Deformation of torus equivariant spectral triples

トーラス同変なスペクトラル三つ組の変形

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DEFORMATION OF TORUS EQUIVARIANT SPECTRAL TRIPLES

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ABSTRACT. We describe a way to deform spectral triples with a torus action and a deformation parameter given by a skew symmetric matrix, motivated by deformation of manifolds by Connes–Landi and Connes–Duboi-Violette. Such deformations are shown to have naturally isomorphic K-theoretic invariants independent of the deformation parameter.

CONTENTS

1. INTRODUCTION

Let M be a compact smooth Riemannian manifold endowed with a smooth action of the 2-torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$. Connes and Landi [8] defined isospectral deformations M_θ of M parametrized by a deformation parameter $\theta \in \mathbb{R}/\mathbb{Z}$ as objects in the framework of noncommutative geometry.

The algebra $C^\infty(M_\theta)$ of "smooth functions over M_θ" is, as a linear space, given by the space $C^\infty(M)$ of smooth functions over M, but endowed with a deformed product

$$f *_\theta g = e^{\pi i\theta(mn'-m'n)} fg$$

when f is a $\mathbb{T}^2$-eigenvector of weight $(m,n) \in \mathbb{Z}^2$ in $C^\infty(M)$ and g is a one of weight (m',n'). In the case where $M = \mathbb{T}^2$ and the action is given by the translation of $\mathbb{T}^2$ on itself, one obtains the well-known noncommutative torus $\mathbb{T}^2_\theta$ whose function algebra is generated by two unitaries u and v subject to the relation $uv = e^{2\pi i\theta} vu$.

They also showed that the notion of metric (or spin) geometry of M continue to make sense over the deformed space M_θ. These are given by representations of $C^\infty(M_\theta)$ on Hilbert spaces accompanied by unbounded self adjoint operators of compact resolvent which satisfy a certain regularity condition on commutators.

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The above construction was generalized to the spectral triples admitting action of an arbitrary dimensional torus and a skew symmetric matrix as the deformation parameter by Connes and Dubois-Violette [7]. The Connes–Dubois-Violette-deformation can be regarded as an adaptation of the deformation quantization for the actions of $\mathbb{R}^n$ on C^*-algebras studied by Rieffel [20,21]. The spectral triples defined this way are known to preserve several regularity of the original one. For example they become "compact quantum metric spaces" in the sense of Rieffel [22], as proved by Li [12].

It turns out that this construction can be easily generalized to any spectral triple $(\mathcal{A}, H, D)$ admitting a smooth action of $\mathbb{T}^2$ with respect to which the "Dirac type" operator D is equivariant. In such a case we obtain a deformed algebra $\mathcal{A}_\theta$ and a new spectral triple given by H and D over $\mathcal{A}_\theta$ exactly as in the case of smooth Riemannian manifolds. The C^*-algebraic closure of $\mathcal{A}_\theta$ is a particular case of the deformation considered by Rieffel [21], where such a deformation is shown to have the same K-group as the original C^*-algebra.

One problem that arises after such deformation is to compute the pairing of the Chern–Connes character (or, more general cyclic cocycles) of the new spectral triple with the K-group of the deformed algebra. In the case of noncommutative torus $\mathbb{T}^2_\theta$, Connes [3,5] made an explicit calculation of the periodic cyclic cohomology group and its pairing with the K_0-group in terms of the connections of projective modules as classified by Rieffel [18].

In this paper we show that 1) any θ-deformation has periodic cyclic cohomology groups isomorphic to the ones of the original algebra, which is compatible with the K-theory isomorphism (Corollary 13), and that 2) the possible values of the pairing between the K-group and the character of the deformed spectral remains the same (Theorem 3), somewhat generalizing the above computations for $\mathbb{T}^2_\theta$ to the general M_θ.

In the course of the proof of the invariance of Chern–Connes characters, we describe the image of the invariant cyclic cocycles under the isomorphism between periodic cyclic cohomology groups due to Elliott, Natsume, and Nest [10]. As a byproduct we obtain a simple description (Theorem 2) of the phenomenon such as

$$\left\langle \tau^{(\theta)}, K_0 C(\mathbb{T}^2_\theta) \right\rangle = \mathbb{Z} + \theta\mathbb{Z}$$

where $\tau^{(\theta)}$ denotes the gauge invariant trace on $C(\mathbb{T}^2_\theta)$. Note that the above formula is sensitive to the value of deformation parameter θ in contrast to Chern–Connes character of the equivariant spectral triples.

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2. Preliminaries

In this section we give a basic definition and related constructions of the Connes–Landi deformation of a spectral triple endowed with an action by the 2-torus. Throughout this paper we consider regular spectral triples [11,17] with an additional assumption on the smoothness of the torus action.

Let $(\mathcal{A}, H, D)$ be an even spectral triple. It means that $H = H^0 \oplus H^1$ is a graded Hilbert space, $\mathcal{A}$ is a $*$-subalgebra of the algebra of the bounded even operators $B(H^0) \oplus B(H^1)$ on H, and D is an odd unbounded self-adjoint operator on H such that $[D, a]$ is bounded and $a(1 + D^2)^{-1/2}$ is compact for any $a \in \mathcal{A}$.

Let δ denote the derivation $T \mapsto [|D|, T]$ on $B(H)$. Recall that $(\mathcal{A}, H, D)$ is said to be *regular* when $\mathcal{A} + [D, \mathcal{A}]$ is contained in $\cap_{k=1}^{\infty} \operatorname{dom} \delta^k$. Let A denote the operator norm closure of $\mathcal{A}$ in $B(H)$. The completion $\mathcal{A}_\delta$ of $\mathcal{A}$ with respect to the seminorms $\|\delta^k(-)\|$ and $\|\delta^k([D, -])\|$ for $k \in \mathbb{N} = \{0, 1, \ldots\}$ is a subalgebra of A stable under holomorphic functional calculus [17, Proposition 16].

Let σ be an action of $\mathbb{T}^2$ on $\mathcal{A}$ by $*$-automorphisms. In the following we assume that it is strongly smooth with respect to the Fréchet topology on $\mathcal{A}$, i.e. for any $a \in \mathcal{A}$, the map $t \mapsto \sigma_t(a)$ is smooth with respect to the seminorms on $\mathcal{A}$ mentioned above. We also assume that σ_t is spatially implemented on H by a strongly continuous even unitary representation $U_t \colon \mathbb{T}^2 \to \mathcal{U}(H^0) \times \mathcal{U}(H^1)$ on H satisfying

$$\sigma_t(a) = \operatorname{Ad}_{U_t}(a), \qquad \operatorname{Ad}_{U_t}(D) = D \tag{1}$$

for any $t \in \mathbb{T}^2$ and $a \in \mathcal{A}$. This implies that σ is isometric with respect to the seminorms on $\mathcal{A}$ mentioned above:

$$\|\delta^k \sigma_t(a)\| = \|\delta^k(a)\|, \qquad \|\delta^k([D, \sigma_t(a)])\| = \|\delta^k([D, a])\|. \tag{2}$$

Put $U_t^{(1)} = U_{(t,0)}$ and $U_t^{(2)} = U_{(0,t)}$. For $i = 1, 2$, let h_i denote the generator

$$h_i \xi = \lim_{t \to 0} \frac{U_t^{(i)} \xi - \xi}{t}$$

of $U_t^{(i)}$ and put $\sigma_t^{(i)} = \operatorname{Ad}_{U_t^{(i)}}|_{\mathcal{A}}$. When $a \in \mathcal{A}$ and $\alpha \in \mathbb{N}^2$, we write $[h, a]^{(\alpha)}$ for the iterated operation

$$\underbrace{[h_1, \cdots, [h_1,}_{\alpha_1 \times} \underbrace{[h_2, \cdots, [h_2,}_{\alpha_2 \times} a] \cdots]] \cdots].$$

The action Ad_U on $B(H)$ preserves A and defines an extension of σ on $\mathcal{A}$. It follows that the extended action on A is again strongly continuous because $\mathcal{A}$ is dense in A, σ is strongly continuous on $\mathcal{A}$ with respect to the operator norm, and Ad_U is isometric on A. By abuse of notation we let σ denote this action on A.

Let $\mathcal{A}^\infty$ be the subalgebra of $\mathcal{A}_\delta$ consisting of the elements a such that the map $t \mapsto \sigma_t(a)$ admit arbitrary order of higher derivatives in $\mathcal{A}_\delta$. By the strong continuity of σ and (2), the element

$$\sigma_f(a) = \int_{\mathbb{T}^2} f(t) \sigma_t(a) dt$$

is contained in $\mathcal{A}^\infty$ for any smooth function f on $\mathbb{T}^2$ and any $a \in \mathcal{A}_\delta$. The algebra $\mathcal{A}$ is contained in $\mathcal{A}^\infty$ by the smoothness assumption on σ. Moreover $\mathcal{A}^\infty$ can be characterized as the closure of $\mathcal{A}$ with respect to the seminorms

$$\nu_{k,\alpha}(a) = \left\|\delta^k([h, a]^{(\alpha)})\right\| + \left\|\delta^k([D, [h, a]^{(\alpha)}])\right\|. \tag{3}$$

for $k \in \mathbb{N}$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$.

Let $\mathcal{A}^{m,n}$ denote the closure of $\mathcal{A}^\infty$ with respect to the norm $\sum_{k \leq m, |\alpha| \leq n} \nu_{k,\alpha}$. It is the joint domain of the closures of the densely defined closable maps

$$a \mapsto \delta^k([h, a]^{(\alpha)}), \qquad a \mapsto \delta^k([D, [h, a]^{(\alpha)}])$$

on A for $k \leq m$ and $|\alpha| \leq n$.

The algebra $\mathcal{A}^{m,n}$ is closed under holomorphic functional calculus inside A. This follows from an argument analogous to the proof of [6, Lemma 6.α.2]: if $\mathcal{A}$ is unital, it is enough to show that $(1 - a)^{-1} \in \mathcal{A}$ whenever $a \in \mathcal{A}$ and $\|a\| < 1$. Under this assumption the partial sums $\sum_{n<k} a^n$ converge to $(1 - a)^{-1}$ inside A as k goes to infinity. Put $N = k + |\alpha|$ and let $\sum_{n \in \mathbb{N}} c_{N,n} x^n$ be the Taylor series expansion of

the derivative of order N of $(1-x)^{-1}$ around $x=0$. Then one has some constant D which depends only on k, α, and a satisfying

$$\left\|\delta^k([h,a^n]^{(\alpha)})\right\| \leq Dc_{N,n-N}\left\|a\right\|^{n-N}.$$

Let us indicate the computation for the case of $k=1$ and $\alpha=(1,0)$. One has

$$\begin{aligned}\delta([h_1,a^n]) = &\sum_{l=0}^{n-1} a^l\delta([h_1,a])a^{n-1-l}\\ &+\sum_{l=0}^{n-2}\sum_{m=0}^{n-2-l} a^l\delta(a)a^m[h_1,a]a^{n-2-l-m}+a^l[h_1,a]a^m\delta(a)a^{n-2-l-m},\end{aligned}$$

which implies

$$\left\|\delta([h_1,a^n])\right\| \leq (n\left\|\delta([h_1,a])\right\|\left\|a\right\|+n(n-1)\left\|\delta(a)\right\|\left\|[h_1,a]\right\|)\left\|a\right\|^{n-2}.$$

By $c_{2,n-2}=n(n-1)$, one has $\left\|\delta([h_1,a^n])\right\| \leq Dc_{2,n-2}\left\|a\right\|^{n-2}$ for the choice of

$$D=\left\|\delta([h_1,a])\right\|\left\|a\right\|+\left\|\delta(a)\right\|\left\|[h_1,a]\right\|.$$

It follows that the series $\sum_{n\in\mathbb{N}} a^n$ is absolutely convergent with respect to the seminorm $\nu_{k,\alpha}$. This implies $(1-a)^{-1}\in\mathcal{A}^{m,n}$ as required. Since $\mathcal{A}^\infty$ can be identified with $\cap_{m\in\mathbb{N},n\in\mathbb{N}}\mathcal{A}^{m,n}$, it follows that the algebra $\mathcal{A}^\infty$ is also stable under holomorphic functional calculus inside A. Nonunital case reduces to the unital case via unitization as in the proof of [17, Proposition 16].

Remark 1. When we define the algebra $\mathcal{A}^\infty$, the functions $t\mapsto\sigma_t(a)$ for $a\in\mathcal{A}^\infty$ were only assumed to have their derivatives of arbitrary order in $\mathcal{A}_\delta$ with respect to the operator norm topology. From this assumption it actually follows that there exist derivatives of $\sigma_t(a)$ with respect to the seminorms $\left\|\delta^k(-)\right\|+\left\|\delta^k([D,-])\right\|$ for $k\in\mathbb{N}$.

Indeed, for $i=1,2$, we know that $\partial_i\sigma_t(a)$ exists with respect to the operator norm, equals $[h_i,\sigma_t(a)]$, and belongs $\mathcal{A}_\delta$. For any $k\in\mathbb{N}$ the operator $\delta^k([h_i,\sigma_t(a)])$ is bounded and equals $[h_i,\sigma_t(\delta^k(a))]$. Now, one has

$$\delta^k(\sigma_t(a))=\delta^k(a)+\int_0^{t_1}\delta^k([h_1,\sigma_r^{(1)}(a)])dr+\int_0^{t_2}\delta^k([h_2,\sigma_s^{(2)}\sigma_{t_1}^{(1)}(a)])ds,$$

which shows $\partial_2\delta^k(\sigma_t(a))=\delta^k([h_2,\sigma_t(a)])$. Thus the map $\sigma_t(a)$ is differentiable by ∂_2 with respect to the seminorm $\left\|\delta^k(-)\right\|$, and that its partial derivative is equal to $[h_2,\sigma_t(a)]$. With an analogous argument one has $\partial_1\sigma_t(a)=[h_1,\sigma_t(a)]$ with respect to this seminorm.

By induction on the order $|\alpha|$ of differentiation, one obtains that the higher order derivatives of $\sigma_t(a)$ exist with respect to the seminorm $\left\|\delta^k(-)\right\|$ and agree with the ones with respect to the operator norm. The case for the seminorms $\left\|\delta^k([D,-])\right\|$ for $k\in\mathbb{N}$ is similar.

By the stability under holomorphic functional calculus, the change of algebras from $\mathcal{A}_\delta$ to $\mathcal{A}^\infty$ does not affect the K_0-group (which is isomorphic to $K_0(A)$) and they have the same K-cycle given by H and D. In the rest of the paper we assume that $\mathcal{A}=\mathcal{A}^\infty$.

The Hilbert space H decomposes into the direct sum of eigenspaces $H_{m,n}$ with weights in $\hat{\mathbb{T}}^2\simeq\mathbb{Z}^2$ characterized by

$$\xi\in H_{m,n}\Leftrightarrow U_t\xi=e^{2\pi i(mt_1+nt_2)}\xi$$

for $t=(t_1,t_2)\in\mathbb{T}^2$.

Similarly, let $B(H)_{m,n}$ denote the subspace of $B(H)$ consisting of the operators T satisfying $\mathrm{Ad}_{U_t}(T) = e^{2\pi i(mt_1+nt_2)}T$. When T is a bounded operator on H, put

$$T_{m,n} = \int_{\mathbb{T}^2} e^{-2\pi i(mt_1+nt_2)}\,\mathrm{Ad}_{U_t}(T)d\mu(t), \tag{4}$$

where μ is the normalized Haar measure on $\mathbb{T}^2$. Thus if T is any element of $B(H)$, the operator $T_{m,n}$ above belongs to $B(H)_{m,n}$. Let $B(H)_{\mathrm{fin}}$ denote the algebraic direct sum $\oplus_{(m,n)\in\mathbb{Z}^2} B(H)_{m,n}$. It is identified with the subspace of $B(H)$ consisting of the operators T where $T_{m,n} = 0$ except for finitely many $(m,n) \in \mathbb{Z}^2$. Finally put

$$\mathcal{A}_{\mathrm{fin}} = \mathcal{A} \cap B(H)_{\mathrm{fin}}, \qquad A_{\mathrm{fin}} = A \cap B(H)_{\mathrm{fin}}.$$

Definition 1. Let θ be an arbitrary real number and $(m,n) \in \mathbb{Z}^2$. Given any operator T in $B(H)_{m,n}$, we define a new bounded operator $T^{(\theta)}$ on H by

$$T^{(\theta)}\xi = e^{\pi i\theta(mn'-m'n)}T\xi$$

for $\xi \in H_{(m',n')}$. We extend this to the operators in $B(H)_{\mathrm{fin}}$ by putting $T^{(\theta)} = \sum T^{(\theta)}_{m,n}$.

Let W denote the unitary operator on H given by the scalar multiplication by $e^{2\pi i(mn'-m'n)\theta}$ on $H_{m,n}$. When $T \in B(H)_{m,n}$, one has

$$T^{(\theta)} = TW, \tag{5}$$

We also have $(T^{(\theta)})^* = (T^*)^{(\theta)}$ and

$$T^{(\theta)}S^{(\theta)} = e^{\pi i(mn'-m'n)\theta}(TS)^{(\theta)}$$

when $T \in B(H)_{m,n}$ and $S \in B(H)_{m',n'}$.

Remark 2. We adopted a presentation of $T^{(\theta)}$ which is slightly different from the one given by Connes and Landi in [8]. Let V be the unitary operator on H characterized by $V\xi = e^{\pi i\theta m'n'}\xi$ for $\xi \in H_{(m',n')}$, and ϕ be the linear transformation of $B(H)_{\mathrm{fin}}$ characterized by $\phi(T) = e^{-\pi i\theta mn}T$ when $T \in B(H)_{m,n}$. Then we have

$$V\phi(T)^{(\theta)}V^*\xi = e^{2\pi i\theta mn'}T\xi,$$

which agrees with their definition of the deformation.

Lemma 1. *Let T be a bounded operator on H. Suppose that the map $t \mapsto \mathrm{Ad}_{U_t}(T)$ admits the derivatives up to the fourth order in $B(H)$ with respect to the weak operator topology. Then the infinite sum $\sum_{(m,n)\in\mathbb{Z}^2} T^{(\theta)}_{m,n}$ is absolutely convergent to a bounded operator $T^{(\theta)}$ with respect to the operator norm.*

Proof. For $i = 1, 2$, let ∂_i denote the partial differentiation in the direction of t_i for functions $F(t_1, t_2)$ defined on $\mathbb{T}^2$. Let (m,n) be any element of $\mathbb{Z}^2$. We claim that the bounded operator $S = \partial_1 \mathrm{Ad}_{U_t}(T)|_{t=(0,0)}$ satisfies $S_{m,n} = 2\pi i m T_{m,n}$. Indeed, when $\xi \in H_{m',n'}$ and $\eta \in H_{m'',n''}$, we have

$$\frac{\left\langle \mathrm{Ad}_{U^{(1)}_s}(T)\xi, \eta\right\rangle - \langle T\xi, \eta\rangle}{s} = \frac{e^{2\pi i(m''-m')s} - 1}{s}\langle T\xi, \eta\rangle .$$

The left hand side converges to $\langle S\xi, \eta\rangle$ as $s \to 0$, while the right hand side converges to $2\pi i(m''-m')\langle T\xi, \eta\rangle$. Hence we have the equality

$$e^{-2\pi i(mt_1+nt_2)}\langle \mathrm{Ad}\, U_t(S)\xi, \eta\rangle = 2\pi i(m''-m')e^{-2\pi i(\tilde{m}t_1+\tilde{n}t_2)}\langle T\xi, \eta\rangle ,$$

where $\tilde{m} = -m - m' + m''$ and $\tilde{n} = -n - n' + n''$. This implies

$$\langle S_{m,n}\xi, \eta\rangle = \begin{cases} 2\pi i(m'' - m')\langle T\xi, \eta\rangle & (m = m'' - m', n = n'' - n') \\ 0 & (\text{otherwise}). \end{cases}$$

This agrees with the value of $2\pi i m \langle T_{m,n}\xi, \eta\rangle$. Hence the bounded operators $S_{m,n}$ and $2\pi i m T_{m,n}$ agree on the linear span of the $H_{m',n'}$ for (m', n') in H. Since this subspace is dense in H, we have established the claim.

Iterating the argument above, we obtain

$$\left((\partial_1^2 + \partial_2^2)^2 \operatorname{Ad}_{U_t}(T)\big|_{t=(0,0)} \right)_{m,n} = 16\pi^4(m^2 + n^2)^2 T_{m,n}$$

for any $(m, n) \in \mathbb{Z}^2$. Since the correspondence $T \mapsto T_{m,n}$ is a contraction, we have

$$\|T_{m,n}\| \leq \frac{1}{16\pi^4(m^2 + n^2)} \left\| (\partial_1^2 + \partial_2^2)^2 \operatorname{Ad}_{U_t}(T)\big|_{t=(0,0)} \right\|.$$

for any $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. Now, the assertion of Lemma follows from the fact that $(m^2 + n^2)^{-2}$ is summable on $\mathbb{Z}^2$ and that $\left\|T_{m,n}^{(\theta)}\right\| = \|T_{m,n}\|$ by (5). □

Lemma 2. *The subspace $\mathcal{A}_\theta \subset B(H)$ of the operators $a^{(\theta)}$ for $a \in \mathcal{A}$ is closed under multiplication.*

Proof. Let a and b be arbitrary elements of $\mathcal{A}$. We then have

$$a^{(\theta)} b^{(\theta)} = \left(\sum_{(m,n),(m',n') \in \mathbb{Z}^2} e^{\pi i \theta(mn' - m'n)} a_{m,n} b_{m',n'} \right)^{(\theta)}. \tag{6}$$

In order to show that the series in the the right hand side defines an element of $\mathcal{A} = \mathcal{A}^\infty$, we must show that it is uniformly convergent with respect to all the seminorms $\nu_{k,\alpha}$ for $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^2$.

By induction on k and $|\alpha|$, one obtains the constants $C_{l,\beta}^{k,\alpha}$ indexed by $k, l \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{N}^2$ which satisfy

$$\delta^k([h, a'b']^{(\alpha)}) = \sum_{l \leq k, \beta \leq \alpha} C_{l,\beta}^{k,\alpha} \delta^l([h, a']^{(\beta)}) \delta^{k-l}([h, b']^{(\alpha - \beta)})$$

for any elements $a', b' \in \mathcal{A}$. Let us fix $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^2$ now. Then, for any $l \leq k$ and $\beta \leq \alpha$, the functions $\sigma_t(\delta^l([h, a]^{(\beta)}))$ and $\sigma_t(\delta^l([h, b]^{(\beta)}))$ admit bounded derivatives in $t \in \mathbb{T}^2$ of order 4. Hence the infinite series

$$\sum_{(m,n) \in \mathbb{Z}^2} \delta^l([h, a_{m,n}]^{(\beta)})$$

is absolutely convergent in operator norm by Lemma 1. Consequently the infinite series

$$\sum_{(m,n),(m',n') \in \mathbb{Z}^2} \delta^l([h, a_{m,n}]^{(\beta)}) \delta^{k-l}([h, b_{m',n'}]^{(\alpha - \beta)})$$

is also absolutely convergent, which implies the convergence of

$$\sum_{(m,n),(m',n') \in \mathbb{Z}^2} e^{\pi i \theta(mn' - m'n)} \delta^l([h, a_{m,n}]^{(\beta)}) \delta^{k-l}([h, b_{m',n'}]^{(\alpha - \beta)}).$$

Combining this for all the indices $l \leq k$ and $\beta \leq \alpha$, one obtains the convergence of the right hand side of (6) with respect to the seminorm $\left\|\delta^k([h, -]^{(\alpha)})\right\|$ on $\mathcal{A}$. We also have the convergence of that with respect to the seminorm $\left\|\delta^k([D, [h, -]^{(\alpha)}])\right\|$ by a similar argument. This proves the assertion of Lemma. □

Definition 2. Let $\mathcal{A}, H, D, \sigma$ and U be as above and θ be an arbitrary real number. The algebra $\mathcal{A}_\theta$ of the operators $a^{(\theta)}$ for $a \in \mathcal{A}$ is called the *Connes–Landi deformation* of $\mathcal{A}$. The operator norm closure A_θ of $\mathcal{A}_\theta$ inside $B(H)$ is called the Connes–Landi deformation of A.

Remark 3. Let A be a C^*-algebra, σ an action of $\mathbb{R}^n$ on A, and J a skew symmetric matrix of size d. Rieffel [20] defined a deformed product

$$a \times_J b = \int \sigma_{Ju}(a)\sigma_v(b)dudv$$

on the σ-smooth part A^∞ of A by means of oscillatory integral. In our setting, the action σ of the 2-torus induces an action of $\mathbb{R}^2$ via the surjection $\mathbb{R}^2 \to \mathbb{T}^2$. For J, consider the following the 2×2 skew symmetric matrix (c.f. [20, Example 10.2])

$$J = \begin{pmatrix} 0 & -\frac{\theta}{2} \\ \frac{\theta}{2} & 0 \end{pmatrix}. \tag{7}$$

Note that $\mathcal{A}$ becomes a dense subalgebra of A^∞ under our assumption. The correspondence $a \mapsto a^{(\theta)}$ gives a representation of $(\mathcal{A}, \times_J)$ on H. Thus the C^*-algebraic closure of $(A^\infty, \times_J)$ is isomorphic to A_θ. Hence the representation of the latter on H as in Definition 2 gives a representation of $(\overline{A^\infty}, \times_J)$ on the graded Hilbert space H as even operators together with a K-cycle given by D.

In the remaining of the section we show that $(\mathcal{A}_\theta, D, \mathrm{Ad}_{U_t})$ satisfies the same conditions assumed for $(\mathcal{A}, D, \sigma_t)$. Note that the action Ad_{U_t} on $\mathcal{A}_\theta$ corresponds to σ on $\mathcal{A}$ via the isomorphism $a^{(\theta)} \leftrightarrow a$ as linear spaces.

Lemma 3. *For any $a \in \mathcal{A}$, $\theta \in \mathbb{R}$, $k \in \mathbb{N}$, and $\alpha \in \mathbb{N}^2$, the operators $\delta^k([h, a^{(\theta)}]^{(\alpha)})$ and $\delta^k([D, [h, a^{(\theta)}]^{(\alpha)}])$ are bounded.*

Proof. Applying Lemma 1 to $T = \delta^k([h, a]^{(\alpha)})$, one has the absolute convergence of $\sum_{m,n} \delta^k([h, a]^{(\alpha)})^{(\theta)}_{m,n}$. From the equality $\delta^k([h, a]^{(\alpha)})^{(\theta)}_{m,n} = \delta^k([h, a^{(\theta)}_{m,n}]^{(\alpha)})$ it follows that the densely defined operator $\delta^k([h, a^{(\theta)}]^{(\alpha)})$ with domain $\operatorname{dom}|D|^k \cap \oplus_{(m,n)\in\mathbb{Z}^2} H_{m,n}$ agrees with the restriction of $\delta^k([h, a]^{(\alpha)})^{(\theta)}$. Hence one has $a^{(\theta)} \in \operatorname{dom}\delta^k([h, -]^{(\alpha)})$. An similar argument holds for $\delta^k([D, [h, a^{(\theta)}]^{(\alpha)}])$. □

Lemma 4. *For any $a \in \mathcal{A}$ and $\theta \in \mathbb{R}$, the operator $a^{(\theta)}(1 + D^2)^{-1/2}$ on H is compact.*

Proof. Fix $(m, n) \in \mathbb{Z}^2$. Applying (5) to the compact operator $a_{m,n}(1 + D^2)^{-1/2}$, one obtains that $a^{(\theta)}_{m,n}(1 + D^2)^{-1/2}$ is compact. Then applying Lemma 1 to the operator $a(1 + D^2)^{-1/2}$, it follows that

$$a^{(\theta)}(1 + D^2)^{-1/2} = \lim_{N\to\infty} \sum_{|m|,|n|<N} a^{(\theta)}_{m,n}(1 + D^2)^{-1/2}$$

is also compact. □

Proposition 5. *The triple $(\mathcal{A}_\theta, H, D)$ is an even regular spectral triple. The action Ad_{U_t} on $\mathcal{A}_\theta$ by $\mathbb{T}^2$ is smooth and $\mathcal{A}_\theta$ is complete with respect to the seminorms $\nu_{k,\alpha}$ for $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}$.*

Proof. Lemmas 3 and 4 imply that $(\mathcal{A}_\theta, H, D)$ is a regular spectral triple and that the restriction of Ad_{U_t} on $\mathcal{A}_\theta$ is smooth.

Let $(a^{(\theta)}_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{A}_\theta$ with respect to the seminorms $\nu_{k,\alpha}$, convergent to a bounded operator S on H. Then the map $t \mapsto \mathrm{Ad}_{U_t}(S)$ is smooth on $\mathbb{T}^2$. Hence we have a bounded operator $S^{(-\theta)}$ on H by Lemma 1. It remains to show that $S^{(-\theta)} \in \mathcal{A}_\delta$, since that will imply $S = (S^{(-\theta)})^{(\theta)} \in \mathcal{A}_\theta$.

Let $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}^2$. As in the proof of Lemma 1, there is a universal constant C such that

$$\left\|\delta^m([h, (a_k^{(\theta)} - a_{k'}^{(\theta)})^{(-\theta)}]^{(\alpha)})\right\| \leq C \left\|\mathrm{Ad}_{U_t}(\delta^m([h, a_k^{(\theta)} - a_{k'}^{(\theta)})]^{(\alpha)})\right\|_{C^4(\mathbb{T}^2)}$$

for any k, k'. Thus we obtain

$$\left\|\delta^m([h, a_k - a_{k'}]^{(\alpha)})\right\| < C \max_{|\alpha'| \leq 4} \nu_{m,\alpha+\alpha'}(a_k^{(\theta)} - a_{k'}^{(\theta)}).$$

There is also a similar estimate for $\|\delta^m([D, a_k - a_{k'}])\|$. Hence the sequence $(a_k)_k$ in $\mathcal{A}$ is a Cauchy sequence for the seminorms $\nu_{k,\alpha}$, which is convergent to $S^{(-\theta)}$. This shows $S^{(-\theta)} \in \mathcal{A}_\delta$. □

Remark 4. By the completeness of $\mathcal{A}_\theta$ with respect to the seminorms $\nu_{k,\alpha}$, the newly obtained spectral triple $(\mathcal{A}_\theta, H, D)$ again satisfies $\mathcal{A}_\theta = \mathcal{A}_\theta^\infty$. Hence one can form $(\mathcal{A}_\theta)_{\theta'}$ for yet another deformation parameter θ', which is identified with $\mathcal{A}_{\theta+\theta'}$. We also have $\mathcal{A}_0 = \mathcal{A}$.

Example 1. Let $\mathcal{A}$ be the algebra $C^\infty(\mathbb{T}^2)$ of smooth functions over the 2-torus, $H = L^2(\mathbb{T}^2, \mu)^{\oplus 2}$ and

$$\not{D} = \begin{bmatrix} 0 & i\partial_1 + \partial_2 \\ i\partial_1 - \partial_2 & 0 \end{bmatrix}.$$

Consider the action of $\mathbb{T}^2$ on $\mathcal{A}$ and H given by translation. Then the spectral triple $(\mathcal{A}, H, \not{D})$ is regular and satisfies $\mathcal{A} = \mathcal{A}^\infty$. Given a real parameter θ, the corresponding regular Connes–Landi deformation $\mathcal{A}_\theta$ is precisely the algebra of Laurent series with rapid decay coefficients over the two unitaries u and v subject to the relation $uv = e^{2\pi i\theta}vu$.

3. K-theory of Connes–Landi deformation

We keep the notations $\mathcal{A}, A, \sigma, \mathcal{A}_\theta$ and A_θ of the previous section. As noted in Remark 3, A_θ is isomorphic to the deformation algebra of A defined by Rieffel [20]. In a subsequent paper [21] he showed that A_θ has the isomorphic K-group as A by means of iterated action by Euclidean spaces. In this section we elaborate a similar crossed product construction in order to show that $\mathcal{A}_\theta$ and $\mathcal{A}$ have isomorphic periodic cyclic cohomology groups and that this isomorphism is compatible with the one between the K-groups.

Let γ denote the gauge action of $\mathbb{T}^2$ on $C(\mathbb{T}^2_\theta)$ given by

$$\gamma_t(u^a v^b) = e^{2\pi i(t_1 a + t_2 b)} u^a v^b.$$

From this we obtain the diagonal product action $\sigma \otimes \gamma$ of $\mathbb{T}^2$ on the minimal tensor product $A \otimes_{\min} C(\mathbb{T}^2_\theta)$.

Proposition 6. *The algebra A_θ is isomorphic to the fixed point subalgebra of $A \otimes_{\min} C(\mathbb{T}^2_\theta)$ with respect to the diagonal action $\sigma \otimes \gamma$ of $\mathbb{T}^2$.*

Proof. When $a \in A$ is a $\mathbb{T}^2$-eigenvector of weight (m, n), the element $a \otimes u^{-m}v^{-n}$ is in the fixed point algebra $(A \otimes_{\min} C(\mathbb{T}^2_\theta))^{\sigma\otimes\gamma}$. The map

$$f\colon \mathcal{A}_\theta \cap B(H)_{\text{fin}} \to (A \otimes_{\min} C(\mathbb{T}^2_\theta))^{\sigma\otimes\gamma}, a^{(\theta)} \mapsto e^{-\pi i mn\theta} a \otimes u^{-m}v^{-n}$$

becomes a $*$-homomorphism.

Let a be any element of $\mathcal{A}$. By the estimate of Lemma 1, one has the absolute convergence of the series $\sum_{m,n} e^{-\pi i mn\theta} n a_{m,n} \otimes u^{-m}v^{-n}$ inside $(A \otimes_{\min} C(\mathbb{T}^2_\theta))^{\sigma\otimes\gamma}$. Hence f extends to a $*$-homomorphism from $\mathcal{A}_\theta$ into $(A \otimes_{\min} C(\mathbb{T}^2_\theta))^{\sigma\otimes\gamma}$ by putting

$$f(a^{(\theta)}) = \sum_{(m,n)\in\mathbb{Z}^2} e^{-\pi i mn\theta} a_{m,n} \otimes u^{-m}v^{-n}.$$

Since $\mathcal{A}_\theta$ is stable under holomorphic functional calculus in A_θ by Remark 4, the spectral radius of $a^{(\theta)}$ in $\mathcal{A}_\theta$ is the same as that in A_θ. Hence f is a contraction with respect to the operator norm and extends to a $*$-homomorphism of A_θ into $(A \otimes_{\min} C(\mathbb{T}^2_\theta))^{\sigma\otimes\gamma}$.

We first prove that f is injective by contradiction. Suppose that x is a nonzero positive element in the ideal $\ker f$ of A_θ. On one hand we have $f(x_{0,0}) = x_{0,0}\otimes 1 \neq 0$ because $x_{0,0}$ is given by $\int \sigma_t(x)d\mu(t)$ whose integrand remains nonzero positive for any t. On the other hand, f is equivariant with respect to the action Ad_{U_t} on A_θ and $\sigma\otimes 1$ on $(A\otimes_{\min} C(\mathbb{T}^2_\theta))^{\sigma\otimes\gamma}$. Hence the ideal $\ker f$ is invariant under Ad_{U_t} and $x_{(0,0)} \in \ker f$. This is a contradiction, hence f is injective.

Next we prove the surjectivity of f. Let E denote the linear span of the elements of the form $a\otimes u^{-m}v^{-n}$, where $(m,n)\in\mathbb{Z}^2$ and $a \in \mathcal{A}\cap B(H)_{m,n}$. By construction E is contained in the image of f. Since any homomorphism between C^*-algebras has a closed image, it is enough to show that E is dense in $(A\otimes_{\min} C(\mathbb{T}^2_\theta))^{\sigma\otimes\gamma}$.

Note that if a sequence $(a_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ converges to a $\mathbb{T}^2$-eigenvector a of weight (m,n) in A, so does the sequence $((a_k)_{m,n})_{k\in\mathbb{N}}$. Hence for any $\mathbb{T}^2$-eigenvector $a\in A$ of weight (m,n), the element $a\otimes u^{-m}v^{-n}$ lies in the closure of E.

Given any element x in $(A\otimes_{\min} C(\mathbb{T}^2_\theta))^{\sigma\otimes\gamma}$, the eigen-decomposition with respect to the action $\sigma\otimes 1$ gives us the $\mathbb{T}^2$-eigenvectors $(a_{m,n})_{(m,n)\in\mathbb{Z}^2}$ satisfying

$$a_{m,n}\otimes u^{-m}v^{-n} = e^{2\pi i(-mt_1-nt_2)}\int_{\mathbb{T}^2}(\sigma_t\otimes 1)(x)d\mu(t).$$

Let $c^{(k)}_{m,n}$ be the sequence of finitely supported coefficients on $\mathbb{Z}^2$ defined by

$$c^{(k)}_{m,n} = \frac{|([m,k+m]\times[n,k+n])\cap([0,k]\times[0,k])\cap\mathbb{Z}^2|}{k^2}.$$

By the Fejér kernel argument as in [2] (see also [1, Theorem 3.1]), the sequence

$$\sum_{(m,n)\in\mathbb{Z}^2} c^{(k)}_{m,n}a_{m,n}\otimes u^{-m}v^{-n}$$

for $k\in\mathbb{N}$ converges to x in norm as k goes to infinity. This shows that E is dense in $(A\otimes_{\min} C(\mathbb{T}^2_\theta))^{\sigma\otimes\gamma}$. □

Let $L^1(\mathbb{T}^2, A;\sigma)$ denote the algebra of A-valued measurable integrable functions on $\mathbb{T}^2$, endowed with the twisted convolution product $(f*g)_t = \int f_s\sigma_s(g_{t-s})ds$. For $i=1,2$, let $\widehat{\sigma^{(i)}}$ be the automorphism of $\mathbb{T}^2\ltimes_\sigma A$ which is dual to $\sigma^{(i)}$. Then the crossed product $\mathbb{Z}\ltimes_{\widehat{\sigma^{(2)}}}\mathbb{T}^2\ltimes_\sigma A$ by $\widehat{\sigma^{(2)}}$ can be described as the C^*-algebraic closure of the normed $*$-algebra of the Laurent polynomials $\sum_k f_k v^k$ where v is a unitary element and the coefficients f_k are in $L^1(\mathbb{T}^2,A;\sigma)$. The product structure is given by the twisted convolution

$$\sum_{m\in\mathbb{Z}} f_m v^m * \sum_{n\in\mathbb{Z}} g_n v^n = \sum_{m,n\in\mathbb{Z}} f_m * \widehat{\sigma^{(2)}}^m(g_n)v^{m+n}.$$

The algebra $\mathbb{Z}\ltimes_{\widehat{\sigma^{(2)}}}\mathbb{T}^2\ltimes_\sigma A$ admits the following two automorphisms of interest. The first one $\widehat{\sigma^{(1)}}$ is given by

$$\widehat{\sigma^{(1)}}(\sum_{k\in\mathbb{Z}} f_k v^k) = \sum_{k\in\mathbb{Z}}\widehat{\sigma^{(1)}}(f_k)v^k,$$

and the second one α_θ is given by

$$\alpha_\theta(\sum_{k\in\mathbb{Z}} f_k v^k) = \sum_{k\in\mathbb{Z}} e^{2\pi i k\theta} f_k v^k.$$

Let $(\widehat{\sigma^{(1)}}, \alpha_\theta)$ denote an automorphism of $\mathbb{Z} \ltimes_{\widehat{\sigma^{(2)}}} \mathbb{T}^2 \ltimes_\sigma A$ given by the composition of these two automorphisms.

Lemma 7. *The algebra A_θ is strongly Morita equivalent to $\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \alpha_\theta} \mathbb{Z} \ltimes_{\widehat{\sigma^{(2)}}} \mathbb{T}^2 \ltimes_\sigma A$.*

Proof. We know that the action $\sigma \otimes \gamma$ is saturated in the sense of Rieffel, by applying [15, Proposition 7.1.9] with $a \in A$ and $b \in C^*(\mathbb{T}^2)_\theta$ in its statement. It follows that $(A \otimes_{\min} C(\mathbb{T}^2_\theta))^{\mathbb{T}^2}$ is strongly Morita equivalent to $\mathbb{T}^2 \ltimes (A \otimes_{\min} C(\mathbb{T}^2_\theta))$ via a bimodule given as a closure of $A \otimes_{\min} C(\mathbb{T}^2_\theta)$ [15, Proposition 7.1.3]. Hence we have the strong Morita equivalence between A_θ and $\mathbb{T}^2 \ltimes (A \otimes_{\min} C(\mathbb{T}^2_\theta))$ by Proposition 6.

The latter algebra is generated by the integrals $\int dt f_t U_t$ of unitaries U_t for $t \in \mathbb{T}^2$ integrated with coefficient functions f in $L^1(\mathbb{T}^2, A \otimes_{\min} C(\mathbb{T}^2_\theta))$. By a proper bookkeeping about the integration of the U_t, we may think of that algebra as being generated by the unitaries $(U_t)_{t \in \mathbb{T}^2}$, A, and the two unitaries u and v satisfying

$$\mathrm{Ad}_{U_t}(a) = \sigma_t(a) \qquad \mathrm{Ad}_{U_t}(u^m v^n) = e^{2\pi i(mt_1 + nt_2)} u^m v^n$$
$$[a, u^m v^n] = 0 \qquad uvu^* = e^{2\pi i\theta} v$$

for any $t \in \mathbb{T}^2$ and $(m,n) \in \mathbb{Z}^2$. These relations are precisely the ones satisfied by the generators of $\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \alpha_\theta} \mathbb{Z} \ltimes_{\widehat{\sigma^{(2)}}} \mathbb{T}^2 \ltimes_\sigma A$. Hence the algebras $\mathbb{T}^2 \ltimes (A \otimes_{\min} C(\mathbb{T}^2_\theta))$ and $\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \alpha_\theta} \mathbb{Z} \ltimes_{\widehat{\sigma^{(2)}}} \mathbb{T}^2 \ltimes_\sigma A$ are isomorphic to each other. □

By abuse of notation, let $\sigma^{(2)}_\theta$ denote the automorphism of $\mathbb{T} \ltimes_{\sigma^{(1)}} A$ characterized by

$$\sigma^{(2)}_\theta(f)_t = \sigma^{(2)}_\theta(f_t) \quad (f \in L^1(\mathbb{T}, A; \sigma), t \in \mathbb{T}).$$

Then we let $(\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta)$ denote the automorphism of $\mathbb{T} \ltimes_{\sigma^{(1)}} A$ given by the composition of $\widehat{\sigma^{(1)}}$ and $\sigma^{(2)}_\theta$.

Lemma 8. *The algebra A_θ is strongly Morita equivalent to $\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta} \mathbb{T} \ltimes_{\sigma^{(1)}} A$.*

Proof. Let $\delta_{(0,\theta)}$ be the delta function at $(0,\theta) \in \mathbb{T}^2$. The integral

$$V = \int_{t \in \mathbb{T}^2} dt \delta_{(r,\theta)}(t) U_t,$$

which makes sense as an element in the multiplier algebra of $\mathbb{T}^2 \ltimes_\sigma A$, can be identified with the unitary operator $U^{(2)}_\theta$. Seen as an element in the multiplier algebra of $\mathbb{Z} \ltimes_{\widehat{\sigma^{(2)}}} \mathbb{T}^2 \ltimes_\sigma A$, it is characterized by the relations

$$V.fv^k = \sigma^{(2)}_\theta(g) v^k, \qquad f_k v^k.V = e^{2\pi k i\theta} g v^k \tag{8}$$

for any $f \in L^1(\mathbb{T}^2, A)$, where g denotes the function $(t_1, t_2) \mapsto f(t_1, t_2 - \theta)$.

Then V is $(\widehat{\sigma^{(1)}}, \alpha_\theta)$ invariant, which means that there is an isomorphism between the crossed products of $\mathbb{Z} \ltimes_{\widehat{\sigma^{(2)}}} \mathbb{T}^2 \ltimes_\sigma A$ by $\mathbb{Z}$, one associated to the automorphism $(\widehat{\sigma^{(1)}}, \alpha_\theta)$ and the other associated to $\mathrm{Ad}_V \circ (\widehat{\sigma^{(1)}}, \alpha_\theta)$.

Now, one computes

$$\mathrm{Ad}_V \circ \widehat{\sigma^{(1)}} \circ \alpha_\theta(fv^k) = \mathrm{Ad}_V(e^{2\pi i k\theta} h v^k) = \sigma^{(2)}_\theta(h) v^k,$$

where $h(t_1, t_2) = e^{2\pi i t_1} f(t_1, t_2)$. Hence the automorphism $\mathrm{Ad}_V \circ (\widehat{\sigma^{(1)}}, \alpha_\theta)$ is equal to the composition of $\widehat{\sigma^{(1)}}$ and $\sigma^{(2)}_\theta$. Thus we have an isomorphism between the crossed products as

$$\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \alpha_\theta} \mathbb{T} \ltimes_{\sigma^{(1)}} \mathbb{Z} \ltimes_{\widehat{\sigma^{(2)}}} \mathbb{T} \ltimes_{\sigma^{(2)}} A \simeq \mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta} \mathbb{T} \ltimes_{\sigma^{(1)}} \mathbb{Z} \ltimes_{\widehat{\sigma^{(2)}}} \mathbb{T} \ltimes_{\sigma^{(2)}} A.$$

The right hand side can be also expressed as

$$\mathbb{Z} \ltimes_{\widehat{\sigma^{(2)}}} \mathbb{T} \ltimes_{\sigma^{(2)}} \mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}},\sigma_\theta^{(2)}} \mathbb{T} \ltimes_{\sigma^{(1)}} A. \tag{9}$$

By Takesaki–Takai duality [23], one has the isomorphism

$$\mathcal{K} \otimes_{\min} \mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}},\sigma_\theta^{(2)}} \mathbb{T} \ltimes_{\sigma^{(1)}} A \simeq \mathbb{Z} \ltimes_{\widehat{\sigma^{(2)}}} \mathbb{T} \ltimes_{\sigma^{(2)}} \mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}},\sigma_\theta^{(2)}} \mathbb{T} \ltimes_{\sigma^{(1)}} A. \tag{10}$$

This proves the assertion of Lemma. □

Remark 5. One may avoid the use of delta function and multiplier algebra in the above proof by simply considering the automorphism Ad_V and working out the isomorphism between the crossed algebras "by hand" using the relations of (8).

Remark 6. Put $X = \mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}},\sigma_\theta^{(2)}} \mathbb{T} \ltimes_{\sigma^{(1)}} A$. The strong Morita equivalence of Lemma 8 can be described in terms of the following X-A_θ-bimodule.

By choosing a projection e of rank 1 in $\mathcal{K}$, one embeds X into the crossed product algebra of (9) by the composition of $x \mapsto e \otimes x$ and the isomorphism of (10). One may choose e so that the embedding is given by $\Psi\colon x \mapsto \sum_{n\in\mathbb{Z}} u_2^n f_n(t)$, where u_2 is is the unitary element implementing the action $\widehat{\sigma^{(2)}}$ on $\mathbb{T} \ltimes_{\sigma^{(2)}} X$, and $f_n(t)$ is the function of $\mathbb{T}$ into X defined by

$$t \mapsto \int_{\mathbb{T}} e^{2\pi i n s} \sigma_{s+t}^{(2)}(x) ds.$$

Thus, when $x \in X$ satisfies $\sigma_t^{(2)}(x) = e^{2\pi i a t} x$, one has $\Psi(x) = u_2^{-a} z_2^a x$, where $z_2^a x$ denotes the element in $\mathbb{T} \ltimes X$ represented by the function $t \mapsto e^{2\pi i a t} x$ of $\mathbb{T}$ into X.

Combining Ψ with the isomorphism of Lemma 8 between the algebra of (9) and

$$Y = \mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}},\alpha_\theta} \mathbb{Z} \ltimes_{\widehat{\sigma^{(2)}}} \mathbb{T}^2 \ltimes_\sigma A,$$

one obtains an embedding Ψ' of X into Y. Suppose that x_a is an element of A that satisfies $\sigma_t^{(2)}(x_a) = e^{2\pi i a t} x_a$. Let $z^k x_a$ denote the function of $\mathbb{T}$ into A given by the function $t \mapsto e^{2\pi i k t} x_a$, and u the implementing unitary of the action $(\widehat{\sigma^{(1)}}, \sigma_\theta^{(2)})$ on $\mathbb{T} \ltimes A$. If $x \in X$ is represented by $u^l z^k x_a$, one has

$$\Psi'(x) = e^{2\pi i a l \theta} w_2^{-a} z_2^a w_1^l z_1^k x_a. \tag{11}$$

By Proposition 6, $A \otimes_{\min} C(\mathbb{T}_\theta^2)$ has a structure of right C^*-A_θ module with an A_θ-valued inner product

$$\langle x, y \rangle = f^{-1} \left(\int_{\mathbb{T}^2} (\sigma \otimes \gamma)_t (x^* y) dt \right).$$

The completion $\mathcal{F}$ of $A \otimes_{\min} C(\mathbb{T}_\theta^2)$ with respect to this inner product becomes a Y-A_θ-bimodule by Lemma 8, which implements the strong Morita equivalence between the two algebras.

The projection e is represented as the operator $w_2^0 z_2^0$ on $A \otimes_{\min} C(\mathbb{T}_\theta^2)$. By $eYe = \Psi'(X)$, the space $\mathcal{E} = e\mathcal{F}$ has a structure of X-A_θ-bimodule via Ψ'. The operators coming from $\mathbb{Z} \ltimes \mathbb{T} \ltimes A$ on $\mathcal{E}$ are precisely the A_θ-compact operators. Hence we obtain

$$\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}},\sigma_\theta^{(2)}} \mathbb{T} \ltimes_{\sigma^{(1)}} A \simeq \mathrm{End}_{A_\theta}^0(\mathcal{E}) \simeq \mathcal{K} \otimes_{\min} A_\theta.$$

Let $\mathcal{E}_0$ be the subspace of $\mathcal{E}$ linearly spanned by the elements $w_1^n w_2^a \otimes x$ for $n, a \in \mathbb{N}$ and $x \in A_{\mathrm{fin}}$ satisfying $\sigma_t^{(2)}(x) = e^{2\pi i a t} x$. For the convenience of notation we regard $\mathcal{E}_0$ as a subspace of $\ell^2\mathbb{Z} \otimes A_{\mathrm{fin}}$ via the correspondence $w_1^n w_2^a \otimes x \mapsto \delta_n \otimes x$. Then the restriction of the A_θ-valued inner product to $\mathcal{E}_0$ is given by

$$\left\langle \delta_k \otimes a_{m,n}, \delta_j \otimes b_{(m',n')} \right\rangle = \delta_{k-m,j-m'} (a^* b)^{(\theta)}.$$

When $x \in A$ is a homogeneous element with respect to the action $\sigma^{(2)}$, and $k, j \in \mathbb{Z}$, the operator $w_1^j z_1^k x$ on $\mathcal{E}$ preserves the subspace $\mathcal{E}_0$. Its action can be described as follows:

$$a_{m,n}.\delta_k \otimes b_{(m',n')} = \delta_k \otimes ab, \tag{12}$$

$$z^j.\delta_k \otimes b_{m,n} = \begin{cases} \delta_k \otimes b & (j = -k + m), \\ 0 & (\text{otherwise}), \end{cases} \tag{13}$$

$$u.\delta_k \otimes b_{m,n} = e^{2\pi i \theta n} \delta_{k+1} \otimes b, \tag{14}$$

where $a_{m,n}, b_{m,n}$ denote any elements of A with weight (m, n) for σ, the element $z^j \in C^*\mathbb{T}$ is represented by the function $t \mapsto e^{2\pi i j t}$, and u is the generating unitary of $\mathbb{Z}$.

Let $(\widehat{\sigma^{(1)}}, \sigma^{(2)}_{\theta t})$ denote the action of $\mathbb{R}$ on $\mathbb{R} \ltimes_{\sigma^{(1)}} A$ given by

$$((\widehat{\sigma^{(1)}}, \sigma^{(2)}_{\theta t})_r f)_s = e^{2\pi r s} \sigma_{(0, \theta t_0)}(f_s).$$

Proposition 9. *The algebra A_θ is strongly Morita equivalent to $\mathbb{R} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_{\theta t}} \mathbb{R} \ltimes_{\sigma^{(1)}} A$.*

Proof. Recall that when B is a C^*-algebra and α is an automorphism of B, the mapping cone $M_\alpha B$ of α is defined to be the algebra of continuous functions from $\mathbb{R}$ to B satisfying $f(t+1) = \alpha(f(t))$ for $t \in \mathbb{R}$. It admits a natural action $\tilde{\alpha}$ by $\mathbb{R}$ called the suspension flow defined by $(\tilde{\alpha}_s f)_t = f(t+s)$. The algebras $\mathbb{R} \ltimes_{\tilde{\alpha}} M_\alpha B$ and $\mathbb{Z} \ltimes_\alpha B$ are strongly Morita equivalent.

When β is an action of $\mathbb{T}$ on a C^*-algebra B', let $\beta_\mathbb{R}$ denote the action of $\mathbb{R}$ given by β and the natural surjection $\mathbb{R} \to \mathbb{T}$. Then the crossed product $\mathbb{R} \ltimes_\beta B'$ is isomorphic to the mapping cone of the automorphism $\hat{\beta}$ of $\mathbb{T} \ltimes_\beta B'$. Moreover the dual action $\widehat{\beta_\mathbb{R}}$ of $\mathbb{R}$ on $\mathbb{R} \ltimes_{\beta_\mathbb{R}} A$ corresponds to the suspension flow.

Next, the mapping cone of $(\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta)$ on $\mathbb{T} \ltimes_{\sigma^{(1)}} A$ is isomorphic to $\mathbb{R} \ltimes_{\sigma^{(1)}} A$. Indeed, the 'untwisting map'

$$\Gamma\colon \mathbb{R} \ltimes_{\sigma^{(1)}} A \to M_{(\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta)} \mathbb{T} \ltimes_{\sigma^{(1)}} A, \quad (\Gamma f)_t = \sigma^{(2)}_{\theta t}(f_t)$$

is an isomorphism between the two algebras.

Combining the above arguments, it follows that there is an isomorphism between $\mathbb{R} \ltimes_{\sigma^{(1)}} A$ and $M_{(\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta)} \mathbb{T} \ltimes_{\sigma^{(1)}} A$ which conjugates the action $(\widehat{\sigma^{(1)}}, \sigma^{(2)}_{\theta t})$ of $\mathbb{R}$ on $\mathbb{R} \ltimes_{\sigma^{(1)}} A$ on the former and the suspension flow on the latter. Hence the crossed product $\mathbb{R} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_{\theta t}} \mathbb{R} \ltimes_{\sigma^{(1)}} A$ is isomorphic to $\mathbb{R} \ltimes M_{(\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta)}(\mathbb{T} \ltimes_{\sigma^{(1)}} A)$, which is strongly Morita equivalent to $\mathbb{Z} \ltimes_{(\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta)} \mathbb{T} \ltimes_{\sigma^{(1)}} A$. Now the assertion of Proposition follows from Lemma 8. $\square$

Corollary 10 ([21]). *The K-groups of A and A_θ are naturally isomorphic to each other.*

Proof. By Proposition 9, the K-group of A_θ can be identified with that of $\mathbb{R} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_{\theta t}} \mathbb{R} \ltimes_{\sigma^{(1)}} A$. By Connes–Thom isomorphism [4], the latter is naturally isomorphic to the K-group of $\mathbb{R} \ltimes_{\sigma^{(1)}} A$ with the charge of parity in the degree. Since this object is independent of θ, we have established the claim. $\square$

3.1. **Smooth KK-equivalence.** Now we turn to the smooth analogue of the above KK-equivalence between the Connes–Landi deformations of C^*-algebras in the previous section. The point is that the strong Morita equivalence of Lemma 8 restricts to an isomorphism between the rapid decay infinite matrix algebra with coefficients in $\mathcal{A}_\theta$ and the "smooth" crossed product $\mathbb{Z} \ltimes_{\hat{\sigma}^{(1)} \sigma^{(2)}(\theta)} \mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A}$.

We first collect the definitions of the relevant smooth algebras. Let $\mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A}$ denote the algebra given by the linear space $C^\infty(\mathbb{T}; \mathcal{A})$ of the $\mathcal{A}$-valued smooth functions endowed with the convolution product twisted by $\sigma^{(1)}$. This algebra has seminorms given by the seminorms $\nu_{k,\alpha}$ on $\mathcal{A}$ for $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^2$ and the derivations of higher order with respect to the real variable on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Next, let $\mathbb{Z} \ltimes_{\hat{\sigma}^{(1)}\sigma^{(2)}(\theta)} \mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A}$ denote the algebra of sequences $(f_n)_{n\in\mathbb{Z}}$ in $\mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A}$ which are rapid decay with respect to the aforementioned seminorms, endowed with the convolution product twisted by $(\widehat{\sigma^{(1)}}, \sigma^{(2)}(\theta))$. We have the inclusions

$$\mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A} \subset \mathbb{T} \ltimes_{\sigma^{(1)}} A, \qquad Z \ltimes_{\hat{\sigma}^{(1)}\sigma^{(2)}(\theta)} \mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A} \subset \mathbb{Z} \ltimes_{\hat{\sigma}^{(1)}\sigma^{(2)}(\theta)} \mathbb{T} \ltimes_{\sigma^{(1)}} A,$$

of these Fréchet algebras into the corresponding C^*-algebras.

Let $\mathcal{K}^\infty$ denote the algebra of the rapid decay matrices on $\mathbb{Z}$: it consists of the matrices $(a_{i,j})_{i,j\in\mathbb{Z}}$ satisfying

$$\sqrt{1+i^2+j^2}^k . a_{i,j} \to 0 \quad (i, j \to \infty)$$

for $k \in \mathbb{N}$. The algebra $\mathcal{K}^\infty$ becomes a Fréchet algebra with respect to the family of (semi-)norms

$$\mu_k((a_{i,j})_{i,j\in\mathbb{Z}}) = \max_{i,j\in\mathbb{Z}} \sqrt{1+i^2+j^2}^k \, |a_{i,j}|\,,$$

for any $k \in \mathbb{N}$. The projective tensor product $\mathcal{K}^\infty \hat{\otimes} \mathcal{A}_\theta$ is identified with the algebra of the $\mathcal{A}_\theta$-valued matrices which are rapid decay with respect to the seminorms on $\mathcal{A}_\theta$ [16, Corollary 2.4].

Proposition 11. *The Fréchet algebras $\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta} \mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A}$ and $\mathcal{K}^\infty \hat{\otimes} \mathcal{A}_\theta$ are isomorphic to each other via the correspondences*

$$\begin{aligned}(15)\quad \mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta} \mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A} \ni u^n f(t) &\mapsto k_{u^n f(t)}(l,m)_{l,m\in\mathbb{Z}} \\ &= \left(\int_{\mathbb{T}^2} e^{2\pi i (mt - (n+m-l)s)} \sigma_{(s-t,(n-l)\theta)}(f(t)) dt ds \right)^{(\theta)}\end{aligned}$$

and

$$\int_{\mathbb{T}} \sigma^{(1)}_t(a) \sum_{m\in\mathbb{Z}} e^{-2\pi i m t} u^m dt \leftarrow\!\shortmid a^{(\theta)} \in \mathcal{A}_\theta, \tag{16}$$

$$u^{l_0 - m_0} e^{-2\pi i m_0 s} \leftarrow\!\shortmid e_{l_0,m_0} = \delta_{(l_0,m_0)}(l,m) \in \mathcal{K}^\infty. \tag{17}$$

Here the terms on the left hand side denote elements in $\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta} \mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A}$ while the ones on the right hand side denote $\mathcal{A}_\theta$-valued matrices in $\mathcal{K}^\infty \hat{\otimes} \mathcal{A}_\theta$.

Proof. These formulae determine well-defined linear maps between the two algebras by the assumption on the smoothness of σ and the description of the smooth crossed products above.

On the other hand, the multiplicativity in the assertion follows from the bimodule $\mathcal{E}$ described in Remark 6. We indicate the case for (15) in the following.

We may take a "basis" $(\delta_k \otimes 1)_{k\in\mathbb{Z}}$ of $\mathcal{E}$ over the right action of $\mathcal{A}_\theta$. Suppose that $f \in C^\infty(\mathbb{T}; \mathcal{A})$ is a function of the form $t \mapsto e^{2\pi i l t} a$ for some integer l and $a \in \mathcal{A}$ of weight (m, n') for σ. On one hand we have

$$u^n f(t).\delta_k \otimes 1 = \begin{cases} e^{2\pi i \theta n n'} \delta_{k+n} \otimes a & (k = m - l) \\ 0 & (\text{otherwise}). \end{cases}$$

On the other hand we have

$$e^{2\pi i \theta n n'} \delta_{k+n} \otimes a = (\delta_{k-m+n} \otimes 1).(e^{2\pi i \theta \{(m-k-n)n' + nn'\}} a)^{(\theta)}.$$

Hence $u^n f(t)$ represents a operator which moves the $(m-l)$-th base to the $(n-l)$-th base multiplied by $e^{2\pi i\theta l n'}a^{(\theta)}$. Thus we need to show that the formula (15) applied to our choice of f gives $e^{2\pi i\theta l n'}a^{(\theta)}\delta_{n-l,m-l}(l',m')$. Now, when l' and m' are arbitrary integers,

$$\begin{aligned} k_{u^n f}(l',m') &= \int_{\mathbb{T}^2} e^{2\pi i(m't-(n+m'-l')s)}\sigma_{(s-t,(n-l')\theta)}(f(t))dtds \\ &= \int_{\mathbb{T}^2} e^{2\pi i\{m't-(n+m'-l')s+(s-t)m+(n-l')\theta n'+lt\}}adtds \\ &= \begin{cases} e^{2\pi i\theta(n-l')n'}a & (l+m'-m=0, m-n-m'+l'=0) \\ 0 & (\text{otherwise}). \end{cases} \end{aligned}$$

In the nontrivial case of $l+m'-m=0$ and $m-n-m'+l'=0$ one has $l'=n-l$ and $(n-l')n'=ln$, which is the desired relation. The rest is proved in a similar way. $\square$

There is a continuous analogue of Proposition 11. Let $\mathcal{K}^\infty_{\mathbb{R}}$ denote the algebra of the compact operators on $L^2(\mathbb{R})$ whose integral kernel belong to the Schwartz class on $\mathbb{R}^2$. The projective tensor product $\mathcal{K}^\infty_{\mathbb{R}}\hat{\otimes}\mathcal{A}_\theta$ is identified with the algebra of the $\mathcal{A}_\theta$-valued integral kernels $I(x,y)$ which satisfy

$$\max_{x,y}\left\|\delta^k((x^m+y^n)(\partial_x^{m'}+\partial_y^{n'})I(x,y))\right\|<\infty$$

for any m,n,m' and n' in $\mathbb{N}$.

Let $\mathcal{S}^*(\mathbb{R};\mathcal{A},\sigma^{(1)})$ be the convolution algebra of the $\mathcal{A}$-valued Schwartz functions with respect to the action σ. The action $(\widehat{\sigma^{(1)}},\sigma^{(2)}_{\theta t})$ restricts to a smooth action on this algebra, hence we may take the convolution algebra

$$\mathcal{S}^*(\mathbb{R};\mathcal{S}^*(\mathbb{R};\mathcal{A},\sigma^{(1)}),(\widehat{\sigma^{(1)}},\sigma^{(2)}_{\theta t}))$$

of the $\mathcal{S}^*(\mathbb{R};\mathcal{A},\sigma^{(1)})$-valued Schwartz functions. Let $\mathbb{R}\ltimes_{\widehat{\sigma^{(1)}},\sigma^{(2)}_{\theta t}}\mathbb{R}\ltimes_{\sigma^{(1)}}\mathcal{A}$ denote this algebra. We write $f(t,\tau)$ for a typical element of this algebra, where for any fixed τ, the function $t\mapsto f(t,\tau)$ is in $\mathcal{S}^*(\mathbb{R};\mathcal{A},\sigma^{(1)})$.

Proposition 12. *The Fréchet algebras $\mathbb{R}\ltimes_{\widehat{\sigma^{(1)}},\sigma^{(2)}_{\theta t}}\mathbb{R}\ltimes_{\sigma^{(1)}}\mathcal{A}$ and $\mathcal{K}^\infty_{\mathbb{R}}\hat{\otimes}\mathcal{A}_\theta$ are isomorphic. The correspondence between these two algebras are given by*

$$\mathbb{R}\ltimes_{\widehat{\sigma^{(1)}},\sigma^{(2)}_{\theta t}}\mathbb{R}\ltimes_{\sigma^{(1)}}\mathcal{A}\ni f_{t,\tau}\mapsto k_{f_{t,\tau}}(\lambda,\mu)= \left(\int_{\mathbb{R}^2} e^{2\pi i(\mu t-(\tau+\mu-\lambda)s)}\sigma_{(s-t,(\tau-\lambda)\theta)}(f_{t,\tau})dtdsd\tau\right)^{(\theta)},$$

and

$$f^k_{s,\tau}=\int_{\mathbb{R}}\sigma^{(1)}_t(k(\lambda,\mu))e^{-2\pi i(\mu-\lambda+\tau)t+\mu s}dtd\lambda d\mu \leftarrow\!\shortmid k(\lambda,\mu)\in\mathcal{K}^\infty_{\mathbb{R}}\hat{\otimes}\mathcal{A}_\theta.$$

Corollary 13. *There is a natural isomorphism $\Phi_\theta\colon \mathrm{HP}^*(\mathcal{A})\to\mathrm{HP}^*(\mathcal{A}_\theta)$ between the periodic cyclic cohomology groups which is compatible with the identification of the K-groups of Corollary 10.*

Proof. On the one hand, by a result of Elliott, Natsume, and Nest [10, Theorem 6.2], one has natural isomorphisms

$$\mathrm{HP}^*(\mathbb{R}\ltimes_{\widehat{\sigma^{(1)}},\sigma^{(2)}_{\theta t}}\mathbb{R}\ltimes_{\sigma^{(1)}}\mathcal{A})\simeq\mathrm{HP}^{*+1}(\mathbb{R}\ltimes_{\sigma^{(1)}}\mathcal{A})\simeq\mathrm{HP}^*(\mathcal{A})$$

compatible with the Connes–Thom isomorphism $K_*(\mathbb{R}\ltimes_{\widehat{\sigma^{(1)}},\sigma^{(2)}_{\theta t}}\mathbb{R}\ltimes_{\sigma^{(1)}}A)\simeq K_*(A)$. On the other hand, by Proposition 12, one has $\mathrm{HP}^*(\mathbb{R}\ltimes_{\widehat{\sigma^{(1)}},\sigma^{(2)}_{\theta t}}\mathbb{R}\ltimes_{\sigma^{(1)}}\mathcal{A})\simeq$

$\mathrm{HP}^*(\mathcal{A}_\theta)$ by the stability of HP^* for $\mathcal{K}^\infty_{\mathbb{R}} \hat{\otimes} -$ [10, Theorem 4.3]. Combining these we have the assertion. □

Remark 7. Although we need the above continuous crossed product presentation of $\mathcal{K}^\infty_{\mathbb{R}} \hat{\otimes} \mathcal{A}_\theta$ later in order to investigate the deformation of cyclic cocycles, the periodic cyclic cohomology isomorphism of Corollary 13 can be deduced from the algebra isomorphism of Proposition 11 and the Pimsner–Voiculescu type 6-term exact sequence for periodic cyclic cohomology groups of the smooth crossed products by $\mathbb{Z}$ due to Nest [14].

4. Preservation of dimension spectrum under deformation

We keep the notations $\mathcal{A}, H, D, \sigma, U$, and the assumptions about them as in the previous sections. In general the regularity of the spectral triple is not guaranteed to be preserved under a deformation as in the case of Podleś sphere studied by Neshveyev and Tuset [13]. We show that the regularity is well preserved in the case of Connes–Landi deformation.

Let $\mathcal{B}$ be the algebra generated by $\delta^k(a)$ for $a \in \mathcal{A} + [D, \mathcal{A}]$ and $k \in \mathbb{N}$. Similarly, let $\mathcal{B}_\theta$ denote the algebra generated by $\delta^k(\mathcal{A}_\theta + [D, \mathcal{A}_\theta])$ for $k \in \mathbb{N}$.

Now suppose that D is n-summable for some positive integer n. Given a bounded operator T on H, the function

$$\zeta_T(s) = \mathrm{Tr}(T\,|D|^{-s})$$

is called the *zeta function* associated to T. This function is a priori holomorphic in the region $\{z \in \mathbb{C} \mid \Re(z) > n\}$. If the functions of the form ζ_T for $T \in \mathcal{B}$ admit meromorphic extension to the whole plane $\mathbb{C}$, the dimension spectrum of $(\mathcal{A}, H, D)$ is defined to be the collection of the poles of the analytic continuation of these functions [9].

Theorem 1. *Suppose that the zeta functions ζ_T for $T \in \mathcal{B}$ admit meromorphic extensions to $\mathbb{C}$. Then the dimension spectrum of the spectral triple $(\mathcal{A}_\theta, H, D)$ is equal to that of $(\mathcal{A}, H, D)$ for any θ.*

Proof. The elements of the algebra $\mathcal{B}$ satisfy the assumption of Lemma 1. Moreover, one has $\delta^k(a)^{(\theta)} = \delta^k(a^{(\theta)})$ for any $k \in \mathbb{N}$ and $a \in \mathcal{A}$ by the proof of Lemma 3. There is also an analogous identity involving $[D, a]$ for $a \in \mathcal{A}$.

Applying Lemma 2 to the algebra $\mathcal{B}$, it follows that the set $X = \{T^{(\theta)} \mid T \in \mathcal{B}\}$ is closed under multiplication. Since X contains $\delta^k(\mathcal{A} + [D, \mathcal{A}])$ as a subspace for any $k \in \mathbb{N}$, $\mathcal{B}_\theta$ is contained in X. Repeating this argument once again for $\mathcal{A}_\theta$ and $(\mathcal{A}_\theta)_{-\theta} = \mathcal{A}$, it follows that the algebra $\mathcal{B}_\theta$ is the collection of the operators of the form $T^{(\theta)}$ for $T \in \mathcal{B}$.

Since D commutes with U_t, the function $\zeta_T(s)$ only depends on $T_{(0,0)}$ for any bounded operator T on H. By the equality $T^{(\theta)}_{(0,0)} = T_{(0,0)}$, we have $\zeta_{T^{(\theta)}}(s) = \zeta_T(s)$. This shows that the dimension spectrum of $(\mathcal{A}_\theta, H, D)$ is the same as that of $(\mathcal{A}, H, D)$. □

5. Deformation of invariant cyclic cocycles

In the rest of the paper we analyze the pairing of the cyclic cocycles and the K-groups of $\mathcal{A}_\theta$. As a consequence of the constructions so far, we obtain the Chern–Connes character of the spectral triple $(\mathcal{A}_\theta, H, D)$ when the original spectral triple $(\mathcal{A}, H, D)$ is finitely summable. These cocycles on different algebras $\mathcal{A}_\theta$ for varying value of θ can be put in the context of the cyclic cocycles which are invariant under the torus action. It turns out that there is a correspondence between the invariant cyclic cocycles on $\mathcal{A}$ and the ones on $\mathcal{A}_\theta$. We shall compare this correspondence with the Connes–Thom isomorphisms of periodic cyclic cohomology groups as in Corollary 13.

We start with a review of the isomorphism $\#_\sigma \colon \mathrm{HC}^n(\mathcal{A}) \to \mathrm{HC}^{n+1}(\mathbb{R} \ltimes_\sigma \mathcal{A})$ of [10]. Let $\mathcal{A}$ be a Fréchet $*$-algebra which is dense and closed under holomorphic functional calculus inside a C^*-algebra A. Let σ be a smooth action of $\mathbb{R}$ on $\mathcal{A}$.

Recall that closed graded traces on graded differential algebras containing $\mathcal{A}$ in the degree 0 give cyclic cocycles on $\mathcal{A}$ [5, Section II.1]. Conversely any cyclic n-cocycle can be represented as a closed graded trace on the universal differential graded algebra $\Omega(\mathcal{A}) = \oplus \Omega(\mathcal{A})_n$ (where $\Omega(\mathcal{A})_0 = \mathcal{A}$ and $\Omega(\mathcal{A})_n = \mathcal{A}^{\hat{\otimes} n} \oplus \mathcal{A}^{\hat{\otimes} n+1}$ for $n > 0$) over $\mathcal{A}$.

Suppose that Ω is a graded differential algebra containing $\mathcal{A}$ in degree 0 and ϕ is a a closed graded trace of degree n. By abuse of notation, we let ϕ denote the corresponding cyclic n-cocycle on $\mathcal{A}$. We shall denote this graded trace by ϕ by abuse of notation. We assume that the action σ extends to an action σ on Ω, and that the differentiation on Ω is equivariant with respect to σ (for example $\Omega(\mathcal{A})$ satisfies these assumptions).

Let $\mathcal{S}^*\mathbb{R}$ be the convolution algebra of the Schwartz class functions on $\mathbb{R}$ and E the direct sum $\Omega(\mathcal{S}^*\mathbb{R})_0 \oplus \Omega(\mathcal{S}^*\mathbb{R})_1$. We construct a differential graded algebra structure on $\Omega \hat{\otimes} E$ containing $\mathbb{R} \ltimes_\sigma \mathcal{A}$ in the degree 0 as follows.

First the space $\Omega \hat{\otimes} E_0$ can be identified with $\mathbb{R} \ltimes_\sigma \Omega = \mathcal{S}^*(\mathbb{R}, \Omega; \sigma)$. There is a derivation $d \colon \Omega \hat{\otimes} E_0 \to \Omega \hat{\otimes} E$ given by

$$d(\omega \otimes f) = (d\omega) \otimes f + (-1)^{\deg \omega} \omega \otimes df$$

for any homogeneous element $\omega \in \Omega$.

Next on $\Omega \hat{\otimes} E_1$, the differentiation is defined by $d(\omega \otimes f dg) = (d\omega) \otimes f dg$. The 1-forms in $\Omega(\mathcal{S}^*\mathbb{R})_1$ act on $\Omega \hat{\otimes} E$ by

$$df(\omega \otimes g) = d(f(\omega \otimes g)) - fd(\omega \otimes g).$$

These operations, together with the product structure on Ω, defines a structure of a graded differential algebra on $\Omega \hat{\otimes} E$.

Then we define a closed graded trace $\#_\sigma \phi$ on $\Omega_n \hat{\otimes} E_1 = \Omega_n \hat{\otimes} \mathcal{S}^*(\mathbb{R}) \hat{\otimes} \mathcal{S}^*(\mathbb{R})$ as follows:

$$\#_\sigma \phi(f) = 2\pi i \int_{-\infty}^{\infty} \int_0^t \phi(\sigma_s f(-t, t)) ds dt. \tag{18}$$

This correspondence of ϕ to $\#_\sigma \phi$ is compatible with the Connes–Thom isomorphism $\Phi_\sigma \colon K_* A \to K_{*+1} \mathbb{R} \ltimes_\sigma A$ [10, Theorem 6.2].

Let ϕ be a σ-invariant cyclic n-cocycle over $\mathcal{A}$, which means

$$\phi(f^0, \ldots, f^n) = \phi(\sigma_t(f^0), \ldots, \sigma_t(f^n)) \quad (\forall t \in \mathbb{R}).$$

Note that under this assumption, the right hand side of (18) reduces to

$$2\pi i \int_{-\infty}^{\infty} t \phi(f(-t, t)) dt. \tag{19}$$

There are two cyclic cocycles associated to ϕ and σ as follows.

First, let X denote the generator

$$X(a) = \lim_{t \to 0} \frac{d\sigma_t(a)}{dt}$$

of σ. We then obtain a new cyclic $n+1$-cocycle $i_X \phi$ [6, Section 3.6.β] by

$$i_X \phi(a_0 da_1 \cdots da_{n+1}) = \sum_{j=1}^{n+1} (-1)^j \phi(a_0 da_1 \cdots X(a_j) \cdots da_{n+1}).$$

Second, noting that the convolution algebra of the $\mathcal{A}$-valued rapid decay functions $\mathcal{S}^*(\mathbb{R};\mathcal{A})$ is identified with $\mathbb{R}\ltimes_\sigma\mathcal{A}$, we obtain the dual cocycle $\hat{\phi}$ over $\mathbb{R}\ltimes_\sigma\mathcal{A}$ by

$$\hat{\phi}(f^0,\dots,f^n)=\int_{\sum_{j=0}^n t_j=0}\phi(f^0_{t_0},\sigma_{t_0}(f^1_{t_1}),\dots,\sigma_{\sum_{j<n}t_j}(f^0_{t_n}))$$

for $f^j\in\mathcal{S}^*(\mathbb{R};\mathcal{A})$.

We identify $\mathrm{HP}^*(\mathbb{R}\ltimes_{\hat{\sigma}}\mathbb{R}\ltimes_\sigma\mathcal{A})$ with $\mathrm{HP}^*(\mathcal{A})$ via the Takesaki–Takai duality type isomorphism [10, Lemma 2.8 and Theorem 4.3]. Regarding these constructions one has the following generalization of [10, Proposition 3.11].

Lemma 14. *Let α and β be smooth actions of $\mathbb{R}$ on a Fréchet algebra $\mathcal{A}$. Suppose that there is an α-cocycle u^t satisfying $\beta_t=\mathrm{Ad}_{u^t}\alpha_t$. When ϕ is a cyclic cocycle on $\mathcal{A}$ which is invariant under both α and β, we have the equality*

$$[i_{X_A}\phi]=[i_{X_B}\phi]$$

in $\mathrm{HP}^*(\mathcal{A})$, where X_A (resp. X_B) is the generator of α (resp. β).

Proof. By Connes's 2×2-matrix trick, we obtain an action $\Phi(\sigma,u_*)$ of $\mathbb{R}$ on $M_2(\mathbb{R}\ltimes_{\hat{\sigma}}\mathbb{R}\ltimes_\sigma\mathcal{A})$ by

$$\Phi(\alpha,u_*)_t\left(\begin{bmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{bmatrix}\right)=\begin{bmatrix}\alpha_t(x_{11})&\alpha_t(x_{12})u_t^*\\u_t\alpha_t(x_{21})&\beta_t(x_{22})\end{bmatrix}.$$

The generator of this action can be written as

$$Y\left(\begin{bmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{bmatrix}\right)=\begin{bmatrix}X_A(x_{11})&X_A(x_{12})-x_{12}h\\hx_{21}+X_A(x_{21})&X_B(x_{22})\end{bmatrix},\tag{20}$$

where h is the generator

$$h=\lim_{t\to0}\frac{u_t-1}{t}$$

of u_t which is in the multiplier algebra of A.

The cyclic cocycle $\phi\otimes\mathrm{Tr}_{M_2}$ on $M_2(\mathcal{A})$ is invariant under $\Phi(\sigma,u_*)$. Thus the derivation Y determines a cyclic $n+1$-cocycle $\psi=i_Y(\phi\otimes\mathrm{Tr}_{M_2})$ on $M_2(\mathcal{A})$.

There are two embeddings of $\mathcal{A}$ into $M_2(\mathcal{A})$, given by

$$\Psi_1(x)=\begin{bmatrix}x&0\\0&0\end{bmatrix},\qquad\Psi_2(x)=\begin{bmatrix}0&0\\0&x\end{bmatrix}.$$

The pullback $\mathrm{HC}^*(M_2(\mathcal{A}))\to\mathrm{HC}^*(\mathcal{A})$ by these two homomorphisms in cyclic cohomology become the same map.

Finally, the pullback of ψ by Ψ_1 is $i_{X_A}\phi$ while the one by Ψ_2 is $i_{X_B}\phi$. Hence these two cocycles determine the same class in the cyclic cohomology group. □

Proposition 15. *Let ϕ be a σ-invariant n-cocycle over $\mathcal{A}$. The class of the cyclic $(n+1)$-cocycle $i_X\phi$ in $\mathrm{HC}^{n+1}(\mathcal{A})$ agrees with that of $\#_{\hat{\sigma}}\hat{\phi}$.*

Proof. Let $b_0,\dots,b_{n+1}$ be elements of $\mathbb{R}\ltimes_\sigma\mathcal{A}$ and $f_0,\dots,f_{n+1}$ be functions in $\mathcal{S}^*\mathbb{R}$. For each $0\le j\le n+1$, $b_j\otimes f_j$ represents the element in $\mathcal{A}\hat{\otimes}E$ which correspond to $t\mapsto b_j\widehat{f_j}(t)\in\mathcal{S}(\mathbb{R};\mathcal{A})$. We then consider the element

$$\omega=(b_0\otimes f_0)d(b_1\otimes f_1)\cdots d(b_{n+1}\otimes f_{n+1})$$

of $\Omega\hat{\otimes}E$.

Each term in the above formula can be expanded as $d(b_j\otimes f_j)=db_j\otimes f_j+b_j\otimes df_j$. In the product of these, the terms containing more than one df_j's vanish. Moreover the value of $\#_{\hat{\sigma}}\hat{\phi}$ depends only on the components of ω in $\Omega_n \hat{\otimes} E_1$. Hence $\#_{\hat{\sigma}}\hat{\phi}(\omega)$ can be expressed as

$$\sum_{j=1}^{n+1} 2\pi i \int_{s_{n+1}=0} t_j \hat{\phi}(\eta_j(t_0, \dots, t_{n+1}))\xi_j(t_0, \dots, t_{n+1}) dt_0 \cdots dt_n, \tag{21}$$

where

$$\begin{aligned} \eta_j(t_0, \dots, t_{n+1}) &= b_0 \hat{\sigma}_{s_0}(db_1) \cdots \hat{\sigma}_{s_{j-1}}(b_j) \cdots \hat{\sigma}_{s_n}(db_{n+1}), \\ \xi_j(t_0, \dots, t_{n+1}) &= f_0(t_0) \cdots \check{f}_j \cdots f_{n+1}(t_{n+1}) df_j(t_j), \\ s_k &= t_0 + \cdots + t_k. \end{aligned}$$

Note that the derivation $\tilde{X}f(t) = tf(t)$ on $\mathcal{S}^*(\mathbb{R}; \mathbb{R} \ltimes \mathcal{A})$ is the generator of the double dual action $\hat{\hat{\sigma}}\colon \mathbb{R} \curvearrowright \mathbb{R} \ltimes_{\hat{\sigma}} \mathbb{R} \ltimes_\sigma \mathcal{A}$. The j-th term of (21) is equal to

$$\hat{\hat{\phi}}(b_0 \otimes f_0) d(b_1 \otimes f_1) \cdots \tilde{X}(b_j \otimes f_j) \cdots d(b_{n+1} \otimes f_{n+1}).$$

Collecting these terms we obtain the equality

$$\#_\sigma \hat{\phi}(\omega) = i_{\tilde{X}} \hat{\hat{\phi}}(\omega). \tag{22}$$

This shows that the cocycle $\#_\sigma \hat{\phi}$ agrees with $i_{\tilde{X}} \hat{\hat{\phi}}$.

Let σ^0 be the action of $\mathbb{R}$ on $\mathbb{R} \ltimes_{\hat{\sigma}} \mathbb{R} \ltimes_\sigma \mathcal{A}$ which corresponds to $\mathrm{Id}_{\mathcal{K}} \otimes \sigma$ on $\mathcal{K}_{\mathbb{R}}^\infty \hat{\otimes} \mathcal{A}$. There is a one-parameter unitary u_t such that the double dual action $\hat{\hat{\sigma}}$ is cocycle conjugate to σ^0 by the formula $\hat{\hat{\sigma}}_t = \mathrm{Ad}_{u_t} \circ \sigma_t^0$.

Note that $\hat{\hat{\phi}}$ is identified to the cyclic cocycle $\phi \otimes \mathrm{Tr}$ on $\mathcal{A} \hat{\otimes} \mathcal{K}_{\mathbb{R}}^\infty$. Hence we may apply Lemma 14 and obtain that $[i_X \phi] = [i_{\tilde{X}} \hat{\hat{\phi}}(\omega)]$ in the periodic cyclic cohomology. Combined with (22), we obtain the equality $[i_X \phi] = [\#_{\hat{\sigma}} \hat{\phi}]$ in $\mathrm{HC}^*(\mathbb{R} \ltimes_{\hat{\sigma}} \mathbb{R} \ltimes_\sigma \mathcal{A})$. □

Now we go back to an even regular spectral triple $(\mathcal{A}, H, D)$ with an action of $\mathbb{T}^2$ satisfying $\mathcal{A}^\infty = \mathcal{A}$ as in the previous sections. Note that we have the estimates

$$\nu_{l,\alpha}\left(\frac{d^k \sigma_t^{(i)}(a)}{dt^k}\right) = \nu_{l,\alpha'}(a)$$

for $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^2$, where $\alpha' = (\alpha_1 + k, \alpha_2)$ or $\alpha' = (\alpha_1, \alpha_2 + k)$ corresponding to the cases $i = 1, 2$. Hence the actions $\sigma^{(i)}$ are smooth on $\mathcal{A}^\infty = \mathcal{A}$ in the sense above.

Let ϕ be a cyclic n-cocycle on $\mathcal{A}$ which is invariant under σ. Then we obtain the dual cocycle $\hat{\phi}$ over the crossed product $\mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A}$ which is invariant under the action $(\widehat{\sigma^{(1)}}, \sigma_\theta^{(2)})$. Thus we obtain a cocycle $\hat{\hat{\phi}}$ on $\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma_\theta^{(2)}} \mathbb{T}_{\sigma^{(1)}} \ltimes \mathcal{A}$.

The cocycle $\hat{\hat{\phi}}$ induces a one $\phi^{(\theta)}$ on $\mathcal{A}_\theta$ via the embedding of $\mathcal{A}_\theta$ into a corner of

$$\mathcal{K}^\infty \hat{\otimes} \mathcal{A}_\theta \simeq \mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma_\theta^{(2)}} \mathbb{T}_{\sigma^{(1)}} \ltimes \mathcal{A}.$$

We record the formula for $\phi^{(\theta)}$ for $a \in \mathcal{A}$:

$$\phi^{(\theta)}(a_0^{(\theta)}, \dots, a_n^{(\theta)}) = \sum_{\substack{m_0 + \cdots + m_n = 0, \\ l_0 + \cdots + l_n = 0}} \phi((a_0)_{m_0, n_0}, b_{(1, m_1, l_1)}, \dots, b_{(n, m_n, l_n)}), \tag{23}$$

where $b_{(k, m_k, l_k)} = e^{2\pi i \theta (\sum_{j<k} m_j) l_k} (a_k)_{m_k, l_k}$. In the particular case of $n = 0$, ϕ is given by a trace on $\mathcal{A}$ and $\phi^{(\theta)}$ is given by the corresponding trace

$$\phi^{(\theta)}(a^{(\theta)}) = \phi(a_{(0,0)}) = \phi(a)$$

on $\mathcal{A}_\theta$.

With an analogous process we obtain another cyclic cocycle on $\mathbb{R} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_{\theta t}} \mathbb{R} \ltimes_{\sigma^{(1)}} \mathcal{A}$ from a σ-invariant cocycle by taking dual twice, which we still denote by $\hat{\hat{\phi}}$ by abuse of notation. This abuse of notation is justified because we have

$$\hat{\hat{\phi}} = \operatorname{Tr} \otimes \hat{\hat{\phi}},$$

where in the left hand side the dual operations are taken over the crossed products by $\mathbb{R}$, while in the right hand side they are taken over the actions of $\mathbb{T}$ and $\mathbb{Z}$. The term $\operatorname{Tr} \otimes -$ stands for the correspondence of the cyclic cocycles over $\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_{\theta}} \mathbb{T}_{\sigma^{(1)}} \ltimes \mathcal{A}$ to the ones over $\mathbb{R} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_{\theta t}} \mathbb{R} \ltimes_{\sigma^{(1)}} \mathcal{A}$ by means of the natural strong Morita equivalence between these algebras (c.f. the proof of Proposition 9).

For $i = 1, 2$, let X_i denote the generator $[h_i, -]$ of the one-parameter group $\sigma^{(i)}$ on $\mathcal{A}$.

Theorem 2. *Let ϕ be a σ-invariant cyclic n-cocycle on $\mathcal{A}$. Then the cyclic n-cocycle $\phi^{(\theta)}$ on $\mathcal{A}_\theta$ corresponds to the nonhomogeneous cyclic cocycle*

$$\phi + \theta i_{X_1} i_{X_2} \phi \tag{24}$$

on $\mathcal{A}$ under the natural isomorphism of Corollary 13.

Proof. It is enough to check that the cyclic n-cocycle $\hat{\hat{\phi}}$ on $\mathbb{R} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_{\theta t}} \mathbb{R} \ltimes_{\sigma^{(1)}} \mathcal{A}$ corresponds to the one of (24) on $\mathcal{A}$ via the Connes–Thom isomorphisms in HP^*.

The action $(\widehat{\sigma^{(1)}}, \sigma^{(2)}_{\theta t})$ of $\mathbb{R}$ on $\mathbb{R} \ltimes_{\sigma^{(1)}} A$ is generated by the derivation

$$\delta^{(\theta)}(f)_s = s f_s + \theta \frac{d\sigma^{(2)}_t(f_s)}{dt}.$$

Let $\widehat{X_1}$ denote the generator of the dual action $\widehat{\sigma^{(1)}}$. The equation above shows that $\delta^{(\theta)} = \widehat{X_1} + \theta X_2$. Applying Proposition 15 to $\hat{\phi}$ and $(\widehat{\sigma^{(1)}}, \sigma^{(2)}_{\theta t})$, we obtain that the cyclic cocycle $\#_{(\widehat{\sigma^{(1)}}, \sigma^{(2)}_{\theta t})} \hat{\phi}$ on $\mathbb{R} \ltimes_{\sigma^{(1)}} \mathcal{A}$ is in the same class as the cocycle

$$i_{\delta^{(\theta)}} \hat{\phi} = i_{\widehat{X_1}} \hat{\phi} + \theta i_{X_2} \hat{\phi}.$$

The first term $i_{\widehat{X_1}} \hat{\phi}$ agrees with $\#_{\sigma^{(1)}} \phi$ as was the case for (22). Hence this term corresponds to the cocycle ϕ on $\mathcal{A}$.

For the second term, one has $i_{X_2} \hat{\phi} = \widehat{i_{X_2} \phi}$. This $n+1$-cocycle on $\mathbb{R} \ltimes_{\sigma^{(1)}} \mathcal{A}$ corresponds to the $n+2$-cocycle $i_{X_1} i_{X_2} \phi$ on $\mathcal{A}$ again by Proposition 15. Combining these two, one obtains the assertion of Theorem. □

Example 2. Let M be a compact smooth manifold, endowed with a smooth action σ of $\mathbb{T}^2$. Then M admits a Riemannian metric which is invariant under σ. The algebra $C^\infty(M)$ of the smooth functions on M, the de Rham complex $\Omega^*(M)$ graded by the degree of forms, the operator $d + d^*$ densely defined on $H = L^2(\Omega^*(M))$, and the induced representation of $\mathbb{T}^2$ on $C^\infty(M)$ and H satisfy the assumptions of this paper.

The volume form dv on M is invariant under σ and it defines an invariant trace $\tau \colon f \mapsto \int_M f dv$ on $C(M)$. Let X_i denote the vector fields on M generating σ_i for $i = 1, 2$. By Theorem 2, the map $K_0(CM_\theta) \to \mathbb{C}$ induced by the trace $\tau^{(\theta)}$ on CM_θ corresponds to the map $K^0(M) \to \mathbb{C}$ induced by the current

$$f + g^0 dg^1 \wedge dg^2 \mapsto \int_M f + \theta(g^0 \langle dg^1, X_1 \rangle \langle dg^2, X_2 \rangle - \langle dg^1, X_2 \rangle \langle dg^2, X_1 \rangle) dv.$$

When E is a vector bundle over M, the above map on the class of E in $K^0(M)$ gives the number

$$\int_M \mathrm{ch}(E) \wedge (dv + \theta i_{X_1} i_{X_2} dv) = \mathrm{vol}(M)\,\mathrm{rk}(E) + \theta \int_M c_1(E) \wedge i_{X_1} i_{X_2} dv.$$

5.1. **Chern–Connes character of deformed triple.** We have a representation of $\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta} \mathbb{T} \ltimes_{\sigma^{(1)}} A$ on $\ell^2(\mathbb{Z}) \overline{\otimes} H$ by

$$z^j a \delta_k \otimes \xi = \begin{cases} \delta_k \otimes a\xi & (a \in A_{m,n}, \xi \in H_{m',n'}, j = -k + m + m') \\ 0 & (\text{otherwise}) \end{cases} \tag{25}$$

and $u \delta_k \otimes \xi = e^{2\pi i n\theta} \delta_{k+1} \otimes \xi$ for any $\xi \in H_{m,n}$, where u and $z^j a$ are as in the last part of Remark 6.

Note that $\mathcal{K}(\ell^2\mathbb{Z}) \otimes_{\min} A_\theta$ admits a representation on $\ell^2(Z) \overline{\otimes} H$ determined by the natural representation of A_θ on H. In the following we make a more detailed analysis of the strong Morita equivalence given by Proposition 9 and Remark 6 in relation to the unbounded selfadjoint operator D.

Proposition 16. *There is a unitary operator V on $\ell^2(\mathbb{Z}) \overline{\otimes} H$ satisfying $V^*(1 \otimes D)V = 1 \otimes D$ and*

$$V^*(\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta} \mathbb{T} \ltimes_{\sigma^{(1)}} A)V = \mathcal{K}(\ell^2\mathbb{Z}) \otimes_{\min} A_\theta.$$

Moreover the above equality restricts to the subalgebras $\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta} \mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A}$ and $\mathcal{K}^\infty \hat{\otimes} \mathcal{A}_\theta$.

Proof. Let V denote the unitary operator on $\ell^2(\mathbb{Z}) \overline{\otimes} H$ determined by

$$V \delta_k \otimes \xi = e^{2\pi i \theta n k} \delta_{k+m} \otimes \xi$$

for any $k \in \mathbb{Z}$ and $\xi \in H_{m,n}$.

The unitary V implements the identification of (15). When $a \in A_{p,q}$ and $\xi \in H_{m,n}$, we have

$$\begin{aligned} V^* z^l a V \delta_k \otimes \xi &= V z^l a e^{2\pi i \theta k n} \delta_{k+m} \otimes \xi \\ &= \begin{cases} e^{2\pi i \theta q(p-k) + pn} \delta_{k-p} \otimes a\xi & (l + k - p = 0) \\ 0 & (\text{otherwise}). \end{cases} \end{aligned}$$

This is the effect of $e^{2\pi i \theta q l} e_{-l,-(l+p)} \otimes a^{(\theta)} \in \mathcal{K}(\ell^2\mathbb{Z}) \otimes_{\min} A_\theta$ on $\delta_k \otimes \xi$, where $e_{m,n}$ denotes the matrix element $\delta_k \mapsto \delta_{n,k} \delta_m$ on $\ell^2\mathbb{Z}$ for any $(m,n) \in \mathbb{Z}^2$.

Similarly one has $V^* w V = v \otimes 1$, where v is the unitary operator $\delta_k \mapsto \delta_{k+1}$ on $\ell^2\mathbb{Z}$. Hence we obtain the inclusion

$$V^*(\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta} \mathbb{T} \ltimes_{\sigma^{(1)}} A)V \subset \mathcal{K}(\ell^2\mathbb{Z}) \otimes_{\min} A_\theta.$$

One can also show that $V(\mathcal{K}(\ell^2\mathbb{Z}) \otimes_{\min} A_\theta)V^* \subset \mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta} \mathbb{T} \ltimes_{\sigma^{(1)}} A$, c.f. (16), (17). □

By Proposition 16, we have

$$\begin{aligned} [\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta} \mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A}, 1 \otimes D] &= \mathrm{Ad}_V([\mathrm{Ad}_{V^*}(\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta} \mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A}), \mathrm{Ad}_{V^*}(1 \otimes D)]) \\ &= \mathrm{Ad}_V([\mathcal{K}^\infty \hat{\otimes} \mathcal{A}_\theta, 1 \otimes D]). \end{aligned}$$

Hence the elements of $\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta} \mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A}$ have bounded commutators with $1 \otimes D$ on $\ell^2(\mathbb{Z}) \overline{\otimes} H$. In particular,

$$(\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_\theta} \mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A}, \ell^2(\mathbb{Z}) \overline{\otimes} H, 1 \otimes D) \tag{26}$$

is a spectral triple.

Suppose that the triple $(\mathcal{A}, H, D)$ is n-summable $(0 < n \in 2\mathbb{N})$. Then the character

$$\mathrm{ch}_D(f_0, \ldots, f_n) = \mathrm{Tr}_s(\gamma f_0 [F, f_1] \cdots [F, f_n]) \quad (F = D\,|D|^{-1})$$

of this triple is a cyclic n-cocycle which is invariant under σ.

The isomorphism of Proposition 11 induces an isomorphism

$$\mathrm{HP}(\mathcal{A}_\theta) \to \mathrm{HP}(\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma_\theta^{(2)}} \mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A}).$$

On one hand, Proposition 16 implies that the image of $\mathrm{ch}_{(\mathcal{A}_\theta, H, D)}$ under this isomorphism is equal to the character of the triple (26). On the other hand, the construction of the representation of $\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma_\theta^{(2)}} \mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A}$ on $\ell^2(\mathbb{Z}) \overline{\otimes} H$ implies the the following description of the Chern–Connes character.

Lemma 17. *The character of the spectral triple* (26) *over* $\mathbb{Z} \ltimes_{\widehat{\sigma^{(1)}}, \sigma_\theta^{(2)}} \mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A}$ *is equal to* $\widehat{\widehat{\mathrm{ch}_{(\mathcal{A}, H, D)}}}$.

Proof. Let $k_0, \ldots, k_n$ be integers and $f_0, \ldots, f_n$ be elements of $\mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A}$. Then one has

$$\mathrm{Tr}_s(u^{k_0} f_0 [F, u^{k_1} f_1] \cdots [F, u^{k_n} f_n]) = 0$$

unless $\sum_{j=0}^n k_j = 0$, c.f. (14). This implies that the character of (26) is equal to the dual of $\mathrm{ch}_{(\mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A}, \ell^2(\mathbb{Z}) \overline{\otimes} H, 1 \otimes D)}$.

It is routine to check that

$$\mathrm{ch}_{(\mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A}, \ell^2(\mathbb{Z}) \overline{\otimes} H, 1 \otimes D)} = \widehat{\mathrm{ch}_{(\mathcal{A}, H, D)}}$$

directory from the definition (25) of the representation of $\mathbb{T} \ltimes_{\sigma^{(1)}} \mathcal{A}$ on $\ell^2(\mathbb{Z}) \overline{\otimes} H$.

Combining the above two, we obtain the assertion of Lemma. □

Remark 8. As a consequence of Lemma 17, or directly from the formula (23), the cocycle $\mathrm{ch}^{(\theta)}_{(\mathcal{A}, H, D)}$ over $\mathcal{A}_\theta$ agrees with $\mathrm{ch}_{(\mathcal{A}_\theta, H, D)}$.

Theorem 3. *Let* $(\mathcal{A}, H, D)$ *be an even regular spectral triple endowed with a smooth action of* $\mathbb{T}^2$ *satisfying* $\mathcal{A} = \mathcal{A}^\infty$. *Then the Chern–Connes characters of* $(\mathcal{A}_\theta, H, D)$ *and* $(\mathcal{A}, H, D)$ *induce the same maps on* $K_0(A_\theta)$ *via the isomorphism of Corollary 10.*

Proof. Put $\phi = \mathrm{ch}_D$. By Remark 8, the character of D over $\mathcal{A}_\theta$ corresponds to the cocycle $\hat{\hat{\phi}}$ on $\mathbb{R} \ltimes_{\widehat{\sigma^{(1)}}, \sigma_{\theta t}^{(2)}} \mathbb{R} \ltimes_{\sigma^{(1)}} \mathcal{A}$. As in the proof of Theorem 2, this cocycle corresponds to the cocycle $i_{\widehat{X_1}} \hat{\phi} + \theta i_{X_2} \hat{\phi}$ on $\mathbb{R} \ltimes \mathcal{A}$.

Given an action α of $\mathbb{R}$ on a C^*-algebra B, let Φ_α denote the Connes–Thom isomorphism $K_* B \to K_{*+1}(\mathbb{R} \ltimes_\alpha B)$. Let y be any element of $K_1(\mathbb{R} \ltimes A)$. Then one has

$$\left\langle \hat{\hat{\phi}}, \left(\Phi_{\widehat{\sigma^{(1)}}, \sigma_{\theta t}^{(2)}}\right)^{-1}(y) \right\rangle = \left\langle i_{\widehat{X_1}} \hat{\phi} + \theta i_{X_2} \hat{\phi}, y \right\rangle = \left\langle i_{\widehat{X_1}} \hat{\phi}, y \right\rangle + \theta \left\langle i_{X_2} \hat{\phi}, y \right\rangle.$$

Since $\hat{\hat{\phi}}$ is equal to a character of a spectral triple, its pairing with any element in $K_0(\mathbb{R} \ltimes_{\widehat{\sigma^{(1)}}, \sigma_{\theta t}^{(2)}} \mathbb{R} \ltimes_{\sigma^{(1)}} A)$ must be an integer. Hence $\left\langle i_{\widehat{X_1}} \hat{\phi}, y \right\rangle + \theta \left\langle i_{X_2} \hat{\phi}, y \right\rangle$ must stay inside $\mathbb{Z}$ regardless of the value of θ, which implies $\left\langle i_{X_2} \hat{\phi}, y \right\rangle = 0$.

Let $\Xi \colon K_0(A_\theta) \to K_0(\mathbb{R} \ltimes_{\widehat{\sigma^{(1)}}, \sigma_{\theta t}^{(2)}} \mathbb{R} \ltimes_{\sigma^{(1)}} A)$ be the isomorphism given by Proposition 9. For any element $x \in K_0(A_\theta)$, one has

$$\langle \mathrm{ch}_D, x \rangle = \left\langle \hat{\hat{\phi}}, \Xi(x) \right\rangle = \left\langle i_{\widehat{X_1}} \hat{\phi}, \Phi_{\widehat{\sigma^{(1)}} \sigma_{\theta t}^{(2)}} \Xi(x) \right\rangle = \left\langle \mathrm{ch}_D, \Phi_{\sigma^{(1)}} \Phi_{\widehat{\sigma^{(1)}} \sigma_{\theta t}^{(2)}} \Xi(x) \right\rangle.$$

Since $x \mapsto \Phi_{\sigma^{(1)}}\Phi_{\widehat{\sigma^{(1)}}\sigma^{(2)}_{\theta t}}\Xi(x)$ is the natural isomorphism $K_0(A_\theta) \to K_0(A)$, one obtains the assertion of Theorem. □

6. Deformation from actions of higher dimensional tori

In this section we generalize the construction of the previous sections to the case of the actions by the torus $\mathbb{T}^n$ of even dimension and the deformation parameter J in the space of the skew symmetric matrices of size n. As in [20], some part of it makes sense for the actions of Euclidean spaces $\mathbb{R}^n$.

Let n be a positive integer, σ an action of $\mathbb{R}^n$ on a C^*-algebra A, and J a skew symmetric matrix of size n. Then we have a deformed algebra A_J as in Remark 3.

As a particular case of this construction, one may consider the case of $A = C_0(\mathbb{R}^n)$ and σ is given by the translation. Then A_J is by n families $(u_i^t)_{i=1}^n$ of unitaries each of which is parametrized by $t \in \mathbb{R}$, satisfying

$$u_i^s u_i^t = u_i^{s+t} \qquad\qquad u_i^s u_j^t u_i^{-s} = e^{4\pi i s t J_{i,j}} u_j^t.$$

For general A and σ, Rieffel [20] showed that A_J is isomorphic to the 'generalized fixed point algebra' $(A \otimes_{\min} C_0(\mathbb{R})_J)^{\mathbb{R}^n}$ of [19] with respect to the tensor product action of σ and the 'gauge action' on $C_0(\mathbb{R})_J$. It follows that there is a strong Morita equivalence between A_J and $\mathbb{R}^n \ltimes (A \otimes_{\min} C_0(\mathbb{R})_J)$ where the action of $\mathbb{R}^n$ on $A \otimes_{\min} C_0(\mathbb{R})_J$ is given by the tensor product of σ and the 'gauge action' on $C_0(\mathbb{R})_J$.

Put $B_0 = \mathbb{R}^n \ltimes_\sigma A$. For $1 \leq k \leq n$, let us inductively construct actions α_k of $\mathbb{R}$ on B_{k-1} and algebras

$$B_k = \mathbb{R} \ltimes_{\alpha_k} B_{k-1} = \mathbb{R} \ltimes_{\alpha_k} \cdots \mathbb{R} \ltimes_{\alpha_1} \mathbb{R}^n \ltimes_\sigma A.$$

For each j, we let u_j^t be the 1-parameter unitary group implementing the action α_j. Thus B_{k-1} is generated by $\mathbb{R} \ltimes_\sigma A$ and the unitaries $u_1^t, \ldots, u_{k-1}^t$ for $t \in \mathbb{R}$. Then the action α_k is determined by

$$\alpha_k^s(u_j^t) = e^{4\pi i s t J_{k,j}} u_j^t$$

for $j < k$, and $\alpha_k^s(x) = \widehat{\sigma^{(k)}}_s(x)$ for $x \in \mathbb{R} \ltimes_\sigma A$.

Lemma 18. *The algebra $\mathbb{R}^n \ltimes (A \otimes_{\min} C_0(\mathbb{R})_J)$ is isomorphic to the iterated crossed product*

$$B_n = \mathbb{R} \ltimes_{\alpha_n} \cdots \mathbb{R} \ltimes_{\alpha_1} \mathbb{R}^n \ltimes_\sigma A. \tag{27}$$

Proof. Let $(v_i^t)_{i=1}^n$ be the generating unitaries of $C^*(\mathbb{R}^n)$, identified to a subalgebra of $\mathbb{R} \ltimes_\sigma A$. Then the algebra (27) is generated by the elements which are generated by the integrable functions from $\mathbb{R}^{2n}$ into A via the correspondence

$$\begin{aligned} f \in L^1(\mathbb{R}^{2n}; A) &\mapsto z_f \\ &= \int_{\mathbb{R}^{2n}} f(t_1, \ldots, t_n, s_1, \ldots, s_n) v_1^{s_1} \cdots v_n^{s_n} u_1^{t_1} \cdots u_n^{t_n} dt_1 \cdots dt_n ds_1 \cdots ds_n. \end{aligned}$$

If f_0 is an integrable function on $\mathbb{R}^n$ and a is an element of A, we can consider an element

$$x_{a,f_0} = a \otimes \int_{\mathbb{R}^{2n}} f_0(t_1, \ldots, t_n) u_1^{t_1} \cdots u_n^{t_n} dt_1 \cdots dt_n$$

of $A \otimes_{\min} C_0(\mathbb{R}^n)_J$. Then, if f_1 is another integrable function on $\mathbb{R}^n$, we obtain the element

$$y_{a,f_0,f_1} = \int_{\mathbb{R}^n} x_{a,f_0} f_1(s_1, \ldots, s_n) v_1^{s_1} \cdots v_n^{s_n} ds_1 \cdots ds_n$$

of $\mathbb{R}^n \ltimes (A \otimes_{\min} C_0(\mathbb{R}^n)_J)$. Let b denote the function

$$b(t_1, \ldots, t_n, s_1, \ldots, s_n) = a f_0(t_1, \ldots, t_n) f_1(s_1, \ldots, s_n)$$

from $\mathbb{R}^{2n}$ into A. Then the correspondence $y_{a,f_0,f_1} \mapsto z_b$ induces the desired isomorphism. □

Let $\mathcal{A}$ be a σ-invariant subalgebra of A endowed with a Fréchet topology with respect to which the restriction of σ is smooth. On one hand we can take

$$\mathcal{A}_J = (\mathcal{A} \hat{\otimes} \mathcal{S}(\mathbb{R}^n)_J)^{\mathbb{R}^n}.$$

On the other hand we can construct subalgebras $\mathcal{B}_k$ of B_k for each k by taking the elements which are represented by the Schwartz class functions from $\mathbb{R}^{n+k}$ into $\mathcal{A}$. The above Lemma shows that there is an isomorphism

$$\mathrm{HP}^*(\mathcal{A}_J) \to \mathrm{HP}^*(\mathbb{R} \ltimes_{\alpha_n-1} \cdots \mathbb{R} \ltimes_{\alpha_0} \mathbb{R}^n \ltimes_\sigma \mathcal{A}).$$

By composing this with the iteration of Elliott–Natsume–Nest isomorphisms, we obtain an isomorphism

$$\Psi^* \colon \mathrm{HP}^*(\mathcal{A}_J) \to \mathrm{HP}^*(\mathcal{A}),$$

which is compatible with the natural isomorphism Ψ in K-theory.

Let ϕ be a cyclic cocycle on $\mathcal{A}$ which is invariant under σ. We apply successive application of the dual cocycle operations as follows: $T_0\phi = \hat{\phi}^{\mathbb{R}^n}$ is the dual cocycle on $\mathcal{B}_0$, and $T_k\phi = \widehat{T_{k-1}\phi}$ is the cocycle on $\mathcal{B}_k$ which is dual to $T_{k-1}\phi$ with respect to the action α_k. In this way we obtain a cocycle $T_n\phi$ on the algebra (27). Through the strong Morita equivalence, we also obtain a cocycle on $\mathcal{A}_J$, which is denoted by $\tilde{\phi}$.

For each $1 \leq i \leq n$, let X_i denote the generator of $\sigma^{(i)}$.

Theorem 4. *Let ϕ be a cyclic cocycle on $\mathcal{A}$ which is invariant under σ. Then the cocycle $\tilde{\phi}$ on $\mathcal{A}_J$ corresponds to the one*

$$(28) \quad \phi + \sum_{i<j} 2J_{i,j} i_{X_i} i_{X_j}\phi + \sum_{\substack{i_1<i_2,i_3<i_4\\ i_2<i_4}} 4J_{i_1,i_2}J_{i_3,i_4} i_{X_{i_1}} i_{X_{i_2}} i_{X_{i_3}} i_{X_{i_4}}\phi + \cdots + \sum_{\substack{i_1<i_2,\ldots,i_{2m-1}<i_{2m}\\ i_2<i_4<\cdots<i_{2m}}} 2^m J_{i_1,i_2}\cdots J_{i_{2m-1},i_{2m}} i_{X_{i_1}}\cdots i_{X_{i_{2m}}}\phi$$

on $\mathcal{A}$ under the isomorphism Ψ^*, where m is the largest integer not exceeding $n/2$.

Proof. We 'untwist $\mathbb{R}$-crossed products' from the cocycle $T_n\phi$ to (28) by induction on n using Lemma 14 and 15. The case for $n = 2$ is exactly Theorem 2.

Now, let n be an arbitrary positive integer and suppose that Theorem was proved for $n-1$. The action α_n on $\mathcal{B}_{n-1}$ is cocycle conjugate to the one

$$\beta_t(f) = \sigma^{(1)}_{2J_{1,n}t} \cdots \sigma^{(n-1)}_{2J_{n-1,n}t} \widehat{\sigma^{(n)}}_t(f) \quad (f \in \mathcal{S}(\mathbb{R}^{n+(n-1)}; \mathcal{A})).$$

By assumption $T_{n-1}\phi$ is also invariant under β. Let $\hat{X}_n$ be the generator of the dual action $\widehat{\sigma^{(n)}}$. By Lemma 14 and Proposition 15, the cocycle $T_n\phi$ corresponds to the cocycle

$$(29) \qquad i_{\hat{X}_n} T_{n-1}\phi + \sum_{j<n} 2J_{j,n} i_{X_j} T_{n-1}\phi$$

on $\mathcal{B}_{n-1}$ under the Elliott–Natsume–Nest isomorphism with respect to α_n.

Let $\mathcal{B}'_0$ denote the crossed product $\mathbb{R}^{n-1} \ltimes_{(\sigma^{(1)},\ldots,\sigma^{(n-1)})} \mathcal{A}$. We can construct actions α'_k and algebras $\mathcal{B}'_k = \mathbb{R} \ltimes_{\alpha'_k} \mathcal{B}'_k$ for $1 \leq k \leq n-1$ exactly as in the case for α_k and $\mathcal{B}_k$. Then $\mathcal{B}_{n-1}$ can be identified with $\mathbb{R} \ltimes_{\sigma^{(n)}} \mathcal{B}'_{n-1}$. Similar to the construction of $T_k\phi$, we also have cyclic cocycles T'_k on $\mathcal{B}'_k$ for each k by successive application of the dual cocycle operation.

The first term $i_{\hat{X}_n} T_{n-1}\phi$ of (29) corresponds to the cocycle $T'_{n-1}\phi$ on $\mathcal{B}'_{n-1}$ by argument in the third paragraph in the proof of Theorem (2). The rest of the terms corresponds to the cocycle

$$\sum_{j<n} 2J_{j,n} i_{X_j} i_{X_n} T'_{n-1}\phi = T'_{n-1} \sum_{j<n} 2J_{j,n} i_{X_j} i_{X_n}\phi$$

on $\mathcal{B}'_{n-1}$ again by Proposition 15. Combining these, (29) corresponds to the cyclic cocycle

$$T'_{n-1}\left(\phi + \sum_{j<n} 2J_{j,n} i_{X_j} i_{X_n}\phi\right)$$

on $\mathcal{B}'_{n-1}$.

We note that $\mathcal{B}'_{n-1}$ is obtained by the iterated crossed construction for the skew symmetric matrix $J^{(1)} = (J_{i,j})_{1\le i,j\le n-1}$ of size $n-1$. By induction hypothesis, we obtain the assertion of Theorem for n. □

Remark 9. One has the anticommutativity $i_X i_Y \phi = -i_Y i_X \phi$ for any commuting derivations X and Y which are generators of actions preserving ϕ. Consequently, if one has $i_j = i_{j'}$ among $(i_1, \ldots, i_{2k})$ for some $j \neq j'$, the term $J_{i_1,i_2} \cdots J_{i_{2k-1},i_{2k}} i_{X_{i_1}} \cdots i_{X_{i_{2k}}}\phi$ in (28) vanishes. It also follows that the formula (28) is invariant under the conjugation by an orthogonal matrix.

6.1. **Deformation of $\mathbb{T}^n$-equivariant spectral triples.** As before, let J be a skew symmetric matrix of size n. Suppose that the action σ is given as a one by $\mathbb{T}^n$. For example, if A is the function algebra $C(\mathbb{T}^n)$ over the n-torus and σ is the translation action, the deformed algebra $C(\mathbb{T}^n)_J$ is generated by n unitaries $(u_i)_{i=1}^n$ satisfying the relation

$$u_i u_j = e^{4\pi i J_{i,j}} u_j u_i.$$

Then there is the 'gauge' action γ of $\mathbb{T}^n$ on $C(\mathbb{T}^n)_J$.

For general algebra A with an action σ of $\mathbb{T}^n$, the deformed algebra A_J can be identified with the fixed point algebra $(A \otimes_{\min} C(\mathbb{T}^n)_J)^{\mathbb{T}^n}$ with respect to the action $\sigma \otimes \gamma$.

In the rest of this section we assume that $(\mathcal{A}, H, D)$ is a regular spectral triple and the action σ is given by an action of $\mathbb{T}^n$. We assume that there is a unitary representation U of $\mathbb{T}^n$ satisfying (1) and that σ satisfies smoothness condition $\mathcal{A}^\infty = \mathcal{A}$ (see the part right after Remark 1). We then obtain that the algebras $\mathbb{T}^n \ltimes (\mathcal{A} \hat{\otimes} C^\infty(\mathbb{T}^n)_J)$ and $\mathcal{A}_J$ are strongly Morita equivalent via an argument analogous to Propositions 11 and 12.

Let $L^2(\mathbb{T}^n_J)$ be the Hilbert space obtained from the GNS-construction for the algebra $C(\mathbb{T}^n)_J$ and its unique gauge invariant trace. Then the algebra $B = A \otimes_{\min} C(\mathbb{T}^n)_J$ is represented on the Hilbert space $\tilde{H} = H \overline{\otimes} L^2(\mathbb{T}^n_J)$, and the commutator of $\tilde{D} = D \otimes \mathrm{Id}_{L^2(\mathbb{T}^n_J)}$ with any element of $\mathcal{B} = \mathcal{A} \hat{\otimes} C^\infty(\mathbb{T}^n)_J$ is bounded. There is a natural unitary representation $\Gamma\colon \mathbb{T}^n \curvearrowright L^2(\mathbb{T}^n_J)$ which satisfies $\mathrm{Ad}_{\Gamma_t}(x) = \gamma_t(x)$ for any $x \in C(\mathbb{T}^n)_J$. Hence one has

$$(\sigma \otimes \gamma)_t(x) = \mathrm{Ad}_{(U\otimes\Gamma)_t}(x) \quad (x \in \mathcal{B}). \tag{30}$$

Moreover the operator $\tilde{D}$ commutes with $(U \otimes \Gamma)_t$ for $t \in \mathbb{T}^n$.

Let $\tilde{H}_0$ be the subspace of $\tilde{H}$ which consists of the invariant vectors under the representation $U \otimes \Gamma$. Then $\tilde{H}_0$ is preserved under the action of the fixed point algebra $A_J = B^{\mathbb{T}^n}$ by (30). As $\tilde{D}$ is $\mathrm{Ad}_{U\otimes\Gamma}$-invariant, it also restricts to $\tilde{H}_0$. Hence one obtains new spectral triple

$$(\mathcal{A}_J, \tilde{D}|_{\tilde{H}_0}, \tilde{H}_0). \tag{31}$$

This operation generalizes the construction of the deformation of Riemannian manifolds endowed with an action of $\mathbb{T}^n$ by Connes–Dubois-Violette [7].

As a consequence of Theorem 4, we obtain the following invariance of the pairing of the Chern character for the different values of J.

Theorem 5. *Let ϕ be the character of the spectral triple* (31) *and x be any element in $K_0(A_J)$. Then one has*

$$\langle \phi, x \rangle = \langle \mathrm{ch}_{(A,D,H)}, \Psi(x) \rangle$$

for the element $\Psi(x) \in K_0(A)$ corresponding to x under the natural isomorphism in the K-theory.

Proof. The cocycle ϕ is equal to $\widetilde{\mathrm{ch}_{(A,D,H)}}$ as in Lemma 17. By Theorem 4, ϕ corresponds to the cocycle $\mathrm{ch}_{(A,D,H)} + \psi_J$ on $\mathcal{A}$, where ψ_J is a cyclic cocycle which continuously depends on J. By the property

$$\langle \mathrm{ch}_{(A,D,H)} + \psi_J, \Psi(x) \rangle = \langle \phi, x \rangle \in \mathbb{Z},$$

we obtain the equality in the assertion. $\square$

7. Concluding Remarks

Remark 10. In the proof of Theorems 3 and 4, we saw that the cyclic cocycles of the form $i_{X_1} \cdots i_{X_{2m}} \mathrm{ch}_D$ on $\mathcal{A}$ pairs trivially with $K_0(A)$. It is very likely that these cocycle are trivial in $\mathrm{HP}^0(\mathcal{A})$.

Note that this phenomenon is specific to the character of K-cycles admitting a unitary representation U_t of $\mathbb{T}^2$ satisfying (1). Otherwise $i_{X_1} i_{X_2} \mathrm{ch}_D$ can pair nontrivially with $K_0(A)$. For example, the trace τ on $\mathbb{T}^2$ given by the Haar integral is the character of a 2-summable even spectral triple over $C^\infty(\mathbb{T}^2)$ and it satisfies $\langle i_{X_1} i_{X_2} \tau, K^0 \mathbb{T}^2 \rangle = \mathbb{Z}$.

Remark 11. After a major portion of the results in this paper was obtained, the author has learned from N. Higson that the invariance of the index pairing as stated in Theorem 3 can be obtained in a purely C^*-algebraic framework as follows.

Let $\mathbb{R}$ act on the C^*-algebra $\mathbb{R} \ltimes_{\sigma^{(1)}} A \otimes_{\min} C[0,1]$ by $(\widehat{\sigma^{(1)}}, \sigma^{(2)}_{\theta t})$ on the copy of $\mathbb{R} \ltimes_{\sigma^{(1)}} A$ at the fiber over $\theta \in [0,1]$. Then the resulting crossed product algebra

$$X = \mathbb{R} \ltimes (\mathbb{R} \ltimes_{\sigma^{(1)}} A \otimes_{\min} C[0,1])$$

act on the Hilbert space $L^2(\mathbb{R}) \mathbin{\overline{\otimes}} H \mathbin{\overline{\otimes}} L^2([0,1])$ determined by the representation of $\mathbb{R} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_{\theta t}} \mathbb{R} \ltimes_{\sigma^{(1)}} A$ on $L^2(\mathbb{R}) \mathbin{\overline{\otimes}} H$ "over the fiber θ" as described in the beginning of Section 5.1.

The pointwise multiplication representation of $C[0,1]$ on $L^2[0,1]$ induces an representation of $C[0,1]$ on $L^2(\mathbb{R}) \mathbin{\overline{\otimes}} H \mathbin{\overline{\otimes}} L^2([0,1])$ as X-endomorphisms. Then one obtains an element

$$\alpha = (L^2(\mathbb{R}) \mathbin{\overline{\otimes}} H \mathbin{\overline{\otimes}} L^2([0,1]), 1_{L^2(\mathbb{R})} \otimes D \otimes 1_{L^2([0,1])}) \in \mathrm{KK}(X, C[0,1]).$$

The evaluation at $\theta \in [0,1]$ induces KK-equivalences

$$\mathrm{ev}_\theta \colon \mathbb{R} \ltimes (\mathbb{R} \ltimes_{\sigma^{(1)}} A \otimes_{\min} C[0,1]) \simeq_{\mathrm{KK}} \mathbb{R} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_{\theta t}} \mathbb{R} \ltimes_{\sigma^{(1)}} A,$$

$$\mathrm{ev}_\theta \colon C[0,1] \simeq_{\mathrm{KK}} \mathbb{C}.$$

These KK-equivalences intertwine α and the element of $K^0(A_\theta)$ given by D, while the composition $\mathrm{ev}_0 \circ \mathrm{ev}_\theta^{-1}$ gives the natural isomorphism between $K_0(A_\theta)$ and $K_0(A)$.

REFERENCES

[1] C. Antonescu and E. Christensen, *Group C^*-algebras, metrics and an operator theoretic inequality*, 2002. preprint available at arXiv:math/0211312v1.

[2] E. Bédos and R. Conti, *On twisted Fourier analysis and convergence of Fourier series on discrete groups*, J. Fourier Anal. Appl. **15** (2009), no. 3, 336–365, available at `arXiv:math/0605145`. MR2511867 (2010f:22008)

[3] A. Connes, *C^* algèbres et géométrie différentielle*, C. R. Acad. Sci. Paris Sér. A-B **290** (1980), no. 13, A599–A604. MR572645 (81c:46053)

[4] A. Connes, *An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of* $\mathbf{R}$, Adv. in Math. **39** (1981), no. 1, 31–55. MR605351 (82j:46084)

[5] A. Connes, *Noncommutative differential geometry*, Inst. Hautes Études Sci. Publ. Math. **62** (1985), 257–360. MR823176 (87i:58162)

[6] A. Connes, *Noncommutative geometry*, Academic Press Inc., San Diego, CA, 1994. MR1303779 (95j:46063)

[7] A. Connes and M. Dubois-Violette, *Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples*, Comm. Math. Phys. **230** (2002), no. 3, 539–579. MR1937657 (2004a:58006)

[8] A. Connes and G. Landi, *Noncommutative manifolds, the instanton algebra and isospectral deformations*, Comm. Math. Phys. **221** (2001), no. 1, 141–159, available at `arXiv:math/0011194`. MR1846904 (2003i:58014)

[9] A. Connes and H. Moscovici, *The local index formula in noncommutative geometry*, Geom. Funct. Anal. **5** (1995), no. 2, 174–243. MR1334867 (96e:58149)

[10] G. A. Elliott, T. Natsume, and R. Nest, *Cyclic cohomology for one-parameter smooth crossed products*, Acta Math. **160** (1988), no. 3-4, 285–305. MR945014 (89h:46093)

[11] J. M. Gracia-Bondía, J. C. Várilly, and H. Figueroa, *Elements of noncommutative geometry*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Boston Inc., Boston, MA, 2001. MR1789831 (2001h:58038)

[12] H. Li, *θ-deformations as compact quantum metric spaces*, Comm. Math. Phys. **256** (2005), no. 1, 213–238, available at `arXiv:math/0311500`. MR2134342 (2006e:46079)

[13] S. Neshveyev and L. Tuset, *A local index formula for the quantum sphere*, Comm. Math. Phys. **254** (2005), no. 2, 323–341, available at `arXiv:math/0309275`. MR2117628 (2005h:58045)

[14] R. Nest, *Cyclic cohomology of crossed products with* $\mathbf{Z}$, J. Funct. Anal. **80** (1988), no. 2, 235–283. MR961899 (90c:46088)

[15] N. C. Phillips, *Equivariant K-theory and freeness of group actions on C^*-algebras*, Lecture Notes in Mathematics, vol. 1274, Springer-Verlag, Berlin, 1987. MR911880 (89k:46086)

[16] N. C. Phillips, *K-theory for Fréchet algebras*, Internat. J. Math. **2** (1991), no. 1, 77–129. MR1082838 (92e:46143)

[17] A. Rennie, *Smoothness and locality for nonunital spectral triples*, K-Theory **28** (2003), no. 2, 127–165. MR1995874 (2004f:46086)

[18] M. A. Rieffel, *Projective modules over higher-dimensional noncommutative tori*, Canad. J. Math. **40** (1988), no. 2, 257–338. MR941652 (89m:46110)

[19] M. A. Rieffel, *Proper actions of groups on C^*-algebras*, Mappings of operator algebras (Philadelphia, PA, 1988), 1990, pp. 141–182. MR1103376 (92i:46079)

[20] M. A. Rieffel, *Deformation quantization for actions of* $\mathbf{R}^d$, Mem. Amer. Math. Soc. **106** (1993), no. 506, x+93. MR1184061 (94d:46072)

[21] M. A. Rieffel, *K-groups of C^*-algebras deformed by actions of* $\mathbf{R}^d$, J. Funct. Anal. **116** (1993), no. 1, 199–214. MR1237992 (94i:46088)

[22] M. A. Rieffel, *Metrics on states from actions of compact groups*, Doc. Math. **3** (1998), 215–229 (electronic). MR1647515 (99k:46126)

[23] H. Takai, *On a duality for crossed products of C^*-algebras*, J. Functional Analysis **19** (1975), 25–39. MR0365160 (51 #1413)

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