega) - r'\rho - \theta' \cdot \omega)} a^\pm(t, r, \theta, \rho, \omega) u(r', \theta') dr' d\theta' d\rho d\omega. \end{aligned}$$

Proposition 2.7.2. *$I^{\pm}_{\mathrm{WKB}}(a^{\pm}(t))$ are bounded on $L^2(\mathbb{R}^d)$ uniformly with respect to $0 \le \pm t \le R_1 t_0$:*

$$||I^{\pm}_{\mathrm{WKB}}(a^{\pm}(t))u||_{L^2(\mathbb{R}^d)} \le C||u||_{L^2(\mathbb{R}^d)}.$$

Proof. For $(r, \theta, \rho, \omega, r', \theta') \in \mathbb{R}^{3d}$ with $r, r' > R_1$ and $0 \le t \le R_1 t_0$, define the map (ρ^1_+, ω^1_+) by

$$(\rho^1_+, \omega^1_+)(t, r, \theta, \rho, \omega, r', \theta') = \int_0^1 (\partial_{r,\theta}\Psi^+)(t, r' + s(r - r'), \theta' + s(\theta - \theta'), \rho, \omega) ds.$$

By (2.7.2), we have

$$\sup_{0 \le t \le R_1 t_0} \left(|\partial^\gamma(\rho^1_+ - \rho)| + R_1^{-1}|\partial^\gamma(\omega^1_+ - \omega)|\right) \le C_\gamma R_1^{-j-j'-|\beta|} t_0,$$

where $\partial^\gamma = \partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta \partial_{r'}^{j'} \partial_{\theta'}^{\alpha'}$ with $\gamma = (j, \alpha, k, \beta, j', \alpha')$. By this estimate and a similar argument as in the proof of Lemma 2.5.9, we obtain that, for all $(r, \theta, r', \theta') \in \mathbb{R}^{2d}$ with $r, r' > R_1$ and $0 \le t \le R_1 t_0$, the map $(\rho, \omega) \mapsto (\rho^1_+, \omega^1_+)(t, r, \theta, \rho, \omega, r', \theta')$ is a diffeomorphism from $\mathbb{R}^d$ onto itself provided that $t_0 > 0$ is small enough. We also see that the corresponding inverse $(\rho, \omega) \mapsto (\rho^2_+, \omega^2_+)(t, r, \theta, \rho, \omega, r', \theta')$ satisfies

$$\sup_{0 \le t \le R_1 t_0} \left(|\partial^\gamma(\rho^2_+ - \rho)| + R_1^{-1}|\partial^\gamma(\omega^2_+ - \omega)|\right) \le C_\gamma R_1^{-j-j'-|\beta|} t_0. \tag{2.7.5}$$

Making the change of variable $(\rho, \omega) \mapsto (\rho^2_+, \omega^2_+)$, $I^+_{\mathrm{WKB}}(a^+(t)) I^+_{\mathrm{WKB}}(a^+(t))^*$ can be regarded as a h-PDO with the amplitude

$$a^+(t, r, \theta, \rho^2_+, \omega^2_+)\overline{a^+(t, r', \theta', \rho^2_+, \omega^2_+)}|\det \partial_{\rho,\omega}(\rho^2_+, \omega^2_+)|.$$

By (2.7.5), this amplitude and its all derivatives with respect to ∂^γ are bounded on $\mathbb{R}^{3d}$ uniformly with respect to $0 \le t \le R_1 t_0$. The assertion then follows from the Calderón-Vaillancourt theorem and the L^2-functional calculus. When $-R_1 t_0 \le t \le 0$, the proof is similar. □

2.7.2 Construction of the parametrix

The main result in this section is the following.

Theorem 2.7.3. *There exists $R_{\mathrm{WKB}} > 0$ large enough and $t_{\mathrm{WKB}} > 0$ small enough such that, for all $R > R_0 > R_1 > R_{\mathrm{WKB}}$, $a^{\pm} \in S_{\mathrm{sc}}(\Gamma^{\pm}(R, U, J, \sigma))$ and $N \ge 0$, we can find*

$$b^{\pm}_h(t) = \sum_{j=0}^N h^j b^{\pm}_j(t)$$

with $(b^{\pm}_j(t))_{0 \le \pm t \le R_1 t_{\mathrm{WKB}}}$ bounded in $S_{\mathrm{sc}}(\Gamma^{\pm}(R_0, U_0, J_0, \sigma_0))$ such that, for all $h \in (0, 1]$ and $0 \le \pm t \le R_1 t_{\mathrm{WKB}}$,

$$||e^{-ith\widehat{P}} \operatorname{Op}_\kappa(a^{\pm}) - \kappa^* I^{\pm}_{\mathrm{WKB}}(b^{\pm}_h(t))\kappa_*||_{\mathcal{L}(L^2(\widehat{M}))} \le C_N h^{N+1}|t|$$

where $C_N > 0$ can be taken uniformly with respect to h, t and R_1.

Remark 2.7.4. The essential point of Theorem 2.7.3 is to construct the parametrix on the time interval $0 \leq |t| \leq R_1 t_{\mathrm{WKB}}$ which allow us to choose the constant $\delta > 0$ in Theorem 2.3.3 independently with respect to R_1. When $|t| > 0$ is small and independent of R_1, such a parametrix construction is basically well known (see [35] for the case of elliptic operators on the Euclidean space, and [4] for the case of the Laplace-Beltrami operator on the asymptotically hyperbolic manifold).

We prove the theorem for the case when $t \geq 0$. Put $B_+(t) = I^+_{\mathrm{WKB}}(b^+_h(t))$. By the Duhamel formula, we have

$$e^{-ith\widehat{P}}\kappa^* B_+(0)\kappa_* = \kappa^* B_+(t)\kappa_* + \frac{i}{h}\int_0^t e^{-i(t-s)hP}\kappa^*(hD_s + h^2\widehat{P}_\kappa)B_+(s)\kappa_* ds,$$

where $D_s = i^{-1}\partial_s$. Since the off-diagonal decay of PDO implies

$$\| \operatorname{Op}_\kappa(a^+) - \kappa^* a^+(r,\theta,hD_r,hD_\theta)\kappa_* \|_{\mathcal{L}(L^2(\widehat{M}))} \leq C_n h^n,$$

for all $n \geq 0$, it suffices to show that there exists $b^+_h(t)$ such that $b^+_h(0) = a^+$, and

$$\|(hD_s + h^2\widehat{P}_\kappa)B_+(s)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_N h^{N+2} \tag{2.7.6}$$

for $h \in (0,1]$ and $0 \leq s \leq R_1 t_{\mathrm{WKB}}$.

Define smooth tensors $x^+(t)$ and $y^+(t)$ by

$$x^+(t) := \partial_{\rho,\omega} p_\kappa(r,\theta,\partial_r\Psi^+(t),\partial_\theta\Psi^+(t)), \quad y^+(t) := (p + p_1)(r,\theta,\partial_r,\partial_\theta)\Psi^+(t).$$

$b^+_h(t)$ satisfying (2.7.6) can be constructed by solving the transport equations.

Lemma 2.7.5. *For sufficiently small $t_0 > 0$, there exist*

$$(b^+_j(t))_{0\leq t\leq R_1 t_0} \text{ bounded in } S_{\mathrm{sc}}(\Gamma^\pm(R_0,U_0,J_0,\sigma_0)),\ j = 0,1,...,N,$$

such that $b^+_j(t)$ solve the transport equations:

$$\begin{cases} \partial_t b^+_0(t) + x^+(t)\cdot\partial_{r,\theta} b^+_0(t) + y^+(t) b^+_0(t) = 0, \\ \partial_t b^+_j(t) + x^+(t)\cdot\partial_{r,\theta} b^+_j(t) + y^+(t) b^+_j(t) + i\widehat{P}_\kappa b^+_{j-1}(t) = 0,\ j \geq 1, \end{cases} \tag{2.7.7}$$

with the initial condition $b^+_0(0) = a^+$, $b^+_j(0) = 0$ for $j = 1,2,...,N$.

Proof. We mimic Bouclet's argument [4, Lemma 6.4]. Choose $R_0', R_0'' > 0$, $U_0', U_0'' \Subset \mathbb{R}^{d-1}$ and $J_0', J_0'' \Subset (0,\infty)$ so that

$$R_0 > R_0' > R_0'' > R_1, \quad U_0 \Subset U_0' \Subset U_0'' \Subset U_1,$$
$$J_0 \Subset J_0' \Subset J_0'' \Subset J_1, \quad \sigma_0 < \sigma_0' < \sigma_0'' < \sigma_1.$$

For $0 \leq s,t \leq R_1 t_0$ and $(r,\theta,\rho,\omega) \in \Gamma^+(R_0'',U_0'',J_0'',\sigma_0'')$, consider the flow generated by $x^+(t)$, that is the solution to the ODE

$$\begin{cases} (\partial_t r^+(t,s), \partial_t\theta^+(t,s)) = x^+(t, r^+(t,s), \theta^+(t,s), \rho, \omega), \\ (r^+(s,s), \theta^+(s,s)) = (r,\theta). \end{cases}$$

Since $x^+(t) = (\partial_r \Psi^+(t), r^{-2}(h^{jk} + a^{jk})\partial_{\theta^k}\Psi^+(t))$, by using (2.7.2), we have

$$\left|\partial_t r^+(t,s)\right| \leq C(1 + r^+(t,s)^{-1}|t|), \quad \left|\partial_t \theta^+(t,s)\right| \leq C r^+(t,s)^{-2}|t|.$$

In particular,

$$\left|\partial_t r^+(s,s)\right| \leq C, \quad \left|\partial_t \theta^+(s,s)\right| \leq C R_0''^{-1} t_0.$$

A same argument as that in the proof of Lemma 2.5.12 then implies that there exists $t_0 > 0$, independent of R_0, such that $(r^+(t,s), \theta^+(t,s))$ is well-defined on $\Gamma^+(R_0'', U_0'', J_0'', \sigma_0'')$ for all $0 \leq s, t \leq R_1 t_0$, and that

$$|r^+(t,s) - r| \leq C|t|, \quad |\theta^+(t,s) - \theta| \leq t_0 \quad \text{on} \quad \Gamma^+(R_0'', U_0'', J_0'', \sigma_0'').$$

In particular, $(r^+(t,s), \theta^+(t,s), \rho, \omega) \in \Gamma^+(R_1, U_1, J_1, \sigma_1)$ for all $0 \leq s, t \leq R_1 t_0$. Moreover, differentiating the integral equation

$$(r^+(t,s), \theta^+(t,s)) = (r, \theta) + \int_s^t x^+(u, r^+(u,s), \theta^+(u,s), \rho, \omega) du$$

with respect to $\partial^\gamma = \partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta$, and by using (2.7.2) and induction on $|\gamma|$, we have

$$|\partial^\gamma (r^+(t,s) - r)| \leq C r^{-j-\beta}|t|, \quad |\partial^\gamma(\theta^+(t,s) - \theta)| \leq C r^{-j-\beta} t_0, \tag{2.7.8}$$

for all $0 \leq s, t \leq R_1 t_0$ and $(r, \theta, \rho, \omega) \in \Gamma^+(R_0'', U_0'', J_0'', \sigma_0'')$. We put

$$\Phi_+(t,s) := (r^+(t,s), \theta^+(t,s), \rho, \omega),$$

and define $b_j^+(t)$ by

$$\begin{cases} b_0^+(t) := a^+(\Phi_+(0,t)) e^{\int_0^t y^+(s, \Phi_+(s,t)) ds}, \\ b_j^+(t) := -\displaystyle\int_0^t (i\widehat{P}_\kappa b_{j-1}^+)(s, \Phi_+(s,t)) e^{\int_s^t y^+(u, \Phi_+(u,t)) du} ds, \quad j = 1, ..., N. \end{cases}$$

We remark that if we choose t_0 small enough, then (2.7.8) implies

$$\begin{aligned} \Phi_+(t,s)(\Gamma^+(R, U, J, \sigma)) &\subset \Gamma^+(R_0, U_0, J_0, \sigma_0), \\ \Phi_+(t,s)(\Gamma^+(R_0, U_0, J_0, \sigma_0)) &\subset \Gamma^+(R_0', U_0', J_0', \sigma_0'), \\ \Phi_+(t,s)(\Gamma^+(R_0', U_0', J_0', \sigma_0')) &\subset \Gamma^+(R_0'', U_0'', J_0'', \sigma_0'') \subset \Gamma^+(R_1, U_1, J_1, \sigma_1), \end{aligned}$$

for all $0 \leq s, t \leq R_1 t_0$. $\Phi_+(s,t)$ is thus well-defined on $\Gamma^+(R_0', U_0', J_0', \sigma_0')$. We also obtain that smooth on $\Gamma^+(R_0, U_0, J_0, \sigma_0)$, and satisfies $\Phi_+(s,t) = \Phi_+(t,s)^{-1}$. Moreover, we obtain

$$\operatorname{supp} b_j^+(t) \subset \Gamma^+(R_0, U_0, J_0, \sigma_0),$$

since $\operatorname{supp} a^+ \subset \Gamma^+(R, U, J, \sigma)$. If we extend $b_j^+(t)$ on $\mathbb{R}^{2d}$ so that

$$b_j^+(t) = 0 \quad \text{outside} \quad \Gamma^+(R_0, U_0, J_0, \sigma_0),$$

then $b_j^+(t)$ are still smooth with respect to $(r, \theta, \rho, \omega)$. Furthermore, by (2.2.3), (2.7.2) and (2.7.8), we see that $(b_j^+(t))_{0 \leq t \leq R_1 t_0}$ is bounded in $S_{\mathrm{sc}}(\Gamma^\pm(R_0, U_0, J_0, \sigma_0))$. Finally, a standard Hamilton-Jacobi theory shows that $b_j^+(t)$ solve (2.7.7) for $0 \leq t \leq R_1 t_0$. □

Proof of Theorem 2.7.1. By the construction, $B_+(0) = a^+(r, \theta, hD_r, hD_\theta)$. Moreover, Ψ^+ solves (2.7.1) on $\Gamma^+(R_1, U_1, J_1, \sigma_1)$ which contains $\Gamma^+(R_0, U_0, J_0, \sigma_0)$, and b_j^+ satisfy (2.7.7) on the latter region. Therefore, a direct computation yields

$$(hD_s + h^2 \widehat{P}_\kappa) B_+(s) = h^{N+2} i \widehat{P}_\kappa b_N^+(s).$$

Since $(\widehat{P}_\kappa b_N^+(s))_{0 \le s \le Rt_0}$ is bounded in $S_{\mathrm{sc}}(\Gamma^\pm(R_0, U_0, J_0, \sigma_0))$ uniformly with respect to R, (2.7.1) implies that $I_{\mathrm{WKB}}^+(i\widehat{P}_\kappa b_j^+(s))$ is bounded on $L^2(\mathbb{R}^d)$ uniformly with respect to $h \in (0, 1]$ and $0 \le s \le Rt_0$. We hence proved (2.7.6). □

2.7.3 Dispersive estimates

We here prove dispersive estimates for the WKB parametrix. Let $R_0 > R_1 > R_{\mathrm{WKB}}$ and $t_{\mathrm{WKB}} > 0$ be as in Theorem 2.7.3.

Theorem 2.7.6. *For any $(b^\pm(t))_{0 \le \pm t \le R_1 t_{\mathrm{WKB}}}$ bounded in $S_{\mathrm{sc}}(\Omega_s^\pm(R_0, U_0, J_0, \sigma_0))$, we can write*

$$I_{\mathrm{WKB}}^\pm(b^\pm(t)) = U_{\mathrm{WKB}}^\pm(t, h) + R_{\mathrm{WKB}}^\pm(t, h),$$

where $U^\pm(t, h)$ satisfy

$$||r^{-\frac{d-1}{2}} U_{\mathrm{WKB}}^\pm(t, h) r^{-\frac{d-1}{2}}||_{\mathcal{L}(L^1(\mathbb{R}^d), L^\infty(\mathbb{R}^d))} \le C|th|^{-d/2}, \qquad (2.7.9)$$

for all $0 < \pm t \le \min(R_1 t_{\mathrm{WKB}}, h^{-1})$, $h \in (0.1]$. Moreover the remainder terms $R_{\mathrm{WKB}}^\pm(t, h)$ are rapidly decaying with respect to h: for any $N \ge 0$,

$$||R_{\mathrm{WKB}}^\pm(t, h)||_{\mathcal{L}(L^2(\mathbb{R}^d))} \le C_N h^N, \quad 0 \le \pm t \le \min(R_1 t_{\mathrm{WKB}}, h^{-1}), \ h \in (0.1]. \qquad (2.7.10)$$

Here constants $C, C_N > 0$ may be taken uniformly with respect to h and R_1.

Proof. Since the proof is similar as the proof of Theorem 2.5.13, we omit details and give the sketch of the proof. We consider the outgoing case only. The distribution kernel of $I_{\mathrm{WKB}}^+(b^+(t))$ is written in the form

$$I_{b^+}(t, h) = (2\pi h)^{-d} \int e^{\frac{i}{h}\psi^+(t, r, \theta, \rho, \omega, r', \theta')} b^+(t, r, \theta, \rho, \omega) d\rho d\omega,$$

where $\psi^+(t, r, \theta, \rho, \omega, r', \theta') = \Psi^+(t, r, \theta, \rho, \omega) - r'\rho - \theta' \cdot \omega$. Let $\chi_\rho \in C_0^\infty(\mathbb{R})$, $\chi_\omega \in C_0^\infty(\mathbb{R}^{d-1})$ be as in the proof of Theorem 2.5.13, and set

$$B^+(t) = \chi_\rho(\partial_\rho \psi^+) \chi_\omega(\partial_\omega \psi^+) b^+(t).$$

We then have $|\partial_\rho \psi^+| < 1$ and $|\partial_\omega \psi^+| < 1$ on $\operatorname{supp} B^+$. We denote the operator having the kernel $I_{B^+}(t, h)$ by $U_{B^+}(t, h)$. By a same proof as that in Lemma 2.5.14, we obtain that $R_{\mathrm{WKB}}^+(t, h) = I_{\mathrm{WKB}}^+(b^+(t)) - U_{B^+}(t, h)$ satisfies (2.7.10).

We next prove (2.7.9) for $U_{\mathrm{WKB}}^+(t, h) := U_{B^+}(t, h)$. Assume $R_1 > 4$ without loss of generality. We first note that $r' \ge R_1/2$ on $\operatorname{supp} B^+$; otherwise, there exists $C_0 > 0$, independent of R_0, such that

$$|\partial_\rho \psi^+| \ge r - r' - C_0 R_1 t_{\mathrm{WKB}} \ge (1/2 - C_0 t_{\mathrm{WKB}}) R_0 > 1$$

if $0 < t_{\mathrm{WKB}} < (4C_0)^{-1}$, since $r > R_0 > R_1$ on $\operatorname{supp} b^+(t)$ and $\partial_\rho \psi = r - r' + O(|t|)$ by (2.7.2). This contradicts the support property of B^+. We thus have

$$r'/4 \le r' - |\partial_\rho \psi^+| - C_0 R_1 t_{\mathrm{WKB}} \le r \le r' + |\partial_\rho \psi^+| + C_0 R_1 t_{\mathrm{WKB}} \le 4r'$$

on the support of B^+. It follows from (2.7.3) that ψ^+ may be written in the form

$$\psi^+(t, r, \theta, \rho, \nu, r', \theta') = (r - r')\rho + (\theta - \theta') \cdot \omega - tp(r, \theta, \rho, \omega) + Q^+(t, r, \theta, \rho, \omega)$$

on $\operatorname{supp} B^+$, where the remainder Q^+ satisfies

$$\left| \partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta Q^+(t, r, \theta, \rho, \omega) \right| \le C r^{-j-|\beta|} |t| t_{\mathrm{WKB}} \quad \text{on} \quad \operatorname{supp} B^+.$$

Let $\chi \in C^\infty(\mathbb{R}^{2d})$ be a smooth cut-off function such that

$$0 \le \chi \le 1, \ \operatorname{supp} \chi \subset \Gamma^+(R_1, U_1, J_1, \sigma_1), \ \chi \equiv 1 \text{ on } \Gamma^+(R_0, U_0, J_0, \sigma_0),$$

and that

$$\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta \chi = O(\langle r \rangle^{-j-|\beta|}).$$

We then define

$$\begin{aligned} &\psi_0^+(t, r, \theta, \rho, \omega, r', \theta') \\ &:= (r - r')\rho + (\theta - \theta') \cdot \omega - tp(r, \theta, \rho, \omega) + \chi(r, \theta, \rho, \omega) Q^+(t, r, \theta, \rho, \omega). \end{aligned}$$

Since $\psi_0^+ \equiv \psi^+$ on $[0, R_1 t_{\mathrm{WKB}}] \times \operatorname{supp} B^+$, making the change of the variable $\omega \mapsto r\nu$, $I_{B^+}(t, h)$ reads

$$I_{B^+}(t, h) = \frac{r^{d-1}}{(2\pi h)^d} \int e^{\frac{i}{h}\psi_0^+(t, r, \theta, \rho, r\nu, r', \theta')} \tilde{B}^+(r, \theta, \rho, \nu, r', \theta') d\rho d\nu,$$

where $\tilde{B}^+(r, \theta, \rho, \nu, r', \theta') := B^+(r, \theta, \rho, r\nu, r', \theta')$ is compactly supported with respect to (ρ, ν), and all derivatives of $\tilde{B}^+$ are bounded on $\mathbb{R}^{3d}$. Since (2.7.9) is obvious for $0 < t < h$ (note that $r'/4 < r < 4r'$), we may assume that $h < t \le R_1 t_{\mathrm{WKB}}$. Set

$$\tilde{\psi}_0^+(t, r, \theta, \rho, \nu, r', \theta') := t^{-1} \psi_0^+(t, r, \theta, \rho, r\nu, r', \theta').$$

Choose $\chi_1 \in C_0^\infty(\mathbb{R})$ so that $\chi_1 \equiv 1$ on $(-1/2, 1/2)$ and $\operatorname{supp} \chi_1 \subset (-1, 1)$, and set

$$\tilde{B}_1 = \chi_1 \left(\frac{r - r'}{C_1 t} \right) \tilde{B}^+, \quad \tilde{B}_2 = \tilde{B} - \tilde{B}_1,$$

with some large $C_1 > 0$. Since $\partial_\rho \tilde{\psi}_0^+ = (r - r')/t - \rho - O(t_{\mathrm{WKB}})$, if $C_1 > 0$ is large enough, then we have $|\partial_\rho \tilde{\psi}_0^+| \gtrsim 1$ on $\operatorname{supp} \tilde{B}_2$. An integration by parts then implies that

$$\left| \frac{1}{(2\pi h)^d} \int e^{\frac{i}{h} t \tilde{\psi}_0^+(t, r, \theta, \rho, \nu, r', \theta')} \tilde{B}_2^+(r, \theta, \rho, \nu, r', \theta') d\rho d\nu \right| \le C_{t_{\mathrm{WKB}}} |th|^{-\frac{d}{2}}$$

for $h \le t \le R_1 t_{\mathrm{WKB}}$, where $C_{t_{\mathrm{WKB}}}$ is independent of R_1. Since

$$\partial_{\rho,\nu}^2 p(r, \theta, \rho, r\nu) = \begin{pmatrix} 1 & 0 \\ 0 & h^{jk}(\theta) + a^{jk}(r, \theta) \end{pmatrix}$$

is bounded from above and below on $\mathbb{R}^d$, we obtain

$$|\partial_\rho^k \partial_\nu^\beta \tilde{\psi}_0^+(t,r,\theta,\rho,\nu,r',\theta')| \le C \quad \text{for} \quad |k+\beta| \ge 2,$$

uniformly with respect to $(r,\theta,\rho,\omega,r',\theta') \in \mathbb{R}^{3d}$ and $0 < t \le R_1 t_{\mathrm{WKB}}$. Moreover, if $t_{\mathrm{WKB}} > 0$, which can be taken uniformly with respect to R_1, is small enough, then we have

$$|\det \partial_{\rho,\nu}^2 \tilde{\psi}_0^+(t,r,\theta,\rho,\nu,r',\theta')| \gtrsim 1, \quad (r,\theta,\rho,\omega,r',\theta') \in \mathbb{R}^{3d},\ 0 < t \le R_1 t_{\mathrm{WKB}}.$$

By a same argument as that in Section 2.5, we can apply the stationary phase theorem, and have the assertion since $r'/4 < r < 4r'$ on the support of $\tilde{B}$. □

2.8 Proof of Theorem 2.3.3

We here prove Theorem 2.3.3. We only consider the case $t \ge 0$, and the proof for the case $t \le 0$ is similar. Recall that $A_h = \sum_\kappa \mathrm{Op}_\kappa^{pr}(a_{\kappa,h})$ is a sum of properly supported h-PDO's. By (2.2.9), (2.2.11) and their adjoint estimates with respect to $\widehat{G}dz$, we have

$$\begin{cases} \|r^{-\frac{d-1}{2}} A_h\|_{\mathcal{L}(L^2(\widehat{M}),L^\infty(\widehat{M}))} + \|A_h^* r^{-\frac{d-1}{2}}\|_{\mathcal{L}(L^1(\widehat{M}),L^2(\widehat{M}))} \le Ch^{-\frac{d}{2}}, \\ \|r^{-\frac{d-1}{2}} A_h r^{\frac{d-1}{2}}\|_{\mathcal{L}(L^\infty(\widehat{M}),L^\infty(\widehat{M}))} + \|r^{\frac{d-1}{2}} A_h^* r^{-\frac{d-1}{2}}\|_{\mathcal{L}(L^1(\widehat{M}),L^1(\widehat{M}))} \le C, \end{cases} \tag{2.8.1}$$

uniformly with respect to $h \in (0,1]$.

Choose $R_0 > 0$ and $\varepsilon_0 > 0$ so that Theorem 2.5.10, 2.5.13, 2.6.1, 2.7.3 and 2.7.6 hold for all $R \ge R_0$ and $0 < \varepsilon \le \varepsilon_0$, and let $N \ge d$.

We first prove (2.3.5). A standard symbolic calculus implies that $\mathrm{Op}_\kappa(b_s^+)^*$ can be replaced by $\mathrm{Op}_\kappa(\tilde{b}_s^+)$ modulo a smoothing term $O(h^{N+1})$ on $L^2(\widehat{M})$, where $\tilde{b}_s^+ \in S_{\mathrm{sc}}(\Omega_s^+(R_2,\tilde{U}_\kappa,J,\varepsilon))$. Moreover, $\mathrm{Op}_\kappa(a_s^+) - \mathrm{Op}_\kappa^{pr}(a_s^+)$ has the $\mathcal{L}(L^2(\widehat{M}))$-norm of order h^{N+1} by (2.2.7). Therefore, Theorem 2.5.10, 2.5.13 with $R = R_2$ and (2.2.11) with $(q,s) = (\infty,(d-1)/2)$ imply that $\mathrm{Op}_\kappa(a_s^+)e^{-ith\widehat{P}}\mathrm{Op}_\kappa(b_s^+)^*$ can be bought to the form

$$U_s^+(t,h,N) + h^{N+1}R_s^+(t,h,N),$$

where $U_s^+(t,h,N)$ and $R_s^+(t,h,N)$ satisfy

$$\|r^{-\frac{d-1}{2}} U_s^+(t,h,N) r^{-\frac{d-1}{2}}\|_{\mathcal{L}(L^1(\widehat{M}),L^\infty(\widehat{M}))} \le C_N |th|^{-\frac{d}{2}},$$
$$\|R_s^+(t,h,N)\|_{\mathcal{L}(L^2(\widehat{M}))} \le C_N,$$

uniformly with respect to $h \in (0,1]$ and $0 < \pm t \le h^{-1}$. By using (2.8.1), we obtain (2.3.5) since $1 \le |th|^{-\frac{d}{2}}$ for $0 < t \le h^{-1}$.

Next, let $R_2 \ge R_0$ and $R_1 > 0$ so that $2R_1 > R_2 > R_1$. Then, there exists $t_{\mathrm{WKB}} > 0$, independent of R_2, such that Theorem 2.7.3 and 2.7.6 with $R = R_2$ hold for $0 < t \le R_2 t_{\mathrm{WKB}}/2$. By a same argument as above, we obtain (2.3.4) with $t_0 = t_{\mathrm{WKB}}/2$.

For any $t_1 > 0$, a same argument as above and Theorem 2.6.1 imply

$$\|\mathrm{Op}_\kappa(a_l^+)e^{-ith\widehat{P}}\mathrm{Op}_\kappa(b_l^+)^*\|_{\mathcal{L}(L^2(\widehat{M}))} \le C_N h^{N+1},$$

for $h \in (0,1]$ and $R_2 t_1 \le t \le h^{-1}$. Combining this estimate with (2.8.1), we obtain (2.3.6). We complete the proof. □

2.9 Proof of Theorem 2.1.2

We here give the sketch of the proof of Theorem 2.1.2. It is sufficient to prove that, for any $K \Subset M$, $\chi_K \in C_0^\infty(M)$ with $\chi_K \equiv 1$ on K and admissible pair (p,q), the following estimate holds under the nontrapping condition:

$$||\chi_K e^{-itP}u_0||_{L^p([0,1];L^q(M))} \le C||u_0||_{L^2(M)}. \qquad (2.9.1)$$

We mimic the Bouclet-Tzvetkov argument [5, Section 5 and 6].

Proposition 2.9.1. *Let $\varphi \in C_0^\infty((0,\infty))$. Then there exist $t_0 > 0$ and $C > 0$ such that for all $h \in (0,1]$ and admissible pair (p,q),*

$$||\chi_K \varphi(h^2P)e^{-itP}u_0||_{L^p([0,t_0h];L^q(M))} \le C||u_0||_{L^2(M)}. \qquad (2.9.2)$$

Proof. Let $\{\kappa : V_\kappa \to U_\kappa\}_\kappa$ be a finite atlas on K and $\{\psi_\kappa\}_\kappa \subset C_0^\infty(M)$ be a partition of unity subordinate to $\{V_\kappa\}$. Let $\tilde{\psi}_\kappa \in C_0^\infty(M)$ so that $\tilde{\psi}_\kappa \equiv 1$ on a neighborhood of $\operatorname{supp}\psi$. For a symbol $a \in C_0^\infty(U_\kappa \times \mathbb{R}^d)$, we define $\operatorname{Op}_\kappa(a)u = \kappa^*(a(z,hD_z)\kappa_*(\tilde{\psi}_\kappa u))$. Since a has a compact support with respect z and ξ, the Schur lemma implies that for $1 \le q \le r \le \infty$ and $h \in (0,1]$,

$$||a(z,hD_z)||_{\mathcal{L}(L^q(\mathbb{R}^d),L^r(\mathbb{R}^d))} \le Ch^{-d(1/q-1/r)}. \qquad (2.9.3)$$

By a same argument as that in [7, Section 2], there exist symbols $a_{\kappa,h} \in C_0^\infty(U_\kappa \times p_\kappa^{-1}(\operatorname{supp}\varphi))$ such that we can approximate $\chi_K\varphi(h^2P)$ by $\sum_\kappa \operatorname{Op}_\kappa(a_{\kappa,h})$ up to $O(h^N)$ on $L^2(M)$ for any $N \ge 0$. In particular if we obtain the following estimate:

$$||\operatorname{Op}_\kappa(b)\operatorname{Op}_\kappa(a)e^{-itP}u_0||_{L^p([0,t_0h];L^q(M))} \le C||u_0||_{L^2(\widehat{M})}, \quad h \in (0,1], \qquad (2.9.4)$$

for any $a, b \in C_0^\infty(U_\kappa \times p_\kappa^{-1}(\operatorname{supp}\varphi))$, then a same argument as that in Section 2.3 implies (2.9.2). (2.9.4) follows from (2.9.3), the TT^*-argument and the WKB parametric approximation of the propagator $e^{-ithP}\operatorname{Op}_\kappa(a)$ for $0 \le t \le t_0$ with sufficiently small $t_0 > 0$. The construction of the WKB parametrix is basically well known and its proof is similar as that in the case of elliptic operators on the Euclidean space. We refer to [35] for details. Dispersive estimates can be proved similarly to the case of compact manifolds without boundaries [7]. □

Proof of (2.9.1). The Duhamel formula implies

$$\varphi(h^2P)\chi_K e^{-itP}u_0 = e^{-itP}\varphi(h^2P)\chi_K u_0 - i\int_0^t e^{-i(t-s)P}\varphi(h^2P)[P,\chi_K]e^{-isP}u_0 ds.$$

By Theorem 2.1.1, Proposition 2.9.1 and a same argument as that in [5, Section 5 and 6], we obtain

$$\begin{aligned}
&||\varphi(h^2P)\chi_K e^{-itP}u_0||_{L^p([0,1];L^q(M))}\\
&\lesssim ||\varphi(h^2P)\chi_K u_0||_{L^2(M)} + h^{-1/2}||\varphi(h^2P)\chi_K e^{-itP}u_0||_{L^2([0,1];L^2(M))}\\
&\quad + h^{1/2}||\varphi(h^2P)[P,\chi_K]e^{-isP}u_0||_{L^2([0,1];L^2(M))}\\
&\lesssim ||\varphi(h^2P)u_0||_{L^2(M)} + h^{1/2}||u_0||_{L^2(M)} + ||\chi_K\varphi(h^2P)e^{-itP}u_0||_{L^2([0,1];H^{1/2}(M))},
\end{aligned}$$

where $H^{1/2}(M)$ is the Sobolev space with the norm $||(1+P)^{1/4}\cdot||_{L^2(M)}$. In the last line we used inequalities

$$||[\varphi(h^2P),\chi_K]||_{\mathcal{L}(L^2(M))}\lesssim h,\ ||[\varphi(h^2P),[P,\chi_K]]||_{\mathcal{L}(L^2(M))}\lesssim 1,$$
$$||[P,\chi_K]\tilde{\chi}_K\varphi(h^2P)||_{\mathcal{L}(H^{1/2}(M),H^{-1/2}(M))}\lesssim 1,$$

where $\tilde{\chi}_K\in C_0^\infty(M)$ is a cut-off function so that $\tilde{\chi}_K\equiv 1$ on $\operatorname{supp}\chi_K$. These inequalities follow from the pseudodifferential approximations of $\chi_K\varphi(h^2P)$ and $(1-\chi_K)\varphi(h^2P)$ such as Lemma 2.2.4 (see also [2]) and the symbolic calculus.

We now use the nontrapping condition of the metric. Under the nontrapping condition, Cardoso-Vodev [8] proved that for sufficiently large $\lambda>0$,

$$||\chi_K(P-\lambda\pm i0)^{-1}\chi_K||_{\mathcal{L}(L^2(M))}\lesssim\langle\lambda\rangle^{-1/2}.$$

By the abstract Kato smooth perturbation theory, this resolvent estimate implies the local smoothing effect:

$$||\chi_K e^{-itP}u_0||_{L^2([0,1];H^{1/2}(M))}\lesssim||u_0||_{L^2(M)}.$$

Using this estimate, we obtain

$$||\varphi(h^2P)\chi_K e^{-itP}u_0||_{L^p([0,1];L^q(M))}\lesssim||\varphi(h^2P)u_0||_{L^2(M)}+h^{1/2}||u_0||_{L^2(M)}$$

uniformly with respect to $h\in(0,1]$. Applying the Littlewood-Paley decomposition proved by [3], we obtain (2.9.1) and conclude the proof. □

Chapter 3

Asymptotic expansions on the line

3.1 Introduction

This chapter is concerned with dispersive estimates for the scattering solution $e^{-itH}P_{ac}u$ to Schrödinger equations

$$i\partial_t u = Hu,$$

where

$$H = -\frac{d^2}{dx^2} + V(x), \quad x \in \mathbb{R}$$

is a one-dimensional Schrödinger operator and P_{ac} is the projection onto the absolutely continuous subspace for H. We assume that $V(x)$ is a real valued potential such that $V \in L^1_1$ at least. Here L^p_γ is the weighted $L^p(\mathbb{R})$ space:

$$L^p_\gamma := \{f \mid \langle x\rangle^\gamma f \in L^p(\mathbb{R})\}, \ ||f||_{L^p_\gamma} := ||\langle x\rangle^\gamma f||_{L^p},$$

where $1 \le p \le \infty$, $\gamma \in \mathbb{R}$. Under the above conditions, H is self-adjoint on $L^2(\mathbb{R})$ with form domain $H^1(\mathbb{R})$ and the absolutely continuous spectrum of H is the half line $[0, \infty)$, the singular continuous spectrum of H is absent, and the eigenvalues of H are strictly negative.

In order to state our results, we introduce a few notations. *The Jost functions* $f_\pm(\lambda, x)$ are the solutions to the equation

$$-f''(\lambda, x) + V(x)f(\lambda, x) = \lambda^2 f(\lambda, x), \quad \lambda, x \in \mathbb{R}$$

satisfying following asymptotic conditions

$$|f_\pm(\lambda, x) - e^{\pm i\lambda x}| \to 0 \text{ as } x \to \pm\infty.$$

It is well known (see [10]) that if $V \in L^1_1$, then the Jost functions are uniquely defined for all $\lambda, x \in \mathbb{R}$. We denote by $W(\lambda)$ their Wronskian

$$W(\lambda) := f_+(\lambda, x) \cdot \partial_x f_-(\lambda, x) - \partial_x f_+(\lambda, x) \cdot f_-(\lambda, x).$$

$W(\lambda)$ is independent of x and does not vanish for $\lambda \neq 0$.

Definition 3.1.1. We say that the potential V is of generic type if $W(0) \neq 0$ and is of exceptional type if $W(0) = 0$. We also say that *zero is a resonance of* H if the potential V is of exceptional type.

We note that V is of exceptional type if and only if there exist a non trivial bounded solution to the equation $Hf = 0$. Hence the trivial potential $V \equiv 0$ is of exceptional type. Our main result is the following:

Theorem 3.1.2. *Let m be a positive integer. Suppose that $V \in L^1_{2m}$ and V is of generic type, or $V \in L^1_{2m+2}$ and V is of exceptional type. Let*

$$s = \begin{cases} 2m-1 & \text{if } V \text{ is of generic type,} \\ 2m & \text{if } V \text{ is of exceptional type.} \end{cases}$$

Then

$$||\langle x\rangle^{-s}(e^{-itH}P_{ac} - P_{m-1})u||_{L^\infty} \leq C|t|^{-\frac{1}{2}-m}||\langle x\rangle^s u||_{L^1}, \quad u \in L^1_s \cap L^2, \tag{3.1.1}$$

for all $t \neq 0$, where P_{m-1} is given by

$$P_{m-1} = \sum_{j=0}^{m-1} t^{-\frac{1}{2}-j} C_{j-1}.$$

Furthermore, the coefficients C_{j-1} satisfy the following:
(1) *If V is of generic type, then*

$$C_{-1} \equiv 0,$$
$$\operatorname{rank} C_{j-1} \leq 2j, \quad j = 1, 2, ..., m-1.$$

Moreover, we have

$$||\langle x\rangle^{-2j+1} C_{j-1} u||_{L^\infty} \leq C||\langle x\rangle^{2j-1}u||_{L^1}, \quad j = 1, 2, ..., m-1.$$

In particular, we obtain the weighted dispersive estimate:

$$||\langle x\rangle^{-1}e^{-itH}P_{ac}u||_{L^\infty} \leq C|t|^{-\frac{3}{2}}||\langle x\rangle u||_{L^1}, \quad t \neq 0.$$

(2) *If V is of exceptional type, then*

$$\operatorname{rank} C_{j-1} \leq 2j+1, \quad j = 0, 1, ..., m-1.$$

Moreover, we have

$$||\langle x\rangle^{-2j} C_{j-1} u||_{L^\infty} \leq C||\langle x\rangle^{2j}u||_{L^1}, \quad j = 0, 1, ..., m-1.$$

Remark 3.1.3. The coefficients C_{j-1} can be computed explicitly. More precisely, the integral kernel of C_{j-1} can be written in the form:

$$\frac{1}{\sqrt{4\pi i}j!(4i)^j}\left(\frac{\partial}{\partial\lambda}\right)^{2j}(T(\lambda)f_-(\lambda,x)f_+(\lambda,y))\bigg|_{\lambda=0},$$

where $f_\pm$ are the Jost functions and $T(\lambda) := \frac{-2i\lambda}{W(\lambda)}$ is the transmission coefficient (cf. Section 3.3). For example, if V is of exceptional type, then

$$C_{-1}u = \frac{1}{\sqrt{4\pi i}}\langle u, f_0\rangle f_0,$$

where f_0 is a non trivial bounded solution to the equation $Hf = 0$ normalized as

$$\lim_{x\to+\infty}\frac{1}{2}(|f_0(x)|^2 + |f_0(-x)|^2) = 1,$$

(see Section 3.4).

It is easily see that Theorem 3.1.2 implies an asymptotic expansion of $e^{-itH}P_{ac}$ in $\mathcal{L}(L^2_s, L^2_{-s})$.

Corollary 3.1.4. *Let m be a positive integer. Suppose that $V \in L^1_{2m}$ and V is of generic type, or $V \in L^1_{2m+2}$ and V is of exceptional type. Let*

$$s > \begin{cases} 2m - \frac{1}{2} & \text{if } V \text{ is of generic type,} \\ 2m + \frac{1}{2} & \text{if } V \text{ is of exceptional type.} \end{cases}$$

Then $e^{-itH}P_{ac}$ has the following asymptotic formula in $\mathcal{L}(L^2_s, L^2_{-s})$:

$$e^{-itH}P_{ac} = \sum_{j=0}^{m-1} t^{-\frac{1}{2}-j}C_{j-1} + O(t^{-\frac{1}{2}-m}), \quad t \to \infty,$$

where C_{j-1} are given by Theorem 3.1.2.

Long time dispersive estimates for Schrödinger equations with potentials have been studied by many authors. Journé-Soffer-Sogge [25] proved usual dispersive estimates:

$$||e^{-itH}P_{ac}u||_{L^\infty} \leq C|t|^{-\frac{d}{2}}||u||_{L^1}, \tag{3.1.2}$$

in dimension $d \geq 3$, under the suitable decay and regularity assumptions for V. Weder [45] proved (3.1.2) for $d = 1$ under the assumption that $V \in L^1_\gamma$ for some $\gamma > \frac{5}{2}$, or else that $V \in L^1_\gamma$, $\gamma > \frac{3}{2}$ and V is of generic type. Later, Goldberg-Schlag [20] proved (3.1.2) under the assumption that $V \in L^1_2$, or else that $V \in L^1_1$ and V is of generic type. In dimensions $d \geq 2$ dispersive estimates (3.1.2) have recently been proved under various assumptions on the potential V and the assumption that zero is neither an eigenvalue nor a resonance of H. Schlag [37] proved dispersive estimates in dimension two. In dimension three, dispersive estimates was proved by Rodnianski-Schlag [36], Goldberg-Schlag [20], Yajima [50], Goldberg [17], [18] and Vodev [42]. In higher dimension, dispersive estimates have been studied Journé-Soffer-Sogge [25], Yajima [47] and Vodev [43]. When zero is either an eigenvalue or a resonance of H, Erdoğan-Schlag [12] and Yajima [50] proved dispersive estimates in dimension three. Moreover, Erdoğan-Schlag [13] proved dispersive estimates for matrix Schrödinger operators in dimension three. On the other hand, the L^p-boundedness of wave operators, which implies the corresponding L^p-L^q estimates, have been studied by Finco-Yajima [14], Jensen-Yajima [24], Yajima [47, 48, 49, 51], Weder

[45], Artbazar-Yajima [1] and D'Ancona-Fanelli [9]. The time decay $t^{-\frac{1}{2}}$ in $d=1$ is not integrable at infinity and is unsuitable for applying to NLS. We hence are interested in a dispersive estimate whose time decay is integrable at infinity. Schlag [38] first proved the estimate (3.1.1) with $m=1$ under the assumptions $V \in L^1_4$ and V is of generic type. Goldberg [19] also proved (3.1.1) with $m=1$ under the assumptions that $V \in L^1_3$ and V is of generic type, or else that $V \in L^1_4$ and V is of exceptional type. Compared to his results, our assumptions on the potential $V(x)$, which are used in Theorem 3.1.2, are weaker. The following non self-adjoint matrix Schrödinger operators

$$\mathcal{H} = \mathcal{H}_0 + V := \begin{pmatrix} -\frac{d^2}{dx^2}+1 & 0 \\ 0 & \frac{d^2}{dx^2}-1 \end{pmatrix} + \begin{pmatrix} U & W \\ -W & -U \end{pmatrix}$$

are considered by Krieger-Schlag [28]. Here U and W are real-valued functions. Such a system arises in the study of the stability or instability of the standing wave to the NLS

$$i\partial_t u + \partial_x^2 u = -F(|u|^2)u$$

where F is a non negative function. They proved some dispersive estimates with time decay $t^{-\frac{1}{2}}$ or $t^{-\frac{3}{2}}$ for the system $\mathcal{H}$ under suitable assumptions for the potentials and the spectrum of $\mathcal{H}$ (*e.g.*, U, W and all derivatives are exponentially decaying and ± 1 are not resonances of $\mathcal{H}$). The proof for the matrix case is similar for the scalar case. We hence expect that the decay and regularity assumptions for the potentials U and W can be relaxed and similar expansions to (3.1.1) hold for the system $\mathcal{H}$, but it is not clear to the author at the moment.

On the other hand, asymptotic expansions of $e^{-itH}P_{ac}$ as $t \to \infty$ in $\mathcal{L}(L^2_s(\mathbb{R}^d), L^2_{-s}(\mathbb{R}^d))$ were proved by Jensen-Kato [23] ($d=3$), Jensen [22] ($d \geq 5$) and Murata [33] ($d \geq 1$). Here $H = -\Delta + V$ in $L^2(\mathbb{R}^d)$ with $|V(x)| \leq C\langle x\rangle^{-\sigma}$ for sufficiently large $\sigma > 0$. Compared to the result [33], our assumptions on the potential $V(x)$ are weaker and the weight in the generic case is better.

We give here the outline of the proof of Theorem 3.1.2. We may assume that $t > 0$ without loss of generality. To prove Theorem 3.1.2, we use the spectral decomposition of $e^{itH}P_{ac}$:

$$\langle e^{-itH}P_{ac}u, v\rangle = \int_0^\infty e^{-it\lambda} d\langle E_{ac}(\lambda)u, v\rangle$$

where $E_{ac}(\lambda)$ is the absolutely continuous part of the spectral measure of H. Since $d\langle E_{ac}(\lambda)u, v\rangle$ satisfies the Stone formula, namely

$$d\langle E_{ac}(\lambda)u, v\rangle = \frac{1}{2\pi i}\langle (R(\lambda + i0) - R(\lambda - i0))u, v\rangle d\lambda,$$

we obtain

$$\langle e^{-itH}P_{ac}u, v\rangle = \frac{1}{2\pi i}\int_0^\infty e^{-it\lambda}\langle (R(\lambda + i0) - R(\lambda - i0))u, v\rangle d\lambda,$$

where $R(\lambda \pm i0) := \lim_{\varepsilon \to +0}(H - (\lambda \pm i\varepsilon))^{-1}$ are the boundary value of the perturbed resolvent. In order to make the change of variables $\lambda \mapsto \lambda^2$, we define an extended resolvent as follows

$$R_\lambda := \begin{cases} R(\lambda^2 + i0), & \lambda > 0, \\ R(\lambda^2 - i0), & \lambda < 0. \end{cases} \tag{3.1.3}$$

Using this definition, we have

$$\langle e^{-itH}P_{ac}u, v\rangle = \frac{1}{\pi i}\int_{\mathbb{R}} e^{-it\lambda^2}\lambda\langle R_\lambda u, v\rangle d\lambda.$$

Let $\tilde{K}(\lambda, x, y)$ be the integral kernel of $-2i\lambda R_\lambda$. Applying the stationary phase method, we have

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{-it\lambda^2}\tilde{K}(\lambda, x, y)d\lambda = \frac{1}{\sqrt{4\pi i}}\sum_{j=0}^{m-1}\frac{t^{-\frac{1}{2}-j}}{j!(4i)^j}(\partial_\lambda^{2j}\tilde{K})(0, x, y) + t^{-\frac{1}{2}-m}\tilde{S}_m(t, \tilde{K}).$$

To estimate the remainder, we split the propagator into high and low energy parts. We prove dispersive estimates for the high energy part in Section 3.2. In Section 3.3, we study some properties of the Jost functions. We give the proof of the low energy part in Section 3.4.

3.2 The High Energy Estimates

In this section we prove a weighted dispersive estimate for the high energy part under assumptions on the potential V and weights that are employed in Theorem 3.1.2. The following proposition was essentially proved by Goldberg-Schlag [20], but we give full details of the proof.

Proposition 3.2.1. *Suppose $V \in L^1_N, N \geq 1$ and set $\lambda_0 := ||V||_{L^1_N}$. Let χ be an even smooth cut-off function such that $\chi(\lambda) = 1$ for $|\lambda| \leq \lambda_0$ and $\chi(\lambda) = 0$ for $|\lambda| \geq 2\lambda_0$. Then for $u \in L^2 \cap L^1_N$,*

$$||\langle x\rangle^{-N}e^{-itH}(1-\chi(\sqrt{H}))u||_{L^\infty} \leq Ct^{-\frac{1}{2}-N}||\langle x\rangle^N u||_{L^1},\ t > 0.$$

Proof. Set $\tilde{\chi}(\lambda) := 1 - \chi(\lambda)$. Let η be an even smooth function on $\mathbb{R}$ such that $\eta(\lambda) = 1$ if $|\lambda| \leq 1$, $\eta(\lambda) = 0$ if $|\lambda| \geq 2$ and let $\tilde{\chi}_L(\lambda) := \eta(\lambda/L)\tilde{\chi}(\lambda)$ for $L \geq 1$. We want to show that

$$\sup_{L\geq 1}\left|\langle e^{-itH}\tilde{\chi}_L(\sqrt{H})u, v\rangle\right| \leq Ct^{-\frac{1}{2}-N}||\langle x\rangle^N u||_{L^1}||\langle x\rangle^N v||_{L^1}$$

for all $t > 0$ and for any Schwartz functions u and v. We recall the Born series expansion of the resolvent $R(\lambda^2 \pm i0)$:

$$R(\lambda^2 \pm i0) = \sum_{n=0}^{\infty} R_0(\lambda^2 \pm i0)(-VR_0(\lambda^2 \pm i0))^n,\ \pm\lambda > 0, \tag{3.2.1}$$

where $R_0(\lambda^2 \pm i0) := (-\frac{d^2}{dx^2} - (\lambda^2 \pm i0))^{-1}$ is the free resolvent which has the distribution kernel

$$R_0(\lambda^2 \pm i0)(x) = -\frac{e^{i\lambda|x|}}{2i\lambda},\ \pm\lambda > 0. \tag{3.2.2}$$

Since

$$||VR_0(\lambda^2 \pm i0)||_{L^1\to L^1} \leq (2|\lambda|)^{-1}||V||_{L^1},$$

we have

$$|\langle R_0(\lambda^2 \pm i0)(-VR_0(\lambda^2 \pm i0))^n u, v\rangle| \leq \frac{||V||_{L^1}^n}{(2|\lambda|)^{n+1}}||u||_{L^1}||v||_{L^1}.$$

Hence the series (3.2.1) converges in the sense of the operator norm from L^1 to L^∞, provided $|\lambda| \geq \lambda_0$. Then using (3.1.3), (3.2.1) and (3.2.2), we can write the kernel of R_λ explicitly as

$$R_\lambda(x,y) = \sum_{n=0}^{\infty} \frac{1}{(-2i\lambda)^{n+1}} \int_{\mathbb{R}^n} e^{i\lambda(|x-x_1|+\sum_{j=2}^n |x_j-x_{j-1}|+|x_n-y|)} \prod_{j=1}^n V(x_j)dx_1...dx_n$$

which converges, provided $|\lambda| \geq \lambda_0$. Applying the above formula to the kernel of R_λ, we have

$$\langle e^{itH}\tilde{\chi}_L(\sqrt{H})u, v\rangle = \frac{1}{\pi i}\sum_{n=0}^{\infty} \frac{1}{(-2i)^{n+1}} \int_{\mathbb{R}^{n+3}} e^{-it\lambda^2+i\lambda\sum_{j=0}^n |x_{j+1}-x_j|} \frac{\tilde{\chi}_L(\lambda)}{\lambda^n}$$

$$\times \; u(x_0)\prod_{j=1}^n V(x_j)v(x_{n+1})d\lambda dx_0...dx_{n+1}.$$

In the previous equality summation and integration can be exchanged because $\tilde{\chi}_L(\lambda)$ are compactly supported. Let us consider the oscillatory integral

$$\Phi(t,a) = \int_{\mathbb{R}} e^{-it\lambda^2+ia\lambda}\frac{\tilde{\chi}_L(\lambda)}{\lambda^n}d\lambda, \; a \in \mathbb{R}.$$

Using integration by parts and the Fourier inversion formula, we have

$$\begin{aligned}\Phi(t,a) &= \frac{1}{(-2it)^N}\int_{\mathbb{R}} e^{-it\lambda^2}P_\lambda^N\left(e^{ia\lambda}\frac{\tilde{\chi}_L(\lambda)}{\lambda^n}\right)d\lambda \\ &= \frac{1}{\sqrt{4\pi i t}}\frac{1}{(-2it)^N}\int_{\mathbb{R}} e^{-\frac{|\xi|^2}{4it}}\mathcal{F}\left[P_\lambda^N\left(e^{ia\lambda}\frac{\tilde{\chi}_L(\lambda)}{\lambda^n}\right)\right](\xi)d\xi,\end{aligned}$$

where $P_\lambda = \frac{\partial}{\partial_\lambda}\frac{1}{\lambda}$ and $\mathcal{F}$ is the Fourier transform with respect to λ. Thus we obtain

$$\begin{aligned}|\Phi(t,a)| &\leq Ct^{-\frac{1}{2}-N}||\mathcal{F}P_\lambda^N(e^{ia\lambda}\tilde{\chi}_L(\lambda)\lambda^{-n})||_{L^1} \\ &\leq Ct^{-\frac{1}{2}-N}\sum_{k=0}^N |a|^{N-k}||\mathcal{F}\partial_\lambda^k(\tilde{\chi}_L(\lambda)\lambda^{-n-N})||_{L^1}.\end{aligned}$$

Since

$$\sum_{j=0}^n |x_{j+1}-x_j| \leq \prod_{j=1}^n (1+|x_j|),$$

if the estimate

$$\sup_{L\geq 1}\sup_{0\leq k\leq N} ||\mathcal{F}\partial_\lambda^k(\tilde{\chi}_L(\lambda)\lambda^{-n-N})||_{L^1} \leq C_N n^N \lambda_0^{-n-N}, \quad n \geq 0, \tag{3.2.3}$$

holds true, then we conclude that

$$\begin{aligned}&\sup_{L\geq 1}|\langle e^{itH}\tilde{\chi}_L(\sqrt{H})u, v\rangle| \\ &\leq Ct^{-\frac{1}{2}-N}\sum_{n=0}^{\infty} 2^{-n}n^N\lambda_0^{-n-N}||(1+|x|)^N V||_{L^1}^n||(1+|x|)^N u||_{L^1}||(1+|x|)^N v||_{L^1} \\ &\leq Ct^{-\frac{1}{2}-N}||\langle x\rangle^N u||_{L^1}||\langle x\rangle^N v||_{L^1}\end{aligned}$$

for all $t > 0$. We now check (3.2.3). To prove this it is sufficient to show that

$$\sup_{L\geq 1} ||\mathcal{F}(\tilde{\chi}_L(\lambda)\lambda^{-n})||_{L^1} \leq \lambda_0^{-n} \text{ for all } n \geq 0. \tag{3.2.4}$$

We note that since $\tilde{\chi}(\lambda) = 1 - \chi(\lambda)$ is not integrable, (3.2.4) with $n = 0$ hence holds in the distribution sense. Let $n = 0$, then

$$\begin{aligned} ||\mathcal{F}\tilde{\chi}_L||_{L^1} &\leq ||\mathcal{F}(\eta(\cdot/L))||_{L^1}(1 + ||\mathcal{F}\chi||_{L^1}) \\ &= ||\mathcal{F}\eta||_{L^1}(1 + ||\mathcal{F}\chi||_{L^1}) < \infty \end{aligned} \tag{3.2.5}$$

uniformly in $L \geq 1$ since $\chi \in C_0^\infty([-2\lambda_0, 2\lambda_0])$. Let $n \geq 1$, then we have

$$||\mathcal{F}(\tilde{\chi}_L(\lambda)\lambda^{-n})(\xi) \cdot \xi^2||_{L^\infty} \leq ||(\tilde{\chi}_L\lambda^{-n})''||_{L^1} \leq C\lambda_0^{-n}$$

where C is independent of n and L. Moreover ,we obtain that

$$\begin{aligned} ||\mathcal{F}(\tilde{\chi}_L(\lambda)\lambda^{-1})||_{L^\infty} &\leq ||\mathcal{F}\tilde{\chi}_L||_{L^1}||\lambda^{-1}||_{L^\infty} < \infty, \\ ||\mathcal{F}(\tilde{\chi}_L(\lambda)\lambda^{-n})||_{L^\infty} &\leq ||\tilde{\chi}_L(\lambda)\lambda^{-n}||_{L^1} \leq C\lambda_0^{-n},\ n \geq 2. \end{aligned}$$

We hence have

$$|\mathcal{F}(\tilde{\chi}_L(\lambda)\lambda^{-n})(\xi)| \leq C\lambda_0^{-n}\frac{1}{1+|\xi|^2},\ n \geq 1 \tag{3.2.6}$$

uniformly in $L \geq 1$. (3.2.4) follows from (3.2.5) and (3.2.6). □

3.3 Properties of Jost Functions

3.3.1 The Jost function

In this section, we collect results on the Jost functions $f_\pm(\lambda, x)$ needed later and improve the Fourier properties of $f_\pm$ obtained by Deift-Trubowitz [10] and D'Ancona-Fanelli [9]. Given a potential $V \in L_1^1$, *the Jost functions* $f_\pm(\lambda, x)$ are the unique solutions to the equation

$$-f''(\lambda, x) + V(x)f(\lambda, x) = \lambda^2 f(\lambda, x)$$

satisfying the asymptotic conditions

$$|f_\pm(\lambda, x) - e^{\pm i\lambda x}| \to 0 \text{ as } x \to \pm\infty. \tag{3.3.1}$$

We recall a few properties of the Jost functions. $f_\pm(\lambda, x)$ satisfies $\overline{f_\pm(\lambda, x)} = f_\pm(-\lambda, x)$. Using the Jost functions and their Wronskian, the kernel of the resolvent R_λ can be written by

$$R_\lambda(x, y) = \begin{cases} \dfrac{f_+(\lambda, y)f_-(\lambda, x)}{W(\lambda)} & \text{for } x < y, \\ \dfrac{f_-(\lambda, y)f_+(\lambda, x)}{W(\lambda)} & \text{for } x > y. \end{cases} \tag{3.3.2}$$

$f_+(\lambda, x)$ and $f_+(-\lambda, x)$ are independent for $\lambda \neq 0$ since their Wronskian

$$\begin{aligned} W[f_+(\lambda,\cdot), f_+(-\lambda,\cdot)] &:= f_+(\lambda,x)\cdot\partial_x f_+(-\lambda,x) - \partial_x f_+(\lambda,x)\cdot f_+(-\lambda,x) \\ &= \lim_{x\to+\infty}[e^{i\lambda x}(-i\lambda)e^{-i\lambda x} - i\lambda e^{i\lambda x}e^{-i\lambda x}] \\ &= -2i\lambda \neq 0. \end{aligned}$$

Similarly $W[f_-(\lambda,\cdot), f_-(-\lambda,\cdot)] = 2i\lambda$. These imply the relations

$$\begin{aligned} T(\lambda)f_-(\lambda,x) &= R_1(\lambda)f_+(\lambda,x) + f_+(-\lambda,x), \\ T(\lambda)f_+(\lambda,x) &= R_2(\lambda)f_-(\lambda,x) + f_-(-\lambda,x), \end{aligned} \tag{3.3.3}$$

where $T(\lambda)$, $R_1(\lambda)$ and $R_2(\lambda)$ are the *transmission* and *reflection* coefficients, respectively. The modified Jost functions are given by $m_\pm(\lambda,x) := e^{\mp i\lambda x}f_\pm(\lambda,x)$. The $m_\pm(\lambda,x)$ are the unique solutions to the Volterra integral equations

$$m_\pm(\lambda,x) = 1 \pm \int_x^{\pm\infty} D_\lambda(\pm(y-x))V(y)m_\pm(\lambda,y)dy,$$

where $D_\lambda(x) = \int_0^x e^{2i\lambda z}dz$. It is well known that if $V \in L_1^1$, then $m_\pm(\cdot,x) - 1$ belongs to the Hardy space $H^{2\pm}$ and if in addition $V \in L_2^1$, then $m_\pm(\lambda,x) \in C^1(\mathbb{R}^2)$ (see [10]). Moreover the following two lemmas hold (see [10], [27] and [1]).

Lemma 3.3.1. *Let $N \in \mathbb{N}$, $N \geq 2$ and suppose $V \in L_N^1$. Then $\partial_\lambda^k m_\pm(\lambda,x)$ exist for $0 \leq k \leq N-1$ and are continuous in $(\lambda,x) \in \mathbb{R}^2$. Moreover, $m_\pm(\lambda,x)$ satisfy*

$$|\partial_\lambda^k m_\pm(\lambda,x)| \leq C(1+\max(\mp x,0))^{k+1},\ (\lambda,x)\in\mathbb{R}^2, \quad 0 \leq k \leq N-1.$$

Lemma 3.3.2. *Suppose $V \in L_1^1$, at least. Then the followings hold:*
(1)

$$\begin{aligned} \frac{1}{T(\lambda)} &= \frac{W(\lambda)}{-2i\lambda} = 1 - \frac{1}{2i\lambda}\int_{\mathbb{R}} V(\sigma)m_+(\lambda,\sigma)d\sigma, \\ \frac{R_1(\lambda)}{T(\lambda)} &= \frac{W[f_-(\lambda,\cdot), f_+(-\lambda,\cdot)]}{-2i\lambda} = \frac{1}{2i\lambda}\int_{\mathbb{R}} e^{-2i\lambda\sigma}V(\sigma)m_-(\lambda,\sigma)d\sigma, \\ \frac{R_2(\lambda)}{T(\lambda)} &= \frac{W[f_-(-\lambda,\cdot), f_+(\lambda,\cdot)]}{-2i\lambda} = \frac{1}{2i\lambda}\int_{\mathbb{R}} e^{2i\lambda\sigma}V(\sigma)m_+(\lambda,\sigma)d\sigma. \end{aligned}$$

(2)

$$\begin{aligned} &|T(\lambda)|^2 + |R_j(\lambda)|^2 = 1, \quad j=1,2, \\ &R_1(\lambda)\overline{T(\lambda)} + \overline{R_2(\lambda)}T(\lambda) = 0, \\ &\overline{T(\lambda)} = T(-\lambda),\ \overline{R_j(\lambda)} = R_j(-\lambda), \quad j=1,2. \end{aligned}$$

(3) *(The generic case) Suppose that V is of generic type and $V \in L_N^1$, $N \geq 1$. Then T, R_1 and $R_2 \in C^{N-1}(\mathbb{R})$ and for $1 \leq k \leq N-1$,*

$$|\partial_\lambda^k T(\lambda)| + |\partial_\lambda^k R_1(\lambda)| + |\partial_\lambda^k R_2(\lambda)| \leq C\langle\lambda\rangle^{-1},\ \lambda\in\mathbb{R}. \tag{3.3.4}$$

Furthermore, we have

$$\begin{aligned} &T(\lambda) = \alpha\lambda + o(1), \quad \alpha \neq 0, \quad \lambda \to 0, \\ &R_1(0) = R_2(0) = -1. \end{aligned}$$

(The exceptional case) Suppose that V is of exceptional type and $V \in L^1_N$, $N \geq 2$. Then T, R_1 and $R_2 \in C^{N-2}(\mathbb{R})$ and (3.3.4) holds for $1 \leq k \leq N-2$. Furthermore, as $\lambda \to 0$, we have

$$T(\lambda) = \frac{2a}{1+a^2} + o(1),$$
$$R_1(\lambda) = \frac{1-a^2}{1+a^2} + o(1),$$
$$R_2(\lambda) = \frac{a^2-1}{1+a^2} + o(1),$$

with $a := \lim_{x \to -\infty} f_+(0,x) \neq 0$.

Since $f_\pm(\lambda, x) = e^{\pm i\lambda x} m_\pm(\lambda, x)$, Lemma 3.3.1 implies

$$|\partial_\lambda^k f_\pm(\lambda, x)| \leq C\langle x\rangle^k (1 + \max(\mp x, 0)), \ (\lambda, x) \in \mathbb{R}^2, \ 0 \leq k \leq N-1. \tag{3.3.5}$$

Using Lemma 3.3.2, we can improve the above estimates. More precisely we prove that if $\lambda \neq 0$, then $\partial_\lambda^k f_\pm(\lambda, x)$ are bounded by $\langle x\rangle^k$, at most. Furthermore if V is of exceptional type or k is an odd number, then this holds for any $\lambda \in \mathbb{R}$.

Lemma 3.3.3. (1) *Suppose that V is of generic type and $V \in L^1_N$, $N \geq 1$. Then*

$$|\partial_\lambda^k (T(\lambda) f_\pm(\lambda, x))| \leq C\langle x\rangle^k, \quad \lambda \neq 0, \ x \in \mathbb{R}, \quad 0 \leq k \leq N-1.$$

If in addition $N \geq 2$, then

$$|\partial_\lambda^k f_\pm(0, x)| \leq C\langle x\rangle^k, \quad x \in \mathbb{R},$$

for $1 \leq k \leq N-1$ and k odd.
(2) *Suppose that V is of exceptional type and $V \in L^1_N$, $N \geq 2$, then*

$$|\partial_\lambda^k f_\pm(\lambda, x)| \leq C\langle x\rangle^k, \quad (\lambda, x) \in \mathbb{R}^2, \quad 0 \leq k \leq N-2.$$

Proof. We give the proof for f_+ only and the proof for f_- is analogous. Suppose V is of generic type. For $\lambda \neq 0$, the assertion follows from (3.3.3), (3.3.5) and Lemma 3.3.2 (3). By (3.3.3),

$$-2i\lambda f_-(\lambda, x) = W(\lambda) R_1(\lambda) f_+(\lambda, x) + W(\lambda) f_+(\lambda, x), \quad \lambda \in \mathbb{R}.$$

Since $f_+(\cdot, x), W, R_1 \in C^{N-1}(\mathbb{R}_\lambda)$ and $W(0) \neq 0$, $R_1(0) = -1$, a direct computation yields

$$-2ik\partial_\lambda^{k-1} f_-(0, x) = \sum_{j=0}^{k-1} \binom{k}{j} \left(\partial_\lambda^{k-j}(WR_1)(0) + (-1)^j \partial_\lambda^{k-j} W(0)\right) \partial_\lambda^j f_+(0, x)$$
$$+ \{-1 + (-1)^k\} W(0) \partial_\lambda^k f_+(0, x),$$

for $1 \leq k \leq N-1$. Hence $\partial_\lambda^k f_+(0, x)$ is a linear combination of $f_+(0, x)$, $\partial_\lambda f_+(0, x)$,..., $\partial_\lambda^{k-1} f_+(0, x)$ and $\partial_\lambda^{k-1} f_-(0, x)$, provided k is an odd number. By induction, we can see that for any $1 \leq k \leq N-1$ and k odd, $\partial_\lambda^k f_+(0, x)$ is a linear combination of $f_\pm(0, x)$, $\partial_\lambda^2 f_\pm(0, x)$, $\partial_\lambda^4 f_\pm(0, x)$,..., and $\partial_\lambda^{k-1} f_\pm(0, x)$. This implies

$$|\partial_\lambda^k f_\pm(0, x)| \leq C\langle x\rangle^k, \ x \in \mathbb{R}.$$

We next consider the exceptional case. Suppose that V is of exceptional type. Since $f_+(\cdot, x), T$ and $R_1 \in C^{N-2}(\mathbb{R}_\lambda)$ and $T(0) \neq 0$, the assertion follows from (3.3.3), (3.3.5) and Lemma 3.3.2, immediately. □

3.3.2 Fourier properties of the Jost function

We next study Fourier properties of the Jost functions. Set

$$B_\pm(\xi,x) := \int_{\mathbb{R}} e^{2i\lambda\xi}(m_\pm(\lambda,x)-1)d\lambda.$$

Since $m_+(\lambda,x)-1 \in H^{2\pm}$, the support of $B_+(\xi,x)$ with respect to ξ is contained in the half line $[0,\infty)$. The function $B_+(\xi,x)$ satisfies the Marchenko equation:

$$B_+(\xi,x) = \int_{x+\xi}^{\infty} V(\sigma)d\sigma + \int_0^{\xi} d\zeta \int_{x+\xi-\zeta}^{\infty} V(\sigma)B_+(\zeta,\sigma)d\sigma \tag{3.3.6}$$

and $B_-(\xi,x)$ also satisfies a corresponding equation. It is well known (see [10]) that if $V \in L^1_1$, then the function $B_+(\xi,x)$ is well defined for $\xi \geq 0$, $x \in \mathbb{R}$ and satisfies the following estimates

$$|B_+(\xi,x)| \leq e^{\gamma(x)}\eta(x+\xi), \quad \xi \geq 0,\ x \in \mathbb{R}, \tag{3.3.7}$$

where $\eta(x) = \int_x^\infty |V(\sigma)|d\sigma$, $\gamma(x) = \int_x^\infty (\sigma - x)|V(\sigma)|d\sigma$. $B_-(\xi,x)$ also satisfies a similar inequalities. Moreover, we obtain the following lemma:

Lemma 3.3.4. *Let $N \in \mathbb{N}$, $N \geq 1$ and suppose $V \in L^1_N$. Then $B_\pm(\xi,x)$ satisfy the estimates*

$$||B_\pm(\cdot,x)||_{L^1_{N-1}} \leq C(1+\max(\mp x,0))^N, \quad x \in \mathbb{R}, \tag{3.3.8}$$

where C depends on $||V||_{L^1_N}$.

We set

$$n_\pm(\lambda,x) := \frac{m_\pm(\lambda,x) - m_\pm(0,x)}{\lambda},$$

and denote by $C_\pm(\xi,x)$ the Fourier transform with respect to λ of $n_\pm$:

$$C_\pm(\xi,x) = \int_{\mathbb{R}} e^{2i\lambda\xi} n_\pm(\lambda,x)d\lambda.$$

Then the following estimate holds for $C_\pm$ as well as $B_\pm$. The proof is obvious by Lemma 3.3.4 and we omit the details.

Corollary 3.3.5. *Let $N \in \mathbb{N}$, $N \geq 2$ and suppose $V \in L^1_N$. Then $C_\pm(\xi,x)$ satisfy the estimates*

$$||C_\pm(\cdot,x)||_{L^1_{N-2}} \leq C(1+\max(\mp x,0))^N, \quad x \in \mathbb{R}, \tag{3.3.9}$$

where C depends on $||V||_{L^1_N}$.

Remark 3.3.6. The estimates (3.3.8) with $N = 1,2$ and (3.3.9) with $N = 2$ were proved by D'Ancona-Fanelli [9].

Proof of Lemma 3.3.4. We prove the estimates for B_+ only and the proof for B_- is similar. We first prove the case $x \geq 0$. Since $\operatorname{supp} B_+(\cdot, x) \subset [0, \infty)$, a direct computation yields

$$
\begin{aligned}
||\xi^{N-1} B_+(\xi, x)||_{L^1(\mathbb{R}_\xi)} &\leq e^{\gamma(x)} \int_0^\infty \xi^{N-1} \int_{x+\xi}^\infty |V(\sigma)| d\sigma d\xi \\
&= \frac{1}{N} e^{\gamma(x)} \int_x^\infty (\sigma - x)^N |V(\sigma)| d\sigma \\
&\leq C e^{||V||_{L^1_1}} ||V||_{L^1_N}
\end{aligned}
$$

for all $N \geq 1$, provided $x \geq 0$. This implies (3.3.8) with $x \geq 0$. The proof for $x < 0$ is by induction on N. Multiplying the Marchenko equation (3.3.6) by ξ^{N-1} and integrating in ξ from 0 to ∞, we have

$$
\begin{aligned}
&||\xi^{N-1} B_+(\xi, x)||_{L^1(\mathbb{R}_\xi)} \\
&\leq \int_0^\infty \xi^{N-1} d\xi \int_{x+\xi}^\infty |V(\sigma)| d\sigma + \int_0^\infty \xi^{N-1} d\xi \int_0^\xi d\zeta \int_{x+\xi-\zeta}^\infty |V(\sigma)| |B_+(\zeta, \sigma)| d\sigma \\
&=: B_1 + B_2. \qquad (3.3.10)
\end{aligned}
$$

It is clear that B_1 is dominated by $C||V||_{L^1_N} \langle x \rangle^N$. Changing the order of integration of B_2, we obtain

$$
\begin{aligned}
B_2 &= \int_0^\infty d\zeta \int_x^\infty d\sigma \int_\zeta^{\zeta+\sigma-x} \xi^{N-1} |V(\sigma)| |B_+(\zeta, \sigma)| d\xi \\
&= \frac{1}{N} \sum_{k=1}^N \binom{N}{k} \int_0^\infty d\zeta \int_x^\infty \zeta^{N-k} (\sigma - x)^k |V(\sigma)| |B_+(\zeta, \sigma)| d\sigma.
\end{aligned}
$$

If $N = 1$, then

$$
B_2 \leq \int_x^\infty (\sigma - x) |V(\sigma)| ||B_+(\cdot, \sigma)||_{L^1} d\sigma.
$$

Since

$$
\int_x^0 \sigma |V(\sigma)| ||B_+(\cdot, \sigma)||_{L^1} d\sigma \leq 0,
$$

and $||B_+(\cdot, \sigma)||_{L^1}$ is bounded uniformly in $\sigma \geq 0$, we obtain

$$
\begin{aligned}
||B_+(\cdot, x)||_{L^1} &\leq C||V||_{L^1_1} \langle x \rangle + C \int_0^\infty \sigma |V(\sigma)| d\sigma + \langle x \rangle \int_x^\infty |V(\sigma)| ||B_+(\cdot, \sigma)||_{L^1} d\sigma \\
&\leq C \langle x \rangle + \langle x \rangle \int_x^\infty |V(\sigma)| ||B_+(\cdot, \sigma)||_{L^1} d\sigma.
\end{aligned}
$$

We now can apply Gronwall's lemma for $x < 0$, and have the bound (3.3.8) for $N = 1$. For $N \geq 2$, we see that

$$
\begin{aligned}
B_2 &\leq C \sum_{k=2}^N \int_x^\infty (\sigma - x)^k |V(\sigma)| ||\zeta^{N-k} B_+(\zeta, \sigma)||_{L^1(\mathbb{R}_\zeta)} d\sigma \\
&\quad + C \int_x^\infty (\sigma - x) |V(\sigma)| ||\zeta^{N-1} B_+(\zeta, \sigma)||_{L^1(\mathbb{R}_\zeta)} d\sigma \\
&=: B_{21} + B_{22}.
\end{aligned}
$$

By hypothesis for the induction and the trivial inequality

$$(\sigma - x)^k \leq |x|^k, \quad x \leq \sigma \leq 0,$$

we have

$$\begin{aligned}\int_x^0 (\sigma - x)^k |V(\sigma)| ||\zeta^{N-k} B_+(\zeta, \sigma)||_{L^1(\mathbb{R}_\zeta)} d\sigma &\leq C \int_x^0 |x|^k |V(\sigma)| \langle\sigma\rangle^{N-k+1} d\sigma \\ &\leq C||V||_{L^1_{N-k+1}} \langle x\rangle^k\end{aligned}$$

for all $2 \leq k \leq N$. By the inequality $(\sigma - x)^k \leq C(|x|^k + |\sigma|^k)$ for $0 \leq \sigma$, we also obtain

$$\int_0^\infty (\sigma - x)^k |V(\sigma)| ||\zeta^{N-k} B_+(\zeta, \sigma)||_{L^1(\mathbb{R}_\zeta)} d\sigma \leq C||V||_{L^1_k} \langle x\rangle^k, \quad 2 \leq k \leq N,$$

since $||\zeta^{N-k} B_+(\zeta, \sigma)||_{L^1(\mathbb{R}_\zeta)}$ is bounded uniformly in $\sigma \geq 0$. We thus have

$$\begin{aligned}B_{21} &\leq C \sum_{k=2}^N \left(||V||_{L^1_{N-k+1}} \langle x\rangle^k + ||V||_{L^1_k} \langle x\rangle^k \right) \\ &\leq C(||V||_{L^1_N}) \langle x\rangle^N.\end{aligned} \tag{3.3.11}$$

On the other hand, since

$$\int_x^0 \sigma |V(\sigma)| ||\zeta^{N-1} B_+(\zeta, \sigma)||_{L^1(\mathbb{R}_\zeta)} d\sigma \leq 0$$

and $||\zeta^{N-1} B_+(\zeta, \sigma)||_{L^1(\mathbb{R}_\xi)}$ is bounded uniformly in $\sigma \geq 0$, B_{22} satisfies

$$\begin{aligned}B_{22} &\leq C \int_0^\infty \sigma |V(\sigma)| ||\zeta^{N-1} B_+(\zeta, \sigma)||_{L^1(\mathbb{R}_\zeta)} d\sigma \\ &\quad - C \int_x^\infty x |V(\sigma)| ||\zeta^{N-1} B_+(\zeta, \sigma)||_{L^1(\mathbb{R}_\zeta)} d\sigma \\ &\leq C(||V||_{L^1_N}) + C\langle x\rangle \int_x^\infty |V(\sigma)| ||\zeta^{N-1} B_+(\zeta, \sigma)||_{L^1(\mathbb{R}_\zeta)} d\sigma.\end{aligned} \tag{3.3.12}$$

By (3.3.10), (3.3.11) and (3.3.12), we obtain

$$\begin{aligned}&\langle x\rangle^{-N} ||\xi^{N-1} B_\pm(\xi, x)||_{L^1(\mathbb{R}_\xi)} \\ &\leq C(||V||_{L^1_N}) + C \int_x^\infty |V(\sigma)| \langle\sigma\rangle^N \langle\sigma\rangle^{-N} ||\zeta^{N-1} B_+(\zeta, \sigma)||_{L^1(\mathbb{R}_\zeta)} d\sigma.\end{aligned}$$

Applying the Gronwall's lemma for $x \leq 0$, we have

$$\langle x\rangle^{-N} ||\xi^{N-1} B_\pm(\xi, x)||_{L^1(\mathbb{R}_\xi)} \leq C(||V||_{L^1_N}) \tag{3.3.13}$$

for $x \leq 0$, $N \geq 2$ and we conclude the proof. □

Using Lemma 3.3.4 and (3.3.4), we obtain the following Lemma which is a natural generalization of Lemma 5 in [20].

Lemma 3.3.7. *Let $\chi \in C_0^\infty(\mathbb{R})$.*
(1)*Suppose that $V \in L^1_N$, $N \geq 1$ and V is of generic type, then*

$$\mathcal{F}(\frac{\chi}{W}) \in L^1_{N-1}.$$

(2)*Suppose that $V \in L^1_N$, $N \geq 2$ and V is of exceptional type, then*

$$\mathcal{F}(\frac{\lambda\chi}{W}) \in L^1_{N-2}.$$

Here $W(\lambda)$ is the Wronskian of the Jost functions.

Proof. We first prove the generic case. By Lemma 3.3.2 (1),

$$\chi(\lambda)W(\lambda) = -2i\lambda\chi(\lambda) + \int_{\mathbb{R}} V(\sigma)\chi(\lambda)m_+(\lambda,\sigma)d\sigma. \tag{3.3.14}$$

Taking the Fourier transform with respect to λ of (3.3.14) and using the Minkowski inequality, we have

$$\begin{aligned}||\mathcal{F}(\chi W)||_{L^1_{N-1}} &\leq 2||\mathcal{F}(\lambda\chi)||_{L^1_{N-1}} + \int_{\mathbb{R}} |V(\sigma)|||\mathcal{F}(\chi m_+(\lambda,\sigma))||_{L^1_{N-1}} d\sigma \\ &\leq C||V||_{L^1_N}.\end{aligned}$$

For the last inequality, we used Lemma 3.3.4 and support property of χ. Now choosing a smooth cut-off $\tilde{\chi}$ such that $\chi\tilde{\chi} \equiv \chi$, we can realize that

$$\frac{\chi}{W} \equiv \frac{\chi}{\tilde{\chi}W}.$$

Since $\mathcal{F}\chi$ and $\mathcal{F}(\tilde{\chi}W)$ are in L^1_{N-1} and $\tilde{\chi}W$ does not vanish in $\operatorname{supp}\chi$, we can apply Wiener's lemma for $\partial_\lambda^{N-1}(\frac{\chi}{\tilde{\chi}W})$ and we conclude that $\mathcal{F}(\frac{\chi}{\tilde{\chi}W}) \in L^1_{N-1}$.

For the exceptional case, we note that V is of exceptional type *i.e.* $W(0) = 0$ if and only if

$$\int_{\mathbb{R}} V(\sigma)m_+(0,\sigma)d\sigma = 0.$$

We set

$$\begin{aligned}p(\lambda) := \frac{1}{T(\lambda)} = \frac{W(\lambda)}{-2i\lambda} &= 1 - \frac{1}{2i\lambda}\int_{\mathbb{R}} V(\sigma)m_+(\lambda,\sigma)d\sigma \\ &= 1 - \frac{1}{2i}\int_{\mathbb{R}} V(\sigma)n_+(\lambda,\sigma)d\sigma\end{aligned} \tag{3.3.15}$$

where $n_+(\lambda,x) = \frac{m_+(\lambda,x)-m_+(0,x)}{\lambda}$. Since $T(\lambda)$ is continuous on $\mathbb{R}$, $p(\lambda) \neq 0$ for all $\lambda \in \mathbb{R}$. Taking the Fourier transform with respect to λ of (3.3.15) and using the Minkowski inequality, we obtain

$$\begin{aligned}||\mathcal{F}(\chi p)||_{L^1_{N-2}} &\leq ||\mathcal{F}\chi||_{L^1_{N-2}} + \frac{1}{2}\int_{\mathbb{R}} |V(\sigma)|||\mathcal{F}(\chi n_+(\lambda,\sigma))||_{L^1_{N-2}} d\sigma \\ &\leq C||V||_{L^1_N}.\end{aligned}$$

For the last inequality, we use Corollary 3.3.5 and support property of χ. By a similar argument as in the generic case, we can see that $\frac{2i\lambda\chi}{W} \equiv \frac{\chi}{\tilde{\chi}p}$. Since $\mathcal{F}\chi$ and $\mathcal{F}(\tilde{\chi}p)$ are in L^1_{N-2} and $\tilde{\chi}p$ does not vanish in $\operatorname{supp}\chi$, we can apply Wiener's lemma and we conclude that $\mathcal{F}(\frac{\chi}{\tilde{\chi}p}) \in L^1_{N-2}$. □

Recall that $f_\pm(\lambda, x) = e^{\pm i\lambda x} m_\pm(\lambda, x)$, $T(\lambda) = -\frac{W(\lambda)}{2i\lambda}$. By the same argument as in the proof of Lemma 3.3.3, we obtain the following.

Corollary 3.3.8. *Let $\chi \in C_0^\infty(\mathbb{R})$. Suppose that $V \in L^1_N$, $N \geq 1$. Then*

$$||\mathcal{F}(\chi(\cdot)f_\pm(\cdot, x))||_{L^1_{N-1}} \leq C\langle x\rangle^{N-1}(1 + \max(\mp x, 0)), \; x \in \mathbb{R}, \tag{3.3.16}$$

Furthermore,
(1) *If V is of generic type, then*

$$||\mathcal{F}(\chi(\cdot)T(\cdot)f_\pm(\cdot, x))||_{L^1_{N-1}} \leq C\langle x\rangle^{N-1}, \; x \in \mathbb{R}.$$

(2) *If V is of exceptional type and $V \in L^1_N$, $N \geq 2$, then*

$$||\mathcal{F}(\chi(\cdot)f_\pm(\cdot, x))||_{L^1_{N-2}} \leq C\langle x\rangle^{N-2}, \; x \in \mathbb{R}.$$

3.4 The Low Energy Estimates

To complete the proof of Theorem 3.1.2, we prove the following.

Proposition 3.4.1. *Let m be a positive integer and let χ be an even smooth cut-off function such that $\chi(\lambda) = 1$ close to zero. Suppose that $V \in L^1_{2m}$ and V is of generic type, or else that $V \in L^1_{2m+2}$ and V is of exceptional type. Let*

$$s = \begin{cases} 2m - 1 & \text{if } V \text{ is of generic type,} \\ 2m & \text{if } V \text{ is of exceptional type.} \end{cases}$$

Then there exists an operator P_{m-1} such that

$$||\langle x\rangle^{-s}(e^{-itH}\chi(\sqrt{H})P_{ac} - P_{m-1})u||_{L^\infty} \leq Ct^{-\frac{1}{2}-m}||\langle x\rangle^s u||_{L^1}, \quad t > 0.$$

Moreover P_{m-1} has the following expansion:

$$P_{m-1} = \sum_{j=0}^{m-1} t^{-\frac{1}{2}-j} C_{j-1}, \; t > 0.$$

The coefficients C_{j-1} are given by

$$C_{j-1}u(x) = \frac{1}{\sqrt{4\pi i}j!(4i)^j} \int_{\mathbb{R}} (\partial_\lambda^{2j} K)(0, x, y)u(y)dy,$$

where $K(\lambda, x, y) := T(\lambda)f_+(\lambda, y)f_-(\lambda, x)$. Moreover, C_{j-1} satisfy the conditions as in Theorem 3.1.2 and Remark 3.1.3.

Proof. We first consider the generic case. Set

$$G(\lambda, x, y) := \frac{K(\lambda, x, y)}{\lambda}.$$

We start from the representation

$$\begin{aligned}\langle e^{-itH}\chi(\sqrt{H})P_{ac}u, v\rangle &= \frac{1}{\pi i}\int_{\mathbb{R}} e^{-it\lambda^2}\lambda\chi(\lambda)\langle R_\lambda u, v\rangle d\lambda \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} e^{-it\lambda^2}\lambda\chi(\lambda)\tilde{G}(\lambda, x, y)d\lambda\right)u(y)\overline{v(x)}dydx,\end{aligned}$$

where $\tilde{G}(\lambda, x, y)$ denotes the kernel of $-2iR_\lambda$ and is given by

$$\tilde{G}(\lambda, x, y) = \begin{cases} G(\lambda, x, y) & \text{for } x < y, \\ G(\lambda, y, x) & \text{for } x > y. \end{cases} \tag{3.4.1}$$

Consider the following oscillatory integral

$$\begin{aligned} I(t, G) &:= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-it\lambda^2}\lambda\chi(\lambda)G(\lambda, x, y)d\lambda \\ &= \frac{1}{4\pi it}\int_{\mathbb{R}} e^{-it\lambda^2}\partial_\lambda(\chi(\lambda)G(\lambda, x, y))d\lambda. \end{aligned} \tag{3.4.2}$$

for $x < y$. The proof for the case $x > y$ is analogous.

The case $m = 1$: It suffice to show that

$$|I(t, G)| \leq Ct^{-\frac{3}{2}}\langle x\rangle\langle y\rangle, \quad x < y. \tag{3.4.3}$$

Using the Fourier inversion formula, we obtain

$$|I(t, G)| \leq Ct^{-\frac{3}{2}}||(\mathcal{F}\partial_\lambda\chi(\cdot)G(\cdot, x, y)||_{L^1}$$

for all $t > 0$ and $x < y$, where $\mathcal{F}$ is the Fourier transform with respect to λ. By Young's inequality, Corollary 3.3.8 and Lemma 3.3.7 (1), we have

$$||(\mathcal{F}\partial_\lambda\chi G)(\cdot, x, y)||_{L^1} \leq C\langle x\rangle\langle y\rangle, \quad x < y.$$

The case $m \geq 2$: Applying the stationary phase theorem to (3.4.2), we have

$$I(t, G) = \frac{1}{\sqrt{\pi i}}\sum_{j=1}^{m-1}\frac{t^{-\frac{1}{2}-j}}{(j-1)!(4i)^j}(\partial_\lambda^{2j-1}G)(0, x, y) + t^{-\frac{1}{2}-m}S_{m-1}(t, G),$$

where the remainder satisfies

$$|S_{m-1}(t, G)| \leq C||(\mathcal{F}\partial_\lambda^{2m-1}\chi G)(\cdot, x, y)||_{L^1} \leq C\langle x\rangle^{2m-1}\langle y\rangle^{2m-1},$$

by Lemma 3.3.7 (1) and Corollary 3.3.8. Considering the fact that

$$\begin{aligned}(\partial_\lambda^{2j-1}G)(0, x, y) &= \frac{1}{2j}(\partial_\lambda^{2j}K)(0, x, y), \\ (\partial_\lambda^{2j}K)(0, x, y) &= (\partial_\lambda^{2j}K)(0, y, x), \quad x, y \in \mathbb{R},\ j = 1, 2, ..., m,\end{aligned}$$

we now define C_{j-1} and P_{m-1} by

$$\begin{aligned} C_{j-1}u(x) &:= \frac{1}{\sqrt{4\pi i}j!(4i)^j} \int_{\mathbb{R}} (\partial_\lambda^{2j} K)(0,x,y)u(y)dy, \ x \in \mathbb{R}, \\ P_{m-1} &:= \sum_{j=0}^{m-1} t^{-\frac{1}{2}-j} C_{j-1}. \end{aligned} \tag{3.4.4}$$

We then have

$$\left| \langle (e^{-itH}\chi(\sqrt{H})P_{ac} - P_{m-1})u, v \rangle \right| \le Ct^{-\frac{1}{2}-m} ||\langle x \rangle^{2m-1} u||_{L^1} ||\langle x \rangle^{2m-1} v||_{L^1}.$$

By the definition and Corollary 3.3.3 (1), we can see that $\operatorname{rank} C_{j-1} \le 2j$, and

$$||\langle x \rangle^{-2j+1} C_{j-1} u||_{L^\infty} \le C ||\langle x \rangle^{2j-1} u||_{L^1}.$$

In particular, $C_{-1} \equiv 0$. These complete the proof of the generic case.

We next consider the exceptional case. Suppose that $V \in L^1_{2m+2}$ and V is of exceptional type. By the stationary phase method, Lemma 3.3.7 (2) and Corollary 3.3.8, we have

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\lambda^2} \chi(\lambda) K(\lambda, x, y) d\lambda = \frac{1}{\sqrt{4\pi i}} \sum_{j=0}^{m-1} \frac{t^{-\frac{1}{2}-j}}{j!(4i)^j} \partial_\lambda^{2j} K(0,x,y) + t^{-\frac{1}{2}-m} S_m(t, K)$$

and

$$|S_m(t,K)| \le C ||(\mathcal{F} \partial_\lambda^{2m} K)(\cdot, x, y)||_{L^1} \le C \langle x \rangle^{2m} \langle y \rangle^{2m},$$

for $x < y$. The same argument as in proof of the generic case implies

$$||\langle x \rangle^{-2m} (e^{-itH}\chi(\sqrt{H})P_{ac} - P_{m-1})u||_{L^\infty} \le Ct^{-\frac{1}{2}-m} ||\langle x \rangle^{2m} u||_{L^1},$$

where P_{m-1} is given by (3.4.4). Since $T(0) \ne 0$, we obtain $\operatorname{rank} C_{j-1} \le 2j+1$ and

$$||\langle x \rangle^{-2j} C_{j-1} u||_{L^\infty} \le C ||\langle x \rangle^{2j} u||_{L^1}.$$

Finally, we define a function $f_0(x)$ by

$$f_+(0,x) = \sqrt{1 + \left(\frac{R_2(0)}{T(0)} + \frac{1}{T(0)} \right)^2} f_0(x),$$

where the coefficient follows from the asymptotic behavior of $f_+(0,x)$:

$$f_+(0,x) \to \begin{cases} 1 & \text{as } x \to +\infty, \\ \frac{R_2(0)}{T(0)} + \frac{1}{T(0)} & \text{as } x \to -\infty, \end{cases}$$

(see (3.3.1), (3.3.3) and (3.3.4)). $f_0(x)$ is a bounded solution to the equation $Hf = 0$ and satisfies the normalized condition

$$\lim_{x \to +\infty} (|f_0(x)|^2 + |f_0(-x)|^2) = 2,$$

and C_{-1} can be written by

$$\begin{aligned} C_{-1}u(x) &= \frac{1}{\sqrt{4\pi i}} \int T(0) f_-(0,x) f_+(0,y) u(y) dy \\ &= \frac{1}{\sqrt{4\pi i}} \int f_0(y) u(y) dy f_0(x). \end{aligned}$$

We complete the proof. □

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