Singular Problems Related to Curvature Flow and Hamilton-Jacobi Equations

（曲率流とハミルトン・ヤコビ方程式における特異問題）

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Notation and Terminology

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space for some $n \in \mathbb{N}$.

We denote by $|\cdot|$ the usual Euclidean norm, i.e.,

$$|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \qquad \text{for all } x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n.$$

For any $p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$ and $q = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \in \mathbb{R}^n$, $p^\top$ denotes the transpose of p, $p \cdot q$ denotes the inner product in $\mathbb{R}^n$, i.e., $p \cdot q = p^\top q = q^\top p = \sum_{i=1}^{n} p_i q_i$, and $p \otimes q$ represents the tensor product in $\mathbb{R}^n$.

For every $z \in \mathbb{R}^n$ and $r > 0$, we denote by $B_r(z)$ the open ball with center at z and radius r.

For any $a, b \in \mathbb{R}$, $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. The bracket $[a]$ stands for the largest integer less than or equal to a, i.e., $[a] = \max\{k \in \mathbb{N} : k \leq a\}$.

For $A, B \subset \mathbb{R}^n$, we denote by $C(A, B)$ and $\mathrm{UC}\,(A, B)$ the space of continuous and uniformly continuous functions on A with values in B, respectively.

We denote by $\mathrm{LSC}\,(A)$, $\mathrm{USC}\,(A)$ and $C^k(A)$ the space of real-valued lower semicontinuous, upper semicontinuous and k-th continuous differentiable functions on A. We write $C^\infty(A) = \bigcap_{k \in \mathbb{N}} C^k(A)$.

For any $p \geq 1$ and Lebesgue measurable $A \subset \mathbb{R}^n$, $L^p(A)$ denotes the usual Lebesgue space, i.e., the set of all measurable functions whose absolute value raised to the p-th power has finite integral; $W^{k,p}(A)$ denotes the usual Sobolev space, i.e., the subset of functions in $L^p(A)$ such that the function and its weak derivatives up to some order k are in $L^p(A)$.

We call a function $m : [0, \infty) \to [0, \infty)$ a *modulus* if it is continuous and nondecreasing and $m(0) = 0$.

Throughout the thesis, we denote the partial derivatives of any function u by $u_t := \partial u / \partial t$ and $\nabla u := (\partial u / \partial x_1, \dots, \partial u / \partial x_n)$. We also denote $\Delta u := \partial^2 u / \partial x_1^2 +, \dots, + \partial^2 u / \partial x_n^2$.

Part I

The Deterministic Discrete Game Approach to PDEs

CHAPTER 1

Introduction

The first part is intended to discuss the discrete game-theoretic approach, which recently attracts large attention, to various partial differential equations. Before studying more precise problems in each chapter, let us quote the expository paper [**36**] for this game approach by L. C. Evans: " It seems to me that within nonlinear analysis we should always be looking for unifying, guiding principles that cut across the boundaries of specific problems." Our general purpose is to interpret all of the equations by optimal control or differential games and compare the details of these interpretations so as to understand more deeply the connections among different equations.

The game method, especially the representation theorems for solutions of PDEs are actually not new. For example, it is wellknown that the solution of a first order linear Hamilton-Jacobi equation in n space dimensions

$$\begin{cases} u_t - f(x) \cdot \nabla u - l(x) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^n \end{cases}$$

can be expressed explicitly as $u(x,t) = u_0(y(t)) + \int_0^t l(y(s))\, ds$, where $y(t) \in \mathbb{R}^n$ is the solution of a state equation

$$\frac{dy}{dt} = f(y) \quad \text{with } y(0) = x.$$

Here u_0, f and l are all smooth functions in $\mathbb{R}^n$ and satisfy some specific conditions. It is tempting to make the equation more complicated so as to meet more applications. An interesting relaxation of equations is

$$u_t + \sup_{a \in A}\{-f(x,a) \cdot \nabla u - l(x,a)\} = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty),$$

where f and l are now functions of x and another argument a taken from a compact set A. The equation can be regarded as a certain supremum of a family of the preceding linear equations and one expects that the solution is some infimum of those corresponding solutions. In fact, we do show that the solution is

$$u(x,t) = \inf_{\alpha \in \mathcal{A}} \left(u_0(y(t)) + \int_0^t l(y(s), \alpha)\, ds \right),$$

where $\mathcal{A}$ stands for the set of all measurable functions α with $\alpha(t) \in A$ for all $t \in (0, \infty)$ and $y(t)$ satisfies the state equation

$$\frac{dy}{dt} = f(y, \alpha) \quad \text{with } y(0) = x.$$

It is consistent with one of the basic ideas of optimal control theory, the so-called dynamic programming principle. The set A is virtually a *control set* for the dynamics and u is called the *value function*. The only problem is that the solution u in the new circumstances is not necessarily differentiable anymore and the theory of viscosity solutions turns out to be the right tool.

Similarly, we can also give the explicit representation formulas for equations like

$$u_t + \sup_{a\in A}\inf_{b\in B}\{-f(x,a,b)\cdot\nabla u - l(x,a,b)\} = 0 \quad \text{in } \mathbb{R}^n\times(0,\infty),$$

which appears in the deterministic differential game theory. The set B is another control set and l is a continuous function representing the running cost. We omit the precise presentation, since it is somewhat standard but the notation is heavier. Consult the book **[5]** for details of continuous differential game theory and its relationship with Hamiltonian-Jacobi equations. Note that the equations are now very general, for a large class of Hamiltonians can be written as $\sup_{a\in A}\inf_{b\in B}\{-f(x,a,b)\cdot\nabla u - l(x,a,b)\}$ with f and l chosen correctly **[35, 40]**.

As for second order equations, we usually add an extra random term with controlled coefficient $\sigma(x,a,b)\in\mathbb{R}^{m\times n}$ in the state equation, which plays a role of diffusion. Taking the expectation in the definition of the value function, we derive the resulting equation in the form of

$$u_t + \sup_{a\in A}\inf_{b\in B}\{-\mathrm{tr}\,(\sigma(x,a,b)^{\top}\sigma(x,a,b)\nabla^2 u) - f(x,a,b)\cdot\nabla u - l(x,a,b)\} = 0, \tag{1.1}$$

which can also be view as a generalization of the classical Feynman-Kac formula, a link between parabolic partial differential equations and stochastic processes. We refer to **[45]** for the (continuous) stochastic optimal control and game theory.

The recent developments of optimal control and differential games are motivated by a further demand of representation formulas for more general equations which may not be written as (1.1). Soner and Touzi **[105]** used the stochastic optimal control to clarify the understanding of the mean curvature flow equation and later a general interpretation for elliptic and parabolic equations was addressed in **[24]**. Meanwhile, another method based on discretization of the usual state equations was advocated. A discrete but deterministic game approximation was conducted by Kohn and Serfaty **[75]** for curvature flow equations

$$\text{(MC)} \qquad \begin{cases} \partial_t u - |\nabla u|\mathrm{div}(\frac{\nabla u}{|\nabla u|}) = 0 & \text{in } \mathbb{R}^n\times(0,\infty),\\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n,\end{cases}$$

and for general elliptic and parabolic equations **[77]** while a discrete stochastic tug-of-war games were proposed to understand the p-Laplacian with $1 < p \leq \infty$ **[96, 95]**.

We focus our attention on the deterministic games in this work. One of the reasons for our preference is that we would like to study several applications later and searching explicit strategies and constructing trajectories are clearly easier on the deterministic occasion.

We are most interested in the so-called Paul and Carol game, which is a variant of pusher-chooser games first introduced by Spencer **[109, 110]**. We review the results of **[75]** below.

A marker, representing the *game state*, is initialized at a position $x \in \overline{\Omega}$ from time 0. The maturity time given is denoted by t. Let the step size for space be $\varepsilon > 0$. Time ε^2 is consumed for every step. Then the total number of game steps N can be regarded as $[\frac{t}{\varepsilon^2}]$. Two players, Paul and Carol participate the game. Paul intends to minimize at the final state an *objective function*, which in our case is u_0, while the other, Carol, is to maximize it. At each round,

(1) Paul chooses in $\mathbb{R}^n$ $n-1$ unit vectors $v_1, v_2, \ldots, v_{n-1}$ pairwise perpendicular, i.e., $v_i \cdot v_j = 0$ for all $1 \le i < j \le n-1$;

(2) Carol has the right to reverse Paul's choice, which determines $b^{(i)} = \pm 1$ for $i = 1, 2, \ldots, n-1$;

(3) The marker is moved from the present state x to $x + \sqrt{2}\varepsilon \sum_{i=1}^{n} b^{(i)} v_i$.

To express the rules in a more mathematical way, set

$$\mathcal{Q} = \{Q = (v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n\times(n-1)} : |v_i| = 1 \text{ and } v_i \cdot v_j = 0, \\ \text{for } 1 \le i < j \le n\}$$

and $\mathcal{B} = \{b = (b^{(1)}, \ldots, b^{(n-1)})^{\top} \in \mathbb{R}^{n-1} : b^{(i)} = \pm 1 \text{ for all } i = 1, 2, \ldots, n-1\}$. Then the inductive *state equation* writes as

$$\begin{cases} y_{k+1} = y_k + \sqrt{2}\varepsilon Q_k b_k, & k = 0, 1, \ldots, N-1; \\ y_0 = x, \end{cases}$$

where $Q_k \in \mathcal{Q}$ and $b_k \in \mathcal{B}$. We denote by α and β respectively the *nonanticipating strategies* of Paul and Carol. Hereafter, for any $x \in \mathbb{R}^n$ and $s \in [0, \infty)$, $y(x, s; \alpha, \beta)$ stands for the game state at the step $[s/\varepsilon^2]$ starting from x under the competing strategies α and β so that our games look like continuous ones. We also use the notation $y(x, s)$ for short if there is no ambiguity in strategies.

Note that we here insist, for simplicity, considering the game in time period $[0, t]$ so that the associated equation is exactly (MC) instead of a backward-in-time one as in [**75**]. No essential changes are made. More precisely, the *value function* is defined, in accordance to [**46**], as

$$u_1^{\varepsilon}(x, t) := \min_{Q_1\in\mathcal{Q}} \max_{b_1\in\mathcal{B}} \ldots \min_{Q_N\in\mathcal{Q}} \max_{b_N\in\mathcal{B}} u_0(y(x, t)), \tag{1.2}$$

or, by using the notation of strategies, briefly and equivalently expressed as

$$u_1^{\varepsilon}(x, t) = \min_{\alpha} \max_{\beta} u_0(y(x, t; \alpha, \beta)). \tag{1.3}$$

By the *dynamic programming*:

$$u_1^{\varepsilon}(x, t) = \min_{Q\in\mathcal{Q}} \max_{b\in\mathcal{B}} u_1^{\varepsilon}(x + \sqrt{2}\varepsilon Q b, t - \varepsilon^2) \tag{1.4}$$

with $u_1^{\varepsilon}(x, 0) = u_0(x)$, we may prove $u_1^{\varepsilon}(x, t)$ converges locally uniformly to the solution u of (MC), as $\varepsilon \to 0$.

On the other hand, an *inverse game value* u_2^{ε} is defined as

$$u_2^{\varepsilon}(x, t) = \max_{Q_1\in\mathcal{Q}} \min_{b_1\in\mathcal{B}} \ldots \max_{Q_N\in\mathcal{Q}} \min_{b_N\in\mathcal{B}} u_0(y(x, t)) = \max_{\alpha} \min_{\beta} u_0(y(x, t; \alpha, \beta)), \tag{1.5}$$

which, via similar arguments, can also be shown to converge to u. A rigorous mathematical statement is as follows.

THEOREM 1.1 (Games for mean curvature flow, [75, Theorem 1.2]). *Assume that u_0 is a bounded and uniformly continuous function in $\mathbb{R}^n$. Let u_1^ε and u_2^ε be the value functions defined by* (1.2) *and* (1.5) *respectively. Then both u_1^ε and u_2^ε converge, as $\varepsilon \to 0$, to the unique viscosity solution of* (MC) *uniformly on compact subsets of $\mathbb{R}^n \times (0, \infty)$.*

A variant for *positive (negative) mean curvature flow* equation is to use

$$\mathcal{Q}' = \{Q = (v_1, \dots, v_{n-1}) \in \mathbb{R}^{n\times(n-1)} : |v_i| \leq 1 \text{ and } v_i \cdot v_j = 0 \quad \text{for } 1 \leq i < j \leq n\}$$

and let

$$\begin{aligned} u_+^\varepsilon(x,t) &= \max_{Q_1\in\mathcal{Q}'} \min_{b_1\in\mathcal{B}} \dots \max_{Q_N\in\mathcal{Q}'} \min_{b_N\in\mathcal{B}} u_0(y(x,t)) \\ &= \max_\alpha \min_\beta u_0(y(x,t;\alpha,\beta)), \end{aligned} \tag{1.6}$$

then we get another convergence theorem for the *signed (positive) mean curvature flow* equation

$$\text{(PMC)} \qquad \begin{cases} \partial_t u - |\nabla u|\left(\operatorname{div}(\frac{\nabla u}{|\nabla u|}) \vee 0\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases}$$

THEOREM 1.2 (Games for signed mean curvature flow, [75, Theorem 1.5]). *Assume that u_0 is a bounded and uniformly continuous function in $\mathbb{R}^n$. Let u_2^ε be the value functions defined by* (1.6). *Then u_+^ε converge, as $\varepsilon \to 0$, to the unique viscosity solution of* (PMC) *uniformly on compact subsets of $\mathbb{R}^n \times (0, \infty)$.*

We remark that the original theorem is only established for u_0 which is constant outside a certain compact set. The extension to our general initial data here seems direct. With the aid of a comparison theorem, the proof for the convergence u_1^ε, for instance, rests on showing the half relaxed limits

$$\overline{u}_1(x,t) := \operatorname*{limsup^*}_{\varepsilon\to 0} u_1^\varepsilon(x,t) = \lim_{\delta\to 0} \sup\{u^\varepsilon(y,s) : 0 < \varepsilon < \delta,\ |x-y| + |t-s| < \delta\}$$

and

$$\underline{u}_1(x,t) := \operatorname*{liminf_*}_{\varepsilon\to 0} u_1^\varepsilon(x,t) = \lim_{\delta\to 0} \inf\{u^\varepsilon(y,s) : 0 < \varepsilon < \delta,\ |x-y| + |t-s| < \delta\}.$$

are respectively a subsolution and a supersolution of (MC).

Heuristically speaking, the core of proofs lies on the following observation for smooth functions. Let $\phi \in C^\infty(\mathbb{R}^n \times (0,\infty))$ in place of u_1^ε satisfy the dynamic programming principle (1.4) for all $x \in \mathbb{R}^n$ and $t \geq \varepsilon^2$. Then by Taylor expansion with ε taken small, we have

$$\varepsilon^2 \partial_t \phi(x) + \min_{Q\in\mathcal{Q}} \max_{b\in\mathcal{B}} \left(\sqrt{2}\varepsilon \nabla\phi(x) \cdot Qb + \varepsilon^2 \left(\nabla^2\phi(x)Qb\right) \cdot Qb\right) = O(\varepsilon^3).$$

It follows immediately that

$$\partial_t \phi - |\nabla\phi| \operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty),$$

on the basis of the fundamental lemma below. (Notice that the replacement of min max by max min does not change the equation.)

LEMMA 1.3. *For any $\xi \in \mathbb{R}^n \setminus \{0\}$ and $X \in \mathbb{R}^{n\times n}$ symmetric, the following inequalities hold:*

$$\min_{Q\in\mathcal{Q}}\max_{b\in\mathcal{B}}\left(\frac{1}{\varepsilon}\xi^\top Qb + (Qb)^\top XQb\right) \le tr\left((I - \frac{\xi\otimes\xi}{|\xi|^2})X\right). \tag{1.7}$$

$$\min_{Q\in\mathcal{Q}}\max_{b\in\mathcal{B}}\left(\frac{1}{\varepsilon}\xi^\top Qb + (Qb)^\top XQb\right) \ge tr\left((I - \frac{\xi\otimes\xi}{|\xi|^2})X\right) - C\varepsilon^2, \tag{1.8}$$

where the constant $C = C(n, X) > 0$ is bounded whenever $X \in \mathbb{R}^{n\times n}$ is bounded. In particular,

$$\min_{Q\in\mathcal{Q}}\max_{b\in\mathcal{B}}\left(\frac{1}{\varepsilon}\xi^\top Qb + (Qb)^\top XQb\right) \to tr\left((I - \frac{\xi\otimes\xi}{|\xi|^2})X\right) \quad as\ \varepsilon\to 0 \tag{1.9}$$

locally uniformly with respect to $\xi \in \mathbb{R}^n \setminus \{0\}$ and $X \in \mathbb{R}^{n\times n}$.

PROOF. Since $\mathcal{Q}$ is invariant under rotation and $\mathcal{B}$ is symmetric with respect to 0, we assume throughout the proof $\xi = (0, \dots, 0, |\xi|)^\top$. Let us first consider the case $|\xi| = 1$. Then we have a convenient equivalence $I - \xi\otimes\xi = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0\end{pmatrix}$, where I_{n-1} denotes the identity in $\mathbb{R}^{(n-1)\times(n-1)}$.

Part 1. To show (1.7), we ake a specific Q such that $Q = \begin{pmatrix}\tilde{Q} \\ 0\end{pmatrix}$, where $\tilde{Q} \in \mathbb{R}^{(n-1)\times(n-1)}$ fulfills $\tilde{Q}^\top = \tilde{Q}^{-1}$, and then we get

$$(Qb)^\top XQb = b^\top\tilde{Q}^\top(I_{n-1}, 0)X\begin{pmatrix} I_{n-1} \\ 0\end{pmatrix}\tilde{Q}b.$$

We choose $\tilde{Q}$ to diagonalize the matrix $(I_{n-1}, 0)X\begin{pmatrix} I_{n-1} \\ 0\end{pmatrix}$ so that

$$(Qb)^\top XQb = \mathrm{tr}\left((I - \frac{\xi\otimes\xi}{|\xi|^2})X\right)$$

and obtain (1.7).

Part 2. To prove (1.8), we claim that there exists $C_n > 0$ depending only on n such that for any $Q = (v_1, \dots, v_{n-1}) \in \mathcal{Q}$ and $\varepsilon > 0$

$$\max_{b\in\mathcal{B}}\left(\frac{1}{\varepsilon}\xi^\top Qb + (Qb)^\top XQb\right) \ge \frac{1}{\varepsilon}C_n|Q^\top\xi| + \sum_{i=1}^{n-1} v_i^\top Xv_i. \tag{1.10}$$

We postpone the proof of this claim. From Part 1, we get $|Q^\top\xi| \le C\varepsilon$ for some constant $C > 0$ depending on n and X; in other words, we have $|v_i\cdot\xi| \le C\varepsilon$ for all $i = 1, ..., n-1$. Take a vector $\eta \in \mathbb{R}^n$ with $|\eta| = 1$ and $\eta\cdot v_i = 0$ for any $i = 1, ..., n-1$. Then we get $|\eta - \xi| \le Cn\varepsilon$, which implies

$$\sum_{i=1}^{n-1}(v_i)^\top Xv_i = \mathrm{tr}X - \eta^\top X\eta \ge \mathrm{tr}\big((I - \xi\otimes\xi)X\big) - C\varepsilon^2 \tag{1.11}$$

with $C = C(n, X) > 0$ updated. Combining (1.10) and (1.11), we are led to (1.8) in the case $|\xi| = 1$.

We next prove the claim (1.10). It is clear that one can take $b_{(1)} = \pm 1$ such that $(\xi \cdot v_1)b_{(1)} = |\xi \cdot v_1|$. Then in this case we have

$$\begin{aligned}&\frac{1}{\varepsilon}\xi^\top Qb + (Qb)^\top XQb \\ =&\frac{1}{\varepsilon}|\xi \cdot v_1| + \sum_{i=1}^{n-1} v_i^\top X v_i + \frac{1}{\varepsilon}\sum_{j=2}^{n-1}(\xi \cdot v_j)b_{(j)} + 2\sum_{j=2}^{n-1}\sum_{i=1}^{j-1} v_i^\top X v_j b_{(i)} b_{(j)}.\end{aligned}$$

Choosing $b_{(j)}(j = 2, ..., n-1)$ one after another, we can make the last two terms in the above inequality nonnegative. We thus reach the conclusion that

$$\max_{b\in\mathcal{B}}\left(\frac{1}{\varepsilon}\xi^\top Qb + (Qb)^\top XQb\right) \geq \frac{1}{\varepsilon}|\xi \cdot v_1| + \sum_{i=1}^{n-1} v_i^\top X v_i.$$

We similarly obtain other inequalities with the first term on the right hand side above replace by $\frac{1}{\varepsilon}|\xi \cdot v_i|$ for all $i = 2, ..., n-1$. It then suffices to take the average of these inequalities to get (1.10).

For the general case that $|\xi| \neq 1$, we take $\lambda = |\xi| > 0$ and then (1.7) and (1.8) hold with ξ/λ and X/λ. It follows that

$$\begin{aligned}&\operatorname{tr}\left(\left(I - \frac{\xi\otimes\xi}{|\xi|^2}\right)X\right) - \lambda C(n, X/\lambda)\varepsilon^2 \\ \leq&\min_{Q\in\mathcal{Q}}\max_{b\in\mathcal{B}}\left(\frac{1}{\varepsilon}\xi^\top Qb + (Qb)^\top XQb\right) \leq \operatorname{tr}\left(\left(I - \frac{\xi\otimes\xi}{|\xi|^2}\right)X\right)\end{aligned}$$

and the rest of our statements hold. □

There is an elliptic version of the games above, which we review briefly in what follows. We need a domain in which solutions of an elliptic equation can be defined. For our particular purpose, we relax the notion of a domain to a more general open set Ω. In fact, there is no obvious reason to restrict our study in a domain, especially from the game-theoretic point of view and we are curious about the solutions on a finite union of open, bounded and connected subsets, say, a set shaped like a figure eight.

The equation we are concerned with is

$$\text{(SC)} \qquad \begin{cases} -|\nabla U|\operatorname{div}(\frac{\nabla U}{|\nabla U|}) - 1 = 0 & \text{in } \Omega. \\ U = 0 & \text{on } \partial\Omega. \end{cases}$$

We follow the same rules as in (1)–(3) above, but this time we are interested in the exit time. Namely, for each $x \in \mathbb{R}^n$, we denote by $T^\varepsilon(x; \alpha, \beta)$ the first time of exit from Ω and by $\hat{T}^\varepsilon(x; \alpha, \beta)$ the first time of exit from $\overline{\Omega}$ under alternate controls $Q \in \mathcal{Q}$ and $b \in \mathcal{B}$ determined by both players. Define

$$U_1^\varepsilon(x) = \max_\alpha \min_\beta \hat{T}^\varepsilon(x; \alpha, \beta) \quad \text{and} \quad U_2^\varepsilon(x) = \min_\alpha \max_\beta T^\varepsilon(x; \alpha, \beta) \tag{1.12}$$

and let

$$\overline{U_i} = \operatorname*{limsup^*}_{\varepsilon\to 0} U_i^\varepsilon \quad \text{and} \quad \underline{U_i} = \operatorname*{liminf_*}_{\varepsilon\to 0} U_i^\varepsilon \quad \text{in } \overline{\Omega} \tag{1.13}$$

for $i = 1, 2$. We then have

THEOREM 1.4 (Games for the elliptic problem). *Suppose Ω is a bounded open set. Then $\overline{U_i}$ and $\underline{U_i}$ defined in* (1.13) *are respectively viscosity subsolutions and supersolutions of* (SC) *for $i = 1, 2$ with the boundary conditions interpreted in the viscosity sense.*

Interestingly, we still get the subsolutions and supersolutions of (SC) when taking $\mathcal{Q}'$ in place of $\mathcal{Q}$, which is different from the former parabolic case. The elliptic problem therefore seems to have more solutions and be more complicated than the parabolic problem. Again, we skip the proof, which follows [75]. See also Propositions 3.18 and 3.19.

We next take the concrete example of evolution of spheres and explain it in terms of games. We give two preliminary lemmas on a very special type of strategies, which we call *concentric sphere strategy*, They essentially play the role of comparison with evolution of spheres and will be often applied later.

LEMMA 1.5 (Concentric sphere strategy of Paul). *For the games with control set $\mathcal{Q}$, Paul has a strategy α_c such that for all $t \geq 0$ and Carol's strategy β the following results hold:*

(i) *There exists a function $\omega_0^1 : [0, \infty) \to \mathbb{R}$ satisfying $y(x, t+\omega_0^1(\varepsilon); \alpha_c, \beta) \in B_{r(t)}(x_0)$ if $x \in B_{r_0}(x_0)$;*

(ii) *There exists a function $\omega_0^2 : [0, \infty) \to \mathbb{R}$ satisfying $y(x, t+\omega_0^2(\varepsilon); \alpha_c, \beta) \in B_{r(t)}(x_0)^c$ if $x \in B_{r_0}(x_0)^c$,*

where $r(t) = (r_0^2 + 2t)^{\frac{1}{2}}$ and $|\omega_0^i(\varepsilon)| \leq \varepsilon^2$ for each $\varepsilon > 0$ and $i = 1, 2$.

PROOF. We ask Paul to choose the directions tangential to the concentric spheres where the marker is located. More specifically, for every step, suppose the marker is at $y \in \mathbb{R}^n$. Then Paul should take $Q \in \mathcal{Q}$ so that $(y - x_0) \cdot Q = 0$. This choice is of feedback and enables us to get

$$|y(x, t, \alpha_c, \beta) - x_0|^2 = r_0^2 + 2\varepsilon^2[t/\varepsilon^2]$$

by inductive applications of the Pythagoras Theorem. Let $\omega_0^1(\varepsilon) = t - [\frac{t}{\varepsilon^2}]\varepsilon^2$ and $\omega_0^2(\varepsilon) = t + \varepsilon^2 - [\frac{t}{\varepsilon^2}]\varepsilon^2$. Then (i) and (ii) follow easily and it is also clear that $|\omega_t^i(\varepsilon)| \leq \varepsilon^2$ for $i = 1, 2$. □

LEMMA 1.6 (Concentric sphere strategy of Carol). *For the games with control set $\mathcal{Q}$, Carol has a strategy β_c such that for all $t \geq 0$ and Paul's strategy α the following results hold:*

(i) *There exists a function $\omega_0^1 : [0, \infty) \to \mathbb{R}$ satisfying $y(x, t+\omega_0^1(\varepsilon); \alpha, \beta_c) \in B_{r(t)}(x_0)$ if $x \in B_{r_0}(x_0)$;*

(ii) *There exists a function $\omega_0^2 : [0, \infty) \to \mathbb{R}$ satisfying $y(x, t+\omega_0^2(\varepsilon); \alpha, \beta_c) \in B_{r(t)}(x_0)^c$ if $x \in B_{r_0}(x_0)^c$,*

where $r(t) = (r_0^2 + 2t)^{\frac{1}{2}}$ and $|\omega_0^i(\varepsilon)| \leq \varepsilon^2$ for each $\varepsilon > 0$ and $i = 1, 2$.

PROOF. The proof is as follows: For every game position y, no matter what $Q \in \mathcal{Q}$ is, take b satisfying $(y - x_0) \cdot Qb \leq 0$, which is certainly possible. Then for every step

$$|y + \sqrt{2}\varepsilon Qb - x_0|^2 - |y - x_0|^2 \leq 2\varepsilon^2,$$

which implies (i) and (ii). The choices of ω_0^1 and ω_0^2 are the same as those in Lemma 1.5. □

Notice that Lemmas 1.5 and 1.6 guarantee that the evolution of radius is $r(t) = (r_0^2 + 2t)^{\frac{1}{2}}$, which completely agrees with the solution of curvature flow equation.

REMARK 1.1. We need $\omega_0^1(\varepsilon)$ and $\omega_0^2(\varepsilon)$, which actually depend on t too, since our computation is on the discrete level and errors are caused by discretization. One can also merely consider the times which can be divided by ε^2 exactly and define the game values by interpolation. In this thesis we choose to add this tiny adjustment in our computation. Hereafter we do not distinguish ω_0^1 and ω_0^2 and only use the notation ω_0 to denote either of them.

REMARK 1.2. It is easy to see that the conclusions in Lemma 1.5 and Lemma 1.6(i) are also true for the modified games in Theorem 1.2 with Paul's control set $\mathcal{Q}$ replaced by $\mathcal{Q}'$, but that of Lemma 1.6(ii) is not.

In spite of the great progress made in the recent years, there are still a lot of questions remain unsolved. Kohn and Serfaty posed two fundamental questions related to the game approach [**75, 76**].

(1) Can we find games for more general PDE problems? Does every equation have a game-theoretic interpretation?
(2) Can we give applications of these games besides the numerics? Can our deterministic control interpretations be used to prove new results about PDE?

Our present work is related to these two directions. In Chapter 2 and Chapter 4, we attempt to show that our control-based approximation work for more problems: we study the Neumann boundary problem for mean curvature flow equation in Chapter 2 and a discrete scheme for the conservation law equation in Chapter 4. Chapter 3 is devoted to the applications of games.

Neumann and oblique problems. We study a deterministic game interpretation for the Neumann boundary problem of the two-dimensional curvature flow equation:

$$\text{(NP1)} \qquad \begin{cases} \partial_t u - |\nabla u| \operatorname{div}(\frac{\nabla u}{|\nabla u|}) = 0 & \text{in } \Omega \times (0, \infty), \\ \nabla u(x,t) \cdot \nu(x) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \overline{\Omega}, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with a C^2 boundary $\partial\Omega$ and $\nu(x)$ is the unit outward normal to $\partial\Omega$ at x.

The Neumann type boundary condition is interpreted in the viscosity sense. This relaxation is first proposed by Lions [**79**]. Applications to deterministic optimal control and differential game theory in [**79**] rely much on a reflecting processes, the solution of the deterministic Skorokhod problem. The Skorokhod problem, especially in its stochastic version, is of great importance in stochastic system theory. Its properties in different circumstances are clarified in a great deal of literature such as [**111, 81, 33**].

Our work serves a two-fold purpose: (a) We show that the game-theoretic approach can be extended to the Neumann boundary problem of mean curvature flow, whose well-posedness is studied in [**100**] for a convex domain and in [**57**] for a nonconvex one. See also [**72, 101**] for more general boundary problems. (b) What is more important is that we give a discrete way of realizing

Neumann boundary condition. Instead of discretizing the Skorokhod problem, we use the billiard dynamics, which is a classical but quite different type of reflections on the boundary $\partial\Omega$. It turns out that billiard motion, usually studied in dynamical systems, can be adapted to our game interpretation with values approximating the solution of (NP1). Moreover, in two dimensions the billiard law seems more explicit than the Skorokhod reflection.

The adoption of the billiard dynamics is not straightforward. It actually requires a little modification so that we can handle its boundary behavior. We follow early study due to Halpern [**59**], Katok and Strelcyn [**74**], and a recent book by Chernov and Markarian [**27**] to define a billiard semiflow on the whole closure of the domain and investigate some of its properties. A more general billiard semiflow for oblique boundary problems is treated in [**82**]. See also a forthcoming work [**83**].

To explain heuristically in more detail how to characterize the solution of (NP1) by a family of games, we first recall the fundamental planar billiard law: the angle of incidence equals to the angle of reflection, based on which we denote by $S^t(x, v)$ the position of unit-speed billiard motion at time t for a starting point $x \in \overline{\Omega}$ and a direction v. Then if we assume:

(D1) $\quad\Omega$ is a bounded and convex domain in $\mathbb{R}^2$ with C^2 boundary $\partial\Omega$,

it then can be shown that

$$S^t(x, v) = x + tv - \alpha^t(x, v) \text{ with } |\alpha^t(x, v)| \leq 2t, \tag{1.14}$$

where $\alpha^t(x, v)$ is an adjusting vector, identified as a (possibly infinite) sum of outward normals to the boundary near x. Such a representation actually demonstrates that billiards are similar in form to the Skorokhod problem, though it is not clear in what sense the solution of Skorokhod problem can be understood as a direct limit of billiards without putting them in our game or optimal control setting as follows.

Let us again play a discrete two-person game with an initial position $x \in \overline{\Omega}$ and the maturity time $t > 0$. We follow out general notation above. One game player, Paul, is supposed to minimize the value $u_0(y(N))$ while the other, Carol, is to maximize it. In the k-th round,

(1) Paul chooses a direction v_k, i.e., $|v_k| = 1$;
(2) Carol has the right to reverse Paul's choice, which determines $b_k = \pm 1$;
(3) The game state is moved from $y(k-1)$ to $y(k) = S^{\sqrt{2}\varepsilon}(y(k-1), b_k v_k)$.

The novelty of our games is that we implement a mirror-like reflection on the boundary so as to keep the game proceeding. The state equations are:

$$\begin{aligned} &y(k) = S^{\sqrt{2}\varepsilon}(y(k-1), b_k v_k), \quad \text{for all } k = 1, 2, \ldots, N; \\ &y(0) = x \in \overline{\Omega}. \end{aligned}$$

For every starting position $x \in \overline{\Omega}$ and time $t \in [0, T]$, we define the value function as

$$u^\varepsilon(x, t) = \inf_{|v_1|=1} \sup_{b_1=\pm 1} \ldots \inf_{|v_N|=1} \sup_{b_N=\pm 1} u_0(y(N)), \tag{1.15}$$

which, in particular, implies $u^\varepsilon(x,t) = u_0(x)$ when $t \in [0, \varepsilon^2)$. By definition, u^ε satisfies the dynamic programming principle:

$$u^\varepsilon(x,t) = \inf_{|v|=1} \sup_{b=\pm 1} u^\varepsilon \left(S^{\sqrt{2}\varepsilon}(x, bv), t - \varepsilon^2 \right) \text{ for all } t \in (0, \infty). \tag{1.16}$$

It follows formally by Taylor's expansion and our billiard expression (1.14) that

$$\begin{aligned} 0 \approx -\varepsilon^2 u_t^\varepsilon + \inf_{|v|=1} \sup_{b=\pm 1} \Big\{ & \nabla u^\varepsilon \cdot \left(\sqrt{2}\varepsilon bv - \alpha^{\sqrt{2}\varepsilon} \right) \\ & + \frac{1}{2} \nabla^2 u^\varepsilon \left(\sqrt{2}\varepsilon bv - \alpha^{\sqrt{2}\varepsilon} \right) \cdot \left(\sqrt{2}\varepsilon bv - \alpha^{\sqrt{2}\varepsilon} \right) \Big\} \quad \text{at } (x,t). \end{aligned} \tag{1.17}$$

To emphasize our particular interest in the boundary condition, we assume $x \in \partial\Omega$. Viewing for the moment that $u^\varepsilon(x,t)$ has bounded derivatives and converges in some sense to a function $u(x,t)$ and

$$\lim_{\varepsilon \to 0} \frac{\alpha^{\sqrt{2}\varepsilon}}{|\alpha^{\sqrt{2}\varepsilon}|} = \nu(x) \quad \text{uniformly in } b \text{ and } v, \tag{1.18}$$

we discuss two cases for every subsequence, still indexed by ε:

1. Boundary condition dominant case: There exists $C > 0$ such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} |\alpha^{\sqrt{2}\varepsilon}| = C.$$

We then divide both sides of (1.17) by ε, pass to the limit $\varepsilon \to 0$ and get via (1.18) that

$$0 = \sqrt{2} \inf_{|v|=1} \sup_{b=\pm 1} |\nabla u(x,t) \cdot bv| - C \nabla u(x,t) \cdot \nu(x).$$

Since the first term on the right hand side is always zero, the classical Neumann boundary condition remains.

2. Mixed type case: Assume on the contrary to the former case

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} |\alpha^{\sqrt{2}\varepsilon}| = 0.$$

Then the same first-order operation as above yields that the "inf sup" is attained at $v = \frac{\nabla^\perp u}{|\nabla u|}$, where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. We here assume that $\nabla u(x,t) \neq 0$ for otherwise we realize the Neumann boundary condition again. Despite uncertainty about the order of $|\alpha^{\sqrt{2}\varepsilon}|$, we can always get the boundary condition desired. Indeed, if $\frac{1}{\varepsilon^2}|\alpha^{\sqrt{2}\varepsilon}| \to \infty$ as $\varepsilon \to 0$, we divide both sides of (1.17) by $|\alpha^{\sqrt{2}\varepsilon}|$ and send $\varepsilon \to 0$ to get

$$\nabla u(x,t) \cdot \nu(x) = 0.$$

If $|\alpha^{\sqrt{2}\varepsilon}|$ is of order $o(\varepsilon^2)$, we in turn use ε^2 as the divisor and obtain the limit equation

$$u_t - \left(\nabla^2 u \frac{\nabla^\perp u}{|\nabla u|} \right) \cdot \frac{\nabla^\perp u}{|\nabla u|} = 0 \text{ at } (x,t),$$

or in other words,

$$u_t - |\nabla u| \mathrm{div} \left(\frac{\nabla u}{|\nabla u|} \right) = 0 \text{ at } (x,t).$$

The most sophisticated case is ascribed to the occasion when $|\alpha^{\sqrt{2}\varepsilon}|$ exactly has the order ε^2. The limit of the divided equation then is

$$u_t(x,t) - \nabla^2 u(x,t) \frac{\nabla^\perp u(x,t)}{|\nabla u(x,t)|} \cdot \frac{\nabla^\perp u(x,t)}{|\nabla u(x,t)|} + M \nabla u(x,t) \cdot \nu(x) = 0,$$

where M is a positive constant. It follows that either

$$u_t - \nabla^2 u \frac{\nabla^\perp u}{|\nabla u|} \cdot \frac{\nabla^\perp u}{|\nabla u|} \geq 0 \text{ and } \nabla u \cdot \nu(x) \leq 0 \text{ at } (x,t)$$

or

$$u_t - \nabla^2 u \frac{\nabla^\perp u}{|\nabla u|} \cdot \frac{\nabla^\perp u}{|\nabla u|} \leq 0 \text{ and } \nabla u \cdot \nu(x) \geq 0 \text{ at } (x,t),$$

which shows a good chance that u fulfills the Neumann boundary condition in the viscosity sense.

The preceding mechanism gives rise to the pivotal result.

THEOREM 1.7 ([**53**]). *Assume that Ω satisfies* (D1). *Assume that u_0 is a continuous function in $\overline{\Omega}$. Let u^ε be the value function of the game defined by* (1.15). *Then u^ε converges, as $\varepsilon \to 0$, to the unique viscosity solution of* (NP1) *uniformly on compact subsets of $\overline{\Omega} \times [0, \infty)$.*

If one keeps above formal argument, translation into viscosity language is similar to [**75**] for $\Omega = \mathbb{R}^2$, although we need several properties of a billiard semiflow. We shall prove in Chapter 2 that the upper relaxed limit $\overline{u} = \limsup^* u^\varepsilon(x,t)$ is a subsolution while the lower relaxed limit $\underline{u} = \liminf_* u^\varepsilon(x,t)$ is a supersolution If there is no terminal layer so that $\overline{u} = \underline{u}$ at $t = 0$, the comparison principle yields $\overline{u} = \underline{u}$, which implies local uniform convergence.

To show the nonexistence of terminal layer, the value functions u^ε are proved in [**75**] to be equicontinuous for a C^2 terminal datum u_0 when $\Omega = \mathbb{R}^2$. However, this method is not directly applicable for our general Ω because of an extra difficulty about the possible discontinuity of billiard motion. We instead use the barrier argument to prove the convergence without assuming that u_0 is C^2. (Our method provides a strong result even when $\Omega = \mathbb{R}^2$.)

With several technical modification, the convergence result in Theorem 1.7 is still valid when Ω is not convex under some additional assumptions on the boundary $\partial\Omega$. Our approach works at least for nonconvex domains which have finite bumps. We here do not intend to give any generalization of equations, dimensions or games, though it is possible as suggested in [**75**], so as to clarify the essential part of the problem. Moreover, applications of this billiard dynamics to the Neumann or general oblique problem of first-order Hamilton-Jacobi equations are also possible and even simpler; see [**83**].

Fattening and comparison principles. We give several applications of the discrete game approach to partial differential equations. We first present a rigorous game-theoretic proof of fattening phenomenon for motion by curvature with figure-eight shaped initial curves without using parabolic PDE theory.

Let us recall in brief the background of the fattening phenomenon. We consider a family $\{\Gamma_t\}_{t\geq 0}$ of compact hypersurfaces embedded in the Euclidean space $\mathbb{R}^n$. The *mean curvature flow* equation is originally written as

$$V = \kappa, \tag{1.19}$$

where V and κ denote respectively the normal velocity and (mean) curvature of Γ_t. With proper initial data Γ_0, the existence and uniqueness of such a smooth solution is well understood. The smooth solution, however, usually exists only in finite time and ends up becoming singular [**47, 58**]. Classical methods cannot be applied after the singularity and other approaches based on a definition of generalized solutions, are necessary. It is now widely known that there are at least three effective approaches of generalized solutions comprising one from geometric measure theory [**18**], another through phase transitions (see, for example, [**39**]) and the third of a level set method [**23, 41**], which is presented in great detail in the book [**51**].

In spite of the perfect existence of all these generalized solutions, the answer to the uniqueness problem remains incomplete. On some occasions, the enhanced varifold solution might be non-unique and the convergence for phase transition approach breaks down. By contrast, the level set method turns the geometric motion into the partial differential equation (MC) and is supposed to ensure uniqueness of solutions because it is established on the ground of comparison principles in parabolic PDE theory, but, unfortunately, a level set at times turns out to develop an interior, which is thereby named *fattening.*

More precisely, let u_0 be a bounded uniformly continuous function such that $\{x \in \mathbb{R}^n : u_0(x) = 0\} = \Gamma_0$. Using the viscosity solution theory, we get a unique solution u of (MC). We are then interested in the situation that the level set $\Gamma_t = \{x \in \mathbb{R}^n : u(x,t) = 0\}$ develops interior even when Γ_0 has empty interior. Examples are given in [**41, 103, 63, 51**] for Cauchy problems and in [**48, 49, 8**] for Neumann boundary problems. Although this difficulty can be overcome by imposing various nonfattening condition on the initial data [**14, 1, 104, 51**] or by adding stochastic perturbation on the equation [**32, 107**], it is not well understood in general.

In contrast to other approximation methods, an important feature of games is that the convergence keeps valid even when level sets develop interior. Our primary motivation is therefore to understand the game interpretation for fattening. For different purposes, we use two game-theoretic methods:

(1) comparing with the *inverse games*; and
(2) perturbing the *objective function.*

The first way, from the viewpoint of PDE, is to substitute u_0 with $-u_0$ and if the geometric flow is *orientation-free*, the solution essentially remains unchanged, but the corresponding game becomes different. We will later see that the optimal trajectories in the original and inverse games can be entirely different. Several examples in two dimensions with explicit game strategies are given in Section 3.1 to show this clearly. These examples of "figure eight" are all known to cause fattening ([**41, 51, 103**], etc). We intend to reveal on some level that fattening can be studied by observing and comparing the distinction between the optimal decisions of players for both types of games.

It is worth remarking that we employ only the game interpretation without using any parabolic PDE theory, which is usually resorted to when one tries to prove the existence of fattening rigorously [**17, 51, 63, 103**]. Our idea is close to the proof of fattening for first order equations in [**14**] and our computation for the examples is elementary.

It is an exclusive property of the second-order games that their inverse versions lead to the same equations. A large class of first-order Hamilton-Jacobi equations possess interpretation of continuous-time and deterministic differential games [**5**], but if one alters the players' objectives, other equations will be derived. Our method is consequently not applicable to the fattening phenomenon for Hamilton-Jacobi equations. Also, when dealing with the geometric flows which are not orientation-free such as (PMC) or those with driving forces, one will find it inconvenient to construct the inverse games. It is easier to use our second method based on perturbed objective function, which is more natural from the PDE point of view. We will not develop the method for parabolic equations. Rather, we would like to apply our point of view to the elliptic equation (SC).

We at first remark that the equation in question should be nonhomogeneous. An example [**96**] shows that the solutions of

$$\begin{cases} |\nabla U|\mathrm{div}(\frac{\nabla U}{|\nabla U|}) = 0 & \text{in } \Omega, \\ U = f & \text{on } \partial\Omega \end{cases}$$

are not unique even in a two-dimensional disk with smooth boundary data f. We thus have to turn our attention to (SC), which looks more reasonable.

The study of (SC) is usually conducted in a bounded and (strictly) *mean convex* domain Ω, initiated by [**41**], in which the existence and uniqueness of continuous solutions are clarified. About the regularity, since the equation is degenerate elliptic, it is hard to apply the Krylov-Safonov theory directly. Ilmanen [**63**] shows that the solution does not belong to the class C^2 in general. For a strictly convex domain, it is recently known that the solution is of class $C^3(\overline{\Omega})$ in two dimensions [**75**] but is not necessarily the case in higher dimensions [**102**].

If Ω is not mean convex, little is known about the wellposedness, since there may be loss of boundary condition. We thus have to relax the Dirichlet boundary condition to a weak sense. The paper [**75**] establishes a family of *exit time* games with values U^ε and shows the *relaxed limits* in $\overline{\Omega}$

$$\limsup_{\varepsilon\to 0}{}^* U^\varepsilon \text{ and } \liminf_{\varepsilon\to 0}{}_* U^\varepsilon$$

are respectively an upper semicontinuous subsolution and a lower semicontinuous supersolution with the Dirichlet boundary condition interpreted in the viscosity sense. By assuming the domain to be *star-shaped* and then establishing a *weak comparison principle*, they prove the existence and uniqueness of solutions which are not necessarily continuous. By weak comparison principle, we mean the property that any subsolution $W_1 \in USC(\overline{\Omega})$ and supersolution

$W_2 \in LSC(\overline{\Omega})$ of (SC) satisfy

$$(W_1)_* \leq W_2 \quad \text{and } W_1 \leq (W_2)^* \quad \text{in } \overline{\Omega},$$

where $(W_1)_*$ and $(W_2)^*$ stand respectively for the lowersemicontinuous envelope of W_1 and uppersemicontinuous envelope of W_2. However, it is an open question whether comparison principle and uniqueness of solutions hold in a more general domain.

We attempt to answer the question by investigating the effect of fattening on its elliptic versions. Heuristically speaking, since the formation of fat level sets is explained as nonuniqueness of solutions, it is tempting to see whether the nonuniqueness for parabolic equations can bring us loss of uniqueness for elliptic cases. The figure-eight shaped region, well analyzed in Section 3.1, turns out to be an immediate counterexample to disprove the existence of comparison principle even in the weak sense.

For a more general Ω, we utilize the corresponding version of our second method, perturbing Ω from inside and outside so that we have value functions $U_+^{\varepsilon,\delta}$ under the same game rules for the equation (PMC), where $\delta \in \mathbb{R}$ is a parameter standing for the perturbation exerted on Ω. Notice that a finite horizon game for (PMC) and an exit-time game for (SC) share exactly the same rules except for the form of cost functions. If fattening happens at a certain $x_0 \in \Omega$ to the signed mean curvature flow (PMC) starting from $\partial\Omega$, we are able to obtain optimal strategies for perturbed games of (PMC) and implement them in the exit time games of (SC) to get

$$\limsup_{\varepsilon\to 0,\ \delta\to 0+}{}^* U_+^{\varepsilon,\delta} > \liminf_{\varepsilon\to 0,\ \delta\to 0-}{}_* U_+^{\varepsilon,\delta}$$

in a neighborhood of x_0. Since the left hand side is a subsolution and the right hand side is a supersolution, we prove the loss of comparison principle.

THEOREM 1.8 ([**85**]). *Assume that Ω is an open bounded subset of $\mathbb{R}^n$. Let u_0 be the signed distance function of Ω (with values being negative in Ω). If the zero level set of the solution of signed mean curvature flow* (PMC) *develops interior in Ω, then the weak comparison principle for* (SC) *fails to hold.*

Any example of fattening becomes a counterexample for the existence of a weak comparison principle. Our counterexample of figure eight is now a special case of Theorem 1.8. As little is known about the positive mean curvature flow, we do not know whether or not the level set of the motion starting from smooth surface $\partial\Omega$ may develop nonempty interior in Ω, although there is evidence to show that the fattening of mean curvature flow (MC) does have chance to take place from a smooth initial surface in dimensions $n \geq 3$ ([**2, 3, 112**]).

On the other hand, we are also curious about the situation when there is no fattening during the evolution. We cannot prove the comparison principle holds in this case but we still obtain the existence of a solution when the motion (PMC) is regular enough.

THEOREM 1.9 ([**85**]). *Assume that Ω is an open bounded subset of $\mathbb{R}^n$. Let u_0 be the signed distance function (with values being negative in Ω) and u be the viscosity solution of* (PMC). *If the evolution* (PMC) *started from $\partial\Omega$ is* *regular in the sense that u satisfies*

$$\overline{\{(x,t) \in \mathbb{R}^n \times [0,\infty) : u(x,t) > 0\}} = \{(x,t) \in \mathbb{R}^n \times [0,\infty) : u(x,t) \geq 0\} \tag{1.20}$$

and

$$\overline{\{(x,t) \in \mathbb{R}^n \times [0,\infty) : u(x,t) < 0\}} = \{(x,t) \in \mathbb{R}^n \times [0,\infty) : u(x,t) \leq 0\}, \tag{1.21}$$

then

$$\left(\limsup_{\varepsilon\to 0,\ \delta\to 0+}{}^{*}\ U_+^{\varepsilon,\delta}\right)_* \leq \liminf_{\varepsilon\to 0,\ \delta\to 0-}{}_{*}\ U_+^{\varepsilon,\delta},\quad \limsup_{\varepsilon\to 0,\ \delta\to 0+}{}^{*}\ U_+^{\varepsilon,\delta} \leq \left(\liminf_{\varepsilon\to 0,\ \delta\to 0-}{}_{*}\ U_+^{\varepsilon,\delta}\right)^* \quad \text{in } \overline{\Omega}.$$

This theorem enables us to get the existence of a solution of (SC) which is continuous except at a nowhere dense subset in $\overline{\Omega}$ and verify the game approximation (in a weaker sense) without using any comparison principle. The regularity assumptions (1.20) and (1.21) say that neither the interior motion nor the exterior motion develops nonempty level sets. The uniqueness of solutions is not clear in this case because comparison results are still unknown.

Since the starshapedness of Ω implies regularity of (PMC) ([**51**, Theorem 4.5.9]), we actually generalizes the game interpretation in [**75**]. Our idea for Theorem 1.9 is inspired by [**106**], in which a game approach for the first order Hamilton-Jacobi equations is exploited.

One may wonder why the motion by positive mean curvature (PMC) instead of the original motion by curvature (MC) is involved. We prefer the former because of a certain monotonicity along trajectories in its game representation. It is not clear whether we can substitute all assumptions on (PMC) in the theorems above with similar ones on (MC) and obtain the same conclusions.

To conclude our introduction, we pose the following two questions:

(1) The loss of weak comparison principle of (SC) is closely related to the fattening for (PMC). Are they actually equivalent?
(2) Does the solution of positive mean curvature flow equation (PMC) have fattening behavior when and only when (MC) does? Note that it is true for the initial curve of a figure-eight.

If we can give affirmative answers to the above, we believe that the weak comparison principle for a smooth domain should hold in two dimensions but does not need to be true in higher dimensions.

We will not tackle other geometric flows but our results here can be extended for more general geometric motions.

A scheme for shocks. We consider a discrete approximation for solutions of the following equation

$$\text{(CL)} \qquad u_t + f(u)u_x = 0 \quad \text{in } \mathbb{R} \times (0,\infty)$$

with initial data

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}. \tag{1.22}$$

It is the so-called one-dimensional conservation law equation. We here start with the one-dimensional cases in order to make the presentation simpler. We assume throughout the note that $f \in C(\mathbb{R})$ is nondecreasing. The equation (CL) can be rewritten as follows:

$$u_t + F(u)_x = 0 \quad \text{in } \mathbb{R} \times (0, \infty),$$

where $F \in C^1(\mathbb{R})$ is assumed to be convex.

Let us follow the method in [77]. Formally, we can write down the state equation:

$$\begin{aligned} \frac{dy}{ds} &= -f(w) \\ \frac{dw}{ds} &= 0 \end{aligned} \tag{1.23}$$

with $y(0) = x$ and $w(0) = z$. We denote the solutions by $y(t; x, z)$ and $w(t; x, z)$. If one then looks into the height function

$$U(x, z, t) = u_0(y(t; x, z)) - w(t; x, z) \quad \text{for all } t \geq 0, \tag{1.24}$$

we have $U(x, z, 0) = u_0(x) - z$ and obtain the dynamical programming principle, saying that for any $0 < s < t$.

$$U(x, z, t) = U(y(s; x, z), w(s; x, z), t - s). \tag{1.25}$$

Let

$$\overline{u}(x, t) = \sup\{z : U(x, z, t) \geq 0\} \text{ and } \underline{u}(x, t) = \inf\{z : U(x, z, t) \leq 0\}.$$

We can show $\overline{u}$ is a subsolution and $\underline{u}$ is a supersolution of (CL) under the usual definitions of viscosity solutions. Indeed, it is clear by the dynamical programming that

$$0 = U(x, \overline{u}(x, t), t) = U(y(s; x, \overline{u}(x, t)), w(s; x, \overline{u}(x, t)), t - s),$$

which implies by definition of $\overline{u}$ that

$$w(s; x, \overline{u}(x, t)) \leq \overline{u}(y(s; x, \overline{u}(x, t)), t - s).$$

By applying the state equations (1.23) and Taylor expansion, we are led, at least on the heuristical level, to

$$\overline{u}(x, t) \leq \overline{u}(x, t) - sf(\overline{u}(x, t))\overline{u}_x(x, t) - s\overline{u}_t(x, t) + o(s).$$

Dividing the above inequality by s and sending $s \to 0$, we have

$$\overline{u}_t + f(\overline{u})\overline{u}_x \leq 0.$$

The supersolution part can be proved in a similar way. If there is a comparison principle, we have $\overline{u} \leq \underline{u}$ and convergence of u^ε is proved.

Unfortunately, the formal formulation above however does not work in practice because U may not describe the graph of a function; in other words, solutions defined above may "overturn" at finite time even if it is initially smooth, as is known to all. One can easily find an example to show that sometimes $\overline{u} > \underline{u}$ in an open set of $\mathbb{R} \times (0, \infty)$.

An idea in [50] to overcome the difficulty is as follows. Treat the graph of u_0 as a curve and use level set method to study the motion of this curve. By adding extra tests in the case when the curve becomes vertical, i.e. formation of shocks takes place, we can prevent the graph from overturning. A graph henceforth keeps being a graph and one ends up obtaining a function, which is a global-in-time solution, called a proper solution. The uniqueness of solutions is guaranteed by comparison theorems. What is more, the extra tests are essentially consistent with the Rankine-Hugoniot condition in our case of conservation law and therefore the notion of proper solutions is equivalent to that of more classical entropy solutions.

We must point out that the existence of solutions in [**50**] is shown by using the vanishing viscosity method instead of the more direct characterization above. Our aim of this part is to provide the existence by means of such kind of natural construction. We mix the method of characteristics (and, more complicated, optimal control and game theory) with the viscosity approach to shocks. From the viewpoint of games, our novelty is that we present a scheme for possibly nonlocal equations.

Our approximation can also be compared with the so-called kinetic formulation (see [**56**] and [**94**], etc) for the conservation law equation. While the solutions are usually approximated in L^1 space, our method is for L^∞ space.

CHAPTER 2

Neumann and Oblique Boundary Problems

In this chapter, we discuss the game interpretation for

$$\text{(NP1)} \qquad \begin{cases} \partial_t u - |\nabla u| \mathrm{div}(\frac{\nabla u}{|\nabla u|}) = 0 & \text{in } \Omega \times (0, \infty), \\ \nabla u(x,t) \cdot \nu(x) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \overline{\Omega}. \end{cases}$$

We introduce convex planar billiards including the definition and properties in Section 2.1. A comparison between billiard and Skorokhod types of reflection is discussed as well. Section 2.2 is the major part, devoted to the rigorous proof of Theorem 1.7 and an extension to nonconvex domains.

2.1. Billiard type reflection

This section provides a discrete system. The study of billiards is of independent interest.

2.1.1. General Planar Billiards. We start with a few preliminaries of classical billiards. A domain $\Omega \subset \mathbb{R}^2$, piecewise C^1 but not necessarily convex, is said to be a *billiard table*. We hereafter use a standard notation $\mathbf{S}^1$ to denote the set of all unit vectors in $\mathbb{R}^2$, i.e., $\mathbf{S}^1 = \{v \in \mathbb{R}^2 : |v| = 1\}$. The billiard flow in Ω, denoted by $T^t : \overline{\Omega} \times \mathbf{S}^1 \to \overline{\Omega}$ $(t \in \mathbb{R})$, describes the billiard motion in the table. To be more precise, imagine there is a little ball (mass point) starting from a point $x \in \overline{\Omega}$ moving at unit speed along a straight line directed by $v \in \mathbf{S}^1$ inside Ω. When it hits the boundary, the tangential component of the original direction v keeps still but its normal component changes sign immediately. After that, the ball goes on its straight-line motion until a next collision takes place. For a fixed pair (x, v), $T^t(x, v)$ represents the ball's position at time t. The set $\{T^t(x, v) \in \overline{\Omega} : t \geq 0\}$ is called a *billiard trajectory* starting from (x, v) and is deemed to record the whole track of movement. We call the hitting points on the boundary *vertices* of the trajectory.

We remark that a billiard flow should be rigourously defined as a group of maps on the phase space $\Omega \times \mathbf{S}^1$ with a parameter $t \in \mathbb{R}$, however, since its second component is discontinuous in t and of little concern here, we will pay more attention to T^t, its natural projection onto the first component, concealing the second one but referring to it as P_2T^t whenever needed.

As a flow, T^t clearly satisfies the group property restricted in $\Omega \times \mathbf{S}^1$ with the identity T^0 and $T^{-t}(x, v) = T^t(x, -v)$ for any $x \in \Omega$ and $v \in \mathbf{S}^1$.

It is easy to understand such a trajectory of a billiard motion but there are three types of situations invalidating its reasonability:

1. **Singularity:** The billiard ball hits a non-differentiable point of $\partial\Omega$.

2. **Entrance:** The billiard trajectory contains an interval in $\partial\Omega$ or in other words, the ball hits an inflection point in a direction right equal to the tangential to $\partial\Omega$.

3. **Termination:** The sequence of vertices $\{p_n\}_{n\geq 1}$ may converge to a point on $\partial\Omega$, called a *terminating point.* For a convex table, an equivalent definition is $\sum_{n\geq 1}|p_{n+1}-p_n| < \infty$, which reveals that the trajectory will break down by finite time. Halpern **[59]** provides an example. We review the results related to it in Appendix A.

From the point of view of dynamical systems, it is unnecessary to define the billiard flow for these three types, just because they rarely happen. It is known that Lebesgue mearsure of their correspondent starting points and directions is zero, provided that Ω is of finite measure (Theorem A.4). Nevertheless, for billiards in $\overline{\Omega}$ and further applications to games, we will fill the gaps by appending proper definitions.

2.1.2. A Modified Convex Billiard. From now on, the ball motion will be turned into another style, which is no longer a billiard in the classical sense. Let us call it a *modified billiard* and use the notation $S^t(x,v)$ to express its position in $\overline{\Omega}$ at time $t \geq 0$ for the motion starting from $x \in \overline{\Omega}$ with the initial direction $v \in \mathbf{S}^1$. It will be seen that S^t is not a flow but a semiflow, i.e., it satisfies associativity only for time $t \geq 0$. We need a better table fulfilling (D1). On this occasion, "singularity", among those situations listed in the last subsection, is immediately ruled out. Moreover, due to the convexity, the situation of entrance hardly takes place unless the motion starts on the boundary.

We hereafter utilize the arc-length parametrization $\Gamma(\cdot):\mathbb{R}\to\mathbb{R}^2$, a function of class C^2, to represent $\partial\Omega$. Its derivative with respect to s is denoted by Γ_s. We also use $\theta(s)$ to stand for the normal angle of $\Gamma(s)$. Its relationship with the normal is described by

$$\nu(\Gamma(s)) = (\cos(\theta(s)),\ \sin(\theta(s))). \tag{2.1}$$

We first prepare two lemmas. The first one is elementary, just like the mean value theorem.

LEMMA 2.1. *Assume that Ω satisfies* (D1) *and is parameterized as $\Gamma(\cdot)$ by its arc length. Then for any $s_2 > s_1$, there exists $\xi \in [s_1, s_2]$ such that*

$$Y(s_1,s_2) = -|Y(s_1,s_2)|\nu(\Gamma(\xi)), \tag{2.2}$$

where $Y(s_1,s_2) = \Gamma(s_2)-\Gamma(s_1)-(s_2-s_1)\Gamma_s(s_1) \in \mathbb{R}^2$; and

$$\Gamma_s(s_1)\cdot\nu(\Gamma(\xi)) \geq 0. \tag{2.3}$$

PROOF. We first fix s_1 and consider the curve locally near $\Gamma(s_1)$ so that the graph of a convex function $f \in C^2(\mathbb{R})$ can be employed to represent it. There is no loss of generality in assuming $\Gamma(s_1) = 0$ and $\Gamma_s(s_1) = (1,0)$, or equivalently, $f(0) = 0$ and $f'(0) = 0$. We thus obtain

$$\Gamma(s_2)-\Gamma(s_1)-(s_2-s_1)\Gamma_s(s_1) = \left(x - \int_0^x \sqrt{1+f'(t)^2}\,dt, f(x)\right),$$

where x is the unique value satisfying

$$\int_0^x \sqrt{1+f'(t)^2}\,dt = s_2 - s_1.$$

Setting the slope

$$k(s_1, s_2) := \frac{f(x)}{x - \int_0^x \sqrt{1+f'(t)^2}\,dt} = \frac{f(x)}{-\int_0^x \frac{f'(t)^2}{1+\sqrt{1+f'(t)^2}}\,dt}$$

and noticing that $\frac{f'(t)}{1+\sqrt{1+f'(t)^2}}$ and $f'(t)$ are both nondecreasing because of the convexity of f, we are led to

$$k(s_1, s_2) \le \frac{f(x)}{-\frac{f'(x)}{1+\sqrt{1+f'(x)^2}} \int_0^x f'(t)\,dt} \le -\frac{1}{f'(x)}.$$

Since the normal angle $\theta(\cdot)$ is continuous and nondecreasing with $\theta(s_1) = -\frac{\pi}{2}$ and $\theta(s_2) = -\arctan\frac{1}{f'(x)}$, the mean value theorem yields the existence of $\xi \in [s_1, s_2]$ such that

$$\tan(\theta(\xi)) = k(s_1, s_2).$$

Consequently we get the equality (2.2) immediately by applying (2.1).

Now it remains to show (2.2) for a global situation. Since there is $\bar{x} > 0$ such that $f'(x) \to \infty$ as $x \to \bar{x}$, the graph representation is not allowed for $s_2 \ge \bar{s} := \int_0^{\bar{x}} \sqrt{1+f'(t)^2}\,dt$. However, the statement we are concerned with still holds. Indeed, in terms of the convexity of the boundary, $\Gamma(s_2)$ must be bounded in $[0,\bar{x}] \times [0,+\infty)$ even if $s_2 \ge \bar{s}$. Therefore $k(s_1, s_2)$ can still be computed to get

$$k(s_1, s_2) \le 0, \quad \text{if } s_2 \ge \bar{s}.$$

Noting that $\theta(s_1) = -\frac{\pi}{2}$ and $\theta(\bar{s}) = 0$, we again have some $\xi \in [s_1, \bar{s}] \subset [s_1, s_2]$ such that $\tan(\theta(\xi)) = k(s_1, s_2)$ and in turn obtain (2.2)

Observing from above that $\theta(\xi) \in [-\frac{\pi}{2}, 0]$, we easily get the inequality (2.3) since $\Gamma_s(s_1) \cdot \nu(\Gamma(\xi)) = \cos(\theta(\xi))$. □

The second important lemma is for the case of termination. It is due to Halpern [**59**] (see Theorem A.1 in Appendix A).

LEMMA 2.2. *Suppose that Ω satisfies* (D1). *If a trajectory terminates at a point $\Gamma(s_\infty) \in \partial\Omega$, with a sequence of vertices $\{\Gamma(s_n)\}_{n\ge1}$ arranged in order, then there exists $N > 0$ such that for $n \ge N$, s_n monotonically converges to s_∞ and $(\Gamma(s_\infty) - \Gamma(s_n))/|s_\infty - s_n|$ converges to a unit tangent, denoted by v_∞, to the boundary at $\Gamma(s_\infty)$.*

We are now in a position to give a definition for our modified billiard as follows.

DEFINITION 2.1. Let Ω satisfy (D1).

(i) If $x \in \partial\Omega$, and v equals to the tangent of $\partial\Omega$, then

$$S^t(x, v) := \Gamma(t), \quad \text{for any } t \ge 0,$$

where $\Gamma(\cdot)$ is the arc-length parametrization of $\partial\Omega$ such that $\Gamma(0) = x$ and $\Gamma_s(0) = v$.

(ii) If $x \in \Omega$ and v is such that $T^t(x,v)$ terminates on $\partial\Omega$ at time t_0, then

$$S^t(x,v) := \begin{cases} T^t(x,v) & \text{if } 0 \le t < t_0, \\ S^{t-t_0}(T^{t_0}(x,v), v_\infty) & \text{if } t \ge t_0, \end{cases}$$

where v_∞ is obtained from Lemma 2.2.

(iii) If $x \in \partial\Omega$ and v points inside Ω, then

$$S^t(x,v) := \begin{cases} x & \text{if } t = 0, \\ S^{t-\varepsilon}(x+\varepsilon v, v) & \text{if } t > 0, \end{cases}$$

where $\varepsilon > 0$ is such that $x + \delta v \in \Omega$ for all $\delta \in (0,\varepsilon)$.

REMARK 2.1. The definition of (i) says that when $x \in \partial\Omega$ and v lies on the tangential line of $\partial\Omega$, the ball will slide along the boundary curve. It corresponds to a special case in (ii) when the value of terminating time t_0 equals to zero.

In (ii), since $T^{t_0}(x,v) \in \partial\Omega$ and v_∞ is a tangent of $\partial\Omega$, we are able to put (i) to use. If the trajectory never terminates on the boundary, we just adhere to the convention $t_0 = \infty$.

Heuristically speaking, although the termination phenomenon stops the trajectory, a final direction remains. The billiard ball is set to inherit the direction and roll along the boundary. Ambiguity may be caused if we abuse the notation $P_2S^t(x,v)$ for $x \in \partial\Omega$. We however can overcome it by always taking the reflected-off direction for the points on $\partial\Omega$, for the natural reason that the billiard law will instantaneously switch any outward pointing unit vector into an inward one. S^t then can be easily verified to be a semiflow in this case.

For non-tangential motion starting on the boundary, it also suffices to give a definition for a reflected-off direction v, pointing inside the domain, as we have done in (iii). Moreover, S^t in (iii) is well defined, since it does not depend on the choice of ε, which is implied by the associativity of the semiflow defined in (ii).

For a convex domain, we have so far completed the definition of S^t. We next present a few basic properties, which will play an essential role in the game interpretation later on.

2.1.3. Basic Properties. We first see that billiard motion in the table can be identified as one conducted outside the table. For $t \ge 0$, $x \in \overline{\Omega}$ and $v \in \mathbf{S}^1$, we set

$$\alpha^t(x,v) = x + tv - S^t(x,v) \tag{2.4}$$

and call it the *boundary adjustor.* We next give a more specific representation of the boundary adjustor of our billiard semiflow. Let $B_r(x)$ denote the closed ball in $\mathbb{R}^2$ centered at x with radius r.

LEMMA 2.3. *Assume that Ω satisfies* (D1). *For any fixed $t \ge 0$, $x \in \overline{\Omega}$ and $v \in \mathbf{S}^1$, let $\alpha^t(x,v)$ be the boundary adjustor of $S^t(x,v)$. Then there exist $d_l \ge 0$ and $y_l \in \partial\Omega \cap B_t(x)$, $l = 1,2,\dots$ such that*

$$\alpha^t(x,v) = \sum_{l=0}^{\infty} d_l \nu(y_l), \tag{2.5}$$

where the convergence on the right hand side is in $\mathbb{R}^2$. *In addition, the following estimates hold:*

$$\left|\alpha^t(x,v)\right| \leq 2t. \tag{2.6}$$

$$\left|\sum_{l=k}^{\infty} d_l\nu(y_l)\right| \leq 4t, \text{ for all } k = 1, 2, \ldots \tag{2.7}$$

$$\sum_{l=1}^{\infty} |y_{l+1} - y_l| \leq 2t. \tag{2.8}$$

Lemma 2.3 is implied by Lemma 2.1 provided that the terminating time t_0 equals to 0. In essence, for a mere boundary sliding case, a singleton $y_0 = \Gamma(\xi)$ given in Lemma 2.1, is sufficient for Lemma 2.3 with no need of finding y_l for $l \geq 1$.

If alternatively the assumption $t_0 > 0$ is made, we will see that for every $l \geq 1$, the boundary point y_l above is actually the vertex $\Gamma(s_l)$ for the billiard trajectory while y_0 is taken as a "mean value point" on the sliding piece over the time t_0. Indeed, we define all the vertices orderly as $y_1, y_2, \ldots$ Suppose $y_1 \neq x$ and $y_l \neq y_{l+1}$ for $l \geq 1$. Correspondingly, we refer to the terminating point, if it exists, as y_∞ as well as $\Gamma(s_\infty)$ in Lemma 2.2. The point y_0 could again be determined through Lemma 2.1. It is then sufficient to show these y_l satisfy (2.5)-(2.8).

It is quite transparent that the broken line connecting them in succession forms the first portion of the trajectory before time t_0. Moreover, to express the length and direction of each segment of the broken line, we set

$$a_0 := |y_1 - x|, \quad v_0 := v = \frac{y_1 - x}{a_0},$$

and for any $k \geq 1$

$$a_k := |y_{k+1} - y_k|, \quad v_k := \frac{y_{k+1} - y_k}{a_k}.$$

The billiard law immediately yields

$$v_{k+1} - v_k = -c_k\nu(y_{k+1}), \text{ for all } k \geq 0, \tag{2.9}$$

where $c_k = 2v_k \cdot \nu(y_{k+1}) \geq 0$. Thanks to Lemma 2.2 about the monotonicity of s_k for large k, it is true that

$$v_k = \frac{y_{k+1} - y_k}{|y_{k+1} - y_k|} = \frac{y_{k+1} - y_k}{s_{k+1} - s_k} \cdot \frac{|s_{k+1} - s_k|}{|y_{k+1} - y_k|} \longrightarrow v_\infty, \text{ as } k \longrightarrow \infty,$$

which through (2.9) amounts to say

$$v_\infty - v = \sum_{k=1}^{\infty}(v_k - v_{k-1}) = -\sum_{k=1}^{\infty} c_{k-1}\nu(y_k). \tag{2.10}$$

PROOF OF LEMMA 2.3. We construct a path via y_l only for the case $x \in \Omega$ and thus $t_0 > 0$. The case $x \in \partial\Omega$ can be handled similarly.

If $t < t_0$, then $S^t(x,v) = T^t(x,v)$ and there exists $n \geq 0$ such that either (a) $t < a_0$ or (b) $\sum_{k=0}^{n} a_k \leq t < \sum_{k=0}^{n+1} a_k$. In case of (a), it is trivial. Both sides of (2.5) are equal to 0. If instead (b) holds, we have

$$S^t(x,v) = x + \sum_{k=0}^{n} a_k v_k + \left(t - \sum_{k=0}^{n} a_k\right) v_{n+1}. \tag{2.11}$$

Observe also that for each $j = 1, 2, \ldots, n$,

$$\sum_{k=0}^{n} a_k v_k = tv_0 + \sum_{k=j}^{n} a_k v_k - \left(t - \sum_{k=0}^{j-1} a_k\right) v_j + \sum_{l=1}^{j} \left(t - \sum_{k=0}^{l-1} a_k\right)(v_l - v_{l-1}).$$

Plugging (2.9) and the equality above with $j = n$ into (2.11), we get

$$S^t(x,v) = x + tv_0 - \sum_{l=1}^{n+1} c_{l-1} \left(t - \sum_{k=0}^{l-1} a_k\right) \nu(y_l), \tag{2.12}$$

which together with the definition of α^t in (2.4) implies (2.5) with $d_l = c_{l-1}(t - \sum_{k=0}^{l-1} a_k)$ for $l = 1, 2, \ldots, n+1$ and $d_l = 0$ for the other l.

Now if $t \geq t_0$, then $\lim_{k\to\infty} s_k = s_\infty$, $\sum_{k=0}^{\infty} a_k = t_0$ and $\sum_{k=0}^{\infty} a_k v_k = y_\infty$. By the argument above, simply taking the limit $n \to \infty$ in (2.12), we are led to

$$\alpha^{t_0}(x,v) = \sum_{l=1}^{\infty} d'_l \nu(y_l), \tag{2.13}$$

where $d'_l = c_{l-1}(t - \sum_{k=0}^{l-1} a_k)$ $(l \geq 1)$. Noticing further that (2.4) gives

$$\alpha^t(x,v) - \alpha^{t_0}(x,v) = (t - t_0)v + S^{t_0}(x,v) - S^t(x,v),$$

we only need to show its right hand side is a sum of normals. Recall the arc-length parametrization $\Gamma(s_l)$ for y_l and assume without loss of generality that $\{s_l\}_{l\geq 1}$ is increasing when l is sufficiently large. We therefore obtain

$$S^t(x,v) - S^{t_0}(x,v) = \Gamma(s_\infty + t - t_0) - \Gamma(s_\infty).$$

In terms of Lemma 2.1, this yields the existence of some constant $C \geq 0$ and a point $\Gamma(\xi)$ on the arc between $S^t(x,v)$ and $S^{t_0}(x,v)$ such that

$$S^{t_0}(x,v) - S^t(x,v) = (t_0 - t)v_\infty + C\nu(\Gamma(\xi)). \tag{2.14}$$

Noting (2.10) and combining the equations (2.13) and (2.14), we finally deduce (2.5) provided that we take $y_0 = \Gamma(\xi)$, $d_0 = C$ and $d_l = d'_l + (t - t_0)c_{l-1}$ for every $l \geq 1$.

The estimate (2.6) is implied directly by an observation that

$$\left|S^t(x,v) - x\right| \leq t.$$

Moreover, if we think of the billiard motion starting from y_k for any $k \geq 1$, we obtain

$$\left|S^{t_k}(y_k, v_k) - y_k\right| \leq 2t_k \leq 2t,$$

where t_k denotes the remaining length the billiard ball needs to cover from y_k to $S^t(x,v)$. It follows that

$$\left|d_0\nu(y_0) + \sum_{l=k}^{\infty} d_l \nu(y_l)\right| \leq 2t,$$

and thus (2.7) is verified since $|d_0|$ is bounded by $2t$.

The estimate (2.8) is obvious due to our choice of y_l. □

We next pause to compare our modified billiard with the deterministic Skorokhod problem, which is known to play a central role in the Neumann boundary problems for continuous optimal control and differential games. Let us review the definition of a solution of the two-dimensional Skorokhod problem for a normal reflection. For any $T > 0$, let $|\psi|(t)$ denote the total variation over the interval $[0, t]$ $(t \leq T)$ whenever ψ is of bounded variation in $[0, T]$, i.e, $\psi \in BV(0, T)$.

DEFINITION 2.2 ([**33**]). For any fixed $T \geq 0$, if for each $w(t) \in C([0, T]; \mathbb{R}^2)$ with $w(0) \in \overline{\Omega}$, there exists a pair of functions $(x(t), \eta(t))$ fulfilling

$$\begin{cases} (1)\ x(t) = w(t) - \eta(t) \text{ for } t \in [0,T] \text{ with } x(0) = w(0), \\ (2)\ x(t) \in \overline{\Omega} \text{ for } t \in [0,T], \\ (3)\ |\eta|(T) < \infty, \\ (4)\ |\eta|(t) = \displaystyle\int_{(0,t]} 1_{\{x(s) \in \partial\Omega\}}\, d|\eta|(s), \\ (5)\ \eta(t) = \displaystyle\int_{(0,t]} \nu(x(s))\, d|\eta|(s), \end{cases}$$

then $(x(t, \eta(t)))$ is said to be the solution of the Skorokhod problem in $[0, T]$ for w with respect to the domain Ω and direction ν.

The dynamics $x(t)$ is determined one part after another by the rule that it is always abiding by $w(t)$ while it lies in Ω but is pushed back in $\overline{\Omega}$ along the inward normal whenever it is about to cross the boundary. In this sense, we convince ourselves that the Skorokhod problem and billiards are very similar. Indeed, our explicit representation in (2.4) and (2.5) is an analogue of the condition (1) in Definition 2.2. Therefore, it is no wonder that differential games based on billiard dynamics can also be linked with Neumann boundary problems.

However, they are not exactly the same:

(1) Definition 2.2 is valid for every continuous function w while billiards are defined only for straight-line move.
(2) An advantage of billiards rests on their explicit form and natural uniqueness. In contrast, the solution of Skorokhod problem in two dimensions usually has no explicit representation and showing its existence is often troublesome if not difficult, though the uniqueness might be easier sometimes (see [**111, 81**]).
(3) The solution of Skorokhod problem enjoys a high level of stability. Indeed, the map $w(\cdot) \mapsto x(\cdot)$ is $1/2$-Hölder continuous on compact sets of $C([0, T]; \mathbb{R}^2)$, as is clarified in [**81**, Theorem 2.2]. By comparison, the lack of stability could be a drawback of our billiard semiflow. We shall give an example in Appendix A, indicating that the billiard trajectories can vary drastically even if the initial position and direction are slightly perturbed. Since our example is made from the terminating phenomenon, it is of question whether the billiard motion is still unstable in a more regular domain with no terminating trajectory in it. Consult Appendix A for such restrictions on the domain, under which the billiard semiflow is promoted to a flow.

(4) The definition of billiards requires C^1 smoothness of the boundary even if the domain is convex (we here assume $\partial\Omega$ is of class C^2). However, the Skorokhod problem is solvable under slightly weaker assumptions for a convex domain [**111**]. Nonconvex cases are complicated for both. As far as our knowledge goes, the Skorokhod problem needs smoothness of $\partial\Omega$ over C^1 class and close to the "positive reach" condition. Our billiard dynamics works for a general domain too but with a stronger assumption on its number of bumps, which will be explained in detail in Section 2.2.2.

We continue to study the properties of our billiards. The next one holds exclusively for convex billiards.

LEMMA 2.4. *Assume that* Ω *satisfies* (D1). *Then*

$$|x_0 - S^t(x,v)| \leq |x_0 - (x+tv)| \text{ for all } x, x_0 \in \overline{\Omega},\ v \in \mathbf{S}^1 \text{ and } t \geq 0. \tag{2.15}$$

REMARK 2.2. It can be considered as a generalization of the separation or support theorem of convex sets in $\mathbb{R}^2$. One can easily find (2.15) is also sufficient for the convexity assumption.

To prove Lemma 2.4, we again divide the argument into two parts, one for the boundary slide and the other for the mirror-like reflection.

LEMMA 2.5. *Let* Ω *be a domain satisfying* (D1). *Assume its boundary is parametrized by arclength as* $\Gamma(\cdot)$ *with* $\Gamma(0) = x \in \partial\Omega$. *Then for every* $x_0 \in \overline{\Omega}$,

$$|x_0 - \Gamma(s)| \leq |x_0 - (x + t\Gamma_s(0))|. \tag{2.16}$$

PROOF. In virtue of Lemma 2.1, we can find $\xi \in [0,t]$ such that

$$x + t\Gamma_s(s) - \Gamma(t) = C\nu(\Gamma(\xi)) \text{ and}$$
$$\Gamma_s(0)\cdot\nu(\Gamma(\xi)) \geq 0. \tag{2.17}$$

According to an elementary separation theorem for convex sets, the tangential line to $\partial\Omega$ at $\Gamma(\xi)$ breaks the plane into halves, one of which has no intersection with $\overline{\Omega}$. We establish orthogonal coordinates and assume, without loss of generality,

$$\Gamma(\xi) = (0,0) \in \mathbb{R}^2 \text{ and } \overline{\Omega} \subset \{(x,y) \in \mathbb{R}^2 : y \geq 0\}.$$

Let us use $\vec{\imath}$ and $\vec{\jmath}$ to denote respectively the unit vectors for x and y positive axes, i.e., $\vec{\imath} = x/|x|$ and $\vec{\jmath} = y/|y|$, and we consequently have $\vec{\jmath} = -\nu(\Gamma(\xi))$. We are led from the convexity of Ω to

$$(\Gamma(\xi) - \Gamma(0))\cdot\vec{\jmath} \geq \xi\Gamma_s(0)\cdot\vec{\jmath},$$

that is, $(x + \xi\Gamma_s(0))\cdot\vec{\jmath} \leq 0$. We then get from (2.17) that $(x + t\Gamma(0))\cdot\vec{\jmath} \leq 0$, since $\xi \leq t$. Hence, it becomes quite evident that

$$(x_0 - \Gamma(t))\cdot\vec{\imath} = (x_0 - x - t\Gamma_s(0))\cdot\vec{\imath};$$
$$\left|(x_0 - \Gamma(t))\cdot\vec{\jmath}\right| \leq \left|(x_0 - x - t\Gamma_s(0))\cdot\vec{\jmath}\right|.$$

In consequence, (2.16) holds. ☐

LEMMA 2.6. *Let Ω satisfy* (D1). *For any $x_0, x \in \overline{\Omega}$, consider the billiard $S^t(x,v)$ with positive terminating time and $t > a_0$, and set*

$$z := x + tv - 2(t - a_0)\,(v \cdot \nu(\Gamma(s_1)))\,\nu(\Gamma(s_1)).$$

Then

$$|x_0 - z| \leq |x_0 - (x + tv)|. \tag{2.18}$$

PROOF. As in the proof of Lemma 2.5, we apply the separation theorem, using the tangential at $\Gamma(s_1)$ to support $\overline{\Omega}$. More precisely, we set $\Gamma(s_1) = (0,0)$ and $\overline{\Omega} \subset \{(x,y) \in \mathbb{R}^2; y \geq 0\}$, and then by taking the unit vectors $\vec{i}$ and $\vec{j}$, we have $(x + tv) \cdot \vec{j} < 0$, for otherwise no collision takes place, which contradicts to the assumption $t > a_0$. Then we easily obtain

$$\left|(x_0 - (x + tv)) \cdot \vec{i}\right| = \left|(x_0 - z) \cdot \vec{i}\right|$$

and

$$\left|(x_0 - x - tv) \cdot \vec{j}\right| \geq \left|(x_0 - z) \cdot \vec{j}\right|,$$

which imply the inequality (2.18). □

We prove Lemma 2.4 below.

PROOF OF LEMMA 2.4. We only need to prove for the cases $t_0 > 0$ and $t > a_0$, since the others are either trivial or given already in Lemma 2.5.

If $t < t_0$, then there exists $n \geq 0$ such that $\sum_{k=0}^{n} a_k \leq t < \sum_{k=0}^{n+1} a_k$. We apply Lemma 2.6 inductively for every k with a change $x \curvearrowright y_k$, $v \curvearrowright v_k$, $t \curvearrowright t - \sum_{j=0}^{k-1} a_j$ and $a_0 \curvearrowright a_k$, and as a result have

$$\begin{aligned} |x_0 - S^t(x,v)| &= \left| x_0 - y_{n+1} + \left(t - \sum_{k=0}^{n} a_k\right) v_{n+1} \right| \\ &\leq \left| x_0 - y_n + \left(t - \sum_{k=0}^{n-1} a_k\right) v_n \right| \leq \ldots \leq |x_0 - x + tv|. \end{aligned}$$

For the case $t \geq t_0$, our discussion in Lemma 2.3 asserts

$$\begin{aligned} S^t(x,v) &= \Gamma(s_\infty + t - t_0), \\ x + tv - \sum_{l=1}^{\infty} d_l \nu(y_l) &= \Gamma(s_\infty) + (t - t_0) v_\infty. \end{aligned}$$

Then by Lemma 2.5, we get

$$\left|x_0 - S^t(x,v)\right| \leq \left| x_0 - \left(x + tv - \sum_{l=1}^{\infty} d_l \nu(y_l)\right)\right|. \tag{2.19}$$

On the other hand, we argue in the same way as in the case $t < t_0$ and have for every $n \in \mathbb{N}$,

$$\left| x_0 - \left(x + tv - \sum_{l=1}^{n} d_l \nu(y_l)\right)\right| \leq |x_0 - (x + tv)|.$$

Sending $n \to \infty$, we complete our proof by taking (2.19) into account. □

2.2. Game approximation for Neumann problems

We now apply our billiard semiflow S^t to games. The billiard involved games for the Neumann boundary problem have been introduced in Section 2.1. This section is devoted to the proof of our main result: Theorem 1.7.

2.2.1. Convex Domains. Let us rewrite the equations as

$$\partial_t u - \Delta u + \left(\nabla^2 u \tfrac{\nabla u}{|\nabla u|}\right) \cdot \tfrac{\nabla u}{|\nabla u|} = 0 \quad \text{in } \Omega \times (0,\infty), \tag{2.20a}$$

$$\nabla u(x,t) \cdot \nu(x) = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{2.20b}$$

$$u(x,0) = u_0(x) \quad \text{in } \overline{\Omega}. \tag{2.20c}$$

We give a definition of a solution of this Neumann boundary problem, handling the boundary condition in the viscosity sense. The reader is referred to [**29**] and [**51**] for more details about the viscosity solution theory.

DEFINITION 2.3. An upper semicontinuous (resp., lower semicontinuous) function u on $\overline{\Omega} \times (0,\infty)$ is a *viscosity subsolution* (resp., *viscosity supersolution*) of (2.20a)-(2.20b) if whenever there are $(\hat{x},\hat{t}) \in \overline{\Omega} \times (0,\infty)$, a neighborhood $\mathcal{O}$ relative to $\overline{\Omega} \times (0,\infty)$ of $(\hat{x},\hat{t})$ and a function $\varphi \in C^2(\mathcal{O})$ such that

$$\max_{\mathcal{O}}(u-\varphi) = (u-\varphi)(\hat{x},\hat{t})$$

$$\left(\text{resp., } \min_{\mathcal{O}}(u-\varphi) = (u-\varphi)(\hat{x},\hat{t})\right),$$

the following holds:

(i) $\hat{x} \in \Omega$ and $\nabla\varphi(\hat{x},\hat{t}) \neq 0$ imply that

$$\partial_t\varphi - \Delta\varphi + \left(\nabla^2\varphi \frac{\nabla\varphi}{|\nabla\varphi|}\right) \cdot \frac{\nabla\varphi}{|\nabla\varphi|} \leq 0 \quad \text{at } (\hat{x},\hat{t})$$

$$\left(\text{resp., } \partial_t\varphi - \Delta\varphi + \left(\nabla^2\varphi \frac{\nabla\varphi}{|\nabla\varphi|}\right) \cdot \frac{\nabla\varphi}{|\nabla\varphi|} \geq 0 \quad \text{at } (\hat{x},\hat{t})\right).$$

(ii) $\hat{x} \in \Omega$, $\nabla\varphi(\hat{x},\hat{t}) = 0$ and $\nabla^2\varphi(\hat{x},\hat{t}) = 0$ imply that

$$\partial_t\varphi(\hat{x},\hat{t}) \leq 0 \quad \left(\text{resp., } \partial_t\varphi(\hat{x},\hat{t}) \geq 0\right).$$

(iii) $\hat{x} \in \partial\Omega$ and $\nabla\varphi(\hat{x},\hat{t}) \neq 0$ imply either

$$\partial_t\varphi - \Delta\varphi + \left(\nabla^2\varphi \frac{\nabla\varphi}{|\nabla\varphi|}\right) \cdot \frac{\nabla\varphi}{|\nabla\varphi|} \leq 0 \quad \text{at } (\hat{x},\hat{t})$$

$$\left(\text{resp., } \partial_t\varphi - \Delta\varphi + \left(\nabla^2\varphi \frac{\nabla\varphi}{|\nabla\varphi|}\right) \cdot \frac{\nabla\varphi}{|\nabla\varphi|} \geq 0 \quad \text{at } (\hat{x},\hat{t})\right)$$

or

$$\nabla\varphi(\hat{x},\hat{t}) \cdot \nu(\hat{x}) \leq 0 \quad \left(\text{resp., } \nabla\varphi(\hat{x},\hat{t}) \cdot \nu(\hat{x}) \geq 0\right).$$

DEFINITION 2.4. A function u on $\overline{\Omega} \times (0,T)$ is called a *viscosity solution* of (2.20a)-(2.20b) if it is both a viscosity subsolution and a viscosity supersolution.

Let us recall the *upper* and *lower relaxed limits* of $\{u^\varepsilon\}_{\varepsilon>0}$ as

$$\begin{aligned}\overline{u}(x,t) :&= \limsup_{\varepsilon\to 0}{}^* u^\varepsilon(x,t)\\ &= \lim_{\delta\to 0}\sup\{u^\varepsilon(y,s) : y\in\overline{\Omega},\ 0<\varepsilon<\delta,\ |x-y|+|t-s|<\delta\}\end{aligned}$$

and

$$\begin{aligned}\underline{u}(x,t) :&= \liminf_{\varepsilon\to 0}{}_* u^\varepsilon(x,t)\\ &= \lim_{\delta\to 0}\inf\{u^\varepsilon(y,s) : y\in\overline{\Omega},\ 0<\varepsilon<\delta,\ |x-y|+|t-s|<\delta\}.\end{aligned}$$

The strategy for the proof of Theorem 1.7 is as follows: We first show the upper and lower relaxed limits coincide at the initial time. We then present two propositions to indicate that they are respectively a subsolution and a supersolution of (2.20a)-(2.20b) and thereby give the locally uniform convergence through the comparison principle established by Sato [**100**].

PROPOSITION 2.7. *Under all the assumptions of Theorem* 1.7,

$$\overline{u}(x,0)=\underline{u}(x,0)=u_0(x) \quad \textit{for all } x\in\overline{\Omega}. \tag{2.21}$$

Results like Proposition 2.7 are often shown via constructing "barrier solutions". We here follow this idea by playing "barrier games". One may think that the proof could be easier if u^ε is "continuous in discrete time", but we cannot count on that since the continuity of our billiard semiflow $S^t(x,v)$ in x is not known.

LEMMA 2.8. *Assume that $x_0, x\in\overline{\Omega}$. Let the game start from x. Then for any $k\in\mathbb{N}$, there exists a strategy for the minimizing player Paul such that the resulting k-step state y_k satisfying, indifferent to the decisions of his opponent Carol,*

$$|x_0-y_k|^2\le|x_0-x|^2+2k\varepsilon^2. \tag{2.22}$$

LEMMA 2.9. *Let $x_0, x\in\overline{\Omega}$ and the game start at x. Then for any $k\in\mathbb{N}$, the maximizing player Carol can adopt a strategy, which leads to the k-step state satisfying*

$$|x_0-y_k|^2\le|x_0-x|^2+2k\varepsilon^2, \tag{2.23}$$

no matter how the minimizing player Paul chooses the directions.

These two lemmas are actually equivalent with Lemmas 1.5 and 1.6. We thus omit the proofs.

PROOF OF PROPOSITION 2.7. We fix an arbitrary point $x_0\in\overline{\Omega}$ and prove in the first place $\overline{u}(x_0,0)\le u_0(x_0)$.

Since u_0 is continuous in $\overline{\Omega}$, we may set a function for every $\lambda>0$,

$$\overline{V}_\lambda(x)=\lambda+u_0(x_0)+C|x-x_0|^2$$

with a sufficiently large constant C depending on λ such that

$$\overline{V}_\lambda(x)\ge u_0(x) \text{ for all } x\in\overline{\Omega}. \tag{2.24}$$

We next play an upper barrier game, starting at $x\in\overline{\Omega}$ and conforming to the same rules but aiming at a new objective $\overline{V}_\lambda$. It is clear that though the exact optimal strategy for Paul is unknown, we can guarantee him a value not more than

$$\lambda + u_0(x_0) + C\left(|x - x_0|^2 + 2t\right)$$

by Lemma 2.8. To see this, we recall that the total number of steps in the game is $[t/\varepsilon^2]$. Then following that strategy, Paul gets to a final position, say y, decided also by Carol's response, such that

$$|x_0 - y|^2 \leq |x_0 - x|^2 + 2\varepsilon^2 \left[\frac{t}{\varepsilon^2}\right] \leq |x_0 - x|^2 + 2t,$$

which implies

$$\overline{V}_\lambda(y) \leq \lambda + u_0(x_0) + C\left(|x - x_0|^2 + 2t\right). \tag{2.25}$$

We also learn from (2.24) that for any such kind of y,

$$u^\varepsilon(x,t) \leq u_0(y) \leq \overline{V}_\lambda(y). \tag{2.26}$$

We plug (2.25) into (2.26) and then have, in terms of the definition of upper relaxed limits,

$$\begin{aligned}\overline{u}(x_0, 0) &\leq \lim_{\substack{|x-x_0|\to 0\\ t\to 0}} \left(\lambda + u_0(x_0) + C\left(|x - x_0|^2 + 2t\right)\right)\\ &= \lambda + u_0(x_0).\end{aligned}$$

Letting $\lambda \downarrow 0$, we get $\overline{u}(x_0, 0) \leq u_0(x_0)$.

Since $\overline{u}(x_0, 0) \geq \underline{u}(x_0, T)$, it remains to prove $\underline{u}(x_0, 0) \geq u_0(x_0)$. We choose for arbitrary $\lambda > 0$ a function

$$\underline{V}_\lambda(x) = -\lambda + u_0(x_0) - C(|x - x_0|^2)$$

with a constant C satisfying $\underline{V}_\lambda(x) \leq u_0(x)$ for $x \in \overline{\Omega}$. We now play a lower barrier game whose terminal data are given by $\underline{V}_\lambda$. For the same reason stated above, use Lemma 2.9 this time to ensure her a value

$$-\lambda + u_0(x_0) - C\left(|x - x_0|^2 + 2t\right).$$

Hence, noting that u^ε is always larger than the value of this lower barrier game, we obtain

$$u^\varepsilon(x,t) \geq -\lambda + u_0(x_0) - C\left(|x - x_0|^2 + 2t\right)$$

and therefore $\underline{u}(x_0, 0) \geq u_0(x_0)$ by sending λ to 0. □

PROPOSITION 2.10. *Assume that* Ω *satisfies* (D1). *Let* $\{u^\varepsilon\}$ *be locally bounded in* $\overline{\Omega} \times (0,\infty)$ *for* $0 < \varepsilon < 1$. *Assume that* u^ε *satisfies the dynamic programming principle* (1.16). *Then* $\overline{u}$ *is a viscosity subsolution of* (2.20a)-(2.20b).

PROOF. We argue by contradiction only for $x_0 \in \partial\Omega$. For an interior point x_0, a contradiction will be reached by the same argument in [**75**]. Suppose to the contrary that there exist a smooth function ϕ and a δ-neighborhood of (x_0, t_0) s.t. $\overline{u} - \phi$ attains a unique maximum at (x_0, t_0) and both relations below hold:

$$\partial_t\phi(x,t) - \Delta\phi(x,t) + \left(\nabla^2\phi(x,t)\frac{\nabla\phi(x,t)}{|\nabla\phi(x,t)|}\right)\cdot\frac{\nabla\phi(x,t)}{|\nabla\phi(x,t)|} \geq \eta_0, \quad \text{for all } (x,t) \in B_\delta((x_0,t_0)) \text{ and } x \in \overline{\Omega}, \tag{2.27}$$

$$\begin{aligned} &\nabla\phi(x,t)\cdot\nu(y)\geq\eta_0,\\ &\qquad\text{for all } (x,t)\in B_\delta((x_0,t_0)),\ x\in\overline{\Omega} \text{ and } y\in B_\delta(x_0)\cap\partial\Omega, \end{aligned} \tag{2.28}$$

where η_0 is a certain positive constant.

We are allowed to take a sequence $\left(x^0_{\varepsilon_n}, t^0_{\varepsilon_n}\right)\to(x_0,t_0)$ such that

$$u^{\varepsilon_n}\left(x^0_{\varepsilon_n}, t^0_{\varepsilon_n}\right)\to\overline{u}(x_0,t_0) \quad\text{and } \varepsilon_n\to 0 \quad \text{as } n\to\infty.$$

Next let us construct a specific path of the game as,

$$X_1=(x_1,t_1)=\left(x^0_{\varepsilon_n}, t^0_{\varepsilon_n}\right),\ \text{and}$$

$$X_{k+1}=(x_{k+1},t_{k+1})=\left(S^{\sqrt{2}\varepsilon_n}\left(x_k,b_kv_k\right), t_k-\varepsilon_n^2\right)$$

with $v_k=\frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|}$ for $k\geq 1$. By (1.16), there is $b_k=\pm1$ such that

$$u^{\varepsilon_n}(X_k)\leq u^{\varepsilon_n}(X_{k+1}),$$

and by induction,

$$u^{\varepsilon_n}(X_1)\leq u^{\varepsilon_n}(X_k). \tag{2.29}$$

We fix such b_k.

Now we estimate $\phi(X_{k+1})-\phi(X_k)$ by applying Taylor's formula. Since $x_k\in\overline{\Omega}$, Lemma 2.3 states that there exist $d_l^{(k)}\geq 0$ and $y_l^{(k)}\in\partial\Omega\cap B_{\sqrt{2}\varepsilon_n}(x_k)$ such that

$$\alpha_n^k=\alpha^{\sqrt{2}\varepsilon_n}(x_k,b_kv_k)=\sum_{l=0}^{\infty}d_l^{(k)}\nu(y_l^{(k)}) \tag{2.30}$$

and $|\alpha_n^k|\leq 2\sqrt{2}\varepsilon_n$. Since $v_k\cdot\nabla\phi(X_k)=0$, we observe that

$$\begin{aligned} \phi(X_{k+1})-\phi(X_k)&=-\nabla\phi(X_k)\cdot\alpha_n^k-\varepsilon_n^2\partial_t\phi(X_k)\\ &+\frac{1}{2}\nabla^2\phi(X_k)\left(\sqrt{2}\varepsilon_nb_kv_k-\alpha_n^k\right)\cdot\left(\sqrt{2}\varepsilon_nb_kv_k-\alpha_n^k\right)+o(\varepsilon_n^2). \end{aligned} \tag{2.31}$$

Assuming for the time being that $X_k\in B_\delta((x_0,t_0))$ and $B_{\sqrt{2}\varepsilon_n}(x_k)\subset B_\delta(x_0)$, which makes sense at least for sufficiently small k, we handle two cases and utilize (2.27) or (2.28) respectively for each k.

Case A. (Boundary condition dominant case) Suppose that for a given k,

$$\liminf_{\varepsilon_n\to 0}\frac{|\alpha_n^k|}{\varepsilon_n}\geq C$$

with some constant C which may depend on k. Then (2.30) gives

$$\left|\nabla\phi(X_k)\cdot\alpha_n^k\right|\geq\sum_{l=0}^{\infty}d_l^{(k)}\eta_0\geq\eta_0|\alpha_n^k|\geq C\eta_0\varepsilon_n. \tag{2.32}$$

We therefore deduce from (2.31)

$$\phi(X_{k+1})-\phi(X_k)\leq -C\eta_0\varepsilon_n \text{ for sufficiently large } n. \tag{2.33}$$

Case B. (Mixed type case) Suppose

$$\liminf_{\varepsilon_n\to 0}\frac{|\alpha_n^k|}{\varepsilon_n}=0,$$

i.e., we can take a subsequence, still denoted by ε_n such that

$$|\alpha_n^k|=o(\varepsilon_n).$$

In this case, we have from (2.28) and (2.30),

$$\nabla\phi(X_k)\cdot\alpha_n^k \geq 0, \tag{2.34}$$

which implies through (2.31) that

$$\begin{aligned}
&\phi(X_{k+1})-\phi(X_k)\\
&\leq -\varepsilon_n^2\partial_t\phi(X_k)+\varepsilon_n^2\left(\nabla^2\phi(X_k)\frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|}\right)\cdot\frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|}+o(\varepsilon_n^2)\\
&\leq -\frac{1}{2}\eta_0\varepsilon_n^2 \text{ for sufficiently large } n,
\end{aligned}\tag{2.35}$$

where the last inequality is obtained by adopting (2.27). (In fact, this inequality can be deduced by the same argument for the case of $x_0\in\Omega$.)

Therefore, it can be concluded that

$$\phi(X_k)\leq\phi(X_1)-\frac{1}{4}k\eta_0\varepsilon_n^2 \tag{$*$}$$

for $n>n_0$ with some n_0 independent of k as long as

$$X_j\in B_\delta((x_0,t_0)) \text{ and } B_{\sqrt{2}\varepsilon_n}(x_j)\subset B_\delta(x_0) \text{ for any } j=1,2,\ldots,k. \tag{2.36}$$

Combining ($*$) with (2.29), we obtain

$$u^{\varepsilon_n}(X_1)-\phi(X_1)\leq u^{\varepsilon_n}(X_k)-\phi(X_k). \tag{2.37}$$

Since ϕ is smooth, there exists $\sigma>0$ satisfying

$$|\phi(x,t)-\phi(x_0,t_0)|\leq\sigma \text{ for all } (x,t)\in B_\delta((x_0,t_0)).$$

We may also assume that $n>n_0$ is sufficiently large such that

$$|\phi(X_1)-\phi(x_0,t_0)|\leq\sigma \text{ and} \tag{2.38}$$

$$\varepsilon_n\leq\min\{\delta/4,1\}. \tag{2.39}$$

Now let k_n be the maximal k satisfying (2.36). (If (2.36) holds for all natural number k, we set $k_n=\infty$.) We claim that $\{k_n\}$ is a nontrivial divergent sequence, or more precisely,

$$\frac{\delta}{4\varepsilon_n}\leq k_n\leq\frac{8\sigma}{\eta_0\varepsilon_n^2}. \tag{2.40}$$

Indeed, by definition, $X_{k_n+1}\notin B_\delta((x_0,t_0))$ or $B_{\sqrt{2}\varepsilon_n}(x_{k_n+1})\backslash B_\delta(x_0)\neq\emptyset$. A direct computation with an application of (2.39) consequently yields

$$|X_{k_n}-(x_0,t_0)|\geq\frac{\delta}{2} \text{ or } |x_{k_n}-x_0|\geq\frac{\delta}{2}. \tag{2.41}$$

By (2.39) our construction of X_k thus implies the lower bound in (2.40). To show k_n's bound from above, we suppose to the contrary that $k_n>\frac{8\sigma}{\eta_0\varepsilon_n^2}$. Then by ($*$) and (2.38), we are led to

$$\phi(X_{k_n})<\phi(x_0,t_0)-\sigma,$$

which contradicts to the fact that $k=k_n$ satisfies (2.36).

We conclude from the definition of k_n and (2.41) that $\{X_k\}$ with $k=k_n$ admits a convergent subsequence in $B_\delta((x_0,t_0))$ such that its limit (x',t') as $n\to\infty$ does not equal to (x_0,t_0). It follows easily that

$$(\overline{u}-\phi)(x_0,t_0)\leq\limsup_{n\to\infty}(u^{\varepsilon_n}-\phi)(X_{k_n})\leq(\overline{u}-\phi)(x',t'),$$

which is a contradiction to the uniqueness of maximizers of $\overline{u} - \phi$ in the δ-neighborhood. □

The distinct two cases above, remarked also in Chapter 1, have a geometric meaning. $|\alpha_n^k|$ essentially quantifies the influence of boundary reflections on the game path commencing from the point in question. The effect indicated by case A is stronger than that in case B. In this sense, case A is supposed to be consistent with the boundary condition while case B may instead give rise to curvature motion in the interior, as is shown in our proof.

We deal with the supersolution part in the same way but the next preliminary fact is needed.

LEMMA 2.11. *Let ξ be a unit vector in $\mathbb{R}^2$ and X be a real symmetric 2×2 matrix. Then there exists a constant $M > 0$ that depends only on the norm of X, such that for any unit vector $v \in \mathbb{R}^2$,*

$$|X\xi^\perp \cdot \xi^\perp - Xv \cdot v| \leq M|\xi \cdot v|, \tag{2.42}$$

where $\xi^\perp$ denotes a unit orthonormal vector of ξ.

PROOF. It is clear that any v can be composed as

$$v = (v \cdot \xi)\xi + (v \cdot \xi^\perp)\xi^\perp,$$

which yields by computation that

$$\begin{aligned} &|X\xi^\perp \cdot \xi^\perp - Xv \cdot v| \\ &= |(v \cdot \xi)^2(X\xi^\perp \cdot \xi^\perp) - (v \cdot \xi)^2(X\xi \cdot \xi) - 2(v \cdot \xi)(v \cdot \xi^\perp)(X\xi \cdot \xi^\perp)| \\ &\leq 4\|X\||v \cdot \xi|, \end{aligned}$$

as desired. □

PROPOSITION 2.12. *Assume that Ω satisfies* (D1). *Let $\{u^\varepsilon\}$ be locally bounded in $\overline{\Omega} \times (0, \infty)$ for $0 < \varepsilon < 1$. Assume that u^ε satisfies the dynamic programming principle* (1.16). *Then $\underline{u}$ is a viscosity supersolution of* (2.20a)-(2.20b).

PROOF. We still assume by contradiction that there exist a constant $\eta_0 > 0$, a smooth function ϕ and a δ-neighborhood of $(x_0, t_0) \in \partial\Omega \times (0, T)$ in which (x_0, t_0) is the unique minimizer of $\overline{u} - \phi$ with

$$\begin{aligned} \partial_t\phi(x,t) - \Delta\phi(x,t) + \left(\nabla^2\phi(x,t)\frac{\nabla\phi(x,t)}{|\nabla\phi(x,t)|}\right) \cdot \frac{\nabla\phi(x,t)}{|\nabla\phi(x,t)|} \leq -\eta_0 < 0, \\ \text{for all } (x,t) \in B_\delta((x_0,t_0)) \text{ and } x \in \Omega, \end{aligned} \tag{2.43}$$

$$\begin{aligned} &\nabla\phi(x,t) \cdot \nu(y) \leq -\eta_0 < 0, \\ &\qquad \text{for all } (x,t) \in B_\delta((x_0,t_0)),\ x \in \overline{\Omega} \text{ and } y \in B_\delta(x_0) \cap \partial\Omega. \end{aligned} \tag{2.44}$$

It is possible to find a sequence $\left(x^0_{\varepsilon_n}, t^0_{\varepsilon_n}\right) \to (x_0, t_0)$ such that

$$u^{\varepsilon_n}\left(x^0_{\varepsilon_n}, t^0_{\varepsilon_n}\right) \to \underline{u}(x_0, t_0) \text{ and } \varepsilon_n \to 0, \text{ as } n \to \infty.$$

We define a sequence of points in $\overline{\Omega} \times (0, T)$ in the following way,

$$X_1 = (x_1, t_1) = \left(x^0_{\varepsilon_n}, t^0_{\varepsilon_n}\right),$$

$$X_{k+1} = (x_{k+1}, t_{k+1}) = \left(S^{\sqrt{2}\varepsilon_n}(x_k, b_k v_k), t_k - \varepsilon_n^2\right) \text{ for } k \geq 1, \tag{2.45}$$

where v_k can be immediately determined through dynamic programming (1.16) so as to satisfy

$$u^{\varepsilon_n}(X_k) \geq \sup_{b_k=\pm 1} u^{\varepsilon_n}(X_{k+1}) - \varepsilon_n^3.$$

Note that $S^t(x,v)$ may not be continuous in x and v. There might be no minimizer v_k achieving the infimum in dynamic programming principle. We thus allow an error ε_n^3 which actually causes no problem in our further calculation. We fix v_k in such a way that

$$u^{\varepsilon_n}(X_k) \geq u^{\varepsilon_n}(X_{k+1}) - \varepsilon_n^3, \text{ for both } b_k = \pm 1, \tag{2.46}$$

leaving b_k to be selected through the estimate of $\phi(X_{k+1}) - \phi(X_k)$ below.

Just as we did in Proposition 2.10, it is desired that there exist $b_k = \pm 1$ leading to

$$\phi(X_{k+1}) - \phi(X_k) \geq w(\varepsilon_n),$$

for a modulus $w(\cdot)$. Again we use Lemma 2.3 and develop $\phi(X_{k+1}) - \phi(X_k)$ as follows,

$$\begin{aligned}\phi(X_{k+1}) - \phi(X_k) &= \nabla\phi(X_k)\cdot(\sqrt{2}\varepsilon_n b_k v_k - \alpha_n^k) - \varepsilon_n^2\partial_t\phi(X_k)\\ &+\frac{1}{2}\left(\nabla^2\phi(X_k)\left(\sqrt{2}\varepsilon_n b_k v_k - \alpha_n^k\right)\right)\cdot\left(\sqrt{2}\varepsilon_n b_k v_k - \alpha_n^k\right) + o(\varepsilon_n^2),\end{aligned} \tag{2.47}$$

where α_n^k is defined as in (2.30). We bear in mind that till now X_k, $d_l^{(k)}$ and $y_l^{(k)}$ are still depending on the choice of b_k, rather than express them too carefully by making our notations heavier.

Noticing (2.6), we again meet two cases for each k.

Case A. (Boundary condition dominant case) If there exists $b_k = 1$ or -1 such that

$$\liminf_{\varepsilon_n\to 0}\frac{|\alpha_n^k|}{\varepsilon_n} \geq C,$$

where $C > 0$ is a constant depending on k. For the same reason as in (2.32) and (2.34), we are led from (2.44) and (2.47) to

$$\sum_{b_k=\pm 1}(\phi(X_{k+1}) - \phi(X_k)) \geq C\eta_0\varepsilon_n + o(\varepsilon_n), \tag{2.48}$$

which enables us to pick $b_k = 1$ or -1 in (2.45) so that

$$\phi(X_{k+1}) - \phi(X_k) \geq \frac{C}{2}\eta_0\varepsilon_n + o(\varepsilon_n).$$

Case B. (Mixed type case) Assume now that

$$\liminf_{\varepsilon_n\to 0}\frac{|\alpha_n^k|}{\varepsilon_n} \to 0, \text{ for both values } b_k = \pm 1.$$

By taking a subsequence, we may replace "lim inf" by "lim" in the above equation. Then in virtue of (2.44) there exists $b_k = \pm 1$ such that

$$\begin{aligned}\phi(X_{k+1}) - \phi(X_k) \geq&\sqrt{2}\varepsilon_n\left|\nabla\phi(X_k)\cdot v_k\right| - \varepsilon_n^2\partial_t\phi(X_k)\\ &+\varepsilon_n^2\left(\nabla^2\phi(X_k)v_k\cdot v_k\right) + o(\varepsilon_n^2).\end{aligned} \tag{2.49}$$

Observing that $\nabla^2\phi(X_k)$ is locally bounded, we employ Lemma 2.11 to learn that the right hand side of the inequality above is greater than

$$\frac{1}{2}\left(-\varepsilon_n^2\partial_t\phi(X_k)+\varepsilon_n^2\left(\nabla^2\phi(X_k)\frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|}\cdot\frac{\nabla^\perp\phi(X_k)}{|\nabla\phi(X_k)|}\right)\right)$$

when ε_n is sufficiently small. Hence, by (2.43), we actually get

$$\phi(X_{k+1})-\phi(X_k)\geq\frac{\eta_0}{2}\varepsilon_n^2 \text{ for } n \text{ sufficiently large.} \tag{2.50}$$

The rest of the proof is almost the same as that of Proposition 2.10. We take a subsequence of ε_n and their correspondent k satisfying the condition (2.36) and having X_k converge to a point $(x', t') \neq (x_0, t_0)$. After summing up both (2.46) and (2.50) for k and sending ε_n to zero, we conclude the proof with a contradiction that

$$\underline{u}(x_0,t_0)-\phi(x_0,t_0)\geq\underline{u}(x',t')-\phi(x',t').$$

□

We eventually complete our proof of Theorem 1.7 by gathering Proposition 2.7, 2.10 and 2.12 and employing a comparison principle presented in [**100**]. We adapt that theorem to our backward mean curvature flow case.

THEOREM 2.13 ([**100**, Theorem 2.1]). *Suppose Ω satisfies* (D1). *Let u and v be, respectively, sub- and supersolutions of* (2.20a)-(2.20b). *If $u^*(x,0)\leq v_*(x,0)$, then for every $T>0$, there is a modulus m such that*

$$u^*(x,t)-v_*(x,t)\leq m(|x-y|), \text{ for all } (x,y,t)\in\overline{\Omega}\times\overline{\Omega}\times(0,T).$$

In particular, $u^*\leq v_*$ on $\overline{\Omega}\times(0,\infty)$.

2.2.2. Relaxation to Nonconvex Domains. We extend the convex billiard semiflow defined in Section 2.1.2 to a nonconvex one so that the deterministic game setting in Section 2.1.3 can be relaxed to connect the Neumann boundary problem (NP1) for more general domains. In what follows, we shall not repeat the whole process but merely note the difference.

We content ourselves by tackling nonconvex domains of special shapes. We write the curvature of $\partial\Omega$ at any point $z\in\partial\Omega$ as $\kappa(z)$. A point $z\in\partial\Omega$ is called an *exit* if any small neighborhood of z contains two points $z_1, z_2\in\partial\Omega$ such that $\kappa(z_1)<0$ and $\kappa(z_2)\geq 0$. Let I denote the set of all the exits. Such a notion is close to that of inflection points, but we reestablish it here for a particular purpose to be explained later. Now we assume

$$\text{(D2)}\quad\begin{cases}\Omega \text{ is a bounded domain in } \mathbb{R}^2, \text{ and its boundary is of class } C^2\\ \text{and has at most finitely many exits.}\end{cases}$$

Implying that the boundary has finite curve bumps, (D2) grants great convenience to classify those zero-curvature points. It prevents the boundary from oscillating too much, so the curvature of $\partial\Omega$ around any $z\in I$, treated as a function, should be negative on one side of z and nonnegative on the other side.

We perceive about billiards that although the convexity of Ω is dropped, terminating takes place still on convex pieces of $\partial\Omega$, or more rigorously stated as the lemma below.

LEMMA 2.14. *Under the assumption* (D2), *if a trajectory terminates at a point* $\Gamma(s_\infty) \in \Omega$ *with collision points in order* $\Gamma(s_1)$, $\Gamma(s_2)$, ..., *then there is* $N > 0$ *such that for all* $n \geq N$,

$$\kappa(\Gamma(s_n)) \geq 0.$$

PROOF. In accordance with the definition of termination, we only place attention to a neighborhood of the limit point, where $\partial\Omega$ can be represented by the graph of a function. More explicitly, we assume without loss that there are $\delta > 0$, a C^2 function $f(\cdot) : [-\delta, \delta] \to \mathbb{R}$ and a sequence $\{x_j\} \subset \mathbb{R}$ such that

$$f(0) = 0,\ f'(0) = 0;$$
$$\Gamma(s_j) = (x_j, f(x_j)) \text{ and } |x_j| \leq \delta,\ j = 1, 2, \ldots;$$
$$\Gamma(s_\infty) = (0, f(0)) = (0, 0).$$

In addition, it is quite clear that our assumption (D2) admits only three possible cases.

(1) $f''(x) \geq 0$ for all $x \in [-\delta, 0) \cup (0, \delta]$;
(2) $f''(x) < 0$ for all $x \in [-\delta, 0) \cup (0, \delta]$;
(3) $z \in I$; or expressed with no loss of generality as, $f''(x) < 0$ if $x \in (0, \delta]$ and $f''(x) \geq 0$ if $x \in [-\delta, 0)$.

(1) directly gives what we need, so it suffices to look into the occasions (2) and (3).

We claim that (2) is impossible. If not the case, there is sufficiently large n such that

$$f''(x_n) < 0, \quad f''(x_{n+1}) < 0,$$

and thus

$$f''(x) \leq 0, \text{ for all } x \in [x_n, x_{n+1}].$$

It follows that $\Gamma(s_{n-1})$ and $\Gamma(s_n)$ are connected by a straight line segment outside $\overline{\Omega}$, contradicting to the definition of a billiard.

At last, in the case of (3), we assume by contradiction that for any large N there is $n > N$ such that $f''(x_n) < 0$. For the same reason explained for the case (2), both $f''(x_{n-1})$ and $f''(x_{n+1})$ are nonnegative, which forces $x_{n-1}, x_{n+1} \in [-\delta, 0)$ and $x_n \in (0, \delta]$. In consequence, after the trajectory hits the boundary at $(x_n, f(x_n))$, its direction v satisfies

$$v \cdot (1, 0) \geq 0.$$

This convinces us that the ball will never hit the arc between $(-\delta, f(-\delta))$ and (0, 0) before it collides on the other part of the boundary, which clearly contradicts to the property $x_{n+1} \in [-\delta, 0)$. □

We have actually shown

LEMMA 2.15. *The statement of Lemma* 2.2 *holds under the assumption of* (D2) *in place of* (D1).

All above amounts to say relaxing a convex domain into such a nonconvex one does not largely affect the billiard motion in the interior, we just need to supplement our definition of S^t for the boundary sliding case. In view of (D2), the tangent to the boundary at each $z \in I$ induces a secant line segment of the domain, whose length will be uniformly bounded by some constant C. Take

$L := \frac{C}{2}$. We just let the sliding ball leave the boundary along the secant and restart a straight line motion with a moving distance no more than L in the interior. This is the exact reason why elements of I are named exits. The following definition of the nonconvex billiard semiflow $\tilde{S}^t$ is an adaptation of Definition 2.1 based on Lemma 2.15.

DEFINITION 2.5. Assume that Ω satisfies (D2).

(i) If $x \in \partial\Omega$, and v equals to the tangent of $\partial\Omega$, then

$$\tilde{S}^t(x,v) := \begin{cases} S^t(x,v) = \Gamma(t) & \text{if } 0 \le t \le \tilde{t}, \\ \Gamma(\tilde{t}) + (t-\tilde{t})\Gamma_s(\tilde{t}) & \text{if } \tilde{t} \le t \le \tilde{t} + L, \end{cases}$$

where $\tilde{t} := \inf\{s \ge 0 : \Gamma(s) \in I\}$ and $\Gamma(\cdot)$ is the arc-length parametrization of $\partial\Omega$ such that $\Gamma(0) = x$ and $\Gamma_s(0) = v$.

(ii) If $x \in \Omega$ and v is such that $T^t(x,v)$ terminates on $\partial\Omega$ at time t_0, then

$$\tilde{S}^t(x,v) = \begin{cases} T^t(x,v) & \text{if } 0 \le t < t_0, \\ \tilde{S}^{t-t_0}(T^{t_0}(x,v), v_\infty) & \text{if } t \ge t_0, \end{cases}$$

where v_∞ is obtained from Lemma 2.15.

(iii) If $x \in \partial\Omega$ and v points inside Ω, then

$$\tilde{S}^t(x,v) := \begin{cases} x & \text{if } t = 0, \\ \tilde{S}^{t-\varepsilon}(x+\varepsilon v, v) & \text{if } t, \end{cases}$$

where $\varepsilon > 0$ is such that $x + \delta v \in \Omega$ for all $\delta \in (0,\varepsilon)$.

Through the foregoing definitions, the orbit of a billiard ball in the domain of (D2) can be clearly understood for all $t \in [0, L]$. This is actually adequate to sustain our application to games. The reason why we impose the assumption (D2) is that we would like to avoid the situation that termination, entrance and exit circulate infinitely many times in a finite time.

The new billiard trajectories keep convex even though the domain is not convex. Since $\tilde{S}^t$ ends up with a straight line segment after an exit, we still have the same results in Lemma 2.1 and hence the key Lemma 2.3 on the boundary adjustor, the only job here is to restrict t in $[0, L]$. We hereafter still use $S^t(x,v)$ to denote our general billiard $\tilde{S}^t(x,v)$ for simplicity in notations.

LEMMA 2.16. *The statement of Lemma 2.3 holds under the assumption* (D2) *in place of* (D1) *but for* $0 \le t \le L$.

We now design a billiard game in the nonconvex domain, following the same rules for Neumann problems given in Introduction 1 and only adding a restraint $0 \le \sqrt{2}\varepsilon \le L$. Since the step size ε will eventually be sent to zero, there is no loss in spite of our restraint. We acquire in the new domain game values approximating the viscosity solution of Neumann boundary problem of motion by curvature.

THEOREM 2.17. *Assume that* Ω *satisfies* (D2). *Assume that* u_0 *is a continuous function on* $\overline{\Omega}$. *Let* u^ε *be the associated value function of the game defined by* (1.15) *with* $0 < \varepsilon < L/\sqrt{2}$ *and* u_0 *be a continuous function in* $\overline{\Omega}$. *Then* u^ε *converges, as* $\varepsilon \to 0$, *to the unique viscosity solution of* (NP1) *uniformly on compacta of* $\overline{\Omega} \times [0,\infty)$.

The proof of Theorem 2.17 is almost the same as that of Theorem 1.7, The next two propositions are variants of Proposition 2.10 and 2.12.

PROPOSITION 2.18. *Assume that* Ω *satisfies* (D2). *Let* $\{u^\varepsilon\}$ *be locally bounded in* $\overline{\Omega} \times (0,\infty)$ *for* $0 < \varepsilon < L/\sqrt{2}$. *Assume that* u^ε *satisfies the dynamic programming principle* (1.16). *Then* $\overline{u}$ *is a viscosity subsolution of* (2.20a)-(2.20b).

PROPOSITION 2.19. *Assume that* Ω *satisfies* (D2). *Let* $\{u^\varepsilon\}$ *be locally bounded in* $\overline{\Omega} \times (0,\infty)$ *for* $0 < \varepsilon < L/\sqrt{2}$. *Assume that* u^ε *satisfies the dynamic programming principle* (1.16). *Then* $\underline{u}$ *is a viscosity supersolution of* (2.20a)-(2.20b).

However, the results in Lemma 2.4 and Proposition 2.7 are no longer true because Ω here is not convex. We have to find another way to construct barrier games. A related problem was solved by Ishii and Sato [**72**] about the construction of sub- and supersolutions when they applied Perron's method to show the existence of solutions for nonlinear oblique boundary problems. It turns out that their results can meet our needs.

THEOREM 2.20 ([**72**, Theorem 4.4]). *Assume that* Ω *is a bounded domain in* $\mathbb{R}^n$ *with* C^1 *boundary. Assume the boundary condition* B *satisfies:*

(B1) $B \in C(\mathbb{R}^n \times \mathbb{R}^n) \cap C^{1,1}(\mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}))$;

(B2) *For each* $x \in \mathbb{R}^n$ *the function* $p \mapsto B(x,p)$ *is positively homogeneous of degree one in* p, *i.e.,*

$$B(x, \alpha p) = \alpha B(x,p), \text{ for all } \alpha \geq 0 \text{ and } p \in \mathbb{R}^n \backslash \{0\};$$

(B3) *There exists a positive constant* θ *such that*

$$\nu(z) \cdot \nabla_p B(z,p) \geq \theta,$$

for all $z \in \partial\Omega$ *and* $p \in \mathbb{R}^n \backslash \{0\}$.

Then there are a function $w \in C^{1,1}(\overline{\Omega} \times \overline{\Omega})$ *and positive constants* C *and* δ *such that for all* $(x,y) \in \overline{\Omega} \times \overline{\Omega}$,

(i) $|x-y|^4 \leq w(x,y) \leq C|x-y|^4$,
$|\nabla_x w(x,y)| \vee |\nabla_y (x,y)| \leq C|x-y|^3$,

(ii) $B(x, \nabla_x w(x,y)) \geq 0$ *if* $y \in \partial\Omega$,
$B(y, -\nabla_y w(x,y)) \leq 0$ *if* $y \in \partial\Omega$,

(iii) $|\nabla_x w(x,y) + \nabla_y w(x,y)| \leq C|x-y|^4$,
$\rho(\nabla_x w(x,y), -\nabla_y w(x,y)) \leq C|x-y|$ *if* $0 < |x-y| \leq \delta$, *and for a.e.* $(x,y) \in \overline{\Omega} \times \overline{\Omega}$,

(iv) $\nabla^2 w(x,y) \leq C\left\{|x-y|^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + |x-y|^4 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right\}$.

Here $\vee$ *and* $\wedge$ *stand respectively for the usual maximizing and minimizing operations, and*

$$\rho(z_1, z_2) := \frac{|z_1 - z_2|}{|z_1| \wedge |z_2|}.$$

Let w be such a function and then for any fixed $x_0 \in \overline{\Omega}$, we see that $W(x) := w(x_0, x)$ belongs to $C^{1,1}(\overline{\Omega})$ and satisfies

$$|x - x_0|^4 \le W(x) \le C|x - x_0|^4, \text{ for all } x \in \overline{\Omega}, \tag{2.51}$$

and

$$\nabla W(y) \cdot \nu(y) \ge 0, \text{ for all } y \in \partial\Omega. \tag{2.52}$$

In order to estimate $\nabla W(x) \cdot (S^{\sqrt{2}\varepsilon}(x, bv) - x)$ for any $x \in \overline{\Omega}$, $b = \pm 1$ and $v \in \mathbf{S}^1$, we calculate $\nabla W(x) \cdot \alpha^t$, where $\alpha^t = \alpha^t(x, bv) = \sum_{l=0}^{\infty} d_l^b \nu(y_l^b)$ with t sufficiently small. We here suppress the superscript b for convenience. Substituting y_l into y in (2.52), we are led to

$$\begin{aligned} &\nabla W(x) \cdot \alpha^t \\ \ge\ & (\nabla W(x) - \nabla W(y_0)) \cdot \alpha^t + (\nabla W(y_0) - \nabla W(y_1)) \cdot \sum_{l=1}^{\infty} d_l \nu(y_l) \\ &+ \sum_{k=2}^{\infty} \left((\nabla W(y_{k-1}) - \nabla W(y_k)) \cdot \sum_{l=k}^{\infty} d_l \nu(y_l) \right). \end{aligned} \tag{2.53}$$

Since W is a $C^{1,1}$ function and Ω is assumed to be bounded, there is a constant M such that

$$|\nabla W(x) - \nabla W(y)| \le M|x - y|, \text{ for all } x, y \in \overline{\Omega}.$$

By Lemma 2.16, we may still use (2.6) and (2.7) to get

$$\begin{cases} |\nabla W(x) - \nabla W(y_0)| \le \sqrt{2}\varepsilon M, & |\alpha^t| \le 2\sqrt{2}\varepsilon, \\ |\nabla W(y_0) - \nabla W(y_1)| \le \sqrt{2}\varepsilon M, & \left|\sum_{l=1}^{\infty} d_l \nu(y_l)\right| \le 4\sqrt{2}\varepsilon, \\ \sum_{k=2}^{\infty} |\nabla W(y_{k-1}) - \nabla W(y_k)| \le 2\sqrt{2}\varepsilon M, & \\ \left|\sum_{l=k}^{\infty} d_l \nu(y_l)\right| \le 4\sqrt{2}\varepsilon \text{ (for all } k \ge 2). & \end{cases}$$

Anyway, we can get from above a constant $C_1 > 0$ to proceed the calculation in (2.53),

$$\nabla W(x) \cdot \alpha^t \ge -C_1 \varepsilon^2.$$

Hence, we conclude

$$\begin{aligned} \nabla W(x) \cdot \left(S^{\sqrt{2}\varepsilon}(x, bv) - x\right) \le & \nabla W(x) \cdot \sqrt{2}\varepsilon bv + C_1 \varepsilon^2, \\ & \text{for all } x \in \overline{\Omega},\ b = \pm 1 \text{ and } v \in \mathbf{S}^1. \end{aligned} \tag{2.54}$$

LEMMA 2.21. *Let u^ε be the associated game value and $(x, t) \in \overline{\Omega} \times [0, \infty)$. Then the following relations hold.*

(i) $u^\varepsilon(x, t + \varepsilon^2) - u^\varepsilon(x, t) \le \sup_{y \in \Omega} \{u^\varepsilon(y, \varepsilon^2) - u_0(y)\}$;

(ii) $u^\varepsilon(x, t + \varepsilon^2) - u^\varepsilon(x, t) \ge \inf_{y \in \Omega} \{u^\varepsilon(y, \varepsilon^2) - u_0(y)\}$.

The lemma above says that comparison between two game values could be postponed whenever they share the same starting point. Its proof, based on the dynamic programming equation, is given in [**52**].

LEMMA 2.22. *Let Ω satisfy* (D2). *Then there exists a constant $K > 0$ such that for all $x \in \overline{\Omega}$, the following two inequalities hold.*

(i) $\inf_{|v|=1} \sup_{b=\pm 1} W\left(S^{\sqrt{2}\varepsilon}(x, bv)\right) - W(x) \le K\varepsilon^2$;

(ii) $\sup_{|v|=1} \inf_{b=\pm 1} W\left(S^{\sqrt{2}\varepsilon}(x, bv)\right) - W(x) \le K\varepsilon^2$.

PROOF. If $\nabla W(x) = 0$, then by Taylor's formula, both inequalities in question are trivial, since Ω is bounded and W is a $C^{1,1}$ function in it.

If otherwise $\nabla W(x) \neq 0$, then for either b, we can take $\hat{v} = \frac{\nabla^\perp W(x)}{|\nabla W(x)|}$ so that Taylor's formula, together with the preceding estimate (2.54), yields

$$\begin{aligned} W\left(S^{\sqrt{2}\varepsilon}(x, b\hat{v})\right) - W(x) &\le \nabla W(x) \cdot \left(S^{\sqrt{2}\varepsilon}(x, b\hat{v}) - x\right) + M\varepsilon^2 \\ &\le \nabla W(x) \cdot \sqrt{2}\varepsilon b\hat{v} + C_1\varepsilon^2 + M\varepsilon^2 \\ &\le (C_1 + M)\varepsilon^2. \end{aligned}$$

Setting $K = C_1 + M$, we conclude (i).

For the same reason, choose $\hat{b}$ for any $v \in \mathbf{S}^1$ such that $\hat{b}\nabla W(x) \cdot v \le 0$ and thus we have

$$W\left(S^{\sqrt{2}\varepsilon}(x, \hat{b}v)\right) - W(x) \le K\varepsilon^2,$$

which implies (ii). □

We now use this W to construct barriers and prove rigorously the consistency of $\overline{u}$ and $\underline{u}$ at the initial time.

PROPOSITION 2.23. *Under the same assumptions in* Theorem 2.17, *there holds* $\overline{u}(x, 0) = \underline{u}(x, 0)$.

PROOF. Fix $x_0 \in \overline{\Omega}$. For any $\lambda > 0$, there is a large constant $\mu > 0$ such that $\overline{V}_\lambda(x) = \lambda + u_0(x_0) + \mu W(x)$ fulfills

$$\overline{V}_\lambda(x) \ge u_0(x), \text{ for all } x \in \overline{\Omega}. \tag{2.55}$$

Let $\overline{U}_\lambda^\varepsilon$ denote the value of the upper barrier game with an objective function $\overline{V}_\lambda$. It is implied by (2.55) that

$$u^\varepsilon \le \overline{U}_\lambda^\varepsilon. \tag{2.56}$$

(We just let Paul follow his optimal decisions in the upper barrier game.) In virtue of Lemma 2.21(i) and the dynamic programming equation, we have

$$\overline{U}_\lambda^\varepsilon(x, t + \varepsilon^2) - \overline{U}_\lambda^\varepsilon(x, t) \le \sup_{y \in \overline{\Omega}} \left\{ \inf_{|v|=1} \sup_{b=\pm 1} \overline{V}_\lambda\left(S^{\sqrt{2}\varepsilon}(y, bv)\right) - \overline{V}_\lambda(y) \right\},$$

which by Lemma 2.22 gives

$$\overline{U}_\lambda^\varepsilon(x, t + \varepsilon^2) - \overline{U}_\lambda^\varepsilon(x, t) \le K\mu\varepsilon^2.$$

We therefore obtain from (2.56)

$$\begin{aligned}&u^\varepsilon(x,t)-u_0(x_0)\\ \le&\overline{U}^\varepsilon_\lambda(x,t)-\overline{U}^\varepsilon_\lambda(x,0)+\lambda+u_0(x_0)+\mu W(x)-u_0(x_0)\\ \le&K\mu N\varepsilon^2+\lambda+\mu W(x)\\ \le&\lambda+K\mu t+C\mu|x-x_0|^4.\end{aligned}$$

So we are led to $\overline{u}(x_0,0)\le\lambda+u_0(x_0)$ for every λ and thus $\overline{u}(x_0,0)\le u_0(x_0)$. On the other hand, for every λ, if we play a lower barrier game with the terminal cost

$$\underline{V}_\lambda(x)=-\lambda+u_0(x_0)-\mu W(x),$$

where μ is such that $\underline{V}_\lambda\le u_0$, then it follows by the same argument but an application of Lemma 2.21(ii) that

$$u^\varepsilon(x,t)\ge-\lambda+u_0(x_0)-K\mu t-C\mu|x-x_0|^4.$$

Hence, we finally obtain $\underline{u}(x_0,0)\ge u_0(x_0)$. □

Our proof of Proposition 2.23 is in essence the same as that of Proposition 2.7. In the case of a convex domain, our auxiliary function $W(x)$ reduces to $|x-x_0|^2$, which in turn implies Lemma 2.4 by the argument in Lemma 2.22.

We here remark that the assumption (D2) is a little bit restrictive. A billiard semiflow is likely to be assigned to more general domains and so are our purely deterministic games. It is then important to understand more complicated billiards with the occurrences of termination, entrance and exit circulating.

2.3. Regularity of billiards

A remaining question is about the regularity, especially the continuity of this semiflow S^t. It turns out that the the function $\overline{\Omega}\times\mathbf{S}^1\ni(x,v)\to S^t(x,v)$ is not continuous in general, especially when termination occurs in the trajectory. However, we can get separately the continuity of $S^t(x,v)$ in $(x,t)\in\overline{\Omega}\times[0,\infty)$ and in $v\in\mathbf{S}^1$ by characterizing the billiard dynamics in the setting of a first-order partial differential equation with the homogeneous Neumann boundary condition

$$\text{(NP2)}\qquad\begin{cases}u_t(x,t)+\sup\limits_{a\in A}\{-f(x,a)\cdot\nabla u(x,t)-l(x,a)\}=0 & \text{in } \Omega\times[0,\infty),\\ \nabla u(x,t)\cdot\nu(x)=0 & \text{on } \partial\Omega\times[0,\infty),\\ u(x,0)=u_0(x) & \text{in } \overline{\Omega}.\end{cases}$$

We establish a discrete system on the basis of the billiard semiflow investigated in Section 2.1. At first assume

$$A \text{ is a compact topological space,}\tag{2.57}$$

$$f:\overline{\Omega}\times A\to\mathbb{R}^2 \text{ satisfies } \sup_{x\in\overline{\Omega},\ a\in A}|f(x,a)|\le M_1,\tag{2.58}$$

and

$$|f(x_1,a)-f(x_2,a)|\le L_1|x_1-x_2| \text{ for } L_1>0 \text{ independent of } a\in A.\tag{2.59}$$

Notice that there exists a function $v_f : \overline{\Omega} \times A \to S^1 = \{v \in \mathbb{R}^2 : |v| = 1\}$ such that for any $(x, a) \in \overline{\Omega} \times A$,

$$f(x, a) = |f(x, a)| v_f(x, a).$$

To formulate our control system, we take the step size $\varepsilon > 0$ and a sequence $y_k, k = 0, 1, 2, \ldots$, which satisfies the following:

$$\begin{cases} y_{k+1} = S^{|f_k|\varepsilon}\left(y_k, v_f(y_k, a_{k+1})\right); \\ y_0 = x, \end{cases} \tag{2.60}$$

where the control variable $a_k \in A$ and $|f_k| = |f(y_k, a_{k+1})|$ for all $k = 0, 1, 2, \ldots$ It seems to be at question whether our definition above is valid since v_f is not uniquely determined when $|f| = 0$. However, there is essentially no problem in the system (2.60) thanks to our billiard structure, which yields a temporary stop whenever $f_k = 0$.

For every $t \geq 0$, let N be the largest integer less than t/ε. Given $x \in \mathbb{R}^2$, $t \geq 0$ and $a = (a_1, \ldots, a_N) \in A^N$, we define a control objective as

$$\begin{aligned} J^\varepsilon(x, t, a) &:= \sum_{k=1}^{N} \varepsilon l(y_{k-1}, a_k) + u_0(y_N), \text{ if } t \geq \varepsilon \text{ and} \\ J^\varepsilon(x, t, a) &:= u_0(x), \text{ if } 0 \leq t < \varepsilon, \end{aligned} \tag{2.61}$$

where $l : \overline{\Omega} \times A \to \mathbb{R}$ stands for the running cost fulfilling

$$\sup_{x \in \overline{\Omega},\ a \in A} |l(x, a)| \leq M_2; \text{ and} \tag{2.62}$$

$$|l(x_1, a) - l(x_2, a)| \leq L_2 |x_1 - x_2| \text{ for } L_2 > 0 \text{ independent of } a \in A. \tag{2.63}$$

and the function $u_0 : \overline{\Omega} \to \mathbb{R}$ is a terminal cost. We next define a value function for every $x \in \overline{\Omega}$ and $t \geq 0$

$$u^\varepsilon(x, t) := \inf_{a \in A^N} J^\varepsilon(x, t, a) \tag{2.64}$$

and it clearly satisfies the dynamic programming equation

$$\begin{aligned} u^\varepsilon(x, t) := \inf_{a \in A} \{u^\varepsilon(S^{|f(x,a)|\varepsilon}(x, v_f(x, a)),\ t - \varepsilon) + \varepsilon l(x, a)\} \\ \text{for all } x \in \overline{\Omega} \text{ and } t \geq \varepsilon. \end{aligned} \tag{2.65}$$

Then we have the following result.

THEOREM 2.24. *Assume* (D1) *and* (2.57)-(2.63). *Let u^ε be the game value in* (2.64) *and u_0 be a continuous function in $\overline{\Omega}$, then u^ε converges, as $\varepsilon \to 0$, uniformly on every compact set of $\overline{\Omega} \times [0, \infty)$ to the unique solution of the Neumann boundary problem of Hamilton-Jacobi equation* (NP2).

Although the proof in the case of first-order equations is supposed to be contained in the former results for second-order equations, we still prefer putting down the whole proof in what follows since our game setting here is more general.

We below present the definition of viscosity solutions of (NP2).

DEFINITION 2.6. An upper semicontinuous (resp., lower semicontinuous) function u on $\overline{\Omega}\times[0,\infty)$ is a *viscosity subsolution* (resp., *viscosity supersolution*) of (NP2) if

$$u(x,0) \leq u_0(x) \quad (\text{resp., } u(x,0) \geq u_0(x))$$

and whenever there are $(\hat{x},\hat{t}) \in \overline{\Omega} \times (0,\infty)$, a neighborhood $\mathcal{O}$ relative to $\overline{\Omega} \times (0,\infty)$ of $(\hat{x},\hat{t})$ and a smooth function $\varphi : \mathcal{O} \to \mathbb{R}$ such that

$$\max_{\mathcal{O}}(u-\varphi) = (u-\varphi)(\hat{x},\hat{t})$$
$$\left(\text{resp., } \min_{\mathcal{O}}(u-\varphi) = (u-\varphi)(\hat{x},\hat{t})\right),$$

the following holds:

(i) If $\hat{x} \in \Omega$, then

$$\partial_t\varphi(\hat{x},\hat{t}) + \sup_{a\in A}\{-f(\hat{x},a)\cdot\nabla\varphi(\hat{x},\hat{t}) - l(\hat{x},a)\} \leq 0$$
$$\left(\text{resp., } \partial_t\varphi(\hat{x},\hat{t}) + \sup_{a\in A}\{-f(\hat{x},a)\cdot\nabla\varphi(\hat{x},\hat{t}) - l(\hat{x},a)\} \geq 0\right).$$

(ii) If $\hat{x} \in \partial\Omega$, then

$$\partial_t\varphi(\hat{x},\hat{t}) + \sup_{a\in A}\{-f(\hat{x},a)\cdot\nabla\varphi(\hat{x},\hat{t}) - l(\hat{x},a)\} \leq 0$$
$$\left(\text{resp., } \partial_t\varphi(\hat{x},\hat{t}) + \sup_{a\in A}\{-f(\hat{x},a)\cdot\nabla\varphi(\hat{x},\hat{t}) - l(\hat{x},a)\} \geq 0\right)$$

or

$$\nabla\varphi(\hat{x},\hat{t})\cdot\nu(\hat{x}) \leq 0 \quad \left(\text{resp., } \nabla\varphi(\hat{x},\hat{t})\cdot\nu(\hat{x}) \geq 0\right).$$

DEFINITION 2.7. A function u on $\overline{\Omega} \times [0,\infty)$ is called a *viscosity solution* of (NP2) if it is both a viscosity subsolution and a viscosity supersolution.

Before we prove Theorem 2.24, let us first introduce the *upper* and *lower relaxed limits* of u^ε as

$$\overline{u}(x,t) := \limsup_{\varepsilon\to 0}{}^*u^\varepsilon(x,t) = \lim_{\delta\to 0}\sup\{u^\varepsilon(y,s) : \varepsilon < \delta,\ |x-y|+|t-s| < \delta\}$$

and

$$\underline{u}(x,t) := \liminf_{\varepsilon\to 0}{}_*u^\varepsilon(x,t) = \lim_{\delta\to 0}\inf\{u^\varepsilon(y,s) : \varepsilon < \delta,\ |x-y|+|t-s| < \delta\}.$$

In what follows, we give our proof of Theorem 2.24 by showing $\overline{u} = \underline{u}$, which consists of three propositions.

PROPOSITION 2.25. $\underline{u}(x,0) = \overline{u}(x,0) = u_0(x)$.

PROOF. We only show $\overline{u}(x,0) \leq u_0(x)$ for every fixed $x \in \overline{\Omega}$. A symmetric argument gives $\underline{u}(x,0) \geq u_0(x)$ and our conclusion is thus reached in virtue of a basic fact that $\underline{u} \leq \overline{u}$. To this end, we adopt a barrier argument since our initial datum u_0 is only continuous and needs regularizing. More precisely, for any $\lambda > 0$, there exists a constant C_λ such that

$$u_0(y) \leq \lambda + u_0(x_0) + C_\lambda|y-x| \text{ for all } y \in \overline{\Omega}.$$

We denote by $\overline{V}_\lambda$ the right hand side of the above inequality and now observe the same discrete-time optimal control problem but only with the terminal cost changed from u_0 to $\overline{V}_\lambda$. Let the value function of this new game be $\tilde{u}^\varepsilon$.

Then, in view of the definition (2.64), the boundedness of functions f and l (i.e., (2.58) and (2.62)) and (2.15) with $x_0 = x$, we directly evaluate for $y \in \overline{\Omega}$ and $t \geq 0$

$$\begin{aligned}\tilde{u}^\varepsilon(y,t) &\leq M_2 N\varepsilon + \lambda + u_0(x) + C_\lambda(|y-x| + M_1 N\varepsilon) \\ &\leq \lambda + u_0(x) + C_\lambda|y-x| + (C_\lambda M_1 + M_2)t.\end{aligned}$$

Noting that game values preserve the order of objectives, that is $u^\varepsilon \leq \tilde{u}^\varepsilon$ in our special case, we are thus led, by the definition of relaxed limits as $y \to x$ and $t \to 0$, to

$$\overline{u}(x,0) \leq \lambda + u_0(x).$$

Sending $\lambda \downarrow 0$, we are done. □

PROPOSITION 2.26. *$\overline{u}$ is a subsolution of* (NP2).

PROOF. We argue by contradiction. Since $\overline{u}$ fulfills the initial data by Proposition 2.25, assume there exist $(x_0, t_0) \in \partial\Omega \times (0,\infty)$ (our argument actually works for the case $x_0 \in \Omega$ as well), a δ-neighborhood B_δ of (x_0, t_0) relative to $\overline{\Omega} \times (0,\infty)$ and a smooth function ϕ on $\overline{\Omega} \times (0,\infty)$ such that

(i) $\overline{u}(x_0,t_0) - \phi(x_0,t_0) > \overline{u}(x,t) - \phi(x,t)$ for all $(x,t) \in B_\delta$;
(ii) $\partial_t\phi(x_0,t_0) + \sup_{a\in A}\{-f(x_0,a)\cdot\nabla\phi(x_0,t_0) - l(x_0,a)\} \geq \eta_0 > 0$; and
(iii) $\nabla\phi(x_0,t_0)\cdot\nu(x_0) \geq \eta_0 > 0$.

Assumption (ii), together with the continuity of f and l in x (i.e., (2.59) and (2.63)), implies the existence of $\overline{a} \in A$ satisfying

$$\partial_t\phi(x,t) - f(x,\overline{a})\cdot\nabla\phi(x,t) - l(x,\overline{a}) \geq \eta_0/2 \text{ for all } (x,t) \in B_\delta \tag{2.66}$$

and (iii) gives rise to

$$\nabla\phi(x,t)\cdot\nu(x) \geq \eta_0/2 \text{ for all } (x,t) \in B_\delta \text{ and } x \in \partial\Omega. \tag{2.67}$$

By the definition of $\overline{u}$, we can take a sequence $(x_0^\varepsilon, t_0^\varepsilon) \to (x_0, t_0)$ with

$$u^\varepsilon(x_0^\varepsilon, t_0^\varepsilon) \to \overline{u}(x_0,t_0).$$

Let us use the constant control $\overline{a}$ to get a sequence of states

$$X_1 \equiv (x_1, t_1) = (x_0^\varepsilon, t_0^\varepsilon);$$

$$X_{k+1} \equiv (x_{k+1}, t_{k+1}) = \left(S^{|f_k|\varepsilon}(x_k, v_f(x_k,\overline{a})), t_k - \varepsilon\right),\ k \geq 1.$$

We assume for the moment that any X_k does not exceed B_δ, which requires that k should not be too large. In terms of the dynamic programming principle (2.65), we have

$$u^\varepsilon(X_k) \leq u^\varepsilon(X_{k+1}) + \varepsilon l(x_k,\overline{a}). \tag{2.68}$$

On the other hand, applying Taylor's formula and the notion of boundary adjustor (2.4), we get

$$\phi(X_{k+1}) = \phi(X_k) - \varepsilon\phi_t(X_k) + \nabla\phi(X_k)\cdot(f(x_k,\overline{a})\varepsilon - \alpha_k^\varepsilon). \tag{2.69}$$

(If x_0 is originally an interior point, taking a sufficient small δ makes all $\alpha_k^\varepsilon = 0$.) Due to the special structure of α as in (2.5) in Lemma 2.3 and an application of (2.67), the above equality yields

$$\phi(X_{k+1}) - \phi(X_k) \leq \varepsilon\left(-\phi_t(X_k) + \nabla\phi(X_k)\cdot f(x_k,\overline{a})\right). \tag{2.70}$$

Combining (2.66), (2.68) and (2.70), we are led to

$$(u^\varepsilon - \phi)(X_{k+1}) - (u^\varepsilon - \phi)(X_k) \geq \frac{\eta_0}{2}\varepsilon$$

and furthermore

$$(u^\varepsilon - \phi)(X_k) - (u^\varepsilon - \phi)(X_1) \geq \frac{(k-1)\eta_0\varepsilon}{2} \text{ for all } k = 1, 2, \ldots.$$

It means that we can take a subsequence of X_k, still indexed by k, in B_δ but converging to $(x', t') \neq (x_0, t_0)$. Hence, by letting $\varepsilon \to 0$, we see

$$(\overline{u} - \phi)(x', t') \geq (\overline{u} - \phi)(x_0, t_0),$$

which is a contradiction to our assumption (i).

□

PROPOSITION 2.27. *$\underline{u}$ is a supersolution of* (NP2).

Showing Proposition 2.27 is not largely different from what has been done for Proposition 2.26. We omit the whole process here.

The proof of Theorem 2.24 is actually completed. The last step in our proof is only a comparison principle to obtain $\overline{u} \leq \underline{u}$ based on our three propositions above. This part of work is classical and elaborated well in [**79**].

We next discuss a most simple special case. The equation is simply

$$\text{(LNP)} \quad \begin{cases} u_t(x,t) - v \cdot \nabla u(x,t) = 0 & \text{in } \Omega \times [0,\infty), \\ \nabla u(x,t) \cdot \nu(x) = 0 & \text{on } \partial\Omega \times [0,\infty), \\ u(x,0) = u_0(x) & \text{in } \overline{\Omega}, \end{cases}$$

where $v \in \mathbf{S}^1$ and $u_0 \in C(\overline{\Omega})$ are given. We apply the optimal control setting above with $f \equiv v$, $l \equiv 0$ and therefore we do not really need to discretize the whole system. Letting $u(x,t) = u_0(S^t(x,v))$, we use Theorem 2.24 to easily obtain the following.

THEOREM 2.28. *u is the unique viscosity solution of* (LNP).

The dynamic programming principle in this case becomes degenerate and obvious.

PROPOSITION 2.29 (Dynamic programming). *For all $(x,t) \in \overline{\Omega} \times [0,\infty)$, $u(x,t) = u(S^\tau(x,v), t-\tau)$ whenever $0 \leq \tau \leq t$.*

We then show some of its immediate consequences. Let us first see the regularity of billiard semiflow as we mentioned.

COROLLARY 2.30. *Assume* (D1). *Fix $v \in \mathbf{S}^1$. Then the mapping $(x,t) \to S^t(x,v)$ is continuous.*

PROOF. We recall the classical result that the solution u is continuous as long as u_0 is continuous. Then $S^t(x,v)$ must be continuous otherwise we can construct u_0 to make u discontinuous. □

COROLLARY 2.31. *Assume* (D1). *Fix $(x,t) \in \overline{\Omega} \times [0,\infty)$. Then the mapping $v \to S^t(x,v)$ is continuous.*

PROOF. Owing to Theorem 2.28, our straightforward proof is based on the stability theory. □

We remark that the difficulty of the original question on continuity of $S^t(x,v)$ in (x,v) is natural because from a PDE viewpoint we are actually seeking for the continuity of solution with the equation perturbed at the same time. It requires more regularity on the domain than (D1), which is reflected by the counterexample below.

Suppose there is a terminating billiard trajectory in $\overline{\Omega}$ with a sequence of vertices $\{y_n\}_{n=1}^{\infty}$ and a terminating point y_∞ on $\partial\Omega$. From the results we have got before, we know as $k \to \infty$, $a_k = |y_{k+1} - y_k| \to 0$ and $v_k = (y_{k+1} - y_k)/a_k \to v_\infty$, where v_∞ is a unit tangent of $\partial\Omega$ at y_∞. Take a point x on the straight line segment $\overline{y_1 y_2}$ such that the distance $\operatorname{dist}(x, \partial\Omega) > 0$. Moreover, it is possible to pick an open interval (x_1, x_2) on the segment $\overline{y_1 y_2}$, satisfying $x \in (x_1, x_2)$ and

$$\inf_{z \in (x_1, x_2)} \operatorname{dist}(z, \partial\Omega) > 0. \tag{2.71}$$

We assume the total length of the billiard trajectory from x to y_∞ is τ_0. Then it is not hard to see that $S^{\tau_0}(\cdot,\cdot)$ is not continuous at $(y_\infty, -v_\infty)$. Indeed, since $y_\infty \in \partial\Omega$ and $-v_\infty$ is a tangent, we have by our definition $S^{\tau_0}(y_\infty, -v_\infty) \in \partial\Omega$. On the other hand, $S^{\tau_0}(y_k, -v_k) \in (x_1, x_2)$ when k is large. So discontinuity is obtained immediately due to (2.71).

Not only can we obtain the regularity of billiards through the above connection, but we may also look to the reverse, making use of billiards' regularity to study the local regularity of PDE solutions.

THEOREM 2.32 ([**27**, Lemma 2.24]). *Assume Ω is of class C^k and $u_0 \in C^{k-1}$ ($k \geq 3$). Then the flow T^t is C^{k-1} smooth at points that experience only regular collisions.*

An immediate consequence follows.

COROLLARY 2.33. *Assume Ω is of class C^k and $u_0 \in C^{k-1}$ ($k \geq 3$). Let u be the solution of* (LNP). *Let $x_0 \in \overline{\Omega}$ and $t_0 \in (0,\infty)$. Then the following statements hold:*

(a) *If $x_0 \in \Omega$, then there exists $0 < \delta < t_0$ such that $u \in C^{k-1}(B_\delta(x_0) \times [t_0 - \delta, t_0 + \delta])$.*
(b) *If $x_0 \in \partial\Omega$ and $v \cdot n(x_0) \neq 0$, then there exists $0 < \delta < t_0$ such that*

$$u \in C([t_0 - \delta, t_0 + \delta], C^{k-1}(B_\delta(x_0))).$$

2.4. Oblique problems in a half plane

This section is devoted to a discrete game interpretation for the oblique boundary problems. To simplify our proofs and emphasize our idea, we again mainly discuss first-order Hamilton-Jacobi equations on the ground of deterministic optimal control theory. The well-posedness of these oblique boundary problems in the viscosity sense is due to [**79**] for first-order cases and [**51, 57, 100, 101**] for second-order ones.

A billiard semiflow is studied in the previous sections. Based on it, discrete deterministic games are constructed so that their value functions converge to the unique solution of the Neumann boundary problem of curve shortening flow equation. We now apply the same method to the first-order Hamilton-Jacobi equations but with oblique type boundary:

$$\text{(OP1)} \qquad \begin{cases} u_t(x,t) + \sup_{a\in A}\{-f(x,a)\cdot\nabla u(x,t) - l(x,a)\} = 0 & \text{in } \Omega\times[0,\infty), \\ \nabla u(x,t)\cdot\gamma(x) = 0 & \text{on } \partial\Omega\times[0,\infty), \\ u(x,0) = u_0(x) & \text{in } \overline{\Omega}, \end{cases}$$

where $\gamma(x)$ is the unit oblique normal, satisfying for every $x \in \partial\Omega$,

$$\gamma(x)\cdot\nu(x) = \theta, \quad 0 < \theta \leq 1.$$

The key is to devise a more general billiard law, as we call the oblique billiard, so as to get differential games for oblique boundary conditions. When creating the obliqueness, we do not imitate the usual billiard law via angles at hitting points, but instead follow the idea of decomposing each incident ray along the normal and tangent and then simply switching the direction of its normal component. Such an operation certainly gives a generalization of the classical billiard but its properties, especially those about its singular phenomena, turn out to be obscure. In this section, without touching too complicated situations, we conduct our game interpretation for the oblique boundary problem of Hamilton-Jacobi equations only in the half plane, where any billiard move hits the boundary at most once.

In order to avoid redundancy, we slightly modify our proof for Neumann boundary problem to adapt it to this oblique boundary case. As a matter of fact, in this case, the similarity between billiard and Skorokhod reflections still holds. Another application of our oblique billiards to the curve shortening equation is quite natural and a formal derivation is included as well in the section. A question remains unsolved how to get any extension of our results or find another type of oblique billiards for more general domains.

We remark that both first and second-order equations can be derived from discrete game settings but their difference is spectacular. In this section, our intention is to deal with the Hamilton-Jacobi equation (OP1) and curve shortening flow equation but only in the domain of a half plane, that is, we assume

$$\Omega = \{(x_1,x_2)\in\mathbb{R}^2 : x_2 > 0\}. \tag{2.72}$$

2.4.1. Hamilton-Jacobi Equation. We first need to seek an oblique billiard, whose existence is suggested by the succeeding simple example.

Under the assumption (2.72), let us denote respectively by $\vec{i}$ and $\vec{j}$ the unit vectors in a pair of coordinates, and then the outward normal of $\partial\Omega$ reduces to a constant vector $-\vec{j}$. Furthermore, for any hitting point x, we suppose,

$$\gamma(x) = -\sqrt{1-\theta^2}\vec{i} - \theta\vec{j}. \tag{2.73}$$

In this domain, a planar oblique billiard trajectory $S_o^t(x,v_1)$ can be set as a straight line with initial data (x,v_1) until it touches $\partial\Omega$. At the hitting moment, varying from the familiar optic reflection law, we linearly decompose the unit vector v_1 according to the vectors γ and $\vec{i}$, only oppose the sign of its component in the γ direction, and in this way get the reflected-off vector v_2 which will lead another straight line move. More precisely, taking a linear transformation

$$R = \begin{pmatrix} 1 & -2\frac{\sqrt{1-\theta^2}}{\theta} \\ 0 & -1 \end{pmatrix}, \tag{2.74}$$

we can express the new type of reflection by

$$v_2 = Rv_1, \tag{2.75}$$

where v_1 satisfies $v_1 \cdot \vec{j} \leq 0$.

It merits mentioning that v_1 and v_2 here should be viewed as speed vectors in stead of directions, because $|v_2| \neq 1$ in general. In other words, the speed shifts at vertices, whose total number is however evidently not more than 1 in this half plane case. For a general domain, the number of vertices could be very large and we know little about the singular phenomena especially the termination. This problem actually obstructs us to handle more complicated domains. It is of great interest if one can generalize Lemma 2.2 for our application in this new circumstance.

In our special domain, the definition of S_o^t is

DEFINITION 2.8. Assume (2.72) and let t_0 be the hitting time, and then define $S_o^t(\cdot,\cdot) : \overline{\Omega} \times S^1 \to \overline{\Omega}$ as

$$S_o^t(x,v) = \begin{cases} x + tv & \text{if } t \leq t_0 \\ x + t_0 v + (t - t_0)Rv & \text{if } t > t_0, \end{cases} \tag{2.76}$$

where R is given in (2.74).

Generalized from normal billiards, this oblique billiard dynamic has an analogue of (2.5) in Lemma 2.3 as

$$\beta^t(x,v) := x + tv - S_o^t(x,v) = C(t)\gamma \text{ and } |\beta^t| = C(t) \leq \frac{2t}{\theta}, \tag{2.77}$$

where $C(t)$ is a constant depending on t.

With all the preparation above, the game corresponding to (OP1) can be established almost the same as we have done in Section 2.2. Merely substituting S^t with S_o^t, one confirms that the new value function u^ε satisfies

$$\begin{aligned} u^\varepsilon(x,t) := \inf_{a\in A}\{u^\varepsilon(S_o^{|f(x,a)|\varepsilon}(x, v_f(x,a)),\, t-\varepsilon) + \varepsilon l(x,a)\} \\ \text{for all } x \in \overline{\Omega} \text{ and } t \geq \varepsilon. \end{aligned} \tag{2.78}$$

Then we get

THEOREM 2.34. *Assume (2.72). Let u^ε be the value function of the game based on oblique billiards above and u_0 be a continuous function in $\overline{\Omega}$. Then u^ε converges, as $\varepsilon \to 0$, uniformly on every compact subset of $\overline{\Omega} \times [0,\infty)$ to the unique viscosity solution of* (OP1).

PROOF. We almost repeat the proof of Theorem (2.28). Indeed, the proof of Proposition 2.25 needs little modification. A crucial point is that, in light of (2.77), for oblique billiard games the distance of each move is still bounded in spite of a necessary alteration of the bound from ε to $(1+2/\theta)\varepsilon$. Our boundary adjustor now is not a series of normals but instead a single oblique one. □

2.4.2. Curve Shortening Flow Equation. Another example of oblique billiards related PDEs is the Neumann boundary problem of two-dimensional curvature flow equation:

$$\text{(OP2)} \qquad \begin{cases} \partial_t u - \Delta u + \left(\nabla^2 u \frac{\nabla u}{|\nabla u|}\right) \cdot \frac{\nabla u}{|\nabla u|} = 0 & \text{in } \Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \overline{\Omega}, \\ \nabla u(x,t) \cdot \gamma(x) = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

where $\gamma(x)$ is the outward oblique normal, defined in the same way as in the Section 2.4.1. We again consider the problem in the half plane, i.e., Ω is assumed to satisfy (2.72).

We next pose the game setting in a concise manner. Start the game at $x \in \overline{\Omega}$ and $t = 0$. We keep in mind that for any $\varepsilon > 0$ each step brings about a movement of length $\sqrt{2}\varepsilon$ but costs time ε^2, which certainly implies that for any time t, the total of game steps N equals the largest integer less than or equal to t/ε^2.

The discrete system now writes

$$y_k = S_o^{\sqrt{2}\varepsilon}(y_{k-1}, b_k v_k), \; b_k = \pm 1 \text{ and } v_k \in \mathrm{S}^1 \text{ for all } k = 1, 2, \ldots, N;$$
$$y_0 = x \in \overline{\Omega},$$

where v and b are control variables of two players who are adverse to each other on the quantity $u_0(y_N)$. Giving the information advantage to the player in charge of control b, we define the value function as

$$u^\varepsilon(x,t) = \inf_{|v_1|=1} \sup_{b_1=\pm 1} \ldots \inf_{|v_N|=1} \sup_{b_N=\pm 1} u_0(y(N)), \tag{2.79}$$

which, in particular, implies $u^\varepsilon(x,t) = u_0(x)$ when $t \in [0, \varepsilon^2)$. As usual, we can show that u^ε satisfies the dynamic programming principle

$$u^\varepsilon(x,t) = \inf_{|v|=1} \sup_{b=\pm 1} u^\varepsilon\left(S_o^{\sqrt{2}\varepsilon}(x, bv), t - \varepsilon^2\right) \text{ for all } t \in [\varepsilon^2, \infty). \tag{2.80}$$

It follows formally by Taylor's formula and the billiard representation (2.77) that at (x,t)

$$\begin{aligned} 0 \approx -\varepsilon^2 u_t^\varepsilon + \inf_{|v|=1} \sup_{b=\pm 1} \Big\{ &\nabla u^\varepsilon \cdot \left(\sqrt{2}\varepsilon bv - \beta^{\sqrt{2}\varepsilon}\right) \\ &+ \frac{1}{2} \nabla^2 u^\varepsilon \left(\sqrt{2}\varepsilon bv - \beta^{\sqrt{2}\varepsilon}\right) \cdot \left(\sqrt{2}\varepsilon bv - \beta^{\sqrt{2}\varepsilon}\right) \Big\}. \end{aligned} \tag{2.81}$$

We explain the heuristics as we did in Chapter 1. Viewing for the moment that $u^\varepsilon(x,t)$ has bounded derivatives and converges in some sense to a function $u(x,t)$, we discuss two cases for every subsequence, still indexed by ε:

1. Boundary condition dominant case: There exists $C > 0$ such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} |\beta^{\sqrt{2}\varepsilon}| = C.$$

We then divide both sides of (2.81) by ε, pass to the limit $\varepsilon \to 0$ and get via (2.77) that

$$0 = \sqrt{2} \inf_{|v|=1} \sup_{b=\pm 1} |\nabla u(x,t) \cdot bv| - C \nabla u(x,t) \cdot \gamma(x).$$

Since the first term on the right hand side is zero, the classical oblique boundary condition remains.

2. Mixed type case: Assume on the contrary to the former case

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}|\beta^{\sqrt{2}\varepsilon}| = 0.$$

Then the same first-order operation as above yields that the "inf sup" is attained at $v = \frac{\nabla^\perp u}{|\nabla u|}$, To avoid directly realizing the oblique boundary again, we additionally assume that $\nabla u(x,t) \neq 0$. If $\frac{1}{\varepsilon^2}|\beta^{\sqrt{2}\varepsilon}| \to \infty$ as $\varepsilon \to 0$, we divide both sides of (2.81) by $|\beta^{\sqrt{2}\varepsilon}|$ and send $\varepsilon \to 0$ to get

$$\nabla u(x,t)\cdot\gamma(x) = 0.$$

If $|\beta^{\sqrt{2}\varepsilon}|$ is of order $o(\varepsilon^2)$, we in turn use ε^2 as the divisor and obtain the limit equation

$$u_t - \nabla^2 u\frac{\nabla^\perp u}{|\nabla u|}\cdot\frac{\nabla^\perp u}{|\nabla u|} = 0 \ \text{ at } (x,t).$$

Even when $|\alpha^{\sqrt{2}\varepsilon}|$ exactly has the order ε^2, the limit of the divided equation then is

$$u_t(x,t) - \nabla^2 u(x,t)\frac{\nabla^\perp u(x,t)}{|\nabla u(x,t)|}\cdot\frac{\nabla^\perp u(x,t)}{|\nabla u(x,t)|} + M\nabla u(x,t)\cdot\gamma(x) = 0,$$

where M is a positive constant. It follows that either

$$u_t - \nabla^2 u\frac{\nabla^\perp u}{|\nabla u|}\cdot\frac{\nabla^\perp u}{|\nabla u|} \geq 0 \ \text{ and } \ \nabla u\cdot\gamma(x) \leq 0 \ \text{ at } (x,t)$$

or

$$u_t - \nabla^2 u\frac{\nabla^\perp u}{|\nabla u|}\cdot\frac{\nabla^\perp u}{|\nabla u|} \leq 0 \ \text{ and } \ \nabla u\cdot\gamma(x) \geq 0 \ \text{ at } (x,t),$$

which reveals that u fulfills the boundary condition in the viscosity sense.

The preceding mechanism gives rise to the following result.

THEOREM 2.35. *Assume that* Ω *satisfies* (2.72) *and* u_0 *is a continuous function in* $\overline{\Omega}$. *Let* u^ε *be the value function of the game defined by* (2.79). *Then* u^ε *converges, as* $\varepsilon \to 0$, *to the unique viscosity solution of* (OP2) *uniformly on compact subsets of* $\overline{\Omega}\times[0,\infty)$.

As the formal deduction is well developed above, a rigorous proof is skipped.

CHAPTER 3

Fattening and Comparison: Applications of Games

This chapter is organized in the following way. In Section 3.1, we prove the fattening phenomenon for two dimensional curvature flow equation with figure-eight initial curves by using the game interpretation. In Section 3.2, we give game-theoretic proofs for some known results on the wellposedness of the equation (SC) in a mean convex domain. In Section 3.3, we use the examples to show that the weak comparison theorem of (SC) is not necessarily true in general and prove Theorems 1.8 and 1.9.

3.1. Game interpretation of fattening

In this section, we present several examples of our game approach to the fattening phenomenon. The initial data are all like figure eight, which are known very well to give rise to fat level sets. Our explanation however is in a very different style.

Throughout this section, we take $n = 2$ and let Γ_t denote the zero level set of a solution of (MC). It is obvious that Γ_t is a closed set.

3.1.1. Crossing Straight Lines. Let us make the first step with a simple example, which is discussed in [41] and [**103**]. We consider the curvature flow initialized from two straight lines crossing perpendicularly in a plane. Without loss, we may think of the lines as x_1-axis and x_2-axis and then denote the origin by O. It is certainly possible for one to endow this initial curve with an initial function u_0, which fulfills the requirement in Theorem 1.1. Indeed, arbitrarily decide a cone from the two open areas divided by axes, say the union of the first and third quadrants, denote it by Ω_- and name the other $\Omega_+ = \mathbb{R}^2 \setminus \overline{\Omega_-}$. Then we may use the signed distance of Ω_-

$$d(x) = \mathrm{dist}(x, \overline{\Omega_-}) - \mathrm{dist}(x, \overline{\Omega_+}) \tag{3.1}$$

to meet our needs, razing those too high and too low places by taking a minimum and a maximum with certain constants, i.e.,

$$u_0(x) = (d(x) \wedge M) \vee (-M), \quad \text{for all } x \in \mathbb{R}^2, \tag{3.2}$$

where $M > 0$ is sufficiently large. We take such a constant M just to assure that u_0 is bounded. It is only the neighborhood of these two axes that really counts.

Such an initial curve is known to develop interior instantly, so our consequence is certainly as follows.

THEOREM 3.1 (Fattening from crossing lines). *Let Ω_- be defined as above and the initial data u_0 of* (MC) *be given as in* (3.2). *Then the zero level set Γ_t of the solution u has nonempty interior for every $t > 0$.*

PROOF. (1) We consider u_1^ε first and take the starting point $x \in \Omega_-$. Without loss of generality, we assume $x = (x_1, x_2)$ with $x_1 > 0$ and $x_2 > 0$. Since Paul tries his best to minimize the value of u_0 where the marker finally is, he probably could use the following strategy, denoted by α_o.

For every step, he picks a feedback control $v = x/|x|$. Let us call this constant strategy *origin-oriented strategy.* It guarantees that the marker never leaves $\Omega_- \cup \{0\}$, no matter which decision is made by Carol. Paul might have better options but this one is enough for us to deduce

$$u_1^\varepsilon(x, t) \leq 0 \quad \text{for all } t > 0. \tag{3.3}$$

(2) On the other hand, suppose that Paul wants to get out of Ω_- as soon as possible, and then he should rely on the strategy α_c described in Lemma 1.5 instead. To be more precise, set $r_0 = x_1 + x_2 + \sqrt{2x_1x_2}$ and $x_0 = (r_0, r_0)$, and then it is clear that the circle $B_{r_0}(x_0)$ passes x. We then use Lemma 1.5(ii) to make sure that despite Carol's best hinderance, the marker can leave Ω_- with consumption of time t_1 at most $r_0^2/2 + \omega_0(\varepsilon)$, noticing that the marker cannot keep staying in Ω_- if it is expelled from $B_{\sqrt{2}r_0}(x_0)$.

One may ask whether the marker can enter Ω_- again after it leaves. The answer is negative. Actually, Paul is supposed to alter his strategy no sooner than the marker exits. The new strategy he adopts is exactly the same as what was described in (1). Assume he is now at $y \in (\Omega_-)^c$, then take $v = y/|y|$ if $|y| \neq 0$ and $v = (0, 1)$ if $y = 0$. It follows easily that the $u_0(y(x, t)) \geq 0$ for $t \geq t_1$. There is nothing his opponent can do about this, again due to the particularity of such strategies.

Paul's strategy is summarized to be a combination of the concentric α_c and the origin-oriented α_o. (See Figure 1.) Since $t_1 \leq r_0^2/2 + \omega_0(\varepsilon)$, we are led to

$$u_2^\varepsilon(x, t) \geq -\sqrt{2}\varepsilon \quad \text{for all } t > t_1 = \frac{1}{2}\left(x_1 + x_2 + \sqrt{2x_1x_2}\right)^2 + \omega_0(\varepsilon). \tag{3.4}$$

It follows from Theorem 1.1 that

$$u(x, t) = 0 \quad \text{for all } x \in A_t^1, \tag{3.5}$$

where $A_t^1 := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 > 0 \text{ and } x_1 + x_2 + \sqrt{2x_1x_2} < \sqrt{2t}\}$. That is to say $A_t^1 \subset \Gamma_t$. Noticing that A_t^1 has interior for every $t > 0$, we have proved in a very concise manner that u has fat level sets. □

Notice further that since Γ_t is closed, we have $\overline{A_t^1} \subset \Gamma_t$, which can also be obtained by repeating the whole argument above for all $x \in \partial A_t^1$. By symmetry, we obtain, without difficulty, that

$$A_t := \{x = (x_1, x_2) \in \mathbb{R}^2 : |x_1| + |x_2| + \sqrt{2|x_1x_2|} < \sqrt{2t}\} \subset \Gamma_t$$

and thus $\overline{A_t} \subset \Gamma_t$ (Figure 2).

Our computation above is straightforward, which is close to the study of characteristics for Hamilton-Jacobi equations. We neither used the parabolic PDE theory as was done in [**103**] and [**63**] nor directly calculated the solution of (MC).

We however are only able to get a lower bound for Γ_t so far because we keep standing on Paul's side; in other words, we only consider suboptimal

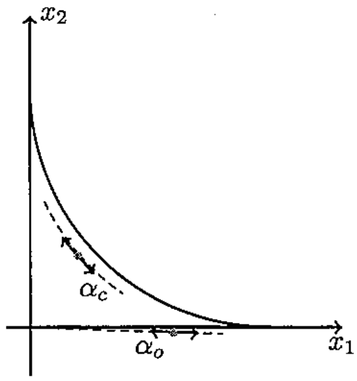

FIGURE 1. Paul's special strategies

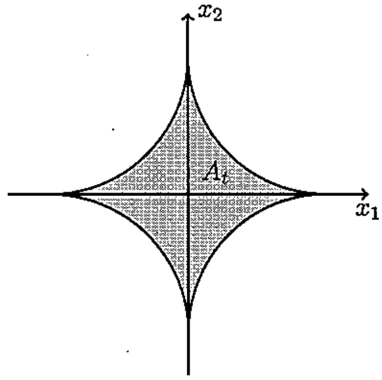

FIGURE 2. Fat level set

strategies. One might think that an upper bound will become available if we turn to seek Carol's optimal strategies. It is true but working on that is harder for the reason that Carol's controls ($b = \pm 1$) are zero-dimensional and hence less powerful than Paul's $(n-1)$-dimensional ones ($|v| = 1$).

3.1.2. Figure Eight. We continue to investigate a little more complicated situation, again proposed by Evans and Spruck [**41**]. The initial surface is a real figure-eight.

Fix a constant $R > 0$. Let $P_1 = (-R, 0)$, $P_2 = (R, 0)$ and $\Omega_- = B_R(P_1) \cup B_R(P_2)$. We take the initial value of (MC) in the same way as in (3.2) and set subsequently $\Omega_+ := \overline{\Omega}_-^c$. In these circumstances, we obtain the following.

THEOREM 3.2 (Fattening from a figure-eight curve). *Assume that* u_0 *is defined in* (3.2) *and* (3.1) *with the above choice of* Ω_-. *Then the zero level set* Γ_t *of the solution* u *of* (MC) *has nonempty interior for every* $0 < t < R^2/2$.

To prove this theorem, it is necessary to estimate the values of u_1^ε and u_2^ε at a fixed time $t > 0$, as is done in Section 3.1.1. The proof is useful for our discussion in Section 3.3 as well.

We find it better to first look into the points lying on the segment between P_1 and the origin $(0, 0)$.

PROPOSITION 3.3. *There exist a constant* $C > 0$ *and* $h \in C[0, +\infty)$ *with* $|h(\rho)| \leq C\rho$ *such that*

$$u_1^\varepsilon((x_1, 0), t - h(\varepsilon)) \leq 0, \quad \textit{for all } t < R^2/2 \textit{ and } -R < x_1 < 0. \tag{3.6}$$

PROOF. Suppose that Paul uses the origin-oriented strategy α_o as in the proof of Theorem 1.1. Carol has three options to run the game. 1

Case A. She moves the marker towards P_1. Then after finite many steps, the marker will reach the nearest point of P_1, which is assumed to be $(-l, 0)$, where $l \in \mathbb{R}$ satisfies $|l - R| < \sqrt{2}\varepsilon$. If Paul turns his strategy to α_c, concentric to P_1, then the estimate desired holds. Indeed, denote the game result $I_a(x, t)$ under the strategy for a starting point x and maturity time t. Let $t_1 = \frac{(x+l)\varepsilon}{\sqrt{2}}$

and $t_2 = t - t_1$. Then by Lemma 1.5(i), we may roughly get

$$I_a((x_1,0),t) \leq 0 \quad \text{if } t_2 < \frac{(R-\sqrt{2}\varepsilon)^2}{2} + \omega_0(\varepsilon). \tag{3.7}$$

We can take a continuous function h_1 with $|h_1(\varepsilon)| \leq C\varepsilon$ for some $C > 0$ so that

$$I_a((x_1,0),t-h_1(\varepsilon)) \leq 0 \quad \text{if } t < \frac{1}{2}R^2. \tag{3.8}$$

Case B. Carol lets the marker get close to P_2. This virtually leads to the same situation with Case A. Paul can take a concentric circle strategy with respect to P_2 this time, which yields again an estimate like (3.8):

$$I_b((x_1,0),t-h_2(\varepsilon)) \leq 0 \quad \text{if } t < \frac{1}{2}R^2. \tag{3.9}$$

Here V_b denotes the game value in this case and h_2 plays the same role as in (3.8).

Case C. Carol has the marker wander between P_1 and P_2. The it is clear that the game value V_c on this occasion satisfies the following.

$$I_c((x_1,0),t) \leq 0 \quad \text{for all } t \geq 0. \tag{3.10}$$

Combining the three cases above and letting $h = \max\{h_1, h_2\}$, we conclude that

$$u_1^\varepsilon((x_1,0),t-h(\varepsilon)) \leq \max\{I_a, I_b, I_c\} \leq 0, \quad \text{for all } t < \frac{1}{2}R^2.$$

□

The additional function h contains not only the error caused by discretization but also the time cost for the origin-oriented strategy.

PROPOSITION 3.4. *For any $-R < x_1 < 0$ and $t > \frac{1}{2}(R^2 - (R+x_1)^2)$, the inverse game value satisfies*

$$u_2^\varepsilon((x_1,0),t) \geq -\sqrt{2}\varepsilon.$$

PROOF. We define a region

$$L := \{(x_1,x_2) \in \mathbb{R}^2 : |x_1| \leq \sqrt{2}\varepsilon\}.$$

Let Paul take α_c as described in Lemma 1.5 with center P_1 and then the marker starting from $x = (x_1, 0)$ will leave the left open disc before the time $t_3 := R^2 - (R-x_1)^2/2 + \omega_0(\varepsilon)$. However there is possibility of getting into the right disc immediately, which we are afraid to see. To prevent it from happening, we devise a new strategy which is described in the following. Set $t_4 := \min\{t \in \mathbb{R} : y(x,t) \in L\}$. For the game after t_4, we ask Paul to implement a strategy α_p "parallel to the x_2-axis;" namely, $v = (0,1)$ for every $t \geq t_4$. Then it follows that

$$y(x,t) \notin \Omega_- \text{ or } y(x,t) \in L, \quad \text{for all } t \geq \min\{t_3, t_4\},$$

indifferent to Carol's decisions. Thus the definitions of u_0 and u_2^ε imply

$$u_2^\varepsilon((x_1,0),t) \geq -\sqrt{2}\varepsilon, \quad \text{for } t \geq \frac{1}{2}\left(R^2 - (R+x_1)^2\right) + \omega_0(\varepsilon).$$

□

REMARK 3.1. We emphasize that if one thinks about the minimal exit time under the same rules, then $U_2^\varepsilon((x_1,0)) \leq \frac{1}{2}(R^2-(R+x_1)^2)+\omega_0(\varepsilon)$ holds. There is no need to consider the maintenance strategy α_p in this case because our game gets over whenever the marker touches the boundary. This is a spectacular difference between games for parabolic and elliptic problems.

It is certain that Proposition 3.4 can be extended for more points on the plane. For instance, we easily observe that the points on the segment between the origin and P_2 own a similar estimate as well. A generalized version is written below without proofs.

PROPOSITION 3.5. *There exist a constant $C>0$ and a continuous function $h \in C[0,+\infty)$ with $|h(\rho)| \leq C\rho$ such that for all $x=(x_1,x_2)\in \mathbb{R}^2 \subset \{0\}$ and $t < \frac{x_1^2R^2}{2(x_1^2+x_2^2)}$*

$$u_1^\varepsilon(x, s-h(\varepsilon)) \leq 0 \quad \textit{whenever } s \leq t. \tag{3.11}$$

PROPOSITION 3.6. *The inverse game value satisfies*

$$u_2^\varepsilon(x,t) \geq -\sqrt{2}\varepsilon,$$

for all $x=(x_1,x_2)\in\mathbb{R}^2$ and $t\geq 0$ satisfying $x_1^2+x_2^2-R|x_1|<0$ and $t > R|x_1| - \frac{1}{2}(x_1^2+x_2^2)+\omega_0(\varepsilon)$.

Putting the above two propositions together and sending $\varepsilon \to 0$ with an application of Theorem 1.1, we get the following lemma.

LEMMA 3.7. *Assume that u is the solution of* (MC) *with initial value u_0. Then*

$$u(x,t)=0, \tag{3.12}$$

for all $0<t<\frac{1}{2}R^2$ and $x=(x_1,x_2)\in E_t^1$, where

$$E_t^1 := \Big\{(x_1,x_2)\in\mathbb{R}^2\setminus\{0\} : x_1^2+x_2^2-R|x_1|<0 \textit{ and } R|x_1|-\frac{1}{2}(x_1^2+x_2^2) < t < \frac{x_1^2R^2}{2(x_1^2+x_2^2)}\Big\}.$$

This lemma amounts to saying that $E_t^1 \subset \Gamma_t$. Notice that for every t, E_t^1 has interior, which already enables us to complete the proof of Theorem 3.2. See Figure 3. However it is still unsatisfactory since the origin $(0,0)$ is supposed to be an interior point of the level set Γ_t. We next show it by means of our game interpretation.

LEMMA 3.8. *Let u be the solution of* (MC) *with initial value u_0. Then $E_t^2 \subset \Gamma_t$ for all $0<t<\frac{1}{2}R^2$, where*

$$E_t^2 := \Big\{(x_1,x_2)\in\mathbb{R}^2\setminus\{0\} :\ x_1^2+\Big(x_2-\frac{R\sqrt{2t}}{\sqrt{R^2-2t}}\Big)^2 > \frac{4t^2}{R^2-2t}, \quad x_1^2+x_2^2<2t \textit{ and } \frac{x_1^2R^2}{x_1^2+x_2^2}<2t\Big\}.$$

To prove the lemma, we show the next two propositions. The first one is obvious.

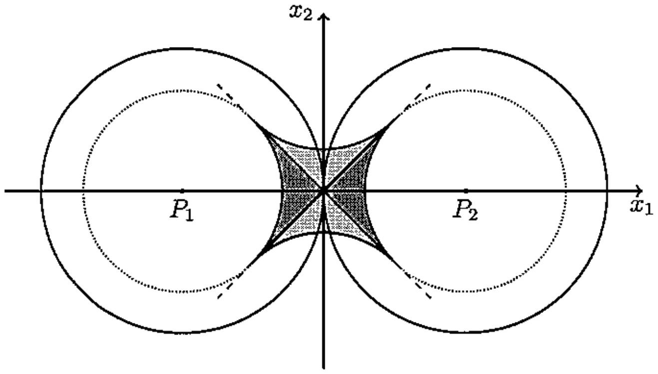

FIGURE 3. Dark gray region E_t^1 and light gray region E_t^2

PROPOSITION 3.9. *For any* $0 < t < \frac{1}{2}R^2$ *and* $x \in E_t^2$, $\limsup^*_{\varepsilon\to 0} u_2^\varepsilon(x,t) \geq 0$.

We skip its proof, which is the same as that of Proposition 3.4. The other proposition is given as follows.

PROPOSITION 3.10. *For any* $0 < t < \frac{1}{2}R^2$ *and* $x \in E_t^2$, $\liminf_{*\varepsilon\to 0} u_1^\varepsilon(x,t) \leq 0$.

PROOF. Take the function h as in Proposition 3.5. Paul may adopt the concentric circle strategy α_c with the center P, a particular point coordinated as $(0, \frac{R\sqrt{2t}}{\sqrt{R^2-2t}})$. If Carol makes choices leading to the existence of $s \leq t - h(\varepsilon)$ such that $y(x, s; \alpha_c, \beta) \notin E_t^2$ and thus $y(x,t) \in \overline{E_t^1}$, then Paul can utilize the strategy described in Proposition 3.3 and its generalization Proposition 3.5 after this moment. We consequently obtain that $u_1^\varepsilon(x, t - h(\varepsilon)) \leq 0$.

If $y(x, s) \in E_t^2$ for all $s \leq t - h(\varepsilon)$, then by Lemma 1.5(ii) Paul's strategy yields, with no regard to Carol's response, that

$$|y(x,t-h(\varepsilon)) - P|^2 > \frac{2R^2t}{R^2-2t} - 4C\varepsilon, \quad \text{for } \varepsilon > 0 \text{ sufficiently small,}$$

which implies that $|y(x, t - h(\varepsilon))|$ has to tend to 0 as $\varepsilon \to 0$. Either of the cases gives $\liminf_{\varepsilon\to 0} u_1^\varepsilon(x, t - h(\varepsilon)) \leq 0$ and our conclusion follows. □

We can take advantage of this point of view to understand fattening for more geometric evolutions, whose game interpretations are given in [**52**], besides the two examples above. The fattening example of motion by curvature proposed in [**63**] for noncompact surfaces can actually be interpreted in the similar way. Another example with Neumann boundary condition, formerly studied by Barles [**8**], is revisited in what follows. In general, however, it is not always easy to find strategies, explicit and nearly optimal.

3.1.3. Neumann boundary problem. Not only can we explain the fattening for Cauchy problems but we can also understand the game-theoretic understanding of fattening for Neumann boundary problems (NP1) by using our billiard games studied in Chapter 2. Here we revisit a very simple example appearing in Barles [**8**].

Let $\Omega = \{x \in \mathbb{R}^2 : |x| < 2\}$ and then take u_0 as those given in (3.2) and (3.1). It is clear that in this situation $u_1^\varepsilon(x,t) \le 0$ for all $x \in \Omega$ and $t > 0$. On the other hand, the estimate for u_2^ε can be given as follows.

LEMMA 3.11. *$u_2^\varepsilon(x,t) \ge -\sqrt{2}\varepsilon$, for all $(x,t) \in \Omega \times (0,\infty)$ such that $|x|^2 + 2t \ge 4$.*

PROOF. Paul's only job in this inverse game is to choose his direction each time on the tangent of the concentric circle around $(0,0)$. Then a direct calculation yields that the marker can reach the boundary $\partial\Omega$ by the time $\frac{1}{2}(4 - |x|^2)$. Notice that after the reaching moment, it will keep colliding the boundary $\partial\Omega$ and hence its distance to the boundary will not exceed $\sqrt{2}\varepsilon$, which deduces our conclusion. □

Applying Theorem 1.7, we thus can understand the development of fattening with ease.

THEOREM 3.12. *Let $\Omega = \{x \in \mathbb{R}^2 : |x| < 2\}$ and u_0 be as in* (3.2) *and* (3.1). *Then the zero level set of the solution u of* (NP1) *has interior for every $t > 0$.*

In fact, the fat zero level-set contains $A_t = \{x \in \Omega : |x|^2 + 2t \ge 4\}$ (Figure 4).

The original example in [8] is for a nonconvex domain $\Omega = \{x \in \mathbb{R}^2 : 1 < |x| < 2\}$, to which Theorem 1.7 cannot be applied. However, since the concave piece of boundary does not result in any additional terminating of billiard trajectories, or in the terminology in Section 2.2.2, there are no exits on the boundary, we can utilize Theorem 2.17 and consequently our interpretation of fattening follows as above. There is another example in [50] for Neumann problem, whose game interpretation will not involve anything other than our preceding study.

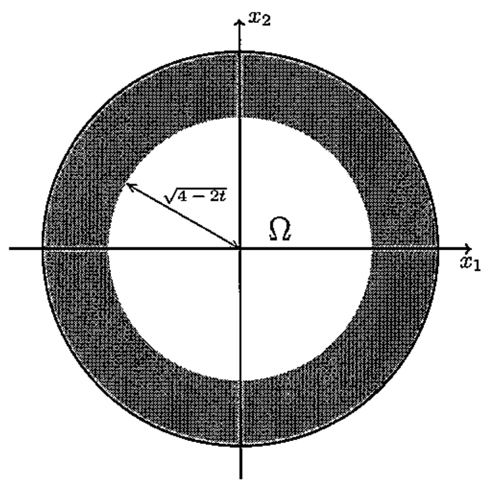

FIGURE 4. Gray region A_t

3.2. The stationary equation and mean convexity

In the next two sections, we would like to study the elliptic equation (SC). We first recall the definitions of viscosity solutions of (SC) with the boundary condition interpreted in the viscosity sense.

DEFINITION 3.1. A bounded and upper semicontinuous function U defined on $\overline{\Omega}$ is called a *viscosity subsolution* of (SC) provided that any test function $\phi \in C^2(\overline{\Omega})$ such that $U - \phi$ attains a unique maximum at $x_0 \in \overline{\Omega}$ satisfies the following:

(1) If $x_0 \in \Omega$, then the viscosity inequalities hold:

$$-|\nabla\phi|\mathrm{div}(\frac{\nabla\phi}{|\nabla\phi|}) - 1 \leq 0 \text{ at } x_0 \tag{3.13}$$

when $\nabla\phi(x_0) \neq 0$ and

$$-1 - \mathrm{tr}\left((I - \xi \otimes \xi)\nabla^2\phi(x_0)\right) \leq 0 \tag{3.14}$$

for some $\xi \in \mathbb{R}^n$ with $\|\xi\| = 1$ when $\nabla\phi(x_0) = 0$.

(2) If $x_0 \in \partial\Omega$, then either $U(x_0) \leq 0$ or the viscosity inequalities hold, i.e., (3.13) holds for $\nabla\phi(x_0) \neq 0$ and (3.14) holds with some $\|\xi\| = 1$ for $\nabla\phi(x_0) = 0$.

DEFINITION 3.2. A bounded and lower semicontinuous function U defined on $\overline{\Omega}$ is called a *viscosity supersolution* of (SC) provided that any test function $\phi \in C^2(\overline{\Omega})$ such that $U - \phi$ attains a unique minimum at $x_0 \in \overline{\Omega}$ satisfies the following:

(1) If $x_0 \in \Omega$, then the viscosity inequalities hold:

$$-|\nabla\phi|\mathrm{div}(\frac{\nabla\phi}{|\nabla\phi|}) - 1 \geq 0 \text{ at } x_0$$

when $\nabla\phi(x_0) \neq 0$ and

$$-1 - \mathrm{tr}\left((I - \xi \otimes \xi)\nabla^2\phi(x_0)\right) \geq 0$$

for some $\xi \in \mathbb{R}^n$ with $\|\xi\| = 1$ when $\nabla\phi(x_0) = 0$.

(2) If $x_0 \in \partial\Omega$, then either $U(x_0) \geq 0$ or the viscosity inequalities hold, i.e., (3.13) holds for $\nabla\phi(x_0) \neq 0$ and (3.14) holds with some $\|\xi\| = 1$ for $\nabla\phi(x_0) = 0$.

DEFINITION 3.3. A bounded function U defined on $\overline{\Omega}$ is said to be the *viscosity solution* of (SC) if U^* is a subsolution and U_* is a supersolution.

REMARK 3.2. It is worth mentioning that by $\phi \in C^2(\overline{\Omega})$ we mean ϕ have C^2 extension in the whole space $\mathbb{R}^n$, following the choice of test functions in the User's Guide [**29**].

We restrict our investigation for a smooth strictly mean convex domain Ω in this section. By strict mean-convexity, we mean that there exists a constant $k > 0$ such that the mean curvature of the boundary $\partial\Omega$ is uniformly greater than κ. As a result, for any sufficiently small $\delta > 0$, there are $k_1, k_2 > 0$ such that

$$0 < k_1 \leq \Delta d(x) \leq k_2, \quad \text{for all } x \in \overline{\Omega} \setminus \Omega_{-\delta}, \tag{3.15}$$

where d is the signed distance of $\partial\Omega$ with positive value in Ω and

$$\Omega_{-\delta} = \{x \in \Omega : d(x) < -\delta\}.$$

The convexity assumption of Ω is a non-degeneracy condition guaranteeing the boundary condition to hold in the classical sense. See results of this type for more general equations in [**30**] etc. The uniqueness of solutions of (SC) is presented in [**41**] by showing the equivalence with the uniqueness of the corresponding parabolic problem. We here instead provide a direct proof for the comparison principle below.

THEOREM 3.13. *Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be bounded and respectively a subsolution and a supersolution of* (SC). *If the boundary condition is realized in the classical sense; in other words, $u \le 0 \le v$ on $\partial\Omega$, then*

$$u \le v \quad in\ \overline{\Omega}.$$

PROOF. For any $0 < \delta < 1$, let $u_\delta = \delta u$. Then $u_\delta \le 0 \le v$ on $\partial\Omega$ and u_δ satisfies in the viscosity sense

$$F(\nabla u_\delta, \nabla^2 u_\delta) \le \delta - 1 < 0,$$

where $F(p, X) = -\mathrm{tr}\left(\left(I - \frac{p\otimes p}{|p|^2}\right) X\right)$. It suffices to prove $u_\delta \le v$ in $\overline{\Omega}$ for any δ close to 1. Assume by contradiction that $\max_{\overline{\Omega}}(u_\delta - v) > 0$. Then there exist $x_\varepsilon, y_\varepsilon \in \overline{\Omega}$ such that

$$\Phi_\varepsilon(x, y) = u_\delta(x) - v(y) - \frac{|x-y|^4}{4\varepsilon}$$

attains a maximum at $(x_\varepsilon, y_\varepsilon)$. Noticing that $\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \ge \max_{\overline{\Omega}}(u_\delta - v)$, we can take a constant $C > 0$ such that

$$\frac{|x_\varepsilon - y_\varepsilon|^4}{4\varepsilon} \le C$$

by the boundedness of u and v. Thus we have

$$|x_\varepsilon - y_\varepsilon| \le C\varepsilon^{1/4}$$

and by taking a subsequence we may let x_ε and y_ε converge to some $z \in \overline{\Omega}$. Now that

$$\begin{aligned}\limsup_{\varepsilon\to 0} \Phi_\varepsilon(x_\varepsilon, y_\varepsilon) &\le \limsup_{\varepsilon\to 0}(u_\delta(x_\varepsilon) - v(y_\varepsilon))\\ &\le u_\delta(z) - v(z)\\ &\le \max_{\overline{\Omega}}(u_\delta - v),\end{aligned}$$

we have

$$\lim_{\varepsilon\to 0} \Phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \max_{\overline{\Omega}}(u_\delta - v) > 0,$$

which implies that $z \notin \partial\Omega$. Now we discuss two cases.

Case 1.$x_\varepsilon \ne y_\varepsilon$ for all $\varepsilon > 0$ sufficiently small. By the standard arguments, for any $\alpha > 0$, we can take $(p, X) \in \overline{J}^{2,+}u_\delta(x_\varepsilon)$ and $(q, Y) \in \overline{J}^{2,-}v(y_\varepsilon)$ such that $p = q = \frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2(x_\varepsilon - y_\varepsilon)$ and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \left(I + \alpha\begin{pmatrix} A_\varepsilon & -A_\varepsilon \\ -A_\varepsilon & A_\varepsilon \end{pmatrix}\right)\begin{pmatrix} A_\varepsilon & -A_\varepsilon \\ -A_\varepsilon & A_\varepsilon \end{pmatrix}$$

with $A_\varepsilon = \frac{1}{\varepsilon}(|x_\varepsilon - y_\varepsilon|^2 I - 2(x_\varepsilon - y_\varepsilon) \otimes (x_\varepsilon - y_\varepsilon))$. It is then not difficult to find that $X \leq Y$. Apply the definitions of subsolutions and supersolutions to u_δ and v at x_ε and y_ε respectively. We obtain

$$F(p, X) \leq \delta - 1;$$
$$F(q, Y) \geq 0.$$

Taking the difference of these two inequalities and using the ellipticity of the equation, we are led to $0 \leq \delta - 1$, which is clearly a contradiction.

Case 2. $x_\varepsilon = y_\varepsilon$ for some subsequence of ε (still indexed by ε). We then only need to use the definition of supersolutions to get

$$F(\xi, 0) \geq 0$$

for some $\xi \in \mathbb{R}^n$. Since $F(\xi, 0) = -1$, we reach a contradiction again. □

We give a game-theoretic proof for a basic theorem in [**41**] on the existence and regularity of solutions of (SC).

THEOREM 3.14. *Assume that Ω is smoothly bounded and strictly mean convex. Then there exists a unique viscosity solution U of* (SC). *Furthermore, there are constants $A, a > 0$ so that*

$$-ad(x) \leq U(x) \leq -Ad(x) \quad \text{for all } x \in \overline{\Omega} \setminus \Omega_{-\delta} \text{ and} \tag{3.16}$$

$$|U(x_1) - U(x_2)| \leq A|x_1 - x_2| \quad \text{for all } x_1, x_2 \in \overline{\Omega}. \tag{3.17}$$

We prove the result on the discrete level.

PROPOSITION 3.15. *Assume that Ω is smoothly bounded and strictly mean convex. Let U^ε be the value functions associated with the games of exit time T^ε. Then there exist $A, a > 0$ such that*

$$-ad(x) \leq U^\varepsilon(x) \leq -Ad(x) \tag{3.18}$$

$$|U^\varepsilon(x_1) - U^\varepsilon(x_2)| \leq A|x_1 - x_2| \quad \text{for all } x, x_1, x_2 \in \Omega \setminus \Omega_{-\delta}. \tag{3.19}$$

PROOF. We apply Taylor expansion to compute for any $x \in \Omega \setminus \Omega_{-\delta}$, $Q \in \mathcal{Q}$ and $b \in \mathcal{B}$

$$d(x + \sqrt{2}\varepsilon Qb) = d(x) + \sqrt{2}\varepsilon \nabla d(x) Qb + \varepsilon^2 (\nabla^2 d(x) Qb) \cdot (Qb) + O(\varepsilon^3).$$

By Lemma 1.3, we obtain constants C^1 and C^2 depending only on n and the bounds of second and third derivatives of d in $\overline{\Omega} \setminus \Omega_{-\delta}$ such that

$$k_1\varepsilon^2 - C_1(\varepsilon^3) \leq \max\min d(x + \sqrt{2}\varepsilon Qb) - d(x) \leq k_2\varepsilon^2 + C_2(\varepsilon^3),$$

which implies that $x + \sqrt{2}\varepsilon Qb \in \mathbb{R}^n \setminus \Omega_{-\delta}$. Moreover, we learn by induction that the minimal and maximal numbers of rounds before the game ends are respectively $[-d(x)/(k_2\varepsilon^2 + C_2(\varepsilon^3))]$ and $[-d(x)/(k_1\varepsilon^2 + C_1(\varepsilon^3))]$, which in turn implies

$$-\frac{1}{2k_2} d(x) \leq U^\varepsilon(x) \leq -\frac{2}{k_1} d(x) \tag{3.20}$$

for all $x \in \Omega \setminus \Omega_{-\delta}$ and $\varepsilon > 0$ sufficiently small. We complete the proof of (3.18) by taking $A = \frac{2}{k_1}$ and $a = \frac{1}{2k_2}$.

To prove the Lipschitz continuity, we notice that for every pair of $x_1, x_2 \in \overline{\Omega}$ with $|x_1 - x_2| \leq \delta$, Paul has a strategy α_0 to finish the game by any time

$\tau > U^\varepsilon(x_2)$. This could be expressed as $y(\tau; x_2, \alpha_0, \beta) \notin \Omega$ for any β, which yields $y(\tau; x_1, \alpha_0, \beta) \notin \Omega_{-\delta}$ due to

$$|y(\tau; x_2, \alpha, \beta) - y(\tau; x_1, \alpha, \beta)| \leq |x_1 - x_2| \leq \delta.$$

We therefore have

$$U^\varepsilon(x_1) \leq U^\varepsilon(x_2) - Ad(x_1) \leq U^\varepsilon(x_2) + A|x_1 - x_2|$$

by dynamic programming principle and (3.20). A symmetric argument yields

$$|U^\varepsilon(x_1) - U^\varepsilon(x_2)| \leq A|x_1 - x_2|.$$

□

PROOF OF THEOREM 3.14. The uniqueness of solutions follows Theorem 3.13. By Theorem 1.4 and Proposition 3.15, the relaxed limits $\overline{U}$ and $\underline{U}$ of U^ε are respectively a subsolution and a supersolution of (SC) and satisfy $\overline{U}(x) = \underline{U}(x) = 0$ for all $x \in \partial\Omega$. Applying Theorem 3.13 again, we get the uniform convergence of U^ε to a unique solution U in $\overline{\Omega}$. Then (3.16) follows (3.18). The proof of (3.17) is not difficult either. Passing to the uniform limit of U^ε in (3.19) as $\varepsilon \to 0$, we have

$$|U(x_1) - U(x_2)| \leq A|x_1 - x_2| \text{ for all } x_1, x_2 \in \overline{\Omega} \text{ such that } |x_1 - x_2| \leq \delta.$$

We thus use the uniform bounded M of U^ε to get

$$\begin{aligned} &\sup_{x_1 \neq x_2} \frac{|U(x_1) - U(x_2)|}{|x_1 - x_2|} \\ \leq &\sup_{0<|x_1-x_2|\leq\delta} \frac{|U(x_1) - U(x_2)|}{|x_1 - x_2|} + \sup_{|x_1-x_2|>\delta} \frac{|U(x_1) - U(x_2)|}{|x_1 - x_2|} \\ \leq &A \vee \left(\frac{M}{\delta}\right). \end{aligned}$$

It remains to take A sufficiently large (or k_1 small enough). □

We have also shown the following convergence of game values.

THEOREM 3.16. *Assume that Ω is smoothly bounded and strictly mean convex. Let U^ε be the value functions associated with the games of exit time T^ε. Then $U^\varepsilon \to U$ uniformly as $\varepsilon \to 0$, where U is the unique solution of* (SC).

This theorem is more general than [**75**, Theorem 1.1], which only treats a (strictly) convex domain in the plane.

Proposition 3.15 guarantees the boundary conditions for $\overline{U}$ and $\underline{U}$ are realized in the classical sense. The game-theoretic interpretation can actually be substituted with a PDE based proof as follows.

We claim that $\overline{U}$ satisfies $\overline{U}(x) \leq -Cd(x)$ for all $x \in \overline{\Omega}$ and some $C > 0$ sufficiently large and independent of x. Indeed, if we assume by contradiction that for any large $C > 0$, there is $x_0 \in \overline{\Omega}$ such that $\overline{U}(x_0) + Cd(x_0) > 0$, then $\overline{U} + Cd$ should attain a maximum at $\hat{x} \in \overline{\Omega} \setminus \overline{\Omega_{-\delta}}$ since $\overline{U}$ is bounded and therefore $\overline{U} + Cd \leq 0$ in $\overline{\Omega_{-\delta}}$ when $C > 0$ is taken large enough. Now we regard $-Cd$ as a test function for $\overline{U}$ and compute

$$C\mathrm{tr}\left(\left(I - \frac{\nabla d(\hat{x}) \otimes \nabla d(\hat{x})}{|\nabla d(\hat{x})|^2}\right)\nabla^2 d(\hat{x})\right) - 1 \geq Ck_1 - 1 > 0$$

for $C > 0$ large, which is a contradiction to the fact that $\overline{U}$ is a subsolution of (SC) in $\overline{\Omega}$. Hence, $\overline{U} \leq 0$ on $\partial\Omega$.

It is obvious that $\underline{U} \geq 0$ on $\partial\Omega$. Our game explanation thereby agrees with the conclusion via PDE methods.

3.3. Connection between fattening and loss of comparison

We have seen in the last section that the weak boundary condition of (SC) could be strengthened and the solutions are unique and continuous if the domain is mean convex. Once the mean convexity is dropped, an example in [**75**] shows that solutions may become discontinuous and the usual comparison principle may not hold. The following weak comparison principle is nearly the best we can expect.

Weak Comparison Principle: If $W_1 \in USC(\overline{\Omega})$ and $W_2 \in LSC(\overline{\Omega})$ are respectively a subsolution and a supersolution of (SC), then

$$(W_1)_* \leq W_2 \quad \text{and } W_1 \leq (W_2)^* \quad \text{in } \overline{\Omega}.$$

Comparison principles of this type are also named *proper comparison principle* and have important applications in other contexts [**50**]. It was left as an open question in [**75**] whether the weak comparison principle holds in any bounded domain. We will later find that the answer is negative if the domain is too general.

Let us again consider the case of figure eight first. Of course, the open region enclosed by the figure eight shaped curve is not really a domain, but if we try to solve the equation (SC) any way, we will lose the comparison principle.

THEOREM 3.17 (Loss of comparison in a figure-eight type region). *Let $\Omega = \Omega_-$ as in* Section 3.1.2. *Then the weak comparison principle fails to hold.*

PROOF. Fix a small $\theta > 0$ and denote $P_\theta = (-\theta, 0) \in \mathbb{R}^2$. Then for any $t < R^2/2$, we may take a positive quantity $\rho < \theta$ small enough to get $B_\rho(P_\theta) \subset E_t^1$, where E_t^1 is given in Lemma 3.7. For every $x \in B_\rho(P_\theta) \cap \Omega$, the game strategies in the proofs of Propositions 3.3, 3.5 and 3.10 yield $U_1^\varepsilon(x) \geq t$ while those in Propositions 3.4, 3.6 and 3.9 give $U_2^\varepsilon(x) \leq \theta + \rho$. We are therefore led to $(\overline{U_1})_*(P_\theta) \geq R^2/2$ and $\underline{U_2}(P_\theta) \leq \theta(R - 2\theta)$. Our assertion hence follows immediately from Theorem 1.4. □

REMARK 3.3. The open set enclosed by the figure-eight type curve is certainly not connected. However, one can let it become a domain by slight modification without changing the essence of the proof above. The weak comparison principle does not hold for the domain Ω in Figure 5.

To generalize Theorem 3.17, we follow the conventional way of characterizing fattening, perturbing the set Ω a little bit before playing the games again. Set, for each $\delta \in \mathbb{R}$,

$$\Omega_\delta := \{x \in \mathbb{R}^n : d(x) < \delta\}.$$

Let $T^{\varepsilon,\delta}$ denote the corresponding exit times from Ω_δ. In contrast to the former sections, since we take the region perturbation into account, it is sufficient to consider the min max games of arrival time towards the boundary only. In order to handle the positive mean curvature flow, we convexify the control set

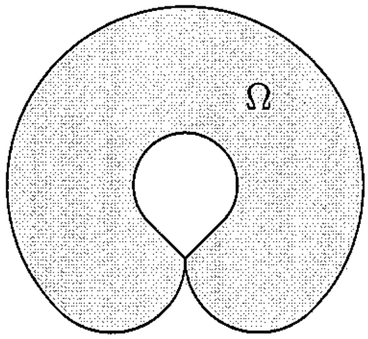

FIGURE 5. A domain Ω in which the weak comparison fails

of Paul; in other words, we use $\mathcal{Q}' = \{Q = (v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n\times(n-1)} : |v_i| \leq 1 \text{ and } v_i \cdot v_j = 0\}$ instead of $\mathcal{Q}$. We define the value functions for every $x \in \mathbb{R}^n$

$$U_+^{\varepsilon,\delta}(x) := \min_{\alpha} \max_{\beta} T^{\varepsilon,\delta}(x; \alpha, \beta).$$

and take relaxed limits

$$V_1 := \operatorname*{limsup^*}_{\varepsilon\to 0,\ \delta\to 0+} U_+^{\varepsilon,\delta} \quad \text{and} \quad V_2 := \operatorname*{liminf_*}_{\varepsilon\to 0,\ \delta\to 0-} U_+^{\varepsilon,\delta}.$$

PROPOSITION 3.18. *V_1 is a viscosity subsolution of* (SC).

PROOF. We first notice that $V_1(x) = 0$ for all $x \in \overline{\Omega}^c$. Assume first that there are $x_0 \in \partial\Omega$ and a function $\phi \in C^2(\overline{\Omega} \cap B_r(x_0))$ with $r > 0$ such that $V_1 - \phi$ attains at x_0 a unique maximum in $\overline{\Omega} \cap B_r(x_0)$ with $V_1(x_0) = \phi(x_0) > 0$. Since ϕ can be extended to a function in $C^2(\mathbb{R}^n)$, we denote this new test function still by ϕ. It is consequently easy to see that $V_1 - \phi$ attains a maximum at x_0 in $B_r(x_0)$. Then by definition of V_1, there exist sequences $\varepsilon_k, \delta_k > 0$ and $x_k \in B_r(x_0)$ such that $\varepsilon_k \to 0$, $\delta_k \to 0$ and $x_k \to x_0$ as $k \to \infty$ and

$$U_1^{\varepsilon_k,\delta_k}(x_k) - \phi(x_k) \geq \sup_{B_r(x_0)} (U_1^{\varepsilon_k,\delta_k} - \phi) - \varepsilon_k^3. \tag{3.21}$$

Denote for brevity $U_1^k = U_1^{\varepsilon_k,\delta_k}$. By the dynamic programming principle, we have

$$U_1^k(x_k) = \min_{Q\in\mathcal{Q}'} \max_{b\in\mathcal{B}} U_1^k(x_k + \sqrt{2}\varepsilon_k Qb) + \varepsilon_k^2,$$

which, combined with (3.21), implies that

$$\min_{Q\in\mathcal{Q}'} \max_{b\in\mathcal{B}} \phi(x_k + \sqrt{2}\varepsilon_k Qb) - \phi(x_k) + \varepsilon_k^2 + \varepsilon_k^3 \geq 0. \tag{3.22}$$

If $\nabla\phi(x_0) \neq 0$, then $\nabla\phi(x_k)$ is bounded away from 0 for all k. Using Taylor expansion and an analogue of Lemma 1.3 and then sending $k \to \infty$, we obtain

$$-|\nabla\phi| \left(\operatorname{div}\Big(\frac{\nabla\phi}{|\nabla\phi|}\Big) \wedge 0\right) - 1 \leq 0 \quad \text{at } x_0,$$

which yields

$$-|\nabla\phi| \operatorname{div}\Big(\frac{\nabla\phi}{|\nabla\phi|}\Big) - 1 \leq 0 \quad \text{at } x_0.$$

(An alternative way to get this is noticing that (3.22) implies

$$\min_{Q\in\mathcal{Q}}\max_{b\in\mathcal{B}} \phi(x_k+\sqrt{2}\varepsilon_k Qb)-\phi(x_k)+\varepsilon_k^2+\varepsilon_k^3\geq 0.$$

and applying Lemma 1.3 directly.)

If, on the other hand, $\nabla\phi(x_0)=0$, then $\nabla\phi(x_k)\to 0$ as $k\to\infty$. We discuss two cases. In the case that $\nabla\phi(x_k)\neq 0$ for some subsequence, Taylor expansion of (3.22) with application of (1.7) in Lemma 1.3 yields

$$-\mathrm{tr}\left(\left(I-\frac{\nabla\phi(x_k)\otimes\nabla\phi(x_k)}{|\nabla\phi(x_k)|^2}\right)\nabla^2\phi(x_k)\right)-1\leq\varepsilon_k.$$

Passing to a subsequence x_{k_j} such that $\nabla\phi(x_{k_j})/|\nabla\phi(x_{k_j})|$ converges to some $\xi\in\mathbb{R}^n$ with $|\xi|=1$, we get (3.14). The remaining case is that $\nabla\phi(x_k)=0$ for all k. We deduce from (3.22) that

$$\mathrm{tr}\left((Q_kb_k)\otimes(Q_kb_k)\nabla^2\phi(x_k)\right)+1\geq-\varepsilon_k,$$

where Q_k and b_k are respectively the minimizer and maximizer among the controls in $\mathcal{Q}'$ and $\mathcal{B}$. Since it is not difficult to find $\xi_k\in\mathbb{R}^n$ such that $\|\xi_k\|=1$ and $\xi_k^\top Q_k=0$, we let $k\to 0$, taking a subsequence if necessary, to get $\xi=\lim_{k\to\infty}\xi_k$, which implies (3.14) again.

We can use the same argument to handle the easier case $x_0\in\Omega$. □

PROPOSITION 3.19. *V_2 is a viscosity supersolution of* (SC).

We skip the proof since the boundary condition is satisfied in the classical sense and thus the proof is almost the same as the subsolution part presented above.

A result more general than Theorem 3.17 is given below.

THEOREM 3.20 (Loss of comparison due to fattening). *Suppose that Ω is a bounded open subset of $\mathbb{R}^n$ and take*

$$u_0=(d(x)\wedge M)\vee(-M)\in BUC(\mathbb{R}^n)$$

as a defining function of $\partial\Omega$, where $M>0$ is a large constant and d is the signed distance of Ω, i.e.,

$$d(x)=dist(x,\overline{\Omega})-dist(x,\Omega^c).$$

Let u be the unique solution of (PMC). *If the zero level set of u fattens at some $x_0\in\Omega$; that is, there exist $\rho>0$ and $t_0>0$ such that*

$$u(x,t_0)=0\quad\textit{for all } x\in B_\rho(x_0)\subset\Omega,$$

then

$$V_1(x)\geq V_2(x)+\frac{3\rho^2}{8}\quad\textit{for all } x\in B_{\rho/2}(x_0). \tag{3.23}$$

The theorem says that the occurrence of fattening in the region enclosed by an initial surface gives rise to the existence of discrepancy between the relaxed limits of minimal exit time with boundary perturbation involved. We therefore can adopt examples of fattening to disprove the general existence of comparison principle for (SC).

COROLLARY 3.21. *Let u_0 be defined as in* Theorem 3.20. *If the zero level set of the viscosity solution of* (PMC) *fattens, then the weak comparison principle for* (SC) *fails to hold.*

On the other hand, we get a solution of (SC) which is continuous except at a nowhere dense subset of $\overline{\Omega}$ provided that the flow is regular.

THEOREM 3.22 (Convergence of game values due to nonfattening). *Under the same assumptions of* Theorem 3.20 *on* Ω *and the choice of* u_0, *let* u *be the unique solution of* (PMC). *If the zero level set of* u *satisfies* (1.20) *and* (1.21), *then*

$$(V_1)_* \leq V_2 \text{ and } V_1 \leq (V_2)^* \quad \text{in } \overline{\Omega}.$$

We are essentially able to prove, as an immediate consequence, the convergence in games with no use of comparison principles.

COROLLARY 3.23. *Assume that the solution* u *of* (PMC) *satisfies* (1.20) *and* (1.21). *Then there exists a possibly discontinuous solution* V *of* (SC) *which satisfies* $(V^*)_* = V_*$ *and* $(V_*)^* = V^*$ *and the game values* $U_+^{\varepsilon,\delta}$ *and* $U_+^{\varepsilon,-\delta}$ *converge as* $\varepsilon, \delta \downarrow 0$ *to* V, *in the sense that*

$$\operatorname*{liminf_*}_{\varepsilon\to 0,\ \delta\to 0-} U_+^{\varepsilon,\delta} = V_* \text{ and } \operatorname*{limsup^*}_{\varepsilon\to 0,\ \delta\to 0+} U_+^{\varepsilon,\delta} = V^*.$$

We next prove Theorems 3.20 and 3.22.

PROOF OF THEOREM 3.20. Since $u(x,t_0) = 0$ and $u_+^\varepsilon \to u$ uniformly in $B_\rho(x_0) \times \{t_0\}$, for any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that

$$u_+^\varepsilon(x,t_0) < \delta \tag{3.24}$$

and

$$u_+^\varepsilon(x,t_0) \geq -\delta \tag{3.25}$$

for all $\varepsilon \leq \varepsilon_0$ and $x \in B_\rho(x_0)$. The first inequality (3.24) means that for every $z \in B_\rho(x_0)$ there exists a strategy β_z satisfying

$$u_0(y(z,t_0;\alpha,\beta_z)) < \delta \tag{3.26}$$

no matter what strategy α Paul adopts. Note further that with this strategy put to use, Carol can also guarantee that the marker never departs from Ω_δ in the whole process, i.e.,

$$u_0(y(z,t;\alpha,\beta_z)) < \delta$$

for all $t \leq t_0$ and α, for otherwise Paul can make the marker stop moving after the departure moment so that (3.26) is violated.

Now if we start games from $x \in B_{\rho/2}(x_0)$, Carol can use the strategy of concentric spheres in Lemma 1.6 to guarantee that the time needed for exit from $B_\rho(x_0)$ is greater than $\tau_0 = 3\rho^2/8 + \omega_0(\varepsilon)$. We therefore must have $y(\tau_0) \in B_\rho(x_0)$ and then we can go on applying the strategy α_z for $z = y(\tau_0)$ to obtain, for all $0 \leq t \leq t_0 + \tau_0$ and $\varepsilon \leq \varepsilon_0$,

$$u_0(y(x,t)) < \delta,$$

which further implies $U_+^{\varepsilon,\delta}(x) \geq t_0 + \tau_0$ and consequently $V_1(x) \geq t_0 + 3\rho^2/8$.

On the other hand, it follows immediately from (3.25) that $U_+^{\varepsilon,-\delta} \leq t_0$ and hence $V_2 \leq t_0$ in $B_\rho(x_0)$. □

PROOF OF THEOREM 3.22. (i) Fix an arbitrary $x_0 \in \overline{\Omega}$. Let $t_0 = V_2(x_0)$. We first claim that $u(x_0, t_0) = 0$. Indeed, there are sequences $x_k \to x_0$, $\varepsilon_k \to 0$ and $\delta_k \to 0$ fulfilling $t_k = U_+^{\varepsilon_k, -\delta_k}(x_k) \to t_0$ as $k \to \infty$, and therefore by definition we have

$$|u_+^{\varepsilon_k}(x_k, t_k) + \delta_k| \leq \sqrt{2}\varepsilon_k.$$

Sending $k \to \infty$, we get $u(x_0, t_0) = 0$ by Theorem 1.2.

Since the level set of u satisfies (1.20), i.e.,

$$\{(x,t) \in \mathbb{R}^n \times [0,\infty) : u(x,t) = 0\} \subset \overline{\bigcup_{\delta>0} \{(x,t) \in \mathbb{R}^n \times [0,\infty) : u(x,t) > \delta\}}.$$

We can take $\delta'_k \to 0$, $y_k \to x_0$ and $s_k \to t_0$ as $k \to \infty$ such that $u(y_k, s_k) > 3\delta'_k$ for all $k \geq 1$. In what follows, we discuss in detail for every k and thus suppress the index k for ease of notation. We again use Theorem 1.2 to find $0 < \varepsilon' \leq \delta'$ so that $u^\varepsilon(y, s) \geq 2\delta'$ for all $\varepsilon \leq \varepsilon'$.

This means that, for every $\varepsilon \leq \varepsilon'$, Paul has a strategy to leave $\Omega_{2\delta'}$ from y by the time s in spite of Carol's obstruction. We follow this strategy to ensure that one is able to exit $\Omega_{\delta'}$ from all $x \in B_{\delta'}(y)$ by the time s, which is expressed as

$$U_+^{\varepsilon,\delta'}(x) \leq s \quad \text{for all } x \in B_{\delta'}(y) \text{ and } \varepsilon \leq \varepsilon'.$$

We then easily get

$$U_+^{\varepsilon,\delta}(x) \leq s \quad \text{for all } x \in B_{\delta'}(y),\ \varepsilon \leq \varepsilon' \text{ and } \delta \leq \delta',$$

which implies $V_1(y) \leq s$. Recalling that $y = y_k$ and $s = s_k$ actually depend on k and passing to the limit $k \to \infty$, we finally obtain $(V_1)_*(x_0) \leq t_0 = V_2(x_0)$.

(ii) We prove $V_1 \leq (V_2)^*$ in $\overline{\Omega}$. Fix $x_0 \in \overline{\Omega}$ and set $t_0 = V_1(x_0)$ this time. We may use the similar argument for (i) to show $u(x_0, t_0) = 0$. The condition (1.21) indicates that

$$\{(x,t) \in \mathbb{R}^n \times [0,\infty) : u(x,t) = 0\} \subset \overline{\bigcup_{\delta>0} \{x \in \mathbb{R}^n \times [0,\infty) : u(x,t) < -\delta\}}$$

and then we can again pick sequences $\delta'_k > 0$ with $\delta'_k \to 0$, $y_k \to x_0$ and $s_k \to t_0$ as $k \to \infty$ satisfying

$$u(y_k, s_k) < -3\delta'_k. \tag{3.27}$$

Again we use brief notation without the index k.

We take $0 < \varepsilon' \leq \delta'$ such that $u_+^\varepsilon(y, s) < -2\delta'$ for all $\varepsilon \leq \varepsilon'$, which indicates that Carol has an effective strategy to make the marker appear in $\Omega_{-2\delta'}$ at the time s. In fact, this strategy also prevents the exit from $\Omega_{-2\delta'}$ before s for the reason that once the exit occurs at any moment before s, Paul will choose $Q = 0$ for every step later on to keep himself outside $\Omega_{-2\delta'}$, which leads to a contradiction to the situation (3.27). Considering the game strategies for starting points $x \in B_{\delta'}(y)$, we are led to

$$U_+^{\varepsilon,-\delta}(x) \geq s \quad \text{for all } x \in B_{\delta'}(y),\ \varepsilon \leq \varepsilon' \text{ and } \delta \leq \delta'.$$

We thus deduce $V_2(y_k) \geq s_k$ and conclude by letting $k \to \infty$. □

REMARK 3.4. Our result shows that the game relaxed limits satisfy the weak comparison principle as long as the interior of the zero level set in parabolic problem keeps empty. The fattening on other levels has no influence on it.

The above proof of (i) actually works for the mean curvature flow equation (MC) too. We state it for a smooth domain in two dimensions.

PROPOSITION 3.24. *Suppose that* Ω *is a smoothly bounded domain in* $\mathbb{R}^2$. *Then*

$$(\overline{U_i})_* \leq \underline{U_i} \text{ in } \overline{\Omega},$$

where $\overline{U_i}$ *and* $\underline{U_i}$ *for* $i = 1, 2$ *are defined in* (1.12) *and* (1.13).

PROOF. As $\partial\Omega$ is compact and smooth, by the results of Gage-Hamilton [**47**] and Grayson [**58**] on curve shortening, we obtain the regularity conditions (1.20) and (1.21) for motion by curvature (MC) and the proof for Theorem 3.22(i) works. □

We obtain merely half of what is supposed to hold by the weak comparison principle. It is not obvious whether the other half is true. Our proof (ii) of Theorem 3.22 is not valid in this case because the regularity condition for (MC) is not sufficient. The issue we cannot get through is that even though we have in hand strategies to guarantee that the marker is in $\Omega_{-\delta}$ ($\delta > 0$) at some time t, we are not sure for each strategy whether or not the marker has already left Ω and just come back into Ω again by the time t. We are looking for a certain strategy to keep it staying in Ω until t_0, which is indispensable in our argument to assert that the minimal exit time cannot be less than t_0. The case of signed curvature flow is simpler at this aspect.

CHAPTER 4

A Discrete Scheme for Shocks

This chapter is intended to give a discrete scheme for the one dimensional conservation law equation:

$$\text{(CL)} \qquad u_t + f(u)u_x = 0 \quad \text{in } \mathbb{R} \times (0, \infty)$$

with initial data $u(x, 0) = u_0(x)$ for all $x \in \mathbb{R}$. We describe in Section 4.1 the scheme in detail and prove the convergence in Section 4.2 under some additional conditions on the initial data.

This chapter is due to the recent unpublished work [**87**].

4.1. The scheme

Our scheme is presented in what follows. For a given $u_0 \in L^\infty(\mathbb{R})$ with $||u_0||_\infty \leq M$, take

$$U_0(x,z) = \begin{cases} 1 & \text{if } z \leq u_0(x) \\ -1 & \text{if } z > u_0(x) \end{cases}$$

Let us fix a small step size $\varepsilon > 0$ and define a value function U^ε inductively via two steps.

Step 1. Run the system for time ε. Namely, for any $x \in \mathbb{R}$, $z \in \mathbb{R}$ and $t \geq \varepsilon$, we denote

$$V^\varepsilon(x, z, t) = U^\varepsilon(x - f(z)\varepsilon, z, t - \varepsilon).$$

Here $V^\varepsilon(x, z, t) = U_0(x, z)$ if $t < \varepsilon$.

Step 2. Rearrange V^ε if it overturns. We first examine whether or not the level set of $V^\varepsilon(\cdot, \cdot, t)$ is still a graph. Define

$$\alpha^\varepsilon(x,t) = \inf\{z : V^\varepsilon(x,z,t) < 0\} \quad \text{and } \beta^\varepsilon(x,t) = \sup\{z : V^\varepsilon(x,z,t) > 0\}.$$

REMARK 4.1. It is quite obvious that if $U^\varepsilon(x, z, t-\varepsilon)$ only takes values ± 1, then $\alpha^\varepsilon \leq \beta^\varepsilon$. We however do not impose any assumption to force $\alpha^\varepsilon = \beta^\varepsilon$. This is different from what was done in [**77**].

Let $I_{x,t}$ denote the interval $[\alpha^\varepsilon(x,t), \beta^\varepsilon(x,t)]$.

In the case that $\alpha^\varepsilon = \beta^\varepsilon$, take $U^\varepsilon(x, z, t) = V^\varepsilon(x, z, t)$. Otherwise, define

$$g(z; I_{x,t}) = -\frac{d}{dz}(\operatorname*{conv}_{I_{x,t}} \int -f)(z),$$

where conv_A denotes the convexification in a set $A \subset \mathbb{R}$, and take

$$U^\varepsilon(x, z, t) = U^\varepsilon(x - g(z; I_{x,t})\varepsilon, z, t - \varepsilon).$$

REMARK 4.2. By inductive application of the definition and Remark 4.1, $U^\varepsilon(x, z, t)$ only takes values ± 1 and $\alpha^\varepsilon(x,t) \leq \beta^\varepsilon(x,t)$ for all $x, z \in \mathbb{R}$ and $t \geq 0$.

REMARK 4.3 (Consistency). If either $z \notin I_{x,t} = [\alpha^\varepsilon(x,t), \beta^\varepsilon(x,t)]$ or $z = \alpha^\varepsilon(x,t) = \beta^\varepsilon(x,t)$, then $g(z; I_{x,t}) = f(z)$. As a result, we have

$$U^\varepsilon(x,z,t) = V^\varepsilon(x,z,t) \quad \text{for all } x \in \mathbb{R},\ z \notin I_{x,t} \text{ and } t \geq 0.$$

Thus, we can write the induction as

$$U^\varepsilon(x,z,t) = U^\varepsilon(x - g(z; I_{x,t})\varepsilon, z, t-\varepsilon),$$

regardless of the difference of these two cases.

Finally, to proceed to our convergence results, we take the relaxed limits

$$\overline{U} = \limsup_{\varepsilon \to 0}{}^* U^\varepsilon \text{ and } \underline{U} = \liminf_{\varepsilon \to 0}{}_* U^\varepsilon.$$

We attempt to show in the next section that

$$\overline{u}(x,t) = \sup\{z : \overline{U}(x,z,t) = 1\} \text{ and } \underline{u}(x,t) = \inf\{z : \underline{U}(x,z,t) = -1\}$$

are respectively a proper subsolution and a proper supersolution.

LEMMA 4.1 (Boundedness). *If $\|u_0\|_\infty \leq M$, then the motion of interface described by the discontinuity of U^ε is bounded in the sense of the following:*

(a) $U^\varepsilon(x,z,t) = -1$ *for all* $x \in \mathbb{R}$, $t \geq 0$ *and* $z > M$.
(b) $U^\varepsilon(x,z,t) = 1$ *for all* $x \in \mathbb{R}$, $t \geq 0$ *and* $z < -M$.

PROOF. By definition, it is obvious that $U^\varepsilon(x,z,t) = -1$ for $z > M$ and $U^\varepsilon(x,z,t) = 1$ for $z < -M$ since $U_0(x,z) = -1$ when $z > M$ and $U_0(x,z) = 1$ when $z < -M$. □

REMARK 4.4. An immediate consequence of the above lemma is that $|\alpha^\varepsilon(x,t)| \leq M$ and $|\beta^\varepsilon(x,t)| \leq M$ for all $x \in \mathbb{R}$ and $t \geq 0$ provided that $\|u_0\|_\infty \leq M$.

Let us next prove that $\overline{u}^\varepsilon$ stays being a function during the evolution.

LEMMA 4.2 (Graph). $U^\varepsilon(x,z,t) \geq U^\varepsilon(x,w,t)$ *for all* $x \in \mathbb{R}$, $t \geq 0$ *and* $w \geq z$.

PROOF. By induction, it suffices to assume the statement holds at the time t and prove it for the moment $t+\varepsilon$. We only need to show that $U^\varepsilon(x,z,t+\varepsilon) < 0$ implies $U^\varepsilon(x,w,t+\varepsilon) < 0$ for all $w \geq z$. Assume the former, then by definition, we have $z \geq \alpha^\varepsilon(x,t+\varepsilon)$. In the case $w > \beta^\varepsilon(x,t+\varepsilon)$, it is clear that

$$U^\varepsilon(x,w,t+\varepsilon) = V^\varepsilon(x,w,t+\varepsilon) < 0.$$

If $z, w \in I_{x,t}$, we get $g(z; I_{x,t+\varepsilon}) \geq g(w; I_{x,t+\varepsilon})$, since $z \mapsto g(z;I)$ is constant (nonincreasing in general) in I. Then we obtain

$$U^\varepsilon(x - g(w)\varepsilon, z, t) < 0,$$

which implies that

$$U^\varepsilon(x,w,t+\varepsilon) = U^\varepsilon(x - g(w)\varepsilon, w, t) < 0.$$

□

LEMMA 4.3 (Monotonicity). *Assume that u_0 is nonincreasing. Then*

$$U^\varepsilon(x,z,t) \geq U^\varepsilon(y,z,t)$$

for all $y \geq x$, $t \geq 0$ and $z \in \mathbb{R}$.

PROOF. We again show this by induction. We first notice that the statement at time t gives

$$V^\varepsilon(y,z,t+\varepsilon) \le V^\varepsilon(x,z,t+\varepsilon).$$

This yields that $\alpha^\varepsilon(y,t+\varepsilon) \le \alpha^\varepsilon(x,t+\varepsilon)$ and $\beta^\varepsilon(y,t+\varepsilon) \le \beta^\varepsilon(x,t+\varepsilon)$ and henceforth $g(z;I_{y,t+\varepsilon}) \le g(z;I_{x,t+\varepsilon})$. Hence, we have

$$\begin{aligned} U^\varepsilon(y,z,t+\varepsilon) &= U^\varepsilon(y-g(z;I_{y,t+\varepsilon})\varepsilon,z,t) \\ &\le U^\varepsilon(x-g(z;I_{x,t+\varepsilon})\varepsilon,z,t) = U^\varepsilon(x,z,t+\varepsilon). \end{aligned}$$

□

4.2. Convergence

We first recall the definition of proper solutions, which is due to [**50**]. We provide the definition for readers' convenience in our particular case.

Let $\{S_t\}_{t\in(0,\infty)}$ be a smooth family of smooth hypersurfaces with unit normal ν.

DEFINITION 4.1. For any upper semicontinuous (resp., lower semicontinuous) function u, we call S_t is an upper (resp., lower) test surface of u at $(x_0,t_0) \in \mathbb{R}\times(0,\infty)$ with level $\mu\in\mathbb{R}$ if

$$u(x,t) \le \mu < u(x_0,t_0) \quad (\text{resp.,} u(x,t) \ge \mu > u(x_0,t_0)),$$

for all $x \in D_t \cap B_r(x_0)$ and $t \in (t_0-r,t_0+r)$, where D_t satisfies $\partial D_t = S_t$ and its inward normal agrees with ν.

DEFINITION 4.2. Let u be a locally bounded subsolution (resp., supersolution) of (CL) in $\mathbb{R}\times(0,\infty)$. The function u is said to be a proper subsolution (resp., proper supersolution) if for any $(x_0,t_0)\in\mathbb{R}\times(0,\infty)$,

$$\begin{aligned} &V(x_0,t_0) - g(u^*(x_0,t_0),[\mu,u^*(x_0,t_0)]) \le 0 \\ &(\text{resp.,} -V(x_0,t_0) - g(u_*(x_0,t_0),[u_*(x_0,t_0),\mu]) \ge 0) \end{aligned}$$

for any upper test surface S_t of u^* (resp., lower test surface S_t of u_*) at (x_0,t_0) with any level $\mu < u^*(x_0,t_0)$ (resp., $\mu > u_*(x_0,t_0)$). Here $V(x_0,t_0)$ denotes the normal velocity of S_t at (x_0,t_0) in the direction $\nu(x_0,t_0)$.

If u is both a proper subsolution and a proper supersolution, then u is called a proper solution.

We now show that $\overline{u}$ is a proper viscosity subsolution.

PROPOSITION 4.4. *Assume that* u_0 *is nonincreasing. Then* $\overline{u}$ *is a viscosity subsolution of (CL).*

PROOF. Assume that there are $(x_0,t_0)\in\mathbb{R}\times(0,\infty)$ and a function $\phi\in C^1(\mathbb{R}\times[0,\infty))$ such that

$$(\overline{u}-\phi)(x_0,t_0) > (\overline{u}-\phi)(x,t) \text{ for all } (x,t)\in\mathbb{R}\times(0,\infty).$$

By Lemma 4.3, it is not difficult to obtain that $\phi_x(x_0,t_0)\le 0$. Set $\Phi(x,z,t) = \phi(x,t)-z$. It is easy to see that $\Phi\in C^1(\mathbb{R}\times\mathbb{R}\times[0,\infty))$ and there exists $r>0$ such that

$$(\overline{U}-\Phi)(x_0,\overline{u}(x_0,t_0),t_0) > (\overline{U}-\Phi)(x,z,t) \text{ for all } (x,z,t)\in B_r(x_0,\overline{u}(x_0,t_0),t_0).$$

Then there exist $x_k \to x_0$, $t_k \to t_0$, $z_k \to \overline{u}(x_0, t_0)$ and $\varepsilon_k \to 0$ as $k \to \infty$ such that

$$(U^\varepsilon - \Phi)(x_k, z_k, t_k) \geq (U^\varepsilon - \Phi)(x, z, t) - \varepsilon_k^2 \quad \text{for all } (x, z, t) \in B_r(x_0, \overline{u}(x_0, t_0), t_0). \tag{4.1}$$

Employing our inductive definition of $\overline{U}^\varepsilon$

$$U^{\varepsilon_k}(x_k, z_k.t_k) = U^{\varepsilon_k}(x_k - g(z_k, I_{x_k,t_k})\varepsilon_k, z_k, t_k - \varepsilon_k),$$

where $I_{x_k,t_k} = [\alpha^{\varepsilon_k}(x_k, t_k), \beta^{\varepsilon_k}(x_k, t_k)]$, and letting $x = x_k - g(z_k, I_{x_k,t_k})\varepsilon_k$, $z = z_k$ and $t = t_k - \varepsilon_k$ in (4.1), we have

$$\Phi(x_k, z_k, t_k) - \Phi(x_k - g(z_k; I_{x_k,t_k})\varepsilon_k, z_k, t_k - \varepsilon_k) - \varepsilon_k^2 \leq 0.$$

Since $\Phi(x, z, t) = \phi(x, t) - z$, we get by Taylor expansion

$$\phi_t(x_k, t_k)\varepsilon_k + g(z_k; I_{x_k,t_k})\phi_x(x_k, t_k)\varepsilon_k \leq \varepsilon_k^2.$$

Dividing the inequality above by ε_k and then sending $k \to \infty$, we obtain either

$$\phi_t(x_0, t_0) + \overline{u}(x_0, t_0)\phi_x(x_0, t_0) \leq 0 \tag{4.2}$$

or

$$\phi_t(x_0, t_0) + g(\overline{u}(x_0, t_0), [\alpha_0, \beta_0])\phi_x(x_0, t_0) \leq 0, \tag{4.3}$$

where α_0 and β_0 are limits of $\alpha^{\varepsilon_k}(x_k, t_k)$ and $\beta^{\varepsilon_k}(x_k, t_k)$ respectively through subsequences.

The inequality (4.2), which corresponds to the case $z_k \notin I_{x_k,t_k}$, is exactly the one desired. We only need to treat the case (4.3) when $z_k \in I_{x_k,t_k}$. We claim that in this case $\beta_0 \leq \overline{u}(x_0, t_0)$ so that (4.2) follows again. Indeed, by definition, we stick to this subsequence and take $\delta_k \geq 0$ such that $\delta_k \to 0$ as $k \to \infty$ and

$$V^{\varepsilon_k}(x_k, \beta^{\varepsilon_k}(x_k, t_k) - \delta_k, t_k) = 1,$$

which implies

$$U^{\varepsilon_k}(x_k - \beta^{\varepsilon_k}(x_k, t_k)\varepsilon_k + \delta_k\varepsilon_k, \beta^{\varepsilon_k}(x_k.t_k) - \delta_k, t_k - \varepsilon_k) = 1.$$

Taking the limit $k \to \infty$, we get

$$\overline{U}(x_0, \beta_0, t_0) = 1,$$

which yields $\beta_0 \leq \overline{u}(x_0, t_0)$. □

PROPOSITION 4.5. *Assume that u_0 is nonincreasing. Then $\overline{u}$ is a proper subsolution of* (CL).

PROOF. Now that we have proved that $\overline{u}$ is a subsolution in the usual viscosity sense, we study the case when $\overline{u}$ is tested from sides. By the monotonicity of u_0 and Lemma 4.3, $\overline{u}$ is nonincreasing. It thus suffices to test the graph of $\overline{u}$ only from right side.

Assume that there is a test smooth surface S_t touching the graph of $\overline{u}$ from right side only at $(x_0, t_0) \in \mathbb{R} \times (0, \infty)$ and only with level $z_0 \leq \overline{u}(x_0, t_0)$. Then there exist $\Phi \in C^\infty(\mathbb{R} \times \mathbb{R} \times (0, \infty))$ and $r > 0$ such that

$$(\overline{U} - \Phi)(x, z, t) < (\overline{U} - \Phi)(x_0, z_0, t_0) = 0 \quad \text{for all } (x, z, t) \in B_r(x_0, z_0, t_0). \tag{4.4}$$

We only need to discuss the situation that $\Phi_z(x_0, z_0, t_0) = 0$.

Then $\Phi_x(x_0, z_0, t_0) \neq 0$, which actually implies $\Phi_x(x_0, z_0, t_0) < 0$ by Lemma 4.3. There exist $\varepsilon_k \to 0$ and $(x_k, z_k, t_k) \to (x_0, z_0, t_0)$ as $k \to \infty$ such that

$$U^{\varepsilon_k}(x_k, z_k, t_k) \to \overline{U}(x_0, z_0, t_0) = 1 \tag{4.5}$$

and

$$(U^{\varepsilon_k} - \Phi)(x_k, z_k, t_k) \geq (U^{\varepsilon_k} - \Phi)(x, z, t) - \varepsilon^2 \quad \text{in } B_r(x_0, z_0, t_0). \tag{4.6}$$

By the definition of U^ε, (4.5) amounts to saying $U^{\varepsilon_k}(x_k, z_k, z_k) = 1$. Also, our definition of U^ε gives

$$U^{\varepsilon_k}(x_k, z_k, t_k) = U^{\varepsilon_k}(x_k - g(z_k; I_{x_k,t_k}), z_k, t_k - \varepsilon_k). \tag{4.7}$$

Denote $\tilde{x}_k = x_k - g(z_k; I_{x_k,t_k})\varepsilon_k$. Then in virtue of (4.6) with $(x, z, t) = (\tilde{x}_k, z_k, t_k - \varepsilon_k)$ and (4.7), we have

$$\Phi(x_k, z_k, t_k) - \Phi(\tilde{x}_k, z_k, t_k - \varepsilon_k) - \varepsilon_k^2 \leq 0, \tag{4.8}$$

which yields by Taylor expansion

$$\Phi_t(x_k, z_k, t_k)\varepsilon_k + \Phi_x(x_k, z_k, t_k) \cdot g(z_k; I_{x_k,t_k})\varepsilon_k - \varepsilon_k^2 \leq 0. \tag{4.9}$$

We discuss two cases for all subsequences.

Case A. $g(z_k; I_{x_k,t_k}) \leq z_k$ for all $k \geq 1$.

Then since $\Phi_x < 0$, (4.9) reduces to

$$\Phi_t(x_k, z_k, t_k)/|\Phi_x(x_k, z_k, t_k)| - z_k \leq 0.$$

Sending $k \to \infty$, we get

$$\Phi_t(x_0, z_0, t_0)/|\Phi_x(x_0, z_0, t_0)| - z_0 \leq 0.$$

Noting that $z_0 \leq g(\overline{u}(x_0, t_0); [z_0, \overline{u}(x_0, t_0)])$, we are led to

$$\Phi_t(x_0, z_0, t_0)/|\Phi_x(x_0, z_0, t_0)| - g(\overline{u}(x_0, t_0); [z_0, \overline{u}(x_0, t_0)]) \leq 0. \tag{4.10}$$

Case B. $g(z_k; I_{x_k,t_k}) > z_k$ for all $k \geq 1$.

Since $z_k \neq g(z_k; I_{x_k,t_k})$, we have $\alpha^{\varepsilon_k}(x_k, t_k) \leq z_k \leq \beta^{\varepsilon_k}(x_k, t_k)$. Hence, passing to a subsequence if necessary, we may assume there exist $\hat{\alpha}$ and $\hat{\beta}$ such that $\alpha^{\varepsilon_k}(x_k, t_k) \to \hat{\alpha}$ and $\beta^{\varepsilon_k}(x_k, t_k) \to \hat{\beta}$ as $k \to \infty$. Moreover, we claim that they must satisfy

$$\hat{\alpha} \leq z_0 \tag{4.11}$$

and

$$\hat{\beta} \leq \overline{u}(x_0, t_0). \tag{4.12}$$

The first inequality (4.11) is clear and we postpone the proof of the second until the end of this proof.

We take the limit of (4.9) as $k \to \infty$ and obtain

$$\Phi_t(x_0, z_0, t_0)/|\Phi_x(x_0, z_0, t_0)| - g(z_0; [\hat{\alpha}, \hat{\beta}]) \leq 0.$$

Here we used the continuity of $g(z; I)$ whenever $z \in I$. Apply (4.11) and (4.12) to the above, we get

$$\Phi_t(x_0, z_0, t_0)/|\Phi_x(x_0, z_0, t_0)| - g(z_0; [z_0, \overline{u}(x_0, t_0)]) \leq 0.$$

The inequality (4.10) follows immediately if one notices that $g(z; I)$ is a constant for $z \in I$.

We finally show the inequality (4.12). Notice that $x_k > \tilde{x}_k + z_k\varepsilon_k$. We easily obtain

$$\beta(x_k, t_k) \le \beta(\tilde{x}_k + z_k\varepsilon_k, t_k),$$

where

$$\beta(\tilde{x}_k + z_k\varepsilon_k, t_k) = \sup\{z \in \mathbb{R} : U^{\varepsilon_k}(\tilde{x}_k + (z_k - z)\varepsilon_k, z, t_k - \varepsilon_k) = 1\}.$$

It is then clear that $\lim_{k\to\infty} \beta(x_k, t_k) \le \overline{u}(x_0, t_0)$, for otherwise, we can take a further subsequence with $\tilde{z}_k$ satisfying

$$\lim_{k\to\infty} \tilde{z}_k > \overline{u}(x_0, t_0) \tag{4.13}$$

such that

$$U^{\varepsilon_k}(\tilde{x}_k + z_k\varepsilon_k - \tilde{z}_k\varepsilon_k, \tilde{z}_k, t_k - \varepsilon_k) = 1.$$

Sending $k \to \infty$ for the equality above, we have by definition of $\overline{U}$

$$\overline{U}(x_0, \lim_{k\to\infty} \tilde{z}_k, t_0) \ge 1,$$

which, by definition of $\overline{u}$, yields $\lim_{k\to\infty} \tilde{z}_k \le \overline{u}(x_0, t_0)$. This is a contradiction to (4.13). □

We thus turn to the consistency of our approximate solution with the initial data.

PROPOSITION 4.6. *If u_0 is bounded, i.e., there exists $M > 0$ such that $\|u_0\|_\infty \le M$, then $\overline{U}(x, z, 0) \le U_0^*(x, z)$ for all $x, z \in \mathbb{R}$. In particular, $\overline{u}(x, 0) \le u_0^*(x)$ for all $x \in \mathbb{R}$.*

PROOF. Since $\overline{U}(x, z, 0) \le 1$, it suffices to prove the statement for $(x, z) \in \mathbb{R}^2$ satisfying $U_0^*(x, z) = -1$. In other words, we discuss the case when $z > u_0^*(x)$. It is clear that there exists $r > 0$ such that

$$U_0(y, w) = -1 \quad \text{for all } (y, w) \in B_r(x, z).$$

Take $\delta > 0$. Then for any $(y, w, s) \in B_\delta(x, z, 0)$ with $s \ge 0$, if we set $s_0 = s$, $y_0 := y$ and inductively $s_{k+1} = s_k - \varepsilon$ and $y_{k+1} = y_k - g(w; I_{y_k, s_k})\varepsilon$ for all $k = 0, 1, \ldots, N_s - 1$, where $N_s = [s/\varepsilon]$, we have

$$U^\varepsilon(y, w, s) = U_0(y_{N_s}, w) = U_0(y_0 - \sum_{k=0}^{N_s - 1} g(w; I_{y_k, s_k})\varepsilon, w).$$

Noticing that $|w - z| \le \delta$ and

$$|g(w; I_{y_k, s_k})| \le M + |z| + \delta,$$

which, by Remark 4.4, implies that

$$|y_{N_s} - x| \le |y - \sum_{k=0}^{N_s - 1} g(w; I_{y_k, s_k}\varepsilon)\varepsilon - x| \le \delta + (M + |z| + \delta)\delta,$$

we are led to

$$U^\varepsilon(y, w, s) = -1 \quad \text{for all } (y, w, s) \in B_\delta(x, z, 0) \text{ and } \varepsilon > 0$$

when δ is taken sufficiently small. Passing to the relaxed limit as $\delta \to 0$, we obtain $\overline{U}(x, z, 0) = -1$. □

We may use the above arguments to show the following.

PROPOSITION 4.7. *Assume that u_0 is nonincreasing. Then $\underline{u}$ is a proper supersolution of* (CL) *with $\underline{u}(x) \geq (u_0)_*(x)$ for all $x \in \mathbb{R}$.*

Since the proper comparison principle holds provided our monotone initial data satisfy some particular condition.

$$\begin{cases} u_0 \text{ is nonincreasing and satisfies the following:} \\ \text{(i) There exists } x_0 \in \mathbb{R} \text{ such that } u_0(x_0) = \max u_0 \text{ or } u_0(x_0) = \min u_0. \\ \text{(ii) } u_0 \text{ is strictly decreasing in } \{x \in \mathbb{R} : u_0(x) \neq \sup u_0, u_0(x) \neq \inf u_0\}. \end{cases} \tag{4.14}$$

THEOREM 4.8 (Proper comparison principle, [50]). *Assume* (4.14). *Assume that $((u_0)^*)_* = (u_0)_*$ and $((u_0)_*)^* = (u_0)^*$. Let u and v be respectively an USC proper subsolution and a LSC proper supersolution of* (CL) *with $u(x, 0) \leq (u_0)^*(x)$ and $v(x, 0) \geq (u_0)_*$. Then $u_* \leq v$ and $u \leq v^*$ in $\mathbb{R} \times [0, \infty)$.*

THEOREM 4.9 (Convergence). *Assume that $f \in C(R)$ is nondecreasing. Assume that u_0 is bounded, satisfies* (4.14) *and satisfies the relations $((u_0)^*)_* = (u_0)_*$ and $((u_0)_*)^* = (u_0)^*$. Then the functions $\overline{u}$ and $\underline{u}$ constructed above satisfy $\overline{u}_* = \underline{u}$ and $\overline{u} = \underline{u}^*$ and are the unique proper solution of* (CL) *and* (1.22).

PROOF. The proof now is easy. Since $\overline{u}$ and $\underline{u}$ are respectively a proper subsolution and a proper supersolution, as shown in Propositions 4.4, 4.5 and 4.7. We may apply Theorem 4.8 to obtain

$$\overline{u}_* \leq \underline{u} \text{ and } \overline{u} \leq \underline{u}^*.$$

The converse inequality is obvious from our construction. □

Remaining Problems. All above is only a very primary step for the (control and game based) approximation for solutions which form discontinuity. Similar results for more general equations are still in progress and there are a lot of interesting related problems, among which I list the most important two.

- For our conservation law equation in one space dimension, if the function f is of class C_1 and $f' \geq c$ for some constant c (in other words, F is uniformly convex), it seems possible to show the entropy condition on the discrete level

$$u^\varepsilon(y, t) - u^\varepsilon(x, t) \leq \frac{1}{ct}(y - x) \text{ for all } y > x \text{ and } t > 0$$

  for general initial data u_0. (In this note, we did not define u^ε, but it is possible.) If it can be proven, we need not assume the monotonicity of u_0 and we are also able to directly show that the limit is an entropy solution without using the equivalence of entropy solutions and proper solutions given in [50].
- Can we generalize the proofs for more equations? It looks easy to apply our method to the equations like

$$u_t + \inf_{a \in A} f(u, a) u_x = 0$$

with spokewise monotone initial data, where A is a compact set and $f \in C(\mathbb{R})$ is nondecreasing. However, the proof will be more complicated if we drop the monotonicity of f. The higher dimensional case is not clear either...

Part II

Large-time Asymptotics for Noncoercive Hamilton-Jacobi Equations

CHAPTER 5

Introduction

We are interested in the large-time behavior of solutions of Hamilton-Jacobi equations

$$u_t + H(x, \nabla u) = 0 \quad \text{in } \mathcal{O} \times (0, \infty),$$

(with proper boundary conditions if needed), where $\mathcal{O} \subset \mathbb{R}^n$ is a domain and the *Hamiltonian* $H : \mathcal{O} \times \mathbb{R}^n \to \mathbb{R}$ is continuous and noncoercive. By the noncoercivity, we mean H does not satisfy the following coercivity property:

$$\inf\{H(x,p) \mid |x| \le R, |p| \ge r\} \to +\infty \text{ as } r \to \infty \text{ for any } R > 0.$$

Let us roughly state our main results. The viscosity solution turns out to have the constant growth rate asymptotically in a subset Ω_e of the whole domain, which we will call the *effective domain*. We prove that there exist a constant $c \in \mathbb{R}$ and $v \in C(\Omega_e)$ such that

$$u(x,t) - ct \to v(x) \text{ in } \Omega_e \text{ as } t \to \infty.$$

On the other hand, outside of the effective domain, the viscosity solution has a greater growth rate; namely,

$$u(x,t) - ct \to +\infty \text{ in } \mathcal{O} \setminus \overline{\Omega_e} \text{ as } t \to \infty.$$

The other feature to be noted is that *gradient grow-up* (or *infinite time gradient blow-up*) of solutions happens. More precisely, the normal derivative with respect to x-variable of $u - ct$ blows up on the boundary $\partial\Omega_e$ of Ω_e as time goes to infinity, i.e.,

$$D\big(u(x,t) - ct\big) \cdot \nu(x) \to +\infty \text{ for all } x \in \partial\Omega_e \text{ as } t \to \infty,$$

which is a natural consequence due to the lack of coercivity. This requires the asymptotic profile v satisfies a *singular Neumann condition* and, in some more singular cases, a *singular Dirichlet* one. We refer to [**97, 108**] and references therein for results on gradient blow-up and grow-up of solutions of parabolic equations.

One of the aims of this part is to investigate the large-time behavior asymptotics of such type. More precisely, we give the formulas of the effective domain and the growth rate and prove that viscosity solutions converge uniformly on the effective domain and that outside of the effective domain they have growth rates which are higher than that on the effective domain.

Our study covers two problems. In Chapter 6, we consider a one-dimensional Cauchy-Dirichlet problem with very general structure of the Hamiltonian H. We take $\mathcal{O} = (0, \infty)$ and add a homogeneous Dirichlet condition at $x = 0$. Chapter 7 is for a Cauchy problem ($\mathcal{O} = \mathbb{R}^n$) without any restriction on the dimensions but we only study a special class of equations which have important applications in the field of crystal growth.

To understand more deeply the novelty of our results, we first recall the known results mainly for the case where H is coercive.

Large-time behavior of solutions. In the last decade, a lot of works have been devoted to the study of large-time behavior of viscosity solutions of Hamilton-Jacobi equations

$$\begin{cases} u_t + H(x, \nabla u) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{5.1}$$

where H is convex and coercive and u_0 is a given function. General convergence results for solutions have been established. More precisely, under some appropriate assumptions on the initial value, for the solution u of (5.1), the convergence

$$u(x,t) - (v(x) - ct) \to 0 \quad \text{local uniformly for } x \in \mathbb{R}^n \text{ as } t \to \infty \tag{5.2}$$

holds, where $(v, c) \in C(\mathbb{R}^n) \times \mathbb{R}$ is a solution of the *additive eigenvalue* or *ergodic* problem

$$H(x, \nabla v(x)) = c \quad \text{in } \mathbb{R}^n. \tag{5.3}$$

Here the additive eigenvalue problem for H is a problem of finding a pair of $v \in C(\mathbb{R}^n)$ and $c \in \mathbb{R}$ such that v is a viscosity solution of (5.3). Namah and Roquejoffre in [**92**] were the first to get general results on this convergence under the following additional assumption:

$$H(x,p) \geq H(x,0) \text{ for all } (x,p) \in \mathcal{M} \times \mathbb{R}^n \text{ and } \max_{\mathcal{M}} H(x,0) = 0, \tag{5.4}$$

where $\mathcal{M}$ is a smooth compact n-dimensional manifold without boundary. Then Fathi [**43**] proves the same type of convergence result by dynamical systems type arguments, so-called weak KAM theory. Contrary to [**92**], the results of [**43**] use strict convexity assumptions on $H(x, \cdot)$, i.e., $\nabla_p^2 H(x,p) \geq \alpha I$ for all $(x,p) \in \mathcal{M} \times \mathbb{R}^n$ and $\alpha > 0$ (and also far more regularity) but do not need (5.4). Afterwards Roquejoffre [**98**] and Davini and Siconolfi [**31**] has refined the approach of Fathi and they study the asymptotic problem for (5.1) on $\mathcal{M}$ or n-dimensional torus. By another approach based on the theory of partial differential equations and viscosity solutions, this type of results has been obtained by Barles and Souganidis in [**15**]. Moreover, we also refer to the literatures [**13, 68, 60, 61, 62**] for the asymptotic problems without the periodic assumptions and the periodic boundary condition and the literatures [**98, 89, 90, 91, 70, 11**] for the asymptotic problems which treat HJ equations under various boundary conditions including three type of boundary conditions, state constraint boundary condition, Dirichlet boundary condition and Neumann boundary condition. We remark that results in [**15, 13, 11**] apply to nonconvex Hamiltonian. It is worth mentioning that as far as we know, only the type of convergence (5.2) of solutions of (5.1) have been investigated for HJ equations (5.1).

We point out that noncoercive ergodicity and homogenization problems are also investigated in [**4, 20**] with *non-resonance conditions* and in [**9**] under partial coercivity assumption, which means $p \mapsto H(x,p)$ is coercive along several directions but not necessarily along the others. In contrast, we will later see that our Hamiltonian is "completely noncoercive" and global ergodic result cannot be expected.

A simple example for the noncoercive case. Our motivation for noncoercive Hamilton-Jacobi equations, to be elaborated in a moment, comes from the theory of crystal growth. Before developing the details, we first give a very primitive example for $\Omega = \mathbb{R}$:

$$\begin{cases} u_t + \dfrac{2}{\pi}\arctan(u_x^2) = |x| & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = 0 & \text{for all } x \in [0,\infty). \end{cases} \tag{5.5}$$

If u is a solution of the one-dimensional Cauchy problem, then the large-time asymptotic behavior of u can be described by

$$u(\cdot,t) \longrightarrow u_\infty \text{ uniformly on } [-1,1],$$

where

$$u_\infty(x) = \int_0^{|x|} (\tan(\frac{\pi}{2}y))^{1/2}\, dy \text{ for all } x \in [-1,1],$$

and

$$u(\cdot,t) \longrightarrow +\infty \text{ uniformly on each compact subset of } (-\infty,-1) \cup (1,\infty)$$

as $t \longrightarrow \infty$. In order to solve this problem by the method of characteristics, we first consider the initial-boundary value problem

$$\begin{cases} v_t + \dfrac{2}{\pi}\arctan(v_x^2) = x & \text{in } (0,\infty) \times (0,\infty), \\ v(0,t) = 0 & \text{for all } t \in [0,\infty), \\ v(x,0) = 0 & \text{for all } x \in [0,\infty). \end{cases} \tag{5.6}$$

We have the Hamiltonian system for this problem

$$\begin{cases} \dfrac{dx(t)}{dt} = \dfrac{4p(t)}{\pi(1+p(t)^4)} & \text{for } t \in (0,\infty), \\ \dfrac{dp(t)}{dt} = 1 & \text{for } t \in (0,\infty), \\ x(0) = x_0 \in \mathbb{R},\ p(0) = 0 \end{cases}$$

and its solution could be explicitly calculated as

$$x(t) = x_0 + \frac{2}{\pi}\arctan(t^2) \text{ and } p(t) = t.$$

Setting $z(t) = v(x(t),t)$, then it is satisfied that

$$\frac{dz(t)}{dt} = \frac{4t^2}{\pi(1+t^4)} + x(t) - \frac{2}{\pi}\arctan(t^2) = \frac{4t^2}{\pi(1+t^4)} + x_0. \tag{5.7}$$

Notice that first term on the last part is integrable and x_0 should be positive when we have $x(t) > 1$ for some $t > 0$. Moreover, an observation is that given any x greater than 1 and t sufficiently large, if we can solve the characteristics inversely from $x(t) = x$ to find x_0, the curve starts from x_0 will cross the line $x = 1$ and stay in the region $x > 1$ for a very long time until it reaches the point x. Putting this observation into (5.7), we find $z(t) = v(x(t),t)$ tends to plus infinity as t goes to infinity if $x > 1$.

As a result, we see that large-time asymptotic profile v_∞ of the solution v of (5.6) is

$$v_\infty(x) = \begin{cases} \int_0^x (\tan(\frac{\pi}{2}y))^{1/2}\,dy & \text{for all } x \in [0,1], \\ +\infty & \text{for all } x \in (1,\infty). \end{cases}$$

Finally, we note that the function u defined by

$$u(x,t) = \begin{cases} v(x,t) & \text{for all } (x,t) \in [0,\infty) \times [0,\infty), \\ v(-x,t) & \text{for all } (x,t) \in (-\infty,0] \times [0,\infty) \end{cases}$$

is a unique viscosity solution of (5.5). We only need to check that u satisfies the equation (5.5) at $x = 0$ in the viscosity sense. Indeed, the subdifferential of u at $x = 0$ and any $t > 0$ is empty set and the superdifferential of u with respect to t-variable is 0. Thus any $p \in \mathbb{R}$ satisfies $0 + (2/\pi)\arctan(p^2) \geq 0$, which implies that u is a viscosity supersolution at $x = 0$.

Therefore, we see that the large-time asymptotic profile u_∞ of the solution u of (5.5) is

$$u_\infty(x) = \begin{cases} \int_0^{|x|} (\tan(\frac{\pi}{2}y))^{1/2}\,dy & \text{for all } x \in [-1,1], \\ +\infty & \text{for all } x \in (-\infty,-1) \cup (1,\infty). \end{cases}$$

From this example we learn that the growth rate of u may depend on x variable explicitly. We emphasize that this phenomenon seems to be new at least from the viewpoint of study for the large-time behavior of solutions of HJ equations. The typical result of study for this asymptotic problem for HJ equations with (*coercive*) Hamiltonian, which has been explained previously, shows that solutions converge (locally) uniformly with the constant growth rate in the whole domain which is considered as time goes to infinity.

Another more persuasive example is the following noncoercive Hamilton-Jacobi equation

$$\begin{cases} u_t - \dfrac{1}{1+|u_x|} - x = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

When $u_0 \equiv 0$, the unique solution is clearly $u(x,t) = xt + \ln(1+t)$, which cannot have the behavior as in (5.2). In fact, one can prove that in this case the solution u should satisfy

$$\lim_{t\to\infty} u(x,t)/t = x \quad \text{and} \quad \lim_{t\to\infty} (u(x,t) - xt)/\ln t = 1 \tag{5.8}$$

for any bounded and continuous initial data u_0. This corresponds to the behavior outside the effective domain, which we revisit in Chapter 8.

One-dimensional Cauchy-Dirichlet problems. We are concerned with large-time behavior of solutions of a Hamilton-Jacobi equation in one space dimension with homogeneous Dirichlet boundary condition:

$$\text{(CD)} \qquad \begin{cases} u_t + H(x,u_x) = 0 & \text{in } (0,\infty) \times (0,\infty), \\ u(0,t) = 0 & \text{for } t \in (0,\infty), \\ u(x,0) = u_0(x) & \text{for } x \in [0,\infty), \end{cases}$$

where u_0 is a locally bounded function, continuous at $x = 0$ with $u_0(0) = 0$. The continuous function $H : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is called a *Hamiltonian*. It is necessary to emphasize that our Dirichlet boundary condition is fulfilled in the classical sense instead of the viscosity sense.

We give a brief presentation of our main results in this chapter. Suppose that H has a limit at $|p| = \infty$; namely, for every $x \in [0, x_c)$

$$\lim_{|p|\to\infty} H(x,p) = c(x) \neq \pm\infty, \tag{5.9}$$

where $c : [0,\infty) \to \mathbb{R}$ is a continuous function which is assumed to have a unique zero x_c, dividing the domain into two parts. The function c satisfies

$$c > 0 \text{ in } [0, x_c) \text{ and } c < 0 \text{ in } (x_c, \infty). \tag{5.10}$$

Assume for the moment that

$$H(x,0) \leq 0 \quad \text{for every } x \in (0, x_c), \tag{5.11}$$

and

$$\sup_{p\in\mathbb{R}} H(x,p) < 0 \quad \text{for all } x \in (x_c, \infty). \tag{5.12}$$

We assert that if the initial data u_0 fulfills a *compatibility condition*, the solution u satisfies

$$u(x,t) \to v(x) \quad \text{for } x \in (0, x_c) \text{ locally uniformly; and} \tag{5.13a}$$

$$u(x,t) \to \infty \quad \text{for } x \in (x_c, \infty) \text{ locally uniformly,} \tag{5.13b}$$

as $t \to \infty$, where v is a solution of the stationary equation

$$\begin{cases} H(x, v_x) = 0 & \text{in } (0, x_c), \\ v(0) = 0. \end{cases} \tag{5.14}$$

Here the compatibility condition is as follows.

$$\begin{cases} u_0 \text{ is continuous at } x = 0 \text{ satisfying } u_0(0) = 0 \\ \text{and } u_0(x) \geq \displaystyle\int_0^x p_1(y)\,dy, \end{cases} \tag{5.15}$$

where $p_1 \in C[0, x_c)$ determined by the Hamiltonian H; see the assumption (H5) below. The asymptotic profiles v are the same for any u_0 satisfying (5.15). If the uniqueness of solutions of the stationary equation (5.14) holds, then we can prove (5.13a) relatively easily. However, it does not hold in general if they are considered in the usual sense. We need to adjust the definition of viscosity solutions and impose extra assumptions to guarantee the uniqueness. It therefore becomes another key issue in this work.

One difficulty for the uniqueness of solutions of (5.14) is about the seemingly missing boundary condition at $x = x_c$. An implicit fact is that our stationary solution satisfies $u_x \to \infty$ as $x \to x_c-$. This means that we need to cope with the *singular Neumann (Dirichlet) boundary condition* at $x = x_c$.

Cauchy problems in arbitrary dimensions. The second topic is the Cauchy problem for Hamilton-Jacobi equation

$$\text{(C)} \qquad \begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), & (5.16) \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n, & (5.17) \end{cases}$$

with *noncoercive Hamiltonian* $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of the form

$$H(x,p) = \sigma(x)m(\|p\|) - f(x). \tag{5.18}$$

Here $\|\cdot\|$ denotes an arbitrary norm in $\mathbb{R}^n$ while $\sigma, f : \mathbb{R}^n \to [0,\infty)$ and $m : [0,\infty) \to [0,1)$ are continuous functions. Moreover $m(r)$ is assumed to be

$$\text{strictly increasing and } m(r) \to 1 \text{ as } r \to \infty.$$

The functions σ, f and m are given functions. The function $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is an unknown function while $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a given initial value which is assumed to be Lipschitz continuous. To be consistent with the theory of crystal growth [**113**] we call σ, f and m a surface supersaturation, external force at point x and a kinetic coefficient, respectively.

By imposing proper assumptions, we assert that there are $\Omega_e \subset \mathbb{R}^n$, $c \in \mathbb{R}$ and $v \in C(\overline{\Omega_e})$ such that

$$u(x,t) - ct \to v \text{ in } \Omega_e \text{ as } t \to \infty$$

and

$$u(x,t) - ct \to +\infty \text{ in } \mathbb{R}^n \setminus \overline{\Omega_e} \text{ as } t \to \infty.$$

It turns out that the asymptotic profile on the effective domain is reduced to stationary problems. However, we again encounter a difficulty related to the boundary-value problem for stationary HJ equations. More precisely, we are led to consider the singular Neumann problem for stationary HJ equations with $\Omega = \Omega_e$

$$\begin{cases} F(x, \nabla u) = h(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = +\infty & \text{on } \partial\Omega, \\ \sup_{x\in\Omega} |u(x)| < +\infty, \end{cases} \tag{5.19}$$

where $F : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ and $h : \Omega \to [0,\infty)$ are given continuous functions which satisfy $F(x,p) \geq F(x,0) = 0$ for any $(x,p) \in \overline{\Omega} \times \mathbb{R}^n$, $F(x,p)$ is *convex* and F is *coercive* with respect to p-variable, i.e.,

$$F(x, \lambda p_1 + (1-\lambda)p_2) \leq \lambda F(x,p_1) + (1-\lambda)F(x,p_2)$$

for any $x \in \overline{\Omega}, p_1, p_2 \in \mathbb{R}^n$ and $\lambda \in [0,1]$,

$$\inf\{F(x,p) \mid x \in \overline{\Omega}, |p| \geq r\} \to +\infty \text{ as } r \to \infty$$

and

$$h(x) \to +\infty \quad \text{as dist}\,(x, \partial\Omega) \to 0 \text{ and } \{x \in \Omega \mid h(x) = 0\} \neq \emptyset,$$

where $\text{dist}\,(x,\partial\Omega) := \min\{|x-y| \mid y \in \partial\Omega\}$ for all $x \in \overline{\Omega}$.

We do not investigate the case of the singular Dirichlet boundary condition in this problem but the generalization is possible if one follows the one-dimensional argument.

Stationary equations with singular boundary conditions. In either problem of the above, we meet with stationary equations (5.14) or (5.19) with singular boundary conditions. Such kind of singular boundary problems are studied by Lasry and Lions [**78**] for a class of viscous Hamilton-Jacobi equations $-\Delta u + |\nabla u|^p + \lambda u = f$ in Ω, where $p > 1, \lambda > 0$ and f is a given smooth function in Ω. Particularly, they proved that if $p > 2$, and f satisfies suitable assumptions, there exists a unique solution of the above equation which is continuous on $\overline{\Omega}$ and satisfies the singular Neumann boundary condition. However, while there are many works on *blow-up problems* (or *singular Dirichlet boundary problem*) for (degenerate) elliptic equations, there seems to be very few works which study singular Neumann problems. It seems that [78] is the only study which investigates the existence and uniqueness of solutions of singular Neumann problem. Moreover, there are few works for first-order Hamilton-Jacobi equations.

For the singular Dirichlet boundary problems of first-order equations, we must point out that the work of Bardi [6], Evans and James [38] and also Bardi and Soravia [7] are closely related to ours. However, our method to deal with (5.14) in Chapter 6 is different from theirs. In contrast to their characterization of free boundary problems to meet their applications in optimal control and differential games, we relax the problem in the whole domain $(0,\infty)$ and permits the solution to take infinity value, which facilitates us very much to handle not only the singular Dirichlet boundary condition but also the Neumann one at the boundary $x = x_c$ fixed by our assumptions.

To prove the uniqueness of the solutions of 5.14, we have to face the difficulty coming from the abstract form of H. Usually, showing the comparison principle is difficult without homogeneity or convexity of H in p (see, e.g., [5, 66]), but, because of (5.9), it is not convenient for us to add such assumptions. Another standard option is to consider special optimal control and game structure compatible with the use of *Kruzkov transform*, which is applied in [7, 38]. This transform however cannot be applied to our problems arising from crystal growth. In fact, we try to obtain results with minimal assumptions on the p-dependence of H. We assume neither convexity nor homogeneity of $H(x,p)$ with respect to the variable p. In the proof of our comparison theorem, we only use an extra monotonicity assumption on $x \mapsto H(x,p)$, which has physics background to be clarified later. Roughly speaking, we assume

$$x \mapsto H(x,p) \text{ is strictly decreasing in } (0,x_c) \text{ for all } p \in \mathbb{R} \tag{5.20}$$

so that we can scale the solutions with respect to x without changing the gradient. This assumption enables us to demand no more assumptions on $p \mapsto H(x,p)$ or the optimal control and game characterization.

The same uniqueness problem for (5.19) requires us to take account of the Aubry-Mather set $\mathcal{A}$ (see [42, 44, 71]). The comparison principle for (5.19) is that if $u \le v$ on $\mathcal{A}$, then $u \le v$ on $\overline{\Omega}$, where $u \in C(\overline{\Omega})$ and $v \in \mathrm{LSC}(\overline{\Omega})$ are a subsolution and a supersolution of (5.19) respectively. It is worth noticing that from the viewpoint of weak KAM theory, Hamiltonian F has a simple dynamical structure, since the Aubry-Mather set $\mathcal{A}$ for $F - h$ is composed of only the *equilibrium point*, i.e.,

$$\mathcal{A} = \{x \in \Omega \mid \min_{p\in\mathbb{R}^n} F(x,p) - h(x) = 0\} = \{x \in \Omega \mid h(x) = 0\}.$$

We also remark that the definitions of stationary solutions to be used in Chapter 6 and Chapter 7 are different. For (5.14) we adjust the usual notion of viscosity solutions in $[x_c,\infty)$ while for (5.19) we apply the definitions first proposed by Lions and Lasry [78]. We however can prove the equivalence of them in some particular cases. See Appendix D.

Physical background. Our present work of asymptotic behavior is more rigorous and general analysis of [113], in which a one-dimensional mathematical model for crystal growth is studied. We present the physical background briefly only for our Cauchy problem in what follows. The motivation for (CD) is similar and we refer the reader to [**113**]. In [**113**] the morphological stability of a growing faceted crystal is discussed by using a model of a Hamilton-Jacobi equation. The molecularly smooth surfaces of crystals cannot grow without the surface kinetics process, such as the lateral motion of steps and the generation of steps. Such a growth mechanism is usually explained by motion of steps of a microscopic height proposed by Burton, Cabrera and Frank [**19**]. Its continuum limit gives a macroscopic model described by a Hamilton-Jacobi equation [**25**]. (This type of derivation is rigorously justified by E and Yip [**34**] and studies on relation between models via step-motion and via macroscopic partial differential equations are still an active research field in various settings; see e.g. [**88**].)

In a macroscopic model the dimensionless growth speed V in the direction normal to a crystal surface Γ_t is generally expressed using the surface supersaturation $\sigma(\geq 0)$ and the dimensionless kinetic coefficient $M(p)$ depending on the modulus p of a local slope (gradient) of the crystal surface as in [**19**] and [**26**] of the form

$$\begin{aligned} V &= M(p)\sigma, \\ M(p) &= \frac{p}{p_s}\tanh(\frac{p_s}{p}), \\ p_s &= \frac{d}{2x_s}. \end{aligned}$$

Here d is the step height and x_s is the mean surface diffusion distance of a molecule of the surface. The quantity p_s is a criterion of local slope which can be regarded as a small parameter and will be denoted by ε. When a surface Γ_t is given as the graph of a height function $z = z^\varepsilon(x, y, t)$, then the equation $V = M(p)\sigma$ is of the form

$$z_t - \sigma(x,y)M_1(\frac{|\nabla z|}{\varepsilon})\sqrt{1+|\nabla z|^2} = 0, \tag{5.21}$$

where $M_1(p) = p\tanh(1/p)$. If one is interested in the behavior of nearly a flat surface, it is natural to assume that σ is independent of z so we have assumed that σ is independent of z in (5.21).

Unfortunately, if initially $z = 0$, then the solution of (5.21) is identically zero and does not grow at all. We need a step source so that the crystal surface Γ_t grows. It is considered that the kinetic coefficient M is valid only outside the region where there are no step sources.

There are a few ways to include this effect. One way is to modify M_1 by a continuous, strictly increasing function m on $[0, \infty)$ which agrees with M_1 except near zero and $m(0) = m_0 > 0$. Another way is to give a growth speed at a particular point of the domain. In [**113**] we took the second point of view by putting the growth speed at the step source at the boundary. More precisely, in [**113**] a one-dimensional Dirichlet problem

$$z_t - \sigma(x)M_1(\frac{z_x}{\varepsilon})\sqrt{1+z_x^2} = 0, \quad x > 0,\ t > 0$$

with the boundary condition

$$z(0,t) = ct, \qquad t > 0$$

is considered, where $c > 0$ is a fixed constant smaller than $\sigma(0)$.

To explain instability of a facet it is reasonable to consider microscopic time approximation as $\varepsilon \to 0$ as studied in [**113**]. If one introduces a new dependent variable

$$\tilde{z}^\varepsilon(x,\tau) = z^\varepsilon(x, \varepsilon\tau)/\varepsilon$$

(called a microscopic height) and a new independent variable $\tau = t/\varepsilon$ (called a microscopic time), then $\tilde{z}^\varepsilon$ converges to a solution of

$$\begin{cases} \tilde{z}_\tau - \sigma(x)M_1(\tilde{z}_x) = 0, & x > 0 \quad \tau > 0 \\ \tilde{z}(0,\tau) = c\tau, & \tau > 0 \end{cases}$$

as is expected. This is rigorously proved in [**113**] at least when initially $z = 0$ and σ is Lipschitz. Note that the Hamiltonian is now noncoercive. When σ is a nonincreasing function, it is shown in [**113**]

$$\tilde{z}(x,\tau) \sim c\tau + b(x) \quad \text{in} \quad (0, x_c) \quad \text{as} \quad \tau \to \infty$$

with some computable function b by a method of characteristic and that outside $(0, x_c)$ such a behavior is not expected. The region $(0, x_c)$ is called a stable region in [**113**], which corresponds to an effective domain in the present paper. Physically speaking, a stable region is a part where a crystal surface stays microscopically flat. So if a stable region covers all crystal surface, the facet is considered to be stable (see the last part of Section 7.2.5). Otherwise, the facet breaks.

In Chapter 6 a more detailed analysis for general initial data as well as for more general noncoercive Hamiltonian is discussed for the Cauchy-Dirichlet problem in a half line. In Chapter 7 the effect of a step source is not given as a boundary condition but included in the Hamiltonian. Namely, we consider the Cauchy problem for

$$z_t - \sigma(x,y)m(\frac{|\nabla z|}{\varepsilon})\sqrt{1+|\nabla z|^2} = 0, \tag{5.22}$$

where $m : [0,\infty) \to (0,\infty)$ is continuous, strictly increasing and equals M_1 except near the origin and $m(0) = m_0 > 0$. After taking a microscopic time approximation (see Appendix 5.1), we obtain

$$\tilde{z}_\tau - \sigma(x,y)m(|\nabla \tilde{z}|) = 0.$$

If one introduces $u = -\tilde{z}$ with $t = \tau$, we end up with

$$u_t + \sigma(x,y)m(|\nabla u|) = 0.$$

In this case we see that the effective domain Ω_e is given by $\Omega_e := \{(x,y) \mid \sigma(x,y) > \overline{\sigma} m_0\}$. Our main result in particular implies that under the convexity and boundedness assumption on Ω_e and the nondegeneracy on $D\sigma$ on $\partial\Omega_e$, i.e., $D\sigma \neq 0$ on $\partial\Omega_e$, we have

$$u(x,y,t) \sim -ct + v_\infty(x,y) \text{ in } \Omega_e \text{ as } t \to \infty,$$

where v_∞ is a solution of the corresponding singular Neumann problem

$$\begin{cases} |\nabla u| = h(x,y) := m^{-1}\Big(\dfrac{\overline{\sigma} m_0}{\sigma(x,y)}\Big) & \text{in } \Omega_e, \\ \dfrac{\partial u}{\partial n} = +\infty & \text{on } \partial\Omega_e, \\ \sup\limits_{(x,y)\in\Omega_e} |u(x,y)| < +\infty. \end{cases}$$

Outside the closure of Ω_e the function u tends to $+\infty$ as $t \to \infty$. Thus, physically Ω_e is a stable region where a crystal surface stays microscopically flat. It is worth mentioning that the Aubry-Mather set of the corresponding stationary problem is considered as the set of step sources which consists of all global maximum points of σ. More precise descriptions are given in Section 7.2.5.

CHAPTER 6

Dirichlet Problem on a Half Line

We are concerned with large-time behavior of solutions of a Hamilton-Jacobi equation in one space dimension with homogeneous Dirichlet boundary condition

$$\text{(CD)} \qquad \begin{cases} u_t + H(x, u_x) = 0 & \text{in } (0,\infty)\times(0,\infty), \\ u(0,t) = 0 & \text{for } t \in (0,\infty), \\ u(x,0) = u_0(x) & \text{for } x \in [0,\infty). \end{cases}$$

We first study the stationary problem in Section 6.1, giving a definition of viscosity solutions and proving the comparison principle. Several examples are also discussed at the end of this section. In Section 6.2, we present our main theorem about the large-time behavior of solutions of the Cauchy-Dirichlet problem and give its proof.

6.1. Stationary equation with singular boundary conditions

Our main purpose of this section is to establish a comparison result for solutions of

$$H(x, u_x) = 0 \text{ in } (0,\infty). \tag{6.1}$$

Let us start with definitions of viscosity solutions and related properties.

6.1.1. Definition and Properties. Basic assumptions we need in this section are as follows.

(H1) $H : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ is a continuous function and there exist a constant $C > 0$ and a modulus ω such that

$$|H(x,p) - H(x,q)| \le C(1+|x|)|p-q|$$

and

$$|H(x,p) - H(y,p)| \le \omega((1+|p|)|x-y|)$$

for all $x, y \in [0,\infty)$ and $p, q \in \mathbb{R}$.

(H2) $\lim_{|p|\to\infty} H(x,p) = c(x)$ locally uniformly in x, where $c(x) \in C([0,\infty))$ such that there is $x_c \in \mathbb{R}$ satisfying

$$c(x) \begin{cases} > 0 & \text{for } x \in [0, x_c); \\ = 0 & \text{for } x = x_c; \\ < 0 & \text{for } x \in (x_c, \infty). \end{cases}$$

We hereafter do not distinguish notations of $H(x, \pm\infty)$ and $c(x)$.

(H3) For any $x \in (x_c,\infty)$, $\sup_{p\in\mathbb{R}} H(x,p) < 0$.

Although our stationary equation looks established in an unbounded domain, we will later see that the assumption (H3) essentially turns it into a problem with the bounded domain $[0, x_c)$.

We next set, for any function $u : [0,\infty) \to \mathbb{R} \cup \{\pm\infty\}$,

$$u^+(x) = \begin{cases} +\infty & \text{if } u(x) = -\infty, \\ u(x) & \text{otherwise} \end{cases} \quad \text{and } u_-(x) = \begin{cases} -\infty & \text{if } u(x) = +\infty, \\ u(x) & \text{otherwise.} \end{cases}$$

We also denote $(u_-)^*$ by $\overline{u}$ and $(u^+)_*$ by $\underline{u}$, where w^* (resp., w_*) stands for the usual *upper* (resp., *lower*) *semicontinuous envelope* of a function w.

DEFINITION 6.1 (Subsolution). A function $u : [0,\infty) \to \mathbb{R} \cup \{\pm\infty\}$ is said to be a subsolution of (6.1) if it satisfies the following:

(i) $\overline{u}(x) < \infty$ for all $x \in (0,\infty) \setminus \{x_c\}$;

(ii) whenever there exist $\varphi \in C^1([0,\infty) \setminus \{x_c\})$ and $\hat{x} \in (0,\infty) \setminus \{x_c\}$ satisfying

$$\max_{x\in[0,\infty)\setminus\{x_c\}} (\overline{u} - \varphi)(x) = (\overline{u} - \varphi)(\hat{x}),$$

then

$$H(\hat{x}, \varphi_x(\hat{x})) \leq 0. \tag{6.2}$$

(iii) if $\overline{u}(x_c) < \infty$, then for every $\varphi \in C^1([0,\infty))$ such that

$$\max_{x\in[0,\infty)} (\overline{u} - \varphi)(x) = (\overline{u} - \varphi)(x_c),$$

the inequality (6.2) holds with $\hat{x} = x_c$, i.e.,

$$H(x_c, \varphi_x(x_c)) \leq 0.$$

DEFINITION 6.2 (Supersolution). A function $u : [0,\infty) \to \mathbb{R} \cup \{\pm\infty\}$ is said to be a supersolution of (6.1) if it satisfies the following:

(i) $\underline{u}(x) > -\infty$ for all $x \in (0,\infty) \setminus \{x_c\}$;

(ii) whenever there exist $\varphi \in C^1([0,\infty) \setminus \{x_c\})$ and $\hat{x} \in (0,\infty) \setminus \{x_c\}$ satisfying

$$\min_{x\in[0,\infty)\setminus\{x_c\}} (\underline{u} - \varphi)(x) = (\underline{u} - \varphi)(\hat{x}),$$

then

$$H(\hat{x}, \varphi_x(\hat{x})) \geq 0. \tag{6.3}$$

(iii) if $\underline{u}(x_c) > -\infty$, then for every $\varphi \in C^1([0,\infty))$ such that

$$\min_{x\in[0,\infty)} (\underline{u} - \varphi)(x) = (\underline{u} - \varphi)(x_c),$$

the inequality (6.3) holds with $\hat{x} = x_c$, i.e.,

$$H(x_c, \varphi_x(x_c)) \geq 0.$$

DEFINITION 6.3 (Solution). A function u is said to be a solution if it is both a subsolution and a supersolution.

In spite of our modification of values in (x_c, ∞), we are unable to prevent a subsolution (resp. a supersolution) from attaining the value $+\infty$ (resp. $-\infty$) at $x = x_c$ a priori. This corresponds to the possibility that solutions of (6.1) with $u(0) = 0$ blow up at $x = x_c$, which we shall discuss in more detail in Section 6.1.3.

As usual comments on viscosity solutions, the maximum and minimum in Definitions 6.1 and 6.2 can both be replaced by a strict maximum and a strict minimum. One can also use the superdifferential $D^+\overline{u}$ and the subdifferential $D^-\underline{u}$ instead of the test function φ to define the solutions. See [**5**, **51**] for instance.

An immediate consequence of Definition 6.1 and assumption (H3) is that our supersolutions do take infinite value in (x_c, ∞).

LEMMA 6.1 (Infinite value of supersolutions in the unstable region). *Assume* (H3). *If u is a supersolution of* (6.1), *then $\underline{u} = +\infty$ in (x_c, ∞).*

PROOF. We argue by contradiction. Suppose there exists $\hat{x} \in (x_c, \infty)$ such that $\underline{u}(\hat{x}) \neq +\infty$. It is obvious by Definition 6.1 that $\underline{u}$ is locally bounded from below; that is, there exist $M \in \mathbb{R}$ and an interval $[\hat{x} - r, \hat{x} + r]$ in which $\underline{u} \geq M$. Moreover, we may let r be so small that $\hat{x} - 2r > x_c$. Now observe that

$$\underline{u}(x) - \underline{u}(\hat{x}) + \frac{1}{2\varepsilon}|x - \hat{x}|^2 \geq M - \underline{u}(\hat{x}) + \frac{1}{8\varepsilon}r^2 \text{ for } x = x_r \pm r$$

and

$$\min_{x \in [\hat{x}-r,\ \hat{x}+r]} \underline{u}(x) - \underline{u}(\hat{x}) + \frac{1}{2\varepsilon}|x - \hat{x}|^2 \leq 0.$$

Then a minimizer x^ε above exists and lies in $(\hat{x} - r, \hat{x} + r)$ when ε is sufficiently small. It follows from (H3) that

$$H(x^\varepsilon, \frac{\hat{x} - x^\varepsilon}{\varepsilon}) < 0,$$

which contradicts to the inequality in Definition 6.2. □

Let us study the regularity of subsolutions by using the assumption (H2).

LEMMA 6.2 (Local Lipschitz continuity of subsolutions). *Assume* (H2). *Let u be a subsolution of* (6.1). *Assume that for every $\varepsilon > 0$ there exists $M > 0$ such that $\overline{u} \leq M$ in $[0, x_c - \varepsilon]$. Then there exists $L > 0$ depending only on ε such that*

$$|\overline{u}(x) - \overline{u}(y)| \leq L|x - y| \textit{ for all } x, y \in (0, x_c - \varepsilon].$$

PROOF. We only show the case

$$\overline{u}(x) - \overline{u}(y) \geq -L|x - y| \tag{6.4}$$

for all $x \in [0, x_c - \varepsilon]$ and $y \in (0, x_c - \varepsilon]$. (The other half can be treated via a symmetric argument.) It suffices to show that for any fixed $\varepsilon > 0$ and $x \in [0, x_c - \varepsilon]$, there is $L > 0$ independent of x such that (6.4) holds for all $y \in (0, x_c - \varepsilon]$.

We argue by contradiction. Assume that there are $\varepsilon > 0$ and $\hat{x} \in [0, x_c - \varepsilon]$ such that there always exists $y \in (0, x_c - \varepsilon]$ satisfying

$$\overline{u}(\hat{x}) - \overline{u}(y) < -L|\hat{x} - y|$$

no matter how large L is. This means

$$\max_{y \in [0, x_c - \varepsilon]} (\overline{u}(y) - \overline{u}(\hat{x}) - L|\hat{x} - y|) > 0 \quad \text{for any } L > 0.$$

Denote by $\hat{y} \in [0, x_c - \varepsilon]$ a maximizer above. Clearly $\hat{y} \neq \hat{x}$. Also, since $\overline{u}$ is bounded from above, we may let L be sufficiently large without depending on $\hat{x}$ so that $\hat{y} \neq 0$ and $\hat{y} \neq x_c - \varepsilon$.

Put $\varphi(y) := \overline{u}(\hat{x}) + L|\hat{x} - y|$. Now that $\overline{u} - \varphi$ attains at $\hat{y} \in (0, x_c - \varepsilon)$ a maximum, we deduce from Definition 6.1 that

$$H(\hat{y}, -L\frac{\hat{x} - \hat{y}}{|\hat{x} - \hat{y}|}) \leq 0,$$

which cannot be true, in view of (H2), when L is large. □

6.1.2. Comparison Principle. To prove comparison principles for the stationary equation, we add an assumption on strict monotonicity in the space variable:

(H4) There exists $l > 0$ such that, for all $x \in (0, x_c]$ and $p \in \mathbb{R}$ satisfying $H(x, p) = 0$, we have

$$H(x_1, p) > H(x_2, p)$$

whenever $x_1, x_2 \in (x - l, x + l) \cap [0, x_c]$ and $x_1 < x_2$.

THEOREM 6.3 (Comparison theorem of (6.1)). *Assume* (H1)–(H4). *Let u and v be respectively a subsolution and a supersolution of* (6.1). *Assume that there exists $M > 0$ such that $\overline{u} \leq M$ and $\underline{v} \geq -M$. If $\overline{u}(0) \leq \underline{v}(0)$, then $\overline{u} \leq \underline{v}$ in $[0, x_c)$.*

REMARK 6.1. We only conduct our comparison in $(0, x_c)$. It is obvious that $\overline{u} \leq \infty = \underline{v}$ in (x_c, ∞). However, it is not necessarily true that $\underline{v}(x_c) \geq \overline{u}(x_c)$. In fact, through the Figure 1, which roughly shows the graph of a solution u, we easily observe that $u(x_c)$ can be modified to be $-\infty$ or any value above $\limsup_{x \to x_c -} u(x)$.

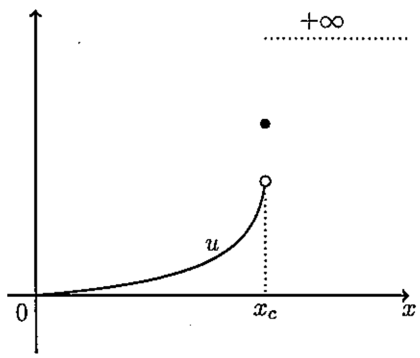

FIGURE 1. Graph of a solution u.

PROOF OF THEOREM 6.3. Without loss of generality, let us assume $\overline{u}(0) \leq \underline{v}(0) \leq 0$. Indeed, we may establish our comparison for $u + C$ and $v + C$ with any constant $C > 0$.

Take $0 < \lambda < 1$. Set $u_\lambda(x) = \frac{1}{\lambda}u(\lambda x)$ and $\overline{u}_\lambda = ((u_\lambda)_-)^*$. It is then obvious that $\overline{u}_\lambda(0) \leq \underline{v}(0)$.

In view of the boundary condition $\overline{u}(0) \leq \underline{v}(0)$, we assume by contradiction that there exist $\xi_0 \in (0, x_c)$ and $\mu > 0$ such that $\overline{u}(\xi_0) - \underline{v}(\xi_0) \geq 2\mu$. Our dilation preserves a positive maximum of $\overline{u} - \underline{v}$; that is,

$$\overline{u}_\lambda(\xi_0) - \underline{v}(\xi_0) \geq \mu \tag{6.5}$$

if we take λ close to 1. Indeed, a direct calculation indicates

$$\overline{u}_\lambda(\xi_0) - \underline{v}(\xi) = \frac{1}{\lambda}\overline{u}(\lambda\xi_0) - \underline{v}(\xi_0) = \overline{u}(\xi_0) - \underline{v}(\xi_0) + \left(\frac{1}{\lambda}\overline{u}(\lambda\xi_0) - \overline{u}(\xi_0)\right),$$

which, together with the continuity of $\overline{u}$ obtained in Lemma 6.2, yields (6.5).

For every $\varepsilon > 0$, we double variables by setting an auxiliary function $\Phi_\varepsilon(x, y) = \overline{u}_\lambda(x) - \underline{v}(y) - \frac{1}{2\varepsilon}|x - y|^2$. It is clear that

$$\sup_{x,y\in[0,\infty)} \Phi_\varepsilon(x, y) \geq \mu.$$

We next claim that the maximum points $(\xi_\varepsilon, \eta_\varepsilon)$ of Φ_ε can be taken in a bounded interval. Usually, to do this, one needs another term to penalize at infinity when proving comparison theorems for an unbounded domain. We however do not, thanks to Lemma 6.1. The term $\frac{1}{2\varepsilon}|x - y|^2$ will in essence play the role of penalizing at space infinity. Following the above idea, we see that Lemma 6.1 yields

$$\sup_{x,y\in[0,\infty)} \Phi_\varepsilon(x, y) = \sup_{\substack{x\in[0,\infty)\\ y\in[0,x_c]}} \Phi_\varepsilon(x, y).$$

Then combined with the upper bound of $\overline{u}$ and $-\underline{v}$, the structure of Φ_ε refrains the supremum from being attained at a very large x when ε is sufficiently small, which implies the existence of ξ_ε and η_ε. In addition, a usual argument gives

$$|\xi_\varepsilon - \eta_\varepsilon| \leq C_1\varepsilon^{\frac{1}{2}} \text{ with } C_1 > 0,$$

and then by taking a subsequence, still indexed by ε, we let ξ_ε and η_ε converge to some $z \in [0, x_c]$. Since $\overline{u}_\lambda(0) \leq \underline{v}(0)$ and Φ_ε is upper semicontinuous in $[0, \infty)^2$, we must have $z > 0$.

Set $\varphi_1(x) := \lambda\underline{v}(\eta_\varepsilon) + \frac{1}{2\varepsilon\lambda}(x - \lambda\eta_\varepsilon)^2$ and $\varphi_2(y) := \overline{u}_\lambda(\xi_\varepsilon) - \frac{1}{2\varepsilon}|\xi_\varepsilon - y|^2$. The maximum of Φ at $(\xi_\varepsilon, \eta_\varepsilon)$ immediately implies that $\overline{u} - \varphi_1$ attains a maximum at $\lambda\xi_\varepsilon$ and $\underline{v} - \varphi_2$ attains a minimum at η_ε. We thus apply our definitions of subsolutions and supersolutions to get

$$H\left(\lambda\xi_\varepsilon, \frac{\xi_\varepsilon - \eta_\varepsilon}{\varepsilon}\right) \leq 0 \tag{6.6}$$

and

$$H\left(\eta_\varepsilon, \frac{\xi_\varepsilon - \eta_\varepsilon}{\varepsilon}\right) \geq 0. \tag{6.7}$$

Owing to (H2), $\frac{1}{\varepsilon}(\xi_\varepsilon - \eta_\varepsilon)$ must be bounded, for otherwise along the subsequence such that it diverges the limit of (6.6) as $\varepsilon \to 0$ gives rise to a contradiction. So we discuss all the converging subsequences for $\frac{1}{\varepsilon}(\xi_\varepsilon - \eta_\varepsilon)$ as $\varepsilon \to 0$. Denote their limits, which depends on λ, by q_λ. By (H1), we then have

$$H(\lambda z, q_\lambda) \leq 0 \tag{6.8}$$

and

$$H(z, q_\lambda) \geq 0. \tag{6.9}$$

Since the Hamiltonian is continuous, there exists $x_\lambda \in [\lambda z, z]$ such that $H(x_\lambda, q_\lambda) = 0$. Take subsequences of x_λ as $\lambda \to 1$ so that there exist $x \in (0, x_c]$ fulfilling $x_\lambda \to x$. Meanwhile, since λz and z are lying in $(x_\lambda - l, x_\lambda + l)$, taking difference of (6.8) and (6.9) and using the assumption (H4) (with $x = x_\lambda$, $x_1 = \lambda z$, $x_2 = z$ and $p = q_\lambda$), we are led to a contradiction

$$0 < H(\lambda z, q_\lambda) - H(z, q_\lambda) \leq 0.$$

□

COROLLARY 6.4 (Uniqueness of solutions). *Assume* (H1)–(H4). *The solutions u of* (6.1) *with* $u(0) = 0$ *are unique in the sense that if u, v are solutions bounded in $[0, x_c)$, then $\overline{u} = \overline{v}$ and $\underline{u} = \underline{v}$ in $[0, x_c) \cup (x_c, \infty)$.*

6.1.3. Comparison Principle for More Singular Solutions.

As we have mentioned, we can also deal with the stationary problem whose solutions blow up on the boundary of the effective domain. However, on this occasion the solutions as in Definition 6.3 may not be unique, as shown in the following simple example.

EXAMPLE 6.5. *When* $H(x,p) = \arctan|p| - x$, *we easily find that*

$$\int_0^x \tan y \, dy \quad \textit{and} \quad - \int_0^x \tan y \, dy \quad (\textit{for } x \in [0, \pi/2))$$

are both solutions of the stationary problem under our present definition. We will see in a moment that the long time limit of our time-dependent problem is actually the former. We therefore choose to only consider the solutions bounded from below.

THEOREM 6.6. *Assume* (H1)–(H4). *Let u and v be respectively a subsolution and a supersolution of (6.1). Assume in addition that u is bounded from below and there exists $\gamma < 0$ such that $\overline{u}(x) - \beta|x - x_c|^\gamma$ is bounded from above for any $\beta > 0$. If $\overline{u}(0) \leq \underline{v}(0)$, then $\overline{u} \leq \underline{v}$ in $[0, x_c)$.*

REMARK 6.2. The hypothesis that $\overline{u}(x) - \beta|x - x_c|^\gamma$ has an upper bound is just technical. We impose it for our convenience to deal with unbounded solutions.

PROOF OF THEOREM 6.6. We only need to modify slightly the proof of Theorem 6.3. Assume as before that there exist $\xi_0 \in (0, x_c)$ and $\mu > 0$ such that

$$\overline{u}_\lambda(\xi_0) - \underline{v}(\xi_0) \geq \mu > 0, \quad (u_\lambda = \frac{1}{\lambda} u(\lambda x), \quad 0 < \lambda < 1).$$

This time, we set

$$\Phi_\varepsilon(x, y) = \overline{u}_\lambda(x) - \underline{v}(y) - \frac{1}{2\varepsilon}|x - y|^2 - \beta|x - \frac{x_c}{\lambda}|^\gamma.$$

Then Φ_ε is bounded from above and greater than $\frac{\mu}{2}$ provided that λ is close to 1 and $\beta > 0$ is small.

Fix such a $\beta > 0$. Let $(\xi_\varepsilon, \eta_\varepsilon)$ be the maximizers of Φ_ε. It is clear, due to Lemma 6.1, that $\eta_\varepsilon \in [0, x_c]$. We stress that this is actually true for arbitrary $\beta > 0$. Since $\overline{u}_\lambda(x) - \underline{v}(y) - \beta|x - \frac{x_c}{\lambda}|^\gamma$ is bounded from above, we have

$$\xi_\varepsilon - \eta_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0.$$

Assume $\xi_\varepsilon, \eta_\varepsilon \to z \in [0, x_c]$. It is easily seen that $z \neq 0$ because of the comparison hypothesis on the boundary. Now we apply our definition of sub- and supersolutions. A direct computation gives

$$\frac{\xi_\varepsilon - \eta_\varepsilon}{\varepsilon} + X_\beta^\varepsilon \in D^+\overline{u}(\lambda\xi_\varepsilon) \text{ and } \frac{\xi_\varepsilon - \eta_\varepsilon}{\varepsilon} \in D^-\underline{v}(\eta_\varepsilon),$$

where $X_\beta^\varepsilon = \beta|\xi_\varepsilon - \frac{x_c}{\lambda}|^{\gamma-2}(\xi_\varepsilon - \frac{x_c}{\lambda})$. Hence we obtain

$$H(\lambda\xi_\varepsilon, \frac{\xi_\varepsilon - \eta_\varepsilon}{\varepsilon} + X_\beta^\varepsilon) \leq 0 \text{ and } H(\eta_\varepsilon, \frac{\xi_\varepsilon - \eta_\varepsilon}{\varepsilon}) \geq 0.$$

Send $\varepsilon \to 0$, passing to subsequences if necessary. Then $X_\beta^\varepsilon \to X_\beta := \beta|z - \frac{x_c}{\lambda}|^{\gamma-2}(z - \frac{x_c}{\lambda})$. If $\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} \to \infty$, a contradiction from (H2) is clear. If otherwise the limit is $q_\lambda \in \mathbb{R}$, then

$$H(\lambda z, q_\lambda + X_\beta) \leq 0 \text{ and } H(z, q_\lambda) \geq 0.$$

Noticing that $z - \frac{x_c}{\lambda} \leq \frac{\lambda-1}{\lambda}x_c$ uniformly in β, we have, with $\beta \to 0$,

$$H(\lambda z, q_\lambda) \leq 0 \text{ and } H(z, q_\lambda) \geq 0,$$

which is a contradiction to (H4) when λ approaches to 1. ☐

COROLLARY 6.7. *Assume* (H1)–(H4). *The solutions u of* (6.1)) *with* $u(0) = 0$ *are unique in the sense that if u, v are solutions bounded from below, and there exists $\gamma < 0$ such that $\overline{u}(x) - \beta|x - x_c|^\gamma$ and $\overline{v}(x) - \beta|x - x_c|^\gamma$ are both bounded from above for any $\beta > 0$ in $[0, x_c)$, then $\overline{u} = \overline{v}$ and $\underline{u} = \underline{v}$ in $[0, x_c) \cup (x_c, \infty)$.*

6.1.4. Examples.

We give several concrete examples for the stationary problem we study.

EXAMPLE 6.8. *Take $p_0 \in \mathbb{R}$ and $\alpha > 0$. It is not hard to verify that*

$$H(x, p) = \max\{2 - x, 0\}\frac{|p - p_0|^\alpha}{1 + |p - p_0|^\alpha} - 1$$

satisfies all the assumption (H1)–(H4) with $x_c = 1$. The unique solution, bounded from below, of (6.1) with Dirichlet condition $u(0) = 0$ is

$$u(x) = \begin{cases} p_0 x - \ln(1 - x), & \text{if } \alpha = 1; \\ p_0 x - \frac{\alpha}{\alpha - 1}[(1 - x)^{1 - 1/\alpha} - 1], & \text{if } \alpha \neq 1. \end{cases}$$

for $x \in (0, 1)$. It is worth noticing that this example contains two different cases. When $0 < \alpha \leq 1$, the solution blows up at $x = 1$. When $\alpha > 1$, the solution is continuous and bounded in $[0, 1]$ but its derivative tends to infinity as $x \to 1$. The uniqueness of solutions is guaranteed either by Theorem 6.3 or by Theorem 6.6.

The following example indicates that even $H(x, p)$ is not bounded, our conclusion can still be true.

EXAMPLE 6.9. *For $H(x,p) = (1-x)^\alpha|p| - 1$, which is coercive only locally away from $x = 1$, if for instance $\alpha = \frac{1}{3}$ or 3, then the unique solution (bounded from below) in the effective domain $(0,1)$ is*

$$u(x) = -\frac{1}{1-\alpha}(1-x)^{1-\alpha} + \frac{1}{1-\alpha} \quad \text{for } x \in (0,1).$$

We do not study this kind of Hamilton-Jacobi equations in the thesis, but such generalization is possible.

We next show that the assumption (H4) is necessary for the uniqueness.

EXAMPLE 6.10. *Consider the following Hamiltonian*

$$H(x,p) = -\frac{1}{1+|p|}(x-1)^2 + \frac{|p|}{1+|p|}(1-x)^\alpha.$$

We take $\alpha = 1$ and then we get two solutions, which in $(0,1)$ are

$$u(x) = \pm\left(\frac{1}{2}x^2 - x\right).$$

The same situation takes place for other choices of α like $7/3$ and 3 as well. The reason for nonuniqueness is that H does not satisfy (H4). Note that $x \mapsto H(x,0)$ is not strictly decreasing at $x = 1$ while $H(1,p) = 0$ for all $p \in \mathbb{R}$.

EXAMPLE 6.11. *Another example of Hamiltonian to show the necessity of (H4) is*

$$H(x,p) = \max\left\{\left(\arctan|p|^2 - x\right), 0\right\} + \min\left\{\left(\frac{\pi}{2} - x\right), 0\right\}.$$

It is not difficult to see that $x_c = \pi/2$ and there are infinitely many bounded solutions including

$$u(x) \equiv 0 \text{ and } u(x) = \pm\int_0^x (\tan y)^{\frac{1}{2}}\,dy.$$

in $(0, \pi/2)$. All of the assumptions except (H4) are satisfied.

6.2. Large-time behavior

We discuss in this section the asymptotic behavior of the solution of (CD) as $t \to \infty$. Let us impose further assumptions:

- (H5) There exists $p_1 \in C([0,x_c)) \cap L^1(0,x_c)$ such that $H(x,p_1(x)) \leq 0$ for all $x \in [0,x_c)$.
- (H6) There exist $\gamma_0 < 0$ and $C_0 > 0$ such that $H(x,p) \geq 0$ for all $x \in [0,x_c)$ and $p \geq C_0|x - x_c|^{\gamma_0 - 1}$.

These assumptions enable us to construct subsolutions and supersolutions for the Cauchy problem and thus to specify the long time profile of the solutions. Another viewpoint is that the assumption (H5) and (H6) give a subsolution and a supersolution for the stationary problem (6.1) with homogeneous Dirichlet condition at $x = 0$ and by Perron's method the existence of stationary solutions, missing in the last section, follows immediately.

6.2.1. Large-time Asymptotics. We adapt our analysis to the setting of solutions which are not necessarily continuous for the following two reasons: (a) to find asymptotic behavior for a discontinuous solution itself is interesting; (b) constructing semicontinuous subsolutions and supersolutions is comparatively easier when we are looking for precise bounds of the solutions of (CD).

Our definition of the solutions of Cauchy-Dirichlet problem is conventional, without the infinity value being involved. For readers' convenience, we provide in what follows the definition of solutions of

$$u_t + H(x, u_x) = 0 \quad \text{in } (0,\infty)\times(0,\infty), \tag{6.10a}$$

$$u(0,t) = 0 \quad \text{for } t \in (0,\infty), \tag{6.10b}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in [0,\infty). \tag{6.10c}$$

This version of definitions is essentially due to Ishii [**65**].

DEFINITION 6.4. Given any locally bounded function $u_0 : [0,\infty) \to \mathbb{R}$, a locally bounded function u on $([0,\infty)\times[0,\infty))$ is called a subsolution (resp., supersolution) of (6.10a)–(6.10b) if

(i) $u^*(0,t) \le 0$ for all $t \in [0,\infty)$
(resp., $u_*(0,t) \ge 0$ for all $t \in [0,\infty)$);

(ii) whenever there exist $\varphi \in C^1([0,\infty))$ and $(\hat{x},\hat{t}) \in (0,\infty)\times(0,\infty)$ satisfying

$$\max_{(x,t)\in[0,\infty)\times[0,\infty)} (u^* - \varphi)(x,t) = (u^* - \varphi)(\hat{x},\hat{t})$$

$$\left(\text{resp.,} \min_{(x,t)\in[0,\infty)\times[0,\infty)} (u_* - \varphi)(x,t) = (u_* - \varphi)(\hat{x},\hat{t})\right),$$

then

$$\varphi_t(\hat{x},\hat{t}) + H(\hat{x},\varphi_x(\hat{x},\hat{t})) \le 0 \tag{6.11}$$

$$\left(\text{resp.}, \varphi_t(\hat{x},\hat{t}) + H(\hat{x},\varphi_x(\hat{x},\hat{t})) \ge 0\right). \tag{6.12}$$

A locally bounded function $u : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is called a solution if u is both a subsolution and a supersolution.

DEFINITION 6.5. We call u a solution of (CD) provided that u is a solution in the sense of Definition 6.4 and $u_{0*}(x) \le u_*(x,0) \le u^*(x,0) \le u_0^*(x)$ for all $x \in [0,\infty)$.

Our main characterization of the large-time behavior is as follows.

THEOREM 6.12 (Main theorem). *Assume* (H1)–(H6). *Assume that u_0 is a locally bounded function which satisfies* (5.15) *with the function p_1 given in* (H6). *Let u be a solution of* (CD). *Then for all $x \in [0,x_c)\cup(x_c,\infty)$, $u(x,t) \to v(x)$ as $t \to \infty$, where v is the unique solution of* (6.1) *with $v(0) = 0$ and takes value $+\infty$ in (x_c,∞). Moreover, the convergence is locally uniform respectively in $[0,x_c)$ and in (x_c,∞).*

REMARK 6.3. The theorem states that the asymptotic profile v is independent of u_0. It is continuous in $[0,x_c)$ even if u_0 is not continuous in $(0,\infty)$. In addition, when $u_0 \in C([0,\infty))$, the solutions u are unique and continuous as well. Theorem 6.12 then reduces to a result of long time behavior for the continuous solution as usual.

The proof, which is standard now, will be given in the next subsections. The point is to build subsolutions and supersolutions, which both have different behavior in $(0, x_c)$ and (x_c, ∞).

EXAMPLE 6.13. *Let us see a simple example:*

$$\begin{cases} u_t + \arctan u_x^2 - x = 0 & \text{in } (0,\infty)\times(0,\infty), \\ u(0,t) = 0 & \text{for all } t \in (0,\infty), \\ u(x,0) = u_0(x) & \text{for } x \in (0,\infty). \end{cases} \tag{6.13}$$

Now that $x_c = \pi/2$*, the explicit solution of the stationary equation is*

$$v(x) = \int_0^{|x|} (\tan y)^{\frac{1}{2}}\, dy$$

for all $x \in [0, \pi/2)$*. Theorem 6.12 gives the long time behavior that as* $t \to \infty$*,* $u(x,t) \to v(x)$ *for* $x \in [0, \pi/2]$ *and* $u(x,t) \to \infty$ *for* $x > \pi/2$*. But we only allow the initial data satisfying* $u_0(0) = 0$ *and the compatibility condition*

$$u_0(x) \geq -\int_0^x (\tan y)^{\frac{1}{2}}\, dy \quad \text{in } (0, \pi/2).$$

6.2.2. Construction of Sub- and Supersolutions. We construct a subsolution and a supersolution of (CD) to bound the solution. Our choice of the lower bound is an upper semicontinuous function in the form of

$$w_-(x,t) = \begin{cases} \int_0^x p_1(z)\, dz, & \text{if } 0 \leq x \leq x_c; \\ W_1(x) - h_1(x)t, & \text{if } x > x_c, \end{cases} \tag{6.14}$$

where $h_1(x) := \sup_{p\in\mathbb{R}} H(x,p) < 0$ for all $x \in (x_c, \infty)$ by (H3) and $W_1 \in USC([0,\infty))$ will be determined later. On the other hand, an upper bound is written as

$$w_+^\delta(x,t) = \begin{cases} W_2(x) & \text{if } 0 \leq x \leq x_c - \delta; \\ W_2(x) - h_2(x)t, & \text{if } x > x_c - \delta, \end{cases} \tag{6.15}$$

where $\delta > 0$ is taken small, $h_2(x) := \inf_{p\in\mathbb{R}} H(x,p) < 0$ for all $x \in (x_c, \infty)$ and W_2 is also to be determined in terms of the initial data u_0.

The construction of a subsolution is simpler.

LEMMA 6.14 (A lower bound). *Assume* (H5). *For any locally bounded lower semicontinuous function* $f : [0,\infty) \to \mathbb{R}$ *which satisfies*

$$f(x) \geq \int_0^x p_1(z)\, dz \quad \text{for all } 0 \leq x \leq x_c,$$

let $W_1 \in C([0,\infty)) \cap C^1((0,\infty))$ *be such that* $W_1(x) = \int_0^x p_1(z)\, dz$ *and* $W_1 \leq f$ *in* (x_c, ∞)*. Then* $w_- \in USC([0,\infty) \times [0,\infty))$ *as defined in* (6.14) *is a subsolution of* (6.10a) *with* $w_-(0,t) = 0$ *and* $w_-(x,0) \leq f(x)$ *for all* $x, t \in [0, \infty)$*.*

PROOF. It is clear that $w_-(x,0) \leq f(x)$ and w_- is continuous except at $x = x_c$. By the definition of p_1, we obtain with great ease that

$$(w_-)_t(x,t) + H(x, (w_-)_x(x,t)) = H(x, p_1(x)) \leq 0, \text{ for } x \in (0, x_c).$$

For every $x \in [x_c, \infty)$, whenever there exist a test function $\varphi \in C^1([0,\infty)^2)$ and $t > 0$ satisfying

$$(w_- - \varphi)(x,t) = \max_{(y,s)\in(0,\infty)^2}(w_- - \varphi)(y,s),$$

we immediately have $\varphi_t(x,t) = -h_1(x)$. It then follows that

$$\varphi_t(x,t) + H(x, \varphi_x(x,t)) \le -h_1(x) + \sup_{p\in\mathbb{R}} H(x,p) = 0.$$

□

LEMMA 6.15 (An upper bound). *Assume* (H6). *Then for any locally bounded upper semicontinuous function $f : [0,\infty) \to \mathbb{R}$ with $f(0) \le 0$, there exists $W_2 \in LSC([0,\infty))$ such that for every small $\delta > 0$, $w_+^\delta \in LSC([0,\infty) \times [0,\infty))$ is a supersolution of* (6.10a) *and satisfies $w_+^\delta(0,t) = 0$ and $w_+^\delta(x,0) \ge f(x)$ for all $x, t \in [0,\infty)$.*

PROOF. Take a constant $\delta < r < x_c$. We build W_2 in a proper way according to f so that w_+^δ has jumps merely on $x = x_c - r$ and $x = x_c - \delta$.

In terms of the assumptions given, it is possible to choose a nondecreasing function $f_1 \in C([0,\infty))$ satisfying $f_1(0) = 0$ and $f_1 \ge f$ as well as

$$f_1(x_c - r) \ge \max_{[x_c - r, x_c]} f. \tag{6.16}$$

We construct W_1 in three intervals.

(i) For $x \in (0, x_c - r]$, we use a variation of the method to regularize a modulus. Consult [**51**, Lemma 2.1.9] for more details. Let

$$f_2(x) = \left(\max_{y\in[x,x_c-r]} \frac{f_1(y)}{y}\right) \quad \text{for all } x \in (0, x_c - r].$$

and $F(x) = \int_{x/2}^{x} 2f_2(y)\,dy$. Then it is easy to show that $F \in C^1((0, x_c - r]) \cap C([0, x_c - r])$ with $F(0) = 0$ and $F(x) \ge f_1(x)$ for $x \in [0, x_c - r]$. However F may not be a stationary supersolution in $(0, x_c - r) \times (0,\infty)$. To overcome this, notice in (H6) that there is $\hat{p} > 0$ such that $\max_{x\in[0,x_c-\delta]} H(x,p) \ge 0$ for all $p \ge \hat{p}$. We next only need to set

$$W_2(x) := \int_0^x \max\left\{\frac{dF}{dy}(y), \hat{p}\right\} dy$$

and it is then obvious that w_+^δ is a supersolution in $(0, x_c - r) \times (0,\infty)$.

(ii) For $x \in (x_c - r, x_c - \delta]$, set $W_2(x) := C_1|x - x_c|^{\gamma_0} + C_2$. It then follows by (H6) that

$$H(x, \frac{d}{dx}W_2(x)) \ge 0 \quad \text{for } x \in (x_c - r, x_c - \delta), \tag{6.17}$$

and $\lim_{x\to(x_c-r)+} W_2(x) \ge \lim_{x\to(x_c-r)-} W_2(x_c - r)$ when C_1, C_2 are sufficiently large without depending on δ. In addition, in view of (6.16) we have

$$w_+^\delta(x,0) \ge f_1(x_c - r) \ge f(x) \quad \text{for } x \in [x_c - r, x_c].$$

(iii) For $x \in (x_c - \delta, \infty]$, we extend W_2 so that $W_2(x) \ge f(x)$ and estimate to get

$$(w_+^\delta)_t + H(x, (w_+^\delta)_x) \ge -h_2(x) + \inf_{p\in\mathbb{R}} H(x,p) = 0$$

for $(x,t) \in (x_c - \delta, \infty) \times (0,\infty)$.

It remains to show w_+^δ is a subsolution at $x = x_c - r$ and $x = x_c - \delta$, where the jumps of value take place. Suppose there is a test function φ touching w_+^δ from below at $x = x_c - r$, then our construction of W_2 yields $\varphi_x(x_c - r, t) \geq \hat{p}$ and hence by (H6)

$$\varphi_t(x_c - r, t) + H(x_c - r, \varphi_x(x_c - r, t)) \geq 0.$$

The same argument, together with an application of (H6), works for the verification at $x = x_c - \delta$ as well. □

We remark that the existence of solutions of (CD) follows easily since we have constructed the subsolution and supersolution associated with the classical boundary condition. More precisely, we can show that the functions

$$b_-(x,t) = w_-(x,t) \vee (u_0(x) - h_1(x)t) \tag{6.18}$$

and

$$b_+(x,t) = w_+^\delta(x,t) \wedge (u_0(x) - h_2(x)t) \tag{6.19}$$

are respectively a subsolution and a supersolution of (CD). Our assumptions (H2) and (H5) yield $b_- \leq b_+$, which enables us to show the existence of solutions of (CD) by Perron's method [**65**].

Before proving our main theorem, we need a comparison theorem for our Cauchy-Dirichlet problem.

THEOREM 6.16 (Comparison principle of (CD)). *Assume* (H1). *Let* u_1 *and* u_2 *be respectively a subsolution and a supersolution of* (6.10a)–(6.10b). *If* ${u_1}^*(x,0) \leq u_{2*}(x,0)$ *for all* $x \in [0,\infty)$, *then* ${u_1}^* \leq u_{2*}$ *in* $[0,\infty) \times [0,\infty)$.

REMARK 6.4. This theorem works for unbounded solutions. It is analogous to the following theorem for Cauchy problem presented in [**64**], which is reproduced in [**5**]. We apply their idea to the setting of semicontinuous solutions.

THEOREM 6.17 ([**5**, Theorem III.3.16]). *Let* $H : \mathbb{R}^{2n} \to \mathbb{R}$ *be continuous and satisfy*

$$|H(x,p) - H(x,q)| \leq K(|x| + 1)|p - q|$$

for all $x, p, q \in \mathbb{R}^n$ *and*

$$|H(x,p) - H(y,p)| \leq \omega(|x - y|, R) + \omega(|x - y||p|, R)$$

for all $p \in \mathbb{R}^n$, $x, y \in B_R(0), R > 0$. *Here* $\omega(\cdot, R)$ *is a modulus of continuity for any* $R > 0$. *If* $u_1, u_2 \in C(\mathbb{R}^n \times [0,T])$ *are respectively a viscosity sub- and supersolution of*

$$u_t + H(x, \nabla u) = 0 \quad \text{in } \mathbb{R}^n \times (0,T)$$

and $u_1(x,0) \leq u_2(x,0)$ *for all* $x \in \mathbb{R}^n$, *then* $u_1 \leq u_2$ *in* $\mathbb{R}^n \times [0,T]$.

The proof of Theorem 6.17, whose details are given in Appendix C, relies on "a cone of dependence" type argument. To prove Theorem 6.16, it consequently suffices to show the comparison theorem in any cone relative to our domain; namely, for any fixed $T > 0$ and $x_0 \geq 0$, let

$$\mathcal{C}_D = \{|x - x_0| \leq C|T - t| \text{ and } x \geq 0\},$$

with part of its boundary, lying on axes, $\Gamma_1 = \{(x,t) \in [0,\infty) \times \{0\} : |x - x_0| \leq CT\}$ and $\Gamma_2 = \{(x,t) \in \{0\} \times [0,\infty) : t \leq T - x_0/C\}$. We recall that $C > 0$ is the Lipschitz constant given in (H1).

LEMMA 6.18 (Local comparison in cones). *Let H satisfy* (H1). *Assume that u_1 and u_2 are respectively a viscosity subsolution and supersolution of* (6.10a)–(6.10b). *If $u_1{}^*(x,t) \leq u_{2*}(x,t)$ for all $(x,t) \in \Gamma_1 \cup \Gamma_2$, then $u_1{}^*(x,t) \leq u_{2*}(x,t)$ in $\mathcal{C}_D$.*

SKETCH OF PROOF. Assume by contradiction that there exist $0 < \delta < T$ and $(\tilde{x}, \tilde{t})$ such that

$$(u_1 - u_2)(\tilde{x}, \tilde{t}) = \delta \text{ and } |\tilde{x} - x_0| \leq C(T - \tilde{t}) - 2\delta$$

and $\sup_{(x,t,y,s)\in \mathcal{C}_D^2}(u_1(x,t) - u_2(y,s)) = N > 0$. Doubling the variables, we define a function

$$\begin{aligned}\Phi(x,y,t,s) =& u_1(x,t) - u_2(y,s) - \frac{|x-y|^2 + |t-s|^2}{2\varepsilon} - k(t+s)\\ &+ g(\langle x - x_0\rangle_\beta - C(T-t)) + g(\langle y - x_0\rangle_\beta - C(T-s)),\end{aligned}$$

where $\varepsilon > 0$, $k > 0$, $\langle x\rangle_\beta := (x^2 + \beta^2)^{\frac{1}{2}}$ for $\beta > 0$ and $g \in C^1(\mathbb{R})$ is such that $g' \leq 0$ in $\mathbb{R}$, $g(x) = 0$ for $x \leq -\delta$ and $g(x) = -3N$ for $x \geq 0$.

The harsh penalization with the utilization of g and comparison conditions along $x = 0$ and $t = 0$ eliminate the possibility of finding maximizers of Φ on any edge of $\mathcal{C}_D$ when k and β are sufficiently small. Then the standard arguments come into play such as showing convergence of maximizers and taking difference of two viscosity inequalities. □

6.2.3. Proof of Main Theorem.

We take the relaxed limits

$$U_1 = \limsup_{t\to\infty}{}^* u \text{ and } U_2 = \liminf_{t\to\infty}{}_* u$$

and show that they have the following properties.

PROPOSITION 6.19 (Properties of half relaxed limits). *Assume that u_0 satisfies* (5.15). *Then U_1 and U_2 have the following properties:*

(i) *U_1 and U_2 are both bounded from below in $[0, x_c]$;*
(ii) *If $\gamma < \gamma_0$, then $U_1 - \beta|x - x_c|^\gamma$ and $U_2 - \beta|x - x_c|^\gamma$ are both bounded from above in $[0, x_c)$ for any $\beta > 0$;*
(iii) *$U_1(0) \leq 0 \leq U_2(0)$.*
(iv) *$U_1 = U_2 = +\infty$ in (x_c, ∞).*

PROOF. Take $f = u_{0*}$ in Lemma 6.14 and $f = u_0{}^*$ in Lemma 6.15 and we therefore note that $u_0(0) = w_-(0,t) = w_+(0,t)$ for all $t \in [0,\infty)$ and

$$w_-(x,0) \leq u_{0*}(x) \leq u_0{}^*(x) \leq w_+(x,0) \quad \text{for all } x \in [0,\infty).$$

Hence, by Theorem 6.16, we have

$$w_- \leq u \leq w_+ \text{ in } [0,\infty) \times [0,\infty).$$

All of the statements thus follow easily by our choices of w_- and w_+. To prove (ii) for example, one can actually show $U_1, U_2 \leq C_1|x - x_c|^{\gamma_0} + C_2$ for any $x \in (0, x_c)$ due to our concrete form of w_+^δ in Lemma 6.15. Then the assertion becomes clear. □

PROPOSITION 6.20 (Viscosity inequalities for half limits). *U_1 and U_2 are respectively a subsolution and a supersolution of (6.1).*

PROOF. Let us prove the part for U_1 first. Since Proposition 6.19(iv) holds, there is no need to apply Definition 6.2 in (x_c, ∞). We thus turn our attention to the interval $(0, x_c]$. By definition, we need to take test functions for $\overline{U_1}$.

For any $\phi \in C^1([0,\infty))$ such that $\overline{U_1}(x) - \phi(x)$ attains a strict minimum at $x = x_c$, by (H1) and (H3), we have

$$H(x_c, \phi_x(x_c)) \leq 0,$$

which means U_1 is a subsolution at $x = x_c$.

We finally handle the case when $\overline{U_1}$ is tested at $x \in (0, x_c)$. Notice that $U_1(x) = \overline{U_1}(x)$ in this case. Put $u^\varepsilon(x,t) = u(x, \frac{t}{\varepsilon})$ for every $x \in [0,\infty)$ and $t \in [0,\infty)$ and then further let

$$u_1(x,t) := (\limsup_{\varepsilon\to 0} {}^*u^\varepsilon)(x,t) \text{ and } u_2(x,t) := (\liminf_{\varepsilon\to 0} {}_*u^\varepsilon)(x,t).$$

It is easily seen that u^ε solves

$$\varepsilon u^\varepsilon_t + H(x, u^\varepsilon_x) = 0 \quad \text{in } (0, x_c) \times (0, \infty),$$

so the stability of viscosity solutions guarantees

$$H(x, (u_1)_x) \leq 0 \quad \text{in } (0, x_c) \times (0, \infty)$$

in the viscosity sense. Indeed, for any $\phi \in C^1([0,\infty) \times [0,\infty))$ such that $u_1 - \phi$ attains a strict maximum at $(\hat{x}, \hat{t}) \in (0, x_c) \times (0, \infty)$, there exist a sequence $\varepsilon_n \to 0$ and $(x_n, t_n) \to (\hat{x}, \hat{t})$ as $n \to \infty$ satisfying

$$(u^{\varepsilon_n})^*(x_n, t_n) \to u_2(\hat{x}, \hat{t});$$
$$((u^{\varepsilon_n})^* - \phi)(x_n, t_n) = \max_{[0,x_c)\times[0,\infty)} ((u^{\varepsilon_n})^* - \phi).$$

(Refer to [**51**, Lemma 2.2.5] for a very general version of this maximizer convergence result.) Then taking into account the fact that u is a solution of the Cauchy-Dirichlet problem, we obtain

$$\varepsilon_n \phi_t(x_n, t_n) + H(x, \phi_x(x_n, t_n)) \leq 0.$$

We conclude by passing to the limit $n \to \infty$ and noticing that $u_1 = U_1$ is independent of the variable t.

The other part of our statements about U_2 can be shown more easily thanks to $U_2 = \underline{U_2}$. □

PROOF OF THEOREM 6.12. Combining Theorem 6.6, Propositions 6.19 and 6.20, we are led to

$$U_1 \leq U_2 \text{ in } [0, x_c).$$

which obviously implies, for $x \in [0, x_c)$, $U_1(x) = U_2(x) = v(x)$ and thus $u(x,t) \to v(x)$ locally uniformly.

On the other hand, observing the concrete form of w_- and w_+^δ, we get $u(\cdot, t) \to +\infty$ locally uniformly in (x_c, ∞) as $t \to \infty$. □

6.3. A remark on the initial data

Since our results are only for the Dirichlet boundary condition in the strict sense, several assumptions should be viewed as the compatibility condition on the boundary. It is certainly interesting to understand the long time behavior when these assumptions are dropped. Let us repeat our assumption on the initial data in Theorem 6.12. We only treat u_0 which satisfies the conditions below.

(a) u_0 is locally bounded in $[0, \infty)$ and is continuous at $x = 0$ with $u_0(0) = 0$.

(b) $u_{0*}(x) \geq \int_0^x p_1(z)\, dz$ for all $x \in [0, x_c)$, where p_1 is given in (H5).

The condition (b) plays an important role and can hardly be relaxed especially when p_1 is taken minimal. To see this, we give a simple example in the following.

EXAMPLE 6.21. *Let us revisit Example 6.13. Suppose* $H(x,p) = \arctan(p^2) - x$. *Then all assumptions (H1)–(H6) are satisfied. In particular,* $x_c = \pi/2$ *and* $p_1(x) = -(\tan(x))^{\frac{1}{2}}$ *in this case. Since such a Hamiltonian is smooth, the equation of its characteristics writes*

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{2p}{1+p^4} \\ \dfrac{dp}{dt} = 1 \end{cases}$$

with prescribed initial data

$$x(0) = x_0 \text{ and } p(0) = p_0.$$

Its solution could be explicitly calculated as

$$x(t) = x_0 + \arctan(p_0 + t)^2 - \arctan p_0^2 \quad \text{and} \quad p(t) = p_0 + t,$$

which evidently demonstrates that the trajectory $x(t)$ *starting from* $x_0 \in (0, \pi/2)$ *will hit the boundary* $x = 0$ *at time* $t = -p_0$ *whenever* $p_0 < p_1(x_0)$. *On this occasion, the solution* u *must violate the classical Dirichlet boundary condition.*

The seemingly particular value $p_1(0) = 0$ in the example above does not actually cause any loss of generality. One may observe analogous examples such as $H(x,p) = \arctan(p^2 - a^2)^2 - x$, where $p_1(0) = -|a|$ with $a \in \mathbb{R}$ arbitrarily chosen.

CHAPTER 7

Cauchy Problem

This chapter is devoted to the study of large time behavior of solutions of the Cauchy problem for Hamilton-Jacobi equations

$$\text{(C)} \qquad \begin{cases} u_t + \sigma(x)m(\|p\|) - f(x) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \quad (7.1) \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^n. \quad (7.2) \end{cases}$$

Here $\|\cdot\|$ denotes an arbitrary norm in $\mathbb{R}^n$ while $\sigma, f : \mathbb{R}^n \to [0,\infty)$ and $m : [0,\infty) \to [0,1)$ are continuous functions. Moreover $m(r)$ is assumed to be

$$\text{strictly increasing and } m(r) \to 1 \text{ as } r \to \infty.$$

The functions σ, f and m are given functions. The function $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is an unknown function while $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a given initial value which is assumed to be Lipschitz continuous.

7.1. Stationary problem with singular Neumann condition

In this section we consider singular Neumann problems

$$\text{(N)} \qquad \begin{cases} F(x, \nabla u) = h(x) & \text{in } \Omega, \quad (7.3) \\ \dfrac{\partial u}{\partial \nu} = +\infty & \text{on } \partial\Omega, \quad (7.4) \\ \sup_{x\in\Omega} |u(x)| < +\infty, & \quad (7.5) \end{cases}$$

and we present the definition, existence, stability and comparison results of solutions of (N). Our definition for the singular boundary condition in this chapter follows the work of Lasry and Lions [**78**], which requires the solutions u to satisfy

$$u - \phi \text{ never has a local minimum on } \overline{\Omega}$$
$$\text{at the boundary } \partial\Omega \text{ for any } \phi \in C^1(\overline{\Omega}). \qquad (7.6)$$

We shall use the following assumptions in this section.

(A1) $F \in C(\overline{\Omega} \times \mathbb{R}^n)$ and $h \in C(\Omega)$.

(A2) F is coercive with respect to p variable uniformly for $x \in \overline{\Omega}$, i.e.,

$$\inf\{F(x,p) \mid x \in \overline{\Omega}, |p| \geq r\} \to +\infty \text{ as } r \to \infty.$$

(A3) For any $x \in \overline{\Omega}$, $p \mapsto F(x,p)$ are convex functions.

(A4) $F(x,p) \geq F(x,0) = 0$ for any $(x,p) \in \overline{\Omega} \times \mathbb{R}^n$.

(A5) The set $\mathcal{A}_h \neq \emptyset$, where $\mathcal{A}_h := \{x \in \overline{\Omega} \mid h(x) = 0\}$.

(A6) $h(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$.

(A7) There exists a modulus ω such that

$$\int_0^1 h(sx + (1-s)y)|x-y|\, ds \leq \omega(|x-y|)$$

for all $x, y \in \Omega$ such that $[x, y] \subset \Omega$.

(A8) Ω is bounded and $\partial\Omega$ is locally represented as the graph of a continuous function, i.e., for each $z \in \partial\Omega$ there exist $r > 0$ and a function $b \in C(\mathbb{R}^{n-1})$ such that —upon relabelling and re-orienting the coordinates axes if necessary— we have

$$\Omega \cap B_r(z) = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x \in B_r(z), x_n > b(x')\}.$$

REMARK 7.1. Assumption (A7) is a kind of growth condition on h. In the case where $N = 1$, if $h \in C((a, b)) \cap L^1(a, b)$ with $a < b$, then (A7) holds. Indeed, setting

$$\omega(r) := \sup_{x \in (a,b)} \int_x^{x+r} \overline{h}(s)\, ds,$$

where $\overline{h} : \mathbb{R} \to \mathbb{R}$ is the function defined by $\overline{h}(x) = h(x)$ for all $s \in (a, b)$ and $\overline{h}(x) = 0$ for all $s \in \mathbb{R} \setminus (a, b)$, we have

$$\int_0^1 |h(sx + (1 - s)y)||x - y|\, ds \le \left| \int_y^x |h(r)|\, dr \right| \le \omega(|x - y|)$$

for any $x, y \in \mathbb{R}$ with $x \neq y$. In the case where $N \ge 2$, the integrability does not imply (A7) in general.

7.1.1. Definition, Existence and Stability. We use the following definition of solutions of (N) introduced in [**78**, Section V.1].

DEFINITION 7.1 (Definition of solutions of (N)). Let u be a function on $\overline{\Omega}$ with values in $\mathbb{R}$. We call u a subsolution of (N) if $u \in \mathrm{USC}\,(\overline{\Omega})$ is a viscosity subsolution of (7.3). We call u a supersolution of (N) if $u \in \mathrm{LSC}\,(\overline{\Omega})$ is a viscosity supersolution of (7.3) and satisfies (7.6). We call u a solution of (N) if $u \in C(\overline{\Omega})$ is a subsolution and a supersolution of (N).

THEOREM 7.1 (Existence of solutions). *Assume that* (A1)–(A6) *hold. There exists a viscosity solution $u \in C(\Omega)$ of* (7.3) *which satisfies* (7.6).

REMARK 7.2. We remark that solutions of (7.3) can blow up at some boundary point, if we do not assume (A7). Indeed, when we consider

$$|u_x| = \frac{1}{x^2} \quad \text{in } (0, 1),$$

then it is clear that the right-hand side does not satisfy (A7) and any solution u blow up at $x = 0$, i.e., $u(0) = +\infty$.

PROOF OF THEOREM 7.1. Take any function $w \in C^1(\Omega)$ such that

$$\begin{aligned} &w(x) \ge 0 \quad \text{for all } x \in \Omega, \\ &w(x) = 0 \quad \text{for all } x \in \mathcal{A}_h, \\ &w(x) \to +\infty \quad \text{as } \operatorname{dist}(x, \partial\Omega) \to 0. \end{aligned}$$

Set

$$w_\varepsilon(x) := \varepsilon w(x) \text{ and } h_\varepsilon(x) := F(x, Dw_\varepsilon(x)).$$

Since $Dw_\varepsilon(x) = \varepsilon Dw(x) \to 0$ locally uniformly in $x \in \Omega$ as $\varepsilon \to 0$ and $F(x, 0) = 0$ for all $x \in \overline{\Omega}$, we have $h_\varepsilon \to 0$ locally uniformly in $x \in \Omega$. It is clear that $h_\varepsilon \in C(\Omega)$ and $h_\varepsilon(x) \ge 0$ for all $x \in \Omega$. Note that

$$F(x, Dw_\varepsilon(x)) = h_\varepsilon(x) \le h(x) + h_\varepsilon(x) \quad \text{in } \Omega. \tag{7.7}$$

We define the functions $u_\varepsilon^\delta \in C(\overline{\Omega}_\delta)$ by

$$u_\varepsilon^\delta(x) := \min\{d_\varepsilon(x,y) + w_\varepsilon(y) \mid y \in \partial\Omega_\delta \cup \mathcal{A}_h\}$$

for small $\delta > 0$, where

$$d_\varepsilon(x,y) := \sup\{v(x) - v(y) \mid F(x, \nabla v) \le h(x) + h_\varepsilon(x) \text{ in } \Omega\},$$
$$\Omega_\delta := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \delta\}.$$

Note that $h_\varepsilon(x) = 0$ for all $x \in \mathcal{A}_h$ and therefore we have $\mathcal{A}_h = \{x \in \Omega \mid h(x) + h_\varepsilon(x) = 0\}$. Since the functions w_ε satisfy the compatibility condition

$$w_\varepsilon(x) - w_\varepsilon(y) \le d_\varepsilon(x,y) \quad \text{for all } x, y \in \partial\Omega_\delta \cup \mathcal{A}_h,$$

in view of [**71**, Theorem 3.3] we see that u_ε^δ is a unique viscosity solution of

$$F(x, \nabla u) = h(x) + h_\varepsilon(x) \quad \text{in } \Omega_\delta \tag{7.8}$$

which satisfies

$$u_\varepsilon^\delta(x) = w_\varepsilon(x) \quad \text{for all } x \in \partial\Omega_\delta \cup \mathcal{A}_h. \tag{7.9}$$

We extend u_ε^δ to a continuous function in Ω and denote it by u_ε^δ again.

Comparison result for (7.8) and (7.9) (see [**90**, Theorem 5.3]) implies that

$$(0 \le) w_\varepsilon \le u_\varepsilon^\delta \quad \text{on } \overline{\Omega}_\delta.$$

Fix $\delta_0 > 0$ so small that $\mathcal{A}_h \subset \Omega_{\delta_0}$. For any $\delta < \delta_0$, we have

$$F(x, \nabla u_\varepsilon^\delta) = h(x) + h_\varepsilon(x) \le \max_{\overline{\Omega}_{\delta_0}} (h + h_\varepsilon)(x) \quad \text{in } \Omega_{\delta_0}$$

in the viscosity sense. In view of the coercivity of F, we have

$$|\nabla u_\varepsilon^\delta(x)| \le C_{\delta_0} \quad \text{in } \Omega_{\delta_0}$$

for some C_{δ_0} and any $\delta \in (0, \delta_0)$. Therefore, we obtain the equi-continuity of $\{u_\varepsilon^\delta\}_{0<\delta<\delta_0}$ in Ω_{δ_0}. Since $u_\varepsilon^\delta = w_\varepsilon = 0$ on $\mathcal{A}_h$, $\{u_\varepsilon^\delta\}_{0<\delta<\delta_0}$ is uniformly bounded in Ω_{δ_0}. Moreover, we even have $u_\varepsilon^{\delta_2} \le u_\varepsilon^{\delta_1}$ on $\overline{\Omega}_{\delta_2}$ for any small δ_1, δ_2 with $\delta_1 < \delta_2$. Therefore, u_ε^δ converges to some $u_\varepsilon \in C(\Omega)$ locally uniformly in Ω as $\delta \to 0$, which is a solution of $F(x, \nabla u) = h(x) + h_\varepsilon(x)$ in Ω. Moreover, u_ε satisfies

$$u_\varepsilon(x) \to +\infty \quad \text{as } \operatorname{dist}(x, \partial\Omega) \to 0,$$

since $u_\varepsilon \ge w_\varepsilon$ in Ω.

Let K be a compact set in Ω. Then we have

$$|\nabla u_\varepsilon| \le C_K \text{ in } \operatorname{int} K \text{ in the viscosity sense and } |u_\varepsilon| \le C_K \text{ on } K$$

for some $C_K > 0$ which depends only on K, which implies that $\{u_\varepsilon\}_{\varepsilon>0}$ is relatively compact in $C(\Omega)$. Therefore, there exists $\{u_{\varepsilon_j}\}_{j\in\mathbb{N}} \subset \{u_\varepsilon\}_{\varepsilon>0}$ such that $u_{\varepsilon_j} \to u$ locally uniformly in Ω as $j \to \infty$ for some $u \in C(\Omega)$, which satisfies $F(x, \nabla u) = h(x)$ in Ω in the viscosity sense. We define the function $\underline{u} \in \mathrm{LSC}(\overline{\Omega})$ by

$$\underline{u}(x) := \begin{cases} u(x) & \text{for all } x \in \Omega \\ \liminf_{y \to x, y \in \Omega} u(y) & \text{for all } y \in \partial\Omega. \end{cases}$$

We finally prove that $\underline{u}$ satisfies (7.6). Let $x_0 \in \partial\Omega$ and then we can distinguish two cases: (i) $\underline{u}(x_0) = +\infty$; (ii) $-\infty < \underline{u}(x_0) < +\infty$. In case (i), it is easy to check that $\underline{u}$ satisfies (7.6).

In case (ii), let $\phi \in C^1(\overline{\Omega})$ and x_0 be a strict minimum over $\overline{\Omega}$ of $\underline{u} - \phi$. Since $u_{\varepsilon_j}(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$, $u_{\varepsilon_j} - \phi$ has a minimum over Ω at some point $x_{\varepsilon_j} \in \Omega$ and $x_{\varepsilon_j} \to x_0$ as $j \to \infty$. We have

$$F(x_{\varepsilon_j}, \nabla\phi(x_{\varepsilon_j})) \geq h(x_{\varepsilon_j}).$$

The above inequality yields a contradiction for a suitable small $\varepsilon > 0$, since $h(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$ and $|\nabla\phi(x)| < +\infty$ for all $x \in \overline{\Omega}$. □

The above idea of using the function w which blows up at any boundary points is used for the proof of approximations of solutions of stationary state constraint problem in [**22**, Theorem VII.3] and we also refer to [**78**] in different problems.

THEOREM 7.2 (Existence of bounded solutions). *Assume that* (A1)–(A8) *hold. There exists a solution* $u \in C(\overline{\Omega})$ *of* (N).

LEMMA 7.3 (Uniform continuity of subsolutions). *Assume that* (A1)–(A3), (A7) *and* (A8) *hold. Let* $u \in \mathrm{USC}(\overline{\Omega})$ *be a subsolution of* (7.3). *Then* u *is uniformly continuous in* Ω.

It is an immediate result of Proposition 7.6 below.

PROOF OF THEOREM 7.2. Let u be the function given in the proof of Theorem 7.1. By Lemma 7.3, u is uniformly continuous in Ω. Therefore, u can be extended uniquely to a function on $\overline{\Omega}$ by continuity and it is a solution of (N) as a continuous function on $\overline{\Omega}$. □

We give a stability result under infimum operation. It is worth mentioning that the stability property, without boundary condition, is the main technical observations in the theory of lower semicontinuous viscosity solutions due to Barron and Jensen [**16**]. We remark that Ishii gives this stability result (see [**69**, Theorem 2.1]) under the oblique Neumann condition.

THEOREM 7.4 (Stability under inf-operation). *Assume that* (A1), (A3) *and* (A6) *hold. Let* $\mathcal{S} \subset C(\overline{\Omega})$ *be a nonempty subset of solutions of* (N). *Set*

$$u(x) := \inf\{v(x) \mid v \in \mathcal{S}\}.$$

If u *is continuous on* $\overline{\Omega}$, *then* u *is a solution of* (N).

PROOF. By the standard stability property of viscosity solutions, we see that u is a supersolution of (7.3). Moreover, in view of the convexity of Hamiltonian, we see that u is a subsolution of (7.3), and therefore we only need to prove that u satisfies the boundary condition. We argue by contradiction. Suppose that there would exist $\phi \in C^1(\overline{\Omega})$ and $z \in \partial\Omega$ such that $u - \phi$ takes a strict minimum at z.

By the definition of u, for any $k \in \mathbb{N}$, there exists $v_k \in \mathcal{S}$ such that $u(z) > v_k(z) - 1/k$. We choose a sequence $\{y_k\}_{k\in\mathbb{N}} \subset \overline{\Omega}$ satisfying $(v_k - \phi)(y_k) = \min_{\overline{\Omega}}(v_k - \phi)$. Since v_k is a solution of (N), we have $\{y_k\}_{k\in\mathbb{N}} \subset \Omega$. Noting that

$$(u - \phi)(z) + \frac{1}{k} > (v_k - \phi)(z) \geq (v_k - \phi)(y_k) \geq (u - \phi)(y_k) \geq (u - \phi)(z),$$

we have $(u-\phi)(y_k) \to (u-\phi)(z)$ as $k \to \infty$. Since $u-\phi$ takes a strict minimum at z, we obtain $y_k \to z$ as $k \to \infty$. The definition of viscosity solutions implies that

$$F(y_k, \nabla\phi(y_k)) \geq h(y_k).$$

We have $h(y_k) \to +\infty$ as $k \to \infty$, which contradicts the boundedness of $F(y_k, \nabla\phi(y_k))$. □

We define the function d on $\Omega \times \Omega$ by

$$d(x,y) := \sup\{v(x) - v(y) \mid F(x, \nabla v) \leq h(x) \text{ in } \Omega \text{ in the viscosity sense}\}.$$

To see that d is well-defined, we set $\mathcal{S}^- := \{v - v(y) \mid v \text{ is a subsolution (7.3)}\}$. Note that, due to (A4), $\mathcal{S}^-$ is a nonempty. By coercivity (A2) $\mathcal{S}^-$ is a family of equi-Lipschitz continuous functions locally in Ω. Note also that $\phi(y) = 0$ for all $\phi \in \mathcal{S}^-$ and that Ω is connected. Therefore, thanks to the Ascoli-Arzelà theorem, $\mathcal{S}^-$ is precompact in the Fréchet space $C(\Omega)$. Thus the function d is a continuous function on $\Omega \times \Omega$ and satisfies $d(x,x) = 0$ for all $x \in \Omega$. Furthermore, by definition, we have $u(x) - u(y) \leq d(x,y)$ for all subsolutions u of (7.3) and $x, y \in \Omega$.

Now, we fix any $y \in \Omega$ and set $u(x) = d(x,y)$ for $x \in \Omega$. We see that u is locally Lipschitz continuous on Ω in view of (A2) and u is a subsolution (7.3). We argue as in the proof of Perron's method for viscosity solutions (see [**65**, **5**, **51**]), to find that u is a solution of (7.3) except at $\{y\}$. Next, we note by the definition of d that $v(x) - v(y) = v(x) - v(z) + v(z) - v(y) \leq d(x,z) + d(x,z)$ for all subsolutions v of (7.3) and $x, y, z \in \Omega$, to conclude that $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in \Omega$. In particular, we see that d is locally Lipschitz continuous on $\Omega \times \Omega$. The following proposition summarizes these observations.

PROPOSITION 7.5. *Assume that* (A1), (A2) *and* (A4) *hold. Then the following statements hold.* (i) *$d(x,x) = 0$ for all $x \in \Omega$ and d is locally Lipschitz continuous on $\Omega \times \Omega$.* (ii) *For all $y \in \Omega$, $d(\cdot, y)$ is a subsolution of* (7.3). (iii) *For all $y \in \Omega$, $d(\cdot, y)$ is a solution of* (7.3) *except at $\{y\}$.* (iv) *$d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in \Omega$.*

It is well-known that we have a variational formula for the function d, i.e.,

$$d(x,y) = \inf\{\int_0^t L(\gamma(s), \dot\gamma(s)) + h(\gamma(s))\, ds \mid t > 0, \gamma \in \mathcal{C}(x,t;y,0)\} \tag{7.10}$$

for any $x, y \in \Omega$, where L is the Lagrangian of F, i.e., $L(x,\xi) = \sup_{p \in \mathbb{R}^n}\{p \cdot \xi - F(x,p)\}$,

$$\mathcal{C}(x,t;y,0) := \{\gamma \in \mathrm{AC}\,([0,t];\Omega) \mid \gamma(t) = x, \gamma(0) = y\}$$

and we denote the set of absolutely continuous functions on $[0,t]$ with values in Ω by $\mathrm{AC}\,([0,t];\Omega)$. For a proof, we refer the reader to [**79**, **5**, **68**].

PROPOSITION 7.6. *Assume that* (A1)–(A3), (A7) *and* (A8) *hold. The function d is uniformly continuous in $\Omega \times \Omega$.*

PROOF OF PROPOSITION 7.6. We fix any $z \in \partial\Omega$, and by (A8) choose $r > 0$ and $b \in C(\mathbb{R}^{n-1})$ so that, after relabelling and re-orienting the coordinates if necessary, we have

$$\Omega \cap B_r(z) = \{(x', x_n) \in \mathbb{R}^n \mid x_n > b(x')\} \cap B_r(z).$$

Here and henceforth, we write $x = (x', x_n)$ for $x \in \mathbb{R}^n$, where $x' \in \mathbb{R}^{n-1}$. Let m_z be the modulus of continuity of b on the ball $\overline{B}_r(z') \subset \mathbb{R}^{n-1}$. We choose a $\delta \in (0, r/2)$ so that $m_z(\delta) < r/4$.

Let $x, y \in \Omega \cap B_{\delta/2}(z)$, and set $\varepsilon := |x-y|$, $\xi := (x', x_n + m_z(\varepsilon))$, and $\eta := (y', y_n + m_z(\varepsilon))$. Consider the line segments $[x,\xi], [y,\eta]$ and $[\xi,\eta]$. Noting that $\xi, \eta \in B_{r/2}(z)$, we see easily that $[x,\xi] \cup [y,\eta] \subset \Omega$ and that $[\xi,\eta] \subset B_{r/2}(z)$. Observe that for any $t \in [0,1]$,

$$\begin{aligned} &b(tx' + (1-t)y') \\ &\le \min\{b(x') + m_z(\varepsilon), b(y') + m_z(\varepsilon)\} \\ &< \min\{x_n, y_n\} + m_z(\varepsilon) \le t(x_n + m_z(\varepsilon)) + (1-t)(y_n + m_z(\varepsilon)), \end{aligned}$$

which reads $b(t\xi' + (1-t)\eta') < t\xi_n + (1-t)\eta_n$. We thus conclude that $[x,\xi] \cup [\xi,\eta] \cup [y,\eta] \subset \Omega$. By [**67**, Proposition 2.1] there exist $\delta, C > 0$ such that

$$L(x,p) \le C \quad \text{for all } (x,p) \in \overline{\Omega} \times B_\rho(0).$$

Set

$$t_1 = t_3 = \frac{m_z(\varepsilon)}{\rho},\ t_2 = \frac{\varepsilon}{\rho} \text{ and } t = t_1 + t_2 + t_3.$$

Define the trajectories $\gamma_i : [0, t_i] \to \Omega$ and $\gamma : [0,t] \to \Omega$ by

$$\gamma_1(s) = y + \frac{\rho(\eta - y)}{m_z(\varepsilon)} s,\ \gamma_2(s) = \eta + \frac{\rho(\xi - \eta)}{\varepsilon} s,\ \gamma_3(s) = \xi + \frac{\rho(x - \xi)}{m_z(\varepsilon)} s$$

and

$$\gamma(s) := \begin{cases} \gamma_1(s) & \text{for all } s \in [0, t_1], \\ \gamma_2(s - t_1) & \text{for all } s \in [t_1, t_1 + t_2], \\ \gamma_3(s - t_1 - t_2) & \text{for all } s \in [t_1 + t_2, t]. \end{cases}$$

Then, we have $\gamma \in \mathcal{C}(x, t; y, 0)$.

By (A7) we calculate that

$$\begin{aligned} d(x,y) \le& \int_0^t L(\gamma(s), \dot\gamma(s)) + h(\gamma(s))\, ds \\ \le&\ Ct + \int_0^t h(\gamma(s))\, ds \\ =&\ Ct + \int_0^{t_1} h(\gamma_1(s))\, ds + \int_0^{t_2} h(\gamma_2(s))\, ds + \int_0^{t_3} h(\gamma_3(s))\, ds \\ =&\ Ct + t_1 \int_0^1 h(y + r(\eta - y))\, dr + t_2 \int_0^1 h(\eta + r(\xi - \eta))\, dr \\ &+ t_3 \int_0^1 h(\xi + r(x - \xi))\, dr \\ =&\ Ct + \frac{1}{\rho}\Big[\int_0^1 h(y + r(\eta - y))|\eta - y|\, dr + \int_0^1 h(\eta + r(\xi - \eta))|\xi - \eta|\, dr \\ &+ \int_0^1 h(\xi + r(x - \xi))|x - \xi|\, dr\Big] \\ \le&\ \frac{C}{\rho}(\varepsilon + 2m_z(\varepsilon)) + \frac{1}{\rho}\big(\omega(\varepsilon) + 2\omega(m_z(\varepsilon))\big). \end{aligned}$$

Setting $\omega_z(\varepsilon) := \frac{C}{\rho}(\varepsilon + 2m_z(\varepsilon)) + \frac{1}{\rho}(\omega(\varepsilon) + 2\omega(m_z(\varepsilon)))$, we have

$$d(x, y) \leq \omega_z(\varepsilon).$$

By the standard compactness argument, we find an open neighborhood V of $\partial\Omega$, relative to Ω, and a modulus ω_0 such that $d(x, y) \leq \omega_0(|x - y|)$ for all $x, y \in V$. We choose a compact neighborhood $W \subset \Omega$ of $\Omega \setminus V$ and observe that u is Lipschitz continuous on $W \times W$. Note that

$$d(y, y) = 0 \text{ and } d(x_1, y) - d(x_2, y) \leq d(x_1, x_2)$$

for all $x_1, x_2, y \in \Omega$. It is now easy to conclude that d is uniformly continuous on $\Omega \times \Omega$. ☐

The function d can be extended uniquely to the function on $\overline{\Omega} \times \overline{\Omega}$ by continuity. We denote it by d again.

Lemma 7.3 is a straightforward result of Proposition 7.6.

PROPOSITION 7.7. *Assume that* (A1)–(A8) *hold. Then the following statements hold.* (i) *For any* $y \in \overline{\Omega}$, $d(\cdot, y)$ *is a supersolution of* (N) *except at* $\{y\}$. (ii) *For any* $y \in \mathcal{A}_h$, $d(\cdot, y)$ *is a solution of* (N).

PROOF. We first prove (i). Fix $y \in \overline{\Omega}$ and set $u(x) := d(x, y)$ for all $x \in \overline{\Omega}$. We only need to prove that $d(\cdot, y)$ satisfies the boundary condition defined by Definition 7.1 except at the point y. We argue by contradiction. Suppose that there would exist a test function $\phi \in C^1(\overline{\Omega})$ such that $u - \phi$ takes a strict minimum at some $z \in \partial\Omega$ with $z \neq y$. We may assume that $(u - \phi)(z) = 0$. Noting that $F(z, \nabla\phi(z)) < +\infty$ and $h(z) = +\infty$, we have

$$F(x, \nabla\phi(x)) \leq h(x) \quad \text{for all } x \in \overline{\Omega} \cap B_r(z),$$
$$y \notin \overline{\Omega} \cap \overline{B}_r(z) \quad \text{for some } r > 0.$$

Set $m := \min_{x \in \overline{\Omega} \cap \partial B_r(z)} (u - \phi)(x)$. Then we see that $m > 0$. For any $t > 0$ and $\gamma \in \mathcal{C}(z, t; y, 0)$, there exists $\tau \in (0, t)$ such that

$$\gamma(\tau) \in \overline{\Omega} \cap \partial B_r(z),$$
$$\gamma(s) \in \overline{\Omega} \cap B_r(z) \quad \text{for all } s \in [\tau, t].$$

We calculate that

$$\begin{aligned}
u(z) = \; & \phi(\gamma(t)) = \phi(\gamma(\tau)) + \int_\tau^t \frac{d\phi(\gamma(s))}{ds}\, ds \\
= \; & \phi(\gamma(\tau)) + \int_\tau^t \nabla\phi(\gamma(s)) \cdot \dot{\gamma}(s)\, ds \\
\leq \; & u(\gamma(\tau)) - m \\
& + \int_\tau^t \big(F(\gamma(s), \nabla\phi(\gamma(s))) - h(\gamma(s))\big) + \big(L(\gamma(s), \dot{\gamma}(s)) + h(\gamma(s))\big)\, ds \\
\leq \; & \int_0^\tau L(\gamma(s), \dot{\gamma}(s)) + h(\gamma(s))\, ds + \int_\tau^t L(\gamma(s), \dot{\gamma}(s)) + h(\gamma(s))\, ds - m \\
= \; & \int_0^t L(\gamma(s), \dot{\gamma}(s)) + h(\gamma(s))\, ds - m.
\end{aligned}$$

Therefore, we get $u(z) \leq u(z) - m$, which is a contradiction.

We next prove (ii). Fix $y \in \mathcal{A}_h$. We just need to prove that d is a viscosity supersolution at the point y. Suppose that there would exist a test function $\phi \in C^1(\overline{\Omega})$ such that

$$d(\cdot, y) - \phi \text{ takes a local minimum at } y \text{ and}$$
$$F(y, \nabla\phi(y)) < h(y).$$

It is easy to see that the above inequality has a contradiction, since $F(x,p) \geq 0$ for all $(x,p) \in \overline{\Omega} \times \mathbb{R}^n$ and $h(y) = 0$. □

7.1.2. Comparison Results. We prove that comparison principle for (N) holds in a special case. In this subsection, we deal with the Hamiltonian defined by

$$F(x,p) = \|p\|, \tag{7.11}$$

where $\|\cdot\|$ is a norm in $\mathbb{R}^n$.

We add the following assumption on h and Ω.

(A9) There exists a constant $\alpha_0 > 0$ such that $\{x \in \Omega \mid h(x) < \alpha\}$ are convex for all $\alpha \geq \alpha_0$.

THEOREM 7.8 (Comparison principle for stationary problem). *Let F be as in* (7.11). *Assume* (A1), (A5)–(A7) *and* (A9) *hold. Let $u \in C(\overline{\Omega})$ and $v \in \mathrm{LSC}(\overline{\Omega})$ be a subsolution and a supersolution of* (N), *respectively. If $u \leq v$ on $\mathcal{A}_h$, then $u \leq v$ on $\overline{\Omega}$.*

For the proof of the above theorem we need an anisotropic Lipschitz extension lemma (Lemma 7.9) and a regularization lemma (Lemma 7.10).

LEMMA 7.9 (Anisotropic Lipschitz extension). *Assume that* (A1) *and* (A9) *hold. Let $u \in C(\overline{\Omega})$ be a subsolution of* (N). *Define the functions $u_k : \mathbb{R}^n \to \mathbb{R}$ by*

$$u_k(x) := \max_{y \in \overline{\Omega}_k}\{u(y) - k\|x-y\|_*\}, \tag{7.12}$$

for any $k \in \mathbb{N}$, where $\Omega_k := \{x \in \Omega \mid h(x) < k\}$ and $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$, i.e.,

$$\|q\|_* := \sup_{p \in \mathbb{R}^n \setminus \{0\}} \frac{|p \cdot q|}{\|p\|}.$$

Then for all $k \geq \alpha_0$ we have

$$u_k(x) = u(x) \quad \text{for all } x \in \overline{\Omega}_k,$$
$$|u_k(x) - u_k(y)| \leq k\|x-y\|_* \quad \text{for all } x, y \in \mathbb{R}^n,$$
$$\|\nabla u_k\| \leq k \quad \text{in } \mathbb{R}^n \text{ in the viscosity sense.}$$

Moreover, u_k are subsolutions of (N) *for all $k \in \mathbb{N}$ with $k \geq \alpha_0$ and $u_k \to u$ as $k \to \infty$ in $C(\overline{\Omega})$.*

REMARK 7.3. When $\|\cdot\| = |\cdot|$, the functions u_k are known as an extension of Lipschitz functions. See Theorem 1 (p. 80) in **[37]**. We use the continuity of u on $\overline{\Omega}$ in the convergence of u_k to u in $C(\overline{\Omega})$.

PROOF. By the definition of Ω_k we have

$$\|\nabla u\| \leq k \quad \text{in } \Omega_k$$

in the viscosity sense. In view of (A9) and Proposition E.3, we have

$$|u(x) - u(y)| \leq k\|x - y\|_* \quad \text{for all } x, y \in \overline{\Omega}_k, \ k \geq \alpha_0,$$

which implies that $u_k(x) \leq u(x)$ for all $x \in \overline{\Omega}_k$. It is clear that $u_k(x) \geq u(x)$ for any $x \in \overline{\Omega}_k$. Thus we get $u_k = u$ in $\overline{\Omega}_k$.

For any $x_1, x_2 \in \mathbb{R}^n$, we have

$$\begin{aligned} u_k(x_1) &= \max_{y \in \overline{\Omega}_k}\{u(y) - k\|x_1 - y\|_*\} \\ &\geq \max_{y \in \overline{\Omega}_k}\{u(y) - k\|x_1 - x_2\|_* - k\|x_2 - y\|_*\} \\ &= u_k(x_2) - k\|x_1 - x_2\|_*. \end{aligned}$$

By symmetry we get

$$|u_k(x_1) - u_k(x_2)| \leq k\|x_1 - x_2\|_*. \tag{7.13}$$

By Proposition E.1 we have

$$\|\nabla u_k(x)\| \leq n \quad \text{in } \mathbb{R}^n \text{ in the viscosity sense.} \tag{7.14}$$

We next prove that u_k are subsolutions of (N) for all $k \geq \alpha_0$. For any $x \in \Omega_k$, we have $u_k(x) = u(x)$. Therefore, we have $\|p\| \leq h(x)$ for any $p \in D^+u_k(x) = D^+u(x)$, since u is a viscosity subsolution in Ω. For any $x \in \Omega \setminus \Omega_k$, we have $h(x) \geq k$. Noting that inequality (7.14) implies that $\|p\| \leq k$ for all $p \in D^+u_k(x)$, we obtain

$$\|p\| - h(x) \leq k - k = 0$$

for all $x \in \Omega \setminus \Omega_k$ and all $p \in D^+u_k(x)$. Finally, since we have $u_k = u$ in Ω_k and $u \in C(\overline{\Omega})$, we see that $u_k \to u$ as $k \to \infty$ in $C(\overline{\Omega})$. □

LEMMA 7.10 (Regularization). *Assume that* (A1) *and* (A9) *hold. Let* $u \in C(\overline{\Omega})$ *be a subsolution of* (N), *and* u_k, Ω_k *be the functions and sets given by Lemma* 7.9 *for any* $k \in \mathbb{N}$. *Define the functions* $u_k^m : \mathbb{R}^n \to \mathbb{R}$ *by* $u_k^m(x) := u_k * \rho_m(x)$, *where* $*$ *denotes the convolution,* $\rho_m(x) := m\rho(mx)$ *for* $m \in \mathbb{N}$ *and* $\rho \in C^\infty(\mathbb{R}^n)$ *is a standard mollification kernel, i.e.,* $\rho \geq 0$, $\operatorname{supp}\rho \subset B_1(0)$ *and* $\int_{\mathbb{R}^n} \rho(x)\,dx = 1$. *Then* $u_k^m \in C^1(\overline{\Omega})$ *and* u_k^m *satisfy*

$$\|\nabla u_k^m(x)\| \leq h(x) + \omega_k(\frac{1}{m}) \quad \textit{for all } x \in \Omega$$

for all $k, m \in \mathbb{N}$ *such that* $1/m \leq \operatorname{dist}(\partial\Omega_{2k}, \Omega_k)$ *and* $\omega_k : [0, \infty) \to [0, \infty)$ *is a modulus for a fixed* $k \in \mathbb{N}$.

PROOF. Let ω_k be a modulus of continuity of h on $\overline{\Omega}_{2k}$. We calculate that for all $x \in \Omega_k$

$$\begin{aligned} 0 &\geq \rho_m * \big(\|\nabla u_k(\cdot)\| - h(\cdot)\big)(x) \\ &= \int_{B_{\frac{1}{m}}(x)} \rho_m(x-y)\big(\|\nabla u_k(y)\| - h(y)\big)\, dy \\ &\geq \int_{B_{\frac{1}{m}}(x)} \rho_m(x-y)\big(\|\nabla u_k(y)\| - h(x) - \omega_k(\frac{1}{m})\big)\, dy \\ &\geq \|\rho_m * \nabla u_k(x)\| - h(x) - \omega_k(\frac{1}{m}) \\ &= \|\nabla u_k^m(x)\| - h(x) - \omega_k(\frac{1}{m}). \end{aligned}$$

We have used Jensen's inequality in the third inequality.

By Lemma 7.9 we have

$$\|\nabla u_k(x)\| \leq n \quad \text{for almost every } x \in \Omega.$$

Therefore, we have

$$\|\nabla v_k^m(x)\| \leq \int_{\mathbb{R}^n} \rho_m(x-y)\|\nabla v_k(y)\|\, dy \leq k \int_{\mathbb{R}^n} \rho_m(x-y)\, dy = k$$

for any $x \in \Omega \setminus \Omega_k$, which implies

$$\|\nabla u_k^m(x)\| - h(x) \leq k - k = 0 \quad \text{for all } x \in \Omega \setminus \Omega_k.$$

□

PROOF OF THEOREM 7.8. For a small constant $\alpha > 0$, there exists $r_\alpha > 0$ such that

$$u(x) \leq v(x) + \alpha \quad \text{on } \mathcal{A}_\alpha := \{x + y \mid x \in \mathcal{A}_h, y \in \overline{B}_{r_\alpha}(0)\}.$$

Set $v_\alpha(x) := v(x) + \alpha$ for $x \in \overline{\Omega}$. We prove that

$$u(x) \leq v_\alpha(x) \quad \text{for all } x \in \overline{\Omega} \setminus \mathcal{A}_\alpha^i,$$

where $\mathcal{A}_\alpha^i$ is the set of interior points of $\mathcal{A}_\alpha$. Suppose that

$$\max_{\overline{\Omega}\setminus\mathcal{A}_\alpha^i}(u - v_\alpha) \geq 5\theta > 0 \quad \text{for some } \theta > 0.$$

For suitable large $n, m \in \mathbb{N}$ such that $1/m \leq \operatorname{dist}(\partial\Omega_{2k}, \Omega_k)$, we may assume that

$$\max_{\overline{\Omega}\setminus\mathcal{A}_\alpha^i}(u_k^m - v_\alpha) \geq 4\theta$$

and

$$u_k^m(x) - v_\alpha(x) \leq \theta \quad \text{for all } x \in \mathcal{A}_\alpha,$$

where u_k^m are the functions in Lemma 7.10. Fix such an $n \in \mathbb{N}$ and we shall send $m \to \infty$ later.

For $\lambda \in (0,1)$, we set

$$u_\lambda(x) := \lambda u_k^m(x) \quad \text{for all } x \in \overline{\Omega}.$$

Fix $\lambda \in (0,1)$ as close to 1 that

$$\max_{\overline{\Omega}\setminus\mathcal{A}_\alpha^i}(u_\lambda - v_\alpha) \geq 3\theta \text{ and } u_\lambda(x) - v_\alpha(x) \leq 2\theta \text{ for all } x \in \mathcal{A}_\alpha. \tag{7.15}$$

Let $z_m(= z(m,n,\lambda)) \in \overline{\Omega} \setminus \mathcal{A}_\alpha^i$ be a point which satisfies $(u_\lambda - v_\alpha)(z_m) = \max_{\overline{\Omega}\setminus\mathcal{A}_\alpha^i}(u_\lambda - v_\alpha)$. Since $u_\lambda \in C^1(\overline{\Omega})$ and v_α is a supersolution, we see that $z_m \in \Omega \setminus \mathcal{A}_\alpha^i$. By the choice of λ we have $z_m \in \Omega \setminus \mathcal{A}_\alpha$. Moreover, we have

$$\|\nabla u_\lambda(z_m)\| \geq h(z_m). \tag{7.16}$$

We may assume that

$$z_m \to \overline{z} \in \overline{\Omega} \setminus \mathcal{A}_\alpha^i \quad \text{as } m \to \infty$$

by taking a subsequence if necessary, since $\overline{\Omega} \setminus \mathcal{A}_\alpha^i$ is a compact set. Suppose that $\overline{z} \in \partial\Omega$. Then we have $h(z_m) \to +\infty$ as $m \to \infty$, which contradicts the inequality $\|\nabla u_\lambda(x)\| \leq \lambda n$ for all $x \in \Omega$. Therefore, we have $\overline{z} \in \Omega$. Moreover, we may assume that

$$\operatorname{dist}(z_m, \partial\Omega) \geq \alpha(k) > 0$$

for all suitable large $m \in \mathbb{N}$ and some constant $\alpha(k) > 0$ which is independent of m. We also have $z_m \notin \partial\mathcal{A}_\alpha$ due to (7.15). Thus, we can choose $r(= r(k,\lambda)) > 0$ which is independent of m such that $B_r(z_m) \subset \Omega^k \setminus \mathcal{A}_\alpha$ for all $m \in \mathbb{N}$, where $\Omega^k := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \alpha(k)/2\}$.

Let us consider the function

$$\Phi(x,y) := u_\lambda(x) - v_\alpha(y) - \frac{|x-y|^2}{2\varepsilon} - |y - z_m|^2 \quad \text{for all } x, y \in \overline{\Omega} \text{ and } \varepsilon > 0.$$

Let $x_\varepsilon, y_\varepsilon \in \overline{\Omega}$ be points which satisfies $\Phi(x_\varepsilon, y_\varepsilon) = \max_{\overline{\Omega}^2} \Phi$. We obtain that $x_\varepsilon, y_\varepsilon \to z_m$ as $\varepsilon \to 0$. Indeed, since $\Phi(x_\varepsilon, y_\varepsilon) \geq \Phi(z_m, z_m)$, we have

$$\frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2 + |y_\varepsilon - z_m|^2 \leq u_\lambda(x_\varepsilon) - v_\alpha(y_\varepsilon) - (u_\lambda - v_\alpha)(z_m).$$

We may assume that $x_{\varepsilon_j}, y_{\varepsilon_j} \to z_0$ as $j \to \infty$ for some $z_0 \in \overline{\Omega}$ by taking subsequences $\{x_{\varepsilon_j}\}_j, \{y_{\varepsilon_j}\}_j$ if necessary, since $\overline{\Omega}$ is compact. We have

$$\limsup_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2 + |y_\varepsilon - z_m|^2\right) \leq (u_\alpha - v_\lambda)(z_0) - (u_\alpha - v_\lambda)(z_m) \leq 0.$$

Thus, we obtain that $z_0 = z_m$ and we may actually assume that $x_\varepsilon, y_\varepsilon \in B_r(z_m)$ for suitable small $\varepsilon \in (0,1)$.

The definition of viscosity solutions immediately implies the following inequalities:

$$\|\frac{1}{\lambda\varepsilon}(x_\varepsilon - y_\varepsilon)\| \leq h(x_\varepsilon) + \omega_k(\frac{1}{m}), \tag{7.17}$$

$$\|\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) - 2(y_\varepsilon - z_m)\| \geq h(y_\varepsilon).$$

Since $x_\varepsilon \in B_r(z_m) \subset \overline{\Omega^k}$ for suitable small $\varepsilon > 0$, we have

$$\frac{1}{\lambda}\|\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon)\| \leq h(x_\varepsilon) + 1 \leq \max_{\overline{\Omega^k}} h + 1.$$

This yields

$$\frac{1}{\lambda\varepsilon}|x_\varepsilon - y_\varepsilon| \leq C_{\lambda,n}$$

for some constant $C_{\lambda,n} > 0$ and we can extract a convergent subsequence $\frac{1}{\varepsilon_j}(x_{\varepsilon_j} - y_{\varepsilon_j})$ and denote its limit point $p \in B_{\lambda C_{\lambda,n}}(0)$.

Sending $j \to \infty$ in the inequality (7.17), we get

$$\frac{1}{\lambda}\|p\| \leq h(z_m) + \omega_k(\frac{1}{m}), \quad \|p\| \geq h(z_m).$$

Therefore we obtain

$$h(z_m) \leq \lambda\Big(h(z_m) + \omega_k(\frac{1}{m})\Big).$$

Sending $m \to \infty$ yields

$$h(\overline{z}) \leq \lambda h(\overline{z}),$$

which yields a contradiction, since $0 < h(\overline{z}) < +\infty$. □

COROLLARY 7.11. *Assume that* (A1), (A5), (A6) *and* (A9) *hold. Let* $u \in C(\overline{\Omega})$ *be a solution of* (N). *Then*

$$u(x) = \min\{d(x,y) + u(y) \mid y \in \mathcal{A}_h\} \quad \textit{for all } x \in \overline{\Omega}. \tag{7.18}$$

PROOF. We denote by $w(x)$ the right hand side of (7.18). We note that w is a solution of (N) by Proposition 7.7 (ii). By the definition of d, we have $u(x) - u(y) \leq d(x,y)$ for all $x, y \in \overline{\Omega}$. Hence we get $u(x) \leq w(x)$ for all $x \in \overline{\Omega}$. Next, by the definition of w, we have $w(x) \leq u(x)$ for all $x \in \mathcal{A}_h$. Thus we have $w(x) \leq u(x)$ for all $x \in \overline{\Omega}$ by Theorem 7.8. Therefore we conclude that $u = w$ on $\overline{\Omega}$. □

REMARK 7.4. Let us consider the equation

$$|u_x|^2 = \tan(|x|) \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \text{ with } u(0) = 0, \tag{7.19}$$

we have the multiplicity of solutions. Indeed, setting

$$u(x) := \int_0^{|x|} (\tan(y))^{1/2}\, dy \text{ for all } x \in [-1,1],$$

$$v(x) := \begin{cases} \int_0^x (\tan(y))^{1/2}\, dy & \text{for all } x \in [0,1], \\ -\int_0^{-x} (\tan(y))^{1/2}\, dy & \text{for all } x \in [-1,0), \end{cases}$$

we see that $\pm u$, $\pm v$ are solutions of (7.19). But u is the only solution of (7.19) which satisfies the singular Neumann condition defined by Definition 7.1.

Although Theorem 7.8 is enough for our application in crystal growth, let us try showing a more general comparison principle with singular Neumann boundary condition. Here by "general", we mean loosening of two restrictions. One is for the structure of equation. We discuss the equation in the generality of (N). Namely, our equation in Ω is

$$F(x, Du) = h(x).$$

The other relaxation is to attempt to weaken the convexity (A9) of high level sets of h.

To these ends, we still assume Ω to be bounded and need extra assumptions:

(A1') There exist a constant $L > 0$ and a modulus ω such that

$$|F(x,p) - F(x,q)| \leq L|p - q|$$

and

$$F(x,p) - F(y,p)| \leq \omega(|x - y|(1 + |p|))$$

for all $x, y \in \overline{\Omega}$ and $p, q \in \mathbb{R}^n$.

(A10) For any fixed $z \in \partial\Omega$, there exist $c, r > 0$ and $\eta \in C(\overline{B_r(z)} \cap \overline{\Omega}; \mathbb{R}^n)$ such that for each $y \in B_r(z) \cap \overline{\Omega}$,
(i) $B_{ct}(y + t\eta(y)) \subset \Omega$;
(ii) $h(x) \leq h(y)$ for all $x \in B_{ct}(y + t\eta(y))$ and $0 < t \leq c$.

The assumption above combines the truncated cone condition, often used in state constraint problems, and a certain local monotonicity along the normal to the boundary of level sets of h. We explain this assumption in brief after the comparison principle below.

THEOREM 7.12 (A general comparison theorem). *Assume* (A1'), (A3)–(A7) *and* (A10). *Let* $u \in C(\overline{\Omega})$ *and* $v \in LSC(\overline{\Omega})$ *be respectively a subsolution and a supersolution of* (N). *If* $u \leq v$ *on* $\mathcal{A}_h$, *then* $u \leq v$ *in* $\overline{\Omega}$.

PROOF. Without loss, assume $u, v \geq 0$. Since F is convex in p and $F(x, 0) = 0$, we can take $\lambda \in (0,1)$ and $u_\lambda := \lambda u$ such that

$$F(x, Du_\lambda) = F(x, \lambda Du) \leq \lambda F(x, Du) + (1-\lambda)F(x, 0) = \lambda h(x) \tag{7.20}$$

in the viscosity sense. It suffices to show $u_\lambda \leq v$ for λ sufficiently close to 1. As u_λ is uniformly continuous in $\overline{\Omega}$, we denote by ω_1 its modulus of continuity.

Suppose by contradiction that it is not true. Then there exist $z \in \overline{\Omega}$ and $\theta > 0$ satisfying

$$\max_{\overline{\Omega}} u_\lambda - v = \lambda u(z) - v(z) = \theta > 0 \tag{7.21}$$

when λ is near 1. It is quite clear that $z \notin \mathcal{A}_h$. We thus divide two cases according to the position of z in the rest of region.

<u>Case 1:</u> $z \in \Omega \setminus \mathcal{A}_h$. This is a comparatively easier case in that the maximizer z lies in the interior of domain. After writing the auxiliary function

$$\Phi(x, y) = u_\lambda(x) - v(y) - \frac{1}{2\varepsilon}|x - y|^2 - |y - z|^2 \tag{7.22}$$

and denoting its maximizers x_ε and y_ε, we jump to the final inequalities

$$\begin{aligned} &F(x_\varepsilon, \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon)) \leq \lambda h(x_\varepsilon) \quad \text{(by (7.20))} \\ &F(y_\varepsilon, \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) - 2(y_\varepsilon - z)) \geq h(y_\varepsilon) \end{aligned} \tag{7.23}$$

with intermediate results that

$$\frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2 \to 0 \text{ and } x_\varepsilon, y_\varepsilon \to z \quad \text{as } \varepsilon \to 0. \tag{7.24}$$

The difference from the proof of Theorem 7.8 is to take difference of these two inequalities before passing to the limit. By (A1'), we obtain

$$-\omega\left(|x_\varepsilon - y_\varepsilon|\left(\frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon| + 1\right)\right) - L|y_\varepsilon - z| \leq \lambda h(x_\varepsilon) - h(y_\varepsilon). \tag{7.25}$$

Letting $\varepsilon \to 0$, we reach a contradiction that the left hand side tends to 0 while the right hand side tends to $(\lambda - 1)h(z) < 0$.

<u>Case 2:</u> $z \in \partial\Omega$. This case is tougher. We established consecutive gradient estimates in the proof of Theorem 7.8 to prevent this case from happening. If this case really occurs, we perhaps can try following some techniques arising in

the circumstances of state constraint problems, from which (A10) comes into play.

Utilize the η given in (A10) and take

$$\Phi(x,y) = u_\lambda(x) - v(y) - \left|\frac{1}{\varepsilon}(x-y) - \eta(z)\right|^2 - (y-z)^2. \tag{7.26}$$

Again, let $x_\varepsilon, y_\varepsilon \in \overline{\Omega}$ denote the pair of maximizers. By definition of supersolutions, $y_\varepsilon \in \Omega$. Moreover, noticing that $\Phi(x_\varepsilon, y_\varepsilon) \geq \Phi(z+\varepsilon\eta(z), z) = \theta - \omega_1(C\varepsilon)$, where $C := \sup_{B_r(z)} |\eta|$, we have

$$\left|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \eta(z)\right|^2 + (y_\varepsilon - z)^2 \leq u_\lambda(x_\varepsilon) - v(y_\varepsilon) - \theta + \omega_1(C\varepsilon). \tag{7.27}$$

By boundedness of $u_\lambda(x) - v(y)$, we get a constant M such that

$$|x_\varepsilon - y_\varepsilon| \leq M\varepsilon. \tag{7.28}$$

We further estimate using (7.27), (7.28) and the uniform continuity of u_λ

$$\begin{aligned} &\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \eta(z)\right)^2 + (y_\varepsilon - z)^2 \\ &\leq (u_\lambda(x_\varepsilon) - u_\lambda(y_\varepsilon)) + (u_\lambda(y_\varepsilon) - v(y_\varepsilon)) + \omega_1(C\varepsilon) - \theta \\ &\leq \omega_1(|x_\varepsilon - y_\varepsilon|) + \omega_1(C\varepsilon). \end{aligned} \tag{7.29}$$

Consequently, we have

$$\left|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \eta(z)\right| \to 0 \text{ and } y_\varepsilon - z \to 0 \text{ as } \varepsilon \to 0, \tag{7.30}$$

and a choice of small ε also implies

$$\begin{aligned} &|\eta(y_\varepsilon) - \eta(z)| \leq \frac{c}{2} \\ &\left|\frac{x_\varepsilon - y_\varepsilon}{x} - \eta(z)\right| \leq \frac{c}{2}, \end{aligned} \tag{7.31}$$

where c appears in (A10). We are thus led to

$$|x_\varepsilon - y_\varepsilon - \varepsilon\eta(y_\varepsilon)| \leq c\varepsilon. \tag{7.32}$$

We now use (A10) to deduce $x_\varepsilon \in \Omega$ and $h(x_\varepsilon) \leq h(y_\varepsilon)$. The definitions of sub- and supersolutions imply

$$\begin{aligned} &F\left(x_\varepsilon, \frac{2}{\varepsilon}\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \eta(z)\right)\right) \leq \lambda h(x_\varepsilon) \quad \text{(again by (7.20))} \\ &F\left(y_\varepsilon, \frac{2}{\varepsilon}\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \eta(z)\right) - 2(y_\varepsilon - z)\right) \geq h(y_\varepsilon). \end{aligned} \tag{7.33}$$

With an application of (A1'), we find that the difference of left hand sides above has a lower bound

$$-M\omega_1(|x_\varepsilon - y_\varepsilon|) - |x_\varepsilon - y_\varepsilon| - 2L|y_\varepsilon - z| \to 0 \quad \text{as } \varepsilon \to 0. \tag{7.34}$$

The difference of right hand sides however is

$$\lambda h(x_\varepsilon) - \lambda h(y_\varepsilon) + (\lambda - 1)h(y_\varepsilon) \leq (\lambda - 1)h(y_\varepsilon) \to -\infty \quad \text{as } \varepsilon \to 0, \tag{7.35}$$

which gives us a contradiction. □

We close this section with a brief explanation about the assumption (A10). A sufficient condition is that high level sets of h are C^1 domains with a uniform C^1 bound, i.e., there exists $a_0 > 0$ such that $\Omega_a := \{x \in \Omega : h(x) \leq a\}$ has C^1 boundary for all $a \geq a_0$ and their derivatives are uniformly bounded.

Indeed, for any y in the small neighborhood of z, we can let η be the unit interior normal to $\partial\Omega_{h(y)}$ and take $c > 0$ small depending on the bound of first derivative of the boundaries so that $B_{ct}(y + t\eta(y)) \subset \Omega_{h(y)}$. This clearly implies that $h(x) \leq h(y)$ for any $x \in B_{ct}(y + t\eta(y))$.

7.2. Large-time behavior

In this section we consider the large-time behavior of solutions of (C). We make the following assumptions throughout this section.

(B1) The function $m : [0, \infty) \to [0, 1)$ is a strictly increasing Lipschitz function with $m(0) =: m_0 \in [0, 1)$, and $m(r) \to 1$ as $r \to \infty$.

(B2) The function f satisfies

$$\{x \in \mathbb{R}^n : f(x) = 0, \sigma(x) = \overline{\sigma}\} \neq \emptyset;$$

(B3) u_0 is Lipschitz continuous in $\mathbb{R}^n$.

We set

$$c := \sup_{x\in\mathbb{R}^n} \min_{p\in\mathbb{R}^n} \big(\sigma(x)m(\|p\|) - f(x)\big) = \overline{\sigma}m_0, \tag{7.36}$$

$$\begin{aligned}\Omega_e &:= \{x \in \mathbb{R}^n : \sup_{p\in\mathbb{R}^n} \big(\sigma(x)m(\|p\|) - f(x)\big) > c\} \\ &= \{x \in \mathbb{R}^n : \sigma(x) - f(x) > c\},\end{aligned} \tag{7.37}$$

$$\begin{aligned}\Omega_d &:= \{x \in \mathbb{R}^n : \sup_{p\in\mathbb{R}^n} \big(\sigma(x)m(\|p\|) - f(x)\big) < c\}, \\ &= \{x \in \mathbb{R}^n : \sigma(x) - f(x) < c\},\end{aligned} \tag{7.38}$$

$$\begin{aligned}\mathcal{A} &:= \{x \in \mathbb{R}^n : \min_{p\in\mathbb{R}^n} \big(\sigma(x)m(\|p\|) - f(x)\big) = c\} \\ &= \{x \in \mathbb{R}^n : f(x) = 0, \sigma(x) = \overline{\sigma}\}.\end{aligned} \tag{7.39}$$

We call the set Ω_e the effective domain for (C) if Ω_e is a domain in $\mathbb{R}^n$. For any $x \in \Omega_e$, $p \in \mathbb{R}^n$ we have $\sigma(x)m(\|p\|) > f(x) + c$, which implies

$$1 > m(\|p\|) > \frac{f(x) + c}{\sigma(x)}.$$

Thus we can define the function $h : \Omega_e \to \mathbb{R}$ by

$$h(x) := m^{-1}\Big(\frac{f(x) + c}{\sigma(x)}\Big) \quad \text{for all } x \in \Omega_e. \tag{7.40}$$

Henceforth, we use the above notations. We notice that

$$\mathcal{A} = \{x \in \mathbb{R}^n \mid h(x) = 0\}.$$

7.2.1. Large-time Asymptotics. We now state the result of the large-time behavior of solutions of (C).

THEOREM 7.13. *Assume* (B1)–(B3). *Let* $u \in C(\mathbb{R}^n \times [0,\infty))$ *be the viscosity solution of* (C). *If the comparison principle taking account of data on* $\mathcal{A}$ *for*

$$(\mathrm{S}) \qquad \begin{cases} \|\nabla u\| = h(x) & \text{in } \Omega_e, \quad (7.41) \\ \dfrac{\partial u}{\partial n} = +\infty & \text{on } \partial\Omega_e, \quad (7.42) \end{cases}$$

holds, then we have

$$u(x,t) + ct \to v_\infty(x) \quad \text{uniformly on each compact subset of } \Omega_e$$

and

$$u(x,t) + ct \to +\infty \quad \text{uniformly on each compact subset of } \Omega_d$$

as $t \to \infty$ *for a solution* $v_\infty \in C(\overline{\Omega}_e)$ *of* (S).

If we add the following technical assumptions

(B4) the set Ω_e is a bounded domain and Ω_d is nonempty,
(B5) h satisfies (A7) and (A9),

then we see that the comparison principle taking account of data on $\mathcal{A}$ holds by Theorem 7.8. In Section 4.5 we discuss functions σ, m which satisfies the assumptions above.

REMARK 7.5. (i) It is worth noticing here that Ω_e and c do not depend on the initial value.
(ii) We set

$$K_c := \{x \in \mathbb{R}^n \mid \sup_{p \in \mathbb{R}^n} \big(H(x,p) - f(x)\big) = c\}.$$

We see that $u + ct$ is bounded on K_c but we do not know if $u + ct$ converges on K_c or not.
(iii) We introduced in Chapter 6 the new definition (see also Definition D.1 in Appendix D) of solutions to the stationary equation

$$\sigma(x)m(\|\nabla u\|) = f(x) + c \quad \text{in } \mathbb{R}^n. \tag{7.43}$$

We present the equivalence between the solutions defined by Definition 7.1 and Definition D.1 precisely in Appendix D.

7.2.2. Construction of Subsolutions and Supersolutions of (C).

PROPOSITION 7.14. *There exist a subsolution* $w^- \in C(\mathbb{R}^n \times [0,\infty))$ *and a supersolution* $w^+ \in C(\mathbb{R}^n \times [0,\infty))$ *of* (C) *which satisfy the followings:*

(i) $|w^\pm(x,t) + ct| \le C$ *for all* $x \in \Omega_e \cup K_c$, $t \in [0,\infty)$ *and some* $C > 0$,
(ii) $w^\pm(x,t) + ct \to +\infty$ *uniformly on each compact subset of* Ω_d *as* $t \to \infty$.

PROOF. We first construct a subsolution. Set

$$G_1(x) := -\sup_{p \in \mathbb{R}^n} \{H(x,p) - f(x)\} (= f(x) - \sigma(x)).$$

By Theorem 7.2 we can choose a solution $U_- \in C(\overline{\Omega}_e)$ of (S). Note that U^-+M is still a solution of (S) for any $M \in \mathbb{R}$. Therefore we may assume that $U_- \leq u_0$ on $\overline{\Omega}_e$. We choose a function $W_1 \in C(\mathbb{R}^n)$ such that

$$W_1(x) \leq u_0(x) \quad \text{for all } x \in \mathbb{R}^n \setminus \overline{\Omega}_e,$$
$$W_1(x) = U_-(x) \quad \text{for all } x \in K_c.$$

Define the function $w^- \in C(\mathbb{R}^n \times [0,\infty))$ by

$$w^-(x,t) := \begin{cases} U_-(x) - ct & \text{for all } (x,t) \in \overline{\Omega}_e \times [0,\infty), \\ W_1(x) + G_1(x)t & \text{for all } (x,t) \in (\mathbb{R}^n \setminus \overline{\Omega}_e) \times [0,\infty). \end{cases}$$

It is easily seen that w^- is a subsolution of (7.1) in $\Omega_e \times (0,\infty)$. Indeed, let $w^- - \phi$ take a maximum at $(x_0,t_0) \in \Omega_e \times (0,\infty)$ for some $\phi \in C^1(\mathbb{R}^n \times (0,\infty))$. Then we have $\phi_t(x_0,t_0) = -c$ and

$$\|\nabla\phi(x_0,t_0)\| \leq h(x_0),$$

which is equivalent as

$$\sigma(x_0)m(\|\nabla\phi(x_0,t_0)\|) \leq f(x_0) + c.$$

Therefore, we get

$$\phi_t(x_0,t_0) + \sigma(x_0)m(\|\nabla\phi(x_0,t_0)\|) \leq f(x_0).$$

Let $w^- - \phi$ take a maximum at $(x_0,t_0) \in (\mathbb{R}^n \setminus \Omega_e) \times (0,T)$ for some $C^1(\mathbb{R}^n \times (0,\infty))$. Noting that $\phi_t(x_0,t_0) = G_1(x_0)$, we have

$$\begin{aligned} &\phi_t(x_0,t_0) + H(x_0,\nabla\phi(x_0,t_0)) - f(x_0) \\ =&G_1(x_0) + H(x_0,\nabla\phi(x_0,t_0)) - f(x_0) \\ =&-\sup_{p\in\mathbb{R}^n}(H(x_0,p) - f(x_0)) + H(x_0,\nabla\phi(x_0,t_0)) - f(x_0) \\ \leq& 0. \end{aligned}$$

Since $G_1(x) = -c$ on K_c and $G_1(x) > -c$ for all $x \in \Omega_d$, we see that w^- satisfies (i)–(ii).

Similarly, we can construct a supersolution. Set

$$G_2(x) := -\inf_{p\in\mathbb{R}^n}\{H(x,p) - f(x)\}(= f(x) - \sigma(x)m_0)$$

and choose a supersolution $U_+ \in C(\overline{\Omega}_e)$ of (N) such that $U_+ \geq u_0$ on $\overline{\Omega}_e$. Choose a function $W_2 \in C(\mathbb{R}^n)$ such that

$$W_2(x) \geq u_0(x) \quad \text{for all } x \in \mathbb{R}^n \setminus \overline{\Omega}_e,$$
$$W_2(x) = U_+(x) \quad \text{for all } x \in K_c.$$

Define the function $w^+ \in C(\mathbb{R}^n \times [0,\infty))$ by

$$w^+(x,t) := \begin{cases} U_+(x) - ct & \text{for all } (x,t) \in \overline{\Omega}_e \times [0,\infty), \\ W_2(x) + G_2(x)t & \text{for all } (x,t) \in (\mathbb{R}^n \setminus \overline{\Omega}_e) \times [0,\infty). \end{cases}$$

Then we see that w^+ is a supersolution of (7.1) which satisfies (i)–(ii). □

7.2.3. Stability and Proof of Theorem 7.13. We define the functions $u^+, u^- : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by

$$u^+(x) := \limsup_{t\to\infty}{}^*(u(x,t)+ct) = \lim_{t\to\infty} \sup\{u(y,s)+cs \mid s \geq t, |x-y| \leq 1/t\},$$
$$u^-(x) := \liminf_{t\to\infty}{}_*(u(x,t)+ct) = \lim_{t\to\infty} \inf\{u(y,s)+cs \mid s \geq t, |x-y| \leq 1/t\}.$$

Since m is Lipschitz continuous, we have

$$|H(x,p) - H(x,q)| \leq C(|x|+1)|p-q|$$

for all $x \in \mathbb{R}^n, p, q \in \mathbb{R}^n$, some $C > 0$. Therefore, the comparison principle for unbounded viscosity solutions of (C) holds. See Theorem C.1 in Appendix C (and also [**64**, Theorem 2.5] or [**5**, Theorem III.3.15]). By Proposition 7.14 we have $u^+ \in \mathrm{USC}\,(\Omega_e \cup K_c, \mathbb{R}), u^- \in \mathrm{LSC}\,(\Omega_e \cup K_c, \mathbb{R})$ and

$$u^+(x) = +\infty \text{ and } u^-(x) = +\infty \text{ for all } x \in \Omega_d.$$

By the standard stability theorem in the theory of viscosity solution, we see that u^+ satisfies

$$H(x, \nabla u^+(x)) \leq f(x) \quad \text{in } \Omega_e$$

in the viscosity sense, which implies

$$\|\nabla u^+(x)\| \leq h(x) \quad \text{in } \Omega_e.$$

In view of Lemma 7.3, the function u^+ is uniformly continuous in Ω_e, and therefore u^+ can be extended uniquely to the function on $\overline{\Omega}_e$ by continuity. We denote this function by $v^+ \in C(\overline{\Omega}_e)$, i.e., we set

$$v^+(x) := \begin{cases} u^+(x) & \text{for all } x \in \Omega_e, \\ \lim_{y\to x, y\in\Omega_e} u^+(y) & \text{for all } x \in \partial\Omega_e. \end{cases}$$

REMARK 7.6. We do not know whether $u^+ = v^+$ on K_c or not.

PROPOSITION 7.15 (Stability for half-relaxed limits). *The functions v^+ and u^- are a subsolution and a supersolution of* (N) *respectively.*

PROOF. We only need to prove that u^- satisfies the boundary condition defined by Definition 7.1. Let $\varepsilon > 0$ and set $u^\varepsilon(x,t) := u(x,t/\varepsilon) + ct/\varepsilon$. Then it is clear that u^ε satisfies

$$\varepsilon u^\varepsilon_t + H(x, \nabla u^\varepsilon) = f(x) + c \quad \text{in } \mathbb{R}^n \times (0,\infty)$$

in the viscosity sense. Set

$$\underline{u}(x,t) := (\liminf_{\varepsilon\to 0}{}_* u^\varepsilon)(x,t)$$

and then we have $\underline{u}(x,t) = u^-(x)$ for all $(x,t) \in \overline{\Omega}_e \times (0,\infty)$.

Suppose that $u^- - \phi$ take a strict minimum over $\overline{\Omega}_e$ at some point $x_0 \in \partial\Omega_e$ and some function $\phi \in C^1(\overline{\Omega}_e)$. Then there exists a function $\psi \in C^1(\mathbb{R}^n \times (0,\infty))$ such that $\underline{u} - \psi$ takes a strict minimum at $(x_0, t_0) \in \partial\Omega_e \times (0,\infty)$ for some $t_0 > 0$.

By standard arguments, there exist $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,1)$ and $\{(x_k,t_k)\}_{k\in\mathbb{N}} \subset \overline{\Omega}_e \times (0,\infty)$ such that

$$\varepsilon_k \to 0, \quad (x_k,t_k) \to (x_0,t_0),$$
$$u^{\varepsilon_k}(x_k,t_k) \to \underline{u}(x_0,t_0) \quad \text{as } n \to +\infty$$

and

$$(u^{\varepsilon_k} - \psi)(x_k, t_k) = \min_{\mathbb{R}^n \times (0,\infty)} (u^{\varepsilon_k} - \psi).$$

Then we have

$$\varepsilon_k \psi_t(x_k, t_k) + H(x_k, \nabla\psi(x_k, t_k)) \geq f(x_k) + c.$$

Sending $k \to \infty$ yields

$$H(x_0, \nabla\psi(x_0, t_0)) \geq f(x_0) + c = \sigma(x_0)$$

Since $|\nabla\psi(x_0, t_0)| < +\infty$ and $m(\|p\|) < 1$ for all $p \in \mathbb{R}^n$, we see that $\sigma(x_0) > \sigma(x_0)m(\|\nabla\psi(x_0, t_0)\|) = H(x_0, \nabla\psi(x_0, t_0))$, which implies a contradiction. □

LEMMA 7.16 (Monotonicity on $\mathcal{A}$). *For any $x \in \mathcal{A}$, the function $t \mapsto u(x,t) + ct$ is nonincreasing in $(0, \infty)$.*

REMARK 7.7. By (B1) we have

$$H(x,p) - f(x) \geq H(x, 0) - f(x) \text{ and } \max_{x \in \mathbb{R}^n}(H(x,0) - f(x)) = c,$$

which is essentially the same as (5.4). In [**92**, Lemma 2.4] it is proved that the same monotonicity result under the assumption that u is Lipschitz continuous on $\mathcal{M} \times [0, \infty)$, where $\mathcal{M}$ is a smooth compact n-dimensional manifold without boundary. It is worthwhile to mention that in the proof of the above lemma we do not use any regularity of u.

PROOF. We prove that

$$u(x, t + \delta) + c(t + \delta) \leq u(x,t) + ct \quad \text{for all } x \in \mathcal{A}, t, \delta > 0.$$

We argue by contradiction. Suppose that there would exist $x_0 \in \mathcal{A}$, $t_0 > 0$ and $\delta_0 > 0$ such that $u(x_0, t_0 + \delta_0) + c(t_0 + \delta_0) > u(x_0, t_0) + ct_0$.

Choose $r > 0$ and $\delta' \in (0, \delta_0)$ such that

$$u(x, t_0 + \delta') + c(t_0 + \delta') > u(x, t_0) + ct_0 \quad \text{for all } x \in \overline{B}_r(x_0).$$

Take $\varepsilon > 0$ so small that we have

$$u(x,t_0) + ct_0 - \frac{\varepsilon}{\delta_0} < u(x, t_0 + \delta') + c(t_0 + \delta') - \frac{\varepsilon}{\delta_0 - \delta'} \quad \text{for all } x \in \overline{B}_r(x_0). \tag{7.44}$$

For any $\alpha > 0$, we define the function $\Phi : B_r(x_0) \times (t_0, t_0 + \delta_0) \to \mathbb{R}$ by

$$\Phi(x,t) := u(x,t) + ct - \frac{\alpha|x - x_0|^2}{|x - x_0|^2 - r^2} - \frac{|x - x_0|^2}{\alpha} - \frac{\varepsilon}{t_0 + \delta_0 - t}.$$

Note that

$$\Phi(x, t_0) < \Phi(x, t_0 + \delta') \quad \text{for all } x \in B_r(x_0) \text{ in view of (7.44)},$$
$$\Phi(x,t) \to -\infty \quad \text{as } t \to t_0 + \delta_0 \text{ for all } x \in B_r(x_0),$$
$$\Phi(x,t) \to -\infty \quad \text{as } |x - x_0| \to r \text{ for all } s \in (t_0, t_0 + \delta_0).$$

Their properties imply that there exists an interior point $(x_\alpha, t_\alpha) \in B_r(x_0) \times (t_0, t_0 + \delta_0)$ such that

$$\Phi(x_\alpha, t_\alpha) = \max_{B_{r_0}(x_0) \times [t_0, t_0 + \delta_0)} \Phi(x,t).$$

Since $\Phi(x_\alpha,t_\alpha)\ge\Phi(x_0,t_\alpha)$ and $u+ct$ is bounded on $\mathcal{A}\times[0,\infty)$, we observe that

$$\begin{aligned}&\frac{\alpha|x_\alpha-x_0|^2}{|x_\alpha-x_0|^2-r^2}+\frac{|x_\alpha-x_0|^2}{\alpha}+\frac{\varepsilon}{t_0+\delta_0-t_\alpha}\\ \le\,& u(x_\alpha,t_\alpha)+ct_\alpha-(u(x_0,t_0)+ct_0)+\frac{\delta}{t_0}\le C\end{aligned}$$

and we have $x_\alpha\to x_0$ as $\alpha\to 0$. We also may assume $t_\alpha\to\bar t$ as $\alpha\to 0$ for some $\bar t(=\bar t(\varepsilon))\in[t_0,t_0+\delta_0)$ by taking a subsequence if necessary.

Thus, the definition of viscosity solutions immediately implies the following inequality:

$$\begin{aligned}\frac{\varepsilon}{(t_0+\delta_0-t_\alpha)^2}-c\le&-H(x_\alpha,\frac{-2\alpha r^2(x_\alpha-x_0)}{(|x_\alpha-x_0|^2-r^2)^2}+\frac{2(x_\alpha-x_0)}{\alpha})+f(x_\alpha)\\ \le&-\sigma(x_\alpha)m_0+f(x_\alpha).\end{aligned}$$

Sending $\alpha\to 0$, we get

$$\frac{\varepsilon}{(t_0+\delta_0-\bar t)^2}-c\le-\sigma(x_0)m_0+f(x_0)=-c,$$

which is a contradiction. □

LEMMA 7.17 (Uniform continuity around $\mathcal{A}$). *There exists a constant $r>0$ such that the solution u of* (C) *is uniformly continuous on $\mathcal{A}_r\times[0,\infty)$, where $\mathcal{A}_r:=\{x+y\mid x\in\mathcal{A},y\in B_r(0)\}$.*

PROOF. We first assume that $u_0\in C^1(\mathbb{R}^n)\cap W^{1,\infty}(\mathbb{R}^n)$ and f is bounded. We set

$$\begin{aligned}\overline{m}&:=\sup_{x\in\mathbb{R}^n}m(\|\nabla u_0(x)\|),\\ v^-(x,t)&:=u_0(x)-\overline{\sigma}\,\overline{m}t.\end{aligned}$$

We note that $\overline{m}<1$. We can easily check that v^- is a subsolution of (C). By the comparison principle for (C) we have $u_0(x)-u(x,t)\le\overline{\sigma}\,\overline{m}$ and moreover by the maximum principle we obtain $u(x,s)-u(x,t+s)\le\overline{\sigma}\,\overline{m}$ for all $x\in\mathbb{R}^n$ and $t,s\ge 0$. Thus, we get

$$u_t(x,t)\ge-\overline{\sigma}\,\overline{m}\text{ in }(0,\infty)\text{ in the viscosity sense}$$

for all $x\in\mathbb{R}^n$ and therefore

$$\sigma(x)m(\|\nabla u\|)\le f(x)+\overline{\sigma}\,\overline{m}\text{ in }\mathbb{R}$$

for all $t\in(0,\infty)$.

Since $\sigma(x)=\overline{\sigma}$, $f(x)=0$ for all $x\in\mathcal{A}$ and $\overline{m}<1$, there exists a constant $r>0$ such that

$$\sigma(x)>f(x)+\overline{\sigma}\,\overline{m}$$

for all $x\in\mathcal{A}_r$. Therefore we see that

$$|\nabla u(x,t)|\le C_1\|\nabla u(x,t)\|\le C_1m^{-1}\Big(\frac{f(x)+\overline{\sigma}\,\overline{m}}{\sigma(x)}\Big)\le C_2\text{ in }\mathcal{A}_r$$

in the viscosity sense for some $C_1,C_2>0$ and all $t\in[0,\infty)$. By [28, Proposition 8.1] we obtain $\{u(\cdot,t)\}_{t\ge0}$ is equicontinuous on $\mathcal{A}_r$. We also see that

$v^+(x,t) := u_0(x) + C_f t$ is a supersolution of (C), where $C_f := \sup f$ and therefore we have $|u(x,t+s) - u(x,s)| \le \overline{\sigma}\,\overline{m} \vee C_f$ for all $x \in \mathbb{R}^n$ and $t, s \ge 0$. Thus, we see that $u \in \mathrm{UC}(\mathcal{A}_r \times [0,\infty))$.

We finally remove the regularity of assumption on u_0 and the boundedness of f. We can choose a sequences functions $\{u_0^k\}_{k\in\mathbb{N}}$ of class $C^1(\mathbb{R}^n)$ and Lipschitz continuous such that u_0^k converges local uniformly to u_0 as $k \to \infty$ and $\{f^k\}_{k\in\mathbb{N}} \subset C(\mathbb{R}^n)$ such that f^k are bounded, $\{f^k(\cdot) = 0\} = \{f(\cdot) = 0\}$ for all $k \in \mathbb{N}$ and f^k converges local uniformly to f as $k \to \infty$. By the maximum principle we see that the solutions $u^k \in \mathrm{UC}(\mathcal{A}_r \times [0,\infty))$ of (C) with $u_0 = u_0^k$ converges local uniformly to u as $k \to \infty$. We note that we can choose such a r is independent of k, since we are assuming that u_0 is Lipschitz. From this observation we see that $u \in \mathrm{UC}(\mathcal{A}_r \times [0,\infty))$. □

COROLLARY 7.18 (Convergence on $\mathcal{A}$). *The function $u(x,t) + ct$ converges uniformly on $\mathcal{A}$ as $t \to \infty$. Moreover, we have*

$$v^+(x) = u^-(x) \quad \text{for all } x \in \mathcal{A}.$$

PROOF. By Lemma 7.17 we see that $v^+(= u^+), u^- \in C(\mathcal{A})$. In view of Dini's theorem we see that $u + ct$ converges uniformly on $\mathcal{A}$ as $t \to \infty$ and by a property of half relaxed limit we obtain $v^+ = u^+ = u^-$ on $\mathcal{A}$. □

PROOF OF THEOREM 7.13. A straightforward consequence of Proposition 7.15 and Corollary 7.18 is $v^+ = u^-$ on $\overline{\Omega}_e$. Thus, we get

$$u^+(x) = u^-(x) =: v_\infty(x) \quad \text{for all } x \in \Omega_e,$$

which implies that

$$u(x,t) + ct \to v_\infty(x) \quad \text{uniformly on each compact subset of } \Omega_e \text{ as } t \to \infty$$

for a solution $v_\infty \in C(\overline{\Omega}_e)$ of (N). It is clear to see that

$$u(x,t) + ct \to +\infty \quad \text{uniformly on each compact subset of } \Omega_d$$

as $t \to \infty$ from Proposition 7.14. □

REMARK 7.8. For generalization, it is worthwhile to say that if we can prove the comparison principle for (N), then we obtain a similar convergence results on viscosity solution of

$$\begin{cases} u_t + \sigma(x) m\big(F(x, \nabla u)\big) = f(x) & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n, \end{cases}$$

where the function F is assumed to satisfy (A1)–(A7) and the functions σ, m, f, u_0 are assumed to satisfy (B1)–(B5).

7.2.4. Asymptotic Profile on the Effective Domain Ω_e. We define the functions $\phi_-, \phi_\infty \in C(\overline{\Omega}_e)$ by

$$\begin{aligned} \phi_-(x) &:= \inf_{t\ge 0}(u(x,t) + ct), \\ \phi_\infty(x) &:= \min\{d(x,y) + \phi_-(y) \mid y \in \mathcal{A}\}, \end{aligned} \tag{7.45}$$

$$d(x,y) := \sup\{v(x) - v(y) \mid v \text{ is a viscosity subsolution of (7.41)}\}.$$

Note that in view of Theorem 7.4 and Proposition 7.7 (ii), we see that ϕ_∞ is a solution of (S).

THEOREM 7.19 (Asymptotic Profile). *We have*

$$\phi_\infty(x) = \lim_{t\to\infty}(u(x,t)+ct) \quad \textit{for all } x \in \Omega_e. \tag{7.46}$$

PROOF. We write $u_\infty(x)$ for the right hand side of (7.46) for $x \in \Omega_e$. By the definition of ϕ_- we have $\phi_-(x) \le u(x,t) + ct$ for all $(x,t) \in \overline{\Omega}_e \times [0,\infty)$, which implies that $\phi_- \le u_\infty$ in Ω_e. By the definition of ϕ_∞ we have $\phi_\infty \le \phi_-$ on $\mathcal{A}$. Therefore, we have $\phi_\infty \le u_\infty$ on $\mathcal{A}$. By Theorem 7.8 we get $\phi_\infty \le u_\infty$ on $\overline{\Omega}_e$. □

7.2.5. A model describing growing faceted crystals. In this subsection we consider the equation for a model of a growing faceted crystal as explained in Introduction (see [**19, 26, 113**] also) and for the reader's convenience we give a simple form of our asymptotic result which is by no means optimal. We consider

$$u_t + \sigma(x)m(\|\nabla u\|) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{7.47}$$

where $m \in C([0,\infty),(0,1))$ which satisfies (B1) and

$$m(0) = m_0 \in (0,1) \text{ and } m(r) = r\tanh(1/r) \text{ if } m_0 \le r\tanh(1/r) \tag{7.48}$$

and $\sigma \in C^1(\mathbb{R}^n,(0,\infty))$ is a function which attains the maximum $\overline{\sigma}$ and satisfies $\sigma(x) < \overline{\sigma}m_0$ for any $x \in \mathbb{R}^n \setminus B_r(0)$ and a suitable large $r > 0$. In this case we have

$$\begin{aligned} &c = \overline{\sigma}m_0,\\ &\Omega_e = \{x \in \mathbb{R}^n \mid \sigma(x) > c\},\\ &\Omega_d = \{x \in \mathbb{R}^n \mid \sigma(x) < c\} \text{ and}\\ &\mathcal{A} := \{x \in \mathbb{R}^n \mid \sigma(x) = \overline{\sigma}\}. \end{aligned}$$

We set

$$h(x) := m^{-1}\left(\frac{c}{\sigma(x)}\right).$$

COROLLARY 7.20. *Assume that m satisfies* (B1), (7.48) *and that* $\sigma \in C^1(\mathbb{R}^n,(0,\infty))$ *satisfies*

$$\Omega_e^\alpha := \{\sigma(\cdot) > c+\alpha\} \textit{ are bounded and convex for all } \alpha \in (0,\alpha_0]$$

for a small $\alpha_0 > 0$ and

$$\nabla\sigma(x) \ne 0 \textit{ on } \partial\Omega_e.$$

Let u be a solution of (7.47) *with $u(\cdot,0) = u_0$ being a Lipschitz continuous function in $\mathbb{R}^n$, and then we have the result of large-time asymptotics given by Theorems* 7.13 *and* 7.19, *i.e.,*

$$u(\cdot,t) + ct \to \phi_\infty \quad \textit{uniformly on each compact subset of } \Omega_e$$

and

$$u(x,t) + ct \to +\infty \quad \textit{uniformly on each compact subset of } \Omega_d$$

as $t \to +\infty$, where ϕ_∞ is the function given by (7.45).

PROOF. We only need to verify that h satisfies (A7). We first notice that by (7.48) we have

$$(0<)1-m(r)\leq e^{-r} \text{ for suitable large } r>0,$$

which implies $m^{-1}(1-e^{-r})\leq r$ for suitable large $r>0$. Thus we have

$$\begin{aligned} h(x)=m^{-1}\Big(\frac{c}{\sigma(x)}\Big)&=m^{-1}\big(1-e^{\log\left(\frac{\sigma(x)-c}{\sigma(x)}\right)}\big)\\ &\leq -\log\Big(\frac{\sigma(x)-c}{\sigma(x)}\Big). \end{aligned}$$

We only prove that h satisfies (A7) in $\{\sigma(\cdot)\leq c+\alpha_0\}$. Otherwise it is easy to see that h satisfies (A7), since h is bounded on $\mathbb{R}^n\setminus\{\sigma(\cdot)>c+\alpha_0\}$. Fix any $x_c\in\partial\Omega_e$ and then for any $x\in\{\sigma(\cdot)\leq c+\alpha_0\}$ we have

$$\begin{aligned} &\int_0^1 h(sx+(1-s)x_c)|x-x_c|\,ds\\ \leq&|x-x_c|\int_0^1\Big[-\log\Big(\frac{s\nabla\sigma(x_c)\cdot(x-x_c)+o(s|x-x_c|)}{\sigma(sx+(1-s)x_c)}\Big)\Big]\,ds\\ \leq&C|x-x_c|\big|\log|x-x_c|\big| \end{aligned}$$

for some $C>0$, where $o:[0,\infty)\to[0,\infty)$ is a function which satisfies $o(r)/r\to 0$ as $r\to 0$. Therefore setting $\omega(r):=Cr|\log(r)|$ for $r>0$, then we see that h satisfies (A7). □

We give an example which we can calculate the function ϕ_∞ given by (7.45) concretely. Let $n=1$ and let us set $u_0\equiv 0$, $\sigma(x)=\overline{\sigma}(1-x^2)_+$ and $m(\|\cdot\|)=m(|\cdot|)$ as considered in **[113]**, where $r_+:=\max\{r,0\}$ for $r\in\mathbb{R}$. Then we have $c=\overline{\sigma}m_0$, $\Omega_e=(-\sqrt{1-m_0},\sqrt{1-m_0})$, $\mathcal{A}=\{0\}$ and $h(x)=m^{-1}(m_0/(1-x^2)_+)$. Note that 0 is a subsolution of (C) at this case and therefore we have

$$0\leq u\leq u+ct \text{ on } \mathbb{R}^n\times[0,\infty),$$

which implies $\phi_-(x)=\inf_{t\geq 0}(u(x,t)+ct)=0$. Therefore we have $\phi_\infty(x)=\min_{y\in\mathcal{A}}\{d(x,y)+\phi_-(y)\}=d(x,0)$. Moreover noting that $L(x,\xi)=0$ if $|\xi|\leq 1$ and $L(x,\xi)=+\infty$ if $|\xi|>1$ for all $x\in\Omega_e$, by (7.10) we have for any $x\in\Omega_e$

$$\begin{aligned} d(x,0)&=\inf\{\int_0^t h(\gamma(s))\,ds\mid t>0,\gamma\in\mathcal{C}(x,t;0,0),|\dot\gamma(s)|\leq 1\}\\ &=\int_0^x h(s)\,ds=\int_0^x m^{-1}\Big(\frac{m_0}{(1-s^2)_+}\Big)\,ds. \end{aligned}$$

Thus we obtain

$$\phi_\infty(x)=\int_0^x m^{-1}\Big(\frac{m_0}{(1-s^2)_+}\Big)\,ds \text{ for all } x\in\overline{\Omega}_e.$$

Finally we note that in the case where $c=\overline{\sigma}m_0<\sigma(x)<\overline{\sigma}$ for all $x\in\mathbb{R}^n$, we have $\Omega_e=\mathbb{R}^n$. In this case the large-time asymptotic of solutions of (C) is similar to that of solutions of coercive HJ equations. More precisely, if we assume the periodicity of σ and u_0, i.e.,

$$\sigma(x+e_i)=\sigma(x),\ u_0(x+e_i)=u_0(x)$$

for all $x \in \mathbb{R}^n$ and $i \in \{1, \dots, n\}$, where $\{e_i\}_i$ is the canonical basis of $\mathbb{R}^n$, then any solution u of (7.47) has the large-time asymptotic

$$u(x,t) + ct \to v(x) \quad \text{uniformly for } x \in \mathbb{R}^n$$

as $t \to \infty$, where $v \in C(\mathbb{R}^n)$ is a viscosity solution of

$$\begin{aligned} &\sigma(x) m(\|\nabla u\|) = c \quad \text{in } \mathbb{R}^n, \\ &u(x + e_i) = u(x) \quad \text{for all } x \in \mathbb{R}^n, i \in \{1, \dots, N\}. \end{aligned} \tag{7.49}$$

We give a sketch of the proof here. Since there exists a periodic viscosity solution of (7.49) in $C(\mathbb{R}^n)$, we see that $u + ct$ is bounded in $\mathbb{R}^n \times [0, \infty)$. Therefore we can define the functions $u^+(x) = \limsup^*_{t\to\infty}(u+ct)$ and $u^-(x) = \liminf_{t\to\infty *}(u + ct)$ for all $x \in \mathbb{R}^n$ and moreover we see that $u^+ = u^-$ on $\mathcal{A}$. By a comparison principle (see [44, Theorem 6.7]) we obtain $u^+ = u^-$ in $\mathbb{R}^n$.

In the theory of crystal growth, it is known that as long as the non-uniformity in supersaturation on the facet is not too large, it can be compensated by a variation of step density along the facet and the faceted crystal can grow in a stable manner. The convergence described above explains this phenomenon from mathematical point of view.

CHAPTER 8

Noncoercive Ergodic Problems

We discuss some ergodic problems for (noncoercive) Hamilton-Jacobi equations in this chapter. Namely, we are concerned with the following equations:

$$\begin{cases} u_t + H(x, \nabla u) = 0, & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x), & \text{in } \mathbb{R}^n. \end{cases} \tag{8.1}$$

and

$$\lambda v_\lambda + H(x, \nabla v_\lambda) = 0, \quad \text{in } \mathbb{R}^n. \tag{8.2}$$

We are interested in the asymptotics of u/t as $t \to \infty$ for (8.1) and of λu as $\lambda \to 0$ for (8.2). It is usually meaningless to study a noncoercive Hamilton equation with the Hamiltonian having finite limit at $|p| = \infty$, since its solution scarcely agrees with any optimal control or game problem with compact control sets. However, the application in crystal growth brings our research value.

Our study in this chapter follows [**86**].

8.1. Erogodicity

Let us recall a simple but typical result for ergodicity; see e.g. [**5**] or [**4**]. If we assume that everything is periodic in space, then there exists a constant c uniquely determined by H such that

$$\lim_{t\to\infty} u(x,t)/t = \lim_{\lambda\to 0} \lambda v_\lambda = c \quad \text{uniformly in } x, \tag{8.3}$$

provided that H is coercive in p:

$$\inf_{x\in\mathbb{R}^n} H(x,p) \to \infty, \quad \text{as } |p| \to \infty.$$

A challenging question is how to study the property (8.3) without the coercivity assumption.

We provide two very easy observations about this question. One is for the ergodic and the other is for the nonergodic. The former is a direct extension of the results by Arisawa and Lions [**4**]. Closely examining their proofs, one will find that the coercivity is used only to give an equicontinuity of solutions, which actually comes from the fact that $\{H(x,\infty) : x \in \mathbb{R}^n\} = \{\infty\}$ and the bounded set $\{H(x,0) : x \in \mathbb{R}^n\}$ have empty intersection.

To be more precise, let us denote

$$\begin{aligned} \mathcal{H}_\infty = \{a \in \mathbb{R} \cup \{\pm\infty\} : H(x_m, p_m) \to a \text{ as } m \to \infty \text{ for sequences} \\ \{x_m\}, \{p_m\} \subset \mathbb{R}^n \text{ with } |p_m| \to \infty \text{ as } m \to \infty\} \end{aligned}$$

and

$$\mathcal{H}_0 = \{H(x,0) : x \in \mathbb{R}^n\}$$

In general, given a closed set $K \subset \mathbb{R}^n$, we denote

$$\mathcal{H}_\infty(K) = \{a \in \mathbb{R} \cup \{\pm\infty\} : H(x_m, p_m) \to a \text{ as } m \to \infty \text{ for sequences } \{x_m\} \subset K, \{p_m\} \subset \mathbb{R}^n \text{ with } |p_m| \to \infty \text{ as } m \to \infty\}$$

and

$$\mathcal{H}_0(K) = \{H(x,0) : x \in K\}.$$

We therefore can believe that the following holds.

THEOREM 8.1. *Assume $H(x,p)$ is periodic in x. Let u_0 be periodic and continuous. Assume that $\mathcal{H}_\infty \cap \mathcal{H}_0 = \emptyset$. Then*

$$\lim_{t\to\infty} u(x,t)/t = \lim_{\lambda\to 0} \lambda v_\lambda = c \quad \textit{uniformly in } x. \tag{8.4}$$

and c is the unique constant such that there exists a periodic solution of

$$H(x, \nabla v(x)) = c. \tag{8.5}$$

PROOF. We follow the proof of [Arisawa-Lions, Theorem II.1].
Step 1 (Equicontinuity). We attempt to show that the solution v_λ of (8.2) satisfies

$$\begin{aligned} &|\nabla v_\lambda| \le L \text{ or } -|\nabla v_\lambda| \ge -L \\ &\text{in the viscosity sense for some } L > 0 \text{ depending only on } H \end{aligned} \tag{8.6}$$

The proof is somewhat standard. Take

$$M_1 = \sup\{H(x,0) : x \in \mathbb{R}^n\} \text{ and } M_2 = \inf\{H(x,0) : x \in \mathbb{R}^n\}.$$

so that we have $\mathcal{H}_0 = [M_2, M_1]$. Then it is easy to see that $-M_1/\lambda$ and $-M_2/\lambda$ are respectively a subsolution and a supersolution of (8.2). By comparison principle, we thus have

$$-M_1 \le \lambda v_\lambda \le -M_2. \tag{8.7}$$

Hence, we deduce that

$$M_2 \le H(x, \nabla v_\lambda) \le M_1 \quad \text{holds in the viscosity sense.} \tag{8.8}$$

Now that (8.8) is obtained, the Lipschitz estimate (8.6) follows easily. Indeed, if it is not the case, then for an arbitrarily large $L > 0$ there are two smooth function ϕ_L^1 and ϕ_L^2 which touch v_λ from above at some $x = x_L^1$ and from below at some $x = x_L^2$ respectively with $|\nabla\phi_L^1(x_L^1)| > L$ and $-|\nabla\phi_L^2(x_L^2)| < -L$. By (8.8), we get

$$H(x_L^1, \nabla\phi_L^1(x_L^1)) \le M_1 \text{ and } H(x_L^2, \nabla\phi_L^2(x_L^2)) \ge M_2.$$

Sending $L \to \infty$, we get a contradiction to $\mathcal{H}_\infty \cap [M_2, M_1] = \emptyset$.

We remark that the solution u of (8.1) does not necessarily satisfy the Lipschitz estimate in x, for the condition $\mathcal{H}_\infty \cap \mathcal{H}_0 = \emptyset$ is not sufficient. One needs a stronger condition like

$$\mathcal{H}_\infty \cap \{H(x,p) : x \in \mathbb{R}^n \text{ and } |p| \le Lip(u_0)\} = \emptyset; \text{ (see Remark 3).}$$

Step 2 (Convergence). In terms of (8.6), it is not hard to show that

$$|v_\lambda(x) - v_\lambda(y)| \le L|x - y| \quad \text{for all } x, y \in \mathbb{R}^n \text{ and } t \ge 0. \tag{8.9}$$

Since the solution v_λ is periodic in x, it follows from Arzela-Ascoli theorem that we can take a subsequence λ_m so that $\lambda_m v_{\lambda_m}$ converges to a constant c. (It should be a constant because the Lipschitz constant $\lambda_m L$ tends to 0.)

Notice that $v_{\lambda_m}(x) - v_{\lambda_m}(0)$ is bounded uniformly and equicontinuous in x. Passing to a further subsequence, we can prove that $v_{\lambda_m}(x) - v_{\lambda_m}(0)$ converges to a continuous periodic function v_0, which is clearly a viscosity solution

$$c + H(x, \nabla v_0) = 0.$$

We next show $\lim_{t\to\infty} u/t = c$ in spite of the absence of equicontinuity for u. To do this, we find a subsolution $v_0(x) + ct - C$ and a supersolution $v_0(x) + ct + C$ with a sufficiently large $C \geq ||u_0||_\infty + ||v_0||_\infty$ such that by comparison principle for (8.1)

$$v_0(x) + ct - C \leq u(x,t) \leq v_0(x) + ct + C$$

and consequently it follows that $\lim_{t\to\infty} u/t = c$.

We finally show that c is the only constant for which one can find a periodic solution v of (8.5). Suppose by contradiction that there are two constants, denoted by c_1 and c_2, and two periodic solutions V_1 and V_2 of the corresponding equations. Without loss we assume $c_1 > c_2$ and by shifting by a constant, we assume $V_1(0) > V_2(0)$. Then it is easily seen that

$$H(x, \nabla V_1) + \lambda V_1 = \lambda V_1 - c_1 \leq \lambda V_2 - c_2 = H(x, \nabla V_2) + \lambda V_2$$

for $\lambda > 0$ small enough. By comparison principle of (8.2), we get $V_1 \leq V_2$ in $\mathbb{R}^n$, which is a contradiction to our assumption $V_1(0) > V_2(0)$. With this conclusion, we see that there is no need to take subsequence λ_m and we have $\lim_{\lambda\to 0} \lambda v_\lambda = c$. □

REMARK 8.1. We assume periodicity mainly for the boundedness of the domain. In our singular Neumann problems, the effective domain is exactly the maximal Ω satisfying $\mathcal{H}_\infty(K) \cap \mathcal{H}_0(K) = \emptyset$ for all compact sets K in Ω. Note that we have assumed the boundedness of the effective domain.

REMARK 8.2. We can use the same argument in Step 1 to show

$$\begin{cases} |\nabla u| \leq L \text{ or } -|\nabla u| \geq -L \\ \text{in the viscosity sense for some } L > 0 \text{ depending only on } H \end{cases} \tag{8.10}$$

provided that

$$\mathcal{H}_\infty \cap \mathcal{H}_{u_0} = \emptyset.$$

where

$$\mathcal{H}_{u_0} = \{H(x,p) : x \in \mathbb{R}^n \text{ and } |p| \leq Lip(u_0)\}$$

We sketch the proof first for the case $u_0 \in C^1(\mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^n)$. Take

$$M_1' = \sup \mathcal{H}_{u_0} \text{ and } M_2' = \inf \mathcal{H}_{u_0}.$$

so that we have $\mathcal{H}_{u_0} = [M_2', M_1']$. Then it is easy to see that $u_0(x) - M_1't$ and $u_0(x) - M_2't$ are respectively a subsolution and a supersolution of (8.1). By comparison principle, we thus have

$$u_0(x) - M_1't \leq u(x,t) \leq u_0(x) - M_2't \tag{8.11}$$

and consequently

$$-M_1'h \leq u(x, t+h) - u(x,t) \leq -M_2'h \tag{8.12}$$

for all $x \in \mathbb{R}$ and $t, h \geq 0$, which roughly says that $-M_1' \leq u_t \leq -M_2'$. We therefore claim that

$$M_2' \leq H(x, \nabla u) \leq M_1' \quad \text{holds in the viscosity sense.} \tag{8.13}$$

We only prove the second inequality in the following and the first one can be similarly proved. Suppose that we have a smooth function ϕ such that there exists $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$ fulfilling

$$u(x,t) - \phi(x,t) \leq u(x_0, t_0) - \phi(x_0, t_0) \quad \text{for all } (x,t) \in \mathbb{R}^n \times (0, \infty).$$

Applying (8.12) with $x = x_0$, we obtain

$$\phi(x_0, t) - \phi(x_0, t_0) \geq u(x_0, t) - u(x_0, t_0) \geq -M_1'(t - t_0), \quad \text{for any } t \geq t_0,$$

which implies

$$\phi_t(x_0, t_0) \geq -M_1'. \tag{8.14}$$

This yields

$$H(x_0, \nabla\phi(x_0, t_0)) \leq M_1'$$

Once one gets (8.13), the rest of proof follows Step 1 (from (8)).

We finally remove the C^1 regularity of assumption on u_0 and the boundedness of f. We can choose a sequences functions $\{u_0^k\}_{k\in\mathbb{N}} \subset C^1(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$ such that u_0^k converges uniformly to u_0 as $k \to \infty$. Then the comparison principle implies that the corresponding solution u^k converges uniformly to the solution u. As our argument above shows, u^k satisfies (8.10). Then so does u in terms of the classical stability of viscosity solutions.

8.2. Nonergodicity

We next turn to a nonergodic problem. By nonergodicity, we mean $\lim_{t\to\infty} u/t = \lim_{\lambda\to 0} \lambda v_\lambda$ is a nonconstant function. In this section, we use nothing but the comparison principle. Assume that

$$\begin{cases} \exists f_0 \in C(\mathbb{R}^n) \text{ satisfying that for } \forall\varepsilon > 0,\ \exists\lambda > 0 \text{ small such that} \\ \text{the following inequalities hold in the viscosity sense for all } x \in \mathbb{R}^n \\ f_0(x) + H(x, \frac{1}{\lambda}\nabla f_0(x)) \leq \varepsilon; \\ f_0(x) + H(x, \frac{1}{\lambda}\nabla f_0(x)) \geq -\varepsilon. \end{cases} \tag{8.15}$$

THEOREM 8.2. *Assume (8.15). Then the solutions u of (8.1) and v_λ of (8.2) satisfy*

$$\lim_{t\to\infty} u(x,t)/t = \lim_{\lambda\to 0} \lambda v_\lambda(x) = f_0(x) \ \textit{uniformly in } x. \tag{8.16}$$

PROOF. The proof is quite elementary. In terms of (8.15), it is not difficult to see that, for any $\varepsilon > 0$, $(f_0 - \varepsilon)/\lambda$ and $(f_0 + \varepsilon)/\lambda$ are respectively a subsolution and a supersolution provided λ is taken sufficiently small. By comparison principle, we have

$$f_0 - \varepsilon \leq \lambda v_\lambda \leq f_0 + \varepsilon.$$

Similarly, we can construct a subsolution and a supersolution for the equation (8.1) and the conclusion follows. □

This result is too easy to be called a theorem. However it seems to contain many interesting facts which we rarely notice. Let us focus our attention on the assumption (8.15). We give a sufficient condition.

PROPOSITION 8.3. *Assume that there exists $h \in C(\mathbb{R}^n)$ such that*

$$H(x,p) \to h(x) \text{ as } |p| \to \infty \text{ uniformly in } x \in \mathbb{R}^n. \tag{8.17}$$

If the viscosity inequalities $|\nabla h| \geq C$ and $-|\nabla h| \leq -C$ hold uniformly in x for some constant $C > 0$, then (8.15) holds with $f_0 = -h$.

PROOF. Take ϕ as a test function for $f_0 = -h$ at $x = x_0$ from above. Then we have $-\nabla\phi(x_0) \geq C$. Since h is a uniform limit of H at $p = \pm\infty$, we get

$$-h(x_0) + H(x_0, \frac{1}{\lambda}\nabla\phi(x_0)) \leq \varepsilon$$

for any $\varepsilon > 0$ as long as λ is small enough, as desired. The verification for the other viscosity inequality in (8.15) can be made analogously. □

A further result in the smooth case is as follows.

PROPOSITION 8.4. *Assume that there exist $h_0, h_1 \in C^1(\mathbb{R}^n)$ such that $|\nabla h_0| \geq C_1$ and $|\nabla h_1| \leq C^2$ for some constants C_1 and C_2 independent of x. Assume*

$$|p|(H(x,p) - h_0(x)) \to h_1(x) \text{ uniformly in } x \in \mathbb{R}^n. \tag{8.18}$$

Then the solutions u and v_λ satisfy

$$\lim_{t\to\infty} \frac{1}{\ln t}(u(x,t) + h_0(x)t) = \lim_{\lambda\to 0} v_\lambda(x) + \frac{1}{\lambda}h_0(x) = -\frac{h_1(x)}{|\nabla h_0(x)|} \text{ uniformly in } x.$$

In particular, if h_1 is bounded, then (8.16) holds with $f_0 = -h_0$.

PROOF. It is easy to verify that for every $\varepsilon > 0$, $-\frac{h_0(x)}{\lambda} - \frac{h_1(x)}{|\nabla h_0(x)|} + \varepsilon$ is a supsolution and $-\frac{h_0(x)}{\lambda} - \frac{h_1(x)}{|\nabla h_0(x)|} - \varepsilon$ is a subsolution of (8.2). We reach the conclusion by comparison principle. For the time-dependent case, the same argument applies. □

REMARK 8.3. Note that the condition (8.18) gives more careful behavior of H at $|p| = \infty$ than (8.17), so we have better description on the asymptotics of solutions than (8.16). We can follow this method to learn the asymptotic behavior up to any degree as long as the corresponding asymptotic expansion of H is known.

Finally we look at two examples.

EXAMPLE 8.5. *Let $H(x,p) = -\frac{1}{1+|p|} - x$. When $u_0 \equiv 0$, the unique solution of (8.1) is clearly $u(x,t) = xt + \ln(1+t)$ and therefore $\lim_{t\to\infty} u/t = x$. In fact, the solution u satisfies*

$$\lim_{t\to\infty} u(x,t)/t = x \quad \text{and} \quad \lim_{t\to\infty}(u(x,t) - xt)/\ln t = 1 \tag{8.19}$$

for any continuous initial data u_0.

The above example is perhaps regarded special because the Hamiltonian is not periodic (and even not bounded) in x. The second example below shows that such kind of large time behavior also appears in the periodic case.

EXAMPLE 8.6. *Let* $H(x,p) = \frac{1-2f(x)}{1+|p|} - f(x)$, *where*

$$f(x) = \begin{cases} x - 2n, & \text{if } 2n \le x < 2n+1; \\ 2n+2-x, & \text{if } 2n+1 \le x < 2n+2. \end{cases}$$

We still obtain (8.15) with $f_0 = f$. *Hence, we have*

$$\lim_{t\to\infty} u(x,t)/t = \lim_{\lambda\to 0} \lambda v_\lambda(x) = f(x) \text{ uniformly in } x.$$

Appendices

APPENDIX A

Singular Phenomena of Billiards

We study in this appendix more details about the particular termination phenomenon and see the sufficient conditions under which billiards can be well defined up to time infinity. Most contents here are taken from [**59**]. We give it here for completeness and readers' convenience.

Let us start with an example to show the phenomenon really exists if no specific assumption is imposed on the table. In this example, it can be shown that the domain is strictly convex but its third derivative is unbounded. We shall take a converging point sequence on a unit circle, and then find a C^2 curve through all those points to make the convergent points vertices of some billiard trajectory. To be more precise, we denote by S the unit circle centered at the origin. Set up polar coordinates (r, θ) and pick a sequence of points $p_n = (r_n, \theta_n) \in S$, $n = 1, 2, \ldots$ with $\theta_n = n^{-\frac{1}{2}}$. It is obvious that p_n converges to $(1, 0)$. We get a broken line by connecting them up and then choose pieces γ_n of different unit circles through p_n such that the broken line satisfies the billiard law at each p_n with respect to γ_n, which are written via polar equations as $r = r_n(\theta)$. We connect these pieces appropriately to obtain a smooth curve γ, whose polar equation indeed fulfills

$$r(\theta) = r_n(\theta) + \alpha_n(\theta)(r_{n+1}(\theta) - r_n(\theta)). \tag{A.1}$$

Here α_n are such that

$$\alpha_n(\theta) = \alpha\left(\frac{\theta - \theta_n}{\theta_n - \theta_{n+1}}\right),$$

where $\alpha : [0, 1] \to [0, 1]$ is an infinitely differentiable function such that $\alpha(t) = 1$ for $t \leq \frac{1}{3}$ and $\alpha(t) = 0$ for $t \geq \frac{2}{3}$. By constructing γ, we have given an example of the trajectory terminating in a C^2 domain.

From now on, we further focus on the regularity of γ, especially the piece near the terminating point $(1, 0)$. We calculate through Taylor's expansion and get estimates, when n is sufficiently large, say larger than some n_0,

$$\delta_n := \theta_n - \theta_{n-1} = a_n n^{-\frac{3}{2}}, \quad \omega_n := \delta_{n+1} - \delta_n = b_n n^{-\frac{5}{2}}, \tag{A.2}$$

where $a_n \to -\frac{1}{2}$ and $b_n \to -\frac{3}{4}$. Then for any $\theta \in [\theta_{n+1}, \theta_n]$,

$$|\alpha_n'(\theta)| \leq 3cn^{\frac{3}{2}}, \quad |\alpha_n''(\theta)| \leq 5cn^3, \tag{A.3}$$

where c is a bound for both $|\alpha'|$ and $|\alpha''|$. We use $r = g(\theta, \omega)$ to denote the unit circle passing through $(1, 0)$ and making an angle of $\frac{\omega}{4}$ with S at $(1, 0)$. It is well-defined and infinitely differentiable on $D = \{(\theta, \omega) : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, -\frac{\pi}{4} \leq \omega \leq \frac{\pi}{4}\}$. We can thus write

$$r_n(\theta) = g(\theta - \theta_n, \omega_n). \tag{A.4}$$

Let us assume there is a bound C for the partial derivatives of g. Then noticing $g(0,\omega) = g(\theta,0) = 1$, we have

$$\begin{aligned} &|g_1(\theta,\omega)| = C|\omega|, \quad |g_{11}(\theta,\omega)| \le C|\omega|, \\ &|g(\theta,\omega) - 1| \le C|\omega||\theta|, \end{aligned} \tag{A.5}$$

where g_1 and g_{11} respectively denote the first and second order partial derivatives of g with respect to its first variable. Consequently from (A.2), (A.4) and (A.5), it is true that for every $\theta \in [\theta_{n+1}, \theta_n]$ and n sufficiently large,

$$\begin{aligned} &|r_n'(\theta)| \le Cn^{-\frac{5}{2}}, \quad |r_n''(\theta)| \le Cn^{-\frac{5}{2}}, \\ &|r_n(\theta) - 1| \le C|\omega_n||\delta_n| \le C|a_n b_n n^{-4}| \le Cn^{-4}. \end{aligned} \tag{A.6}$$

Now if we take a large n_0 and redefine γ_{n_0} to be given by $r_{n_0}(\theta) = 1$ and only consider the piece of γ corresponding to $\theta \in [-\pi,\pi)\backslash(0,\theta_{n_0})$. Then we are led from (A.1), (A.3) and (A.6) to

$$\begin{aligned} &|r(\theta) - 1| \le 3Cn^{-4}, \quad |r'(\theta)| \le (3C + 6cC)n^{-\frac{5}{2}}, \\ &|r''(\theta)| \le 6cCn^{-1}, \quad \left|\frac{r''(\theta)}{\theta}\right| \le \frac{6cCn^{-1}}{(n+1)^{-\frac{1}{2}}}, \end{aligned}$$

for $n \ge n_0$ and $\theta \in [\theta_{n+1}, \theta_n]$. It follows that $\lim_{\theta\to 0} r(\theta) = 1$, $\lim_{\theta\to 0} r'(\theta) = 0$, $\lim_{\theta\to 0} r''(\theta) = 0$. Therefore $r'(0)$ and $r''(0)$ exist and r, r' and r'' are continuous. Moreover, as $\theta \to 0$,

$$\frac{r''(\theta) - r''(0)}{\theta} = \frac{r''(\theta)}{\theta} \to 0.$$

Hence $r'''(0)$ exists. We also see that if n_0 is picked sufficiently large then the curvature of γ can be made arbitrarily close to the curvature of S and hence never vanishes. By Theorem A.3 below, the third derivative of $\partial\Omega$ must be unbounded.

There are also several sufficient conditions for nonoccurrence of termination given below for convex billiards. We assume the billiard table Ω is bounded and convex, and consider the Poincaré map of a billiard flow T^t. Given a point $p \in \partial\Omega$ and an angle θ, $0 < \theta < \pi$, set $G(p,\theta) = (p',\theta')$ where p' is the other intersection of the directed line L which passes through p making an angle θ with the tangent to $\partial\Omega$ at p ($\partial\Omega$ is oriented counterclockwise) and θ' is the positive angle between L and the tangent to $\partial\Omega$ at p'. That is, if the billiard ball leaves from p making an angle θ with $\partial\Omega$ then p' is the next point of contact with the boundary $\partial\Omega$ and θ' is the next angle of reflection. Since Ω is convex, $G : \partial\Omega \times (0,\pi) \to \partial\Omega \times (0,\pi)$ is well defined. Set $G^n = G \circ G \ldots \circ G$ n times. Let $P_1 : \partial\Omega \times (0,\pi) \to \partial\Omega$ and $P_2 : \partial\Omega \times (0,\pi) \to (0,\pi)$ be the natural projections.

THEOREM A.1 ([**59**, Theorem 1]). *If Ω is any billiard table and* $\lim_{n\to\infty} P_1 \circ G^n(p,\theta) = q$ *then*

(a) $\lim_{n\to\infty} P_2 G^n(p,\theta) = 0$ *or* π.
(b) *Either* $\sum_{n=1}^{\infty} P_2 G^n(p,\theta) < \infty$ *or* $\sum_{n=1}^{\infty} (\pi - P_2 G^n(p,\theta)) < \infty$.
(c) $\sum_{n=0}^{\infty} |P_1 G^{n+1}(p,\theta) - P_1 G^n(p,\theta)| < \infty$.

PROOF. Set up a Cartesian coordinate system with the origin at q, the positive x-axis in the direction of the tangent to γ at q and the positive y-axis in the direction of the normal of γ at q. We may express the curve γ locally about q as $y = g(x)$ for $|x| < \varepsilon$ where $\varepsilon > 0$ and g is continuously differentiable and satisfies $g(0) = 0$, $g'(0) = 0$. We denote this portion of γ by $\overline{\gamma}$. By picking ε smaller if necessary we may further require that $|g'(x)| \leq 10^{-6}$ for $|s| < \varepsilon$. Since $\lim_{n\to\infty} P_1 G^n(p,\theta) = q$, there is an n_0 such that $q_n = P_1 G^n(p,\theta) \in \overline{\gamma}$ for $n \geq n_0$. By replacing (p,θ) by $G^{n_0}(p,\theta)$ we may assume $n_0 = 0$. Set $\theta_n = P_2 G^n(p,\theta)$ and x_n and y_n the x and y coordinates of q_n. It follows from the mean value theorem that the slope s of the line $\overline{q_n q_{n+1}}$ must satisfy $|s| \leq 10^{-6}$.

We may now show that the sequence x_n is monotone. for if the x_n's should change direction $x_n < x_{n+1} > x_{n+2}$ or $x_n > x_{n+1} < x_{n+2}$ for some n, then clearly the tangent to γ at q_{n+1} must be nearly vertical. But this is ruled out because $|g'(x)| \leq 10^{-6}$ for each x, $|x| < \varepsilon$. Hence the sequence x_n is monotone. By reversing the direction of the x-axis if necessary we may assume that x_n is monotone increasing. Since $\lim_{n\to\infty} x_n = 0$, we must have $x_n \leq 0$. Since the direction of motion monotonically changes by $2\theta_i$ at each point of contact, $\sum_{i=1}^{\infty} 2\theta_i$ is equal to the angle between the initial direction of motion and the tangent at q. Conclusions (a), (b), and

$$\sum_{n=1}^{\infty} |q_{n+1} - q_n| \leq \text{arclength of } \gamma \text{ from } q_1 \text{ to } q,$$

and (c) follows easily. □

The next corollary is a direct consequence of the above theorem.

COROLLARY A.2 ([**59**, Corollary 2]). *Given* $(p,\theta) \in \partial\Omega \times (0,\pi)$ *and* p_0 *strictly between* p *and* $q = P_1 G(p,\theta)$ *on the straight line segment from* p *to* q*, set* $v = \frac{q-p}{|q-p|}$*, then the following are equivalent:*

(a) $T^t(p_0, v)$ *is well defined for all* $t \geq 0$.
(b) $\sum_{n=0}^{\infty} |P_1 G^{n+1}(p,\theta) - P_1 G^n(p,\theta)| = \infty$.
(c) $P_1 G^n(p,\theta)$ *diverges.*

An important sufficient condition of non-terminating is as follows.

THEOREM A.3 ([**59**, Theorem 3]). *If* $\partial\Omega$ *has a bounded third derivative and nowhere vanishing curvature, then* $T^t(x,v)$ *is well defined for all* $(x,v) \in \Omega \times \mathbf{S}^1$.

It is well known that G preserves the measure μ on $\partial\Omega \times (0,\pi)$:

$$d\mu = \sin\theta \, dx d\theta.$$

We then get to understand that termination rarely happens in measure.

THEOREM A.4 ([**59**, Theorem 4]). *On any billiard table* Ω*, for almost all initial conditions* $(x,v) \in \Omega \times \mathbf{S}^1$*,* $T^t(x,v)$ *is well defined for all* $t \geq 0$.

The proofs of Theorem A.3 and Theorem A.4 are omitted here.

APPENDIX B

Derivation of Noncoercive Hamilton-Jacobi Equations

In this appendix we show the derivation of the equation (7.1) with H given as in (5.18). Suppose the evolution of a hypersurface $\{\Gamma_t\}_{t\geq 0} \subset \mathbb{R}^{n+1}$ is governed by the law of propagation

$$V_\varepsilon(x, x_{n+1}, t) = \overline{\sigma}(x, x_{n+1}) m(\frac{\|p\|}{\varepsilon}) - \overline{f}(x, x_{n+1}) \quad \text{on } \Gamma_t, \tag{B.1}$$

where $\varepsilon > 0$, V_ε is the normal growth rate at the surface, p is the step density and $\overline{\sigma}, \overline{f} : \mathbb{R}^{N+1} \to \mathbb{R}$ are given functions which satisfy

$$\overline{\sigma}(x, 0) = \sigma(x), \ \overline{f}(x, 0) = f(x) \text{ for all } x \in \mathbb{R}^n.$$

We refer to the literatures [**19, 25, 26, 34**] for the background of the physical model in crystal growth. Let us consider the graph representation of the above evolution and therefore we introduce the function v^ε which satisfies $\Gamma_t = \{(x, -v^\varepsilon(x,t)) \mid x \in \mathbb{R}^n\}$. Then the step density and the growth rate perpendicular to x-axis are expressed by the gradient of v^ε, i.e., $p = \nabla v^\varepsilon(x)$ and v^ε_t, respectively. Thus, the above surface evolution equation can be written by

$$v^\varepsilon_t + H_\varepsilon(x, v^\varepsilon, \nabla v^\varepsilon) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{B.2}$$

where

$$H_\varepsilon(x, r, p) := \big(\overline{\sigma}(x, -r) m(\frac{\|p\|}{\varepsilon}) - \overline{f}(x, -r)\big)\sqrt{|p|^2 + 1}.$$

We approximate v^ε by using the *microscopic time variable*, i.e., $\tau = t/\varepsilon$ so that

$$v^\varepsilon(x, \varepsilon\tau) = \varepsilon u(x, \tau) + o(\varepsilon) \quad \text{as } \varepsilon \to 0,$$

where $o : [0, \infty) \to [0, \infty)$ is a function which satisfies $o(r)/r \to 0$ as $r \to 0$. By replacing τ by t, we see that formally u solves (C). We present a rigorous argument in the next Proposition.

PROPOSITION B.1. *Assume that u_0 and $\overline{\sigma}, \overline{f}$ are bounded in $\mathbb{R}^n$ and $\mathbb{R}^{N+1}$, respectively. Let v^ε be the viscosity solutions of* (B.2) *with the initial value u_0. Then,*

$$u^\varepsilon(x, t) := \frac{1}{\varepsilon} v^\varepsilon(x, \varepsilon t)$$

converges to the viscosity solution of (C) *uniformly on every compact set of $\mathbb{R}^n \times [0, \infty)$ as $\varepsilon \to 0$.*

PROOF. It is easily seen that u^ε satisfies

$$\begin{cases} u^\varepsilon_t + \tilde{H}_\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u^\varepsilon(x, 0) = u_0(x) & \text{for all } x \in \mathbb{R}^n, \end{cases} \tag{B.3}$$

where

$$\tilde{H}_\varepsilon(x, p) := \big(\overline{\sigma}(x, -\varepsilon r) m(\|p\|) - \overline{f}(x, -\varepsilon r)\big)\sqrt{\varepsilon |p|^2 + 1}.$$

Define the functions $u^-, u^+ : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ by

$$u^-(x,t) := -\sup_{x\in\mathbb{R}^n, r\in\mathbb{R}} |\overline{\sigma}(x,r)m_0 - \overline{f}(x,r)|t - C,$$
$$u^+(x,t) := \sup_{x\in\mathbb{R}^n, r\in\mathbb{R}} |\overline{\sigma}(x,r)m_0 - \overline{f}(x,r)|t + C,$$

where $C > 0$ is the constant with $C \geq \sup|u_0|$. Then it is easy to check that u^-, u^+ are viscosity subsolution and viscosity supersolution of (B.3) with the initial value u_0, respectively.

By a comparison principle for (B.3) we have

$$u^- \leq u_\varepsilon \leq u^+ \quad \text{on } \mathbb{R}^n \times (0,\infty).$$

If one takes half relaxed limits

$$\overline{u}(x,t) := \limsup_{t\to\infty}{}^* u^\varepsilon(x,t),$$
$$\underline{u}(x,t) := \liminf_{t\to\infty}{}_* u^\varepsilon(x,t),$$

we observe that $\overline{u}$, $\underline{u}$ are viscosity subsolution and viscosity supersolution of (C), respectively, by the usual stability result of viscosity solution, since $\tilde{H}_\varepsilon$ converges to $H(x,p) - f(x)$ locally uniformly for $(x,p) \in \mathbb{R}^{2N}$ as $\varepsilon \to 0$. By the comparison principle for (C) again, we have

$$\overline{u} \leq \underline{u} \quad \text{on } \mathbb{R}^n \times [0,\infty),$$

which implies that $\overline{u} = \underline{u}$ and u^ε converges to the viscosity solution of (C) uniformly on every compact set of $\mathbb{R}^n \times [0,\infty)$ as $\varepsilon \to 0$. □

Theorems 7.13 and 7.19 (see also Corollary 7.20) now give a clear view of the solution v^ε on the effective domain Ω_e of (7.1) with the initial value u_0. We have

$$\begin{aligned} v^\varepsilon(x,t) &= \varepsilon u(x, \frac{t}{\varepsilon}) + o(\varepsilon) \\ &= \varepsilon\phi_\infty(x) + ct + o(\varepsilon) \text{ for all } x \in \Omega_e, \end{aligned}$$

where $o : [0,\infty) \to [0,\infty)$ is a function which satisfies $o(r)/r \to 0$ as $r \to 0$ and c, ϕ_∞ are the constant and the function given by (7.36) and (7.45). Therefore, we see that roughly speaking, the growing facet moving according to (B.1) is flat up to order ε with speed c on the effective domain Ω_e.

REMARK B.1. We can consider the case where the surface supersaturation depends on the curvature. More precisely, let us add the curvature term to (B.1) and consider

$$V_\varepsilon(x, x_{N+1}, t) = \big(\sigma(x, x_{N+1}) - \operatorname{div}(n(x, x_{N+1}))\big) m\big(\frac{\|p\|}{\varepsilon}\big) - f(x, x_{N+1}) \quad \text{on } \Gamma_t. \tag{B.4}$$

Then the microscopic time scale approximated equation of the graph represented equation of (B.4) is the same as (7.1). Indeed, let v^ε be a viscosity solution of the graph represented equation of (B.4) and set $u^\varepsilon(x,t) = (1/\varepsilon)v^\varepsilon(x,\varepsilon t)$.

Then it is easy to check (see [51] for instance) that u^ε satisfies

$$u_t^\varepsilon + \tilde{H}_\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) - m(\|\nabla u^\varepsilon\|)\mathrm{div}\left(\frac{\varepsilon \nabla u^\varepsilon}{\sqrt{\varepsilon|\nabla u|^2+1}}\right)\sqrt{\varepsilon|\nabla u|^2+1} = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

By the same argument of the proof of Proposition B.1, we see that u^ε converges a solution of (7.1) locally uniformly in $\mathbb{R}^n \times (0, \infty)$.

APPENDIX C

Local Comparison Principle for Unbounded Solutions

In this Appendix, we prove a comparison result for evolutive Hamilton-Jacobi equations, which we used in Chapter 6 and Chapter 7. With this comparison principle at hand, we do not need any hypotheses on boundedness or growth rates of the solutions but have to pay the price that $p \to H(x,p)$ is assumed to be Lipschitz. The result is originally due to Ishii [**64**] and we place it here for the reader's convenience.

THEOREM C.1. *Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be continuous and satisfy*

$$|H(x,p) - H(x,q)| \le K(|x|+1)|p-q|$$

with some $K > 0$ for all $x, p, q \in \mathbb{R}^n$ and

$$|H(x,p) - H(y,p)| \le \omega(|x-y|, R) + \omega(|x-y||p|, R)$$

for all $p \in \mathbb{R}^n$, $x, y \in B_R(0), R > 0$. Here $\omega(\cdot, R)$ is a modulus of continuity for any $R > 0$. If $u_1 \in USC(\mathbb{R}^n \times [0,T])$ and $u_2 \in LSC(\mathbb{R}^n \times [0,T])$ are respectively a viscosity sub- and supersolution of

$$u_t + H(x, \nabla u) = 0 \quad \text{in } \mathbb{R}^n \times (0,T) \tag{C.1}$$

and $u_1(x,0) \le u_2(x,0)$ for all $x \in \mathbb{R}^n$, then $u_1 \le u_2$ in $\mathbb{R}^n \times [0,T]$.

Our proof follows that of [**5**, Theorem III.3.16]. We first give a local comparison principle in the cone

$$\mathcal{C} := \{(x,t) \in \mathbb{R}^n \times (0,T) : |x| < C(T-t)\}$$

for some $C > 0$ and $T > 0$.

THEOREM C.2. *Let $H : \overline{B}_{CT}(0) \times \mathbb{R}^n \to \mathbb{R}$ be continuous and satisfy*

$$|H(x,p) - H(x,q)| \le K|p-q| \tag{C.2}$$

for all $x, p, q \in \mathbb{R}^n$ and

$$|H(x,p) - H(y,p)| \le \omega(|x-y|) + \omega(|x-y||p|) \tag{C.3}$$

for all $x, y \in B_{CT}(0)$, $p, q \in \mathbb{R}^n$, where ω is a modulus. If $u_1 \in USC(\overline{\mathcal{C}})$ and $u_2 \in LSC(\overline{\mathcal{C}})$ are respectively a sub- and supersolution of (C.1) and $u_1(x,0) \le u_2(x,0)$ for all $x \in B_{CT}(0)$, then $u_1 \le u_2$ in $\mathcal{C}$.

PROOF. We localize the problem in the cone $\mathcal{C}$. Assume by contradiction that there exists $0 < \delta < T$ and $(\hat{x}, \hat{t})$ such that

$$(u_1 - u_2)(\hat{x}, \hat{t}) = \delta \text{ and } |\hat{x}| \le C(T - \hat{t}) - 2\delta. \tag{C.4}$$

Take $M > \sup_{\{} |u_1(x,t) - u_2(y,s)| : (x,t,y,s) \in \mathcal{C}^2\} \ge \delta$ and $h \in C^1(\mathbb{R})$ such that $h' \le 0$, $h(r) = 0$ for $r \le -\delta$, $h(r) = -3M$ for $r \ge 0$. Now define

$\langle x\rangle_\beta := (|x|^2+\beta^2)^{1/2}$ and

$$\Phi(x,y,t,s) := u_1(x,t) - u_2(y,s) - \frac{|x-y|^2+|t-s|^2}{2\varepsilon} - \eta(t+s) + h(\langle x\rangle_\beta - C(T-t)) + h(\langle y\rangle_\beta - C(T-s)),$$

where ε, η, β are positive parameters. Let $(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon) \in \mathcal{C}^2$ be a maximizer of Φ. We claim that either $t_\varepsilon = 0$ or $s_\varepsilon = 0$ or $(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon) \in \mathcal{C}^2$ for β and η small enough. In fact, if $|x_\varepsilon| = C(T-t_\varepsilon)$ or $|y_\varepsilon| = C(T-s_\varepsilon)$ we have $\Phi(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon) \le M - 3M = -2M$ by the definition of h, because $\langle x\rangle_\beta > |x|$ for all x and β. On the other hand, by (C.4) and the definition of h, for any $\beta < \delta$ and $\eta < \delta/(4\hat{t})$,

$$\max_{\overline{\mathcal{C}}^2} \Phi \ge \Phi(\hat{x},\hat{x},\hat{t},\hat{t}) = \delta - 2\eta\hat{t} + 2h(\langle \hat{x}\rangle_\beta - C(T-\hat{t})) \ge \delta/2, \tag{C.5}$$

which proves the claim.

From the inequality $\Phi(x_\varepsilon, x_\varepsilon, t_\varepsilon, t_\varepsilon) + \Phi(y_\varepsilon, y_\varepsilon, s_\varepsilon, s_\varepsilon) \le 2\Phi(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon)$ we get

$$\frac{|x_\varepsilon - y_\varepsilon|^2 + |t_\varepsilon - s_\varepsilon|^2}{\varepsilon} \le u_1(x_\varepsilon, t_\varepsilon) - u_1(y_\varepsilon, s_\varepsilon) + u_2(x_\varepsilon, t_\varepsilon) - u_2(y_\varepsilon, s_\varepsilon),$$

and thus

$$|x_\varepsilon - y_\varepsilon|^2 + |t_\varepsilon - s_\varepsilon|^2 \le 2M\varepsilon. \tag{C.6}$$

Suppose there exist $\bar{x} \in \overline{\mathcal{C}}$ and $\bar{t} \in [0,T]$ such that there exist subsequences satisfying $x_\varepsilon, y_\varepsilon$ converge to $\bar{x}$ and $t_\varepsilon, s_\varepsilon$ converge to $\bar{t}$ as $\varepsilon \to 0$. Now noticing that $\Phi(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon) \ge \Phi(\bar{x},\bar{x},\bar{t},\bar{t})$, we have

$$\frac{|x_\varepsilon - y_\varepsilon|^2 + |t_\varepsilon - s_\varepsilon|^2}{\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0 \tag{C.7}$$

by the upper semicontinuity of u_1 and $-u_2$. We next show that neither t_ε nor s_ε can be zero if ε is small. Indeed, if $t_\varepsilon = 0$, we get

$$\Phi(x_\varepsilon, y_\varepsilon, 0, s_\varepsilon) \le u_1(x_\varepsilon, 0) - u_2(x_\varepsilon, 0) + u_2(x_\varepsilon, 0) - u_2(y_\varepsilon, s_\varepsilon).$$

Since $u_1(x_\varepsilon, 0) \le u_2(x_\varepsilon, 0)$, the right hand side of the above inequality cannot be positive as $\varepsilon \to 0$, which gives a contradiction. The proof for the case that $s_\varepsilon = 0$ is analogous.

Now we define the test functions

$$\phi(x,t) := \frac{|x-y_\varepsilon|^2 + |t-s_\varepsilon|^2}{2\varepsilon} + \eta(t+s_\varepsilon) - h(\langle x\rangle_\beta - C(T-t))$$

$$\psi(y,s) := -\frac{|x_\varepsilon - y|^2 + |t_\varepsilon - s|^2}{2\varepsilon} - \eta(t_\varepsilon + s) + h(\langle y\rangle_\beta - C(T-s)),$$

so that $u_1 - \phi$ has a maximum at $(x_\varepsilon, t_\varepsilon)$ and $u_2 - \psi$ has a minimum at $(y_\varepsilon, s_\varepsilon)$. We compute the partial derivatives of ϕ and ψ, set $X = \langle x_\varepsilon\rangle_\beta - C(T-t_\varepsilon)$, $Y = \langle y_\varepsilon\rangle_\beta - C(T-s_\varepsilon)$, and use the definition of viscosity sub- and supersolution to get

$$2\eta \le C(h'(Y) + h'(X)) + H\left(y_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + h'(Y)\frac{y_\varepsilon}{\langle y_\varepsilon\rangle_\beta}\right) - H\left(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - h'(X)\frac{x_\varepsilon}{\langle x_\varepsilon\rangle_\beta}\right).$$

We now use (C.2), (C.3) and the fact that $h' \leq 0$ to estimate the right hand side of the above and obtain

$$2\eta \leq \omega(|x_\varepsilon - y_\varepsilon|) + \omega\left(\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} + |x_\varepsilon - y_\varepsilon||h'(Y)|\right) + C\left|h'(Y)\frac{y_\varepsilon}{\langle y_\varepsilon\rangle_\beta} + h'(X)\frac{x_\varepsilon}{\langle x_\varepsilon\rangle_\beta}\right|. \tag{C.8}$$

Using (C.6) and (C.7), we get that the right hand side of (C.8) tends to 0 as $\varepsilon \to 0$, which is a contradiction. We thus complete the proof. □

REMARK C.1. We can easily verify that the above theorem holds for the cone

$$\{(x,t) : 0 < t < T \text{ and } |x - x_0| < C(T-t)\}$$

centered at any $x_0 \in \mathbb{R}^n$ provided that (C.2) and (C.3) hold for all $x, y \in B_{CT}(x_0)$ and $u_1 \leq u_2$ in $B_{CT}(x_0) \times \{0\}$.

Now we are in a position to prove Theorem C.1.

PROOF OF THEOREM C.1. We may assume without loss of generality that $T < 1/K$, because the proof of the general case is obtained by iterating the following arguments on time intervals of fixed length smaller than $1/K$. We fix $x_0 \in \mathbb{R}^n$ and define $r := KT(|x_0| + 1)/(1 - KT) > 0$, so that

$$r = KT(|x_0| + 1 + r) =: CT. \tag{C.9}$$

We define the cone

$$\mathcal{C}_{x_0} := \{(x,t) : 0 < t < T \text{ and } |x - x_0| < K(|x_0| + 1 + r)(T - t)\}.$$

If $x \in B_{CT}(x_0)$, then $|x| < |x_0| + r$ by (C.9) we can take $C = K(|x_0| + 1 + r)$ as a Lipschitz constant for H with respect to the variable p in $B_{CT}(x_0) \times \mathbb{R}^n$ and apply Theorem C.2 and Remark C.1 to get $u_1 \leq u_2$ in $\mathcal{C}_{x_0}$. Since $\mathbb{R}^n \times (0,T) = \bigcup_{x_0 \in \mathbb{R}^n} \mathcal{C}_{x_0}$, the proof is complete. □

APPENDIX D

Equivalence of Definitions

We used two kinds of definitions respectively in Chapter 6 and Chapter 7 for solutions of the singular boundary problems. Our definition in Chapter 6 turns out suitable for the noncoercive equation (7.43) while the coercive equation (S) in Chapter 7 is easier to be understood by applying the Definition 7.1. The equations (7.43) and (S) look like the same up to a composition of a monotone function. We rigorously prove the equivalence of them in this appendix. The following definition is a multidimensional extension of Definitions 6.1–6.3.

DEFINITION D.1. Let u be a function on $\mathbb{R}^n$ to $\mathbb{R} \cup \{\pm\infty\}$. We call u a subsolution (resp., supersolution) of (7.43) if $\overline{u}(x) < \infty$ (resp., $\underline{u}(x) > -\infty$) for all $x \in \mathbb{R}^n$ and $\overline{u}$ is a viscosity subsolution (resp., $\underline{u}$ is a viscosity supersolution). We call u a solution if u is a subsolution and supersolution. Here we denote $(u_-)^*$ (resp., $(u_+)_*$) by $\overline{u}$ (resp., $\underline{u}$), where

$$u_+(x) := \begin{cases} +\infty & \text{if } u(x) = -\infty, \\ u(x) & \text{if } u(x) > -\infty, \end{cases}$$

$$u_-(x) := \begin{cases} -\infty & \text{if } u(x) = +\infty, \\ u(x) & \text{if } u(x) < +\infty, \end{cases}$$

and v^* (resp., v_*) is the upper-semicontinuous envelope (resp., lower semicontinuous envelope) of a function v, i.e.,

$$v^*(x) := \lim_{r \to 0} \sup\{v(y) \mid y \in \mathbb{R}^n, |x - y| \le r\},$$
$$(\text{resp., } v_*(x) := \lim_{r \to 0} \inf\{v(y) \mid y \in \mathbb{R}^n, |x - y| \le r\}.)$$

PROPOSITION D.1. *Let u be a supersolution of* (7.43) *in the sense of Definition D.1. Then we have*

$$\underline{u}(x) = \infty \quad \text{for all } x \in \Omega_d.$$

PROOF. Suppose that there exists $\hat{x} \in \Omega_d$ such that $\underline{u}(\hat{x}) \neq \infty$. Note that Ω_d is an open set. Since $\underline{u}$ is locally bounded from below, there exist $r > 0$ and $M > 0$ such that $\underline{u}(x) \ge -M$ for all $x \in B_r(\hat{x}) \subset \Omega_d$.

It is easily seen that

$$\min_{\overline{B}_r(\hat{x},r)} \{\underline{u}(x) - \underline{u}(\hat{x}) + \frac{1}{2\varepsilon}|x - \hat{x}|^2\} \le 0.$$

Let x_ε be a point which gives the minimum of the above. We may assume that $x_\varepsilon \in B_r(\hat{x})$ for a suitable small $\varepsilon > 0$. Then we have

$$\sigma(x_\varepsilon)m(\|\frac{x_\varepsilon - \hat{x}}{\varepsilon}\|) < \sigma(x_\varepsilon) < f(x_\varepsilon) - c,$$

which contradicts that u is a supersolution of (7.43). □

PROPOSITION D.2. *Let $u : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a subsolution (resp., a supersolution) of* (7.43) *in the sense of Definition* D.1. *Assume that $u \in C(\overline{\Omega}_e)$. Then u is a subsolution (resp., supersolution) of* (S) *in the sense of Definition* 7.1.

PROOF. Since it is clear that a subsolution of (7.43) in the sense of Definition D.1 is a subsolution of (S) in the sense of Definition 7.1, we only prove that a supersolution of (7.43) in the sense of Definition D.1 is a supersolution of (S) in the sense of Definition 7.1.

Suppose that $u - \phi$ take a minimum at $\hat{x} \in \partial\Omega_e$ for some $\phi \in C^1(\overline{\Omega}_e)$. Extend ϕ to the function $\tilde{\phi}$ defined on $\mathbb{R}^n$ such that $\tilde{\phi} \in C^1(\mathbb{R}^n)$ and $\tilde{\phi} = \phi$ on $\overline{\Omega}_e$. Then $\underline{u} - \tilde{\phi}$ takes minimum at $\hat{x} \in \partial\Omega_e$ by Proposition D.1. Therefore, we have

$$\sigma(\hat{x})m(\|\phi(\hat{x})\|) < \sigma(\hat{x}) = f(\hat{x}) - c,$$

which contradicts that u is a supersolution in the sense of Definition 7.1. □

PROPOSITION D.3. *Let $u \in C(\overline{\Omega}_e)$ (resp., $v \in C(\overline{\Omega}_e)$) be a solution in the sense of Definition* 7.1. *Extend u to the function on $\mathbb{R}^n$ such that $\tilde{u} = u$ on $\overline{\Omega}_e$ and $\tilde{u}(x) = \infty$ for all $x \in \mathbb{R}^n \setminus \overline{\Omega}_e$. Then $\tilde{u}$ is a subsolution (resp., supersolution) of* (7.43) *in the sense of Definition* 7.1.

PROOF. By the definition of supersolution we have $D^-(\underline{\tilde{u}})(x) = D^-\tilde{u}(x) = \emptyset$ for all $x \in \partial\Omega_e$. Thus, it is clear that u is a supersolution of (7.43) in the sense of Definition D.1.

For any $p \in D^+(\overline{\tilde{u}})(x) = D^+\tilde{u}(x)$ and $x \in \partial\Omega_e$, we have

$$\sigma(x)m(\|p\|) < \sigma(x) = f(x) - c,$$

which implies that u is a subsolution of (7.43) in the sense of Definition D.1. □

APPENDIX E

Lipschitz Estimates and Derivative Bounds

We consider the Lipschitz constant of viscosity solutions of

$$\|\nabla u(x)\| \leq C \quad \text{in } \Omega, \tag{E.1}$$

where C is a positive constant and Ω is a domain in $\mathbb{R}^n$.

PROPOSITION E.1. *If* $|u(x) - u(y)| \leq C\|x - y\|_*$ *for all* $x, y \in \Omega$, *then* $\|\nabla u\| \leq C$ *in* Ω *in the viscosity sense.*

PROOF. Note that every norm in a finite dimensional space is equivalent. In view of Rademacher's theorem, for almost every $y \in \Omega$, there exists $p_y \in \mathbb{R}^n$ such that

$$\frac{|u(x) - u(y) - p_y \cdot (x - y)|}{\|x - y\|_*} = o(1) \quad \text{as } y \to x,$$

where $o(1) \to 0$ as $y \to x$.

Choose $q_y \in \mathbb{R}^n \setminus \{0\}$ such that $\|p_y\| = |p_y \cdot q_y|/\|q_y\|_*$ and set $x_\varepsilon := y + \varepsilon q_y/\|q_y\|_*$ for $\varepsilon > 0$. Then we have

$$\begin{aligned}\|p_y\| &= \left|\frac{u(x_\varepsilon) - u(y) - p_y \cdot (x_\varepsilon - y)}{\|x_\varepsilon - y\|_*} - \frac{u(x_\varepsilon) - u(y)}{\|x_\varepsilon - y\|_*}\right| \\ &\leq \left|\frac{u(x_\varepsilon) - u(y) - p_y \cdot (x_\varepsilon - y)}{\|x_\varepsilon - y\|_*}\right| + C \\ &\to C \quad \text{as } \varepsilon \to 0.\end{aligned}$$

Therefore, we have

$$\|\nabla u(x)\| \leq C \quad \text{for almost every } x \in \Omega.$$

Since the map $p \mapsto \|p\|$ is convex, we have

$$\|\nabla u\| \leq C \quad \text{in } \Omega \text{ in the viscosity sense.}$$

□

LEMMA E.2. *Let* u *be a viscosity solution of* (E.1). *For any* $y \in \Omega$ *and* $r > 0$ *such that* $B_r(y) \subset \Omega$, *we have*

$$|u(x) - u(y)| \leq C\|x - y\|_* \quad \textit{for all } x \in B_{r/4}(y).$$

PROOF. Let $\varepsilon > 0$ and consider a function $\phi \in C^1(B_{r/2}(y) \setminus \{y\}) \cap C^0(B_{r/2}(y))$ which satisfies

$$\begin{gathered}\phi(x) = u(y) + (C + \varepsilon)\|x - y\|_* \quad \text{for all } x \in B_{r/4}(y), \\ \|\nabla \phi(x)\| \geq C + \varepsilon \text{ in } B_{r/4}(y) \setminus \overline{B}_{r/4}(y) \quad \text{and} \\ u(x) < \phi(x) \text{ for all } x \in \partial B_{r/2}(y).\end{gathered}$$

Let $z \in \overline{B}_{r/2}(y)$ be a point such that $\max_{\overline{B}_{r/2}(y)}(u-\phi) = (u-\phi)(z)$. Noting that $\max_{\overline{B}_{r/2}(y)}(u-\phi) \geq 0$, we see $z \in B_{r/2}(y)$. Suppose that $z \in B_{r/2}(y) \setminus \{y\}$. Then by the definition of viscosity subsolution, we have

$$\|\nabla\phi(z)\| \leq C.$$

Meanwhile, we have $\|\nabla\phi(x)\| \geq C+\varepsilon$ for all $x \in B_{r/2}(y) \setminus \{y\}$, which contradicts the above. Therefore, z needs to be y. Consequently,

$$(u-\phi)(x) \leq (u-\phi)(y) = 0 \quad \text{for all } x \in B_{r/4}(y),$$

which implies that

$$u(x) - u(y) \leq (C+\varepsilon)\|x-y\|_* \quad \text{for all } x \in B_{r/4}(y).$$

Sending $\varepsilon \to 0$, we get a conclusion. □

PROPOSITION E.3. *Assume that Ω is convex and let u be a viscosity solution of* (E.1). *Then we have*

$$|u(x)-u(y)| \leq C\|x-y\|_* \quad \textit{for any } x, y \in \overline{\Omega}.$$

REMARK E.1. We remark that if we assume that Ω is convex, then the Lipschitz constant of solutions of (E.1) coincides with the constant in (E.1).

PROOF. Fix $x, y \in \Omega$. In view of the convexity of Ω, we have $[x,y] \subset \Omega$. Choose $r > 0$ such that $\bigcup_{z\in[x,y]} B_r(z) \subset \Omega$. There exist $m \in \mathbb{N}$ and $\{\xi_i\}_{i=1,\dots,m} \subset [x,y]$ such that $\xi_1 = x, \xi_m = y$, $\xi_i \in B_{r/4}(\xi_{i+1})$ for all $i = 1,\dots,m-1$ and $|x-y| = \sum_{i=1}^{m-1}|\xi_i - \xi_{i+1}|$. By Lemma E.2 we have

$$|u(x)-u(y)| \leq \sum_{i=1}^{m-1} C|u(\xi_i) - u(\xi_{i+1})| \leq \sum_{i=1}^{m-1} C\|\xi_i - \xi_{i+1}\|_* = C\|x-y\|_*.$$

Fix $x, y \in \partial\Omega$. Choose $\{x_j\}_{j\in\mathbb{N}}, \{y_j\}_{j\in\mathbb{N}}$ such that $x_j \to x$ and $y_j \to y$ as $j \to \infty$. We have $|u(x_j) - u(y_j)| \leq C\|x_j - y_j\|_*$. Sending $j \to \infty$, we get $|u(x)-u(y)| \leq C\|x-y\|_*$. □

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Bibliography

[1] S. J. Altschuler, S. B. Angenent and Y. Giga, *Mean curvature flow through singularities for surfaces of rotation*, J. Geom. Analysis, 5 (1995), 293–358.

[2] S. B. Angenent, T. Ilmanen and D. L. Chopp, *A computed example of nonuniqueness of mean curvature flow in* $\mathbb{R}^3$, Comm. Partial Differential Equations, 20 (1995), 1937–1958.

[3] S. B. Angenent, T. Ilmanen and J. J. L. Velázquez, *Fattening from smooth initial data in mean curvature flow*, unpublished preprint.

[4] M. Arisawa and P. L. Lions, *On ergodic stochastic control*, Comm. Partial Differential Equations, 23 (1998), 2187-2217.

[5] M. Bardi and I. Capuzzo-Dolcetta, "Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations", Systems and Control: Foundations and Applications, Birkhäuser Boston, Boston, 1997.

[6] M. Bardi, *A boundary value problem for the minimum-time function*, SIAM J. Control Optim., 27 (1989), 776–785.

[7] M. Bardi and P. Soravia, *Hamilton-Jacobi equations with singular boundary conditions on a free boundary and applications to differential games*, Trans. Amer. Math. Soc., 325 (1991), 205–229.

[8] G. Barles, *Nonlinear Neumann boundary conditons for quasilinear degenerate elliptic equations and applications*, J. Differential Equations, 154 (1999), 191–224.

[9] G. Barles, *Some homogenization results for non-coercive Hamilton-Jacobi equations*, Calc. Var. Partial Differential Equations, 30 (2007), 449–466.

[10] G. Barles and F. Da Lio, *On the generalized Dirichlet problem for viscous Hamilton-Jacobi equations*, J. Math. Pures Appl. (9), 83 (2004), 53–75.

[11] G. Barles and H. Mitake, *A PDE approach to large-time asymptotics for boundary-value problems for nonconvex Hamilton-Jacobi equations*, preprint.

[12] G. Barles and B. Perthame, *Discontinuous solutions of deterministic optimal stopping time problems*, RAIRO Model. Math. Anal. Numer., 21 (1987), 557–579.

[13] G. Barles and J.-M. Roquejoffre, *Ergodic type problems and large time behaviour of unbounded solutions of Hamilton-Jacobi equations*, Comm. Partial Differential Equations, 31 (2006), 1209–1225.

[14] G. Barles, H. M. Soner and P. E. Souganidis, *Front propagation and phase field theory*, SIAM J. Control Optim. 31 (1993), 439–469.

[15] G. Barles and P. E. Souganidis, *On the large time behavior of solutions of Hamilton-Jacobi equations*, SIAM J. Math. Anal., 31 (2000), 925–939.

[16] E. N. Barron and R. Jensen, *Semicontinuous viscosity solutions for Hamilton-Jacobi equations with convex Hamiltonians*, Comm. Partial Differential Equations, 15 (1990), 1713–1742.

[17] G. Bellettini and M. Paolini, *Two examples of fattening for the curvature flow with a driving force*, Atti Accad. Naz. Lincei Cl. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 5 (1994), 229–236.

[18] K. Brakke, *The motion of a surface by its mean curvature*, Mathematical Notes 20, Princeton Univ. Press, Princeton, New Jersey.

[19] W. K. Burton, N. Cabrera, F. C. Frank, *The growth of crystals and the equilibrium structure of their surfaces*, Phils. Trans. Roy. Soc. London. Ser. A., 243 (1951), 299–358.

[20] P. Cardaliaguet, *Ergodicity of Hamilton-Jacobi equations with a non coercive non convex Hamilton in* $\mathbb{R}^2/\mathbb{Z}^2$, Ann. Inst. H. Poincare Anal. Non Lineaire, 27 (2010), 837C856.

[21] F. Catté, F. Dibos and G. Koepfler, *A morphological scheme for mean curvature motion and applications to anistropic diffusion andmotion of level sets*, SIAM J. Numer. Anal., Vol. 32 (1995), 1895-1909.

[22] I. Capuzzo Dolcetta and P.-L. Lions, *Hamilton-Jacobi equations with state constraints*, Trans. Amer. Math. Soc., 318 (1990), 643–683.

[23] Y. G. Chen, Y. Giga and S. Goto, *Uniqueness and existence of viscosity solitions of generalized mean curvature flow equations*, J. Differential Geom., Vol. 33 (1991), 749-786.

[24] P. Cheridito, H. M. Soner, N. Touzi and N. Victoir, *Second-order backward stochastic differential equations and fully nonlinear parabolic PDEs*, Comm. Pure Appl. Math., 60 (2007), 1081–1110.

[25] A. A. Chernov, *Application of the method of characteristics to the theory of the growth from of crystals*, Soviet Phys. - Crystal., 8 (1964), 401-405.

[26] A. A. Chernov, *Stability of faceted shapes*, J. Crystal Growth, 24/25 (1974), 11-31.

[27] N. Chernov and R. Markarian, "Chaotic Billiards", Mathematical Surveys and Monographs 127, American Mathematical Society, 2006.

[28] D. Chopp, L. C. Evans and H. Ishii, *Waiting time effects for Gauss curvature flows*, Indiana Univ. Math. J., 48 (1999), 311–344.

[29] M. Crandall, H. Ishii and P. L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N. S.), Vol. 27 (1992), 1-67.

[30] F. Da Lio, *Strong comparison results for quasilinear equations in annular domains and applications*, Comm. Partial Differential Equations, 27 (2002), 283–323.

[31] A. Davini and A. Siconolfi, *A generalized dynamical approach to the large time behavior of solutions of Hamilton-Jacobi equations*, SIAM J. Math. Anal., 38 (2006), 478–502.

[32] N. Dirr, S. Luckhaus and M. Novaga, *A stochastic selection principle in case of fattening for curvature flow*, Calc. Var. Partial Differential Equations, 13 (2001), 405–425.

[33] P. Dupuis and H. Ishii, *SDEs with oblique reflection on nonsmooth domains*, Ann. Prob., 21 (1993), 554-580.

[34] W. E and N. K. Yip, *Continuum theory of epitaxial crystal growth. I*, J. Statistical Physics, 104 (2001), 221–253.

[35] L. C. Evans, *Some min-max methods for the Hamilton-Jacobi equation*, Indiana U. Math J, 33 (1984), 31C50.

[36] L. C. Evans, *The 1-Laplacian, the ∞-Laplacian and differential games*, in Perspectives in Nonlinear Partial Differential Equations, Contemp. Math., 446 (2007), 245–254.

[37] L. C. Evans and R. F. Gariepy, "Measure theory and fine properties of functions", Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[38] L. C. Evans and M. R. James, *The Hamiltonian-Jacobi-Bellman equation for time-optimal control*, SIAM J. Control Optim., 6 (1989), 1477–1489.

[39] L. C. Evans and H. M. Soner and P. E. Souganidis, *Phase transitions and generalized motion by mean curvature*, Comm. Pure Appl. Math., 45 (1992), 1097–1123.

[40] L. C. Evans and P. E. Souganidis, *Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations*, Indiana Univ. Math. J., 33 (1984), 773-797.

[41] L. C. Evans and J. Spruck, *Motion of level sets by mean curvature*, I., J. Differential Geom., Vol. 33 (1991), 635-681.

[42] A. Fathi, *Théorème KAM faible et théorie de Mather sur les systèmes lagrangiens*, C. R. Acad. Sci. Paris Sér. I Math., 324 (1997), 1043–1046.

[43] A. Fathi, *Sur la convergence du semi-groupe de Lax-Oleinik*, C. R. Acad. Sci. Paris Sér. I Math., 327 (1998), 267–270.

[44] A. Fathi and A. Siconolfi, *PDE aspects of Aubry-Mather theory for quasiconvex Hamiltonians*, Calc. Var. Partial Differential Equations, 22 (2005), 185–228.

[45] W. H. Fleming and H. M. Soner, "Controlled Markov processes and viscosity solutions", Springer, New York, 1993.

[46] A. Friedman, "Differential Games", Pure and Applied Mathematics 25, Wiley, New York, 1971.

[47] M. Gage and R. Hamilton, *The shrinking of convex plane curves by the heat equation*, J. Diff. Geom., 23 (1986), 69–96.

[48] Y. Giga, *Evolving curves with boundary conditions*, Curvature Flows and Related Topics, Mathematical Sciences and Applications, 5 (1995), Gakuto, Tokyo, 99-109.

[49] Y. Giga, *A level set method for surface evolution equations* (in Japanese), Sūgaku, 47 (1995), 321–340. English translation: Sugaku Expositions 10 (1999), 217–241.

[50] Y. Giga, *Viscosity solutions with shocks*, Comm. Pure Appl. Math., 55 (2002), 431–480.

[51] Y. Giga, "Surface evolution equations, a level set approach", Monographs in Mathematics 99, Birkhäuser Verlag, Basel, 2006.

[52] Y. Giga and Q. Liu, *A remark on the discrete deterministic game approach for curvature flow equations*, Nonlinear phenomena with energy dissipation: Mathematical analysis, modeling and simulation, 29 (2008), 103–115.

[53] Y. Giga and Q. Liu, *A billiard-based game interpretation of the Neumann problem for the curve shortening equation*, Adv. Differential Equations, 14 (2009), 201–240.

[54] Y. Giga, Q. Liu and H. Mitake, *Large-time asymptotics for one-dimensional Dirichlet problems for Hamilton-Jacobi equations with noncoercive Hamiltonians*, in preparation.

[55] Y. Giga, Q. Liu and H. Mitake, *Singular Neumann problems and large-time behavior of solutions of noncoercive Hamilton-Jacobi equations*, submitted.

[56] Y. Giga and T. Miyakawa, *A kinetic construction of global solutions of first order quasilinear equations*, Duke Math. J. 50 (1983), 505–515.

[57] Y. Giga and M.-H. Sato, *Neumann problem for singular degenerate parabolic equations*, Differential Integral Equations, Vol. 6 (1993), 1217-1230.

[58] M. Grayson, The heat equation shrinks embedded plane curves to points, J. Diff. Geom., 26 (1987), 285–314.

[59] B. Halpern, *Strange billiard tables*, Tran. Amer. Math. Soc., Vol. 232 (1977), 297-305.

[60] N. Ichihara and H. Ishii, *Asymptotic solutions of Hamilton-Jacobi equations with semi-periodic Hamiltonians*, Comm. Partial Differential Equations, 33 (2008), 784–807.

[61] N. Ichihara and H. Ishii, *The large-time behavior of solutions of Hamilton-Jacobi equations on the real line*, Methods Appl. Anal., 15 (2008), 223–242.

[62] N. Ichihara and H. Ishii, *Long-time behavior of solutions of Hamilton-Jacobi equations with convex and coercive Hamiltonians*, Arch. Ration. Mech. Anal., 194 (2009), 383–419.

[63] T. Ilmanen, *Generalized flow of sets by mean curvature on a manifold*, Indiana J. Math., 41 (1992), 671–705.

[64] H. Ishii, *Uniqueness of unbounded viscosity solution of Hamilton-Jacobi equations*, Indiana Univ. Math. J., 33 (1984), 721–748.

[65] H. Ishii, *Perron's method for Hamilton-Jacobi equations*, Duke Math. J., 55 (1987), 369–384.

[66] H. Ishii, *A simple, direct proof of uniqueness for solutions of the Hamilton-Jacobi equations of Eikonal type*, Proc. Amer. Math. Soc., 100 (1987), 247–251.

[67] H. Ishii, *A boundary value problem of the Dirichlet type for Hamilton-Jacobi equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 16 (1989), 105–135.

[68] H. Ishii, *Asymptotic solutions for large time of Hamilton-Jacobi equations in Euclidean n space*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 25 (2008), 231–266.

[69] H. Ishii, *Weak KAM aspects of convex Hamilton-Jacobi equations with Neumann type boundary conditions*, preprint. Available at http://www.edu.waseda.ac.jp/ ishii/.

[70] H. Ishii, *Long-time asymptotic solutions of convex Hamilton-Jacobi equations with Neumann type boundary conditions*, preprint.

[71] H. Ishii and H. Mitake, *Representation formulas for solutions of Hamilton-Jacobi equations with convex Hamiltonians*, Indiana Univ. Math. J., 56 (2007), 2159–2183.

[72] H. Ishii and M.-H. Sato, *Nonlinear oblique derivative problems for a singular degenreate parabolic equations on a general domain*, Nonlinear Analysis, Vol. 57 (2004), 1077-1098.

[73] K. Kasai, *Representation of solutions for nonlinear parabolic equations via two-person game with interest rate*, preprint.

[74] A. Katok and J. Strelcyn, "Invariant manifolds, entropy and billiards; smooth maps with singularities", Lecture Notes in Mathematics 1222, Springer, Berlin, 1986.

[75] R. V. Kohn and S. Serfaty, *A deterministic-control-based approach to motion by curvature*, Comm. Pure Appl. Math., Vol. 59 (2006), 344-407.

[76] R.V. Kohn and S. Serfaty, *Second-order PDE's and deterministic games*, in 6th International Congress on Industrial and Applied Math. – Zurich, Switzerland, 16-20 July 2007 – Invited Lectures, R. Jeltsch and G. Wanner eds, Euro Math Soc (2009), 239–249.

[77] R. V. Kohn and S. Serfaty, *A deterministic-control-based approach to fully nonlinear parabolic and elliptic equations*, Comm. Pure Appl. Math. 63 (2010), 1298–1350.

[78] J.-M. Lasry and P.-L. Lions, *Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem*, Math. Ann., 283 (1989), 583–630.

[79] P. L. Lions, *Neumann type boundary conditions for Hamilton-Jacobi equations*, Duke Math. J., Vol. 52 (1985), 793-820.

[80] P. L. Lions, G. Papanicolaou and S. R. S. Varadhan, *Homogenization of Hamilton-Jacobi equations*, preprint.

[81] P. L. Lions and A. S. Sznitman, *Stochastic differential equations with reflecting boundary conditions*, Comm. Pure Appl. Math., Vol. 37 (1984), 511-537.

[82] Q. Liu, *On billiards for a game interpretation of the Neumann problems for curvature flows*, M.S. Thesis, The University of Tokyo, 2008.

[83] Q. Liu, *On game interpretations for the curvature flow equation and its boundary problems*, RIMS Kokyuroku, Kyoto University, 1633 (2009), 138–150.

[84] Q. Liu, *On an elementary approach to optimal control with Neumann boundary condition*, RIMS Kokyuroku, Kyoto University, 1695 (2010), 56–64.

[85] Q. Liu, *Fattening and comparison principle for level-set equations of mean curvature type*, submitted.

[86] Q. Liu, *Noncoercive ergodicity*, unpublished note, 2010.

[87] Q. Liu, *A discrete scheme for shocks*, unpublished note, 2010.

[88] D. Margetis and R. V. Kohn, *Continuum relaxation of interacting steps on crystal surfaces in 2+1 dimensions*, Multiscale Model. Simul., 5 (2006), 729-758.

[89] H. Mitake, *Asymptotic solutions of Hamilton-Jacobi equations with state constraints*, Appl. Math. Optim., 58 (2008), 393–410.

[90] H. Mitake, *The large-time behavior of solutions of the Cauchy-Dirichlet problem for Hamilton-Jacobi equations*, NoDEA Nonlinear Differential Equations App. 15 (2008), 347–362.

[91] H. Mitake, *Large time behavior of solutions of Hamilton-Jacobi equations with periodic boundary data*, Nonlinear Anal., 71 (2009), 5392–5405.

[92] G. Namah and J.-M. Roquejoffre, *Remarks on the long time behaviour of the solutions of Hamilton-Jacobi equations*, Comm. Partial Differential Equations, 24 (1999), 883–893.

[93] D. Pasquignon, *Approximation of viscosity solutions by morphological filters*, ESAIM Control Optim. Calc. Var., Vol. 4 (1999), 335-359.

[94] B. Perthame, "Kinetic formulation of conservation laws", Oxford Lecture Series in Mathematics and its Applications, 21, Oxford, 2002.

[95] Y. Peres, O. Schramm, S. Sheffield and D. Wilson, *Tug-of-war and the infinity Laplacian*, J. Amer. Math. Soc., 22 (2009), 167–210.

[96] Y. Peres and S. Sheffield, *Tug of war with noise: a game theoretic view of the p-Laplacian*, Duke Math. J., 145 (2008), 91–120.

[97] P. Quittner and P. Souplet, "Superlinear parabolic problems, blow-up, global existence and steady states", Birkhauser (2007).

[98] J. M. Roquejoffre, *Convergence to steady states or periodic solutions in a class of Hamilton-Jacobi equations*, J. Math. Pures Appl. 80 (2001), 85–104.

[99] H. M. Soner, *Optimal control with state-space constraint I.*, SIAM J. Control Optim., 24 (1986), 552–561.

[100] M.-H. Sato, *Interface evolution with Neumann boundary condition*, Adv. Math. Sci. Appl., Vol. 4 (1994), 249-264.

[101] M.-H. Sato, *Capillary problem for singular degenerate parabolic equations on a half space*, Differential Integral Equations, Vol. 9 (1996), 1213-1224.

[102] N. Sesum, *Rate of convergence of the mean curvature flow*, Comm. Pure Appl. Math., 61 (2008), 464–485.

[103] H. M. Soner, *Motion of a set by the curvature of its boundary*, J. Differential Equations, 101 (1993), 313–372.

[104] H. M. Soner and P. E. Souganidis, *Singularities and uniqueness of cylindrically symmetric surfaces moving by mean curvature*, Comm. Partial Differential Equations, 18 (1993), 859–894.

[105] H. M. Soner and N. Touzi, *A stochastic representation for mean curvature type geometric flows*, Ann. Probab., 31 (2003), 1145–1165.

[106] P. Soravia, *Generalized motion of a front propagating along its normal direction: a differential games approach*, Nonlinear Anal., 22 (1994), 1247–1262.

[107] P. E. Souganidis and N. K. Yip, *Uniqueness of motion by mean curvature perturbed by stochastic noise*, Ann. Inst. H. Poincare Anal. Non Lineaire 21 (2004), 1–23.

[108] P. Souplet, Q. S. Zhang, *Global solutions of inhomogeneous Hamilton-Jacobi equations*, J. Anal. Math., 99 (2006), 355–396.

[109] J. Spencer, *Balancing games*, J. Combinatorial Theory Ser. B 23 (1977), 68C74.

[110] J. Spencer, *Balancing vectors in the max norm*, Combinatorica 6 (1986), 55C65.

[111] H. Tanaka, *Stochastic differential equations with reflecting boundary conditions in convex regions*, Hiroshima Math. J., Vol. 9 (1979), 163-177.

[112] B. White, *Evolution of curves and surfaces by mean curvature*, Proceedings of the International Congress of Mathematicians, Beijing, 2002, 525–538.

[113] E. Yokoyama, Y. Giga and P. Rybka, *A microscopic time scale approximation to the behavior of the local slope on the faceted surface under a nonuniformity in supersaturation*, Phys. D, 237 (2008), 2845–2855.