Hydrodynamic limit and equilibrium fluctuation for nongradient systems

（非勾配型の系に対する流体力学極限と平衡揺動）

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Introduction

One of the most fundamental problems in statistical physics is to derive a macroscopic evolutional equation describing natural phenomena like the dynamics of fluids from a complicated interacting system of microscopic objects such as atoms or molecules. This microscopic system has a large number of degrees of freedom, and called a large scale interacting system. On the basis of stochastic analysis, some scaling limits for the large scale interacting systems have been extensively studied to establish a mathematical foundation of statistical physics. Hydrodynamic limit is the most typical and important one among them, in which the averaging caused by the local equilibrium plays an essential role. It is a limiting procedure in an adequate space-time scaling, which enables us to derive a partial differential equation describing a macroscopic evolution from a stochastic process describing a microscopic evolution. The limiting equation is called a hydrodynamic equation. The equilibrium fluctuation is also one of such limiting procedures under a space-time scaling. It enables us to derive a stochastic differential equation which describe macroscopic time dependent fluctuation around the hydrodynamic limit starting from an equilibrium state. From the viewpoint of the probability theory, the hydrodynamic limit is a law of large numbers for macroscopic parameters and the equilibrium fluctuation is a central limit theorem for them.

A large scale interacting system which satisfies the so-called gradient condition is said to be gradient. As for reversible gradient models, the hydrodynamic limit and the equilibrium fluctuation are proved in principle using the entropy method introduced by Guo, Papanicolou and Varadhan in [11] or the relative entropy method by Yau in [29]. To apply these methods to general (nongradient) models, Varadhan proposed a clever method called *gradient replacement* in [26], which has actually been applied to many nongradient models (e.g. [21, 15, 10, 28, 16]). This method, however, requires several model-dependent estimates, there are still many open problems related to scaling limits for nongradient models. In this thesis, we study scaling limits for three types of nongradient models, which are interesting from physical points of view but have their respective difficulties of applying Varadhan's method.

In Chapter 1, we introduce a new class of nonreversible and nongradient lattice gas models and prove the hydrodynamic limit for this model under the diffusive space-time scaling. The hydrodynamic equation is a certain nonlinear diffusion equation and its diffusion coefficient is characterized by a variational formula.

The hydrodynamic limit for a nonreversible and nongradient system under the diffusive scaling is first proved by Xu in [28] where the hydrodynamic behavior of a one-dimensional mean-zero zero-range process is studied. Later, Komoriya proved the hydrodynamic limit for a mean-zero exclusion process in [16]. A crucial step for extending the entropy method first developed for reversible systems to nonreversible systems consists in controlling the asymmetric part of the generator by the symmetric one. This is related to the so-called sector condition. In [28, 27, 25], some versions of the sector condition are proved using the idea called *loop decomposition* first introduced in [28]. The idea depends deeply on the mean-zero property of random walks considered there. In this thesis, we do not use *loop decomposition*, and instead we use the parity of the system to show a version of the sector condition. This new method can be applied to Hamiltonian systems, and indeed we use this in Chapter 2.

We are interested in this model not only because it is nonreversible and nongradient but also because it can be considered as an intermediate between two well-studied lattice gas models, namely the totally asymmetric exclusion process (TASEP) and the simple symmetric exclusion process (SSEP).

The model considered here is characterized by a positive real number γ which represents the strength of eternal forces of the system. If γ takes 0, the system evolves as same as TASEP. On the other hand, if γ takes ∞ heuristically, the system evolves as same as SSEP where the hydrodynamic equation under the diffusive scaling is the heat equation with a constant diffusion coefficient 1/2. From this point of view, we study the asymptotic behavior of the diffusion coefficient of the hydrodynamic equation as γ goes to 0 or ∞ and obtain some results which show that the asymptotic behavior of the diffusion coefficient is consistent with the asymptotic behavior of the microscopic evolution.

In Chapter 2, we consider a chain of anharmonic oscillators with a stochastic perturbation preserving total energy. We study a fluctuation of a spatial distribution of energy at equilibrium and prove the equilibrium fluctuation under the diffusive space-time scaling. The limit fluctuation process is governed by a generalized stationary Ornstein-Uhlenbeck process, whose covariances are given by a variational formula. Chapter 2 is based on a joint work with Professor Stefano Olla.

The derivation of the diffusion equation for a macroscopic evolution of energy through a diffusive space-time scaling limit from a microscopic Hamilton dynamics, is one of the most important problem in non-equilibrium statistical mechanics ([5]). The main difficulty of this problem relates to the origin of the diffusive behavior in classical physics. One dimensional chains of oscillators have been used as simple models for this study. As deterministic dynamics lacks good ergodicity properties, the derivation of macroscopic evolution equations from Hamilton systems has not been justified mathematically so far. Therefore, stochastically perturbed Hamilton systems have been considered instead. But since these systems are usually nonreversible and nongradient, it is still difficult to prove scaling limits for these systems under diffusive scaling by Varadhan's method. In this thesis, we overcome this difficulty by using the method developed in Chapter 1 and obtain the diffusive behavior of energy.

In Chapter 3, we consider a multi-species generalization of the symmetric simple exclusion process in homogeneous and non-homogeneous hypercubes of $\mathbf{Z}^d$. We show some estimates of the spectral gap (the absolute value of the second largest eigenvalue of the generator), which plays an important role in the proof of the hydrodynamic limit for nongradient models. Chapter 3 is based on a joint work with Professor Yukio Nagahata.

The spectral gap has been estimated in recent works by Caputo in [6] and by Dermoune and Heinrich in [8] for non-homogeneous multi-color exclusion processes, in which the multi-color particles evolve in accordance with the identical dynamics. A distinctive feature of the multi-color exclusion process is that the spectral gap depends on the density of vacant sites, which is not the case for the one-species simple exclusion process. In particular, the spectral gap vanishes as the density of vacant sites approaches 0. This degeneracy of the spectral gap was first shown by Quastel in [21] for the simple exclusion process with color, which was introduced by himself. In [6] and [8], the non-homogeneous multi-color exclusion process was considered and they estimated the dependence of the spectral gap on the density of vacant sites in detail.

The aim of our study is to extend the previous results to a multi-*species* exclusion process. Namely, we consider a system of several species of particles having their own dynamics, or precisely, their own jump rates and jump ranges. In physical point of view, it is a system of several constituents having physically different properties. In our model, the hyperplanes of configurations with given numbers of particles of each species are not necessarily irreducible. We give a sufficient condition of the dynamics to make them irreducible. Also, assuming the irreducibility of them, we show that the spectral gap is bounded from below by $C\rho_0/n^2$ where ρ_0 is a density of vacant sites, n is the length of each side of a lattice space and C is a positive constant independent of n and ρ_0.

This paper is organized as follows: In Chapter 1, we prove the hydrodynamic limit for exclusion processes with velocity. In Chapter 2, we prove the equilibrium fluctuation for a chain of anharmonic oscillators with a stochastic perturbation. In Chapter 3, we give detailed estimates of the spectral gap for multi-species exclusion processes.

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Chapter 1

Hydrodynamic limit for exclusion processes with velocity

1.1 Introduction

One of the main open problems in nonequilibrium statistical physics is the derivation of diffusion equations from the microscopic Hamiltonian dynamics. The main difficulty of this problem relates to the origin of diffusivity in classical physics. One way to approach the problem is through the entropy method introduced by Varadhan [26], but because of the lack of analytic tools, it is difficult to apply this method for nonreversible and nongradient Hamiltonian systems under diffusive scaling. Therefore, as a first step, in the present paper, we study the hydrodynamic limit for nonreversible and nongradient lattice gas models under diffusive scaling.

The model we consider is a system of particles with velocity on the one-dimensional discrete lattice $\mathbb{Z}$ under the constraint that at most one particle can occupy each site. A set of possible velocities is $\{1, -1\}$ and the state space of the process is $\{1, 0, -1\}^{\mathbb{Z}}$. Its elements (called configurations) are denoted by $\omega = (\omega(x), x \in \mathbb{Z})$ with $\omega(x) = 0$ or 1 or -1 depending on whether $x \in \mathbb{Z}$ is empty or occupied by a particle with velocity 1 or a particle with velocity -1, respectively. A particle at site x with velocity 1 (resp. -1) waits for an exponential time at rate one and then jumps to $x+1$ (resp. $x-1$) provided the site is not occupied. If the site is occupied by a particle with velocity 1 (resp. -1), then the jump is suppressed. On the other hand, if the site is occupied by a particle with velocity -1 (resp. 1), then the particle at x collides to the particle at $x+1$ (resp. $x-1$), namely the particle cannot jump to $x+1$ but instead the velocities of these two particles are exchanged.

The changes of velocities also happen due to random external factors. The sign of velocity of each particle flips (from 1 to -1 or from -1 to 1) with exponential law with a positive constant rate γ. Flips of the sign of each particle's velocity happen independently of each other. Moreover, jumps or collisions of each particle and flips of the sign of each particle's velocity are all independent.

It is easy to see that the number of particles is a unique conserved quantity for such process. We prove the hydrodynamic limit for the density of particles and obtain a variational formula for the diffusion coefficient of the hydrodynamic equation, assuming the continuity of the diffusion coefficient. The only assumption required for an initial condition is that a law of large numbers holds for the distribution of the density of particles. In particular, the macroscopic evolution of the density of particles does not depend on the distribution of the initial velocities of particles, but only on the distribution of the initial position of particles.

The hydrodynamic limit for a nonreversible and nongradient system is first considered in [28] where the hydrodynamic behavior of a one-dimensional mean-zero zero-range process is studied. Later, Komoriya studied the hydrodynamic behavior of a mean-zero exclusion process in [16]. A crucial step for extending the entropy method first developed for reversible systems to nonreversible systems consists in controlling the asymmetric part of the generator by the symmetric one. This is related to the so-called sector condition. In [28], [27] and [25], some versions of the sector condition are proved using the idea called *loop decomposition* first introduced in [28]. The idea depends deeply on the mean-zero property of random walks considered there. In the present paper, we do not use *loop decomposition*, and instead we use the parity of the system to show a version of the sector condition. This new method can be applied to Hamiltonian systems.

We are interested in the model defined above not only because it is nonreversible and nongradient but also because it can be considered as an intermediate between the totally asymmetric exclusion process (TASEP) and the simple symmetric exclusion process (SSEP). If we consider the case with $\gamma = 0$ and the initial condition satisfying that the velocities of all particles are same (e.g. 1), the system evolves as same as TASEP. In this situation, since there is a transport in the system, we have to consider the hyperbolic space-time scaling instead of the diffusive scaling to know the time evolution of the density of particles. On the other hand, if we consider the model with $\gamma = \infty$ heuristically, each particle jumps to right or left with probability 1/2 under the exclusive constraint after exponential waiting time with rate one. Therefore the system evolves as same as SSEP where the density of particles evolves according to the heat equation with a constant diffusion coefficient 1/2 under the diffusive scaling. From this point of view, we study the asymptotic behavior of the diffusion coefficient of our model as γ goes to 0 or ∞ and obtain some results which show that the asymptotic behavior of the diffusion coefficient is consistent with the asymptotic behavior of the evolution of the model. Specifically, we show that the diffusion coefficient, denoted by $D^{\gamma}(\rho)$ as a function of the density of particles ρ, is strictly bigger that 1/2 for all positive γ and $\rho \in [0,1]$, and it converges to 1/2 as γ goes to ∞ for all $\rho \in [0,1]$. We also show that $D^{\gamma}(\rho) = O(\frac{1}{\gamma})$ as γ goes to 0 for $\rho \in [0,1)$. On the other hand, for $\rho = 1$, we show that $D^{\gamma}(\rho) \leq O(\frac{1}{\sqrt{\gamma}})$. The difference of the order between $\rho \in [0,1)$ and $\rho = 1$ implies that $\rho = 1$ has some special property. Actually, we can transform a configuration of our model with full density (i.e. every site is occupied by a particle) to a configuration of the usual exclusion process by corresponding a site occupied by a particle with velocity 1 to an occupied site and a site occupied by a particle with velocity -1 to a vacant site. By this transformation, we obtain an exclusion process where a particle jumps only to one direction with a Glauber term at rate γ at each site. Therefore, as γ goes to 0, we obtain TASEP as a limit again formally. In [24], we study a high-dimensional version of the model we consider here and show that in the case $d \geq 3$, the diffusion coefficient $D^{\gamma}(\rho)$ of the hydrodynamic equation goes to ∞ for $\rho \in [0,1)$ but remains finite for $\rho = 1$ as γ goes to 0. Especially, we conjecture that $\lim_{\gamma\downarrow 0} D^{\gamma}(1)$ relates to the diffusion coefficient for TASEP, which is well studied and diverges for $d = 1,2$ but remains finite for $d \geq 3$ (see e.g. [18], [19]) .

This chapter is organized as follows: In Section 1.2 we introduce our model and state main results. In Section 1.3, we give the proof of Theorem 1.1, which is divided into several subsections. The proof of a version of the sector condition is in Subsection 1.3.4 and the detailed estimates of the diffusion coefficient are obtained in Subsections 1.3.5 and 1.3.6. In Section 1.4 we give a spectral gap estimate. In Section 1.5, we characterize the class of closed forms.

1.2 Model and results

The exclusion process with velocity we consider is a Markov process ω_t on a configuration space $\chi_N = \{-1, 0, 1\}^{\mathbb{T}_N}$, where $\mathbb{T}_N = (\mathbb{Z}/N\mathbb{Z})$ is a one-dimensional discrete torus. To avoid technical things, we work on a space with the periodic boundary condition. The dynamics is defined by means of an infinitesimal generator L_N^γ acting on functions $f : \chi_N \to \mathbb{R}$ as

$$(L_N^\gamma f)(\omega) = \sum_{x \in \mathbb{T}_N} \{L_x^+ f(\omega) + L_x^- f(\omega) + \gamma L_x^v f(\omega)\},$$

where

$$L_x^+ f(\omega) = 1_{\{\omega_x = 1\}}\{f(\omega^{x,x+1}) - f(\omega)\},$$
$$L_x^- f(\omega) = 1_{\{\omega_x = -1\}}\{f(\omega^{x,x-1}) - f(\omega)\},$$

and

$$L_x^v f(\omega) = 1_{\{\omega_x \neq 0\}}\{f(\omega^x) - f(\omega)\}.$$

In the above formula, ω^x and $\omega^{x,y} \in \chi^N$ stand for

$$\omega_z^{x,y} = \begin{cases} \omega_z & \text{if } z \neq x, y, \\ \omega_y & \text{if } z = x, \\ \omega_x & \text{if } z = y, \end{cases} \qquad \omega_z^x = \begin{cases} \omega_z & \text{if } z \neq x, \\ -\omega_x & \text{if } z = x, \end{cases}$$

respectively. We shall use the notation $\eta_x = 1_{\{\omega_x \neq 0\}}$ so that the variable $\eta \in \{0, 1\}^{\mathbb{T}_N}$ denotes the configuration of occupied sites associated to ω.

The process is invariant with respect to the following one-parameter family of translation invariant product measures ν_ρ.

Definition 1.1. *For each fixed $\rho \in [0, 1]$, let ν_ρ be a product measure on χ_N with marginal given by*

$$\nu_\rho\{\omega_x = 1\} = \frac{\rho}{2}, \quad \nu_\rho\{\omega_x = 0\} = 1 - \rho \quad \text{and} \quad \nu_\rho\{\omega_x = -1\} = \frac{\rho}{2}$$

for all $x \in \mathbb{T}_N$.

The index ρ stands for the density of particles, namely $E_{\nu_\rho}[\eta_0] = \rho$. We will abuse the same notation ν_ρ for the product measures on the configuration spaces χ_N or $\chi = \{-1, 0, 1\}^{\mathbb{Z}}$, namely on the torus or on the infinite lattice. The expectation with respect to ν_ρ will be sometimes denoted by

$$\int f(\omega)\nu_\rho(d\omega) = \langle f \rangle_\rho.$$

From the definition, our model is nonreversible with respect to the measure ν_ρ. By the simple computations, we obtain that

$$(L_N^{\gamma,*}f)(\omega) = \sum_{x\in\mathbb{T}_N} \{L_x^{*,+}f(\omega) + L_x^{*,-}f(\omega) + \gamma L_x^v f(\omega)\},$$

where $L_N^{\gamma,*}$ stands for the adjoint operator of L_N^γ with respect to ν_ρ,

$$L_x^{*,+}f(\omega) = 1_{\{\omega_x=1\}}\{f(\omega^{x,x-1}) - f(\omega)\},$$

and

$$L_x^{*,-}f(\omega) = 1_{\{\omega_x=-1\}}\{f(\omega^{x,x+1}) - f(\omega)\}.$$

Denote by $L_N^{\gamma,S}$ the symmetric part of L_N^γ and by L_N^A the anti-symmetric part of L_N^γ:

$$(L_N^{\gamma,S}f)(\omega) = \sum_{x\in\mathbb{T}_N} \{L_{x,x+1}^{ex}f(\omega) + \gamma L_x^v f(\omega)\}, \quad (L_N^A f)(\omega) = \sum_{x\in\mathbb{T}_N} L_x^A f(\omega),$$

where

$$L_{x,x+1}^{ex}f(\omega) = \frac{1}{2}(1_{\{\eta_x=1\}} + 1_{\{\eta_{x+1}=1\}})\{f(\omega^{x,x+1}) - f(\omega)\}$$

and

$$L_x^A f(\omega) = \frac{1}{2}\omega_x[\{f(\omega^{x,x+1}) - f(\omega)\} - \{f(\omega^{x,x-1}) - f(\omega)\}]..$$

Note that L_N^A does not depend on γ.

Here and after, we call f a cylinder function on χ if f depends on the configurations only through a finite set of coordinates.

For any $x \in \mathbb{Z}$ and cylinder functions f, g, let us define $\mathcal{D}_{x,x+1}(\nu_\rho; f, g)$, $\mathcal{D}_{x,x+1}(\nu_\rho; f)$, $\mathcal{D}_x(\nu_\rho; f, g)$ and $\mathcal{D}_x(\nu_\rho; f)$ by

$$\mathcal{D}_{x,x+1}(\nu_\rho; f, g) := \langle -(L_{x,x+1}^{ex})f, g\rangle_\rho, \quad \mathcal{D}_{x,x+1}(\nu_\rho; f) := \mathcal{D}_{x,x+1}(\nu_\rho; f, f),$$

$$\mathcal{D}_x(\nu_\rho; f, g) := \langle -(L_x^v)f, g\rangle_\rho \quad \text{and} \quad \mathcal{D}_x(\nu_\rho; f) := \mathcal{D}_x(\nu_\rho; f, f),$$

where $\langle \cdot, \cdot \rangle_\rho$ stands for the inner product in $L^2(\nu_\rho)$. The reversibility of $L_{x,x+1}^{ex}$ and L_x^v implies

$$\mathcal{D}_{x,x+1}(\nu_\rho; f, g) = \frac{1}{2}\langle (\nabla_{x,x+1}f)(\nabla_{x,x+1}g)\rangle_\rho$$

and

$$\mathcal{D}_x(\nu_\rho; f, g) = \frac{1}{2}\langle (\nabla_x f)(\nabla_x g)\rangle_\rho$$

where

$$\nabla_{x,x+1}f := 1_{\{\eta_x=1\}}\{f(\omega^{x,x+1}) - f(\omega)\}$$

and

$$\nabla_x f := 1_{\{\eta_x=1\}}\{f(\omega^x) - f(\omega)\}.$$

Let τ_x be the shift operator acting on the set $A \subset \mathbb{Z}$ and cylinder functions f as well as configurations ω as follows:

$$\tau_x A := x + A, \quad (\tau_x\omega)_z := \omega_{z+x} \quad \text{and} \quad \tau_x f(\omega) = f(\tau_x\omega) \quad \text{for} \quad x, z \in \mathbb{Z}.$$

For every cylinder function $g : \chi \to \mathbb{R}$, consider the formal sum

$$\Gamma_g := \sum_{x\in\mathbb{Z}} \tau_x g$$

which does not make sense but for which

$$\nabla_{0,1}\Gamma_g := 1_{\{\eta_0=1\}}\{\Gamma_g(\omega^{0,1}) - \Gamma_g(\omega)\}$$

and

$$\nabla_0\Gamma_g := 1_{\{\eta_0=1\}}\{\Gamma_g(\omega^0) - \Gamma_g(\omega)\}$$

are well defined.

We are now in a position to define the diffusion coefficient. For each $\rho \in (0,1)$, the diffusion coefficient of the hydrodynamic equation for the exclusion processes with velocity $D^\gamma(\rho)$ is given as

$$(1.2.1) \qquad \begin{aligned} D^\gamma(\rho) = \frac{1}{2} + \frac{1}{\chi(\rho)} \inf_{f\in\mathcal{C}_0} \sup_{g\in\mathcal{C}_0} [&\mathcal{D}_{0,1}(\nu_\rho;\Gamma_f) + \gamma\mathcal{D}_0(\nu_\rho;\Gamma_f) \\ &+ 2\langle W_{0,1}^A - L^A f, \Gamma_g\rangle_\rho - \mathcal{D}_{0,1}(\nu_\rho;\Gamma_g) - \gamma\mathcal{D}_0(\nu_\rho;\Gamma_g)] \end{aligned}$$

where $\mathcal{C}_0$ is a subspace of cylinder functions defined in Subsection 1.3.2 below and $W_{0,1}^A$ is a cylinder function defined in (1.3.2) below. Note that $\langle W_{0,1}^A - L^A f, \Gamma_g\rangle_\rho$ is well defined by the definition of $\mathcal{C}_0$. In this formula $\chi(\rho)$ stands for the so-called static compressibility which in our case is equal to

$$\chi(\rho) = \langle \eta_0^2\rangle_\rho - \langle\eta_0\rangle_\rho^2 = \rho(1-\rho).$$

According to the later argument, we can show that $D^\gamma(\rho)$ is continuous in $\rho \in [0,1)$ without any assumption, but to prove Theorem 1.1, we assume a little stronger condition that $D^\gamma(\rho)$ is continuous in $\rho \in [0,1]$ throughout the paper. Note that it can be also shown that $D^\gamma(\rho)$ is Lipshitz continuous in $\gamma > 0$, though it is not necessary for the proofs of our main theorems.

For a probability measure μ^N on χ_N, we denote by $\mathbb{P}_{\mu^N}$ the distribution on the path space $D(\mathbb{R}_+, \chi_N)$ of the Markov process $\omega(t) = \{\omega_x(t), x \in \mathbb{T}_N\}$ with generator $N^2 L_N^\gamma$, which is accelerated by a factor N^2, and the initial measure μ^N. Hereafter $\mathbb{E}_{\mu^N}$ stands for the expectation with respect to $\mathbb{P}_{\mu^N}$.

With these notations our main results are stated as follows:

Theorem 1.1. *Let $(\mu^N)_{N\geq 1}$ be a sequence of probability measures on χ_N such that the corresponding initial density fields satisfy*

$$\lim_{N\to\infty} \mu^N[|\frac{1}{N}\sum_{x\in\mathbb{T}_N} G(\frac{x}{N})\eta_x - \int_{\mathbb{T}=[0,1)} G(u)\rho_0(u)du| > \delta] = 0$$

for every $\delta > 0$, every continuous function $G : \mathbb{T} \to \mathbb{R}$ and some measurable function $\rho_0 : \mathbb{T} \to [0,1]$. Then, for every $t > 0$,

$$\limsup_{N\to\infty} \mathbb{P}_{\mu^N}[|\frac{1}{N}\sum_{x\in\mathbb{T}_N} G(\frac{x}{N})\eta_x(t) - \int_{\mathbb{T}} G(u)\rho(t,u)du| > \delta] = 0$$

for every $\delta > 0$ and every continuous function $G : \mathbb{T} \to \mathbb{R}$, where $\rho(t,u)$ is the unique weak solution of the following nonlinear parabolic equation:

$$\begin{cases} \partial_t \rho(t,u) = \dfrac{\partial}{\partial u}\{D^\gamma(\rho(t,u))\dfrac{\partial \rho}{\partial u}(t,u)\} \\ \rho(0,\cdot) = \rho_0(\cdot). \end{cases} \tag{1.2.2}$$

Moreover, for each $\rho \in (0,1)$ and $\gamma > 0$, $D^\gamma(\rho)$ satisfies the inequality:

$$\frac{1}{2} + \frac{1-\rho}{2\gamma} \leq D^\gamma(\rho) \leq \frac{1}{2} + \frac{2-\rho}{4\gamma}. \tag{1.2.3}$$

Corollary 1.1.1. *By (1.2.3), we obtain*

$$\limsup_{\rho\to 0} D^\gamma(\rho) = \liminf_{\rho\to 0} D^\gamma(\rho) = \frac{1}{2} + \frac{1}{2\gamma}$$

which proves that $D^\gamma(\rho)$ is continuous at $\rho = 0$. Also, we have

$$\lim_{\gamma\to 0} D^\gamma(\rho) = \infty \quad \text{for} \quad \rho \in [0,1) \quad \text{and} \quad \lim_{\gamma\to\infty} D^\gamma(\rho) = \frac{1}{2} \quad \text{for} \quad \rho \in [0,1]$$

where the second formula for $\rho = 1$ means

$$\lim_{\gamma\to\infty} \liminf_{\rho\to 1} D^\gamma(\rho) = \lim_{\gamma\to\infty} \limsup_{\rho\to 1} D^\gamma(\rho) = \frac{1}{2}$$

precisely.

Theorem 1.2. *For each $\gamma > 0$,*

$$\limsup_{\rho \to 1} D^{\gamma}(\rho) \le \frac{1}{2} + \frac{1}{2\gamma + 2\sqrt{\gamma^2 + 2\gamma}}.$$

In particular, $\limsup_{\rho \to 1} D^{\gamma}(\rho) \le O(\frac{1}{\sqrt{\gamma}})$ *as γ goes to 0.*

Remark 1.1. *Since we work on the one-dimensional torus, to prove the uniqueness of weak solutions of the Cauchy problem (1.2.2), we can apply the proof of Theorem A.2.4.4 in [14] straightforwardly.*

1.3 Proofs of the main theorems

The strategy of the proof is as follows. First, in Subsection 1.3.1, we reduce the problem stated in the theorem to the replacement of the current by a gradient, and give the proof of Theorem 1.1. Second, in Subsection 1.3.2, with the usual non-gradient method we prove that the computation of a central limit theorem variance associated to the symmetric part of the generator is sufficient for the proof of this replacement and show a variational formula for this central limit theorem variance. These two steps are essentially the same as given for the generalized exclusion process in [14], which is based on the method first established by Varadhan in [26]. The main argument of this part is obtaining the estimate of the spectral gap and the characterization of the closed forms, which are presented in Sections 1.4 and 1.5. Then, in Subsection 1.3.3, we obtain the decomposition of a Hilbert space equipped with the inner product defined by the central limit theorem variances. Here, we use the sector condition in the sense of this Hilbert norm (1.5.8), which is shown in Subsection 1.3.4. In Subsection 1.3.5, we obtain two variational formula for the diffusion coefficient and its estimates from above and below. Finally, in Subsection 1.3.6, we give the proof of Theorem 1.2.

1.3.1 Replacement of the current by a gradient

We start with considering a class of martingales associated with the empirical measure. We take $T > 0$ arbitrarily and fix it in the rest of this section. For each smooth function $H : \mathbb{T} \to \mathbb{R}$, let $M^{H,N}(t) = M^H(t)$ be the martingale defined by

$$M^H(t) = \langle \pi_t^N, H \rangle - \langle \pi_0^N, H \rangle - \int_0^t N^2 L_N^{\gamma} \langle \pi_s^N, H \rangle ds,$$

where π_t^N stands for the empirical measure associated with $\eta(t)$, namely

$$\pi_t^N(du) = \frac{1}{N}\sum_{x\in\mathbb{T}_N}\eta_x(t)\delta_{\frac{x}{N}}(du), \quad 0\le t\le T, \quad u\in\mathbb{T}, \tag{1.3.1}$$

and $\langle\pi_t^N, f\rangle$ stands for the integration of f with respect to π_t^N.

A simple computation shows that the expected value of the quadratic variation of $M^H(t)$ vanishes as $N\uparrow\infty$, and therefore by Doob's inequality, for every $\delta>0$, we have

$$\lim_{N\to\infty}\mathbb{P}_{\mu^N}[\sup_{0\le t\le T}|M^H(t)|\ge\delta]=0.$$

A spatial summation by parts permits to rewrite the martingale $M^H(t)$ as

$$M^H(t) = \langle\pi_t^N, H\rangle - \langle\pi_0^N, H\rangle - \int_0^t\sum_{x\in\mathbb{T}_N}(\partial_u^N H)(\frac{x}{N})\tau_x W_{0,1}(\omega_s)ds,$$

where $W_{0,1}(\omega)=W_{0,1}^S(\omega)+W_{0,1}^A(\omega)$ represents the instantaneous current from 0 to 1 with

$$W_{0,1}^S(\omega)=\frac{1}{2}(\eta_0-\eta_1), \quad W_{0,1}^A(\omega)=\frac{1}{2}(\omega_0(1-\eta_1)+\omega_1(1-\eta_0)) \tag{1.3.2}$$

and $\partial_u^N H$ represents the discrete derivative of H:

$$(\partial_u^N H)(\frac{x}{N}) = N[H(\frac{x+1}{N})-H(\frac{x}{N})].$$

Here, $W_{0,1}^S$ (resp.$W_{0,1}^A$) is the instantaneous current from 0 to 1 associated to the symmetric (resp. anti-symmetric) part of the generator respectively.

Next we show that the current $W_{0,1}$ can be decomposed into a linear combination of the gradient $\eta_1-\eta_0$ and a function in the range of the generator L_N^γ: $W_{0,1}+D^\gamma(\rho)[\eta_1-\eta_0]=L_N^\gamma\mathfrak{f}$ for a certain cylinder function $\mathfrak{f}$ and the function $D^\gamma(\rho)$ that depends on the density defined by (1.2.1), see Theorem 1.3 and Corollary 1.3.1 for more precise statement.

For positive integers l, N, a function H in $C^2(\mathbb{T})$ and a cylinder function $\mathfrak{f}$ on χ, let

$$X_{N,l}^{\mathfrak{f}}(H,\omega)=\sum_{x\in\mathbb{T}_N}H(\frac{x}{N})\tau_x V^{\mathfrak{f},l}(\omega),$$

where

$$V^{\mathfrak{f},l}(\omega)=W_{0,1}(\omega)+D^\gamma(\eta^l(0))[\eta^l(1)-\eta^l(0)]-L_N^\gamma\mathfrak{f}(\omega),$$

and

$$\eta^l(x) = \frac{1}{(2l+1)} \sum_{|y-x|\le l} \eta_y, \quad x \in \mathbb{T}_N.$$

Theorem 1.3. *Fix $\rho \in (0,1)$ arbitrarily. Then, for every function H in $C^2(\mathbb{T})$, we have*

$$\inf_{\mathfrak{f}\in\mathcal{C}} \limsup_{\varepsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}_{\nu_\rho^N}[\exp\{N|\int_0^T X^{\mathfrak{f}}_{N,\varepsilon N}(H,\omega(s))ds|\}] = 0$$

where $\mathcal{C}$ stands for the set of cylinder functions on χ.

The proof of Theorem 1.3 is postponed to the rest of this section. This theorem implies the following corollary. For a positive integer l and a function H in $C^2(\mathbb{T})$, let

$$Y_{N,l}(H,\omega) = \sum_{x\in\mathbb{T}_N} H(\frac{x}{N})\{W_{x,x+1}(\omega) + D^\gamma(\eta^l(x))[\eta^l(x+1) - \eta^l(x)]\}.$$

Corollary 1.3.1. *For every function H in $C^2(\mathbb{T})$,*

$$\limsup_{\varepsilon\to 0} \limsup_{N\to\infty} \mathbb{E}_{\mu^N}[|\int_0^T Y_{N,\varepsilon N}(H,\omega(s))ds|] = 0.$$

To prove this corollary, we can follow the argument in the proof of Corollary 7.1.2 in [14] straightforwardly. In particular, the $L_N^\gamma \mathfrak{f}$ term is negligible. We have now all elements to prove the hydrodynamic behavior of our nongradient system.

Proof of Theorem 1.1. Recall that the empirical measure π_t^N is defined by (1.3.1). Denote by Q_{μ^N} the distribution on the path space $D([0,T],\mathcal{M}(\mathbb{T}))$ of the process π_t^N where $\mathcal{M}(\mathbb{T})$ stands for the space of measures on $\mathbb{T}$ endowed with the weak topology.

Following the same argument as for the generalized exclusion process in Section 7.6 in [14] it is easy to prove that the sequence $\{Q_{\mu^N}, N \ge 1\}$ is weakly relatively compact and that every limit point Q^* is concentrated on absolutely continuous paths $\pi_t(du) = \pi(t,u)du$ with density bounded by 1: $0 \le \pi(t,u) \le 1$.

From Remark 1.1, there exists at most one weak solution of (1.2.2). Therefore, to conclude the proof of the theorem, it remains to show that all limit points of the sequence $\{Q_{\mu^N}, N \ge 1\}$ are concentrated on absolutely continuous trajectories $\pi(t,du) = \pi(t,u)du$ whose densities are weak solutions of the equation (1.2.2). With Theorem 1.3, it is done exactly same way in the proof of Theorem 7.0.1 in [14]. □

1.3.2 Central limit theorem variances

To state the main theorem of this subsection, first we introduce some notation. For a fixed positive integer l we define $\Lambda_l := \{-l, -l+1, ..., l-1, l\}$ and $L^{\gamma}_{\Lambda_l}$, $L^{\gamma,S}_{\Lambda_l}$ the restriction of the generator L^{γ}_N, $L^{\gamma,S}_N$ to Λ_l respectively. We denote the set of cylinder functions on χ by $\mathcal{C}$. For Ψ in $\mathcal{C}$, denote by s_Ψ the smallest positive integer s such that Λ_s contains the support of Ψ and define $\Lambda_\Psi := \Lambda_{s_\Psi}$. Let $\mathcal{C}_0$ be the space of cylinder functions with mean zero with respect to all canonical invariant measures:

$$\mathcal{C}_0 = \{\ g \in \mathcal{C}\ ; \langle g \rangle_{\Lambda_g, K} = 0 \text{ for all } 0 \le K \le |\Lambda_g|\ \}.$$

Here, for a finite subset Λ of $\mathbb{Z}$, we denote by $|\Lambda|$ the cardinality of Λ and by $\langle \cdot \rangle_{\Lambda,K}$ the expectation with respect to the canonical measure $\nu_{\Lambda,K} := \nu_\rho(\ \cdot \mid \sum_{x\in\Lambda} \eta_x = K)$ for $0 \le K \le |\Lambda|$ which is indeed independent of the choice of ρ. For a finite set Λ and a canonical measure $\nu_{\Lambda,K}$, denote by $\langle \cdot, \cdot \rangle_{\Lambda,K}$ (resp.$\langle \cdot, \cdot \rangle_\rho$) the inner product in $L^2(\nu_{\Lambda,K})$ (resp. $L^2(\nu_\rho)$).

With these notations, we reduce the proof of Theorem 1.3 to the following theorem:

Theorem 1.4.

$$\inf_{\mathfrak{f}\in\mathcal{C}} \lim_{l\to\infty} \sup_K 2l \langle (-L^{\gamma,S}_{\Lambda_l})^{-1} \tilde{V}^{\mathfrak{f},l}, \tilde{V}^{\mathfrak{f},l} \rangle_{l,K} = 0 \tag{1.3.3}$$

where

$$\begin{aligned}\tilde{V}^{\mathfrak{f},l}(\omega) &= (2l'+1)^{-1} \sum_{|y|\le l'} \tau_y W_{0,1}(\omega) \\ &+ D^{\gamma}(\eta^l(0))[\eta^{l'}(1) - \eta^{l'}(0)] - (2l_{\mathfrak{f}}+1)^{-1} \sum_{y\in\Lambda_{l_{\mathfrak{f}}}} (\tau_y L^{\gamma}_{\Lambda_{s_{\mathfrak{f}}+1}} \mathfrak{f})(\omega),\end{aligned}$$

$l' = l-1$ and $l_{\mathfrak{f}} = l - s_{\mathfrak{f}} - 1$ so that $\tau_y L_{\Lambda_{s_{\mathfrak{f}}+1}} \mathfrak{f}$ is $\mathcal{F}_{\Lambda_l}$-measurable for every y in $\Lambda_{l_{\mathfrak{f}}}$.

By the general argument, Theorem 1.4 is enough to conclude the proof of Theorem 1.3. The precise argument for proving Theorem 1.3 from (1.3.3) should be omitted here since it is very similar to that in Section 7.2 and 7.3 in [14], so here we just show the sketch of the argument: First, using the Feynman-Kac formula, we reduce the proof of Theorem 1.3 to the estimation of the largest eigenvalue of the symmetric operator $N^2 L^{S,\gamma}_N + N X^{\mathfrak{f}}_{N,\varepsilon N}$. Then, to localize the eigenvalue problem, we use the usual procedure of the proof of the two-blocks estimate and an integration by parts formula (see Section 7.2 in [14]) and obtain that it is enough to estimate the largest eigenvalue of the symmetric operator $N^2 L_N^\gamma + N\tilde{X}_{N,l}^{\mathfrak{f}}$ for each fixed $l \in \mathbb{N}$ instead of $X_{N,\varepsilon N}^{\mathfrak{f}}$ where

$$\tilde{X}_{N,l}^{\mathfrak{f}}(H,\omega) := \sum_{x\in\mathbb{T}_N} H(\frac{x}{N})\tau_x \tilde{V}^{\mathfrak{f},l}(\omega)$$

and $\tilde{V}^{\mathfrak{f},l}(\omega)$ is the function defined just above. In the proof of the two-blocks estimate, we need the assumption that $\mathcal{D}^\gamma(\rho)$ is continuous in ρ not only on the open interval $(0,1)$ but also at the boundary. Finally, with the usual procedure of the proof of the one-block estimate and the general estimate of the spectrum of reversible Markov processes (see Appendix 3.1.1 in [14]), we conclude that to show (1.3.3) is sufficient.

For the beginning of the proof of (1.3.3) we obtain a variational formula for this variance. We start with introducing a semi-norm on $\mathcal{C}_0$, which is closely related to the central limit theorem variance. For cylinder functions g, h in $\mathcal{C}_0$, let

$$\ll g,h \gg_{\rho,*} = \sum_{x\in\mathbb{Z}} \langle g, \tau_x h\rangle_\rho \quad \text{and} \quad \ll g \gg_{\rho,**} = \sum_{x\in\mathbb{Z}} x\langle g, \eta_x\rangle_\rho.$$

$\ll g,h \gg_{\rho,*}$ and $\ll g \gg_{\rho,**}$ are well defined because g and h belong to $\mathcal{C}_0$ and therefore all but a finite number of terms vanish. For h in $\mathcal{C}_0$, define the semi-norm $\ll h \gg_{\rho,\gamma}^{\frac{1}{2}}$ by

$$\begin{aligned}\ll h \gg_{\rho,\gamma} &= \sup_{g\in\mathcal{C}_0}\{2 \ll g,h \gg_{\rho,*} + \frac{2\ll h \gg_{\rho,**}^2}{\chi(\rho)} - \frac{1}{2}\langle(\nabla_{0,1}\Gamma_g)^2\rangle_\rho - \frac{\gamma}{2}\langle(\nabla_0\Gamma_g)^2\rangle_\rho\}\\ &= \sup_{g\in\mathcal{C}_0, a\in\mathbb{R}}\{2\ll g,h\gg_{\rho,*} + 2a \ll h \gg_{\rho,**} - \mathcal{D}_{0,1}(\nu_\rho; a\eta_0 + \Gamma_g) - \gamma\mathcal{D}_0(\nu_\rho; a\eta_0 + \Gamma_g)\}.\end{aligned}$$

We investigate several properties of the semi-norm $\ll \cdot \gg_{\rho,\gamma}^{\frac{1}{2}}$ in the next section, while in this section we prove that the variance

$$(2l)^{-1}\langle(-L_{\Lambda_l}^{\gamma,S})^{-1}\sum_{|x|\le l_\psi}\tau_x\psi, \sum_{|x|\le l_\psi}\tau_x\psi\rangle_{l,K_l}$$

of any cylinder function ψ in $\mathcal{C}_0$ converges to $\ll \psi \gg_{\rho,\gamma}$, as $l \uparrow \infty$ and $\frac{K_l}{2l} \to \rho$. Here l_ψ stands for $l - s_\psi$ so that the support of $\tau_x\psi$ is included in Λ_l for every $x \in \Lambda_{l_\psi}$.

We are now in a position to state the main result of this section.

Proposition 1.1. *Consider a cylinder function ψ in $\mathcal{C}_0$ and a sequence of integers K_l such that $0 \le K_l \le 2l+1$ and $\lim_{l\to\infty} \frac{K_l}{2l} = \rho$. Then,*

$$\lim_{l\to\infty} (2l)^{-1} \langle (-L^{\gamma,S}_{\Lambda_l})^{-1} \sum_{|x|\le l_\psi} \tau_x \psi, \sum_{|x|\le l_\psi} \tau_x \psi \rangle_{l,K_l} = \ll \psi \gg_{\rho,\gamma} .$$

Proof. The key of the proof is the fact that any germ of a closed form, which is defined and studied in Section 1.5, can be decomposed as a sum of germs in a proper way. This is proved in Theorem 1.9 in Section 1.5, see there for more precise statement. Once the theorem is established, the proof of Proposition 1.1 is the same as that of Theorem 7.4.1 in [14] since the proof depends on the specific model only through an integration by parts formula and the equivalence of ensembles, which are easily shown. □

We conclude this section proving that for each ψ in $\mathcal{C}_0$ the function $\ll \psi \gg_{\cdot,\gamma}$: $[0,1] \to \mathbb{R}_+$ that associates to each density ρ the value $\ll \psi \gg_{\rho,\gamma}$ is continuous and that the convergence of the finite volume variances to $\ll \cdot \gg_{\rho,\gamma}$ is uniform on $[0,1]$. For each l in $\mathbb{N}$ and $0 \le K \le 2l+1$, denote by $V^\psi_l(\frac{K}{2l+1})$ the variance of $(2l+1)^{-1} \sum_{|x|\le l_\psi} \tau_x \psi$ with respect to $\nu_{l,K}$:

$$V^\psi_l\Big(\frac{K}{2l+1}\Big) = (2l)^{-1} \langle (-L^{\gamma,S}_{\Lambda_l})^{-1} \sum_{|x|\le l_\psi} \tau_x \psi, \sum_{|x|\le l_\psi} \tau_x \psi \rangle_{l,K}.$$

We may interpolate linearly to extend the definition of V^ψ_l to the whole interval $[0,1]$. With this definition V^ψ_l is continuous. Proposition 1.1 asserts that V^ψ_l converges, as $l \uparrow \infty$, to $\ll \psi \gg_{\rho,\gamma}$, for any sequence K_l such that $\frac{K_l}{2l+1} \to \rho$. In particular, $\lim_{l\to\infty} V^\psi_l(\rho_l) = \ll \psi \gg_{\rho,\gamma}$ for any sequence $\rho_l \to \rho$. This implies that $\ll \psi \gg_{\rho,\gamma}$ is continuous and that $V^\psi_l(\cdot)$ converges uniformly to $\ll \psi \gg_{\cdot,\gamma}$ as $l \uparrow \infty$. We have thus proved the following theorem.

Theorem 1.5. *For each fixed h in $\mathcal{C}_0$, $\ll h \gg_{\rho,\gamma}$ is continuous as a function of the density ρ on $[0,1]$. Moreover, the variance*

$$(2l)^{-1} \langle (-L^{\gamma,S}_{\Lambda_l})^{-1} \sum_{|x|\le l_h} \tau_x h, \sum_{|x|\le l_h} \tau_x h \rangle_{l,K_l}$$

converges uniformly to $\ll h \gg_{\rho,\gamma}$ as $l \uparrow \infty$ and $\frac{K_l}{2l+1} \to \rho$. In particular,

$$\lim_{l\to\infty} \sup_{0\le K\le 2l+1} (2l)^{-1} \langle (-L^{\gamma,S}_{\Lambda_l})^{-1} \sum_{|x|\le l_h} \tau_x h, \sum_{|x|\le l_h} \tau_x h \rangle_{l,K} = \sup_{0\le\rho\le 1} \ll h \gg_{\rho,\gamma} .$$

1.3.3 Hilbert space

We investigate here the main properties of the semi norm $\ll \cdot \gg_{\rho,\gamma}$ introduced in the previous section. We first define from $\ll \cdot \gg_{\rho,\gamma}$ a semi-inner product on $\mathcal{C}_0$ through polarization:

$$\ll g, h \gg_{\rho,\gamma} = \frac{1}{4}\{\ll g+h \gg_{\rho,\gamma} - \ll g-h \gg_{\rho,\gamma}\}. \tag{1.3.4}$$

It is easy to check that (1.3.4) defines a semi-inner product on $\mathcal{C}_0$. Denote by $\mathcal{N}_{\rho,\gamma}$ the kernel of the semi-norm $\ll \cdot \gg_{\rho,\gamma}^{\frac{1}{2}}$ on $\mathcal{C}_0$. Since $\ll \cdot \gg_{\rho,\gamma}$ is a semi-inner product on $\mathcal{C}_0$, the completion of $\mathcal{C}_0|_{\mathcal{N}_{\rho,\gamma}}$, denoted by $\mathcal{H}_{\rho,\gamma}$, is a Hilbert space.

Here and after, we consider generators L^γ, $L^{\gamma,S}$ and L^A acting on functions f in χ as

$$L^\gamma f = \sum_{x\in\mathbb{Z}}\{L_x^+ f + L_x^- f + \gamma L_x^v f\}, \quad L^{\gamma,S} f = \sum_{x\in\mathbb{Z}}\{L_{x,x+1}^{ex} f + \gamma L_x^v f\}, \quad L^A f = \sum_{x\in\mathbb{Z}} L_x^A f.$$

Simple computations show that the linear space generated by $W_{0,1}^S$ and $L^{\gamma,S}\mathcal{C}_0 = \{L^{\gamma,S}g;\ g\in\mathcal{C}_0\}$ are subsets of $\mathcal{C}_0$. The first main result of this subsection consists in showing that $\mathcal{H}_{\rho,\gamma}$ is the completion of $L^{\gamma,S}\mathcal{C}_0|_{\mathcal{N}_{\rho,\gamma}} + \{W_{0,1}^S\}$, in other words, that all elements of $\mathcal{H}_{\rho,\gamma}$ can be approximated by $aW_{0,1}^S + L^{\gamma,S}g$ for some a in $\mathbb{R}$ and g in $\mathcal{C}_0$. To prove this result we derive two elementary identities:

$$\ll h, L^{\gamma,S}g \gg_{\rho,\gamma} = -\ll h, g \gg_{\rho,*}, \quad \ll h, W_{0,1}^S \gg_{\rho,\gamma} = -\ll h \gg_{\rho,**} \tag{1.3.5}$$

for all h, g in $\mathcal{C}_0$.

By Proposition 1.1 and (1.3.4), the semi-inner product $\ll h, g \gg_{\rho,\gamma}$ is the limit of the covariance $(2l)^{-1}\langle(-L_{\Lambda_l}^{\gamma,S})^{-1}\sum_{|x|\le l_g}\tau_x g, \sum_{|x|\le l_h}\tau_x h\rangle_{l,K_l}$ as $l\uparrow\infty$ and $\frac{K_l}{2l}\to\rho$. In particular, if $g = L^{\gamma,S}g_0$, for some cylinder function g_0, the inverse of the generator cancels with the generator. Therefore, $\ll h, L^{\gamma,S}g_0 \gg_{\rho,\gamma}$ is equal to

$$-\lim_{l\to\infty}(2l)^{-1}\langle \sum_{|x|\le l_{g_0}}\tau_x g_0, \sum_{|x|\le l_h}\tau_x h\rangle_{l,K_l} = \ll g_0, h \gg_{\rho,*}.$$

The second identity is proved by similar way with the elementary relation $L_{\Lambda_l}^{\gamma,S}\sum_{x\in\Lambda_l}x\eta_x = \sum_{x,x+1\in\Lambda_l}W_{x,x+1}^S$.

The identities of (1.3.5) permit to compute these elementary relations

$$\ll W_{0,1}^S, L^{\gamma,S}h \gg_{\rho,\gamma} = 0$$

for all $h \in \mathcal{C}_0$ and

$$\ll W_{0,1}^S, W_{0,1}^S \gg_{\rho,\gamma} = \frac{1}{2}\chi(\rho).$$

Recall that $\chi(\rho)$ stands for the static compressibility and is equal to $\langle \eta_0^2 \rangle_\rho - \langle \eta_0 \rangle_\rho^2$. Furthermore,

$$\ll aW_{0,1}^S + L^{\gamma,S} g \gg_{\rho,\gamma} = \frac{a^2}{2}\chi(\rho) + \frac{1}{2}\langle (\nabla_{0,1}\Gamma_g)^2 \rangle_\rho + \frac{\gamma}{2}\langle (\nabla_0 \Gamma_g)^2 \rangle_\rho$$

for a in $\mathbb{R}$ and g in $\mathcal{C}_0$. In particular, the variational formula for $\ll h \gg_{\rho,\gamma}$ writes

$$(1.3.6) \quad \ll h \gg_{\rho,\gamma} = \frac{2 \ll h, W_{0,1}^S \gg_{\rho,\gamma}^2}{\chi(\rho)} + \sup_{g \in \mathcal{C}_0} \{ -2 \ll h, L^{\gamma,S} g \gg_{\rho,\gamma} - \ll L^{\gamma,S} g \gg_{\rho,\gamma} \}.$$

Proposition 1.2. *Recall that we denote by $L^{\gamma,S}\mathcal{C}_0$ the space $\{L^{\gamma,S}g;\ g \in \mathcal{C}_0\}$. Then, for each $0 < \rho < 1$, we have*

$$\mathcal{H}_{\rho,\gamma} = \overline{L^{S,\gamma}\mathcal{C}_0}|_{\mathcal{N}_{\rho,\gamma}} \oplus \{W_{0,1}^S\}.$$

Proof. We can apply the proof of Proposition 7.5.2 in [14] straightforwardly. □

Next, to replace the space $L^{\gamma,S}\mathcal{C}_0$ by $L^\gamma \mathcal{C}_0$, we show some useful lemmas.

Lemma 1.5.1. *For all $g, h \in \mathcal{C}_0$ and $0 < \rho < 1$, $\ll L^{\gamma,S}g, L^A h \gg_{\rho,\gamma} = - \ll L^A g, L^{\gamma,S} h \gg_{\rho,\gamma}$. Especially, $\ll L^{\gamma,S}g, L^A g \gg_{\rho,\gamma} = 0$.*

Proof. By the first identity of (1.3.5),

$$\begin{aligned} \ll L^{\gamma,S}g, L^A h \gg_{\rho,\gamma} &= - \ll g, L^A h \gg_{\rho,*} = -\sum_{x\in\mathbb{Z}} \langle \tau_x g, L^A h \rangle_\rho \\ &= \sum_{x\in\mathbb{Z}} \langle L_A \tau_x g, h \rangle_\rho = \sum_{x\in\mathbb{Z}} \langle \tau_x L_A g, h \rangle_\rho \\ &= \sum_{x\in\mathbb{Z}} \langle L_A g, \tau_{-x} h \rangle_\rho = \sum_{x\in\mathbb{Z}} \langle L_A g, \tau_x h \rangle_\rho = - \ll L^A g, L^{\gamma,S} h \gg_{\rho,\gamma}. \end{aligned}$$

This concludes the proof. □

Lemma 1.5.2. *For all $g \in \mathcal{C}_0$ and $0 < \rho < 1$, $\ll L^{\gamma,S}g, W_{0,1}^A \gg_{\rho,\gamma} = - \ll L^A g, W_{0,1}^S \gg_{\rho,\gamma}$.*

Proof. By the first identity of (1.3.5),

$$\ll L^{\gamma,S}g, W_{0,1}^A \gg_{\rho,\gamma} = - \ll g, W_{0,1}^A \gg_{\rho,*} = -\sum_{x\in\mathbb{Z}} \langle \tau_x g, W_{0,1}^A \rangle_\rho$$

$$= -\sum_{x\in\mathbb{Z}} \langle g, W^A_{x,x+1}\rangle_\rho = -\sum_{x\in\mathbb{Z}} x\langle g, W^A_{x-1,x} - W^A_{x,x+1}\rangle_\rho$$
$$= -\sum_{x\in\mathbb{Z}} x\langle g, L^A\eta_x\rangle_\rho = \sum_{x\in\mathbb{Z}} x\langle L^A g, \eta_x\rangle_\rho = - \ll L^A g, W^S_{0,1} \gg_{\rho,\gamma} .$$

It concludes the proof. □

Lemma 1.5.3. *For all $a \in \mathbb{R}$ and $g \in \mathcal{C}_0$ and $0 < \rho < 1$, $\ll aW^S_{0,1} + L^{\gamma,S}g, aW^A_{0,1} + L^A g \gg_{\rho,\gamma} = 0$.*

Proof. By the second identity of (1.3.5), it is easy to see that $\ll W^S_{0,1}, W^A_{0,1} \gg_{\rho,\gamma}= 0$. Then, Lemma 1.5.1 and Lemma 1.5.2 concludes the proof straightforwardly. □

Proposition 1.3. *There exists a positive constant C_γ such that for all $g \in \mathcal{C}_0$ and $0 \le \rho \le 1$, $\ll L^A g \gg_{\rho,\gamma} \le C_\gamma \ll L^{\gamma,S} g \gg_{\rho,\gamma}$.*

Proof. By Lemma 1.5.2, $\ll L^A g, W^S_{0,1} \gg^2_{\rho,\gamma} = \ll L^{\gamma,S}g, W^A_{0,1} \gg^2_{\rho,\gamma} \le \ll L^{\gamma,S} g \gg_{\rho,\gamma} \ll W^A_{0,1} \gg_{\rho,\gamma}$. On the other hand, by Lemma 1.5.8 in the next subsection, $| \ll L^A g, L^{\gamma,S} f \gg_{\rho,\gamma} | \le \frac{1}{2} \ll L^{\gamma,S} f \gg_{\rho,\gamma} + \frac{1}{2\gamma} \ll L^{\gamma,S} g \gg_{\rho,\gamma}$ for all $f \in \mathcal{C}_0$. In particular, for all $f \in \mathcal{C}_0$, $-2 \ll L^A g, L^{\gamma,S} f \gg_{\rho,\gamma} - \ll L^{\gamma,S} f \gg_{\rho,\gamma} \le \frac{1}{\gamma} \ll L^{\gamma,S} g \gg_{\rho,\gamma}$. Therefore, by variational formula for $\ll L^A g \gg_{\rho,\gamma}$ in (1.3.6),

$$\ll L^A g \gg_{\rho,\gamma} \le \frac{2 \ll L^{\gamma,S} g \gg_{\rho,\gamma} \ll W^A_{0,1} \gg_{\rho,\gamma}}{\chi(\rho)} + \frac{1}{\gamma} \ll L^{\gamma,S} g \gg_{\rho,\gamma} .$$

Moreover, by the inequality (1.3.12), we have $\frac{\ll W^A_{0,1} \gg_{\rho,\gamma}}{\chi(\rho)} \le \frac{1}{2\gamma}$ for $0 \le \rho \le 1$, therefore we can conclude the proof with $C_\gamma = \frac{2}{\gamma}$. □

Now, we have all elements to show the desired decomposition of the Hilbert spaces $\mathcal{H}_{\rho,\gamma}$.

Proposition 1.4. *Denote by $L^\gamma \mathcal{C}_0$ the space $\{L^\gamma g;\ g \in \mathcal{C}_0\}$. Then, for each $0 < \rho < 1$, we have*

$$\mathcal{H}_{\rho,\gamma} = \overline{L^\gamma \mathcal{C}_0}|_{\mathcal{N}_{\rho,\gamma}} + \{W^S_{0,1}\}.$$

Proof. Since $\{W^S_{0,1}\}$ and $L^\gamma \mathcal{C}_0$ are contained in $\mathcal{C}_0$ by definition, $\mathcal{H}_{\rho,\gamma}$ contains the right hand space. To prove the converse inclusion, let $h \in \mathcal{H}_{\rho,\gamma}$ so that $\ll h, W^S_{0,1} \gg_{\rho,\gamma}= 0$ and $\ll h, L^\gamma g \gg_{\rho,\gamma}= 0$ for all $g \in \mathcal{C}_0$. Then by assumption and Proposition 1.2, $h = \lim_{k\to\infty} L^{\gamma,S} h_k$ in $\mathcal{H}_{\rho,\gamma}$ for some $h_k \in \mathcal{C}_0$. Especially $\ll h \gg_{\rho,\gamma} = \lim_{k\to\infty} \ll L^{\gamma,S} h_k, L^{\gamma,S} h_k \gg_{\rho,\gamma} = \lim_{k\to\infty} \ll L^{\gamma,S} h_k, L^\gamma h_k \gg_{\rho,\gamma}$ since

$\ll L^{\gamma,S}h_k, L^A h_k \gg_{\rho,\gamma}= 0$ by Lemma 1.5.1. On the other hand, by assumption $\ll h, L^\gamma h_k \gg_{\rho,\gamma}= 0$ for all k. Also, by Proposition 1.3, $\sup_k \ll L^\gamma h_k \gg_{\rho,\gamma}\leq (C_\gamma + 1)\sup_k \ll L^{\gamma,S}h_k \gg_{\rho,\gamma}:= C_h$ is finite. Therefore,

$$\begin{aligned}\ll h \gg_{\rho,\gamma} &= \lim_{k\to\infty} \ll L^{\gamma,S}h_k, L^\gamma h_k \gg_{\rho,\gamma}\\ &= \lim_{k\to\infty} \ll L^{\gamma,S}h_k - h, L^\gamma h_k \gg_{\rho,\gamma}\leq \lim_{k\to\infty}\sqrt{C_h \ll L^{\gamma,S}h_k - h \gg_{\rho,\gamma}} = 0.\end{aligned}$$

This concludes the proof. □

Lemma 1.5.4. *For each $0 < \rho < 1$, we have*

$$\mathcal{H}_{\rho,\gamma} = \overline{L^\gamma \mathcal{C}_0}|_{\mathcal{N}_{\rho,\gamma}} \oplus \{W^S_{0,1}\}.$$

Proof. Let a sequence $g_k \in \mathcal{C}_0$ satisfy $\lim_{k\to\infty} L^\gamma g_k = aW^S_{0,1}$ in $\mathcal{H}_{\rho,\gamma}$ for some $a \in \mathbb{R}$. By the similar argument of the proof of Proposition 1.4, $\limsup_{k\to\infty} \ll L^{\gamma,S}g_k, L^{\gamma,S}g_k \gg_{\rho,\gamma}= \limsup_{k\to\infty} \ll L^\gamma g_k, L^{\gamma,S}g_k \gg_{\rho,\gamma}= \limsup_{k\to\infty} \ll L^\gamma g_k - aW^S_{0,1}, L^{\gamma,S}g_k \gg_{\rho,\gamma}= 0$ since $\ll W^S_{0,1}, L^{\gamma,S}g_k \gg_{\rho,\gamma}= 0$ for all k. On the other hand, by Proposition 1.3, $\ll L^\gamma g_k \gg_{\rho,\gamma}\leq (C_\gamma + 1) \ll L^{\gamma,S}g_k \gg_{\rho,\gamma}$, then $a = 0$. □

Recall that $W^S_{0,1} = \frac{1}{2}(\eta_0 - \eta_1)$. Then we obtain the following decomposition:

Corollary 1.5.1. *For each $g \in \mathcal{C}_0$, there exists a unique constant $a \in \mathbb{R}$ such that*

$$g - a(\eta_1 - \eta_0) \in \overline{L^\gamma \mathcal{C}_0} \quad in \quad \mathcal{H}_{\rho,\gamma}.$$

1.3.4 Sector condition in $\mathcal{H}_{\rho,\gamma}$

In this section, to obtain the sector condition in $\mathcal{H}_{\rho,\gamma}$, we study the special structure of the space $\mathcal{C}_0$. Roughly speaking the space $\mathcal{C}_0$ is divided in the countable spaces which are orthogonal to each other in $L^2(\nu_\rho)$.

First, we define some subsets of $\mathcal{C}$ indexed by nonnegative integers by

$$\tilde{\mathbb{L}}_i = \{f \in \mathcal{C}; f = \Pi^i_{k=1}\omega_{x_k} g(\eta), x_k \neq x_l \text{ if } k \neq l, g \in \mathcal{C}\}$$

and $\mathbb{L}_i$ be the linear space generated by $\tilde{\mathbb{L}}_i$. Here, g is a cylinder function depending only on a configuration η instead of ω. In other words, $\mathbb{L}_i$ is the eigenspace of the operator L^v with respect to the eigenvalue $-2i$.

To consider the relation to the inner product introduced in the last section, we restrict these spaces to $\mathcal{C}_0$: $\mathbb{M}_i := \{f \in \mathbb{L}_i; f \in \mathcal{C}_0\}$. It is easily shown that

$\mathcal{C} = \oplus_{i\in\mathbb{N}_{\geq 0}}\mathbb{L}_i$, $\mathcal{C}_0 = \oplus_{i\in\mathbb{N}_{\geq 0}}\mathbb{M}_i$ and $\mathbb{M}_i = \mathbb{L}_i$ for $i \geq 1$. Moreover, $\mathbb{L}_i$ and $\mathbb{L}_j$ are orthogonal in $L^2(\nu_\rho)$ if $i \neq j$.

Now, we prepare two easy but useful lemmas.

Lemma 1.5.5. *For all $f, g \in \mathcal{C}_0$,*

$$\ll L^{\gamma,S} f, L^A g \gg_{\rho,\gamma} = -\frac{1}{2}\langle \Gamma_f \nabla_0^A \Gamma_g \rangle_\rho = \frac{1}{2}\langle \Gamma_g \nabla_0^A \Gamma_f \rangle_\rho$$

where $\nabla_x^A f := w_x\{f(\omega^{x,x+1}) - f(\omega^{x,x-1})\}$.

Proof. By the first identity of (1.3.5),

$$\begin{aligned}
- \ll L^{\gamma,S} f, L^A g \gg_{\rho,\gamma} &= \ll f, L^A g \gg_{\rho,*} = \sum_{x\in\mathbb{Z}} \langle \tau_x f, L^A g\rangle_\rho = \frac{1}{2}\sum_{x,y\in\mathbb{Z}} \langle \tau_x f, \nabla_y^A g\rangle_\rho \\
&= \frac{1}{2}\sum_{x,z\in\mathbb{Z}} \langle \tau_x f, \nabla_{z-x}^A g\rangle_\rho = \frac{1}{2}\sum_{x,z\in\mathbb{Z}} \langle \tau_x f, \tau_x(\nabla_z^A \tau_{-x} g)\rangle_\rho = \frac{1}{2}\sum_{x,z\in\mathbb{Z}} \langle f, \nabla_z^A \tau_{-x} g\rangle_\rho \\
&= \frac{1}{2}\sum_{z\in\mathbb{Z}} \langle f, \nabla_z^A \Gamma_g\rangle_\rho = \frac{1}{2}\sum_{z\in\mathbb{Z}} \langle \tau_z f, \nabla_0^A \Gamma_g\rangle_\rho = \frac{1}{2}\langle \Gamma_f \nabla_0^A \Gamma_g\rangle_\rho.
\end{aligned}$$

□

Lemma 1.5.6. *For all nonnegative integers i, $f \in \mathbb{M}_i$ and $x \in \mathbb{Z}$, $L^{\gamma,S} f, \tau_x f \in \mathbb{M}_i$ and $L^A f \in \mathbb{M}_{i-1} \oplus \mathbb{M}_{i+1}$. Here $\mathbb{M}_{-1} := \phi$. Therefore, for all $f, g \in \mathcal{C}_0$,*

$$\begin{aligned}
\ll L^{\gamma,S} f \gg_{\rho,\gamma} &= \sum_{i=0}^{\infty} \ll L^{\gamma,S} f_i \gg_{\rho,\gamma}, \\
\ll L^{\gamma,S} f, L^A g \gg_{\rho,\gamma} &= \sum_{i=0}^{\infty} \ll L^{\gamma,S} f_i, L^A g_{i+1} \gg_{\rho,\gamma} + \sum_{i=0}^{\infty} \ll L^{\gamma,S} f_{i+1}, L^A g_i \gg_{\rho,\gamma} \\
&= -\sum_{i=0}^{\infty} \ll L^{\gamma,S} g_{i+1}, L^A f_i \gg_{\rho,\gamma} + \sum_{i=0}^{\infty} \ll L^{\gamma,S} f_{i+1}, L^A g_i \gg_{\rho,\gamma}
\end{aligned}$$

where f_i and g_i are the projection of f and g to the space $\mathbb{M}_i$ respectively.

Proof. Straightforward. □

Next lemma gives us the essential estimates to prove our main result in this subsection.

Lemma 1.5.7. *For all nonnegative integers i, $f_{i+1} \in \mathbb{M}_{i+1}$, $g_i \in \mathbb{M}_i$ and any positive number $A > 0$,*

$$|\langle \Gamma_{f_{i+1}} \nabla_0^A \Gamma_{g_i} \rangle_\rho| \leq \frac{1}{2A}\langle (\nabla_0 \Gamma_{f_{i+1}})^2 \rangle_\rho + \frac{A}{2}\langle (\nabla_{0,1}\Gamma_{g_i})^2\rangle_\rho.$$

Proof. For any f_{i+1} we have the unique decomposition of $\Gamma_{f_{i+1}}$ such that

$$\Gamma_{f_{i+1}} = \sum_{\Lambda \subset \mathbb{Z}_+, 0 \in \Lambda, |\Lambda| = i+1} \Gamma_{f_{i+1,\Lambda}}$$

where $f_{i+1,\Lambda}(\omega) = \Pi_{x \in \Lambda} \omega_x \phi(\eta)$ for some cylinder function ϕ that depends only on η. Since we take the index set as $\{\Lambda \subset \mathbb{Z}_+, 0 \in \Lambda, |\Lambda| = i+1\}$, we obtain the uniqueness of the decomposition. Note that all but a finite number of $f_{i+1,\Lambda}$ are 0. Therefore

$$\begin{aligned}
\langle \Gamma_{f_{i+1}} \nabla_0^A \Gamma_{g_i} \rangle_\rho &= \sum_\Lambda \langle \Gamma_{f_{i+1,\Lambda}} \omega_0 [\Gamma_{g_i}(\omega^{0,1}) - \Gamma_{g_i}(\omega^{0,-1})] \rangle_\rho \\
&= \sum_\Lambda \langle \sum_{z \in \Lambda} \tau_{-z} f_{i+1,\Lambda} \omega_0 [\Gamma_{g_i}(\omega^{0,1}) - \Gamma_{g_i}(\omega^{0,-1})] \rangle_\rho
\end{aligned}$$

since for all $z \notin \Lambda$ and $x \in \mathbb{Z}$, $\langle \tau_{-z} f_{i+1,\Lambda} \omega_0 [\tau_x g_i(\omega^{0,1}) - \tau_x g_i(\omega^{0,-1})] \rangle_\rho = 0$.

By Schwarz inequality, the last expression is bounded from above by

$$\begin{aligned}
&\sqrt{\langle \eta_0 (\sum_\Lambda \sum_{z \in \Lambda} \tau_{-z} f_{i+1,\Lambda})^2 \rangle_\rho \langle \eta_0 [\Gamma_{g_i}(\omega^{0,1}) - \Gamma_{g_i}(\omega^{0,-1})]^2 \rangle_\rho} \\
&\leq \frac{2}{A} \langle \eta_0 (\sum_\Lambda \sum_{z \in \Lambda} \tau_{-z} f_{i+1,\Lambda})^2 \rangle_\rho + \frac{A}{8} \langle \eta_0 [\Gamma_{g_i}(\omega^{0,1}) - \Gamma_{g_i}(\omega^{0,-1})]^2 \rangle_\rho \\
&\leq \frac{1}{2A} \langle (\nabla_0 \Gamma_{f_{i+1}})^2 \rangle_\rho + \frac{A}{2} \langle (\nabla_{0,1} \Gamma_{g_i})^2 \rangle_\rho.
\end{aligned}$$

Here we use the relation that

$$\langle (\nabla_0 \Gamma_{f_{i+1}})^2 \rangle_\rho = \langle (\nabla_0 \sum_\Lambda \Gamma_{f_{i+1,\Lambda}})^2 \rangle_\rho = \langle \eta_0 (-2 \sum_\Lambda \sum_{z \in \Lambda} \tau_{-z} f_{i+1,\Lambda})^2 \rangle_\rho.$$

and the inequality that

$$\begin{aligned}
&\langle \eta_0 [\Gamma_{g_i}(\omega^{0,1}) - \Gamma_{g_i}(\omega^{0,-1})]^2 \rangle_\rho \\
&\quad \leq 2 \langle \eta_0 [\Gamma_{g_i}(\omega^{0,1}) - \Gamma_{g_i}(\omega)]^2 \rangle_\rho + 2 \langle \eta_0 [\Gamma_{g_i}(\omega) - \Gamma_{g_i}(\omega^{0,-1})]^2 \rangle_\rho = 4 \langle (\nabla_{0,1} \Gamma_{g_i})^2 \rangle_\rho.
\end{aligned}$$

□

Now we show the main result in this subsection.

Lemma 1.5.8 (sector condition). *For all $f, g \in C_0$,*

$$| \ll L^{\gamma,S} f, L^A g \gg_{\rho,\gamma} | \leq \frac{1}{2} \ll L^{\gamma,S} f \gg_{\rho,\gamma} + \frac{1}{2\gamma} \ll L^{\gamma,S} g \gg_{\rho,\gamma}.$$

Proof. By Lemma 1.5.5,

$$
\begin{aligned}
&| \ll L^{\gamma,S} f, L^A g \gg_{\rho,\gamma} | \\
&= | - \sum_{i=0}^{\infty} \ll L^{\gamma,S} g_{i+1}, L^A f_i \gg_{\rho,\gamma} + \sum_{i=0}^{\infty} \ll L^{\gamma,S} f_{i+1}, L^A g_i \gg_{\rho,\gamma} | \\
&\leq \frac{1}{2} \sum_{i=0}^{\infty} |\langle \Gamma_{g_{i+1}} \nabla_0^A \Gamma_{f_i} \rangle_\rho| + \frac{1}{2} \sum_{i=0}^{\infty} |\langle \Gamma_{f_{i+1}} \nabla_0^A \Gamma_{g_i} \rangle_\rho|.
\end{aligned}
$$

Then, by Lemma 1.5.7, the last expression is bounded from above by

$$
\begin{aligned}
&\frac{1}{2} \sum_{i=0}^{\infty} [\frac{1}{2} \langle (\nabla_0 \Gamma_{g_{i+1}})^2 \rangle_\rho + \frac{1}{2} \langle (\nabla_{0,1} \Gamma_{f_i})^2 \rangle_\rho] + \frac{1}{2} \sum_{i=0}^{\infty} [\frac{\gamma}{2} \langle (\nabla_0 \Gamma_{f_{i+1}})^2 \rangle_\rho + \frac{1}{2\gamma} \langle (\nabla_{0,1} \Gamma_{g_i})^2 \rangle_\rho] \\
&\quad \leq \frac{1}{2} \ll L^{\gamma,S} f \gg_{\rho,\gamma} + \frac{1}{2\gamma} \ll L^{\gamma,S} g \gg_{\rho,\gamma}
\end{aligned}
$$

since $\ll L^{\gamma,S} f \gg_{\rho,\gamma} = \sum_{i=0}^{\infty} [\frac{1}{2} \langle (\nabla_{0,1} \Gamma_{f_i})^2 \rangle_\rho + \frac{\gamma}{2} \langle (\nabla_0 \Gamma_{f_i})^2 \rangle_\rho]$. □

1.3.5 Diffusion coefficient

We now start to describe the diffusion coefficient of the hydrodynamic equation. From Corollary 1.5.1, there exists a unique number $D^\gamma(\rho)$ such that

$$
W_{0,1} + D^\gamma(\rho)(\eta_1 - \eta_0) \in \overline{L^\gamma \mathcal{C}_0} \quad \text{in} \quad \mathcal{H}_{\rho,\gamma}.
$$

Our purpose now is to obtain the explicit formula for D^γ. To do this, we follow the argument in [17].

Lemma 1.5.9. *For each $0 < \rho < 1$, we have*

$$
\mathcal{H}_{\rho,\gamma} = \overline{L^\gamma \mathcal{C}_0}|_{\mathcal{N}_\rho} \oplus \{W_{0,1}\} = \overline{L^{\gamma,*} \mathcal{C}_0}|_{\mathcal{N}_\rho} \oplus \{W_{0,1}^*\}
$$

where $W_{0,1}^* := W_{0,1}^S - W_{0,1}^A$.

Proof. We shall prove the first decomposition; the same arguments apply to the second one. Since we already prove in Lemma 1.5.4 that $\overline{L^\gamma \mathcal{C}_0}|_{\mathcal{N}_\rho}$ has a one-dimensional complementary subspace in $\mathcal{H}_{\rho,\gamma}$, it is sufficient to show that $\mathcal{H}_{\rho,\gamma}$ is generated by $\overline{L^\gamma \mathcal{C}_0}$ and the current. Let $h \in \mathcal{H}_{\rho,\gamma}$ so that $\ll h, W_{0,1} \gg_{\rho,\gamma} = 0$ and $\ll h, L^\gamma g \gg_{\rho,\gamma} = 0$ for all $g \in \mathcal{C}_0$. By Proposition 1.2, $h = \lim_{k\to\infty}(aW_{0,1}^S + L^{\gamma,S} h_k)$ in $\mathcal{H}_{\rho,\gamma}$ for some $a \in \mathbb{R}$ and $h_k \in \mathcal{C}_0$. Especially $\ll h \gg_{\rho,\gamma} = \lim_{k\to\infty} \ll aW_{0,1}^S + L^{\gamma,S} h_k, aW_{0,1}^S +$

$L^{\gamma,S}h_k \gg_{\rho,\gamma}= \lim_{k\to\infty} \ll aW_{0,1}^S + L^{\gamma,S}h_k, aW_{0,1} + L^\gamma h_k \gg_{\rho,\gamma}$ since $\ll aW_{0,1}^S + L^{\gamma,S}h_k, aW_{0,1}^A + L^A h_k \gg_{\rho,\gamma}= 0$ by Lemma 1.5.3. On the other hand, by assumption $\ll h, aW_{0,1} + L^\gamma h_k \gg_{\rho,\gamma}= 0$ for all k. Also, by Proposition 1.3, $\sup_k \ll aW_{0,1} + L^\gamma h_k \gg_{\rho,\gamma} \leq 2a^2 \ll W_{0,1} \gg_{\rho,\gamma} + 2(C_\gamma + 1) \sup_k \ll L^{\gamma,S}h_k \gg_{\rho,\gamma} := C_h$ is finite. Therefore, $\ll h \gg_{\rho,\gamma}= \lim_{k\to\infty} \ll aW_{0,1}^S + L^{\gamma,S}h_k, aW_{0,1} + L^\gamma h_k \gg_{\rho,\gamma}= \lim_{k\to\infty} \ll aW_{0,1}^S + L^{\gamma,S}h_k - h, aW_{0,1} + L^\gamma h_k \gg_{\rho,\gamma} \leq \limsup_{k\to\infty} \sqrt{C_h \ll aW_{0,1}^S + L^{\gamma,S}h_k - h \gg_{\rho,\gamma}} = 0$. This concludes the proof. □

Now, we can define bounded linear operators $T : \mathcal{H}_\rho \to \mathcal{H}_\rho$ and $T^* : \mathcal{H}_\rho \to \mathcal{H}_\rho$ as

$$T(aW_{0,1} + L^\gamma f) := aW_{0,1}^S + L^{\gamma,S} f, \quad T^*(aW_{0,1}^* + L^{\gamma,*} f) := aW_{0,1}^S + L^{\gamma,S}$$

since $\ll aW_{0,1} + L^\gamma f \gg_{\rho,\gamma}=\ll aW_{0,1}^* + L^{\gamma,*} f \gg_{\rho,\gamma}=\ll aW_{0,1}^S + L^{\gamma,S} f \gg_{\rho,\gamma} + \ll aW_{0,1}^A + L^A f \gg_{\rho,\gamma}$. We can easily show that T^* is the adjoint operator of T and also we have the relations

$$\ll TW_{0,1}^S, L^{\gamma,*} f \gg_{\rho,\gamma} =\ll T^* W_{0,1}^S, L^\gamma f \gg_{\rho,\gamma}= 0,$$
$$\ll TW_{0,1}^S, W_{0,1}^* \gg_{\rho,\gamma} =\ll T^* W_{0,1}^S, W_{0,1} \gg_{\rho,\gamma}= \frac{\chi(\rho)}{2}$$

for all $f \in \mathcal{H}_\rho$. Especially, $\mathcal{H}_{\rho,\gamma} = \overline{L^{\gamma,*}\mathcal{C}_0}|_{\mathcal{N}_\rho} \oplus \{TW_{0,1}^S\}$ and there exists a unique number $Q^\gamma(\rho)$ such that

$$W_{0,1}^* - Q^\gamma(\rho) TW_{0,1}^S \in \overline{L^{\gamma,*}\mathcal{C}_0} \quad \text{in} \quad \mathcal{H}_{\rho,\gamma}.$$

Proposition 1.5.

$$Q^\gamma(\rho) = \frac{\chi(\rho)}{2 \ll TW_{0,1}^S \gg_{\rho,\gamma}} = \frac{2}{\chi(\rho)} \inf_{f\in\mathcal{C}_0} \ll W_{0,1}^* - L^* f \gg_{\rho,\gamma} . \tag{1.3.7}$$

Proof. First identity follows from the fact that

$$\ll TW_{0,1}^S, W_{0,1}^* - Q^\gamma(\rho) TW_{0,1}^S \gg_{\rho,\gamma}= \frac{\chi(\rho)}{2} - Q^\gamma(\rho) \ll TW_{0,1}^S \gg_{\rho,\gamma}= 0.$$

Second identity is obtained by the expression

$$\inf_{f\in\mathcal{C}_0} \{\ll W_{0,1}^* - Q^\gamma(\rho) TW_{0,1}^S - L^{\gamma,*} f \gg_{\rho,\gamma}\} = 0$$

since

$$\inf_{f\in\mathcal{C}_0} \{\ll W_{0,1}^* - Q^\gamma(\rho) TW_{0,1}^S - L^{\gamma,*} f \gg_{\rho,\gamma}\}$$

$$= \inf_{f\in C_0}\{\ll W^*_{0,1} - L^{\gamma,*}f \gg_{\rho,\gamma}\} - Q^\gamma(\rho)\chi(\rho) + Q^\gamma(\rho)^2 \ll TW^S_{0,1} \gg_{\rho,\gamma}$$
$$= \inf_{f\in C_0}\{\ll W^*_{0,1} - L^{\gamma,*}f \gg_{\rho,\gamma}\} - Q^\gamma(\rho)\chi(\rho) + \frac{Q^\gamma(\rho)\chi(\rho)}{2}.$$

□

By a simple computation, we can show that $\ll Tg, g \gg_{\rho,\gamma} = \ll Tg, Tg \gg_{\rho,\gamma}$ for all $g \in \mathcal{H}_\rho$, and therefore $W^S_{0,1} - TW^S_{0,1} \in \overline{L^{\gamma,*}\mathcal{C}_0}$ since $W^S_{0,1} - TW^S_{0,1}$ is orthogonal to $TW^S_{0,1}$. By the fact we obtain the following variational formula for $\ll TW^S_{0,1} \gg_{\rho,\gamma}$:

Proposition 1.6.

$$\ll TW^S_{0,1} \gg_{\rho,\gamma} = \inf_{f\in C_0} \ll W^S_{0,1} - L^{\gamma,*}f \gg_{\rho,\gamma}. \tag{1.3.8}$$

Proof. By the similar argument with the proof of Proposition 1.5, we have

$$\inf_{f\in C_0}\{\ll W^S_{0,1} - TW^S_{0,1} - L^{\gamma,*}f \gg_{\rho,\gamma}\} = 0$$

and

$$\inf_{f\in C_0}\{\ll W^S_{0,1} - TW^S_{0,1} - L^{\gamma,*}f \gg_{\rho,\gamma}\} = \inf_{f\in C_0}\{\ll W^S_{0,1} - L^{\gamma,*}f \gg_{\rho,\gamma}\} - \ll TW^S_{0,1} \gg_{\rho,\gamma}$$

which concludes the proof. □

Theorem 1.6.

$$D^\gamma(\rho) = \frac{1}{\chi(\rho)} \inf_{f\in C_0} \ll W^*_{0,1} - L^*f \gg_{\rho,\gamma} = \frac{\chi(\rho)}{4\inf_{f\in C_0} \ll W^S_{0,1} - L^*f \gg_{\rho,\gamma}}. \tag{1.3.9}$$

Proof. By the definition, $W_{0,1} - 2D^\gamma(\rho)W^S_{0,1} \in \overline{L^\gamma \mathcal{C}_0}$ and therefore

$$\ll W_{0,1} - 2D^\gamma(\rho)W^S_{0,1}, T^*W^S_{0,1} \gg_{\rho,\gamma} = \frac{\chi(\rho)}{2} - 2D^\gamma(\rho) \ll TW^S_{0,1} \gg_{\rho,\gamma} = 0.$$

So, $D^\gamma(\rho) = \frac{Q^\gamma(\rho)}{2}$ and we obtain two variational formula from (1.3.7) and (1.3.8). □

Theorem 1.7. *$D^\gamma(\rho)$ is continuous in $\rho \in (0,1)$.*

Proof. Since $\ll g \gg_{\rho,\gamma}$ is continuous in ρ for all $g \in \mathcal{H}_{\rho,\gamma}$, $D^\gamma(\rho)$ is upper semi-continuous and lower semi-continuous in $\rho \in (0,1)$ by variational formula (1.3.9). □

Remark 1.2. *We can rewrite the first variational formula in (1.3.9) for $D^\gamma(\rho)$ as*

$$D^\gamma(\rho) = \frac{1}{\chi(\rho)} \inf_{f\in C_0}\{\ll W^S_{0,1} \gg_{\rho,\gamma} + \ll L^{\gamma,S}f \gg_{\rho,\gamma} + \ll W^A_{0,1} - L^A f \gg_{\rho,\gamma}\}$$

$$(1.3.10) \quad = \frac{1}{2} + \frac{1}{\chi(\rho)} \inf_{f\in C_0}\{\ll L^{\gamma,S}f \gg_{\rho,\gamma} + \ll W^A_{0,1} - L^A f \gg_{\rho,\gamma}\}$$

$$(1.3.11) \quad = \frac{1}{2} + \frac{1}{\chi(\rho)} \inf_{f\in C_0}\sup_{g\in C_0}\{\ll L^{\gamma,S}f \gg_{\rho,\gamma} - 2 \ll W^A_{0,1} - L^A f, L^{\gamma,S}g \gg_{\rho,\gamma} - \ll L^{\gamma,S}g \gg_{\rho,\gamma}\}.$$

The last expression is rewritten in the explicit form as

$$\frac{1}{2} + \frac{1}{\chi(\rho)} \inf_f \sup_g [\mathcal{D}_{0,1}(\nu_\rho;\Gamma_f) + \gamma\mathcal{D}_0(\nu_\rho;\Gamma_f) + 2\langle W^A_{0,1} - L^A f, \Gamma_g\rangle_\rho - \mathcal{D}_{0,1}(\nu_\rho;\Gamma_g) - \gamma\mathcal{D}_0(\nu_\rho;\Gamma_g)].$$

Here, we use the fact that in the variational formula (1.3.10), it is enough to take infimum in the set $\mathbb{M}_{even} := \cup_{i=0}^\infty \mathbb{M}_{2i}$ and for all $f \in \mathbb{M}_{even}$, $W^A_{0,1} - L^A f \in \mathbb{M}_{odd} := \cup_{i=0}^\infty \mathbb{M}_{2i+1}$, therefore $\ll W^A_{0,1} - L^A f, W^S_{0,1} \gg_{\rho,\gamma} = 0$.

Proposition 1.7.

$$D^\gamma(\rho) \le \frac{1}{2} + \frac{2-\rho}{4\gamma}.$$

Proof. Take $f = 0$ in the variational formula (1.3.10), then we have

$$D^\gamma(\rho) \le \frac{1}{2} + \frac{\ll W^A_{0,1} \gg_{\rho,\gamma}}{\chi(\rho)}.$$

Since we have the variational formula (1.3.6) for $\ll W^A_{0,1} \gg_{\rho,\gamma}$ and $W^A_{0,1} \in \mathbb{M}_1$,

$$\ll W^A_{0,1} \gg_{\rho,\gamma} = \sup_{f\in\mathbb{M}_1}\Big\{ -2 \ll W^A_{0,1}, L^{\gamma,S}f \gg_{\rho,\gamma} - \ll L^{\gamma,S}f \gg_{\rho,\gamma} \Big\}.$$

Especially, since $f = \tau_x f$ in $\mathcal{H}_\rho$ for any $f \in \mathbb{M}_1$ and $x \in \mathbb{Z}$, we can assume that $f = \omega_0 \tilde{f}(\eta)$ with $\tilde{f}$ which depends only on the values $\{\eta_x; |x| \ge 1\}$. Then, by the first identity of (1.3.5)

$$\ll W^A_{0,1}, L^{\gamma,S}f \gg_{\rho,\gamma} = -\sum_{x\in\mathbb{Z}} \langle W^A_{x,x+1}, f\rangle_\rho$$

$$= -\frac{1}{2}\sum_{x\in\mathbb{Z}}\langle \omega_x(1-\eta_{x+1}) + \omega_{x+1}(1-\eta_x), \omega_0\tilde{f}(\eta)\rangle_\rho = -\frac{1}{2}\langle \eta_0(2-\eta_1-\eta_{-1})\tilde{f}(\eta)\rangle_\rho.$$

Therefore,

$$
\begin{aligned}
-2 \ll W^A_{0,1}, L^{\gamma,S} f \gg_{\rho,\gamma} - \frac{\gamma}{2}\langle(\nabla_0 \Gamma_f)^2\rangle_\rho &= \langle \eta_0(2-\eta_1-\eta_{-1})\tilde{f}(\eta)\rangle_\rho - 2\gamma\langle \eta_0 \tilde{f}(\eta)^2\rangle_\rho \\
&= -2\gamma\langle \eta_0\{\tilde{f}(\eta) - \frac{1}{4\gamma}(2-\eta_1-\eta_{-1})\}^2\rangle_\rho + \frac{1}{8\gamma}\langle \eta_0(2-\eta_1-\eta_{-1})^2\rangle_\rho.
\end{aligned}
$$

So, it is shown that

(1.3.12)

$$
\ll W^A_{0,1} \gg_{\rho,\gamma} = \sup_{f \in \mathbb{M}_1} \Big\{ -2 \ll W^A_{0,1}, L^{\gamma,S} f \gg_{\rho,\gamma} - \ll L^{\gamma,S} f \gg_{\rho,\gamma} \Big\} \le \frac{2-\rho}{4\gamma}\chi(\rho).
$$

□

Proposition 1.8.

$$
D^\gamma(\rho) \ge \frac{1}{2} + \frac{1-\rho}{2\gamma}.
$$

Proof. We take $g = a\omega_0$ in the variational formula (1.3.11) for $a \in \mathbb{R}$ and obtain the inequality that

$$
\begin{aligned}
D^\gamma(\rho) \ge \frac{1}{2} + \frac{1}{\chi(\rho)} \inf_{f \in \mathcal{C}_0} \sup_{a \in \mathbb{R}} \{ &\ll L^{\gamma,S} f \gg_{\rho,\gamma} \\
&- 2a \ll W^A_{0,1} - L^A f, L^{\gamma,S}\omega_0 \gg_{\rho,\gamma} - a^2 \ll L^{\gamma,S}\omega_0 \gg_{\rho,\gamma}\}.
\end{aligned}
$$

By simple computations and the fact $\ll L^A f, L^{\gamma,S}\omega_0 \gg_{\rho,\gamma} = 0$, the last expression is equal to

$$
\begin{aligned}
&\frac{1}{2} + \frac{1}{\chi(\rho)} \inf_{f \in \mathcal{C}_0} \sup_{a \in \mathbb{R}} \{ \ll L^{\gamma,S} f \gg_{\rho,\gamma} + 2a\rho(1-\rho) - a^2 2\gamma\rho \} \\
&= \frac{1}{2} + \frac{1}{\chi(\rho)} \frac{\rho(1-\rho)^2}{2\gamma} = \frac{1}{2} + \frac{1-\rho}{2\gamma}.
\end{aligned}
$$

□

Proposition 1.9.

$$
\inf_{\mathfrak{f} \in \mathcal{C}_0} \sup_{0 \le \rho \le 1} \ll W_{0,1} + D^\gamma(\rho)(\eta_1 - \eta_0) - L^\gamma \mathfrak{f}(\omega) \gg_{\rho,\gamma} = 0.
$$

Proof. Essentially, we use three facts that $D^\gamma(\rho)$ is continuous in $\rho \in (0,1)$, $D^\gamma(\rho)$ is uniformly bounded in $\rho \in [0,1]$ and the sector condition. Note that the continuity of the diffusion coefficient at boundary is not necessary.

Take arbitrary $\varepsilon > 0$ and fix it. Then by Proposition 1.7 and its proof, we have $0 < \delta_\varepsilon < \frac{1}{2}$ such that for all $\rho \in [0, \delta_\varepsilon] \cup [1-\delta_\varepsilon, 1]$,

$$
\ll W_{0,1} + D^\gamma(\rho)(\eta_1 - \eta_0) \gg_{\rho,\gamma} = \ll (1 - 2D^\gamma(\rho))W^S_{0,1} + W^A_{0,1} \gg_{\rho,\gamma}
$$

$$= (1-2D^\gamma(\rho))^2\frac{\chi(\rho)}{2} + \ll W_{0,1}^A \gg_{\rho,\gamma} \le [\frac{1}{2\gamma^2}+\frac{1}{2\gamma}]\chi(\rho) \le \varepsilon.$$

On the other hand, by definition of $D^\gamma(\rho)$, for each ρ in $(\frac{\delta_\varepsilon}{2}, 1-\frac{\delta_\varepsilon}{2})$, we can define a function $H(\rho,\omega) \in \mathcal{C}_0$ such that

$$\ll W_{0,1} + D^\gamma(\rho)(\eta_1-\eta_0) - L^\gamma H(\rho,\omega) \gg_{\rho,\gamma} \le \varepsilon.$$

Since by Theorem 1.7 $\ll h \gg_{\rho,\gamma}$ is continuous in ρ for all h in $\mathcal{C}_0$ and $D^\gamma(\rho)$ is continuous in $\rho \in (0,1)$, for each ρ_0 in $(\frac{\delta_\varepsilon}{2}, 1-\frac{\delta_\varepsilon}{2})$, there exists a neighborhood $\mathcal{O}_{\rho_0}$ of ρ_0 such that $\ll W_{0,1} + D^\gamma(\rho_0)(\eta_1-\eta_0) - L^\gamma H(\rho_0,\omega) \gg_{\rho,\gamma} \le 2\varepsilon$ and also $|D^\gamma(\rho_0) - D^\gamma(\rho)|^2 \le \varepsilon$ for ρ in $\mathcal{O}_{\rho_0}$. To obtain an open covering of the compact set $[0,1]$, for ρ in $[0,\frac{\delta_\varepsilon}{2}]$, we define $\mathcal{O}_\rho$ by $[0,\delta_\varepsilon)$ and for ρ in $[1-\frac{\delta_\varepsilon}{2},1]$, define it by $(1-\delta_\varepsilon,1]$. Also for ρ in $[0,\frac{\delta_\varepsilon}{2}] \cup [1-\frac{\delta_\varepsilon}{2},1]$ we take $H(\rho,0)\equiv 0$. Then, by these definition, for all $\rho_0 \in [0,1]$ and $\rho \in \mathcal{O}_{\rho_0}$, we have the inequality that

$$\ll W_{0,1} + D^\gamma(\rho)(\eta_1-\eta_0) - L^\gamma H(\rho_0,\omega) \gg_{\rho,\gamma} \le 5\varepsilon.$$

Now, since there exists a finite subcovering $\{\mathcal{O}_{\rho_k}, 1 \le k \le n\}$, it is possible to define by interpolation a function $H^\varepsilon(\rho,\omega)$ so that

$$\sup_{0\le\rho\le1} \ll W_{0,1} + D^\gamma(\rho)(\eta_1-\eta_0) - L^\gamma H^\varepsilon(\rho,\omega) \gg_{\rho,\gamma} \le 5\varepsilon.$$

with the following two condition: (i) For each $\rho \in [0,1]$, $H(\rho,\cdot)$ is a mean-zero cylinder function with uniform support. (ii) For each configuration ω, $H(\cdot,\omega)$ is a smooth function of class $C^2([0,1])$. In order to remove the dependence on ρ, we define $\mathfrak{f}_l$ by $\mathfrak{f}_l(\omega) := H^\varepsilon(\eta^l(0),\omega)$. Then, for sufficiently large l, $\mathfrak{f}_l$ belongs to $\mathcal{C}_0$ and

$$\sup_{0\le\rho\le1} \ll W_{0,1} + D^\gamma(\rho)(\eta_1-\eta_0) - L^\gamma \mathfrak{f}_l \gg_{\rho,\gamma} \le \sup_{0\le\rho\le1} \ll L^\gamma[\mathfrak{f}_l - H^\varepsilon(\rho,\omega)] \gg_{\rho,\gamma} + 5\varepsilon.$$

By Proposition 1.3,

$$\sup_{0\le\rho\le1} \ll L^\gamma[\mathfrak{f}_l - H^\varepsilon(\rho,\omega)] \gg_{\rho,\gamma} \le (C_\gamma+1)\sup_{0\le\rho\le1} \ll L^{\gamma,S}[\mathfrak{f}_l - H^\varepsilon(\rho,\omega)] \gg_{\rho,\gamma}$$
$$= (C_\gamma+1)\sup_{0\le\rho\le1}[\frac{1}{2}\langle(\nabla_{0,1}\sum_{x\in\mathbb{Z}}\tau_x[H^\varepsilon(\eta^l(0),\omega) - H^\varepsilon(\rho,\omega)]^2\rangle_\rho$$
$$+ \frac{\gamma}{2}\langle(\nabla_0\sum_{x\in\mathbb{Z}}\tau_x[H^\varepsilon(\eta^l(0),\omega) - H^\varepsilon(\rho,\omega)]^2\rangle_\rho]$$

and now we can apply the method used in Lemma 2.1 in [10] directly to obtain that the last term goes to 0 as l goes to infinity. □

This result together with (1.3.3), the definition of $\tilde{V}_i^{\mathfrak{f},l}$ and Theorem 1.5 concludes the proof of Theorem 1.3.

1.3.6 Proof of Theorem 1.2

In this subsection, we obtain a detailed estimate of the diffusion coefficient at the boundary $\rho = 1$. Especially, we show that the asymptotic behavior of it as γ goes to 0 is different from that for $\rho \in [0,1)$. Moreover, by the proof of this estimate, we can conclude that this asymptotic behavior for $\rho = 1$ depends on the dimension of the space.

First, we define a subset of $\mathcal{C}$, cylinder functions, indexed by the density ρ:

$$\mathcal{A}_\rho := \{g \in \mathcal{C};\ \ g = \sum_{\Lambda \Subset \mathbb{Z}, \Lambda \not\ni 0} g_\Lambda \Pi_{x\in\Lambda}(\eta_x - \rho),\ g_\Lambda \in \mathbb{R}\}.$$

Note that all but a finite number of g_Λ are 0 for $g \in \mathcal{A}_\rho$ since g is a cylinder function. In the proof of Proposition 1.7, we obtain the inequality

$$D^\gamma(\rho) \le \frac{1}{2} + \frac{\ll W_{0,1}^A \gg_{\rho,\gamma}}{\chi(\rho)}$$

and the variational formula

$$\begin{aligned}\ll W_{0,1}^A \gg_{\rho,\gamma} &= \sup_{f\in\mathbb{M}_1} \Big\{ -2 \ll W_{0,1}^A, L^{\gamma,S} f \gg_{\rho,\gamma} - \ll L^{\gamma,S} f \gg_{\rho,\gamma} \Big\} \\ &= \sup_{g\in\mathcal{A}_\rho} \Big\{ -2 \ll W_{0,1}^A, L^{\gamma,S}(\omega_0 g) \gg_{\rho,\gamma} - \ll L^{\gamma,S}(\omega_0 g) \gg_{\rho,\gamma} \Big\}.\end{aligned}$$

By the first identity of (1.3.5),

$$-2 \ll W_{0,1}^A, L^{\gamma,S} f \gg_{\rho,\gamma} = \langle \eta_0(2-\eta_1-\eta_{-1})g(\eta)\rangle_\rho = \chi(\rho)\{2g_\phi - g_{\{1\}}\rho - g_{\{-1\}}\rho\}$$

and

$$\begin{aligned}\ll L^{\gamma,S}(\omega_0 g) \gg_{\rho,\gamma} &= \frac{\gamma}{2}\langle(\nabla_0\Gamma_{\omega_0 g})^2\rangle_\rho + \frac{1}{2}\langle(\nabla_{0,1}\Gamma_{\omega_0 g})^2\rangle_\rho \\ &= 2\gamma\rho \sum_{\Lambda\Subset\mathbb{Z},\Lambda\not\ni 0} g_\Lambda^2 \chi(\rho)^{|\Lambda|} + \frac{1}{2}\sum_{i\in\mathbb{Z}, i\neq 0,1} \langle \eta_0\eta_i\big(\tau_i g(\eta^{0,1}) - \tau_i g(\eta)\big)^2\rangle_\rho \\ &\quad + \frac{1}{2}\langle\eta_0\eta_1\big(g(\eta^{0,1}) - \tau_1 g(\eta)\big)^2\rangle_\rho + \frac{1}{2}\langle\eta_0\big(\tau_1 g(\eta^{0,1}) - g(\eta)\big)^2\rangle_\rho \\ &\ge 2\gamma\rho\{g_\phi^2 + \sum_{i\in\mathbb{Z},i\neq 0} g_{\{i\}}^2\chi(\rho)\} + \frac{1}{2}\sum_{i\in\mathbb{Z},i\neq 0,1}\langle\eta_0\eta_i\big(\tau_i g(\eta^{0,1}) - \tau_i g(\eta)\big)^2\rangle_\rho \\ &\quad + \frac{1}{2}\langle\eta_0\big(\tau_1 g(\eta^{0,1}) - g(\eta)\big)^2\rangle_\rho.\end{aligned}$$

Then, by simple computations, we obtain

$$\langle\eta_0\eta_i\big(\tau_i g(\eta^{0,1}) - \tau_i g(\eta)\big)^2\rangle_\rho$$

$$= \rho\langle \eta_0 \Big(\sum_{\Lambda \Subset \mathbb{Z}, \Lambda \not\ni 0, -i, -i+1} (g_{\Lambda \cup \{-i\}} - g_{\Lambda \cup \{-i+1\}})(\eta_1 - \eta_0) \Pi_{x \in \tau_i \Lambda} (\eta_x - \rho) \Big)^2 \rangle_\rho$$
$$= \chi(\rho)\rho \Big\{ \sum_{\Lambda \Subset \mathbb{Z}, \Lambda \not\ni 0, -i, -i+1} (g_{\Lambda \cup \{-i\}} - g_{\Lambda \cup \{-i+1\}})^2 \chi(\rho)^{|\Lambda|} \Big\}$$
$$\geq \chi(\rho)\rho (g_{\{-i\}} - g_{\{-i+1\}})^2$$

and

$$\langle \eta_0 \big(\tau_1 g(\eta^{0,1}) - g(\eta)\big)^2 \rangle_\rho = \rho \langle \Big(\sum_{\Lambda \Subset \mathbb{Z}, \Lambda \not\ni 0,1} (g_{\tau_{-1}\Lambda} - g_\Lambda) \Pi_{x \in \Lambda} (\eta_x - \rho)$$
$$+ \sum_{\Lambda \Subset \mathbb{Z}, \Lambda \not\ni 0, \Lambda \ni 1} (g_{\tau_{-1}\Lambda \setminus \{0\} \cup \{-1\}} - g_\Lambda) \Pi_{x \in \Lambda} (\eta_x - \rho) \Big)^2 \rangle_\rho$$
$$= \rho \Big\{ \sum_{\Lambda \Subset \mathbb{Z}, \Lambda \not\ni 0,1} (g_{\tau_{-1}\Lambda} - g_\Lambda)^2 \chi(\rho)^{|\Lambda|} + \sum_{\Lambda \Subset \mathbb{Z}, \Lambda \not\ni 0, \Lambda \ni 1} (g_{\tau_{-1}\Lambda \setminus \{0\} \cup \{-1\}} - g_\Lambda)^2 \chi(\rho)^{|\Lambda|} \Big\}$$
$$\geq \sum_{i \in \mathbb{Z}, i \neq -1, 0} \rho \chi(\rho) (g_{\{i\}} - g_{\{i+1\}})^2.$$

Therefore, we have

$$\frac{\ll W^A_{0,1} \gg_{\rho,\gamma}}{\chi(\rho)} = \frac{\sup_{g \in \mathcal{A}_\rho} \Big\{ -2 \ll W^A_{0,1}, L^{\gamma,S}(\omega_0 g) \gg_{\rho,\gamma} - \ll L^{\gamma,S}(\omega_0 g) \gg_{\rho,\gamma} \Big\}}{\chi(\rho)}$$
$$\leq \sup_{g \in \mathcal{A}_\rho} \Big\{ 2 g_\phi - g_{\{1\}}\rho - g_{\{-1\}}\rho - \frac{2\gamma g_\phi^2}{1-\rho} - 2\gamma\rho \sum_{i \in \mathbb{Z}, i \neq 0} g_{\{i\}}^2 - \rho \sum_{i \in \mathbb{Z}, i \neq -1, 0} (g_{\{i\}} - g_{\{i+1\}})^2 \Big\}$$
$$= \frac{1-\rho}{2\gamma} + 2\rho \sup_{\{a_i\}_{i=1}^\infty} \Big\{ -a_1 - 2\gamma \sum_{i=1}^\infty a_i^2 - \sum_{i=1}^\infty (a_i - a_{i+1})^2 \Big\}$$

where $\sup_{\{a_i\}_{i=1}^\infty}$ is taken over all finite sequences of real numbers. Now, by the next lemma we obtain the desired inequality.

Lemma 1.7.1.

$$\sup_{\{a_i\}_{i=1}^\infty} \Big\{ -a_1 - 2\gamma \sum_{i=1}^\infty a_i^2 - \sum_{i=1}^\infty (a_i - a_{i+1})^2 \Big\} = \frac{1}{4\gamma + 4\sqrt{\gamma^2 + 2\gamma}}$$

where $\sup_{\{a_i\}_{i=1}^\infty}$ *is taken over all finite sequences of real numbers.*

Proof. Define an $n \times n$ symmetric matrix A^n by setting $A^n_{1,1} = 2\gamma + 1$, $A^n_{i,i} = 2\gamma + 2$ for $2 \leq i \leq n$, $A^n_{i,i+1} = A^n_{i+1,i} = -1$ for $1 \leq i \leq n-1$ and other components are

0. Let P^n be an orthogonal matrix satisfying that $D^n := (P^n)^{-1}A^nP^n$ is diagonal. Then,

$$\sup_{\{a_i\}_{i=1}^{\infty}} \Big\{ -a_1 - 2\gamma\sum_{i=1}^{\infty} a_i^2 - \sum_{i=1}^{\infty}(a_i - a_{i+1})^2 \Big\} = \sup_n \sup_{\mathbf{a}\in\mathbb{R}^n} \Big\{ -^t\mathbf{e}_1\mathbf{a} -^t\mathbf{a}A^n\mathbf{a} \Big\}$$

where $\mathbf{e}_1 =^t (1,0,\dots,0) \in \mathbb{R}^n$. Now, by the change of variable $\mathbf{a} = P^n\mathbf{b}$ and the simple argument of the linear algebra,

$$\begin{aligned}
&\sup_{\mathbf{a}\in\mathbb{R}^n} \{ -^t\mathbf{e}_1\mathbf{a} -^t\mathbf{a}A^n\mathbf{a}\} = \sup_{\mathbf{b}\in\mathbb{R}^n} \{ -^t\mathbf{e}_1P^n\mathbf{b} -^t\mathbf{b}D^n\mathbf{b}\} \\
&= \sup_{\mathbf{b}\in\mathbb{R}^n} \{ -^t(\mathbf{b} + \frac{1}{2}(P^nD^n)^{-1}\mathbf{e}_1)D^n(\mathbf{b} + \frac{1}{2}(P^nD^n)^{-1}\mathbf{e}_1)\} + \frac{1}{4}{}^t\mathbf{e}_1P^n(D^n)^{-1}(P^n)^{-1}\mathbf{e}_1 \\
&= \frac{1}{4}{}^t\mathbf{e}_1(A^n)^{-1}\mathbf{e}_1
\end{aligned}$$

since A^n is positive definite. Define an $n \times n$ symmetric matrix B^n by setting $B^n_{i,i} = 2\gamma + 2$ for $1 \le i \le n$, $B^n_{i,i+1} = B^n_{i+1,i} = -1$ for $1 \le i \le n-1$ and other components are 0, then we have

$${}^t\mathbf{e}_1(A^n)^{-1}\mathbf{e}_1 = \frac{\det B^{n-1}}{(2\gamma+1)\det B^{n-1} - \det B^{n-2}} = \frac{1}{2\gamma + 1 - b_{n-1}}$$

where $b_n = \frac{\det B^{n-1}}{\det B^n}$. Since ${}^t\mathbf{e}_1(A^n)^{-1}\mathbf{e}_1$ is positive and increasing by definition, $\{b_n\}_{n=1}^{\infty}$ is a bounded increasing sequence of positive numbers and therefore $b_\infty = \lim_{n\to\infty} b_n$ exists. By the definition of B^n, we have an equation $\det B^n = (2\gamma + 2)\det B^{n-1} - \det B^{n-2}$. By taking a limit, we obtain that $\frac{1}{b_\infty} = 2\gamma + 2 - b_\infty$ and since $b_\infty \le 2\gamma + 1$, we have $b_\infty = \gamma + 1 - \sqrt{\gamma^2 + 2\gamma}$ which concludes the proof. □

1.4 Spectral gap

In this section, we prove the spectral gap for the exclusion process with velocity on finite one-dimensional cubes, which is used in the proof of Theorem 1.9 in the next section. We use the notation Λ_l and $L^{\gamma,S}_{\Lambda_l}$ defined in Subsection 1.3.2. Also as in the previous sections, we denote $\nu_{\Lambda_l,K}$ by $\nu_{l,K}$, and expected values with respect to the measure $\nu_{l,K}$ by $\langle\cdot\rangle_{l,K}$.

The main purpose of this section is to prove that the generator $L^{\gamma,S}_{\Lambda_l}$ in $L^2(\nu_{l,K})$ has a spectral gap of order at least l^{-2}.

Theorem 1.8. *There exists a positive constant $C = C(\gamma)$ such that for every positive integer l, every integer $0 \le K \le |\Lambda_l|$ and every function f in $L^2(\nu_{l,K})$ satisfying $\langle f \rangle_{l,K} = 0$,*

$$\langle f^2 \rangle_{l,K} \le Cl^2 \langle -L_{\Lambda_l}^{\gamma,S} f, f \rangle_{l,K}.$$

Proof. As in Subsection 1.3.4, we can decompose f as $f = \sum_{i=0}^{m} f_i$ with some positive integer m and f_i in $\mathbb{M}_i$. Then, it is obvious that

$$\langle f^2 \rangle_{l,K} = \sum_{i=0}^{m} \langle f_i^2 \rangle_{l,K} \quad \text{and} \quad \langle -L_{\Lambda_l}^{\gamma,S} f, f \rangle_{l,K} = \sum_{i=0}^{m} \langle -L_{\Lambda_l}^{\gamma,S} f_i, f_i \rangle_{l,K}.$$

By simple computation, for $i \ge 1$,

$$\langle f_i^2 \rangle_{l,K} \le 2i \langle f_i, f_i \rangle_{l,K} = \langle -\sum_{x=-l}^{l} L_x^v f_i, f_i \rangle_{l,K} \le \frac{1}{\gamma} \langle -L_{\Lambda_l}^{\gamma,S} f_i, f_i \rangle_{l,K}.$$

On the other hand, f_0 is a function depends only on a configuration η instead of ω. Therefore we can apply the result for simple symmetric exclusion process (cf. [10]) to obtain a positive constant C such that for every positive integer l, every integer $0 \le K \le |\Lambda_l|$ and every function f_0 in $L^2(\nu_{l,K})$ and $\mathbb{M}_0$ satisfying $\langle f_0 \rangle_{l,K} = 0$,

$$\langle f_0^2 \rangle_{l,K} \le Cl^2 \sum_{x=-l}^{l-1} \langle 1_{\{\eta_x = 1, \eta_{x+1} = 0\}} (f_0(\eta^{x,x+1}) - f_0(\eta))^2 \rangle_{l,K}.$$

Then, with this constant C, it is easily shown that

$$\begin{aligned} \langle f_0^2 \rangle_{l,K} &\le Cl^2 \sum_{x=-l}^{l-1} \langle 1_{\{\eta_x = 1\}} (f_0(\omega^{x,x+1}) - f_0(\omega))^2 \rangle_{l,K} \\ &= 2Cl^2 \sum_{x=-l}^{l-1} \langle -L_{x,x+1}^{ex} f_0, f_0 \rangle_{l,K} \le 2Cl^2 \langle -L_{\Lambda_l}^{\gamma,S} f_0, f_0 \rangle_{l,K}. \end{aligned}$$

□

1.5 Closed forms

In this section, to complete the proof of Proposition 1.1, we introduce the notion of closed forms.

Consider the configuration space $\chi = \{-1, 0, 1\}^{\mathbb{Z}}$. For two configurations ω and $\xi \in \chi$, define a function $D(\omega, \xi)$ as follows. $D(\omega, \xi) = 1$ if one of the following two conditions is satisfied: (i) There exists a unique point $x \in \mathbb{Z}$ such that $\omega^{x,x+1} = \xi$ and $(\omega_x, \omega_{x+1}) = (1,0)$ or $(0,1)$, (ii) There exists a unique point $x \in \mathbb{Z}$ such that $\omega = \xi^x$ and $\omega \neq \xi$, and $D(\eta, \xi) = 0$ otherwise.

Let $\mathcal{H}_{x,x+1}$ and $\mathcal{H}_x$ be subsets of $\chi = \{-1,0,1\}^{\mathbb{Z}}$ such that $\mathcal{H}_{x,x+1} := \{\omega\,; (\omega_x, \omega_{x+1}) = (1,0)\}$ and $\mathcal{H}_x := \{\omega\,; \omega_x = 1\}$. Consider a family of $\mathbb{R}^2$-valued continuous functions $\mathrm{u} = (\mathrm{u}^1, \mathrm{u}^2) = (\mathrm{u}^1_x, \mathrm{u}^2_x)_{x\in\mathbb{Z}}$ where $\mathrm{u}^1_x : \mathcal{H}_{x,x+1} \to \mathbb{R}$ and $\mathrm{u}^2_x : \mathcal{H}_x \to \mathbb{R}$. For an ordered pair (ω, ξ) satisfying $D(\omega, \xi) = 1$, define a one-step integral $I_{(\omega,\xi)}$ of u by

$$I_{(\omega,\xi)}(\mathrm{u}) = \begin{cases} \mathrm{u}^1_x(\omega) & \text{if } \omega^{x,x+1} = \xi \text{ and } \omega \in \mathcal{H}_{x,x+1}, \\ -\mathrm{u}^1_x(\omega^{x,x+1}) & \text{if } \omega^{x,x+1} = \xi \text{ and } \xi \in \mathcal{H}_{x,x+1}, \\ \mathrm{u}^2_x(\omega) & \text{if } \omega^x = \xi \text{ and } \omega \in \mathcal{H}_x, \\ -\mathrm{u}^2_x(\omega^x) & \text{if } \omega^x = \xi \text{ and } \xi \in \mathcal{H}_x. \end{cases}$$

By the definition of $D(\omega, \xi)$, the one-step path integral $I_{(\omega,\xi)}$ of u is well-defined. Next, we consider more general paths. A path $\Gamma(\omega, \xi) = (\omega = \omega^0, \omega^1, ..., \omega^{m-1}, \omega^m = \xi)$ from ω to ξ is defined as a sequence of configurations ω^j such that every two successive configurations satisfies $D(\omega^j, \omega^{j+1}) = 1$. A path integral can be naturally extended to paths of any length as $I_{\Gamma(\omega,\xi)}(\mathrm{u}) := \sum_{j=0}^{m-1} I_{(\omega^j,\omega^{j+1})}(\mathrm{u})$.

Now, we introduce a notion of closed forms.

Definition 1.2. *A family of $\mathbb{R}^2$-valued continuous functions $\mathrm{u} = (\mathrm{u}^1_x, \mathrm{u}^2_x)_{x\in\mathbb{Z}}$ with $\mathrm{u}^1_x : \mathcal{H}_{x,x+1} \to \mathbb{R}$ and $\mathrm{u}^2_x : \mathcal{H}_x \to \mathbb{R}$ is called an closed form if it satisfies all of the following conditions:*
(i) $\mathrm{u}^1_x(\omega) + \mathrm{u}^1_y(\omega^{x,x+1}) = \mathrm{u}^1_y(\omega) + \mathrm{u}^1_x(\omega^{y,y+1})$ for all $|x-y| \geq 2 \in \mathbb{Z}$ and $\omega \in \mathcal{H}_{x,x+1} \cap \mathcal{H}_{y,y+1}$,
(ii) $\mathrm{u}^2_x(\omega) + \mathrm{u}^2_y(\omega^x) = \mathrm{u}^2_y(\omega) + \mathrm{u}^2_x(\omega^y)$ for all $x, y \in \mathbb{Z}$ and $\omega \in \mathcal{H}_x \cap \mathcal{H}_y$,
(iii) $\mathrm{u}^1_x(\omega) + \mathrm{u}^2_y(\omega^{x,x+1}) = \mathrm{u}^2_y(\omega) + \mathrm{u}^1_x(\omega^y)$ for all $x, x+1 \neq y \in \mathbb{Z}$ and $\omega \in \mathcal{H}_{x,x+1} \cap \mathcal{H}_y$.

Proposition 1.10. *If a family of $\mathbb{R}^2$-valued continuous functions $\mathrm{u} = (\mathrm{u}^1_x, \mathrm{u}^2_x)_{x\in\mathbb{Z}}$ is an closed form, then for all closed path $\Gamma(\omega, \xi)$, $I_{\Gamma(\omega,\xi)}(\mathrm{u}) = 0$ where a path $\Gamma(\omega, \xi)$ is called closed if $\omega = \xi$.*

Proof. Consider a closed path $\Gamma(\omega, \omega) = (\omega = \omega^0, \omega^1, ..., \omega^{m-1}, \omega^m = \omega)$. We prove that the path integral along this path vanishes. The strategy consists in constructing a new path with length $m-2$ and same path integral.

First, we assume that there exists $0 \leq i \leq m-1$ such that $(\omega^i)^x = \omega^{i+1}$ for some $x \in \mathbb{Z}$. We take i_0 as the first time when this happen: $i_0 := \min\{i \geq 0; \exists x \in$

$\mathbb{Z}$ s.t. $(\omega^i)^x = \omega^{i+1}\}$ and denote by x_0 the unique site which satisfies $(\omega^{i_0})^{x_0} = \omega^{i_0+1}$. Obviously, $\omega_{x_0}^{i_0} = 1$ or -1. We consider these two cases separately.

Assume that $\omega_{x_0}^{i_0} = 1$ namely $\omega_{x_0}^{i_0+1} = -1$. Then, by the definition of i_0, $\omega_{x_0}^0 \neq -1$ and since Γ is a closed path, later the velocity of the particle at x_0 must be changed again. Let i_1 be the first time when this happen: $i_1 := \min\{i \geq i_0 + 1; (\omega^i)^{x_0} = \omega^{i+1}\}$. If $i_1 = i_0 + 1$, a new closed path of length $m - 2$ can be constructed as $(\omega = \omega^0, \omega^1, ..., \omega^{i_0-1}, \omega^{i_0} = \omega^{i_0+2}, \omega^{i_0+3}, ..., \omega^{m-1}, \omega^m = \omega)$. If $i_1 \geq i_0 + 2$, then since $\omega_{x_0}^j = -1$ for all $i_0 + 1 \leq j \leq i_1$, especially for $j = i_1 - 1$ and i_1, there exists a unique site $y \neq x_0, x_0 - 1$ such that $\omega^{i_1} = (\omega^{i_1-1})^y$ or $(\omega^{i_1-1})^{y,y+1}$, which immediately concludes that $\omega^{i_1+1} = \left((\omega^{i_1-1})^y\right)^{x_0} = \left((\omega^{i_1-1})^{x_0}\right)^y$ or $\omega^{i_1+1} = \left((\omega^{i_1-1})^{y,y+1}\right)^{x_0} = \left((\omega^{i_1-1})^{x_0}\right)^{y,y+1}$. By the definition of a closed form, the path $(\omega = \omega^0, \omega^1, ..., \omega^{i_1-1}, (\omega^{i_1-1})^{x_0}, \omega^{i_1+1}, ..., \omega^{m-1}, \omega^m = \omega)$ has a same length and path integral with the original path $\Gamma(\omega, \omega)$. Repeating this argument $i_1 - i_0 - 1$ times we obtain that the path integral along $\tilde{\Gamma}(\omega, \omega) = (\omega = \omega^0, \omega^1, ..., \omega^{i_0}, \omega^{i_0+1}, (\omega^{i_0+1})^{x_0}, ..., (\omega^{i_1-2})^{x_0}, (\omega^{i_1-1})^{x_0}, \omega^{i_1+1}, ..., \omega^{m-1}, \omega^m = \omega)$ also has a same length and path integral with the original one. By the definition, $\omega^{i_0+1} = (\omega^{i_0})^{x_0}$, a new closed path with length $m - 2$ can be constructed as $(\omega = \omega^0, \omega^1, ..., \omega^{i_0} = (\omega^{i_0+1})^{x_0}, (\omega^{i_0+2})^{x_0} ..., (\omega^{i_1-2})^{x_0}, (\omega^{i_1-1})^{x_0}, \omega^{i_1+1}, ..., \omega^{m-1}, \omega^m = \omega)$.

Now, we turn to the case with $\omega_{x_0}^{i_0} = -1$. Then, by the definition of i_0, $\omega_{x_0}^0 = -1$, and again since Γ is a closed path, later the velocity of the particle at x_0 must be changed. This time, let i_1 be the last time when this happen: $i_1 := \max\{i \leq m - 1; (\omega^i)^{x_0} = \omega^{i+1}\}$. If $i_0 = 0$ and $i_1 = m - 1$, a new closed path of length $m - 2$ can be constructed as $(\omega^1 = \omega^{x_0}, \omega^2, ..., \omega^{m-2}, \omega^{m-1} = \omega^{x_0})$ and it is easily shown that the path integral of this new path is same as that of the original one. If $i_0 \geq 1$, since $\omega_{x_0}^j = -1$ for $0 \leq j \leq i_0$, especially for $i_0 - 1$ and i_0, we can construct a new path with same length and path integral with original one as $(\omega = \omega^0, (\omega^0)^{x_0}, (\omega^1)^{x_0} ..., (\omega^{i_0-1})^{x_0}, \omega^{i_0+1}, ..., \omega^{m-1}, \omega^m = \omega)$ by the same way for the last case. Also, if $i_1 \leq m - 2$ then $\omega_{x_0}^j = -1$ for $i_1 + 1 \leq j \leq m$ and therefore $(\omega = \omega^0, \omega^1, ..., \omega^{i_1}, (\omega^{i_1+2})^{x_0}, \omega^{i_1+2}, ..., \omega^{m-1}, \omega^m = \omega)$ has same length and path integral with original one. Repeating this argument we obtain a new path with $i_0 = 0$ and $i_1 = m - 1$ from which we can construct a path of length $m - 2$ easily.

Finally, we consider the case in which for all $0 \leq i \leq m - 1$, $\omega^{i+1} = (\omega^i)^{x,x+1}$ for some $x \in \mathbb{Z}$. For this case, we just need to consider the usual jumps of particles. Without loss of generality, we can assume that $(\omega_{x_0}^0, \omega_{x_0+1}^0) = (1, 0)$ and

$\omega^1 = (\omega^0)^{x_0,x_0+1}$. By the special structure of one-dimensional space $\mathbb{Z}$ and the assumption that Γ is a closed path, there exists a unique time $1 \le i_0 \le m-1$ such that $i_0 = \min\{i \ge 1, \omega^{i+1} = (\omega^i)^{x_0,x_0+1}, (\omega^i_{x_0}, \omega^i_{x_0+1}) = (0,1)\}$. Let i_1 be the last time before i_0 that a particle jumps from x_0 to x_0+1: $i_1 = \max\{i \le i_0, \omega^{i+1} = (\omega^i)^{x_0,x_0+1}, (\omega^i_{x_0}, \omega^i_{x_0+1}) = (1,0)\}$. Here, we shall assume without loss of generality that between i_1 and i_0, there are no jumps from some site x_1 to x_1+1 (resp. x_1-1) and then a jump from site x_1+1 (resp.x_1-1) to x. Otherwise, we can repeat the same argument to x_1 in place of x_0. By this assumption, we can show that for any $i_1 < j < i_0$, $\omega^{j+1} = (\omega^j)^{y,y+1}$ for $y \ne x_0-1$, x_0 and x_0+1. Therefore, since u is a closed form, we can invert the order of jumps and construct a new path ($\omega = \omega^0, \omega^1, ..., \omega^{i_1}, \omega^{i_1+1}, (\omega^{i_1+1})^{x_0,x_0+1} ..., (\omega^{i_0-2})^{x_0,x_0+1}, (\omega^{i_0-1})^{x_0,x_0+1}, \omega^{i_0+1}, ..., \omega^{m-1}, \omega^m$ $= \omega$) with same path integral with original one. Then, since $\omega^{i_1+1} = (\omega^{i_1})^{x_0,x_0+1}$, we can construct a new closed path of length $m-2$ as ($\omega = \omega^0, \omega^1, ..., \omega^{i_1} = (\omega^{i_1+1})^{x_0,x_0+1}, ..., (\omega^{i_0-2})^{x_0,x_0+1}, (\omega^{i_0-1})^{x_0,x_0+1}$, ω^{i_0+1}, ... $,\omega^{m-1}$, $\omega^m = \omega$) which concludes the proof. □

Next corollary follows form this result:

Corollary 1.8.1. *Let Λ be a connected finite subset of $\mathbb{Z}$. All closed forms on $\{-1,0,1\}^\Lambda$ are exact forms. Precisely, if u is a closed form, then there exists a function $F : \{-1,0,1\}^\Lambda \to \mathbb{R}$ such that $u^1_x(\omega) = F(\omega^{x,x+1}) - F(\omega)$ for all $\omega \in \mathcal{H}_{x,x+1}$ and $u^2_x(\omega) = F(\omega^x) - F(\omega)$ for all $\omega \in \mathcal{H}_x$.*

Now, we extend the definition of closed forms. Let $\mathcal{G}_x$ be a subset of $\chi = \{-1,0,1\}^{\mathbb{Z}}$ such that $\mathcal{G}_x := \{\omega\ ; \omega_x = 1 \text{ or } -1\} = \{\omega\ ; \eta_x = 1\}$.

Definition 1.3. *A family of $\mathbb{R}^2$-valued continuous functions $u = (u^1_x, u^2_x)_{x\in\mathbb{Z}}$ with $u^i_x : \mathcal{G}_x \to \mathbb{R}$ for $i = 1,2$ is called an closed form if it satisfies all of the following conditions:*

(i) The restriction of $u = (u^1_x|_{\mathcal{H}_{x,x+1}}, u^2_x|_{\mathcal{H}_x})$ is a closed form in the sense of Definition 1.2,

(ii) $u^2_x(\omega) = -u^2_x(\omega^x)$ for all $x \in \mathbb{Z}$ and $\omega_x = -1$,

(iii) $u^1_x(\omega) = -u^2_x(\omega^x) + u^1_x(\omega^x) + u^2_{x+1}((\omega^x)^{x,x+1})$ for all $x \in \mathbb{Z}$ and $(\omega_x, \omega_{x+1}) = (-1,0)$,

(iv) $u^1_x(\omega) = u^2_x(\omega^{x+1}) - u^2_{x+1}(\omega^{x+1})$ for all $x \in \mathbb{Z}$ and $(\omega_x, \omega_{x+1}) = (1,-1)$,

(v) $u^1_x(\omega) = -u^1_x(\omega^{x,x+1})$ for all $x \in \mathbb{Z}$ and $\omega \in \mathcal{G}_x \cap \mathcal{G}_{x+1}$.

By the definition, the set of closed forms of $\mathfrak{u} = (\mathfrak{u}_x^1, \mathfrak{u}_x^2)_{x\in\mathbb{Z}}$ with $\mathfrak{u}_x^i : \mathcal{G}_x \to \mathbb{R}$ and the set of their regulations $\mathfrak{u} = (\mathfrak{u}_x^1|_{\mathcal{H}_{x,x+1}}, \mathfrak{u}_x^2|_{\mathcal{H}_x})$ is one-to-one. Especially, the following holds:

Proposition 1.11. *Let Λ be a connected finite subset of $\mathbb{Z}$. All closed forms on $\{-1,0,1\}^\Lambda$ are exact forms. Precisely, if $\mathfrak{u}$ is a closed form, then there exists a function $F : \{-1,0,1\}^\Lambda \to \mathbb{R}$ such that $\mathfrak{u}_x^1(\omega) = F(\omega^{x,x+1}) - F(\omega)$ and $\mathfrak{u}_x^2(\omega) = F(\omega^x) - F(\omega)$ for all $\omega \in \mathcal{G}_x$.*

Proof. By Corollary 1.8.1 and the definition of closed forms, it is easy to show. □

Let us introduce the notion of a germ of closed form. A pair of continuous functions $\mathfrak{g} = (\mathfrak{g}^1, \mathfrak{g}^2)$ where $\mathfrak{g}^i : \mathcal{G}_0 \to \mathbb{R}$ is a germ of closed form if $\mathfrak{u} = (\mathfrak{u}_x)_{x\in\mathbb{Z}} := (\tau_x \mathfrak{g})_{x\in\mathbb{Z}}$ is an closed form. For a pair of $L^2(\nu_\rho)$-functions $\mathfrak{g} = (\mathfrak{g}^1, \mathfrak{g}^2)$ where $\mathfrak{g}^i : \mathcal{G}_0 \to \mathbb{R}$, we call it a germ of closed form if $\mathfrak{u} = (\mathfrak{u}_x)_{x\in\mathbb{Z}} := (\tau_x \mathfrak{g})_{x\in\mathbb{Z}}$ satisfies all of conditions as a closed form in $L^2(\nu_\rho)$ sense. Main theorem of this section is formulated as follows:

Theorem 1.9. *For every germ of closed form $\mathfrak{g} = (\mathfrak{g}^1, \mathfrak{g}^2)$ with $\mathfrak{g}^i \in L^2(\nu_\rho)$ for $i = 1, 2$, there exists a sequence of cylinder functions h_n and a constant c such that*

$$\mathfrak{g}^1 = \lim_{n\to\infty} (c\,\eta_0(1-\eta_1) + \nabla_{0,1}\Gamma_{h_n}) \quad in \quad L^2(\nu_\rho) \quad and \quad \mathfrak{g}^2 = \lim_{n\to\infty} (\nabla_0 \Gamma_{h_n}) \quad in \quad L^2(\nu_\rho).$$

The strategy of the proof is essentially the same as given for the generalized exclusion process in the proof of Theorem A.3.4.14 in [14]. We first project the closed form $\{\mathfrak{g}_x, x \in \mathbb{Z}\}$ on finite subsets and apply Corollary 1.8.1 to obtain the cylinder function h_n. Then, we divide the vector $((2n)^{-1}\nabla_{0,1}\Gamma_{h_n}, (2n)^{-1}\nabla_0\Gamma_{h_n})$ into the boundary term and the non boundary term, and show that the latter term converges strongly in $L^2(\nu_\rho)$ to the germ $\mathfrak{g}$. Finally, we prove that the boundary term is a weakly relatively compact sequence and all limit points of the sequence belong to the linear space generated by $(\eta_0(1-\eta_1), 0)$.

For each positive integer n, denote by $\mathcal{F}_n$ the σ-algebra generated by $\{\omega(x), |x| \leq n\}$ and let $\mathfrak{g}_x^{i,n} = E_\rho[\mathfrak{g}_x^i | \mathcal{F}_n]$. It is easy to check that $\{\mathfrak{g}_x^{i,n}\}$ is a closed form on $\{-1,0,1\}^{\Lambda_n}$. By Corollary 1.8.1, there exists a $\mathcal{F}_n$-measurable function $\tilde{\psi}_n$ such that $\nabla_{x,x+1}\tilde{\psi}_n = \mathfrak{g}_n^1$ for all $-n \leq x \leq n-1$, $\nabla_x \tilde{\psi}_n = \mathfrak{g}_n^2$ for all $-n \leq x \leq n$ and $E[\tilde{\psi}_n | \sum_{|x| len} \eta_x = K] = 0$ for all $0 \leq K \leq 2n+1$. Then for each n, let

$\psi_n := E[\tilde{\psi}_{3n}|\mathcal{F}_n]$, $R_n^1 = (2n)^{-1}\nabla_{0,1}\Gamma_{\psi_n}$ and $R_n^2 = (2n)^{-1}\nabla_0\Gamma_{\psi_n}$. $R_n = (R_n^1, R_n^2)$ can be decomposed as

$$(2n)^{-1}\nabla \sum_{x=-n+1}^{n} \tau_x\psi_n + (2n)^{-1}\nabla\tau_{-n}\psi_n + (2n)^{-1}\nabla\tau_{n+1}\psi_n$$

where $\nabla = (\nabla^1, \nabla^2) := (\nabla_{0,1}, \nabla_0)$.

Lemma 1.9.1.

$$(2n)^{-1}\nabla^i \sum_{x=-n+1}^{n} \tau_x\psi_n = (2n)^{-1} \sum_{x=-n+1}^{n} \tau_x\mathfrak{g}_{-x}^{i,n}$$

converges to $\mathfrak{g}^i$ *in* $L^2(\nu_\rho)$ *for* $i = 1, 2$.

Proof. By the martingale convergence theorem, it is easy to show. □

Next, we prove that the second component of the boundary term $(2n)^{-1}\nabla\tau_{-n}\psi_n$ (resp. $(2n)^{-1}\nabla\tau_{n+1}\psi_n$) converges to 0.

Lemma 1.9.2.

$$(2n)^{-1}\nabla^2\tau_{-n}\psi_n = (2n)^{-1}\tau_{-n}\mathfrak{g}_n^{2,n}$$

converges to 0 *in* $L^2(\nu_\rho)$ *and*

$$(2n)^{-1}\nabla^2\tau_{n+1}\psi_n = 0$$

for all n.

Proof.

$$E_\rho[(\tau_{-n}\mathfrak{g}_n^{2,n})^2] = E_\rho[(E[\tau_n\mathfrak{g}^2|\mathcal{F}_n])^2] \leq E_\rho[(\mathfrak{g}^2)^2]$$

concludes the proof. □

Next, we consider the first component of the boundary terms. Let $\phi_n := E_\rho[\psi_n|\mathcal{G}_n]$ where $\mathcal{G}_n$ is a σ-algebra generated by $\{\eta_x, |x| \leq n\}$. We first prove that the difference between the term $(2n)^{-1}\nabla^1\tau_{-n}\psi_n$ (resp. $(2n)^{-1}\nabla^1\tau_{n+1}\psi_n$) and $(2n)^{-1}\nabla^1\tau_{-n}\phi_n$ (resp. $(2n)^{-1}\nabla^1\tau_{n+1}\phi_n$) converges to 0 in $L^2(\nu_\rho)$.

Lemma 1.9.3.

$$(2n)^{-1}\nabla^1\tau_{-n}(\psi_n - \phi_n) \quad \textit{and} \quad (2n)^{-1}\nabla^1\tau_{n+1}(\psi_n - \phi_n)$$

converges to 0 *in* $L^2(\nu_\rho)$.

Proof. Since $\nabla^1 \tau_{-n}(\psi_n - \phi_n) = \tau_{-n}\nabla_{n,n+1}(\psi_n - \phi_n)$ and $\nabla^1\tau_{n+1}(\psi_n - \phi_n) = \tau_{n+1}\nabla_{-n-1,-n}(\psi_n - \phi_n)$, it is enough to show that $n^{-2}E_\rho[(\psi_n - \phi_n)^2]$ converges to 0.

Let F be a $\mathcal{F}_n$-measurable function and $F_k := E_\rho[F|\mathcal{F}_{k,n}]$ where $\mathcal{F}_{k,n}$ is a σ-algebra generated by $\{\omega_x, -n \le x \le k\}$ and $\{\eta_x, k+1 \le x \le n\}$ for $-n-1 \le k \le n$. Note that $\mathcal{F}_n = \mathcal{F}_{n,n} \supset \mathcal{F}_{n-1,n} \supset \dots \supset \mathcal{F}_{-n,n} \supset \mathcal{F}_{-n-1,n} = \mathcal{G}_n$. Then, we obtain the equality that

$$E_\rho[(F - E[F|\mathcal{G}_n])^2] = E_\rho[\{\sum_{k=-n}^{n}(F_k - F_{k-1})\}^2] = \sum_{k=-n}^{n} E_\rho[(F_k - F_{k-1})^2].$$

On the other hand, by simple computations, we can show that

$$E_\rho[(F_k - F_{k-1})^2] = E_\rho[(\frac{F_k(\omega) - F_k(\omega^k)}{2})^2] = \frac{1}{4}E_\rho[(\nabla_k F_k)^2] \le \frac{1}{4}E_\rho[(\nabla_k F)^2].$$

Therefore, we obtain the general inequality

$$E_\rho[(F - E[F|\mathcal{G}_n])^2] \le \frac{1}{4}\sum_{k=-n}^{n} E_\rho[(\nabla_k F)^2].$$

Now, we apply this to ψ_n and obtain the inequality

$$E_\rho[(\psi_n - \phi_n)^2] \le \frac{1}{4}\sum_{k=-n}^{n} E_\rho[(\nabla_k\psi_n)^2] \le \frac{1}{4}\sum_{k=-n}^{n} E_\rho[(\tau_k \mathfrak{g}^2)^2] = \frac{2n+1}{4}E_\rho[(\mathfrak{g}^2)^2]$$

with the relation that $\nabla_k\psi_n = \mathfrak{g}_k^{2,n}$, which concludes the proof. □

Let $\tilde{\phi}_n := E[\tilde{\psi}_n|\mathcal{G}_n]$. Then we can apply the method of the proof of Lemma A.4.15 in [14] to show that there exist finite constants $C_1(\rho)$ and $C_2(\rho)$ such that

$$E_\rho[(\nabla_{n,n+1}\phi_n)^2] \le C_1(\rho)n^{-1}E_\rho[\tilde{\phi}_{3n}^2] + C_2(\rho)n\sum_{x=-3n}^{3n-1}\mathcal{D}_{x,x+1}(\nu_\rho; \tilde{\phi}_{3n})$$

and

$$E_\rho[(\nabla_{-n-1,-n}\phi_n)^2] \le C_1(\rho)n^{-1}E_\rho[\tilde{\phi}_{3n}^2] + C_2(\rho)n\sum_{x=-3n}^{3n-1}\mathcal{D}_{x,x+1}(\nu_\rho; \tilde{\phi}_{3n}).$$

Especially, with the spectral gap estimates proved in the last section, we obtain that

$$\sup_n E_\rho[\{(2n)^{-1}\nabla_{n,n+1}\phi_n\}^2] < \infty \quad \text{and} \quad \sup_n E_\rho[\{(2n)^{-1}\nabla_{-n-1,-n}\phi_n\}^2] < \infty$$

which shows that the sequences $(2n)^{-1}\nabla_{0,1}\tau_{-n}\phi_n$ and $(2n)^{-1}\nabla_{0,1}\tau_{n+1}\phi_n$ are weakly relatively compact. To prove that all limit points belong to the linear space generated by $\eta_0(1-\eta_1)$, consider a weakly convergent subsequence and denote by $\mathfrak{b}_+$ (resp. $\mathfrak{b}_-$) the weak limit of the positive (resp. negative) boundary term.

Lemma 1.9.4. $\mathfrak{b}_\pm$ *depend on* ω *only through* η_0 *and* η_1.

Proof. We prove this statement fo $\mathfrak{b}_-$; same arguments apply to $\mathfrak{b}_+$. By the definition, $(2n)^{-1}\nabla_{0,1}\tau_{-n}\phi_n$ is measurable with respect to the σ-algebra generated by $\{\eta_x, x \leq 1\}$ and the weak limit $\mathfrak{b}_-$ inherits this property. Now, we prove that $\nabla_{x,x+1}\mathfrak{b}_- = 0$ for all $x \leq -2$. Since $\nabla_{x,x+1}$ is continuous with respect to the weak topology, it is enough to show that the sequence $(2n)^{-1}\nabla_{x,x+1}\nabla_{0,1}\tau_{-n}\phi_n$ converges to 0 in $L^2(\nu_\rho)$. Actually, for any $x \leq -2$ and any $2n \geq |x|$,

$$\begin{aligned}\nabla_{x,x+1}\nabla_{0,1}\tau_{-n}\phi_n &= \nabla_{0,1}\nabla_{x,x+1}\tau_{-n}\phi_n = \nabla_{0,1}\tau_{-n}\nabla_{x+n,x+n+1}E[\tilde{\psi}_{3n}|\mathcal{G}_n] \\ &= \nabla_{0,1}\tau_{-n}E[\nabla_{x+n,x+n+1}\tilde{\psi}_{3n}|\mathcal{G}_n] = \nabla_{0,1}\tau_{-n}E[\tau_{x+n}\mathfrak{g}^1|\mathcal{G}_n].\end{aligned}$$

Therefore, for each fixed $x \leq -2$,

$$\limsup_{n\to\infty} E[\{(2n)^{-1}\nabla_{x,x+1}\nabla_{0,1}\tau_{-n}\phi_n\}^2] \leq \limsup_{n\to\infty} n^{-2}E[(\mathfrak{g}^1)^2] = 0.$$

Now, we can apply the argument in the proof of Theorem A.3.4.14 step 5 to conclude the proof. $\square$

Finally, by the definition, we can see that $(2n)^{-1}\nabla_{0,1}\tau_{-n}\phi_n = 0$ and $(2n)^{-1}\nabla_{0,1}\tau_{n+1}\phi_n = 0$ if $(\eta_0, \eta_1) = (0,0), (0,1)$ or $(1,1)$ and so it is also true for $\mathfrak{b}_\pm$. Then immediately we conclude that $\mathfrak{b}_\pm(\omega) = C_\pm\eta_0(1-\eta_1)$.

Chapter 2

Macroscopic energy diffusion for a chain of anharmonic oscillators

2.1 Introduction

The deduction of the heat equation or the Fourier law for the macroscopic evolution of the energy through a diffusive space-time scaling limit from a microscopic dynamics given by Hamilton or Schrödinger equations, is one of the most important problem in non-equilibrium statistical mechanics ([5]). One dimensional chains of oscillators have been used as simple models for this study. In the context of the classical (Hamiltonian) dynamics, it is clear that non-linear interactions are crucial for the diffusive behavior of the energy. In fact, in a chain of harmonic oscillators the energy evolution is ballistic ([23]). In this linear system, the energy of each mode of vibration is conserved. Non-linearities introduce interactions between different modes and destroy these conservation laws and give a certain ergodicity to the microscopic dynamics.

In order to describe the mathematical problem, let us introduce some notation we will use in the rest of the paper. We study a system of N anharmonic oscillators with a periodic boundary condition. The particles are denoted by $j = 1, \cdots, N$, and we identify $N+1 = 1$. We denote by $\{q_j\}_{j=1,\cdots,N}$ their positions, and by $\{p_j\}_{j=1,\cdots,N}$ the corresponding momenta (which are equal to their velocities since we assume that the mass of all particles are unity). Each pair of consecutive particles $(i, i+1)$ are connected by an anharmonic spring. The interaction is described by a potential energy $V(q_{i+1} - q_i)$. We assume that V is a positive symmetric smooth function satisfying

$$Z_\beta := \int_{\mathbb{R}} e^{-\beta V(r)} dr < \infty$$

for all $\beta > 0$. Let a be the equilibrium inter-particle distance, where V attains its minimum that we assume to be 0 : $V(a) = 0$. It is convenient to work with inter-particle distances as coordinates, rather than absolute particle positions, so we define $r_j = q_j - q_{j-1} - a, j = 1, ..., N$, with $q_0 = q_N$. We replace $V(\cdot)$ with $V(\cdot + a)$ hereafter. Namely, we assume $V(0) = 0$. The configuration of the system is given by $p_j, r_j, j = 1, \cdots, N \in \mathbb{R}^{2N}$, and energy function (Hamiltonian) defined for each configuration is given by

$$H = \sum_{j=1}^{N} \mathcal{E}_j, \qquad \mathcal{E}_j = \frac{1}{2} p_j^2 + V(r_j), \quad j = 1, \ldots, N.$$

The choice of $\mathcal{E}_j$ as the energy of each oscillator is a bit arbitrary, because we associate the potential energy of the bond $V(r_j)$ to the particle j. Different choices can be made, but this one is notationally convenient.

We consider the Hamiltonian dynamics:

$$\begin{aligned} r_j'(t) &= p_j(t) - p_{j-1}(t), \quad j = 1, ..., N, \\ p_j'(t) &= V'(r_{j+1}(t)) - V'(r_j(t)), \quad j = 1, ..., N. \end{aligned} \tag{2.1.1}$$

We are interested in the macroscopic evolution of the energy empirical density profile under a diffusive macroscopic space-time scaling. More precisely, we study the limit as $N \to \infty$, of the energy distribution on the circle $\mathbb{T}$ of length 1 defined by

$$\frac{1}{N} \sum_{i=1}^{N} \mathcal{E}_i(N^2 t) \delta_{i/N}(dy). \tag{2.1.2}$$

The total energy is not the only conserved quantity under the dynamics (2.1.1). The total length $\sum_{i=1}^N r_i$ and the total momentum $\sum_{i=1}^N p_i$ are also integral of the motion that survive to the limit as $N \to \infty$. In one dimensional system, even for anharmonic interaction, generically we expect a superdiffusion of the energy, essentially because of the momentum conservation ([20, 1]). Adding a pinning potential $\sum_{i=1}^N U(q_i)$ to the Hamiltonian will break the translation invariance of the system and the momentum conservation, and we expect a diffusive behavior for the energy, i.e. the energy profile defined by (2.1.2) would converge to the solution of a heat equation

$$\partial_t e(t, u) = \partial_y D(e(t, u)) \partial_y e(t, u)$$

under specific conditions on the initial configuration. The diffusivity $D = D(e)$ is defined by the Green-Kubo formula associated to the corresponding infinite dynamics in equilibrium at average energy e (see below the definition).

As the deterministic problem is out of reach mathematically, it has been proposed an approach that models the chaotic effects of the non-linearities by a stochastic perturbations of the dynamics such that energy is conserved. A random exchange of momenta of nearest neighbor particles that conserve total energy but total momentum has been studied in the harmonic case [4, 9, 3]. Stochastic exchanges that also conserve total momentum have been considered in [1, 2], where a divergence of the diffusivity is proven for unpinned harmonic chains. The stochastic perturbations considered in these papers are very degenerate (of hypoelliptic type), since they act only on the momenta of the particles, and not on the positions. In particular these stochastic terms conserve also the total length $\sum_j r_j$.

In this article we want to deal with anharmonic chains with noise that conserves total energy. For reasons we will explain in a moment, we need more elliptic stochastic perturbations that acts also on the positions. In the case of one-dimensional unpinned chains there is a way to define these perturbation locally (see the next section) just using squares of vector fields that compose the Liouville vector field that generates the Hamiltonian dynamics. It results a dynamics that conserves *only* the total energy. So it has a one-parameter family of invariant measures that can be described as follows.

For any $\beta > 0$, the grand canonical measure ν_β^N defined by

$$\nu_\beta^N = \prod_{j=1}^{N} \frac{e^{-\beta \mathcal{E}_i}}{\sqrt{2\pi\beta^{-1}} Z_\beta} dp_j dr_j$$

is stationary for this dynamics. The distribution is called grand canonical Gibbs measure at temperature $T = \beta^{-1}$. Notice that $r_1, ..., r_N, p_1, ..., p_N$ are independently distributed under this probability measure. The ergodic measures of the dynamics are the corresponding conditioned measures on the energy surfaces (microcanonical Gibbs measures). As $N \to \infty$ the microcanonical measure of energy e converges to the corresponding $\nu_{\beta(e)} (= \nu_{\beta(e)}^\infty)$, in the sense of the finite dimensional distribution (equivalence of ensembles), with corresponding inverse temperature $\beta(e)$ given by the usual thermodynamic relation. So we can consider the system starting with the distribution ν_β^N as the system in equilibrium at temperature $T = \beta^{-1}$.

If the system is in equilibrium at $\beta(e)$, then standard central limit theorem for independent variables tell us that as $N \to \infty$ energy has Gaussian fluctuations, i.e. the energy fluctuation field

$$Y^N = \frac{1}{\sqrt{N}} \sum_i \delta_{i/N} \{\mathcal{E}_i(0) - e\}$$

converges in law to a delta correlated centered Gaussian field Y

$$\mathbb{E}[Y(F)Y(G)] = \chi(\beta(e)) \int_{\mathbb{T}} F(y)G(y)dy$$

where $\chi(\beta)$ is the variance of $\mathcal{E}_0$ under ν_β.

In this thesis we prove that these *macroscopic* energy fluctuations evolve diffusively in time (after a diffusive space-time scaling) , i.e. that the time dependent distribution

$$Y_t^N = \frac{1}{\sqrt{N}} \sum_i \delta_{i/N} \{\mathcal{E}_i(N^2t) - e\}$$

converges in law to the solution of the linear SPDE

$$\partial_t Y = D(\beta(e))\partial_y^2 Y \, dt + \sqrt{2D(\beta(e))\chi(\beta(e))}\partial_y B(y,t)$$

where B is standard normalized space-time white noise. In this sense energy fluctuation in equilibrium follows linearized heat equation.

The main point in the proof of this result is the following. Since total energy is conserved, locally the energy of each particle is changed by the energy currents with its neighbors, i.e. applying the generator L of the process to the energy $\mathcal{E}_i$ we obtain

$$L\mathcal{E}_i = W_{i-1,i} - W_{i,i+1} \tag{2.1.3}$$

where $W_{i,i+1} = -p_i V'(r_{i+1}) + W_{i,i+1}^S$. Here $-p_i V'(r_{i+1})$ is the instantaneous energy current associated to the Hamiltonian mechanism, while $W_{i,i+1}^S$ is the instantaneous energy current due to the stochastic part of the dynamics. While (2.1.3) provides automatically one space derivative already at the microscopic level, $W_{i,i+1}$ is not a space-gradient. In this sense this model falls in the class of the non-gradient models, and some of them have been studied with a method introduced by Varadhan [26]. The main point is to prove that $W_{i,i+1}$ can be approximated by a fluctuation-dissipation decomposition

$$W_{i,i+1} \sim D\nabla\mathcal{E}_i + LF$$

for a properly chosen sequence of smooth local functions F. In the harmonic case, with noise conserving energy, this decomposition is exact for every configuration, i.e. there exist a local second order polynomial F such that $W_{i,i+1} = D\nabla\mathcal{E}_i + LF$ for a given constant D (cf. [4]). In the anharmonic case such decomposition can be only approximated by a sequence of local function F_K and in the sense that the difference has small space-time variance respect to the dynamics in equilibrium at given temperature (consequently D is a function of this temperature).

It is in order to do such decomposition that we have to use Varadhan's approach to non-gradient systems [26] and the generalization to non-reversible dynamics [28, 16]. The main ingredients of the methods are a *spectral gap* for the stochastic part of the dynamics, and a sector condition for the generator L of the dynamics. It is in order to prove these properties that we need such elliptic noise acting also on the positions.

This chapter is organized as follows: In Section 2.2 we introduce our model and state main results. In Section 2.3, we give the strategy of the proof of the main theorem. The proof is divided into several sections: Section 2.4, 2.5 and 2.6. The proof of a version of the sector condition is in Subsection 2.6.1 and the detailed estimates of the diffusion coefficient are obtained in Subsections 2.6.2 and 2.6.3. In Section 2.7, we give a spectral gap estimate which is used in Section 2.8, where we characterize the class of closed forms.

2.2 Model and results

We will now give a precise description of the model. We consider a system of N particles in one-dimensional space evolving under an interacting random mechanism. Let $\mathbb{T} := (0, 1]$ be the 1-dimensional torus, and for a positive integer N denote by $\mathbb{T}_N$ the lattice torus of length N : $\mathbb{T}_N = \{1, \dots N\}$. The configuration space is denoted by $\Omega^N = (\mathbb{R}^2)^{\mathbb{T}_N}$ and a typical configuration is denoted by $\omega = (p_i, r_i)_{i\in\mathbb{T}_N}$ where r_i represents the inter-particle distance between the particle $i-1$ and i, and p_i represents the velocity of the particle i. The configuration changes with time and, as a function of time undergoes a diffusion in $\mathbb{R}^{2N}$. The diffusion mentioned above have as an infinitesimal generator the following operator

$$L_N^\gamma = A_N + \gamma S_N$$

where

$$A_N = \sum_{i\in\mathbb{T}_N}(X_i - Y_{i,i+1}), \quad S_N = \frac{1}{2}\sum_{i\in\mathbb{T}_N}\{(X_i)^2 + (Y_{i,i+1})^2\}$$

and

$$Y_{i,j} = p_i\partial_{r_j} - V'(r_j)\partial_{p_i}, \qquad X_i = Y_{i,i}.$$

We assume that the function $V : \mathbb{R} \to \mathbb{R}_+$ satisfies the following four properties: (i) $V(r)$ is a smooth symmetric function. (ii) $V(r)$ is strictly increasing in $\mathbb{R}_+$. (iii) There exist some constants d_+ and d_- such that

$$0 < d_- \le \frac{\sqrt{2V(r)}}{V'(r)} \le d_+ < \infty$$

for all $r > 0$. (iv) The pair of constants d_+ and d_- in (iii) satisfies $d_-/d_+ > (3/4)^{1/16}$.

Remark 2.1. *By the assumptions (i), (ii) and (iii), it is easy to show that*

$$\frac{r^2}{2d_+} \le V(r) \le \frac{r^2}{2d_-}$$

for all $r \in \mathbb{R}$.

Remark 2.2. *The assumption (iv) is quite technical and required only in the proof of the spectral gap estimate in Section 2.7.*

We denote the energy associated to the particle i by $\mathcal{E}_i = \frac{p_i^2}{2} + V(r_i)$ and the total energy defined by $\mathcal{E} = \sum_{i\in\mathbb{T}_N}\mathcal{E}_i$ which denotes the Hamiltonian of the original dynamics. Observe that the total energy satisfies $L_N^\gamma(\mathcal{E}) = 0$, i.e. total energy is a conserved quantity.

Recall that ν_β^N on Ω^N is defined by

$$\nu_\beta^N(dpdr) = \prod_{i=1}^N \frac{\exp(-\beta(\frac{p_i^2}{2} + V(r_i))}{\sqrt{2\pi\beta^{-1}}Z_\beta}dp_i dr_i$$

where

$$Z_\beta := \int_{\mathbb{R}} e^{-\beta V(r)}dr < \infty.$$

Denote by $L^2(\nu_\beta^N)$ the Hilbert space of functions f on Ω^N such that $\nu_\beta^N(f^2) < \infty$. S_N is formally symmetric on $L^2(\nu_\beta^N)$ and A_N is formally antisymmetric on $L^2(\nu_\beta^N)$.

In fact, it is easy to see that for smooth functions f and g in a core of the operator S_N and A_N, we have for all $\beta > 0$

$$\int_{\mathbb{R}^{2N}} S_N(f) g \nu_\beta^N(dpdr) = \int_{\mathbb{R}^{2N}} f S_N(g) \nu_\beta^N(dpdr),$$

and

$$\int_{\mathbb{R}^{2N}} A_N(f) g \nu_\beta^N(dpdr) = -\int_{\mathbb{R}^{2N}} f A_N(g) \nu_\beta^N(dpdr).$$

In particular, the diffusion is invariant with respect to all the measures ν_β^N. The distribution is called grand canonical Gibbs measure at temperature $T = \beta^{-1}$. Notice that $r_1, ..., r_N, p_1, ..., p_N$ are independently distributed under this probability measure.

On the other hand, for every $\beta > 0$ the Dirichlet form of the diffusion with respect to ν_β^N is given by

$$\mathcal{D}_{N,\beta}(f) = \frac{\gamma}{2} \int_{\mathbb{R}^{2N}} \sum_{i \in \mathbb{T}_N} \{[X_i(f)]^2 + [Y_{i,i+1}(f)]^2\} \nu_\beta^N(dpdr).$$

We will use the notation ν_β for the product measures on the configuration spaces $\Omega := (\mathbb{R}^2)^{\mathbb{Z}}$, namely on the infinite lattice with marginal given by $\nu_\beta|_{\{1,2,...,N\}} = \nu_\beta^N$. The expectation with respect to ν_β will be sometimes denoted by

$$\int_\Omega f \nu_\beta(dpdr) = \langle f \rangle_\beta.$$

Denote by $\{\omega(t) = (p(t), r(t)); t \geq 0\}$ the Markov process generated by $N^2 L_N$ (the factor N^2 correspond to an acceleration of time). Let $C(\mathbb{R}_+, \Omega^N)$ be the space of continuous trajectories on the configuration space. Fixed a time $T > 0$ and for a given measure μ^N on Ω^N, the probability measure on $C([0,T], \Omega^N)$ induced by this Markov process starting in μ^N will be denoted by $\mathbb{P}_{\mu^N}$. As usual, expectation with respect to $\mathbb{P}_{\mu^N}$ will be denoted by $\mathbb{E}_{\mu^N}$. The diffusion generated by $N^2 L_N^\gamma$ can also be described by the following system of stochastic differential equations

$$\begin{aligned} dp_i(t) &= N^2[V'(r_{i+1}) - V'(r_i) - \frac{\gamma p_i}{2}\{V''(r_i) + V''(r_{i+1})\}]dt \\ &\qquad + \sqrt{\gamma} N \{V'(r_{i+1}) dB_i^1 - V'(r_i) dB_i^2\}, \\ dr_i(t) &= N^2[p_i - p_{i-1} - \gamma V''(r_i)]dt + \sqrt{\gamma} N \{-p_{i-1} dB_{i-1}^1 + p_i dB_i^2\} \end{aligned}$$

where $\{B_i^1, B_i^2\}_{i \in \mathbb{T}_N}$ are $2N$-independent standard Brownian motions.

Then, by Ito's formula, we have

$$d\mathcal{E}_i(t) = N^2[W_{i-1,i} - W_{i,i+1}]dt + N\{\sigma_{i-1,i}dB^1_{i-1} - \sigma_{i,i+1}dB^1_i\}$$

where $W_{i,i+1} = W^A_{i,i+1} + W^S_{i,i+1}$, $W^A_{i,i+1} = -p_iV'(r_{i+1})$, $W^S_{i,i+1} = \frac{\gamma}{2}\{p_i^2V''(r_{i+1}) - V'(r_{i+1})^2\}$ and $\sigma_{i,i+1} = -\sqrt{\gamma}p_iV'(r_{i+1})$.

We can think of $W_{i,i+1}$ as being the instantaneous microscopic current of energy between i and $i+1$. Observe that the current $W_{i,i+1}$ cannot be written as the gradient of a local function, neither by an exact fluctuation-dissipation equation, i.e. as the sum of a gradient and a dissipative term of the form $L^\gamma_N(\tau_i h)$. That is, we are in the nongradient case. The collective behavior of the system is described thanks to empirical measures. With this purpose let us introduce the energy empirical measure associated to the process defined by

$$\pi^N_t(\omega, du) = \frac{1}{N}\sum_{i\in\mathbb{T}_N}\mathcal{E}_i(t)\delta_{\frac{i}{N}}(du), \quad 0 \le t \le T, \quad u \in \mathbb{T},$$

and $\langle \pi^N_t, f\rangle$ stands for the integration of f with respect to π^N_t. To investigate equilibrium fluctuations of the empirical measure π^N we fix $\beta > 0$ and consider the system in the equilibrium ν^N_β. Denote by Y^N_t the empirical energy fluctuation field acting on smooth functions $H : \mathbb{T} \to \mathbb{R}$ as

$$Y^N_t(H) = \frac{1}{\sqrt{N}}\sum_{i\in\mathbb{T}_N} H(\frac{i}{N})\{\mathcal{E}_i(t) - (\frac{1}{2\beta} + \psi(\beta))\}$$

where $\psi(\beta) := \langle V(r_0)\rangle_\beta$. On the other hand, let $\{Y_t\}_{t\le 0}$ be the stationary generalized Ornstein-Uhlenbeck process with zero mean and covariances given by

$$\mathbb{E}[Y_t(H_1)Y_s(H_2)] = \frac{\chi(\beta)}{\sqrt{4\pi(t-s)D(\beta)}}\int_{\mathbb{R}} du \int_{\mathbb{R}} dv \bar{H}_1(u)\exp\{-\frac{(u-v)^2}{4(t-s)D(\beta)}\}\bar{H}_2(v)$$

for every $0 \le s \le t$. Here $\chi(\beta)$ stands for the thermal capacity given by $\chi(\beta) = \langle \mathcal{E}_0^2\rangle_\beta - \langle \mathcal{E}_0\rangle^2_\beta = \frac{1}{2\beta^2} - \psi'(\beta)$ and $\bar{H}_1(u)$ (resp. $\bar{H}_2(u)$) is the periodic extension to the real line of the smooth function H_1 (resp H_2), and $D(\beta)$ is the diffusion coefficient determined later.

Consider for $k > \frac{3}{2}$ the Sobolev space $\mathcal{H}_{-k}$. Denote by $\mathbb{Q}_N$ the probability measure on $C([0,T], \mathcal{H}_{-k})$ induced by the energy fluctuation field Y^N_t and the Markov process $\{\omega^N(t), t \ge 0\}$ defined at the beginning of this section, starting from the equilibrium probability measure ν^N_β. Let $\mathbb{Q}$ be the probability measure on the space $C([0,T], \mathcal{H}_{-k})$ corresponding to the generalized Ornstein-Uhlenbeck process Y_t defined above. We are now ready to state the main result of this work.

Theorem 2.1. *The sequence of the probability measures $\{\mathbb{Q}_N\}_{N\geq 1}$ converges weakly to the probability measure $\mathbb{Q}$.*

Remark 2.3. *For each $H \in C^\infty(\mathbb{T})$,*

$$M_t^{D,H} := Y_t(H) - Y_0(H) - \int_0^t Y_s(D(\beta)\Delta H)ds, \tag{2.2.1}$$

and

$$N_t^{D,H} := (M_t^{D,H})^2 - 2\chi(\beta)D(\beta)\langle (H')^2\rangle_{L^2(\mathbb{T})} \tag{2.2.2}$$

are $L^1(\mathbb{Q})$-martingale.

2.3 Strategy of the proof of the main theorem

We follow the argument in Section 11 in [14]. According to their argument, Theorem 2.1 follows from the following three properties: (i) $\{\mathbb{Q}_N\}_{N\geq 1}$ is tight, (ii) the restriction of all limit points $\mathbb{Q}^*$ to $\mathcal{F}_0$ are Gaussian fields with covariance given by

$$\mathbb{E}[Y(H_1)Y(H_2)] = \chi(\beta)\langle H_1, H_2\rangle_{L^2(\mathbb{T})},$$

(iii) all limit points $\mathbb{Q}^*$ of convergent subsequence of $\{\mathbb{Q}_N\}_{N\geq 1}$ solves the martingale problem (2.2.1) and (2.2.2).

For the proof of (i), we can apply the argument in [12] directly. (ii) is straightforward.

To prove (iii), for a given smooth function $H : \mathbb{T} \to \mathbb{R}$, we begin by rewriting $Y_t^N(H)$ as

$$Y_t^N(H) = Y_0^N(H) + \int_0^t \sqrt{N}\sum_{i\in\mathbb{T}_N} \nabla^N H(\frac{i}{N})W_{i,i+1}ds + M^{H,N}(t)$$

where $\nabla^N H$ represents the discrete derivative of H:

$$(\nabla^N H)(\frac{i}{N}) = N[H(\frac{i+1}{N}) - H(\frac{i}{N})]$$

and the martingale $M^{H,N}(t)$ is

$$M^{H,N}(t) = \int_0^t \frac{1}{\sqrt{N}}\sum_{i\in\mathbb{T}_N} \nabla^N H(\frac{i}{N})\sigma_{i,i+1}dB_i^1.$$

Then, we can decompose this as follows:

$$Y_t^N(H) = Y_0^N(H) + \int_0^t Y_s^N(D(\beta)\Delta^N H)ds + I_{N,F,t}^1(H) + I_{N,F,t}^2(H) + D(\beta)\chi(\beta)\beta^2 I_{N,t}^3(H) + M_{N,F,t}^1(H) + M_{N,F,t}^2(H)$$

where

$$I_{N,F,t}^1(H) = \int_0^t \sqrt{N}\sum_{i\in\mathbb{T}_N} \nabla^N H(\frac{i}{N})[W_{i,i+1} + D(\beta)\chi(\beta)\beta^2(p_{i+1}^2 - p_i^2) - L_N^\gamma(\tau_i F)]ds,$$

$$I_{N,F,t}^2(H) = \int_0^t \sqrt{N}\sum_{i\in\mathbb{T}_N} \nabla^N H(\frac{i}{N}) L_N^\gamma(\tau_i F)ds,$$

$$I_{N,t}^3(H) = \int_0^t \frac{1}{\sqrt{N}}\sum_{i\in\mathbb{T}_N} \Delta^N H(\frac{i}{N})[(p_i^2 - \frac{1}{\beta}) - \frac{1}{\chi(\beta)\beta^2}\{\mathcal{E}_i - (\frac{1}{2\beta} + \psi(\beta))\}],$$

$$M_{N,F,t}^1(H) = \int_0^t \frac{1}{\sqrt{N}}\sum_{i\in\mathbb{T}_N} \nabla^N H(\frac{i}{N})[(\sigma_{i,i+1} + \sqrt{\gamma}Y_{i,i+1}(\Gamma_F))dB_i^1 - \sqrt{\gamma}X_i(\Gamma_F)dB_i^2],$$

$$M_{N,F,t}^2(H) = \int_0^t \frac{1}{\sqrt{N}}\sum_{i\in\mathbb{T}_N} \nabla^N H(\frac{i}{N})[-\sqrt{\gamma}Y_{i,i+1}(\Gamma_F)dB_i^1 + \sqrt{\gamma}X_i(\Gamma_F)dB_i^2].$$

The proof of (iii) is reduced to the following lemmas:

Lemma 2.1.1. *For every smooth function $H : \mathbb{T} \to \mathbb{R}$ and local function F in the Schwartz space,*

$$\lim_{N\to\infty} \mathbb{E}_{\nu_\beta^N}[\sup_{0\le t\le T}(I_{N,F,t}^2(H) + M_{N,F,t}^2(H))^2] = 0.$$

Lemma 2.1.2. *For every smooth function $H : \mathbb{T} \to \mathbb{R}$ and $t > 0$,*

$$\lim_{N\to\infty} \mathbb{E}_{\nu_\beta^N}[(I_{N,t}^3(H))^2] = 0.$$

Lemma 2.1.3. *There exists a sequence of local functions F_K in the Schwartz space such that, for every smooth function $H : \mathbb{T} \to \mathbb{R}$ and $t > 0$*

$$\lim_{K\to\infty}\lim_{N\to\infty} \mathbb{E}_{\nu_\beta^N}[(I_{N,F_K,t}^1(H))^2] = 0.$$

Moreover, for this sequence F_K,

$$\lim_{K\to\infty} E_{\nu_\beta}[(\sigma_{0,1} + \sqrt{\gamma}Y_{0,1}(\Gamma_{F_K}))^2 + (\sqrt{\gamma}X_0(\Gamma_{F_K}))^2] = 2D(\beta)\chi(\beta) = \frac{2\tilde{D}(\beta)}{\beta^2}.$$

where $\tilde{D}(\beta) := D(\beta)\chi(\beta)\beta^2$. Note that

$$I_{N,F,t}^1(H) = \int_0^t \sqrt{N}\sum_{i\in\mathbb{T}_N} \nabla^N H(\frac{i}{N})[W_{i,i+1} + \tilde{D}(\beta)(p_{i+1}^2 - p_i^2) - L_N(\tau_i F)]ds.$$

Now we proceed to give a proof of Lemma 2.1.1.

Proof of Lemma 2.1.1. Let us define

$$\zeta_{N,F}(t) = \frac{1}{N^{\frac{3}{2}}} \sum_{i \in \mathbb{T}_N} \nabla^N H(\frac{i}{N}) \tau_i F(\omega_t^N).$$

From the Ito's formula we obtain

$$\zeta_{N,F}(t) = \zeta_{N,F}(0) + I^2_{N,F,t}(H) \\ + \int_0^t \frac{1}{\sqrt{N}} \sum_{i \in \mathbb{T}_N} \nabla^N H(\frac{i}{N}) \sqrt{\gamma} \sum_{j \in \mathbb{T}_N} [-Y_{j,j+1}(\tau_i F) dB^1_j + X_j(\tau_i F) dB^2_j].$$

Therefore,

$$(I^2_{N,F,t}(H) + M^2_{N,F,t}(H))^2 \leq 2(\zeta_{N,F}(t) - \zeta_{N,F}(0))^2 \\ +2\Big\{ \int_0^t \frac{1}{\sqrt{N}} \sum_{i,j \in \mathbb{T}_N} \nabla^N H(\frac{i}{N}) \sqrt{\gamma} [-Y_{j,j+1}(\tau_i F) dB^1_j + X_j(\tau_i F) dB^2_j] - M^2_{N,F,t}(H) \Big\}^2.$$

Since F is bounded and H is smooth, it is easy to see that the first term is of order $\frac{1}{N}$. Using additionally the fact that F is local and in the Schwartz class, we can prove that second term is also of order $\frac{1}{N}$. □

2.4 Boltzmann-Gibbs principle

In this section, we prove Lemma 2.1.2. First, recall that for each $e > 0$, $\beta(e)$ is defined by the relation $\mathbb{E}_{\nu_{\beta(e)}}[\mathcal{E}_1] = e$. Then, by simple calculations, we have

$$\frac{d}{de} \langle p_1^2 \rangle_{\beta(e)} = \frac{d}{de} \Big(\frac{1}{\beta(e)} \Big) = \frac{-1}{\beta(e)^2} \Big\{ - \frac{1}{2\beta(e)^2} + \psi'(\beta(e)) \Big\}^{-1} = \frac{1}{\chi(\beta(e))\beta(e)^2}.$$

Now, we can rewrite the term $I^3_{N,t}(H)$ as

$$\int_0^t \frac{1}{\sqrt{N}} \sum_{i \in \mathbb{T}_N} \Delta^N H(\frac{i}{N}) [p_i^2 - h(e) - h'(e)(\mathcal{E}_i - e)]$$

where $h(e) = \frac{1}{\beta(e)} = \langle p_1^2 \rangle_{\beta(e)}$ and $h'(e)$ stands for the derivative of $h(e)$ with respect to e. Then, Lemma 2.1.2 follows from standard arguments (cf. [14]).

2.5 Central limit theorem variances

In this section, as the beginning of the proof of Lemma 2.1.3, we study the central limit theorem variances. First, we introduce some notation. We denote $\mathcal{C}$ the set of smooth local functions f on $\Omega = (\mathbb{R}^2)^{\mathbb{Z}}$ satisfying that

$$p^{\alpha_1} r^{\alpha_2} D_p^{\alpha_3} D_r^{\alpha_4} f(p,r) \in L^2(\nu_\beta)$$

for any multi-indices α_1, α_2, α_3 and α_4 and for any $\beta > 0$. Here and after, we consider operators L^γ, S^γ and A acting on functions f in $\mathcal{C}$ as

$$L^\gamma f = S^\gamma f + Af, \quad S^\gamma f = \frac{\gamma}{2}\sum_{i\in\mathbb{Z}}\{(X_i)^2 f + (Y_{i,i+1})^2 f\}, \quad Af = \sum_{i\in\mathbb{Z}} X_i f - Y_{i,i+1} f.$$

For a fixed positive integer l, we define $\Lambda_l := \{-l, -l+1, ..., l-1, l\}$ and $L^\gamma_{\Lambda_l}$, $S^\gamma_{\Lambda_l}$ the restriction of the generator L^γ, S^γ to Λ_l respectively. For Ψ in $\mathcal{C}$, denote by $\tilde{s_\Psi}$ the smallest positive integer s such that Λ_s contains the support of Ψ. For a technical reason, we define $s_\Psi = \tilde{s_\Psi} + 1$.

Let $\mathcal{C}_0$ be a subspace of cylinder functions defined as follows:

$$\mathcal{C}_0 = \{\ f; f = \sum_{i\in\Lambda}[X_i(F_i) + Y_{i,i+1}(G_i)] \text{ for some } \Lambda \subset\subset \mathbb{Z} \text{ and} \{F_i\}_{i\in\Lambda}, \{G_i\}_{i\in\Lambda} \in \mathcal{C}\}.$$

For a finite subset Λ of $\mathbb{Z}$, we denote by $|\Lambda|$ the cardinality of Λ and by $\langle\cdot\rangle_{\Lambda,E}$ the expectation with respect to the canonical measure $\nu_{\Lambda,E} := \nu_\beta(\ \cdot \mid \sum_{i\in\Lambda}\mathcal{E}_i = |\Lambda|E)$ for $E > 0$ which is indeed independent of the choice of β. For a finite set Λ and a canonical measure $\nu_{\Lambda,E}$, denote by $\langle\cdot,\cdot\rangle_{\Lambda,E}$ (resp. $\langle\cdot,\cdot\rangle_\beta$) the inner product in $L^2(\nu_{\Lambda,E})$ (resp. $L^2(\nu_\beta)$).

First, we note some useful properties of the space $\mathcal{C}_0$.

Lemma 2.1.4. *(i) For any $f \in \mathcal{C}_0$, $N \geq s_f$ and $E > 0$, $\langle f\rangle_{\Lambda_N,E} = 0$.*
(ii) $W^S_{0,1}$, $W^A_{0,1}$ and $p_1^2 - p_0^2$ are elements of $\mathcal{C}_0$.
(iii) For any $F \in \mathcal{C}$, $L^\gamma F$, $S^\gamma F$ and AF are elements of $\mathcal{C}_0$.
(iv) For any $f \in \mathcal{C}_0$, there exists a constant $C_{f,E}$ depending only on f and $E > 0$ such that for any $u \in D(L^\gamma_{\Lambda_N})$,

$$\int_{(\mathbb{R}^2)^{\Lambda_N}} u(\sum_{|i|\leq N-s_f-1}\tau_i f)\, d\nu_{\Lambda_N,E} \leq C_{f,E}\sqrt{N\mathcal{D}_{N,E}(u)}$$

where $\mathcal{D}_{N,E}(u) := \int_{(\mathbb{R}^2)^{\Lambda_N}}\sum_{|k|\leq N}(X_k u)^2 d\nu_{\Lambda_N,E} + \int_{(\mathbb{R}^2)^{\Lambda_N}}\sum_{k=-N}^{N-1}(Y_{k,k+1}u)^2 d\nu_{\Lambda_N,E}$.

Proof. (i) and (iii) are straightforward.

(ii): We have $W^S_{0,1} = \frac{\gamma}{2}\{p_0^2 V''(r_1) - V'(r_1)^2\} = \frac{\gamma}{2} Y_{0,1}(p_0 V'(r_1))$, $W^A_{0,1} = -p_0 V'(r_1) = Y_{0,1}(-V(r_1))$ and $p_1^2 - p_0^2 = X_1\{(p_0+p_1)r_1\} - Y_{0,1}\{(p_0+p_1)r_1\}$.

(iv): By the definition, it is easy to see that there exist some functions F_i, $G_i \in \mathcal{C}$ such that $f = \sum_{|i|\le s_f} X_i F_i + Y_{i,i+1} G_i$. Then, we have

$$\begin{aligned}
&\int_{(\mathbb{R}^2)^{\Lambda_N}} u\Big(\sum_{|i|\le N-s_f-1} \tau_i f\Big) d\nu_{\Lambda_N,E} \\
&= \int_{(\mathbb{R}^2)^{\Lambda_N}} u \sum_{|i|\le N-s_f-1} \tau_i\Big\{\sum_{|j|\le s_f} X_j F_j + Y_{j,j+1} G_j\Big\} d\nu_{\Lambda_N,E} \\
&= \int_{(\mathbb{R}^2)^{\Lambda_N}} u \sum_{|i|\le N-s_f-1} \sum_{|j|\le s_f} \{X_{i+j}\tau_i F_j + Y_{i+j,i+j+1}\tau_i G_j\} d\nu_{\Lambda_N,E} \\
&= -\int_{(\mathbb{R}^2)^{\Lambda_N}} \sum_{|i|\le N-s_f-1} \sum_{|j|\le s_f} \{(X_{i+j}u)\tau_i F_j + (Y_{i+j,i+j+1}u)\tau_i G_j\} d\nu_{\Lambda_N,E} \\
&= -\int_{(\mathbb{R}^2)^{\Lambda_N}} \sum_{|k|\le N-1} \sum_{\substack{|j|\le s_f \\ |k-j|\le N-s_f-1}} \{(X_k u)\tau_{k-j} F_j + (Y_{k,k+1}u)\tau_{k-j} G_j\} d\nu_{\Lambda_N,E} \\
&\le \int_{(\mathbb{R}^2)^{\Lambda_N}} \sqrt{\sum_{|k|\le N-1} \{(X_k u)^2 + (Y_{k,k+1}u)^2\}} \sqrt{\sum_{|k|\le N-1} \{(F^{(k)})^2 + (G^{(k)})^2\}} d\nu_{\Lambda_N,E}
\end{aligned}$$

where $F^{(k)} = \sum_{|j|\le s_f, |k-j|\le N-s_f-1} \tau_{k-j} F_j$ and $G^{(k)} = \sum_{|j|\le s_f, |k-j|\le N-s_f-1} \tau_{k-j} G_j$. By Schwartz inequality, the last expression is bound from above by

$$\begin{aligned}
&\sqrt{\mathcal{D}_{N,E}(u) \int_{(\mathbb{R}^2)^{\Lambda_N}} \sum_{|k|\le N-1} \{(F^{(k)})^2 + (G^{(k)})^2\} d\nu_{\Lambda_N,E}} \\
&\le \sqrt{\mathcal{D}_{N,E}(u)(2s_f+1) \int_{(\mathbb{R}^2)^{\Lambda_N}} \sum_{\substack{|k|\le N-1, |j|\le s_f, \\ |k-j|\le N-s_f-1}} [(\tau_{k-j}F_j)^2 + (\tau_{k-j}G_j)^2] d\nu_{\Lambda_N,E}} \\
&= \sqrt{\mathcal{D}_{N,E}(u)(2s_f+1) \int_{(\mathbb{R}^2)^{\Lambda_N}} \sum_{\substack{|k|\le N-1, |j|\le s_f, \\ |k-j|\le N-s_f-1}} (F_j)^2 + (G_j)^2] d\nu_{\Lambda_N,E}} \\
&\le \sqrt{\mathcal{D}_{N,E}(u)(2s_f+1)(2N-1) \int_{(\mathbb{R}^2)^{\Lambda_N}} \sum_{|j|\le s_f} [(F_j)^2 + (G_j)^2] d\nu_{\Lambda_N,E}}
\end{aligned}$$

and by the equivalence of ensembles, we obtain the desired result. □

Next, we study the variance

$$(2N)^{-1}\langle(-S^{\gamma}_{\Lambda_N})^{-1}\sum_{|i|\le N_\psi}\tau_i\psi,\sum_{|i|\le N_\psi}\tau_i\psi\rangle_{\Lambda_N,E}$$

for $\psi\in\mathcal{C}_0$ where $N_\psi=N-s_\psi-1$. We start with introducing a semi-norm on $\mathcal{C}_0$, which is closely related to the central limit theorem variance. For cylinder functions g, h in $\mathcal{C}_0$, let

$$\ll g,h\gg_{\beta,*}=\sum_{i\in\mathbb{Z}}\langle g,\tau_i h\rangle_\beta \quad\text{and}\quad \ll g\gg_{\beta,**}=\sum_{i\in\mathbb{Z}}i\langle g,\mathcal{E}_i\rangle_\beta.$$

$\ll g,h\gg_{\beta,*}$ and $\ll g\gg_{\beta,**}$ are well defined because g and h belong to $\mathcal{C}_0$ and therefore all but a finite number of terms vanish. For h in $\mathcal{C}_0$, define the semi-norm $\ll h\gg^{\frac{1}{2}}_{\beta,\gamma}$ by

$$\ll h\gg_{\beta,\gamma}=\sup_{g\in\mathcal{C},a\in\mathbb{R}}\{2\ll g,h\gg_{\rho,*}+2a\ll h\gg_{\rho,**}$$
$$-\frac{\gamma}{2}\langle(ap_0V'(r_1)+Y_{0,1}\Gamma_g)^2\rangle_\beta-\frac{\gamma}{2}\langle(X_0\Gamma_g)^2\rangle_\beta\}.$$

We investigate several properties of the semi-norm $\ll\cdot\gg^{\frac{1}{2}}_{\beta,\gamma}$ in the next section, while in this section we prove that the variance

$$(2N)^{-1}\langle(-S^{\gamma}_{\Lambda_N})^{-1}\sum_{|i|\le N_\psi}\tau_i\psi,\sum_{|i|\le N_\psi}\tau_i\psi\rangle_{\Lambda_N,E_N}$$

of any cylinder function ψ in $\mathcal{C}_0$ converges to $\ll\psi\gg_{\beta,\gamma}$, as $N\uparrow\infty$ and $E_N\to\langle\mathcal{E}_0\rangle_\beta=\frac{1}{2\beta}+\psi(\beta)$. We are now in a position to state the main result of this section.

Proposition 2.1. *Consider a cylinder function ψ in $\mathcal{C}_0$ and a sequence of integers E_N such that $\lim_{N\to\infty}E_N=\frac{1}{2\beta}+\psi(\beta)$. Then,*

$$\lim_{N\to\infty}(2N)^{-1}\langle(-S^{\gamma}_{\Lambda_N})^{-1}\sum_{|i|\le N_\psi}\tau_i\psi,\sum_{|i|\le N_\psi}\tau_i\psi\rangle_{\Lambda_N,E_N}=\ll\psi\gg_{\beta,\gamma}.$$

The proof is divided into two lemmas.

Lemma 2.1.5. *Consider a cylinder function ψ in $\mathcal{C}_0$ and a sequence of integers E_N such that $\lim_{N\to\infty} E_N = \frac{1}{2\beta} + \psi(\beta)$. Then,*

$$\liminf_{N\to\infty}(2N)^{-1}\langle(-S^{\gamma}_{\Lambda_N})^{-1}\sum_{|i|\le N_\psi}\tau_i\psi,\sum_{|i|\le N_\psi}\tau_i\psi\rangle_{\Lambda_N,E_N}\ge\ll\psi\gg_{\beta,\gamma}.$$

Lemma 2.1.6. *Consider a cylinder function ψ in $\mathcal{C}_0$ and a sequence of integers E_N such that $\lim_{N\to\infty} E_N = \frac{1}{2\beta} + \psi(\beta)$. Then,*

$$\limsup_{N\to\infty}(2N)^{-1}\langle(-S^{\gamma}_{\Lambda_N})^{-1}\sum_{|i|\le N_\psi}\tau_i\psi,\sum_{|i|\le N_\psi}\tau_i\psi\rangle_{\Lambda_N,E_N}\le\ll\psi\gg_{\beta,\gamma}.$$

Proof of Lemma 2.1.5. For the beginning of the proof, we calculate some variances and covariances. Define $A_N := \sum_{i=-N}^{N-1}\tau_i W^S_{0,1}$ and for $F\in\mathcal{C}$, let $H^F_N := \sum_{|i|\le N-s_F-1}\tau_i S^\gamma F$. It is easy to see that

$$\begin{aligned}
&\lim_{N\to\infty}(2N)^{-1}\langle(-S^{\gamma}_{\Lambda_N})^{-1}\sum_{|i|\le N_\psi}\tau_i\psi, A_N\rangle_{\Lambda_N,E_N} = -\ll\psi\gg_{\beta,*},\\
&\lim_{N\to\infty}(2N)^{-1}\langle(-S^{\gamma}_{\Lambda_N})^{-1}\sum_{|i|\le N_\psi}\tau_i\psi, H^F_N\rangle_{\Lambda_N,E_N} = -\ll\psi,F\gg_{\beta,**},\\
&\lim_{N\to\infty}(2N)^{-1}\langle(-S^{\gamma}_{\Lambda_N})^{-1}(aA_N+H^F_N), aA_N+H^F_N\rangle_{\Lambda_N,E_N}\\
&\qquad=\frac{\gamma}{2}\langle(ap_0V'(r_1)+Y_{0,1}\Gamma_F)^2\rangle_\beta+\frac{\gamma}{2}\langle(X_0\Gamma_F)^2\rangle_\beta.
\end{aligned}$$

Then, obviously,

$$\begin{aligned}
&\liminf_{N\to\infty}(2N)^{-1}\langle(-S^{\gamma}_{\Lambda_N})^{-1}\sum_{|i|\le N_\psi}\tau_i\psi,\sum_{|i|\le N_\psi}\tau_i\psi\rangle_{\Lambda_N,E_N}\\
&\quad\ge\liminf_{N\to\infty}[2(2N)^{-1}\langle(-S^{\gamma}_{\Lambda_N})^{-1}\sum_{|i|\le N_\psi}\tau_i\psi,-(aA_N+H^F_N)\rangle_{\Lambda_N,E_N}\\
&\qquad-(2N)^{-1}\langle(-S^{\gamma}_{\Lambda_N})^{-1}(aA_N+H^F_N),(aA_N+H^F_N)\rangle_{\Lambda_N,E_N}]\\
&=2\ll\psi,F\gg_{\beta,*}+2a\ll\psi\gg_{\beta,**}-\frac{\gamma}{2}\langle(ap_0V'(r_1)+Y_{0,1}\Gamma_F)^2\rangle_\beta-\frac{\gamma}{2}\langle(X_0\Gamma_F)^2\rangle_\beta.
\end{aligned}$$

Then, taking the supremum of $a\in\mathbb{R}$ and F we obtain the desired inequality. □

Proof of Lemma 2.1.6. The key of the proof is the characterization of closed forms which is proved in Section 2.8. Once Theorem 2.7 in Section 2.8 is established, the proof of Lemma 2.1.6 is same as that of Theorem 7.4.1 in [14] since the proof depends on the specific model only through an integration by parts formula and the equivalence of ensembles, which are easily shown. □

We conclude this section proving that for each ψ in $\mathcal{C}_0$ the function $\ll \psi \gg_{\cdot,\gamma}$: $\mathbb{R}_{>0} \to \mathbb{R}_+$ that associates to each energy β the value $\ll \psi \gg_{\beta,\gamma}$ is continuous and that the convergence of the finite volume variances to $\ll \cdot \gg_{\beta,\gamma}$ is uniform on any compact set in $\mathbb{R}_{>0}$. For each N in $\mathbb{N}$ and $E > 0$, denote by $V_N^\psi(E)$ the variance of $(2N+1)^{-1}\sum_{|i|\le N_\psi} \tau_i\psi$ with respect to $\nu_{\Lambda_N,E}$:

$$V_N^\psi(E) = (2N)^{-1}\langle(-S_{\Lambda_N}^\gamma)^{-1}\sum_{|i|\le N_\psi}\tau_i\psi, \sum_{|i|\le N_\psi}\tau_i\psi\rangle_{\Lambda_N,E}.$$

With this definition V_N^ψ is continuous. Proposition 2.1 asserts that V_N^ψ converges, as $N\uparrow\infty$, to $\ll \psi \gg_{\beta,\gamma}$, for any sequence E_N such that $E_N \to \frac{1}{2\beta}+\psi(\beta)$. In particular, $\lim_{N\to\infty} V_N^\psi(E_N) = \ll \psi \gg_{\beta,\gamma}$ for any sequence $E_N \to \frac{1}{2\beta}+\psi(\beta)$. This implies that $\ll \psi \gg_\beta$ is continuous and that $V_N^\psi(\cdot)$ converges uniformly on any compact set to $\ll \psi \gg_{\cdot,\gamma}$ as $N\uparrow\infty$. We have thus proved the following theorem.

Theorem 2.2. *For each fixed h in $\mathcal{C}_0$, $\ll h \gg_{\beta,\gamma}$ is continuous as a function of β on $\mathbb{R}_+$. Moreover, the variance*

$$(2N)^{-1}\langle(-S_{\Lambda_N}^\gamma)^{-1}\sum_{|i|\le N_h}\tau_i h, \sum_{|i|\le N_h}\tau_i h\rangle_{\Lambda_N,E_N}$$

converges uniformly to $\ll h \gg_{\beta,\gamma}$ as $N\uparrow\infty$ and $E_N \to \frac{1}{2\beta}+\psi(\beta)$.

Now by the argument in [12], in order to prove Lemma 2.1.3, all we have to show is that there exists a sequence of functions $\{F_k\}_k$ in $\mathcal{C}$ satisfying that $\lim_{k\to\infty} \ll W_{0,1} + \tilde{D}(\beta)(p_1^2 - p_0^2) - L^\gamma F_k \gg_{\beta,\gamma} = 0$ and for this sequence F_K,

$$\lim_{K\to\infty} E_{\nu_\beta}[(\sigma_{0,1} + \sqrt{\gamma}Y_{0,1}(\Gamma_{F_K}))^2 + (\sqrt{\gamma}X_0(\Gamma_{F_K}))^2] = \frac{2\tilde{D}(\beta)}{\beta^2}.$$

We will prove this in the next section and also give some detailed estimates of $\tilde{D}(\beta)$.

2.6 Hilbert space

In this section, we investigate the main properties of the semi norm $\ll \cdot \gg_{\beta,\gamma}$ introduced in the previous section. We first define from $\ll \cdot \gg_{\beta,\gamma}$ a semi-inner product on $\mathcal{C}_0$ through polarization:

$$\ll g,h \gg_{\beta,\gamma} = \frac{1}{4}\{\ll g+h \gg_{\beta,\gamma} - \ll g-h \gg_{\beta,\gamma}\}. \tag{2.6.1}$$

It is easy to check that (2.6.1) defines a semi-inner product on $\mathcal{C}_0$. Denote by $\mathcal{N}_{\beta,\gamma}$ the kernel of the semi-norm $\ll \cdot \gg_{\beta,\gamma}^{\frac{1}{2}}$ on $\mathcal{C}_0$. Since $\ll \cdot \gg_{\beta,\gamma}$ is a semi-inner product on $\mathcal{C}_0$, the completion of $\mathcal{C}_0|_{\mathcal{N}_{\beta,\gamma}}$, denoted by $\mathcal{H}_{\beta,\gamma}$, is a Hilbert space.

By Lemma 2.1.4, the linear space generated by $W_{0,1}^S$ and $S^\gamma\mathcal{C} := \{S^\gamma g;\ g \in \mathcal{C}\}$ are subsets of $\mathcal{C}_0$. The first main result of this subsection consists in showing that $\mathcal{H}_{\beta,\gamma}$ is the completion of $S^\gamma\mathcal{C}|_{\mathcal{N}_{\beta,\gamma}} + \{W_{0,1}^S\}$, in other words, that all elements of $\mathcal{H}_{\beta,\gamma}$ can be approximated by $aW_{0,1}^S + S^\gamma g$ for some a in $\mathbb{R}$ and g in $\mathcal{C}$. To prove this result we derive two elementary identities:

$$(2.6.2) \qquad \ll h, S^\gamma g \gg_{\beta,\gamma} = - \ll h, g \gg_{\beta,*}, \quad \ll h, W_{0,1}^S \gg_{\beta,\gamma} = - \ll h \gg_{\beta,**}$$

for all h in $\mathcal{C}_0$ and g in $\mathcal{C}$.

By Proposition 2.1 and (2.6.1), the semi-inner product $\ll h, g \gg_{\beta,\gamma}$ is the limit of the covariance $(2N)^{-1}\langle(-S_{\Lambda_N}^\gamma)^{-1}\sum_{|i|\le N_g}\tau_i g, \sum_{|i|\le N_h}\tau_i h\rangle_{\Lambda_N,E_N}$ as $N \uparrow \infty$ and $E_N \to \frac{1}{2\beta} + \psi(\beta)$. In particular, if $g = S^\gamma g_0$, for some g_0 in $\mathcal{C}$, the inverse of the operator S^γ cancels with the operator S^γ. Therefore, $\ll h, S^\gamma g_0 \gg_{\beta,\gamma}$ is equal to

$$-\lim_{N\to\infty}(2N)^{-1}\langle \sum_{|i|\le N_{g_0}} \tau_i g_0, \sum_{|i|\le N_h} \tau_i h\rangle_{N,E_N} = \ll g_0, h \gg_{\beta,*} .$$

The second identity is proved by similar way with the elementary relation $S_{\Lambda_N}^\gamma \sum_{i\in\Lambda_N} i\mathcal{E}_i = \sum_{i,i+1\in\Lambda_N} W_{i,i+1}^S$.

The identities of (2.6.2) permit to compute the following elementary relations

$$\begin{aligned}\ll W_{0,1}^S, S^\gamma h \gg_{\beta,\gamma} &= -\sum_{i\in\mathbb{Z}} i\langle \mathcal{E}_i S^\gamma h\rangle_\beta \\ &= -\gamma\sum_{i\in\mathbb{Z}} i\langle \mathcal{E}_i[(Y_{i-1,i})^2 + (Y_{i,i+1})^2]h\rangle_\beta = \gamma\langle p_0 V'(r_1) Y_{0,1}\Gamma_h\rangle_\beta,\end{aligned}$$

$$\ll p_1^2 - p_0^2, S^\gamma h \gg_{\beta,\gamma} = 0$$

for all $h \in \mathcal{C}$, and

$$\ll W_{0,1}^S, W_{0,1}^S \gg_{\beta,\gamma} = \frac{\gamma}{2}\langle(p_0V'(r_1))^2\rangle_\beta, \quad \ll W_{0,1}^S, p_1^2 - p_0^2 \gg_{\beta,\gamma} = -\frac{1}{\beta^2}.$$

Furthermore,

$$\ll aW_{0,1}^S + S^\gamma g \gg_{\beta,\gamma} = \frac{\gamma}{2}\langle(ap_0V'(r_1) + Y_{0,1}\Gamma_g)^2\rangle_\beta + \frac{\gamma}{2}\langle(X_0\Gamma_g)^2\rangle_\beta$$

for a in $\mathbb{R}$ and g in $\mathcal{C}$. In particular, the variational formula for $\ll h \gg_{\beta,\gamma}$ is reduced to the expression

$$\ll h \gg_{\beta,\gamma} = \sup_{a\in\mathbb{R}, g\in\mathcal{C}}\{-2 \ll h, aW_{0,1}^S + S^\gamma g \gg_{\beta,\gamma} - \ll aW_{0,1}^S + S^\gamma g \gg_{\beta,\gamma}\}.$$

Proposition 2.2. *Recall that we denote by $S^\gamma \mathcal{C}$ the space $\{S^\gamma g;\ g \in \mathcal{C}\}$. Then, for each $\beta > 0$, we have*

$$\mathcal{H}_{\beta,\gamma} = \overline{S^\gamma \mathcal{C}}|_{\mathcal{N}_{\beta,\gamma}} \oplus \{W_{0,1}^S\}.$$

Proof. We can apply the proof of Proposition 7.5.2 in [14] straightforwardly. □

Corollary 2.2.1. *For each $\beta > 0$, we have*

$$\mathcal{H}_{\beta,\gamma} = \overline{S^\gamma \mathcal{C}}|_{\mathcal{N}_{\beta,\gamma}} \oplus \{p_1^2 - p_0^2\}.$$

Proof. Straightforward. □

Next, to replace the space $S^\gamma \mathcal{C}$ by $L^\gamma \mathcal{C}$, we show some useful lemmas.

Lemma 2.2.1. *For all $g, h \in \mathcal{C}$ and $\beta > 0$, $\ll S^\gamma g, Ah \gg_{\rho,\gamma} = - \ll Ag, S^\gamma h \gg_{\beta,\gamma}$. Especially, $\ll S^\gamma g, Ag \gg_{\beta,\gamma} = 0$.*

Proof. By the first identity of (2.6.2),

$$\begin{aligned}
\ll S^\gamma g, Ah \gg_{\beta,\gamma} &= - \ll g, Ah \gg_{\beta,*} = - \sum_{i\in\mathbb{Z}} \langle \tau_i g, Ah \rangle_\beta \\
&= \sum_{i\in\mathbb{Z}} \langle A\tau_i g, h \rangle_\beta = \sum_{i\in\mathbb{Z}} \langle \tau_i Ag, h \rangle_\beta \\
&= \sum_{i\in\mathbb{Z}} \langle Ag, \tau_{-i} h \rangle_\beta = \sum_{i\in\mathbb{Z}} \langle Ag, \tau_i h \rangle_\beta = - \ll Ag, S^\gamma h \gg_{\beta,\gamma} .
\end{aligned}$$

This concludes the proof. □

Lemma 2.2.2. *For all $g \in \mathcal{C}$ and $\beta > 0$, $\ll S^\gamma g, W_{0,1}^A \gg_{\beta,\gamma} = - \ll Ag, W_{0,1}^S \gg_{\beta,\gamma}$.*

Proof. By the first identity of (2.6.2),

$$\begin{aligned}
\ll S^\gamma g, W_{0,1}^A \gg_{\beta,\gamma} &= - \ll g, W_{0,1}^A \gg_{\beta,*} = - \sum_{i\in\mathbb{Z}} \langle \tau_i g, W_{0,1}^A \rangle_\beta \\
&= - \sum_{i\in\mathbb{Z}} \langle g, W_{i,i+1}^A \rangle_\beta = - \sum_{i\in\mathbb{Z}} i \langle g, W_{i-1,i}^A - W_{i,i+1}^A \rangle_\beta \\
&= - \sum_{i\in\mathbb{Z}} i \langle g, A\mathcal{E}_i \rangle_\beta = \sum_{i\in\mathbb{Z}} i \langle Ag, \mathcal{E}_i \rangle_\rho = - \ll Ag, W_{0,1}^S \gg_{\beta,\gamma} .
\end{aligned}$$

This concludes the proof. □

Lemma 2.2.3. *For all $a \in \mathbb{R}$ and $g \in \mathcal{C}$ and $\beta > 0$,*

$$\ll aW_{0,1}^S + S^\gamma g, aW_{0,1}^A + Ag \gg_{\beta,\gamma} = 0.$$

Proof. By the second identity of (2.6.2), it is easy to see that $\ll W_{0,1}^S, W_{0,1}^A \gg_{\beta,\gamma}= 0$. Then, Lemma 2.2.1 and Lemma 2.2.2 concludes the proof straightforwardly. □

Proposition 2.3. *There exists a positive constant* $C_{\beta,\gamma}$ *such that for all* $g \in \mathcal{C}$, $\ll Ag \gg_{\beta,\gamma} \leq C_{\beta,\gamma} \ll S^\gamma g \gg_{\beta,\gamma}$.

Proof. By Proposition 2.2, we have the following variational formula for $\ll Ag \gg_{\beta,\gamma}$,

$$\begin{aligned}\ll Ag \gg_{\beta,\gamma} &= \sup_{a\in\mathbb{R}, h\in\mathcal{C}} \{\frac{\ll Ag, aW_{0,1}^S + S^\gamma h \gg_{\beta,\gamma}^2}{\ll aW_{0,1}^S + S^\gamma h \gg_{\beta,\gamma}}\} \\ &= \max\left\{ \sup_{h\in\mathcal{C}}\{\frac{\ll Ag, S^\gamma h \gg_{\beta,\gamma}^2}{\ll S^\gamma h \gg_{\beta,\gamma}}\}, \sup_{a\neq 0, h\in\mathcal{C}}\{\frac{\ll Ag, aW_{0,1}^S + S^\gamma h \gg_{\beta,\gamma}^2}{\ll aW_{0,1}^S + S^\gamma h \gg_{\beta,\gamma}}\}\right\} \\ &= \max\left\{ \sup_{h\in\mathcal{C}}\{\frac{\ll Ag, S^\gamma h \gg_{\beta,\gamma}^2}{\ll S^\gamma h \gg_{\beta,\gamma}}\}, \sup_{h\in\mathcal{C}}\{\frac{\ll Ag, W_{0,1}^S + S^\gamma h \gg_{\beta,\gamma}^2}{\ll W_{0,1}^S + S^\gamma h \gg_{\beta,\gamma}}\}\right\}.\end{aligned}$$

By Lemma 2.2.8 in the next subsection, there exists a positive constant C_γ such that $\ll Ag, S^\gamma h \gg_{\beta,\gamma}^2 \leq C_\gamma \ll S^\gamma h \gg_{\beta,\gamma} \ll S^\gamma g \gg_{\beta,\gamma}$ for all $g, h \in \mathcal{C}$. Therefore, we have

$$\sup_{h\in\mathcal{C}}\{\frac{\ll Ag, S^\gamma h \gg_{\beta,\gamma}^2}{\ll S^\gamma h \gg_{\beta,\gamma}}\} \leq C_\gamma \ll S^\gamma g \gg_{\beta,\gamma} .$$

On the other hand, by Lemma 2.2.2, we have $\ll Ag, W_{0,1}^S \gg_{\beta,\gamma}^2 = \ll S^\gamma g, W_{0,1}^A \gg_{\beta,\gamma}^2 \leq \ll S^\gamma g \gg_{\beta,\gamma} \ll W_{0,1}^A \gg_{\beta,\gamma}$. Therefore,

$$\begin{aligned}&\sup_{h\in\mathcal{C}}\{\frac{\ll Ag, W_{0,1}^S + S^\gamma h \gg_{\beta,\gamma}^2}{\ll W_{0,1}^S + S^\gamma h \gg_{\beta,\gamma}}\} \\ &\qquad \leq \ll S^\gamma g \gg_{\beta,\gamma} \sup_{h\in\mathcal{C}}\{\frac{2 \ll W_{0,1}^A \gg_{\beta,\gamma} + 2C_\gamma \ll S^\gamma h \gg_{\beta,\gamma}}{\ll W_{0,1}^S + S^\gamma h \gg_{\beta,\gamma}}\}.\end{aligned}$$

Now, we only have to show that

$$\sup_{h\in\mathcal{C}}\{\frac{1}{\ll W_{0,1}^S + S^\gamma h \gg_{\beta,\gamma}}\} < \infty, \quad \sup_{h\in\mathcal{C}}\{\frac{\ll S^\gamma h \gg_{\beta,\gamma}}{\ll W_{0,1}^S + S^\gamma h \gg_{\beta,\gamma}}\} < \infty.$$

The first inequality follows from Proposition 2.2. To prove the second identity, since we have the first inequality, it is enough to show that

$$\sup_{\substack{t\geq 2, h\in\mathcal{C} \\ \ll S^\gamma h \gg_{\beta,\gamma} = t \ll W_{0,1}^S \gg_{\beta,\gamma}}} \left\{\frac{\ll S^\gamma h \gg_{\beta,\gamma}}{\ll W_{0,1}^S + S^\gamma h \gg_{\beta,\gamma}}\right\} < \infty.$$

The triangle inequality shows that

$$\sqrt{\ll W_{0,1}^S + S^\gamma h \gg_{\beta,\gamma}} \geq \sqrt{\ll S^\gamma h \gg_{\beta,\gamma}} - \sqrt{\ll W_{0,1}^S \gg_{\beta,\gamma}} = (\sqrt{t}-1)\sqrt{\ll W_{0,1}^S \gg_{\beta,\gamma}}$$

for any h satisfying $\ll S^\gamma h \gg_{\beta,\gamma} = t \ll W_{0,1}^S \gg_{\beta,\gamma}$. Then, we obtain that

$$\sup_{\substack{t\geq 2, h\in\mathcal{C} \\ \ll S^\gamma h\gg_{\beta,\gamma}=t\ll W_{0,1}^S\gg_{\beta,\gamma}}} \left\{\frac{\ll S^\gamma h \gg_{\beta,\gamma}}{\ll W_{0,1}^S + S^\gamma h \gg_{\beta,\gamma}}\right\} \leq \sup_{t\geq 2}\{\frac{t}{(\sqrt{t}-1)^2}\} < \infty.$$

□

Now, we have all elements to show the desired decomposition of the Hilbert spaces $\mathcal{H}_{\beta,\gamma}$.

Proposition 2.4. *Denote by $L^\gamma\mathcal{C}$ the space $\{L^\gamma g;\ g \in \mathcal{C}\}$. Then, for each $\beta > 0$, we have*

$$\mathcal{H}_{\beta,\gamma} = \overline{L^\gamma\mathcal{C}}|_{\mathcal{N}_{\beta,\gamma}} + \{p_1^2 - p_0^2\}.$$

Proof. Since $\{p_1^2 - p_0^2\}$ and $L^\gamma\mathcal{C}$ are contained in $\mathcal{C}_0$ by definition, $\mathcal{H}_{\beta,\gamma}$ contains the right hand space. To prove the converse inclusion, let $h \in \mathcal{H}_{\beta,\gamma}$ so that $\ll h, p_1^2 - p_0^2 \gg_{\beta,\gamma} = 0$ and $\ll h, L^\gamma g \gg_{\beta,\gamma} = 0$ for all $g \in \mathcal{C}$. By Corollary 2.2.1, $h = a(p_1^2 - p_0^2) + \lim_{k\to\infty} S^\gamma h_k$ in $\mathcal{H}_{\beta,\gamma}$ for some $a \in \mathbb{R}$ and $h_k \in \mathcal{C}$ and by the assumption $a = 0$. Namely, $\ll h \gg_{\beta,\gamma} = \lim_{k\to\infty} \ll S^\gamma h_k, S^\gamma h_k \gg_{\beta,\gamma} = \lim_{k\to\infty} \ll S^\gamma h_k, L^\gamma h_k \gg_{\beta,\gamma}$ since $\ll S^\gamma h_k, A h_k \gg_{\beta,\gamma} = 0$ by Lemma 2.2.1. On the other hand, by the assumption $\ll h, L^\gamma h_k \gg_{\beta,\gamma} = 0$ for all k. Also, by Proposition 2.3, $\sup_k \ll L^\gamma h_k \gg_{\beta,\gamma} \leq (C_{\beta,\gamma} + 1) \sup_k \ll S^\gamma h_k \gg_{\beta,\gamma} := C_h$ is finite. Therefore, $\ll h \gg_{\beta,\gamma} = \lim_{k\to\infty} \ll S^\gamma h_k, L^\gamma h_k \gg_{\beta,\gamma} = \lim_{k\to\infty} \ll S^\gamma h_k - h, L^\gamma h_k \gg_{\beta,\gamma} \leq \lim_{k\to\infty} \sqrt{C_h \ll S^\gamma h_k - h \gg_{\beta,\gamma}} = 0$. This concludes the proof. □

Lemma 2.2.4. *For each $\beta > 0$, we have*

$$\mathcal{H}_{\beta,\gamma} = \overline{L^\gamma\mathcal{C}}|_{\mathcal{N}_{\beta,\gamma}} \oplus \{p_1^2 - p_0^2\}.$$

Proof. Let a sequence $g_k \in \mathcal{C}$ satisfy $\lim_{k\to\infty} L^\gamma g_k = a(p_1^2 - p_0^2)$ in $\mathcal{H}_{\beta,\gamma}$ for some $a \in \mathbb{R}$. By the similar argument of the proof of Proposition 2.4, $\limsup_{k\to\infty} \ll S^\gamma g_k, S^\gamma g_k \gg_{\beta,\gamma} = \limsup_{k\to\infty} \ll L^\gamma g_k, S^\gamma g_k \gg_{\beta,\gamma} = \limsup_{k\to\infty} \ll L^\gamma g_k - a(p_1^2 - p_0^2), S^\gamma g_k \gg_{\beta,\gamma} = 0$ since $\ll p_1^2 - p_0^2, S^\gamma g_k \gg_{\beta,\gamma} = 0$ for all k. On the other hand, by Proposition 2.3, $\ll L^\gamma g_k \gg_{\beta,\gamma} \leq (C_{\beta,\gamma} + 1) \ll S^\gamma g_k \gg_{\beta,\gamma}$, then $a = 0$. □

Corollary 2.2.2. *For each $g \in \mathcal{C}_0$, there exists a unique constant $a \in \mathbb{R}$ such that*

$$g + a(p_1^2 - p_0^2) \in \overline{L^\gamma\mathcal{C}} \quad \textit{in} \quad \mathcal{H}_{\beta,\gamma}.$$

2.6.1 Sector condition in $\mathcal{H}_{\beta,\gamma}$

In this section, to obtain the sector condition in $\mathcal{H}_{\beta,\gamma}$, we study the special structure of the space $\mathcal{C}_0$. Roughly speaking the space $\mathcal{C}_0$ is divided in the countable spaces which are orthogonal to each other in $L^2(\nu_\beta)$.

First, for each finite subset of $\mathbb{Z}$, we define a subspace $\mathbb{L}_\Lambda$ of $\mathcal{C}$ as a set of functions f satisfying that f is an odd function as a function of p_i for all $i \in \Lambda$ and f is an even function as a function of p_i for all $i \notin \Lambda$:

$$\mathbb{L}_\Lambda = \{f \in \mathcal{C}; f((p^i, r)) = -f((p,r)) \text{ for all } i \in \Lambda, f((p^i,r)) = f((p,r)) \text{ for all } i \notin \Lambda\}$$

where $p^i = (..., p_{i-2}, p_{i-1}, -p_i, p_{i+1}, p_{i+2}...)$. Then, it is obvious that $\mathcal{C} = \oplus_{|\Lambda|<\infty}\mathbb{L}_\Lambda$, and $\mathbb{L}_\Lambda$ and $\mathbb{L}_{\tilde{\Lambda}}$ are orthogonal in $L^2(\nu_\beta)$ if $\Lambda \neq \tilde{\Lambda}$. Next, we define $\mathbb{L}_i$ as a direct sum of $\mathbb{L}_\Lambda$ satisfying $|\Lambda| = i$: $\mathbb{L}_i = \oplus_{|\Lambda|=i}\mathbb{L}_\Lambda$. Obviously, $\mathbb{L}_i$ and $\mathbb{L}_j$ are orthogonal in $L^2(\nu_\beta)$ if $i \neq j$.

Now, we prepare two easy but useful lemmas.

Lemma 2.2.5. *For all $f, g \in \mathcal{C}$, $\ll S^\gamma f, Ag \gg_{\beta,\gamma} = -\langle \Gamma_f (X_0 - Y_{0,1})\Gamma_g\rangle_\beta = \langle \Gamma_g(X_0 - Y_{0,1})\Gamma_f\rangle_\beta$.*

Proof. By the first identity of (2.6.2),

$$\begin{aligned}
- \ll S^\gamma f, Ag \gg_{\beta,\gamma} &= \ll f, Ag \gg_{\beta,*} = \sum_{i\in\mathbb{Z}} \langle \tau_i f, Ag\rangle_\beta = \sum_{i,j\in\mathbb{Z}} \langle \tau_i f, (X_j - Y_{j,j+1})g\rangle_\beta \\
&= \sum_{i,k\in\mathbb{Z}} \langle \tau_i f, (X_{k+i} - Y_{k+i,k+i+1})g\rangle_\beta = \sum_{i,k\in\mathbb{Z}} \langle \tau_i f, \tau_i((X_k - Y_{k,k+1})\tau_{-i}g)\rangle_\beta \\
&= \sum_{k\in\mathbb{Z}} \langle f, (X_k - Y_{k,k+1})\Gamma_g\rangle_\beta = \sum_{k\in\mathbb{Z}} \langle \tau_k f, (X_0 - Y_{0,1})\Gamma_g\rangle_\beta = \langle \Gamma_f(X_0 - Y_{0,1})\Gamma_g\rangle_\beta.
\end{aligned}$$

□

Lemma 2.2.6. *For all nonnegative integers i, $f \in \mathbb{L}_i$ and $k \in \mathbb{Z}$, $S^\gamma f, \tau_k f \in \mathbb{L}_i$ and $Af \in \mathbb{L}_{i-1} \oplus \mathbb{L}_{i+1}$. Here $\mathbb{L}_{-1} := \phi$. Therefore, for all $f, g \in \mathcal{C}$,*

$$\begin{aligned}
\ll S^\gamma f \gg_{\beta,\gamma} &= \sum_{i=0}^{\infty} \ll S^\gamma f_i \gg_{\beta,\gamma}, \\
\ll S^\gamma f, Ag \gg_{\beta,\gamma} &= \sum_{i=0}^{\infty} \ll S^\gamma f_i, Ag_{i+1} \gg_{\beta,\gamma} + \sum_{i=0}^{\infty} \ll L^{\gamma,S} f_{i+1}, L^A g_i \gg_{\beta,\gamma} \\
&= -\sum_{i=0}^{\infty} \ll S^\gamma g_{i+1}, Af_i \gg_{\beta,\gamma} + \sum_{i=0}^{\infty} \ll S^\gamma f_{i+1}, Ag_i \gg_{\beta,\gamma}
\end{aligned}$$

where f_i and g_i are the projections of f and g to the space $\mathbb{L}_i$ respectively.

Proof. Straightforward. □

Next lemma gives us the essential estimates to prove our main result in this subsection.

Lemma 2.2.7. *There exists a positive constant C such that for all nonnegative integers i, $f_{i+1} \in \mathbb{L}_{i+1}$ and $g_i \in \mathbb{L}_i$,*

$$|\langle \Gamma_{f_{i+1}}(X_0 - Y_{0,1})\Gamma_{g_i}\rangle_\beta| \le C\sqrt{\langle (X_0\Gamma_{f_{i+1}})^2\rangle_\beta \langle (X_0\Gamma_{g_i})^2 + (Y_{0,1}\Gamma_{g_i})^2\rangle_\beta}.$$

Proof. For any f_{i+1} we have the unique decomposition of $\Gamma_{f_{i+1}}$ such that

$$\Gamma_{f_{i+1}} = \sum_{\Lambda \subset \mathbb{Z}_+, 0\in\Lambda, |\Lambda|=i+1} \Gamma_{f_{i+1,\Lambda}}$$

where $f_{i+1,\Lambda} \in \mathbb{L}_\Lambda$. Since we take the index set as $\{\Lambda \subset \mathbb{Z}_+, 0 \in \Lambda, |\Lambda| = i+1\}$, we obtain the uniqueness of the decomposition. Note that all but a finite number of $f_{i+1,\Lambda}$ are 0. Therefore

$$\begin{aligned}\langle \Gamma_{f_{i+1}}(X_0 - Y_{0,1})\Gamma_{g_i}\rangle_\beta &= \sum_\Lambda \langle \Gamma_{f_{i+1,\Lambda}}(X_0 - Y_{0,1})\Gamma_{g_i}\rangle_\beta \\ &= \sum_\Lambda \langle \sum_{z\in\Lambda} \tau_{-z} f_{i+1,\Lambda}(X_0 - Y_{0,1})\Gamma_{g_i}\rangle_\beta\end{aligned}$$

since for all $z \notin \Lambda$ and $k \in \mathbb{Z}$, $\langle \tau_{-z} f_{i+1,\Lambda}(X_0 - Y_{0,1})\tau_k g_i\rangle_\beta = 0$.

By Schwarz inequality, the last expression is bounded from above by

$$\sqrt{\langle (\sum_\Lambda \sum_{z\in\Lambda} \tau_{-z} f_{i+1,\Lambda})^2\rangle_\beta \langle [(X_0 - Y_{0,1})\Gamma_{g_i}]^2\rangle_\beta}.$$

Since $\sum_\Lambda \sum_{z\in\Lambda} \tau_{-z} f_{i+1,\Lambda}$ is an odd function as a function of p_0, by the spectral gap estimate for the one-site dynamics, the last expression is bounded from above by

$$\begin{aligned}&\le C\sqrt{\langle \{X_0(\sum_\Lambda \sum_{z\in\Lambda} \tau_{-z} f_{i+1,\Lambda})\}^2\rangle_\beta \langle (X_0\Gamma_{g_i})^2 + (Y_{0,1}\Gamma_{g_i})^2\rangle_\beta} \\ &\le C\sqrt{\langle (X_0\Gamma_{f_{i+1}})^2\rangle_\beta \langle (X_0\Gamma_{g_i})^2 + (Y_{0,1}\Gamma_{g_i})^2\rangle_\beta}\end{aligned}$$

for some positive constant C. Here, the second inequality follows from the relation

$$\langle (X_0\Gamma_{f_{i+1}})^2\rangle_\beta = \langle (X_0 \sum_\Lambda \Gamma_{f_{i+1,\Lambda}})^2\rangle_\beta$$

$$
\begin{aligned}
&= \langle \{X_0(\sum_{\Lambda}\sum_{z\in\Lambda}\tau_{-z}f_{i+1,\Lambda}) + X_0(\sum_{\Lambda}\sum_{z\notin\Lambda}\tau_{-z}f_{i+1,\Lambda})\}^2\rangle_\beta \\
&= \langle \{X_0(\sum_{\Lambda}\sum_{z\in\Lambda}\tau_{-z}f_{i+1,\Lambda})\}^2\rangle_\beta + \langle \{X_0(\sum_{\Lambda}\sum_{z\notin\Lambda}\tau_{-z}f_{i+1,\Lambda})\}^2\rangle_\beta \\
&\geq \langle \{X_0(\sum_{\Lambda}\sum_{z\in\Lambda}\tau_{-z}f_{i+1,\Lambda})\}^2\rangle_\beta
\end{aligned}
$$

since $X_0(\sum_\Lambda \sum_{z\in\Lambda}\tau_{-z}f_{i+1,\Lambda}) \in \mathbb{L}_i$ and $X_0(\sum_\Lambda \sum_{z\notin\Lambda}\tau_{-z}f_{i+1,\Lambda}) \in \mathbb{L}_{i+2}$. □

Now we show the main result in this subsection.

Lemma 2.2.8 (sector condition). *There exists a positive constant C_γ such that for all $f, g \in \mathcal{C}$,*

$$
| \ll S^\gamma f, Ag \gg_{\beta,\gamma} | \leq C_\gamma \sqrt{\ll S^\gamma f \gg_{\beta,\gamma} \ll S^\gamma g \gg_{\beta,\gamma}}.
$$

Proof. By Lemma 2.2.5 and 2.2.7,

$$
\begin{aligned}
| \ll S^\gamma f, Ag \gg_{\beta,\gamma} | &= | -\sum_{i=0}^{\infty} \ll S^\gamma g_{i+1}, Af_i \gg_{\beta,\gamma} + \sum_{i=0}^{\infty} \ll S^\gamma f_{i+1}, Ag_i \gg_{\beta,\gamma} | \\
&\leq \sum_{i=0}^{\infty} |\langle \Gamma_{g_{i+1}}(X_0 - Y_{0,1})\Gamma_{f_i}\rangle_\rho| + \sum_{i=0}^{\infty} |\langle \Gamma_{f_{i+1}}(X_0 - Y_{0,1})\Gamma_{g_i}\rangle_\rho| \\
&\leq C\sum_{i=0}^{\infty} \sqrt{\langle (X_0\Gamma_{g_{i+1}})^2\rangle_\beta \langle (X_0\Gamma_{f_i})^2 + (Y_{0,1}\Gamma_{f_i})^2\rangle_\beta} \\
&+ C\sum_{i=0}^{\infty} \sqrt{\langle (X_0\Gamma_{f_{i+1}})^2\rangle_\beta \langle (X_0\Gamma_{g_i})^2 + (Y_{0,1}\Gamma_{g_i})^2\rangle_\beta} \\
&\leq C\frac{4}{\gamma}\sqrt{\ll S^\gamma f \gg_{\beta,\gamma} \ll S^\gamma g \gg_{\beta,\gamma}}
\end{aligned}
$$

since $\ll S^\gamma f \gg_{\beta,\gamma} = \sum_{i=0}^{\infty} \frac{\gamma}{2}[\langle (X_0\Gamma_{f_i})^2\rangle_\beta + \langle (Y_{0,1}\Gamma_{f_i})^2\rangle_\beta]$. □

2.6.2 Diffusion coefficient

We now start to describe the diffusion coefficient. From Corollary 2.2.2, there exists a unique number $\tilde{D}^\gamma(\beta)(= \tilde{D}(\beta)$ in Lemma 2.1.3) such that

$$
W_{0,1} + \tilde{D}^\gamma(\beta)(p_1^2 - p_0^2) \in \overline{L^\gamma \mathcal{C}} \quad \text{in} \quad \mathcal{H}_{\beta,\gamma}.
$$

Our purpose now is to obtain the explicit formula for $\tilde{D}^\gamma$. To do this, we follow the argument in [17].

Lemma 2.2.9. *For each $\beta > 0$, we have*

$$\mathcal{H}_{\beta,\gamma} = \overline{L^\gamma \mathcal{C}}|_{\mathcal{N}_\beta} \oplus \{W_{0,1}\} = \overline{L^{\gamma,*}\mathcal{C}}|_{\mathcal{N}_\beta} \oplus \{W^*_{0,1}\}$$

where $W^*_{0,1} := W^S_{0,1} - W^A_{0,1}$ *and* $L^{\gamma,*} = S^\gamma - A$.

Proof. We shall prove the first decomposition since the same arguments apply to the second one. Because we already prove in Lemma 2.2.4 that $\overline{L^\gamma \mathcal{C}}|_{\mathcal{N}_\beta}$ has a one-dimensional complementary subspace in $\mathcal{H}_{\beta,\gamma}$, it is sufficient to show that $\mathcal{H}_{\beta,\gamma}$ is generated by $\overline{L^\gamma \mathcal{C}}$ and the current. Let $h \in \mathcal{H}_{\beta,\gamma}$ so that $\ll h, W_{0,1} \gg_{\beta,\gamma} = 0$ and $\ll h, L^\gamma g \gg_{\beta,\gamma} = 0$ for all $g \in \mathcal{C}$. By Proposition 2.2, $h = \lim_{k\to\infty}(aW^S_{0,1} + S^\gamma h_k)$ in $\mathcal{H}_{\beta,\gamma}$ for some $a \in \mathbb{R}$ and $h_k \in \mathcal{C}$. Especially $\ll h \gg_{\beta,\gamma} = \lim_{k\to\infty} \ll aW^S_{0,1} + S^\gamma h_k, aW^S_{0,1} + S^\gamma h_k \gg_{\beta,\gamma} = \lim_{k\to\infty} \ll aW^S_{0,1} + S^\gamma h_k, aW_{0,1} + L^\gamma h_k \gg_{\beta,\gamma}$ since $\ll aW^S_{0,1} + S^\gamma h_k, aW^A_{0,1} + Ah_k \gg_{\beta,\gamma} = 0$ by Lemma 2.2.3. On the other hand, by assumption $\ll h, aW_{0,1} + L^\gamma h_k \gg_{\beta,\gamma} = 0$ for all k. Also, by Proposition 2.3, $\sup_k \ll aW_{0,1} + L^\gamma h_k \gg_{\beta,\gamma} \le 2a^2 \ll W_{0,1} \gg_{\beta,\gamma} + 2(C_{\beta,\gamma}+1)\sup_k \ll S^\gamma h_k \gg_{\beta,\gamma} := C_h$ is finite. Therefore, $\ll h \gg_{\beta,\gamma} = \lim_{k\to\infty} \ll aW^S_{0,1} + S^\gamma h_k, aW_{0,1} + L^\gamma h_k \gg_{\beta,\gamma} = \lim_{k\to\infty} \ll aW^S_{0,1} + S^\gamma h_k - h, aW_{0,1} + L^\gamma h_k \gg_{\beta,\gamma} \le \limsup_{k\to\infty} \sqrt{C_h \ll aW^S_{0,1} + S^\gamma h_k - h \gg_{\beta,\gamma}} = 0$. This concludes the proof. □

Now, we can define bounded linear operators $T : \mathcal{H}_{\beta,\gamma} \to \mathcal{H}_{\beta,\gamma}$ and $T^* : \mathcal{H}_{\beta,\gamma} \to \mathcal{H}_{\beta,\gamma}$ as

$$T(aW_{0,1} + L^\gamma f) := aW^S_{0,1} + S^\gamma f, \quad T^*(aW^*_{0,1} + L^{\gamma,*} f) := aW^S_{0,1} + S^\gamma f$$

since $\ll aW_{0,1} + L^\gamma f \gg_{\beta,\gamma} = \ll aW^*_{0,1} + L^{\gamma,*} f \gg_{\beta,\gamma} = \ll aW^S_{0,1} + S^\gamma f \gg_{\beta,\gamma} + \ll aW^A_{0,1} + Af \gg_{\beta,\gamma}$. We can easily show that T^* is the adjoint operator of T and also we have the relations

$$\ll T(p_1^2 - p_0^2), W^*_{0,1} \gg_{\beta,\gamma} = \ll T^*(p_1^2 - p_0^2), W_{0,1} \gg_{\beta,\gamma} = -\frac{1}{\beta^2},$$

and

$$\ll T(p_1^2 - p_0^2), L^{\gamma,*} f \gg_{\beta,\gamma} = \ll T^*(p_1^2 - p_0^2), L^\gamma f \gg_{\beta,\gamma} = 0$$

for all $f \in \mathcal{H}_{\beta,\gamma}$. Especially, $\mathcal{H}_{\beta,\gamma} = \overline{L^{\gamma,*}\mathcal{C}}|_{\mathcal{N}_{\beta,\gamma}} \oplus \{T(p_1^2 - p_0^2)\}$ and there exists a unique number $Q^\gamma(\beta)$ such that

$$W^*_{0,1} + Q^\gamma(\beta) T(p_1^2 - p_0^2) \in \overline{L^{\gamma,*}\mathcal{C}} \quad \text{in} \quad \mathcal{H}_{\beta,\gamma}.$$

Proposition 2.5.

$$Q^{\gamma}(\beta) = \frac{1}{\beta^2 \ll T(p_1^2 - p_0^2) \gg_{\beta,\gamma}} = \beta^2 \inf_{f \in \mathcal{C}} \ll W_{0,1}^* - L^* f \gg_{\beta,\gamma} . \tag{2.6.3}$$

Proof. First identity follows from the fact that

$$\ll T(p_1^2 - p_0^2), W_{0,1}^* + Q^{\gamma}(\beta) T(p_1^2 - p_0^2) \gg_{\beta,\gamma} = -\frac{1}{\beta^2} + Q^{\gamma}(\beta) \ll T(p_1^2 - p_0^2) \gg_{\beta,\gamma} = 0.$$

Second identity is obtained by the expression

$$\inf_{f \in \mathcal{C}} \{ \ll W_{0,1}^* + Q^{\gamma}(\beta) T(p_1^2 - p_0^2) - L^{\gamma,*} f \gg_{\beta,\gamma} \} = 0$$

since

$$\begin{aligned} &\inf_{f \in \mathcal{C}} \{ \ll W_{0,1}^* + Q^{\gamma}(\beta) T(p_1^2 - p_0^2) - L^{\gamma,*} f \gg_{\beta,\gamma} \} \\ &= \inf_{f \in \mathcal{C}} \{ \ll W_{0,1}^* - L^{\gamma,*} f \gg_{\beta,\gamma} \} - \frac{2Q^{\gamma}(\beta)}{\beta^2} + Q^{\gamma}(\beta)^2 \ll T(p_1^2 - p_0^2) \gg_{\beta,\gamma} \\ &= \inf_{f \in \mathcal{C}} \{ \ll W_{0,1}^* - L^{\gamma,*} f \gg_{\beta,\gamma} \} - \frac{2Q^{\gamma}(\beta)}{\beta^2} + \frac{Q^{\gamma}(\beta)}{\beta^2}. \end{aligned}$$

□

By a simple computation, we can show that $\ll Tg, g \gg_{\beta,\gamma} = \ll Tg, Tg \gg_{\beta,\gamma}$ for all $g \in \mathcal{H}_{\beta,\gamma}$, and therefore $(p_1^2 - p_0^2) - T(p_1^2 - p_0^2) \in \overline{L^{\gamma,*}\mathcal{C}_0}$ since $(p_1^2 - p_0^2) - T(p_1^2 - p_0^2)$ is orthogonal to $T(p_1^2 - p_0^2)$. By the fact we obtain the variational formula for $\ll T(p_1^2 - p_0^2) \gg_{\beta,\gamma}$:

Proposition 2.6.

$$\ll T(p_1^2 - p_0^2) \gg_{\beta,\gamma} = \inf_{f \in \mathcal{C}} \ll p_1^2 - p_0^2 - L^{\gamma,*} f \gg_{\beta,\gamma} . \tag{2.6.4}$$

Proof. By the similar argument with the proof of Proposition 2.5, we have

$$\inf_{f \in \mathcal{C}} \{ \ll p_1^2 - p_0^2 - T(p_1^2 - p_0^2) - L^{\gamma,*} f \gg_{\beta,\gamma} \} = 0$$

and

$$\begin{aligned} &\inf_{f \in \mathcal{C}} \{ \ll p_1^2 - p_0^2 - T(p_1^2 - p_0^2) - L^{\gamma,*} f \gg_{\beta,\gamma} \} \\ &= \inf_{f \in \mathcal{C}} \{ \ll p_1^2 - p_0^2 - L^{\gamma,*} f \gg_{\beta,\gamma} \} - \ll T(p_1^2 - p_0^2) \gg_{\beta,\gamma} \end{aligned}$$

which concludes the proof. □

Theorem 2.3.

$$\tilde{D}^{\gamma}(\beta) = \beta^2 \inf_{f \in \mathcal{C}} \ll W^*_{0,1} - L^* f \gg_{\beta,\gamma} = \frac{1}{\beta^2 \inf_{f \in \mathcal{C}} \ll p_1^2 - p_0^2 - L^* f \gg_{\beta,\gamma}}.$$

Proof. By the definition, $W_{0,1} + \tilde{D}^{\gamma}(\beta)(p_1^2 - p_0^2) \in \overline{L^{\gamma}\mathcal{C}}$ and therefore

$$\ll W_{0,1} + \tilde{D}^{\gamma}(\beta)(p_1^2 - p_0^2), T^*(p_1^2 - p_0^2) \gg_{\beta,\gamma} = -\frac{1}{\beta^2} + \tilde{D}^{\gamma}(\beta) \ll T(p_1^2 - p_0^2) \gg_{\beta,\gamma} = 0$$

So, $\tilde{D}^{\gamma}(\beta) = Q^{\gamma}(\beta)$ and we obtain two variational formula from (2.6.3) and (2.6.4). □

Theorem 2.4. *For any sequences F_K in $\mathcal{C}$ such that* $\lim_{K \to \infty} \ll W_{0,1} + \tilde{D}^{\gamma}(\beta)(p_1^2 - p_0^2) - L^{\gamma} F_K \gg_{\beta,\gamma} = 0$,

$$\lim_{K \to \infty} [\frac{\gamma}{2} \langle (p_0 V'(r_1) - Y_{0,1}\Gamma_{F_K})^2 \rangle_{\beta} + \frac{\gamma}{2} \langle (X_0 \Gamma_{F_K})^2 \rangle_{\beta}] = \frac{\tilde{D}^{\gamma}(\beta)}{\beta^2}.$$

Proof. By the assumption, $\lim_{K \to \infty} \ll T\{W_{0,1} + \tilde{D}^{\gamma}(\beta)(p_1^2 - p_0^2) - L^{\gamma} F_K\} \gg_{\beta,\gamma} = 0$ and therefore

$$\lim_{K \to \infty} \ll W^S_{0,1} - S^{\gamma} F_K \gg_{\beta,\gamma} = \tilde{D}^{\gamma}(\beta)^2 \ll T(p_1^2 - p_0^2) \gg_{\beta,\gamma}.$$

Then, since $\tilde{D}^{\gamma}(\beta) = Q^{\gamma}(\beta) = \frac{1}{\beta^2 \ll T(p_1^2 - p_0^2) \gg_{\beta,\gamma}}$ and $\ll W^S_{0,1} - S^{\gamma} F_K \gg_{\beta,\gamma} = \frac{\gamma}{2} \langle (p_0 V'(r_1) - Y_{0,1}\Gamma_{F_K})^2 \rangle_{\beta} + \frac{\gamma}{2} \langle (X_0 \Gamma_{F_K})^2 \rangle_{\beta}$, we completes the proof. □

Next, we give an explicit expression of the variational formula for the diffusion coefficient.

Recall that $\mathcal{C}$ is divided into two orthogonal spaces $\mathbb{L}_e := \cup_{i=0}^{\infty} \mathbb{L}_{2i}$ and $\mathbb{L}_o := \cup_{i=0}^{\infty} \mathbb{L}_{2i+1}$. Consider two subspaces of $\mathcal{H}_{\beta,\gamma}$ defined as $\mathcal{H}^e_{\beta,\gamma} := \overline{S^{\gamma}\mathbb{L}_e}|_{\mathcal{N}_{\beta,\gamma}} \oplus \{W^S_{0,1}\}$ and $\mathcal{H}^o_{\beta,\gamma} := \overline{S^{\gamma}\mathbb{L}_o}|_{\mathcal{N}_{\beta,\gamma}}$.

Lemma 2.4.1. *For each $\beta > 0$, we have*

$$\mathcal{H}_{\beta,\gamma} = \mathcal{H}^e_{\beta,\gamma} \oplus \mathcal{H}^o_{\beta,\gamma}$$

and they are orthogonal to each other. Moreover, $W^A_{0,1} \in \mathcal{H}^o_{\beta,\gamma}$, $Af \in \mathcal{H}^o_{\beta,\gamma}$ if $f \in \mathbb{L}_e$ and $Af \in \mathcal{H}^e_{\beta,\gamma}$ if $f \in \mathbb{L}_o$.

Proof. Straightforward. □

Proposition 2.7.

$$\tilde{D}^\gamma(\beta) = \beta^2 \inf_{f\in\mathbb{L}_e} \sup_{g\in\mathbb{L}_o}\{\gamma[\frac{1}{2}\langle(p_0V'(r_1) - Y_{0,1}\Gamma_f)^2\rangle_\beta + \frac{1}{2}\langle(X_0\Gamma_f)^2\rangle_\beta] \tag{2.6.5}$$
$$+2\langle(W_{0,1}^A - Af)\Gamma_g\rangle_\beta - \gamma[\frac{1}{2}\langle(Y_{0,1}\Gamma_g)^2\rangle_\beta + \frac{1}{2}\langle(X_0\Gamma_g)^2\rangle_\beta]\}.$$

Proof. We can rewrite the first variational formula for $\tilde{D}^\gamma(\beta)$ in (2.3) as

$$\begin{aligned}
&\beta^2 \inf_{f\in\mathcal{C}}\{\ll W_{0,1}^S - S^\gamma f \gg_{\beta,\gamma} + \ll W_{0,1}^A - Af \gg_{\beta,\gamma}\} \\
&= \beta^2 \inf_{f_e\in\mathbb{L}_e} \inf_{f_o\in\mathbb{L}_o}\{\ll W_{0,1}^S - S^\gamma f_e \gg_{\beta,\gamma} + \ll S^\gamma f_o \gg_{\beta,\gamma} \\
&\quad + \ll W_{0,1}^A - Af_e \gg_{\beta,\gamma} + \ll Af_o \gg_{\beta,\gamma}\} \\
&= \beta^2 \inf_{f\in\mathbb{L}_e}\{\ll W_{0,1}^S - S^\gamma f \gg_{\beta,\gamma} + \ll W_{0,1}^A - Af \gg_{\beta,\gamma}\} \\
&= \beta^2 \inf_{f\in\mathbb{L}_e} \sup_{g\in\mathbb{L}_o}\{\ll W_{0,1}^S - S^\gamma f \gg_{\beta,\gamma} -2 \ll W_{0,1}^A - Af, S^\gamma g \gg_{\beta,\gamma} - \ll S^\gamma g \gg_{\beta,\gamma}\} \\
&= \beta^2 \inf_{f\in\mathbb{L}_e} \sup_{g\in\mathbb{L}_o}\{\gamma[\frac{1}{2}\langle(p_0V'(r_1) - Y_{0,1}\Gamma_f)^2\rangle_\beta + \frac{1}{2}\langle(X_0\Gamma_f)^2\rangle_\beta] + 2\langle(W_{0,1}^A - Af)\Gamma_g\rangle_\beta \\
&\quad - \gamma[\frac{1}{2}\langle(Y_{0,1}\Gamma_g)^2\rangle_\beta + \frac{1}{2}\langle(X_0\Gamma_g)^2\rangle_\beta]\}.
\end{aligned}$$

□

2.6.3 Detailed estimates of the diffusion coefficient

In this subsection, we give some detailed estimates of the diffusion coefficient as a function of γ. Note that they are not necessary to prove our main theorem.

Proposition 2.8.

$$\tilde{D}^\gamma(\beta) \leq \frac{\gamma}{2}\langle V''(r_0)\rangle_\beta + \frac{3}{2\gamma}.$$

Proof. Take $f = 0$ in the variational formula (2.6.5), then we have

$$\begin{aligned}
\tilde{D}^\gamma(\beta) &\leq \beta^2 \sup_{g\in\mathbb{L}_o}\{\frac{\gamma}{2}\langle p_0^2V'(r_1)^2\rangle_\beta + 2\langle W_{0,1}^A\Gamma_g\rangle_\beta - \gamma[\frac{1}{2}\langle(Y_{0,1}\Gamma_g)^2\rangle_\beta + \frac{1}{2}\langle(X_0\Gamma_g)^2\rangle_\beta]\} \\
&= \frac{\gamma}{2}\langle V''(r_0)\rangle_\beta + \frac{\beta^2}{\gamma} \sup_{g\in\mathbb{L}_o}\{2\langle W_{0,1}^A\Gamma_g\rangle_\beta - \frac{1}{2}\langle(Y_{0,1}\Gamma_g)^2\rangle_\beta - \frac{1}{2}\langle(X_0\Gamma_g)^2\rangle_\beta\}.
\end{aligned}$$

Since $W_{0,1}^A = Y_{0,1}(\frac{p_0^2}{2})$,

$$\sup_{g\in\mathbb{L}_o}\{2\langle W_{0,1}^A\Gamma_g\rangle_\beta - \frac{1}{2}\langle(Y_{0,1}\Gamma_g)^2\rangle_\beta - \frac{1}{2}\langle(X_0\Gamma_g)^2\rangle_\beta\}$$

$$= \sup_{g\in\mathbb{L}_o}\{\langle p_0^2, Y_{0,1}\Gamma_g\rangle_\beta - \frac{1}{2}\langle (Y_{0,1}\Gamma_g)^2\rangle_\beta - \frac{1}{2}\langle (X_0\Gamma_g)^2\rangle_\beta\}$$
$$= \sup_{g\in\mathbb{L}_o}\{-\frac{1}{2}\langle (Y_{0,1}\Gamma_g - p_0^2)^2\rangle_\beta + \frac{1}{2}\langle p_0^4\rangle_\beta - \frac{1}{2}\langle (X_0\Gamma_g)^2\rangle_\beta\} \leq \frac{1}{2}\langle p_0^4\rangle_\beta.$$

□

Proposition 2.9.

$$\tilde{D}^\gamma(\beta) \geq \frac{\gamma}{4\beta\langle r_0^2\rangle_\beta}.$$

Proof. By the variational formula (2.6.5)

$$\tilde{D}^\gamma(\beta) \geq \gamma\beta^2 \inf_{f\in\mathbb{L}_e}\{[\frac{1}{2}\langle (p_0V'(r_1) + Y_{0,1}\Gamma_f)^2\rangle_\beta + \frac{1}{2}\langle (X_0\Gamma_f)^2\rangle_\beta]\}.$$

Since $\frac{1}{\beta^2} = \langle p_0V'(r_1), p_0r_1\rangle_\beta$ and $\langle p_0r_0, X_0(\Gamma_f)\rangle_\beta - \langle p_0r_1, Y_{0,1}(\Gamma_f)\rangle_\beta = \langle V'(r_0)r_0 - V'(r_1)r_1, \Gamma_f\rangle_\beta = 0$ for any $f \in \mathbb{L}_e$, we have

$$\frac{1}{\beta^2} = \langle p_0V'(r_1) - Y_{0,1}(\Gamma_f), p_0r_1\rangle_\beta + \langle p_0r_0, X_0(\Gamma_f)\rangle_\beta$$

for any $f \in \mathbb{L}_0$. Then, by Schwarz inequality,

$$\frac{1}{\beta^4} \leq \inf_{f\in\mathbb{L}_0}\langle (p_0V'(r_1) - Y_{0,1}(\Gamma_f))^2 + (X_0(\Gamma_f))^2\rangle_\beta\langle (p_0r_1)^2 + (p_0r_0)^2\rangle_\beta$$
$$= \frac{2}{\beta}\langle r_0^2\rangle_\beta \inf_{f\in\mathbb{L}_0}\langle (p_0V'(r_1) - Y_{0,1}(\Gamma_f))^2 + (X_0(\Gamma_f))^2\rangle_\beta.$$

□

2.7 Spectral gap

In this section, we prove the spectral gap estimates for the process of finite oscillators without the periodic boundary condition, which is used in the proof of Theorem 2.7 in the next section. We use the following notation:

$$E_{\nu_{N,E}}[\,\cdot\,] := E_{\nu_\beta^N}[\,\cdot\,|\frac{1}{N}\sum_{i=1}^N(\frac{p_i^2}{2} + V(r_i)) = E].$$

Recall that we assume $V(r)$ to be strictly increasing in $\mathbb{R}_+$ and satisfy

$$0 < d_- \leq \frac{\sqrt{2V(r)}}{V'(r)} \leq d_+ < \infty$$

for all $r > 0$. Under these assumptions, we can operate the change of variables $(p, r) \to (\mathcal{E}, \theta)$ as $\sqrt{\mathcal{E}} \cos\theta = \frac{p}{\sqrt{2}}$ and $\sqrt{\mathcal{E}} \sin\theta = sgn(r)\sqrt{V(r)}$, and we obtain that

$$\int_{\mathbb{R}^2} f(p,r) d\nu_\beta^1 = \frac{1}{\sqrt{2\pi\beta^{-1}} Z_\beta} \int_0^\infty \int_0^{2\pi} \tilde{f}(\mathcal{E},\theta) e^{-\beta\mathcal{E}} q(\mathcal{E},\theta) d\mathcal{E} d\theta$$

for some $q : \mathbb{R}_+ \times [0, 2\pi] \to \mathbb{R}_+$, which satisfies $d_- \leq q(\mathcal{E},\theta) \leq d_+$ for all $\mathcal{E}$ and θ. Here, $\tilde{f}(\mathcal{E},\theta) := f(p(\mathcal{E},\theta), r(\mathcal{E},\theta))$.

Let $h_\beta(x)dx$ be the probability distribution on $\mathbb{R}_+$ of $p^2/2 + V(r)$ under $d\nu_\beta^1$, i.e.

$$\int_{\mathbb{R}^2} g(p^2/2 + V(r)) d\nu_\beta^1 = \int_0^\infty g(x) h_\beta(x) dx$$

for any $g : \mathbb{R}_+ \to \mathbb{R}$. Then, since $h_\beta(x) = \frac{1}{\sqrt{2\pi\beta^{-1}} Z_\beta} \int_0^{2\pi} e^{-\beta x} q(x,\theta) d\theta$, we obtain

$$\frac{d_-}{d_+} e^{-\beta x} \leq h_\beta(x) \leq \frac{d_+}{d_-} e^{-\beta x}$$

for all $x > 0$.

With these notations, we prepare two lemmas before we state the main result of this section.

Lemma 2.4.2. *There exists a positive constant C such that*

$$E_{\nu_{1,E}}[(f - E_{\nu_{1,E}}[f])^2] \leq C E_{\nu_{1,E}}[(X_1 f)^2]$$

for every $E > 0$, and every differentiable function f.

Proof. By simple computations with the change of variable,

$$E_{\nu_{1,E}}[(f - E_{\nu_{1,E}}[f])^2] = \frac{\int_0^{2\pi} (\tilde{f}(E,\theta) - E_{\nu_{1,E}}[f])^2 q(E,\theta) d\theta}{\int_0^{2\pi} q(E,\theta) d\theta}$$

and

$$E_{\nu_{1,E}}[(X_1 f)^2] = \frac{\int_0^{2\pi} \{q(E,\theta)^{-1} \partial_\theta \tilde{f}(E,\theta)\}^2 q(E,\theta) d\theta}{\int_0^{2\pi} q(E,\theta) d\theta}.$$

Therefore, it sufficient to show that there exists a positive constant C such that

$$\int_0^{2\pi} (\tilde{f}(E,\theta) - E_{\nu_{1,E}}[f])^2 q(E,\theta) d\theta \leq C \int_0^{2\pi} (\partial_\theta \tilde{f}(E,\theta))^2 q(E,\theta)^{-1} d\theta$$

for every $E > 0$ and every differentiable function f. Then, since $d_- \leq q(E,\theta) \leq d_+$ for all $E > 0$ and θ, and

$$\int_0^{2\pi} (\tilde{f}(E,\theta) - E_{\nu_{1,E}}[f])^2 q(E,\theta)d\theta \leq \int_0^{2\pi} \Big(\tilde{f}(E,\theta) - (\int_0^{2\pi} \tilde{f}(E,\theta)d\theta)\Big)^2 q(E,\theta)d\theta$$

holds for every $E > 0$ and every differentiable function f, the desired inequality follows from the Poincaré inequality. □

Lemma 2.4.3. *There exist positive constants* $0 < c \leq C < \infty$ *such that*

$$c\,E \leq \alpha_i(E) \leq CE$$

for all $E > 0$ *and for* $i = 1, 2$ *where* $\alpha_1(E) := E_{\nu_{1,E}}[p_1^2]$ *and* $\alpha_2(E) := E_{\nu_{1,E}}[V'^2(r_1)]$.

Proof. By the change of variables introduced above,

$$E_{\nu_{1,E}}[p_1^2] = \frac{\int_0^{2\pi} 2E\cos\theta^2 q(E,\theta)d\theta}{\int_0^{2\pi} q(E,\theta)d\theta}$$

and it is easy to show that $\frac{d_-}{d_+}E \leq E_{\nu_{1,E}}[p_1^2] \leq 2E$. Similarly,

$$E_{\nu_{1,E}}[V'^2(r_1)] \leq \frac{2}{d_-^2} E_{\nu_{1,E}}[V(r_1)] = \frac{2\int_0^{2\pi} E\sin\theta^2 q(E,\theta)d\theta}{d_-^2 \int_0^{2\pi} q(E,\theta)d\theta} \leq \frac{2E}{d_-^2}$$

and

$$E_{\nu_{1,E}}[V'^2(r_1)] \geq \frac{2}{d_+^2} E_{\nu_{1,E}}[V(r_1)] = \frac{2\int_0^{2\pi} E\sin\theta^2 q(E,\theta)d\theta}{d_+^2 \int_0^{2\pi} q(E,\theta)d\theta} \geq \frac{d_- E}{d_+^3}.$$

□

The following is the main theorem in this section.

Theorem 2.5. *There exists a positive constant* C *such that*

$$E_{\nu_{N,E}}[f^2] \leq C \sum_{k=1}^{N} E_{\nu_{N,E}}[(X_k f)^2] + CN^2 \sum_{k=1}^{N-1} E_{\nu_{N,E}}[(Y_{k,k+1} f)^2] \tag{2.7.1}$$

for every positive integer N, *every* $E > 0$, *and every differentiable function* f *satisfying* $E_{\nu_{N,E}}[f] = 0$.

Proof. We start the proof by the usual martingale decomposition. Let $\mathcal{G}_k$ be the σ-field generated by variables $\{\mathcal{E}_1, \dots, \mathcal{E}_k, p_{k+1}, r_{k+1}, \cdots, p_N, r_N\}$. Define $f_k := E_{\nu_{N,E}}[f|\mathcal{G}_k]$ for $k = 0, 1, \cdots, N$. Note that $f_0 = f$ and $f_N = f_N(\mathcal{E}_1, \dots, \mathcal{E}_N)$. Then, we obtain

$$E_{\nu_{N,E}}[f^2] = \sum_{k=0}^{N-1} E_{\nu_{N,E}}[(f_k - f_{k+1})^2] + E_{\nu_{N,E}}[f_N^2].$$

We analyze each term separately.

By Lemma 2.4.2, for any k

$$E_{\nu_{N,E}}[(f_k - f_{k+1})^2|\mathcal{G}_k] \leq C E_{\nu_{N,E}}[(X_{k+1} f_k)^2|\mathcal{G}_k]$$

and therefore we have

$$\begin{aligned} E_{\nu_{N,E}}[f^2] &\leq C \sum_{k=1}^{N} E_{\nu_{N,E}}[(X_k f_{k-1})^2] + E_{\nu_{N,E}}[f_N^2] \\ &\leq C \sum_{k=1}^{N} E_{\nu_{N,E}}[(X_k f)^2] + E_{\nu_{N,E}}[f_N^2]. \end{aligned}$$

So we are left to estimate $E_{\nu_{N,E}}[f_N^2]$ in terms of the Dirichlet form $\sum_{k=1}^{N-1} E_{\nu_{N,E}}[(Y_{k,k+1} f_N)^2]$.

Observe that $Y_{k,k+1} f_N = p_k V'(r_{k+1}) \left(\partial_{\mathcal{E}_k} - \partial_{\mathcal{E}_{k+1}}\right) f_N(\mathcal{E}_1, \dots, \mathcal{E}_N)$. Since $\nu_{N,E}$ is the conditional probability of the product measure ν_β^N,

$$E_{\nu_{N,E}}[p_k^2 V'(r_{k+1})^2|\mathcal{G}_N] = E_{\nu_{1,\mathcal{E}_k}}[p^2] E_{\nu_{1,\mathcal{E}_{k+1}}}[V'^2(r)] = \alpha_1(\mathcal{E}_k)\alpha_2(\mathcal{E}_{k+1}).$$

By Lemma 2.4.3, the Dirichlet form $\sum_{k=1}^{N-1} E_{\nu_{N,E}}[(Y_{k,k+1} f_N)^2]$, is equivalent to

$$\sum_{k=1}^{N-1} E_{\nu_{N,E}} \left[\mathcal{E}_k \mathcal{E}_{k+1} \left\{\left(\partial_{\mathcal{E}_k} - \partial_{\mathcal{E}_{k+1}}\right) f_N\right\}^2\right].$$

Now the problem is reduced to the estimates of the spectral gap for the energy dynamics depending only on variables $\mathcal{E}_1, \dots, \mathcal{E}_N$. Since we can write the probability distribution $\nu_{N,E}(\cdot|\mathcal{G}_N)$ on $\{(\mathcal{E}_1, \dots, \mathcal{E}_N) : \sum_i \mathcal{E}_i = NE\}$ as the product measure $\prod_{i=1}^N h_1(x_i)dx_i$ (or $\prod_{i=1}^N h_\beta(x_i)dx_i$ for any β) conditioned on the same surface, Theorem 2.6 in the next subsection completes the proof. $\square$

2.7.1 Spectral gap for the energy dynamics

Consider the product measure $\prod_{i=1}^N h_1(x_i)dx_i$ on $\mathbb{R}_+^N$ and $d\mu_{N,E}$ the conditional distribution of it on the surface $\Sigma_{N,E} = \{\sum_{i=1}^N x_i = NE\}$. We have the following expression

$$d\mu_{N,E} = \prod_{i=1}^N h(x_i) d\lambda_{N,E}(x_1, \ldots, x_N)$$

where $d\lambda_{N,E}$ is the uniform measure on the surface $\Sigma_{N,E}$.

Theorem 2.6. *There exists a positive constant C such that*

$$E_{\mu_{N,E}}[g^2] \le CN^2 \sum_{i=1}^{N-1} E_{\mu_{N,E}} \left[x_i x_{i+1} \left(\partial_{x_i} g - \partial_{x_{i+1}} g \right)^2 \right]$$

for every positive integer N, every $E > 0$ and every differentiable function $g : \Sigma_{N,E} \to \mathbb{R}$ satisfying $E_{\mu_{N,E}}[g] = 0$.

To prove this, we first refer Caputo's result (Example 3.1 in [6]). Let $E_{i,j}$ and $D_{i,j}$ be operators defined by $E_{i,j} f = E_{\mu_{N,E}}[f | \mathcal{F}_{i,j}]$ and $D_{i,j} f = E_{i,j} f - f$ where $\mathcal{F}_{i,j}$ is the σ-algebra generated by variables $\{x_k\}_{k \neq i,j}$.

Lemma 2.6.1 (Caputo, [6]). *If $d_-/d_+ > (3/4)^{1/16}$, then there exists a positive constant C such that*

$$E_{\mu_{N,E}}[g^2] \le \frac{C}{N} \sum_{i,j=1}^N E_{\mu_{N,E}}[(D_{i,j} g)^2]$$

for every $E > 0$, every positive integer N and every differentiable function $g : \Sigma_{N,E} \to \mathbb{R}$ satisfying $E_{\mu_{N,E}}[g] = 0$.

Next, we show that we can take a telescopic sum.

Lemma 2.6.2. *There exists a positive constant C such that*

$$\frac{1}{N} \sum_{i,j=1}^N E_{\mu_{N,E}}[(D_{i,j} g)^2] \le CN^2 \sum_{i=1}^{N-1} E_{\mu_{N,E}}[(D_{i,i+1} g)^2]$$

for every $E > 0$, every positive integer N and every differentiable function $g : \Sigma_{N,E} \to \mathbb{R}$.

Proof. First, we rewrite the term $E_{i,j}g$ in an integral form:

$$E_{i,j}g(x) = \frac{1}{\Xi_{x_i+x_j}} \int_0^1 g(R^t_{i,j}x)h((x_i+x_j)t)h((x_i+x_j)(1-t))dt$$

where $\Xi_a = \int_0^1 h(at)h(a(1-t))dt$ and $R^t_{i,j}x \in \mathbb{R}^N_+$ is a configuration defined by

$$(R^t_{i,j}x)_k = \begin{cases} x_k & \text{if } k \neq i,j, \\ (x_i+x_j)t & \text{if } k = i, \\ (x_i+x_j)(1-t) & \text{if } k = j. \end{cases}$$

Then, by Schwarz's inequality we have

$$\begin{aligned}(D_{i,j}g(x))^2 &= (\frac{1}{\Xi_{x_i+x_j}} \int_0^1 \{g(R^t_{i,j}x) - g(x)\}h((x_i+x_j)t)h((x_i+x_j)(1-t))dt)^2 \\ &\leq \frac{1}{\Xi_{x_i+x_j}} \int_0^1 \{g(R^t_{i,j}x) - g(x)\}^2 h((x_i+x_j)t)h((x_i+x_j)(1-t))dt.\end{aligned}$$

Now, we introduce operators $\pi^{i,j}$, $\sigma^{i,j}$ and $\tilde{\sigma}^{i,j} : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ for $i < j$ as

$$(\pi^{i,j}x)_k = \begin{cases} x_k & \text{if } k \neq i,j, \\ x_j & \text{if } k = i, \\ x_i & \text{if } k = j, \end{cases}$$

$\sigma^{i,j} := \pi^{j-1,j} \circ \pi^{j-2,j-1} \cdots \circ \pi^{i,i+1}$ and $\tilde{\sigma}^{i,j} := \pi^{i,i+1} \circ \pi^{i+1,i+2} \cdots \circ \pi^{j-1,j}$. With these notations, for any $i < j$, we can rewrite the term $g(R^t_{i,j}x) - g(x)$ as

$$\begin{aligned}g(R^t_{i,j}x) - g(x) = &\{g(\tilde{\sigma}^{i,j-1}(R^t_{j-1,j}(\sigma^{i,j-1}x))) - g(R^t_{j-1,j}(\sigma^{i,j-1}x))\} \\ &+ \{g(R^t_{j-1,j}(\sigma^{i,j-1}x)) - g(\sigma^{i,j-1}x)\} + \{g(\sigma^{i,j-1}x) - g(x)\}.\end{aligned}$$

Therefore, we can bound the term $E_{\mu_{N,E}}[(D_{i,j}g(x))^2]$ from above by

$$\begin{aligned}&3E_{\mu_{N,E}}[\frac{1}{\Xi_{x_i+x_j}} \int_0^1 \{g(\tilde{\sigma}^{i,j-1}(R^t_{j-1,j}(\sigma^{i,j-1}x))) - g(R^t_{j-1,j}(\sigma^{i,j-1}x))\}^2 \\ &\qquad h((x_i+x_j)t)h((x_i+x_j)(1-t))dt] \\ &+3E_{\mu_{N,E}}[\frac{1}{\Xi_{x_i+x_j}} \int_0^1 \{g(R^t_{j-1,j}(\sigma^{i,j-1}x)) - g(\sigma^{i,j-1}x)\}^2 \\ &\qquad h((x_i+x_j)t)h((x_i+x_j)(1-t))dt] \\ &+3E_{\mu_{N,E}}[\frac{1}{\Xi_{x_i+x_j}} \int_0^1 \{g(\sigma^{i,j-1}x) - g(x)\}^2 h((x_i+x_j)t)h((x_i+x_j)(1-t))dt].\end{aligned} \tag{2.7.2}$$

We estimate three terms separately. The last term of equation (2.7.2) is equal to

$$3E_{\mu_{N,E}}[\{g(\sigma^{i,j-1}x) - g(x)\}^2]$$

and therefore bounded form above by

$$3N\sum_{k=i}^{j-2} E_{\mu_{N,E}}[\{g(\pi^{k,k+1}x) - g(x)\}^2].$$

By simple computations, we obtain that

$$\begin{aligned}
&E_{\mu_{N,E}}[\{g(\pi^{k,k+1}x) - g(x)\}^2] \\
&= E_{\mu_{N,E}}[\{g(\pi^{k,k+1}x) - (E_{k.k+1}g)(\pi^{k,k+1}x) + (E_{k.k+1}g)(x) - g(x)\}^2] \\
&\leq 2E_{\mu_{N,E}}[\{g(\pi^{k,k+1}x) - (E_{k.k+1}g)(\pi^{k,k+1}x)\}^2] + 2E_{\mu_{N,E}}[\{(E_{k.k+1}g)(x) - g(x)\}^2] \\
&= 4E_{\mu_{N,E}}[(D_{k.k+1}g)^2].
\end{aligned}$$

By the change of variable with $y = \sigma^{i,j-1}x$, we can rewrite the second term of equation (2.7.2) as

$$\begin{aligned}
&3E_{\mu_{N,E}}[\frac{1}{\Xi_{y_{j-1}+y_j}}\int_0^1 \{g(R^t_{j-1,j}y) - g(y)\}^2 h((y_{j-1}+y_j)t)h((y_{j-1}+y_j)(1-t))dt] \\
&\quad = 3E_{\mu_{N,E}}[E_{j,j+1}(g^2) - 2gE_{j.j+1}g + g^2] \\
&\quad = 6E_{\mu_{N,E}}[g^2 - (E_{j.j+1}g)^2] = 6E_{\mu_{N,E}}[(D_{j,j+1}g)^2].
\end{aligned}$$

Similarly, the first term of equation (2.7.2) is rewritten as

$$\begin{aligned}
3E_{\mu_{N,E}}[\frac{1}{\Xi_{y_{j-1}+y_j}}\int_0^1 &\{g(\tilde{\sigma}^{i,j-1}(R^t_{j-1,j}y)) - g(R^t_{j-1,j}y)\}^2 \\
&h((y_{j-1}+y_j)t)h((y_{j-1}+y_j)(1-t))dt] \\
= 3E_{\mu_{N,E}}[E_{j,j+1}(\{g\circ\tilde{\sigma}^{i,j-1} - g\}^2)] &= 3E_{\mu_{N,E}}[\{g\circ\tilde{\sigma}^{i,j-1} - g\}^2].
\end{aligned}$$

As same for the first term of (2.7.2), it is bounded from above by $12N\sum_{k=i}^{j-2} E_{\mu_{N,E}}[(D_{k,k+1}g)^2]$. Therefore, we complete the proof. □

Lemma 2.6.3. *There exists a constant C such that*

$$E_{\mu_{2,E}}[(D_{1,2}g)^2] \leq CE_{\mu_{2,E}}[x_1x_2(\partial_{x_1}g - \partial_{x_2}g)^2]. \tag{2.7.3}$$

for every $E > 0$ and every differentiable function $g : \Sigma_{2,E} \to \mathbb{R}$.

Proof. Since the both sides of (2.7.3) do not change if we replace g with $g+a$ for any constant a, it is sufficient to show that the inequality holds for every differentiable function $g : \Sigma_{2,E} \to \mathbb{R}$ satisfying $E_{\lambda_{2,E}}[g] = 0$. In particular, since $E_{\mu_{2,E}}[(D_{1,2}g)^2] \leq E_{\mu_{2,E}}[g^2]$, it is sufficient to show that

$$E_{\mu_{2,E}}[g^2] \leq CE_{\mu_{2,E}}[x_1x_2(\partial_{x_1}g - \partial_{x_2}g)^2].$$

Note that for any positive function $f : \Sigma_{2,E} \to \mathbb{R}_+$ and for any $E > 0$,

$$(\frac{d_-}{d_+})^4 E_{\mu_{2,E}}[f] \leq E_{\lambda_{2,E}}[f] \leq (\frac{d_+}{d_-})^4 E_{\mu_{2,E}}[f].$$

In fact,

$$E_{\mu_{2,E}}[f] = \frac{1}{\Xi_E}\int_0^1 f(tE,(1-t)E)h(tE)h((1-t)E)dt$$

where $\Xi_E = \int_0^1 h(tE)h((1-t)E)dt$. Then, since $\frac{d_-}{d_+}e^{-x} \leq h(x) \leq \frac{d_+}{d_-}e^{-x}$, the above estimate holds. Now, all we have to show is that, there exists a constant C such that

$$E_{\lambda_{2,E}}[g^2] \leq CE_{\lambda_{2,E}}[x_1x_2(\partial_{x_1}g - \partial_{x_2}g)^2]$$

for every $E > 0$ and every differentiable function $g : \Sigma_{2,E} \to \mathbb{R}$ satisfying $E_{\lambda_{2,E}}[g] = 0$. It is the same statement of this lemma with $V(r) = \frac{r^2}{2}$, or $h(x) = e^{-x}$. For this case, it is well-known that the estimate holds (see [12]). □

Lemma 2.6.4. *There exists a positive constant C such that*

$$E_{\mu_{N,E}}[(D_{i,i+1}g)^2] \leq CE_{\mu_{N,E}}[x_ix_{i+1}(\partial_{x_i}g - \partial_{x_{i+1}}g)^2]$$

for every positive integer N, $i = 1,\dots,N-1$, and every differentiable function $g : \Sigma_{N,E} \to \mathbb{R}$.

Proof. By Lemma 2.6.3,

$$E_{\mu_{N,E}}[(D_{i,i+1}g)^2|\mathcal{F}_{i,i+1}] \leq CE_{\mu_{N,E}}[x_ix_{i+1}(\partial_{x_i}g - \partial_{x_{i+1}}g)^2|\mathcal{F}_{i,i+1}]$$

holds. Then, by taking the expectation, we complete the proof. □

2.8 Closed forms

In this section, to complete the proof of Lemma 2.1.6, we introduce the notion of closed forms and give a characterization of them.

Let us define $\mathcal{A} = \cup_{k\geq 1}\mathcal{A}_k$, where $\mathcal{A}_k$ is the space of smooth functions F depending only on the variables $(p_i, r_i)_{-k\leq i\leq k}$. Given $F \in \mathcal{A}_k$ we consider the formal sum

$$\widetilde{F}(p,r) = \sum_{j=-\infty}^{\infty} \tau_j F(p,r)$$

and for every $i \in \mathbb{Z}$ the expressions

$$\frac{\partial \widetilde{F}}{\partial p_i}(p,r) = \sum_{i-k\leq j\leq i+k} \frac{\partial}{\partial p_i}\tau_j F(p,r)$$

and

$$\frac{\partial \widetilde{F}}{\partial r_i}(p,r) = \sum_{i-k\leq j\leq i+k} \frac{\partial}{\partial r_i}\tau_j F(p,r)$$

are well defined. The formal invariance $\widetilde{F}(\tau_i(p,r)) = \widetilde{F}(p,r)$ leads us to the relation

$$\frac{\partial \widetilde{F}}{\partial p_i}(p,r) = \frac{\partial \widetilde{F}}{\partial p_0}(\tau_i(p,r)). \tag{2.8.1}$$

Remember that $Y_{i,j} = p_i\partial_{r_j} - V'(r_j)\partial_{p_i}$ and $X_i := Y_{i,i}$. Given $F \in \mathcal{A}$ and $i \in \mathbb{Z}$, $X_i(\widetilde{F})$ and $Y_{i,i+1}(\widetilde{F})$ are well defined and satisfy

$$X_i(\widetilde{F})(p,r) = \tau_i X_0(\widetilde{F})(p,r), \quad Y_{i,i+1}(\widetilde{F})(p,r) = \tau_i Y_{0,1}(\widetilde{F})(p,r).$$

Now we consider the following set

$$\mathcal{B}_\beta = \{(X_0(\widetilde{F}), Y_{0,1}(\widetilde{F})) \in L^2(\nu_\beta) \times L^2(\nu_\beta) : F \in \mathcal{A}\}.$$

We denote by $\mathfrak{H}_\beta$ the linear space generated by the closure of $\mathcal{B}_\beta$ in $L^2(\nu_\beta) \times L^2(\nu_\beta)$ and $(0, p_0V'(r_1))$. First, we observe that defining a vector-valued function $\xi = (\xi^0, \xi^1)$ as $(X_0(\widetilde{F}), Y_{0,1}(\widetilde{F}))$ for $F \in \mathcal{A}$ or $(0, p_0V'(r_1))$, the following properties are satisfied:

i) $X_i(\tau_j\xi^0) = X_j(\tau_i\xi^0)$ for all $i, j \in \mathbb{Z}$,

ii) $Y_{i,i+1}(\tau_j\xi^1) = Y_{j,j+1}(\tau_i\xi^1)$ for all $i, j \in \mathbb{Z}$, *iii)* $X_i(\tau_j\xi^1) = Y_{j,j+1}(\tau_i\xi^0)$ if $\{i\} \cap \{j, j+1\} = \emptyset$,

iv) $p_i[X_i(\tau_i\xi^1) - Y_{i,i+1}(\tau_i\xi^0)] = V'(r_{i+1})\tau_i\xi^0 - V'(r_i)\tau_i\xi^1$ for all $i \in \mathbb{Z}$,

iv) $V'(r_{i+1})[X_{i+1}(\tau_i\xi^1) - Y_{i,i+1}(\tau_{i+1}\xi^0)] = V''(r_{i+1})p_{i+1}\tau_i\xi^1 - V''(r_{i+1})p_i\tau_{i+1}\xi^0$ for all $i \in \mathbb{Z}$.

Now we can claim the desired characterization.

Theorem 2.7. *If $\xi = (\xi^0, \xi^1) \in L^2(\nu_\beta) \times L^2(\nu_\beta)$ satisfies conditions i) to v) in a weak sense then $\xi \in \mathfrak{H}_\beta$.*

Proof. The goal is to find a sequence $(F_N)_{N\geq 1}$ in $\mathcal{A}$ such that $(\xi^0 - X_0(\widetilde{F_N}), \xi^1 - Y_{0,1}(\widetilde{F_N}))$ converge to $(0, cp_0V'(r_1))$ in $L^2(\nu_\beta) \times L^2(\nu_\beta)$ for some constant c.
First, observe that for a smooth function $F \in \mathcal{A}_k$ we can rewrite $X_0(\widetilde{F})$ and $Y_{0,1}(\widetilde{F})$, by using (2.8.1), as

$$\sum_{i=-k}^{k} X_i(F)(\tau_{-i}(p,r)) \tag{2.8.2}$$

and

$$\sum_{i=-k}^{k-1} Y_{i,i+1}(F)(\tau_{-i}(p,r)) - \left(V'(r_{k+1})\frac{\partial F}{\partial p_k}\right)(\tau_{-k}p) + \left(p_{-k-1}\frac{\partial F}{\partial r_{-k}}\right)(\tau_{k+1}(p,r)) \tag{2.8.3}$$

respectively.

We define

$$\xi_i^{m,(N)} = \mathbf{E}_{\nu_\beta}[\xi_i^m|\mathcal{F}_N]\varphi\left(\frac{1}{2N+1}\sum_{i=-N}^{N}\{\frac{p_i^2}{2} + V(r_i)\}\right)$$

for $m = 0, 1$ where $\xi_i^m(p,r) = \tau_i\xi^m(p,r)$, $\mathcal{F}_N$ is the sub σ-field of Ω generated by $(p_i, r_i)_{i=-N}^{N}$ and φ is a smooth function with compact support such that $\varphi(\frac{1}{2\beta} + \psi(\beta)) = 1$ (we need this cutoff in order to do uniform bounds later). Because ν_β is a product measure and φ satisfies that

$$X_i\varphi\left(\frac{1}{2N+1}\sum_{i=-N}^{N}\{\frac{p_i^2}{2} + V(r_i)\}\right) = 0$$

for $-N \leq i \leq N-1$ and

$$Y_{i,i+1}\varphi\left(\frac{1}{2N+1}\sum_{i=-N}^{N}\{\frac{p_i^2}{2} + V(r_i)\}\right) = 0$$

for $-N \le i \le N-1$, the set of functions $\{(\xi_i^{0,N})\}_{-N\le i\le N}$ and $\{(\xi_i^{1,N})\}_{-N\le i\le N-1}$ even satisfies the conditions i) and v) on the finite set $\{-N, -N+1, \dots, N\}$. Therefore, by the similar argument in Appendix B in [12], there exists a $\mathcal{F}_N$-measurable function $g^{(N)}$ such that

$$(2.8.4)\qquad \begin{aligned} X_i(g^{(N)}) &= \xi_i^{0,(N)} \quad \textit{for} \quad -N \le i \le N, \\ Y_{i,i+1}(g^{(N)}) &= \xi_i^{1,(N)} \quad \textit{for} \quad -N \le i \le N-1. \end{aligned}$$

Because $\mathbf{E}_{\nu_\beta}[g^{(N)}|\mathcal{E}_{-N} + \cdots + \mathcal{E}_N]$ is radial and the integration was performed over spheres, $\widetilde{g}^{(N)} = g^{(N)} - \mathbf{E}_{\nu_\beta}[g^{(N)}|\mathcal{E}_{-N} + \cdots + \mathcal{E}_N]$ is still a solution of the system (2.8.4). So we can suppose that $\mathbf{E}_{\nu_\beta}[g^{(N)}|\mathcal{E}_{-N} + \cdots + \mathcal{E}_N = E] = 0$ for every $E \in \mathbb{R}^+$.
Define

$$g^{(N,k)} = \frac{\beta}{2(N+k)\phi_\beta}\mathbf{E}_{\nu_\beta}[p^2_{-N-k-1}V'(r_{N+k+1})^2 g^{(2N)}|\mathcal{F}_{N+k}]$$

and

$$\widehat{g}^N = \frac{4}{N}\sum_{k=N/2}^{3N/4} g^{(N,k)}$$

where $\phi_\beta := \mathbf{E}_{\nu_\beta}[V'(r_0)^2]$.

Using (2.8.2) and (2.8.3) for $g^{(N,k)}$ and then averaging over k we obtain that

$$X_0\left(\sum_{j=-\infty}^{\infty} \tau_j \widehat{g}^N\right) = \xi^0 + \frac{\beta}{\phi_\beta}[I_N^1 + I_N^2 + I_N^3 + I_N^4]$$

and

$$Y_{0,1}\left(\sum_{j=-\infty}^{\infty} \tau_j \widehat{g}^N\right) = \xi^1 + \frac{\beta}{\phi_\beta}[J_N^1 + J_N^2 + J_N^3 - R_N^1 + R_N^2],$$

where

$$I_N^1 = \sum_{k=N/2}^{\widehat{3N/4}} \sum_{i=-N-k}^{\widehat{N+k-1}} \tau_{-i}\mathbf{E}_{\nu_\beta}[V'(r_{N+k+1})^2 p^2_{-N-k-1}(\xi_i^{0,(2N)} - \xi_i^{0,(N+k)})\varphi(\mathcal{E}_{-2N,2N})|\mathcal{F}_{N+k}],$$

$$I_N^2 = \sum_{k=N/2}^{\widehat{3N/4}} \sum_{i=-N-k}^{\widehat{N+k-1}} \tau_{-i}\{(\xi_i^{0,(N+k)} - \xi_i^0)\mathbf{E}_{\nu_\beta}[V'(r_{N+k+1})^2 p^2_{-N-k-1}\varphi(\mathcal{E}_{-2N,2N})|\mathcal{F}_{N+k}]\},$$

$$I_N^3 = \sum_{k=N/2}^{\widehat{3N/4}} \sum_{i=-N-k}^{\widehat{N+k-1}} \xi^0(p,r)\tau_{-i}\mathbf{E}_{\nu_\beta}[V'(r_{N+k+1})^2 p^2_{-N-k-1}(\varphi(\mathcal{E}_{-2N,2N}) - 1)|\mathcal{F}_{N+k}],$$

$$I_N^4 = \sum_{k=N/2}^{\widehat{3N/4}} \frac{1}{2(N+K)}\tau_{-N-k}\mathbf{E}_{\nu_\beta}[V'(r_{N+k+1})^2 p^2_{-N-k-1}\xi_{N+k}^{0,(2N)}\varphi(\mathcal{E}_{-2N,2N})|\mathcal{F}_{N+k}],$$

$$J_N^1 = \widehat{\sum_{k=N/2}^{3N/4}} \widehat{\sum_{i=-N-k}^{N+k-1}} \tau_{-i}\mathbf{E}_{\nu_\beta}[V'(r_{N+k+1})^2 p_{-N-k-1}^2 (\xi_i^{1,(2N)} - \xi_i^{1,(N+k)})\varphi(\mathcal{E}_{-2N,2N})|\mathcal{F}_{N+k}],$$

$$J_N^2 = \widehat{\sum_{k=N/2}^{3N/4}} \widehat{\sum_{i=-N-k}^{N+k-1}} \tau_{-i}\{(\xi_i^{1,(N+k)} - \xi_i^1)\mathbf{E}_{\nu_\beta}[V'(r_{N+k+1})^2 p_{-N-k-1}^2 \varphi(\mathcal{E}_{-2N,2N})|\mathcal{F}_{N+k}]\},$$

$$J_N^3 = \widehat{\sum_{k=N/2}^{3N/4}} \widehat{\sum_{i=-N-k}^{N+k-1}} \xi^1(p,r)\tau_{-i}\mathbf{E}_{\nu_\beta}[V'(r_{N+k+1})^2 p_{-N-k-1}^2 (\varphi(\mathcal{E}_{-2N,2N}) - 1)|\mathcal{F}_{N+k}],$$

$$R_N^1 = \widehat{\sum_{k=N/2}^{3N/4}} \tau_{-N-k}\{V'(r_{N+k+1})\frac{\partial}{\partial p_{N+k}} g^{(N,k)}\},$$

$$R_N^2 = \widehat{\sum_{k=N/2}^{3N/4}} \tau_{N+k+1}\{p_{-N-k-1}\frac{\partial}{\partial r_{-N-k}} g^{(N,k)}\}.$$

Here the hat over the sum symbol means that it is in fact an average, and $\mathcal{E}_{-2N,2N}$ is equal to $\frac{1}{4N+1}\sum_{i=-2N}^{2N}\mathcal{E}_i$.

The proof of the theorem will be concluded in the following way. First we show that the middle terms $I_N^1, I_N^2, I_N^3, I_N^4$ and J_N^1, J_N^2, J_N^3 tend to zero in $L^2(\nu_\beta)$. Then, the proof will be concluded by showing the existence of a subsequence of $\{-R_N^1 + R_N^2\}_{N\geq 1}$ weakly convergent to $cp_0V'(r_1)$ with some constant c.

For the sake of clarity, the proof is divided in three steps. Before that, let us state two remarks.

Remark 2.4. *We know that for $m = 0, 1$, $\boldsymbol{E}_{\nu_\beta}[\xi^m|\mathcal{F}_N] \xrightarrow{L^2} \xi^m$, i.e given $\epsilon > 0$ there exist $N_0 \in \mathbb{N}$ such that*

$$\boldsymbol{E}_{\nu_\beta}[|\xi^m - \xi^{m,(N)}|^2] \leq \epsilon \quad \textit{if} \quad N \geq N_0.$$

Moreover, by the translation invariance we have

$$\boldsymbol{E}_{\nu_\beta}[|\xi_i^m - \xi_i^{m,(N)}|^2] \leq \epsilon \quad \textit{if} \quad [-N_0 - i, N_0 + i] \subseteq [-N, N].$$

In fact, given $\tau_{-i}A \in \mathcal{F}_N$

$$\int_A \xi_i^{m,(N)}(\tau_{-i}(p,r))\nu_\beta(dpdr) = \int_{\tau_{-i}(A)} \xi_i^{m,(N)}(p,r)\nu_\beta(dpdr) = \int_{\tau_{-i}(A)} \xi_i^m(p,r)\nu_\beta(dpdr)$$
$$= \int_A \xi_i^m(\tau_{-i}(p,r))\nu_\beta(dpdr) = \int_A \xi^m(p,r)\nu_\beta(dpdr).$$

In addition, since $\xi_i^{m,(N)}(\tau_{-i}) \in \mathcal{F}_{-N-i}^{N-i}$ *we have*

$$\xi_{i,i+1}^{(N)}(\tau_{-i}) = \boldsymbol{E}_{\nu_\beta}[\xi_{0,1}|\mathcal{F}_{-N-i}^{N-i}]$$

and therefore

$$\boldsymbol{E}_{\nu_\beta}[|\xi_i^m - \xi_i^{m,(N)}|^2] = \boldsymbol{E}_{\nu_\beta}[|\xi^m - \xi_i^{m,(N)}(\tau_{-i})|^2] \leq \boldsymbol{E}_{\nu_\beta}[|\xi^m - \xi_0^{m,(N_0)}|^2].$$

Remark 2.5. *Besides a Strong law of large numbers for* $(p_i^2 V'(r_i)^2)_{i\in\mathbb{Z}}$ *we have*

$$\boldsymbol{E}_{\nu_\beta}\left[\left(\frac{1}{N}\sum_{i=1}^{N} p_i^2 V'(r_i)^2 - \frac{\phi_\beta}{\beta}\right)^2\right] \leq \frac{C_\beta}{N}$$

for some finite constant C_β.

Step 1. The convergence of the middle terms to 0. The convergence to zero as N tends to infinity of I_N^1 and I_N^2 in $L^2(\nu_\beta)$ follows from Schwarz inequality, Remark 2.4 and the fact that φ is a bounded function.

Using the symmetry of the measure about exchanges of variables, I_N^3 can be rewritten as

$$\xi^0(p,r)\sum_{\widehat{k=N/2}}^{\widehat{3N/4}}\sum_{\widehat{i=-N-k}}^{\widehat{N+k-1}} \mathbf{E}_{\nu_\beta}[\sum_{\widehat{j=1}}^{\widehat{N-k}} V'(r_{N+k+j})^2 p_{-N-k-j}^2(\varphi(\mathcal{E}_{-2N,2N}) - 1)|\mathcal{F}_{N+k}](\tau_{-i}(p,r))$$

and then we decompose it as $I_N^5 + \frac{\phi_\beta}{\beta} I_N^6$, where I_N^5 and I_N^6 are respectively

$$\xi^0(p,r)\sum_{\widehat{k=N/2}}^{\widehat{3N/4}}\sum_{\widehat{i=-N-k}}^{\widehat{N+k-1}}$$
$$\mathbf{E}_{\nu_\beta}[\sum_{\widehat{j=1}}^{\widehat{N-k}}\{V'(r_{N+k+j})^2 p_{-N-k-j}^2 - \frac{\phi_\beta}{\beta}\}(\varphi(\mathcal{E}_{-2N,2N}) - 1)|\mathcal{F}_{N+k}](\tau_{-i}(p,r))$$

and

$$\xi^0(p,r)\sum_{\widehat{k=N/2}}^{\widehat{3N/4}}\sum_{\widehat{i=-N-k}}^{\widehat{N+k-1}} \mathbf{E}_{\nu_\beta}[\varphi(\mathcal{E}_{-2N,2N}) - 1|\mathcal{F}_{N+k}](\tau_{-i}(p,r)).$$

For the first term, observe that

$$|I_N^5|^2 \leq |\xi^0(p,r)|^2 \sum_{\widehat{k=N/2}}^{\widehat{3N/4}}\sum_{\widehat{i=-N-k}}^{\widehat{N+k-1}} \mathbf{E}_{\nu_\beta}\left[\left(\sum_{\widehat{j=1}}^{\widehat{N-k}}\{V'(r_{N+k+j})^2 p_{-N-k-j}^2 - \frac{\phi_\beta}{\beta}\}\right)^2\right],$$

and the expectation inside the last expression is bounded by $\frac{C_\beta}{N-k}$, so

$$||I_N^5||^2_{L^2(\nu_\beta)} \leq \frac{C_\beta}{N}||\xi^0||^2_{L^2(\nu_\beta)}.$$

For the second term, written explicitly the conditional expectation we see that $|I_N^6|^2$ is bounded by

$$|\xi^0(p,r)|^2 \sum_{k=N/2}^{\widehat{3N/4}} \sum_{i=-N-k}^{\widehat{N+k-1}} \int |\varphi(\frac{1}{4N+1}\sum_{|j|>N+k}\mathcal{E}'_j + \frac{1}{4N+1}\sum_{|j|\leq N+k}\mathcal{E}_{j+i}) - 1|^2 d\nu_\beta.$$

We rewrite the integral part as

$$\int |\varphi(\frac{1}{4N+1}\sum_{|j|>N+k}(\mathcal{E}'_j - E_\beta) + \frac{1}{4N+1}\sum_{|j|\leq N+k}(\mathcal{E}_{j+i} - E_\beta) + E_\beta) - 1|^2 d\nu_\beta$$

where $E_\beta = \frac{1}{2\beta} + \psi(\beta)$. Using the fact that φ is a Lipschitz positive function bounded by 1 such that $\varphi(E_\beta) = 1$, we obtain that $|I_N^6|^2$ is bounded from above by

$$|\xi^0(p,r)|^2 \sum_{k=N/2}^{\widehat{3N/4}} \sum_{i=-N-k}^{\widehat{N+k-1}} 1 \wedge \int |\frac{1}{4N+1}\sum_{|j|>N+k}(\mathcal{E}'_j - E_\beta) + \frac{1}{4N+1}\sum_{|j|\leq N+k}(\mathcal{E}_{j+i} - E_\beta)|^2 d\nu_\beta$$

where $a \wedge b$ denote the minimum of $\{a, b\}$. So, taking expectation and using the Strong law of large numbers together with the dominated convergence theorem, the convergence to zero as N tends to infinity of I_N^3 in $L^2(\nu_\beta)$ is proved.

Same arguments can be applied for J_N^1, J_N^2 and J_N^3. For I_N^4, we can bound the L^2-norm of the term from above by $\frac{C_\beta}{N}||\xi^0||^2_{L^2(\nu_\beta)}$ for some constant C_β.

Step 2. The uniform bound of the $L^2(\nu_\beta)$ norms of the boundary terms.

Remember that R_N^1 is defined as

$$\sum_{k=N/2}^{\widehat{3N/4}} \frac{1}{2(N+k)}\tau_{-N-k}\{V'(r_{N+k+1})\mathbf{E}_{\nu_\beta}[p^2_{-N-k-1}V'(r_{N+k+1})^2 \frac{\partial}{\partial p_{N+k}} g^{(2N)}|\mathcal{F}_{N+k}]\}$$

$$= -\sum_{k=N/2}^{\widehat{3N/4}} \frac{1}{2(N+k)}$$

$$\tau_{-N-k}\{V'(r_{N+k+1})\mathbf{E}_{\nu_\beta}[p^2_{-N-k-1}V'(r_{N+k+1})Y_{N+k,N+k+1}g^{(2N)}|\mathcal{F}_{N+k}]\}$$

$$+\sum_{k=N/2}^{\widehat{3N/4}} \frac{1}{2(N+k)}$$

$$\tau_{-N-k}\{p_{N+k}V'(r_{N+k+1})\mathbf{E}_{\nu_\beta}[p^2_{-N-k-1}V'(r_{N+k+1})\frac{\partial}{\partial r_{N+k+1}}g^{(2N)}|\mathcal{F}_{N+k}]\}.$$

By Schwarz inequality and (2.8.4), we can see that the $L^2(\nu_\beta)$ norm of the first term in the right hand side of the last equality is bounded by $\frac{C_\beta}{N}||\xi^1||_{L^2(\nu_\beta)}$ for some constant C_β. After an integration by parts, the second term can be written as

(2.8.5)

$$\widehat{\sum_{k=N/2}^{3N/4}}\frac{1}{2(N+k)}$$

$$\tau_{-N-k}\{p_{N+k}V'(r_{N+k+1})\mathbf{E}_{\nu_\beta}[p^2_{-N-k-1}(\beta V'(r_{N+k+1})^2-V''(r_{N+k+1}))g^{(2N)}|\mathcal{F}_{N+k}]\}.$$

Using the symmetry of the measure again, the conditional expectation appearing in the last expression can be rewritten as

$$\mathbf{E}_{\nu_\beta}[p^2_{-N-k-1}\widehat{\sum_{j=N+k+1}^{2N}}(\beta V'(r_j)^2-V''(r_j))(g^{(2N)}\circ\pi_r^{j,N+k+1})|\mathcal{F}_{N+k}]\ ,$$

where $\pi_r^{j,N+k+1}$ stands for the exchange operator of r_j and r_{N+k+1}. After that, we decompose the last expression as the sum of the following two terms,

$$\mathbf{E}_{\nu_\beta}[p^2_{-N-k-1}\widehat{\sum_{j=N+k+1}^{2N}}(\beta V'(r_j)^2-V''(r_j))g^{(2N)}|\mathcal{F}_{N+k}],$$

and

$$\mathbf{E}_{\nu_\beta}[p^2_{-N-k-1}\widehat{\sum_{j=N+k+1}^{2N}}(\beta V'(r_j)^2-V''(r_j))(g^{(2N)}\circ\pi_r^{j,N+k+1}-g^{(2N)})|\mathcal{F}_{N+k}].$$

The square of the last expressions are respectively bounded from above by

$$C_\beta N^{-1}\mathbf{E}_{\nu_\beta}[(g^{(2N)})^2|\mathcal{F}_{N+k}],\quad C_\beta\mathbf{E}_{\nu_\beta}[\widehat{\sum_{j=N+k+1}^{2N}}(g^{(2N)}\circ\pi_r^{j,N+k+1}-g^{(2N)})^2|\mathcal{F}_{N+k}]$$

for some constant C_β. Using Schwarz inequality we can see that the square of each term of the sum is respectively bounded from above by

$$\frac{C_\beta}{N^3}\mathbf{E}_{\nu_\beta}[(\widehat{\sum_{k=N/2}^{3N/4}}p^2_{N+k})(g^{(2N)})^2],$$

and

$$\frac{C'_\beta}{N^2}\widehat{\sum_{k=N/2}^{3N/4}}\mathbf{E}_{\nu_\beta}\big[p^2_{N+k}\widehat{\sum_{j=N+k+1}^{2N}}(g^{(2N)}\circ\pi_r^{j,N+k+1}-g^{(2N)})^2\big]$$

$$\leq\frac{C'_\beta}{N^2}\widehat{\sum_{k=N/2}^{3N/4}}\mathbf{E}_{\nu_\beta}\big[p^2_{N+k}\widehat{\sum_{j=N+k+1}^{2N}}2\{j-(N+k+1)\}\sum_{i=N+k+1}^{j-1}(g^{(2N)}\circ\pi_r^{i,i+1}-g^{(2N)})^2\big]$$

$$\leq\frac{C'_\beta}{N}\widehat{\sum_{k=N/2}^{3N/4}}\mathbf{E}_{\nu_\beta}\big[p^2_{N+k}\sum_{i=3N/2+1}^{2N}(g^{(2N)}\circ\pi_r^{i,i+1}-g^{(2N)})^2\big]$$

for some constants C_β and C'_β. One can now estimate $\widehat{\sum_{k=N/2}^{3N/4}}p^2_{N+k}$ uniformly because of the cutoff. Using the spectral gap estimate (2.7.1) proved in the last section, we can bound the first expression by a constant.

Finally, we state that we can bound the term $\mathbf{E}_{\nu_\beta}\big[(g^{(2N)}\circ\pi_r^{i,i+1}-g^{(2N)})^2\big]$ by the Dirichlet form of $g^{(2N)}$ which concludes the proof.

Proposition 2.10. *There exists some constant C such that for all $f:\Omega\to\mathbb{R}$,*

$$\boldsymbol{E}_{\nu_\beta}\big[(f\circ\pi_r^{i,i+1}-f)^2\big]\leq C\{\boldsymbol{E}_{\nu_\beta}\big[(X_i f)^2\big]+\boldsymbol{E}_{\nu_\beta}\big[(Y_{i,i+1}f)^2\big]\}.$$

Proof. The change of variables used in the last section and simple computations conclude the proof. □

Step 3. The existence of a weakly convergent subsequence of $\{R^1_N\}_{N\geq1}$. Firstly, observe that the expression (2.8.5) is equal to

$$p_0V'(r_1)h^1_N(p_0,r_0,\dots,p_{-7N/2},r_{-7N/2})$$

where

$$h^1_N=\widehat{\sum_{k=N/2}^{3N/4}}\frac{1}{2(N+k)}\tau^{-N-k}\mathbf{E}_{\nu_\beta}[p^2_{-N-k-1}(\beta V'(r_{N+k+1})^2-V''(r_{N+k+1}))g^{(2N)}|\mathcal{F}_{N+k}].$$

On the other hand, we had proved in **Step 2** that $\{p_0V'(r_1)h^1_N\}_{N\geq1}$ is bounded in $L^2(\nu_\beta)$, therefore it contains a weakly convergent subsequence $\{p_0V'(r_1)h_{N'}\}_{N'}$. We can conclude in a similar way that $\{h^1_N\}_{N\geq1}$ is bounded in $L^2(\nu_\beta)$, therefore $\{h^1_{N'}\}_{N'}$ contains a weakly convergent subsequence, whose limit will be denoted by h. It is easy to see that

$$||X_ih^1_N||_{L^2(\nu_\beta)}\leq\frac{C}{N}||\xi^0||_{L^2(\nu_\beta)}\qquad for\qquad i\in\{0,-1,-2,\cdots\}$$

and

$$||Y_{i,i+1}h_N^1||_{L^2(Q_y)} \leq \frac{C}{N}||\xi^1||_{L^2(Q_y)} \qquad \textit{for} \qquad \{i, i+1\} \subseteq \{0, -1, -2, \cdots\}$$

which implies that $X_i h = 0$ for $i \in\subseteq \{0, -1, -2, \cdots\}$ and $Y_{i,i+1}h = 0$ for $\{i, i+1\} \subseteq \{0, -1, -2, \cdots\}$. Since the function h depends only on $\{p_0, r_0, p_{-1}, r_{-1}, p_{-2}, r_{-2} \cdots\}$ one can show that h is a constant function, let's say c. Taking suitable test functions, we can conclude that in fact $\{p_0 V'(r_1) h_{N'}^1\}_{N'}$ converges weakly to $cp_0 V'(r_1)$. This proves that for every weakly convergent subsequence of $\{R_N^1\}_{N\geq 1}$ there exist a constant c such that the limit is $cp_0 V'(r_1)$. Exactly the same can be said about $\{R_N^2\}_{N\geq 1}$. □

Chapter 3

Spectral gap for multi-species exclusion processes

3.1 Introduction

A key estimate needed for hydrodynamic limits is a sharp upper bound on the relaxation time (inverse of the spectral gap) of a process (cf. [14]). In recent works, the spectral gap for the multi-color exclusion process has been studied (e.g. [6], [8]). A distinctive feature of the multi-color exclusion process is that the spectral gap depends on the density of vacant sites, which is not the case for the one-species simple exclusion process. In particular, the spectral gap vanishes as the density of vacant sites approaches 0. This degeneracy of the spectral gap was first shown by Quastel in [21] for the simple exclusion process with color, which was introduced by himself. In [6] and [8], the non-homogeneous multi-color exclusion process was considered and they estimated the dependence of the spectral gap on the density of vacant sites in detail. The aim of this paper is to extend the previous results to a multi-*species* exclusion process. Namely, we consider a system of several species of particles having their own dynamics, or precisely, their own jump rates and jump ranges. In physical point of view, it is a system of several constituents having physically different properties. We study a process in homogeneous and non-homogeneous hypercubes of $\mathbb{Z}^d$ both.

The homogeneous multi-species exclusion process is defined as follows. Let us consider the number of species $r \geq 2$ and the d-dimensional cube $\Lambda_n := \{-n, -n+1, ..., n-1, n\}^d$ with $n \geq 1$ and $d \geq 1$. A configuration is denoted by $\eta \in \Sigma_n := \{0, 1, 2, \dots, r\}^{\Lambda_n}$ with interpretation that $\eta(x) = 0$ means that the site x is empty, whereas $\eta(x) = i$ for $i \in \{1, 2, \dots, r\}$ means that x is occupied by a particle of the i-th species. Each particle of the i-th species at x jumps to y at rate $g(i)p_i(x, y)$ if y is empty. Here $g(i)$ is a positive constant representing a speed of the i-th species and $p_i(\cdot, \cdot)$ is a transition probability of the i-th species on $\mathbf{Z}^d$. p_i is assumed to be finite range, translation invariant, irreducible and symmetric for all i.

It is obvious that the numbers of particles of each species are conserved under the dynamics. We define $\Sigma_{n,k}$ for $k = (k_0, k_1, \dots, k_r)$ with $\sum_{i=0}^{r} k_i = |\Lambda_n|$ as a set of configurations with k_i particles of the i-th species and k_0 empty sites: $\Sigma_{n,k} := \{\eta \in \Sigma_n; \sum_{x\in\Lambda_n} 1_{\{\eta_x = i\}} = k_i \text{ for all } 0 \le i \le r\}$.

One of the difficulties in studying the multi-species exclusion process (or even the multi-color exclusion process) is that there exist some choices of $\{p_i\}_{i=1}^{r}$ where for the dynamics defined by $\{p_i\}_{i=1}^{r}$, $\Sigma_{n,k}$ is not an ergodic component even for large n and $k_0 \ge 1$. We give two examples of such $\{p_i\}_{i=1}^{r}$ below (Example 3.1, 3.2). We also give a sufficient condition for the local ergodicity, or precisely, the following to hold: for large enough n, $\Sigma_{n,k}$ is an ergodic component if $k_0 \ge 1$ (Proposition 3.1).

Under the assumption of the local ergodicity (Assumption 3.1), we prove that the spectral gap for the homogeneous multi-species exclusion process is of order $\frac{\rho_0}{n^2}$ where $\rho_0 := \frac{k_0}{|\Lambda_n|}$ is the density of vacant sites (Theorem 3.1). Moreover, we show that Assumption 3.1 is crucial for the estimate (Proposition 3.2).

The non-homogeneous multi-species exclusion process is a multi-species exclusion process on a disordered lattice. Let $\{q_x\}_{x\in\mathbf{Z}^d}$ be a collection of occupation probabilities satisfying $\varepsilon \le q_x \le 1-\varepsilon$ for some $\varepsilon \in (0, \frac{1}{2}]$. In the non-homogeneous dynamics, each particle of the i-th species at site x jumps to y with rate $g(i)p_i(x, y)(1-q_x)q_y$ if y is empty.

In [6], Caputo considered the mean-field type non-homogeneous multi-color exclusion process, namely the case with $g(i) = 1$ and $p_i(x, y) \equiv 1/|\Lambda_n|$ for all $1 \le i \le r$. He showed that the spectral gap is of order ρ_0.

In [8], Dermoune and Heinrich considered the non-homogeneous nearest-neighbor multi-color exclusion process, namely the case with $g(i) = 1$ and $p_i(x, y) = 1_{\{\max_{1\le j\le d}|x_j - y_j| = 1\}}$ for all $1 \le i \le r$. They showed that assuming $\{q_x\}_{x\in\mathbf{Z}^d}$ to be periodic, the spectral gap is bounded from below by $C\rho_0 n^{-2}$ with a positive constant C.

In this thesis, we study the case with more general $g(i)$ and p_i assuming the local ergodicity (Assumption 3.1). To obtain a lower bound of the spectral gap, we give two sufficient conditions for $\{q_x\}_{x\in\mathbf{Z}^d}$ (Assumption 3.2). First condition is a weaker version of the condition assumed in [8]. Second condition is a rather different condition. Assuming either of them, we show that the spectral gap is bounded from below by $C\rho_0 n^{-2}$ (Theorem 3.2). It is worth to point out that the second condition will hold almost surely for realizations of some i.i.d. random disorder.

This chapter is organized as follows: In Section 3.2 we introduce our model and give two examples of jump ranges for which the local ergodicity does not hold. Then, we state our main results. In Section 3.3, we give proofs of main results. In Section 3.4, we give proofs of lemmas used in Section 3.3.

3.2 Model and results

Let $g : \{0, 1, \ldots, r\} \to \mathbf{R}$ be a function satisfying $g(0) = 0$, $g(i) > 0$ for all $i \in \{1, \ldots, r\}$ and $\{p_i\}_{1\le i\le r}$ be finite range, translation invariant, irreducible symmetric transition probabilities on $\mathbf{Z}^d$. The generator acting on functions $f : \Sigma_n \to \mathbf{R}$ as

$$(L_n f)(\eta) := \sum_{x,y\in\Lambda_n} p_{\eta_x}(x,y) g(\eta_x) 1_{\{\eta_y=0\}} \pi^{x,y} f(\eta)$$

defines a Markov process on Σ_n called multi-species exclusion process with parameters $(g, \{p_i\}_{1\le i\le r})$ where $p_0(x,y) \equiv 0$ by convention. Here, $\pi^{x,y}$ is an operator defined by

$$\pi^{x,y} f(\eta) = f(\eta^{x,y}) - f(\eta)$$

and $\eta^{x,y}$ is the configuration obtained from η letting η_x and η_y be exchanged:

$$(\eta^{x,y})_z = \begin{cases} \eta_y & \text{if } z = x \\ \eta_x & \text{if } z = y \\ \eta_z & \text{otherwise.} \end{cases}$$

One may expect that if n is large enough, then each of $\Sigma_{n,k}$ for $k = (k_0, k_1, \ldots, k_r)$ with $\sum_{i=0}^{r} k_i = |\Lambda_n|$ and $k_0 \ge 1$ is an ergodic component of L_n. But there is a counter example as follows:

Example 3.1. *Let $d = 2$ and $r = 2$. Suppose that the range of p_1 is the set of nearest neighbor sites and that of p_2 is given by the legal knight-moves, i.e., $\{x \in \mathbf{Z}^2 : p_1(x) > 0\} = \{x \in \mathbf{Z}^2 : \|x\|_1 = 1\}$ and $\{x \in \mathbf{Z}^2 : p_2(x) > 0\} = \{(\pm 2, \pm 1), (\pm 1, \pm 2)\}$*

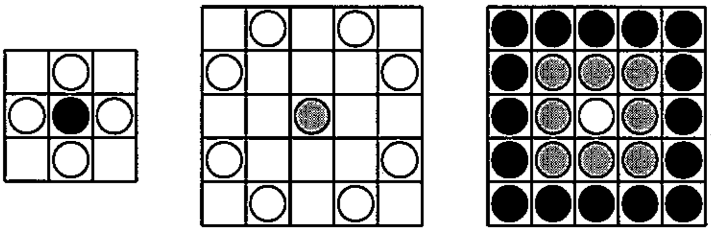

Figure 3.1: This figure corresponds to Example 3.1. The black circle corresponds to a first-species particle and the gray circle to a second-species particle. In the left and center figures, the white circles correspond to the range of the first-species particles and the second-species particles respectively. In the right figure, the white circle corresponds to a vacant site.

where $\|x\|_1 := \sum_{i=1}^{d} |x_i|$ *for* $x = (x_1, x_2, \ldots, x_d) \in \mathbf{Z}^d$. *First, we consider the ergodicity of a hyperplane* $\Sigma_{n,k}$ *with* $n \geq 2$ *and* $k_0 = 1$. *If there is no particle at origin, the sites* $\{x = (x_1, x_2) \in \mathbf{Z}^2 : \max\{|x_1|, |x_2|\} = 1\}$ *are occupied by second-species particles, the sites* $\{x = (x_1, x_2) \in \mathbf{Z}^2 : \max\{|x_1|, |x_2|\} = 2\}$ *are occupied by first-species particles, and each of other sites is occupied by one of a first or second species particle, then any particle cannot move. It implies that* $\Sigma_{n,k}$ *should be divided into several ergodic components. Note that in this case,* $k_0 = 1$ *is not essential. Let* $\{x_i = (x_{i,1}, x_{i,2})\}_{i=1}^{k} \subset \Lambda_n$ *be a sequence satisfying* $|x_{i,1} - x_{j,1}| \geq 4$ *and* $|x_{i,2} - x_{j,2}| \geq 4$ *for all* $1 \leq i < j \leq k$. *Suppose that there is no particle at* $\{x_i\}_{i=1}^{k}$, *the sites* $\{x_i + z : 1 \leq i \leq k, z = (z_1, z_2), \max\{|z_1|, |z_2|\} = 1\} \cap \Lambda_n$ *are occupied by second-species particles, the sites* $\{x_i + z : 1 \leq i \leq k, z = (z_1, z_2), \max\{|z_1|, |z_2|\} = 2\} \cap \Lambda_n$ *are occupied by first-species particles, and each of other sites is occupied by one of a first or second species particle. Then any particle cannot move again. This yields, if* $k_0 \leq \frac{(2n+1)^2}{16}$, $\Sigma_{n,k}$ *should be divided into several ergodic components.*

One may still expect that if $p_1 = p_2 = \ldots = p_r$ and n is large enough, then each of $\Sigma_{n,k}$ for $k = (k_0, k_1, \ldots, k_r)$ with $\sum_{i=0}^{r} k_i = |\Lambda_n|$ and $k_0 \geq 1$ is an ergodic component of L_n. But there is also a counter example as follows:

Example 3.2. *Let* $d = 2$, $r = 2$ *and* $p_1 = p_2$ *satisfy the condition* $\{x \in \mathbf{Z}^2 : p_1(x) > 0\} = \{x \in \mathbf{Z}^2 : p_2(x) > 0\} = \{(\pm 1, \pm 1), (-2, 1), (-1, 2), (1, -2), (2, -1)\}$. *Consider the ergodicity of a hyperplane* $\Sigma_{n,k}$ *with* $k_0 = 1$ *and* $k_1, k_2 \geq 1$. *For each configuration in such* $\Sigma_{n,k}$, *the particle placed at upper right corner is frozen at the corner. Precisely, the particle placed at upper right corner can move only to its left lower site, but there is one vacant site so if the particle tries to move again, then*

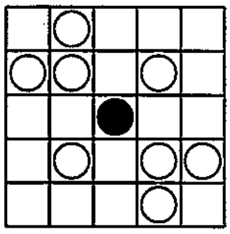

Figure 3.2: This figure corresponds to Example 3.2. The black circle corresponds to a first-species or a second-species particle and the white circles correspond to the range of these particles.

it has to go back to the upper right corner. (Also, the particle placed at lower left corner is frozen at the corner.) Therefore $\Sigma_{n,k}$ *should be divided into three ergodic components if* $k_1 = 1$ *or* $k_2 = 1$*, and into four if* $k_1, k_2 \geq 2$*. Note that, in this case,* $k_0 = 1$ *is crucial. It is not difficult to see that if* $k_0 \geq 2$*, then* $\Sigma_{n,k}$ *is an ergodic component for* n *large enough.*

To avoid the case where the spectral gap vanishes, from now on we make the following assumption:

Assumption 3.1. *There exists* n_0 *such that for each* $n \geq n_0$ *and* $k = (k_0, k_1, \ldots, k_r)$ *with* $\sum_{i=0}^{r} k_i = |\Lambda_n|$ *and* $k_0 \geq 1$*,* $\Sigma_{n,k}$ *is an ergodic component of* L_n*.*

It should be possible to replace the condition in Assumption 3.1 by a condition for the ranges of $\{p_i\}_{1 \leq i \leq r}$. However it seems difficult to give an equivalent condition so far, hence we give a following sufficient condition:

Proposition 3.1. *Suppose that one of the following two conditions is satisfied:*
(i) $d \geq 2$ *and* $p_i(x) > 0$ *for all* $\|x\|_1 = 1$ *and* $i \in \{1, 2, \ldots, r\}$*,*
(ii) $p_i(x) > 0$ *for all* $\|x\|_1 = 1$ *and* $i \in \{1, 2, \ldots, r\}$*, and for each* $i \in \{1, 2, \ldots, r\}$*, there exists* $l_i \in \mathbf{Z}^d$ *such that* $\|l_i\|_1 \geq 2$*,* $p_i(l_i) > 0$*.*
Then Assumption 3.1 holds.

Next, we define the spectral gap precisely. Let $\nu_{n,k}$ be the uniform probability measure on $\Sigma_{n,k}$ for $k = (k_0, k_1, \ldots, k_r)$ with $\sum_{i=0}^{r} k_i = |\Lambda_n|$. It is easy to see that L_n is reversible with respect to $\nu_{n,k}$. Under the Assumption 3.1, for $n \geq n_0$, the restriction of L_n on $\Sigma_{n,k}$ which is denoted by $L_{n,k}$ is irreducible if $k_0 \geq 1$. The spectral gap of $-L_{n,k}$ is given by

$$\lambda = \lambda(n,k) := \inf_f \left\{ \frac{E_{\nu_{n,k}}[f(-L_n)f]}{E_{\nu_{n,k}}[f^2]} \;\middle|\; E_{\nu_{n,k}}[f] = 0 \right\}. \tag{3.2.1}$$

In what follows C, C' etc. represent constants that do not depend on n nor k.

Theorem 3.1. *Suppose that Assumption 3.1 holds. Then, there exist positive constants C and C' such that for all $n \geq n_0$ and $k = (k_0, k_1, \dots, k_r)$ satisfying $\sum_{i=0}^r k_i = |\Lambda_n|$ and $k_i, k_j > 0$ for some $1 \leq i < j \leq r$,*

$$C\frac{\rho_0}{n^2} \leq \lambda(n,k) \leq C'\frac{\rho_0}{n^2}.$$

Remark 3.1. *Consider a hyperplane $\Sigma_{n,k}$ satisfying $k_i = 0$ for all $1 \leq i \leq r$ but one. Then the process restricted on $\Sigma_{n,k}$ turns out to be a symmetric simple exclusion process. Hence the spectral gap does not depend on the density of vacant sites and it is of order n^{-2}. This estimate is true even for the non-homogeneous case defined below (See [22]).*

We emphasize that Assumption 3.1 is essential to the estimate of the dependence on the density of vacant sites:

Proposition 3.2. *There exists a multi-species exclusion process satisfying the followings: (i) For large enough n, $\Sigma_{n,k}$ is an ergodic component of L_n if $k_0 \geq 2$,*
(ii) For all $n \in \mathbf{N}$, $\Sigma_{n,k}$ is divided into several ergodic components if $k_0 = 1$ and $k_i, k_j > 0$ for some $1 \leq i < j \leq r$ (especially, Assumption 3.1 does not hold),
(iii) the spectral gap $\lambda(n,k) \leq C \min\{\frac{\rho_0}{n^2}, \rho_0^2\}$ if $k_0 \geq 2$ and $k_i, k_j > 0$ for some $1 \leq i < j \leq r$.

Now we move on to the non-homogeneous case. We shall use the notation

$$\xi_x(\eta) = 1_{\{\eta_x \geq 1\}}$$

for $x \in \Lambda_n$ so that $\xi(\eta) \in \{0,1\}^{\Lambda_n}$ denotes the configuration of occupied sites associated to η. Let $\{q_x, x \in \mathbf{Z}^d\}$ be a collection of occupation probabilities. All throughout this paper, we assume that for some $\varepsilon \in (0, \frac{1}{2}]$, we have

$$\varepsilon \leq q_x \leq 1 - \varepsilon \quad \text{for all } x \in \mathbf{Z}^d.$$

Given a collection of occupation probability $\{q_x, x \in \mathbf{Z}^d\}$, we define the jump rate c by

$$c_{x,y}(\eta) = \begin{cases} q_x(1-q_y) & \text{if } (\xi_x(\eta), \xi_y(\eta)) = (0,1), \\ q_y(1-q_x) & \text{if } (\xi_x(\eta), \xi_y(\eta)) = (1,0), \\ 0 & \text{otherwise.} \end{cases}$$

The generator L_n acting on functions $f : \Sigma_n \to \mathbf{R}$ as

$$(L_n f)(\eta) := \sum_{x,y \in \Lambda_n} p_{\eta_x}(x,y) g(\eta_x) c_{x,y}(\eta) \pi^{x,y} f(\eta)$$

defines a Markov process on Σ_n called non-homogeneous multi-species exclusion process with parameter $(g, \{p_i\}_{i=1}^r, \{q_x\}_{x \in \mathbf{Z}^d})$ where $p_0(x,y) \equiv 0$ by convention. When q_x does not depend on x, we say that we are in the homogeneous case. Actually, in that case,

$$g(\eta_x) c_{x,y} = g(\eta_x) q(1-q) 1_{\{\eta_y = 0\}}$$

for some q, therefore the notation L_n is consistent with previous definition except multiple constant $q(1-q)$.

We now describe reversible measures of the process. Let $(s_i)_{1 \le i \le r}$ be a probability distribution on the set $\{1, ..., r\}$ and μ be the product probability measure on Σ_n defined by

$$\mu(\eta) = \prod_{i=1}^r s_i^{N_i(\eta)} \prod_{x \in \Lambda_n} \left(q_x^{\xi_x(\eta)} (1-q_x)^{1-\xi_x(\eta)} \right).$$

where $N_i(\eta)$ is the number of particles of the i-th species on Λ_n for a configuration η. We denote the canonical measure associated to $k = (k_0, k_1, ..., k_r)$ by $\nu_{n,k}(\cdot) := \mu(\cdot | \Sigma_{n,k})$ which is indeed independent of the choice of $(s_i)_{1 \le i \le r}$. In the homogeneous case, $\nu_{n,k}$ is uniform measure on $\Sigma_{n,k}$ so that the notation is consistent with the previous definition.

It is easy to see that the generator L_n is reversible w.r.t. μ and the same is true for the measure $\nu_{n,k}$ since $\nu_{n,k}(\eta) c_{x,y}(\eta) = \nu_{n,k}(\eta^{x,y}) c_{y,x}(\eta^{x,y})$ for all x, y and η. In the same way as the homogeneous case, under Assumption 3.1, for $n \ge n_0$, the restriction of L_n on $\Sigma_{n,k}$ which is denoted by $L_{n,k}$ is irreducible if $k_0 \ge 1$ and the spectral gap of $-L_{n,k}$ is given by (3.2.1).

To give a lower bound of the spectral gap, we need the following assumption for $\{q_x\}_{x \in \mathbf{Z}^d}$:

Assumption 3.2. *Either of them holds:*
(i) There exists $q^0 \in [\varepsilon, 1-\varepsilon]$ *and* $T \in \mathbf{N}$ *such that for any* $x \in \mathbf{Z}^d$, $\{y \in 2Tx + \Lambda_T :$

$q_y = q^0\}$ *is not empty.*
(ii) There exists $M \in \mathbf{N}$ *such that* $\#\{q_x : x \in \mathbf{Z}^d\} = M$.

Theorem 3.2. *Suppose that Assumptions 3.1 and 3.2 hold. Then, there exists a positive constant* C *such that for all* $n \geq n_0$ *and* $k = (k_0, k_1, \ldots, k_r)$ *satisfying* $\sum_{i=0}^{r} k_i = |\Lambda_n|$,

$$\lambda(n,k) \geq C\frac{\rho_0}{n^2}.$$

3.3 Proofs of the main results

3.3.1 Proof of Theorem 3.2

First we recall the following result due to Caputo [6]. (See also references therein and [7]). Given a collection of occupation probability $\{q_x\}_{x\in\mathbf{Z}^d}$, we define the mean-field type non-homogeneous multi-color exclusion process by the generator

$$L_n^m f(\eta) = \frac{1}{|\Lambda_n|}\sum_{x,y\in\Lambda_n} c_{x,y}(\eta)\pi^{x,y}f(\eta),$$

where $c_{x,y}$ is given by (3.2). Then it is easy to see that L_n^m is also reversible w.r.t. $\nu_{n,k}$ defined in the previous section. The restriction of L_n^m on $\Sigma_{n,k}$ which is denoted by $L_{n,k}^m$ is irreducible if $k_0 \geq 1$. The spectral gap of $-L_{n,k}^m$ is given by

$$\lambda_m = \lambda_m(n,k) := \inf_f \left\{ \frac{E_{\nu_{n,k}}[f(-L_n^m)f]}{E_{\nu_{n,k}}[f^2]} \;\middle|\; E_{\nu_{n,k}}[f] = 0 \right\}.$$

Proposition 3.3. *([6]) There exist constants* $C_1 = C_1(\varepsilon)$ *and* $C_2 = C_2(\varepsilon)$ *such that for all* n *and* $k = (k_0, k_1, \ldots, k_r)$ *satisfying* $\sum_{i=0}^{r} k_i = |\Lambda_n|$,

$$C_1\rho_0 \leq \lambda_m(n,k) \leq C_2\rho_0.$$

Note that Proposition 3.3 was proved without Assumption 3.2.

According to this result, to get the lower bound of the spectral gap in our main theorems, it is enough to bound the term $E_{\nu_{n,k}}[f(-L_n^m)f]$ from above by the Dirichlet form of our process multiplied by n^2 and some constant. In order to do it, we use following two lemmas.

Lemma 3.2.1. *Suppose that Assumption 3.2 holds. Then there exists a positive constant $C = C(\varepsilon, d, T)$ or $C(\varepsilon, d, M)$ such that for all n, $k = (k_0, k_1, \ldots, k_r)$ satisfying $\sum_{i=0}^{r} k_i = |\Lambda_n|$ and $f : \Sigma_{n,k} \to \mathbf{R}$*

$$(3.3.1) \qquad E_{\nu_{n,k}}[f(-L_n^m)f] \leq Cn^2 \sum_{x \in \Lambda_n} \sum_{\|z\|_1 = 1,2; x+z \in \Lambda_n} E_{\nu_{n,k}}[1_{\{\eta_{x+z}=0\}}(\pi^{x,x+z} f(\eta))^2].$$

It follows immediately from this assertion that we have the desired bound if $g(i) = 1$ and $\{x \in \mathbf{Z}^d : \|x\|_1 = 1, 2\} \subset \{x \in \mathbf{Z}^d : p_i(x) > 0\}$ for all $1 \leq i \leq r$.

The next lemma claims that according to Assumption 3.1, we can bound each term of the RHS of (3.3.1) corresponding to x from above by the Dirichlet form of the generator restricted to a certain neighborhood of x multiplied by some universal constant. To state the next lemma, we introduce some notations. Define a shift operator τ_x by

$$(\tau_x \eta)_z = \eta_{x+z} \quad \text{for } x, z \in \mathbf{Z}^d,$$
$$\tau_x A = x + A \quad \text{for } x \in \mathbf{Z}^d \text{ and } A \subset \mathbf{Z}^d.$$

For $n \geq n_0$ and $x \in \Lambda_n$, we define $\bar{x} = \bar{x}(x) \in \Lambda_n$ by

$$\bar{x}_i = \begin{cases} \min\{n - n_0, x_i\} & \text{if } x_i \geq 0, \\ \max\{n_0 - n, x_i\} & \text{if } x_i < 0. \end{cases}$$

Obviously, $\tau_{\bar{x}} \Lambda_{n_0} \subset \Lambda_n$ and $\{x + z; \|z\|_1 = 1, 2, \ x + z \in \Lambda_n\} \subset \tau_{\bar{x}} \Lambda_{n_0}$, since we can regard n_0 to be greater than or equal to two. Note that if $\|x - z\|_1 > n_0$ for all $z \notin \Lambda_n$ then $\bar{x} = x$.

Lemma 3.2.2. *Suppose that Assumption 3.1 holds. Then, there exists a positive constant $C = C(\varepsilon, d, n_0)$ such that for all n, $k = (k_0, k_1, \ldots, k_r)$ satisfying $\sum_{i=0}^{r} k_i = |\Lambda_n|$, $f : \Sigma_{n,k} \to \mathbf{R}$ and $x \in \Lambda_n$*

$$\sum_{z: x+z \in \Lambda_n, \|z\| = 1,2} E_{\nu_{n,k}}[1_{\{\eta_{x+z}=0\}}(\pi^{x,x+z} f(\eta))^2]$$
$$\leq C \sum_{w,v \in \tau_{\bar{x}} \Lambda_{n_0}} E_{\nu_{n,k}}[p_{\eta_w}(w, v) 1_{\{\eta_v = 0\}}(\pi^{w,v} f(\eta))^2].$$

We will prove these lemmas in the next section.

Proof of Theorem 3.2. From Proposition 3.3, Lemma 3.2.1 and Lemma 3.2.2,

$$V_{\nu_{n,k}}[f^2] \leq C \frac{1}{\rho_0} E_{\nu_{n,k}}[f(-L_n^m)f]$$

$$\leq C\frac{n^2}{\rho_0}\sum_{x\in\Lambda_n}\sum_{\|z\|_1=1,2;x+z\in\Lambda_n} E_{\nu_{n,k}}[1_{\{\eta_{x+z}=0\}}(\pi^{x,x+z}f(\eta))^2]$$

$$\leq C\frac{n^2}{\rho_0}\sum_{x\in\Lambda_n}\sum_{w,v\in\tau_{\bar{x}}\Lambda_{n_0}} E_{\nu_{n,k}}[p_{\eta_w}(w,v)1_{\{\eta_v=0\}}(\pi^{w,v}f(\eta))^2].$$

Since for all $w \in \Lambda_n$, $|\{x; w \in \tau_{\bar{x}}\Lambda_{n_0}\}| \leq (4n_0+1)^d$, the last term is bounded from above by

$$C\frac{n^2}{\rho_0}\sum_{w,v\in\Lambda_n} E_{\nu_{n,k}}[p_{\eta_w}(w,v)1_{\{\eta_v=0\}}(\pi^{w,v}f(\eta))^2].$$

Here a constant C changes from line to line. Now, since

$$p_{\eta_w}(w,v)1_{\{\eta_v=0\}} \leq \max_{1\leq i\leq r}\frac{1}{g(i)}\frac{c_{w,v}(\eta)}{\varepsilon^2}p_{\eta_w}(w,v)g(\eta_w)$$

it is easy to see that

$$V_{\nu_{n,k}}[f^2] \leq C\frac{n^2}{\rho_0}\sum_{w,v\in\Lambda_n} E_{\nu_{n,k}}[p_{\eta_w}(w,v)g(\eta_w)c_{w,v}(\eta)(\pi^{w,v}f(\eta))^2]$$

$$= C\frac{n^2}{\rho_0}E_{\nu_{n,k}}[f(-L_n)f]$$

for some positive constant C. □

3.3.2 Proof of Theorem 3.1

Proof of Theorem 3.1. We have only to prove the upper bound estimate, since Theorem 3.2 is valid for the homogeneous case. For each fixed n and $k = (k_0, k_1, \ldots, k_r)$ satisfying $\sum_{i=0}^{r} k_i = |\Lambda_n|$, we pick and fix a pair (i,j) such that $1 \leq i,j \leq r$, $i \neq j$ and $0 < k_i \leq k_j$. We define F by

$$F(\eta) = \sum_{x\in\Lambda_n} x_1 1_{\{\eta_x=i\}}$$

where x_1 denotes the first component of x. Then we have

$$\begin{aligned} E_{\nu_{n,k}}[F] &= 0, \\ E_{\nu_{n,k}}[F^2] &\asymp n^2|\Lambda_n|\rho_i(1-\rho_i), \\ E_{\nu_{n,k}}[F(-L_n)F] &\asymp |\Lambda_n|\rho_i\rho_0. \end{aligned}$$

where $\rho_i = \frac{k_i}{|\Lambda_n|}$ and $f \asymp g$ means that there exist positive constants C and C' such that $Cf \le g \le C'f$. Since $\rho_i \le 1/2$, there exists a constant C such that

$$\lambda(n,k) \le \frac{E[F(-L_n)F]}{E[F^2]} \le \frac{C\rho_0}{n^2}.$$

□

3.3.3 Proof of Proposition 3.1

Proof of Proposition 3.1.

First we suppose that the condition (i) is satisfied. In particular, we can assume that $d \ge 2$ and $\{x \in \mathbf{Z}^d : p_i(x) > 0\} = \{x \in \mathbf{Z}^d : \|x\|_1 = 1\}$ for all $1 \le i \le r$. In this case, on the one hand, it is not difficult to see that $\Sigma_{n,k}$ is an ergodic component if $k_0 \ge 2$. On the other hand it is well known that if $k_0 = 1$ and $\max_{1 \le i \le r} k_i = 1$, i.e., all particles are distinguishable, then $\Sigma_{n,k}$ is divided into two ergodic components. Furthermore, for such $\Sigma_{n,k}$, if $\eta \in \Sigma_{n,k}$, $x, y \in \Lambda_n$ satisfying $\|x - y\|_1 = 1$ and $\eta_x, \eta_y \ne 0$, then η and $\eta^{x,y}$ belong to different ergodic components (see e.g. Section 7.4 in [13]). This yields that though $k_0 = 1$, if $\max_{1 \le i \le r} k_i > 1$ then $\Sigma_{n,k}$ is an ergodic component since we have at least one pair of particles of same species. Hence we take $n_0 = \min\{n \in \mathbf{N} : |\Lambda_n| \ge r + 2\}$. Then for any $n \ge n_0$, we have $\max_{0 \le i \le r} k_i > 1$, which completes the proof of the first case.

Second we suppose that $d = 1$, $p_i(x) > 0$ for all $|x| = 1$ for all $1 \le i \le r$ and there exists $l_0 \ge 2$ such that $p_i(x) > 0$ for all $|x| = l_0$ and $1 \le i \le r$. It is a stricter condition than the condition (ii). In this case, it is not difficult to see that if $\max_{0 \le i \le r} k_i \ge l_0 - 1$ and $k_0 \ge 1$, then $\Sigma_{n,k}$ is an ergodic component. Hence we take $n_0 = \min\{n \in \mathbf{N} : 2n+1 \ge (r+1)(l_0-2)+1\}$ and then we have $\max_{0 \le i \le r} k_i \ge l_0 - 1$ for any $n \ge n_0$.

Finally we suppose that the condition (ii) is satisfied. We only need to prove the case where $d = 1$, $p_i(x) > 0$ for all $|x| = 1$ and $1 \le i \le r$, and there exists $l = (l_1, l_2, \dots, l_r) \in \mathbf{Z}^r$ such that $|l_i| \ge 2$, $p_i(l_i) > 0$ for all $1 \le i \le r$. In this case, set $l_0 = l_0' + 1$ where l_0' is the least common multiple of $l_1 - 1, l_2 - 1, \dots, l_r - 1$. Then one can see that if $\max_{0 \le i \le r} k_i \ge l_0 - 1$ and $k_0 \ge 1$, then $\Sigma_{n,k}$ is an ergodic component, similarly to the second case. Hence we take $n_0 = \min\{n \in \mathbf{Z} : 2n + 1 \ge (r+1)(l_0 - 2) + 1\}$ and then we have $\max_{0 \le i \le r} k_i \ge l_0 - 1$ for any $n \ge n_0$, which proves the proposition. □

3.3.4 Proof of Proposition 3.2

Proof of Proposition 3.2.

Let $d \geq 2$, $r = 2$ and $q_x \equiv q$. We define a set $A \subset \mathbf{Z}^d$ by

$$\begin{aligned} A &= A_1 \cup A_4, \\ \Lambda_1 &= \{x = (x_1, x_2, \ldots, x_d) \in \mathbf{Z}^d; \max_{1 \leq i \leq d} |x_i| \leq 1\}, \\ A_1 &= \{(\pm 1, \pm 1, \ldots, \pm 1) \in \mathbf{Z}^d\}, \\ A_2 &= A_1 \setminus \{(1,1,\ldots,1), (-1,-1,\ldots,-1) \in \mathbf{Z}^d\}, \\ A_3 &= \{x \in \mathbf{Z}^d; \|x\|_1 = 1\}, \\ A_4 &= (\cup_{x \in A_2} A_3 + x) \setminus \Lambda_1. \end{aligned}$$

Suppose that the range of p_1 and p_2 are same and satisfy $\{x \in \mathbf{Z}^d : p_1(0,x) > 0\} = \{x \in \mathbf{Z}^d : p_2(0,x) > 0\} = A$. This model with $d = 2$ corresponds to Example 3.2. Just as shown in Example 3.2, if $k_0 = 1$, then the particles placed upper right corner, $(n, n, \ldots, n)$ and lower left corner $(-n, -n, \ldots, -n)$ are frozen at the corner for all n. Furthermore if $k_0 \geq 2$, $\Sigma_{n,k}$ is an ergodic component for n large enough. To simplify our model, we suppose that $g(i) = 1$ for $i = 1, 2$.

The upper bound estimate given in Section 3.3.2 is valid for this model. To obtain another upper bound of the spectral gap, we define G by

$$\begin{aligned} G(\eta) &= 1_{\{\eta_{y_n} = i\}} + 1_{\{\eta_{y_n} = 0, \eta_{y_{n-1}} = i\}} \\ &\quad - 1_{\{\eta_{-y_n} = i\}} - 1_{\{\eta_{-y_n} = 0, \eta_{-y_{n-1}} = i\}} \end{aligned}$$

for $i = 1$ or 2 satisfying $0 < \rho_i \leq \frac{1}{2}$ where $y_n = (n, n, \ldots, n) \in \mathbf{Z}^d$. Then by simple computations, we have

$$\begin{aligned} E_{\nu_{n,k}}[G] &= 0, \\ E_{\nu_{n,k}}[G^2] &\geq C' \rho_i, \\ E_{\nu_{n,k}}[G(-L_n)G] &\leq C'' \rho_0^2 \rho_i. \end{aligned}$$

for some constants C' and C''. Hence there is a constant C such that

$$\lambda(n,k) \leq C \min\{\frac{\rho_0}{n^2}, \rho_0^2\}.$$

Therefore we conclude that Assumption 3.1 is crucial for the estimate of the spectral gap. □

Without giving the proof, we comment that lower bound estimate for this model is given as follows; there is a constant C such that

$$\lambda(n,k) \geq C\frac{\rho_0^2}{n^2}.$$

3.4 Proof of lemmas

Here and after, a constant C may change from line to line.
Proof of Lemma 3.2.1.

First we suppose that Assumption 3.2 (i) holds. We extend the argument due to Dermoune et al. (Section 4 in [8]). For each fixed $x \in \mathbf{Z}^d$, we can pick and fix

$$y = y(x) \text{ such that } y(x) \in \tau_{2Tx}\Lambda_T \text{ and } q_{y(x)} = q^0.$$

For $n \geq 4T$, we define

$$\Lambda_n^0 := \bigcup_{x \in \Lambda_{[n/2T]-1}} \{y(x)\},$$

where $[\cdot]$ denotes Gauss' symbol. It is easy to see that $\Lambda_n^0 \subset \Lambda_n$. We also define

$$\begin{aligned}
A_n^0(z) &:= \{u \in \Lambda_n^0 : \|u-z\|_1 \leq 5dT\} \\
A_n^1(z) &:= \{(u,v) \in (\Lambda_n)^2 : u,v \in A_n^0(z) \cup \{z\}\} \\
A_n^1(z,w) &:= \{(u,v) \in (\Lambda_n)^2 : u \in A_n^0(z), v \in A_n^0(w)\}
\end{aligned}$$

for $z, w \in \Lambda_n$.

Note that $\#A_n^0(z) \geq 1$. By Schwarz inequality, for any $z, w \in \Lambda_n$,

$$\begin{aligned}
&E_{\nu_{n,k}}[1_{\{\eta_w=0\}}(\pi^{z,w}f(\eta))^2] \\
&\leq 3\frac{(1-\varepsilon)^4}{\varepsilon^4}\Big\{E_{\nu_{n,k}}[\sum_{(u,v)\in A_n^1(z)} 1_{\{\eta_v=0\}}(\pi^{u,v}f(\eta))^2] \\
&\quad +E_{\nu_{n,k}}[\sum_{(u,v)\in A_n^1(w)} 1_{\{\eta_v=0\}}(\pi^{u,v}f(\eta))^2] \\
&\quad +E_{\nu_{n,k}}[\sum_{(u,v)\in A_n^1(z,w)} 1_{\{\eta_v=0\}}(\pi^{u,v}f(\eta))^2]\Big\}.
\end{aligned}$$

Since there is a positive constant $C_{T,d}$ such that $|\{z \in \Lambda_n : (u,v) \in A_n^1(z)\}| \leq C_{T,d}$ and $|\{(z,w) \in (\Lambda_n)^2 : (u,v) \in A_n^1(z,w)\}| \leq C_{T,d}$ for any pair $(u,v) \in (\Lambda_n)^2$, we have

$$E_{\nu_{n,k}}[f(-L_n^m)f] \;\leq\; CE_{\nu_{n,k}}[\sum_{u\in\Lambda_n^0}\sum_{v\in\Lambda_n:\|u-v\|_1\leq 10dT} 1_{\{\eta_v=0\}}(\pi^{u,v}f(\eta))^2]$$

$$+CE_{\nu_{n,k}}[\frac{1}{|\Lambda_n|}\sum_{u,v\in\Lambda_n^0} 1_{\{\eta_v=0\}}(\pi^{u,v}f(\eta))^2].$$

Since $q_y = q^0$ for all $y \in \Lambda_n^0$, by a standard argument, we obtain that

$$\begin{aligned}
&E_{\nu_{n,k}}[\frac{1}{|\Lambda_n|}\sum_{u,v\in\Lambda_n^0} 1_{\{\eta_v=0\}}(\pi^{u,v}f(\eta))^2] \\
\leq\ & Cn^2 E_{\nu_{n,k}}[\sum_{z,w\in\Lambda_{[n/2T]}:\|z-w\|_1\leq 2} 1_{\{\eta_{y(w)}=0\}}(\pi^{y(z),y(w)}f(\eta))^2] \\
\leq\ & Cn^2 E_{\nu_{n,k}}[\sum_{u,v\in\Lambda_n:\|u-v\|_1\leq 2} 1_{\{\eta_v=0\}}(\pi^{u,v}f(\eta))^2].
\end{aligned}$$

The second inequality holds because $\|y(z)-y(w)\|_1 \leq 4T + 2Td$ if $\|z-w\|_1 \leq 2$. It is also standard to see that the first term of (3.4.1) is less than or equal to

$$CE_{\nu_{n,k}}[\sum_{u,v\in\Lambda_n:\|u-v\|_1\leq 2} 1_{\{\eta_v=0\}}(\pi^{u,v}f(\eta))^2],$$

which completes the proof.

Second we suppose that Assumption 3.2 (ii) holds. Given $x, y \in \Lambda_n$ we set $\gamma(x,y) \subset \Lambda_n$ the canonical path from x to y, which denotes the nearest neighbor path that goes from x to y, moving successively as far as it has to in each of the coordinate directions, following the natural order for the different coordinate directions. In order to prove Lemma 3.2.1, we have only to show that

$$\begin{aligned}
&E_{\nu_{n,k}}[1_{\{\eta_y=0\}}(\pi^{x,y}f(\eta))^2] \\
&\leq C\|x-y\|_1 E_{\nu_{n,k}}[\sum_{u,v\in\gamma(x,y),\|u-v\|_1\leq 2} 1_{\{\eta_v=0\}}(\pi^{u,v}f(\eta))^2].
\end{aligned} \tag{3.4.1}$$

To simplify our notation, we shall prove this inequality for $d = 1$ and $y < x$. The strategy is similar to the one used in [22].

Step 1. Given $K \in \mathbb{N}$, we define $z_i = z_i^K$ for $0 \leq i \leq 4K-3$ by

$$z_i := \begin{cases} i & \text{if } 0 \leq i \leq K, \\ K-2-j & \text{if } i = K+2j+1, 0 \leq j \leq K-2, \\ K-j & \text{if } i = K+2j, 0 \leq j \leq K-1, \\ i-3K+3 & \text{if } 3K-3 \leq i \leq 4K-3. \end{cases}$$

Note that

$$|z_i - z_{i-1}| = 1 \text{ or } 2 \text{ for } 1 \leq i \leq 4K-3, \tag{3.4.2}$$

$$\max_{u \in \mathbf{Z}} \#\{i : 0 \le i \le 4K-3, z_i = u\} \le 4.$$

Given a sequence of sites $\{a(i) : 0 \le i \le N\} \subset \Lambda_n$ satisfying if $0 \le i < j \le N$, then $a(i) < a(j)$, we define $S_i = S_i^a : \Sigma_n \to \Sigma_n$ for $0 \le i \le 4N-3$ by

$$S_i\eta := \begin{cases} \eta & \text{if } i = 0 \\ \Xi_i \circ S_{i-1}\eta & \text{if } 1 \le i \le 4N-3 \end{cases}$$
$$\Xi_i\eta = \eta^{a(z_{i-1}^N),a(z_i^N)}.$$

Then it is easy to see that $S_{4N-3}\eta = \eta^{a(0),a(N)}$ and if $\eta_{a(0)} = 0$ then $(S_i\eta)_{a(z_i)} = 0$ for all $0 \le i \le 4N-3$. Therefore, by using Schwarz inequality

$$\begin{aligned} & 1_{\{\eta_{a(0)}=0\}}(\pi^{a(0),a(N)}f(\eta))^2 \\ = & \left\{ \sum_{i=1}^{4N-3} 1_{\{(S_{i-1}\eta)_{a(z_{i-1})}=0\}} \pi^{a(z_i),a(z_{i-1})} f(S_{i-1}\eta) \right\}^2 \\ \le & \left(\sum_{i=1}^{4N-3} |a(z_i) - a(z_{i-1})| \right) \\ & \left(\sum_{i=1}^{4N-3} \frac{1}{|a(z_i) - a(z_{i-1})|} 1_{\{(S_{i-1}\eta)_{a(z_{i-1})}=0\}} (\pi^{a(z_i),a(z_{i-1})} f(S_{i-1}\eta))^2 \right). \end{aligned}$$

By the definition, we have $\sum_{i=1}^{4N-3} |a(z_i) - a(z_{i-1})| \le 5|a(N) - a(0)|$. Therefore we have

$$\begin{aligned} & E_{\nu_{n,k}}[1_{\{\eta_{a(0)}=0\}}(\pi^{a(0),a(N)}f(\eta))^2] \\ \le & \; C|a(N) - a(0)| \sum_{i=1}^{4N-3} \sum_{\eta \in \Sigma_{n,k}} \frac{1}{|a(z_i) - a(z_{i-1})|} \\ & 1_{\{(S_{i-1}\eta)_{a(z_{i-1})}=0\}} (\pi^{a(z_i),a(z_{i-1})} f(S_{i-1}\eta))^2 \nu_{n,k}(S_{i-1}\eta) \prod_{j=1}^{i-1} \frac{\nu_{n,k}(S_{j-1}\eta)}{\nu_{n,k}(S_j\eta)}. \end{aligned}$$

Suppose that there exists a constant C^* such that

$$\sup_{1 \le i \le 4N-3} \prod_{j=1}^{i-1} \frac{\nu_{n,k}(S_{j-1}\eta)}{\nu_{n,k}(S_j\eta)} \le C^*.$$

Then by using change of variables, we have

$$E_{\nu_{n,k}}[1_{\{\eta_{a(0)}=0\}}(\pi^{a(0),a(N)}f(\eta))^2]$$

$$
\begin{aligned}
&\leq \; C|a(N)-a(0)| \sum_{i=1}^{4N-3} \frac{1}{|a(z_i)-a(z_{i-1})|} \\
&\qquad \times E_{\nu_{n,k}}[1_{\{(\eta)_{az_{i-1}}=0\}}(\pi^{a(z_i),a(z_{i-1})}f(\eta))^2].
\end{aligned}
$$

Since

$$
E_{\nu_{n,k}}[1_{\{\eta_u=0\}}(\pi^{u,v}f(\eta))^2] \leq \frac{1-\varepsilon}{\varepsilon} E_{\nu_{n,k}}[1_{\{\eta_v=0\}}(\pi^{u,v}f(\eta))^2]
$$

for all $u, v \in \Lambda_n$, and (3.4.2), we have

$$
\begin{aligned}
&E_{\nu_{n,k}}[1_{\{\eta_{a(0)}=0\}}(\pi^{a(0),a(N)}f(\eta))^2] \\
&\leq \; C|a(N)-a(0)| \sum_{0\leq i<j\leq N, |i-j|\leq 2} \frac{1}{|a(j)-a(i)|} \\
&\qquad \times E_{\nu_{n,k}}[1_{\{\eta_{a(i)}=0\}}(\pi^{a(i),a(j)}f(\eta))^2], \\
&E_{\nu_{n,k}}[1_{\{\eta_{a(N)}=0\}}(\pi^{a(0),a(N)}f(\eta))^2] \\
&\leq \; C|a(N)-a(0)| \sum_{0\leq i<j\leq N, |i-j|\leq 2} \frac{1}{|a(j)-a(i)|} \\
&\qquad \times E_{\nu_{n,k}}[1_{\{\eta_{a(i)}=0\}}(\pi^{a(i),a(j)}f(\eta))^2].
\end{aligned}
$$

Step 2. If $|x-y| \leq K$ for some constant K, then applying *Step 1* in case where $\{a(i)\} = \{y, y+1, \ldots, x\}$ together with $\sup_{1\leq j\leq 4N-3} \frac{\nu_{n,k}(S_{j-1}\eta)}{\nu_{n,k}(S_j\eta)} \leq (\frac{1-\varepsilon}{\varepsilon})^2$, we have the inequality (3.4.1).

Step 3. In this step we apply *Step 1* inductively as follows:

Due to Assumption 3.2 (ii), there is a pair of a constant M and a set $\{q^i : 1 \leq i \leq M\}$ such that $\{q^i : 1 \leq i \leq M\} = \{q_x : x \in \mathbf{Z}\}$. Given $u, v \in \Lambda_n, u > v$ and $1 \leq k \leq M$, we set

$$
\begin{aligned}
B_0 &= B_0(u,v,k) := \{w : v \leq w \leq u, q_w = q^k\} \\
B_1 &= B_1(u,v,k) = \{b(i)\}_{i=0}^N = \{b(i;u,v,k)\}_{i=0}^N := B_0 \cup \{u,v\},
\end{aligned}
$$

where $N = N(u,v,k) = \#B_1 - 1$ and if $0 \leq i < j \leq N$ then $b(i) < b(j)$.

(I) Set $\{a_1(i)\} = B_1(x,y,1)$. If $\{a_1(z_{j-1}), a_1(z_j)\} \cap \{x,y\} = \emptyset$, then $\frac{\nu_{n,k}(S_{j-1}\eta)}{\nu_{n,k}(S_j\eta)} = 1$. This and (3.4.2) yield

$$
\sup_{1\leq i\leq 4N-3} \prod_{j=1}^{i-1} \frac{\nu_{n,k}(S_{j-1}\eta)}{\nu_{n,k}(S_j\eta)} \leq \left(\frac{1-\varepsilon}{\varepsilon}\right)^{32}.
$$

Therefore we can apply *Step 1* and we have

$$
\frac{1}{|x-y|} E_{\nu_{n,k}}[1_{\{\eta_y=0\}}(\pi^{x,y}f(\eta))^2]
$$

$$\leq C \sum_{0\leq i<j\leq N, |i-j|\leq 2} \frac{1}{|a_1(j)-a_1(i)|} E_{\nu_{n,k}}[1_{\{\eta_{a_1(i)}=0\}}(\pi^{a_1(i),a_1(j)}f(\eta))^2].$$

Note that

$$\max\{\#\{u : a(i) \leq u \leq a(j), q_u = q^1\} : i<j, |j-i| \leq 2\} \leq 3,$$
$$\max\{\#\{(i,j) : i<j, |j-i| \leq 2, a(i) \leq u \leq a(j)\} : y \leq u \leq x\} \leq 3.$$

(II) For each pair of $a_1(k), a_1(l)$ for $k<l$ and $|l-k| \leq 2$ in (I), we set $\{a_2(i)\} = \{a_2(i;k,l)\} := B_1(a_1(l), a_1(k), 2)$. Similarly, *Step 1* is applicable and we have

$$\begin{aligned} &\frac{1}{|a_1(l)-a_1(k)|} E_{\nu_{n,k}}[1_{\{\eta_{a_1(l)}=0\}}(\pi^{a_1(k),a_1(l)}f(\eta))^2] \\ &\leq C \sum_{0\leq i<j\leq N, |i-j|\leq 2} \frac{1}{|a_2(j)-a_2(i)|} E_{\nu_{n,k}}[1_{\{\eta_{a_2(i)}=0\}}(\pi^{a_2(i),a_2(j)}f(\eta))^2]. \end{aligned}$$

Note that the constants C do not depend on i, j, k nor l. We also note that

$$\begin{aligned} &\max\{\max\{\#\{u : a(i;k,l) \leq u \leq a(j;k,l), q_u = q^n\} : n=1,2\} \\ &: i<j, |j-i| \leq 2, k<l, |l-k| \leq 2\} \leq 3, \\ &\max\{\#\{(i,k,j,l) : i<j, |j-i| \leq 2, k<l, |l-k| \leq 2, \\ &a(i;k,l) \leq u \leq a(j;k,l)\} : y \leq u \leq x\} \leq 3^3. \end{aligned}$$

(III) Inductively we set

$$\begin{aligned} \{a_m(i)\} &= \{a_m(i;k^m,l^m)\} \\ &:= B_1(a_{m-1}(l;k^{m-1},l^{m-1}), a_{m-1}(k;k^{m-1},l^{m-1}), m) \end{aligned}$$

for $3 \leq m \leq M$, $k^{m-1}, l^{m-1} \in \mathbf{N}^{m-2}$, $k^m, l^m \in \mathbf{N}^{m-1}$ s.t. $k^m_n < l^m_n$, $|l^m_n - k^m_n| \leq 2$ for all $1 \leq n \leq m-1$, $k^m_n = k^{m-1}_n$, $l^m_n = l^{m-1}_n$ for all $1 \leq n \leq m-2$, $k<l$ and $|l-k| \leq 2$. Similarly, *Step 1* is applicable and we have

$$\begin{aligned} &\frac{1}{|a_{m-1}(l)-a_{m-1}(k)|} E_{\nu_{n,k}}[1_{\{\eta_{a_{m-1}(l)}=0\}}(\pi^{a_{m-1}(k),a_{m-1}(l)}f(\eta))^2] \\ &\leq C \sum_{0\leq i<j\leq N, |i-j|\leq 2} \frac{1}{|a_m(j)-a_m(i)|} E_{\nu_{n,k}}[1_{\{\eta_{a_m(i)}=0\}}(\pi^{a_m(i),a_m(j)}f(\eta))^2]. \end{aligned}$$

Note that the constants C do not depend on i, j, k^m nor l^m. We also note that

$$\begin{aligned} &\max\{\max\{\#\{u : a(i;k^m,l^m) \leq u \leq a(j;k^m,l^m), q_u = q^n\} : n \leq m\} \\ &: i<j, |j-i| \leq 2, k^m_v < l^m_v, |l^m_v - k^m_v| \leq 2 \text{ for } 1 \leq v \leq m-1\} \leq 3, \end{aligned} \tag{3.4.3}$$

and

$$\begin{aligned}(3.4.4)\qquad &\max\{\#\{(i,k^m,j,l^m) : i<j, |j-i|\le 2, k_v^m < l_v^m,\\ &|l_v^m - k_v^m| \le 2 \text{ for } 1\le v\le m-1, a(i;k^m,l^m)\le u\le a(j;k^m,l^m)\}\\ &: y\le u\le x\} \le \overbrace{3^{(3^{(3\cdots)})}}^{m-\text{fold}}.\end{aligned}$$

Step 4. By (3.4.3), we have

$$\begin{aligned}&\max\{\{|a(j;k^M,l^M) - a(i;k^M,l^M)|\\ &: i<j, |j-i|\le 2, k_v^M < l_v^M, |l_v^M - k_v^M| \le 2 \text{ for } 1\le v\le M-1\} \le 3M.\end{aligned}$$

Plugging *Step 2* into *Step 3*, together with this and (3.4.4), we have inequality (3.4.1). □

Proof of Lemma 3.2.2.

Let us denote $T^{x,y}\eta = \eta^{x,y}$. Given a number N, a sequence $\{x_i \in \Lambda_n : 0 \le i \le N\}$ and a configuration $\eta \in \Sigma_n$, we define

$$\eta^0 = \eta, \text{ and } \eta^i := T^{x_{i+1},x_i} \circ T^{x_i,x_{i-1}} \circ \ldots \circ T^{x_1,x_0}\eta \text{ for } 1 \le i \le N.$$

Due to Assumption 3.1, $\Sigma_{n_0,k}$ with $k_0 \ge 1$ is an ergodic component of L_{n_0}. Therefore for each fixed $x, y \in \Lambda_{n_0}$ and $\zeta \in \Sigma_{n_0,k}$ with $k_0 \ge 1$ and $\zeta_y = 0$, there is a pair of a number $N = N(\zeta, x, y)$ and a sequence $\{x_i(\zeta, x, y) \in \Lambda_{n_0} : 0 \le i \le N, x_0 = y, x_N = x\}$ such that $T^{x_N,x_{N-1}} \circ T^{x_{N-1},x_{N-2}} \circ \ldots \circ T^{x_1,x_0}\eta = \eta^{x,y}$, $(\zeta^i)_{x_i} = 0$ for all $0 \le i \le N$ and $p_{(\zeta^{i-1})_{x_i}}(x_i, x_{i-1}) > 0$ for all $1 \le i \le N$. By taking the spatial shift, for each fixed $n \ge n_0$, $x \in \Lambda_n$, $x + z \in \Lambda_n$ with $\|z\|_1 = 1, 2$, and $\eta \in \Sigma_{n,k}$ with $\eta_{x+z} = 0$, we define $N(\eta, x, x+z)$ by $N(\eta, x, x+z) := N(\tau_{-\bar{x}}\eta|_{\Lambda_{n_0}}, x - \bar{x}, x + z - \bar{x})$ and $\{x_i(\eta, x, x+z) \in \tau_{\bar{x}}\Lambda_{n_0} : 0 \le i \le N, x_0 = x+z, x_N = x\}$ by $x_i(\eta, x, x+z) = x_i(\tau_{-\bar{x}}\eta|_{\Lambda_{n_0}}, x - \bar{x}, x + z - \bar{x}) + \bar{x}$. Note that $N(\eta, x, x+z)$ and $\{x_i(\eta, x, x+z)\}_i$ depend only on the occupation variables of the sites $\tau_{\bar{x}}\Lambda_{n_0}$ and satisfy that $T^{x_N,x_{N-1}} \circ T^{x_{N-1},x_{N-2}} \circ \ldots \circ T^{x_1,x_0}\eta = \eta^{x,x+z}$, $(\eta^{i-1})_{x_{i-1}} = 0$ for all $1 \le i \le N$ and $p_{(\eta^{i-1})_{x_i}}(x_i, x_{i-1}) > 0$ for all $1 \le i \le N$. Since Σ_{n_0} is a finite set, N is bounded above by a constant, i.e., there is a constant $N_0 < \infty$ such that $\sup_n \max_{\eta\in\Sigma_n} \max_{x\in\Lambda_n, \|z\|=1,2} N(\eta, x, x+z) = \max_{\zeta\in\Sigma_{n_0}} \max_{x,y\in\Lambda_{n_0}} N(\zeta, x, y) = N_0$. Since $p_{(\eta^{i-1})_{x_i}}(x_i, x_{i-1}) > 0$ for all $1 \le i \le N$, we have

$$1_{\{\eta_{x+z}=0\}}(\pi^{x,x+z}f(\eta))^2$$

$$\leq C\left\{\sum_{i=1}^{N} 1_{\{(\eta^{i-1})_{x_{i-1}}=0\}}\pi^{x_i,x_{i-1}} f(\eta^{i-1})\right\}^2$$
$$\leq C\left\{\sum_{i=1}^{N}\{p_{(\eta^{i-1})_{x_i}}(x_i,x_{i-1})\}^{1/2} 1_{\{(\eta^{i-1})_{x_{i-1}}=0)\}}\pi^{x_i,x_{i-1}} f(\eta^{i-1})\right\}^2$$

By using Schwarz inequality, right hand side above is less than or equal to

$$CN\sum_{i=1}^{N} p_{(\eta^{i-1})_{x_i}}(x_i,x_{i-1}) 1_{\{(\eta^{k-1})_x=0\}}(\pi^{x_i,x_{i-1}} f(\eta^{i-1}))^2.$$

Hence we have

$$\begin{aligned}
&E_{\nu_{n,k}}[1_{\{\eta_{x+z}=0\}}(\pi^{x,x+z}f(\eta))^2] \\
&\leq CE_{\nu_{n,k}}[N(\eta,x,x+z)\sum_{i=1}^{N(\eta,x,x+z)} p_{(\eta^{i-1})_{x_i}}(x_i,x_{i-1}) \\
&\quad 1_{\{(\eta^{i-1})_{x_{i-1}}=0\}}(\pi^{x_i,x_{i-1}} f(\eta^{i-1}))^2] \\
&\leq CN_0 E_{\nu_{n,k}}[\sum_{i=1}^{N(\eta,x,x+z)} p_{(\eta^{i-1})_{x_i}}(x_i,x_{i-1}) 1_{\{(\eta^{i-1})_{x_{i-1}}=0\}}(\pi^{x_i,x_{i-1}} f(\eta^{i-1}))^2].
\end{aligned}$$

Since $N(\eta,x,x+z)$ and $\{x_i(\eta,x,x+z)\}_i$ depend only on the occupation variables of the sites in $\tau_{\bar{x}}\Lambda_{n_0}$, we have

$$\begin{aligned}
&E_{\nu_{n,k}}[\sum_{i=1}^{N(\eta,x,x+z)} p_{(\eta^{i-1})_{x_i}}(x_i,x_{i-1}) 1_{\{(\eta^{i-1})_{x_{i-1}}=0\}}(\pi^{x_i,x_{i-1}} f(\eta^{i-1}))^2] \\
&= \sum_{\zeta\in\Sigma_{n_0}} E_{\nu_{n,k}}[1_{\{\tau_{-\bar{x}}\eta|_{\Lambda_{n_0}}=\zeta\}} \sum_{i=1}^{N(\zeta,x-\bar{x},x+z-\bar{x})} p_{(\eta^{i-1})_{x_i}}(x_i,x_{i-1}) \\
&\quad 1_{\{(\eta^{i-1})_{x_{i-1}}=0\}}(\pi^{x_i,x_{i-1}} f(\eta^{i-1}))^2] \\
&= \sum_{\zeta\in\Sigma_{n_0}} \sum_{i=1}^{N(\zeta,x-\bar{x},x+z-\bar{x})} E_{\nu_{n,k}}[1_{\{\tau_{-\bar{x}}\eta|_{\Lambda_{n_0}}=\zeta\}} p_{(\eta^{i-1})_{x_i}}(x_i,x_{i-1}) \\
&\quad 1_{\{(\eta^{i-1})_{x_{i-1}}=0)\}}(\pi^{x_i,x_{i-1}} f(\eta^{i-1}))^2] \\
&\leq \sum_{\zeta\in\Sigma_{n_0}} \sum_{i=1}^{N(\zeta,x-\bar{x},x+z-\bar{x})} \sum_{w,v\in\tau_{\bar{x}}\Lambda_{n_0}} E_{\nu_{n,k}}[1_{\{\tau_{-\bar{x}}\eta|_{\Lambda_{n_0}}=\zeta\}} p_{(\eta^{i-1})_w}(w,v) \\
&\quad 1_{\{(\eta^{i-1})_v=0)\}}(\pi^{w,v} f(\eta^{i-1}))^2].
\end{aligned}$$

Under the condition $\tau_{-\bar{x}}\eta|_{\Lambda_{n_0}} = \zeta$, we can perform the change of valuable and obtain that for all $\zeta \in \Sigma_{n_0}$, $w, v \in \tau_{\bar{x}}\Lambda_{n_0}$ and $1 \le i \le N(\zeta, x - \bar{x}, x + z - \bar{x})$,

$$\begin{aligned}
& E_{\nu_{n,k}}[1_{\{\tau_{-\bar{x}}\eta|_{\Lambda_{n_0}}=\zeta\}} p_{(\eta^{i-1})_w}(w,v) 1_{\{(\eta^{i-1})_v=0)\}} (\pi^{w,v} f(\eta^{i-1}))^2] \\
\le \; & \sup_\eta \frac{\nu_{n,k}(\{\eta\})}{\nu_{n,k}(\{\eta^i\})} E_{\nu_{n,k}}[1_{\{\tau_{-\bar{x}}\eta|_{\Lambda_{n_0}}=\zeta^{i-1}\}} p_{\eta_w}(w,v) 1_{\{\eta_v=0)\}} (\pi^{w,v} f(\eta))^2] \\
\le \; & C_\varepsilon E_{\nu_{n,k}}[p_{\eta_w}(w,v) 1_{\{\eta_v=0\}} (\pi^{w,v} f(\eta))^2]
\end{aligned}$$

where $\sup_\eta \frac{\nu_{n,k}(\{\eta\})}{\nu_{n,k}(\{\eta^i\})} \le (\frac{1-\varepsilon}{\varepsilon})^{|\Lambda_{n_0}|} =: C_\varepsilon$. Therefore, we obtain that

$$\begin{aligned}
& E_{\nu_{n,k}}[\sum_{i=1}^{N(\eta,x,x+z)} p_{(\eta^{i-1})_{x_i}}(x_i, x_{i-1}) 1_{\{(\eta^{i-1})_{x_{i-1}}=0\}} (\pi^{x_i,x_{i-1}} f(\eta^{i-1}))^2] \\
\le \; & C N_0^2 (r+1)^{|\Lambda_{n_0}|} \sum_{w,v \in \tau_{\bar{x}}\Lambda_{n_0}} E_{\nu_{n,k}}[p_{\eta_w}(w,v) 1_{\{\eta_v=0\}} (\pi^{w,v} f(\eta))^2],
\end{aligned}$$

which completes the proof. □

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