Regularity of two dimensional steady capillary gravity water waves

（二次元定常表面張力重力波の正則性）

張 光輝（Zhang Guanghui）

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Guanghui Zhang

Contents

Chapter 1

Introduction

Consider a two-dimensional inviscid incompressible fluid acted on by gravity and with a free surface.

While recent years have seen great progress in the study of well-posedness of the initial-value problem (see [26] and the references therein and the references therein), we confine ourselves to traveling-wave solutions.we confine ourselves to traveling-wave solutions.

Using the stream function ψ of the normalized velocity vector field (cf. Section 2.4), we obtain the free boundary problem

$$\begin{aligned} &\Delta\psi = -\gamma(\psi) \quad \text{in} \quad \Omega \cap \{\psi > 0\} \\ &|\nabla\psi|^2 + 2gy - 2\sigma\kappa = \text{const} \quad \text{on} \quad \Omega \cap \partial\{\psi > 0\}; \end{aligned} \tag{1.1}$$

here g is the gravitation constant, $\sigma > 0$ is the surface tension coefficient and $\gamma(\psi)$ is the vorticity of the fluid.

The free boundary problem describes both *water waves*, in which case we would add homogeneous Neumann boundary conditions on a bottom $y = -d$ combined with periodicity in the x-direction or some condition at $x = \pm\infty$, and the equally physical problem of the equilibrium state of a fluid when pumping in water from one lateral boundary and sucking it out at the other lateral boundary. In the latter setting we would consider a finite domain with an inhomogeneous Neumann boundary condition at the lateral boundary, and the bottom could be a non-flat surface.

While most of the literature on existence of waves is based on perturbation methods (see for example [22], [13], [19], [20], [3], [4], [25]), and the regularity of the solutions so obtained is not an issue, in recent years there have been also studies based on variational methods (see for example [8], [6], [18]). These have the potential to lead to the existence of large amplitude waves, which is also supported by numerical studies; there is a large amount of numerical research, see for example [14] and [23].It is therefore natural to ask whether there are singularities.

In the case of zero surface tension ($\sigma = 0$), study of traveling waves goes back to Stokes, who in 1880 made the famous conjecture that the free surface of a wave of maximal amplitude is not smooth at a free surface point of maximal height, but forms a *sharp crest with an angle of* 120°. The conjecture has been proved rigorously in [2] and [21] for isolated singularities satisfying further structural assumptions, and in [24] for a more general setting.

When adding surface tension, physical intuition suggests that the corner singularities just mentioned should disappear. Mathematically we are aware of only two results in that direction: in [7], he authors prove that irrotational waves with surfaces that are $W^{2,2}$-graphs, are analytic. In [12], the authors prove for irrotational waves in three dimensions that $C^{2,\alpha}$-solutions are analytic.

In Theorem 4.2.3 we prove that this physical intuition remains true for a larger class of weak solutions allowing for example the possibility of corners and cusps (see Section 2.4 for our notion of solutions). On a technical level, however, the situation is not that obvious: in the case of zero vorticity, solutions turn out to be critical points of the Mumford-Shah functional. For the Mumford-Shah problem it is known that there exist energy minimising crack tips, opening the possibility of *cusps* in our water wave problem (see [1] for an overview on the Mumford-Shah problem and [15] for a related problem in two dimensions). The major task is then to exclude cusps. On the way there, however, there are a number of technical difficulties: while the Mumford-Shah problem has been extensively studied, many results are confined to minimisers, so that we cannot use them. In particular, in the case of non-minimizers it is not obvious whether limits of weak solutions are again weak solutions. We would want a *concentration compactness* result making it possible to pass to the limit in the nonlinear curvature term. Also, for non-zero vorticity, our problem is not any more equivalent to the Mumford-Shah problem, as the velocity potential ceases to exist. Different from the previous results [7] and [13] for irrotational waves, our results hold for rotational waves and do not use much initial regularity (see the Assumption in Section 2.4 and Remark 2.4.3). Rotational waves are currently a topic of interest (see for example [10]). Our results *may be extended to the Jordan curve case provided that the air phase satisfies a uniform exterior sphere condition* (cf.Remark 2.4.4. Last, let us mention that our methods are not based on transformations in the complex plane, but we work with the original variables.

Our paper is composed as follows:

In Chapter 2 we derive a variational formula for the two-dimensional steady water waves. In Section 2.2 we introduce the governing equations and the equivalent free boundary problem. In Section 2.3, we prove a variational formula based on a domain variation/inner variation of the energy

$$E(\psi) = \int_\Omega (|\nabla\psi|^2 - 2G(\psi) - \alpha y \chi_{\{\psi>0\}}) - \beta\mathcal{H}^1(\partial\{\psi > 0\}).$$

Note that the negative sign in front of the Hausdorff measure makes for a drastic difference to the positive sign in [5] and means that we cannot use their methods. Weak solutions will be defined in Section 2.4.

In Chapter 3 we study the blow-up limit of the stream function. In Section 3.1 we extend Bonnets monotonicity formula to our problem. This will in Section 3.2 lead to a growth estimate for the Dirichlet part of the energy as a preliminary characterization of blow-up limits. Next we have to deal with the curvature term in the equation. In Section 3.3 we will use the one-sided curvature bound included in the equation to control the length of the free surface, show existence of one-sided tangents and prove convergence of the curvature term when passing to blow-up limits. This leads to more information about the blow-up limits (Section 3.4): it turns out that the only possibility of singularities is cusps pointing into the water phase.

Chapter 4 will be devoted to the regularity of the free boundary. Before investigating that possibility of cusp type singularity, we show in Section 4.1 that outside the locally finite set of cusp points, the free surface is regular. Last, using the regularity of the free boundary outside cusp points we prove in Section 4.2 that cusps do not exist.

In Chapter 5 we let the surface tension go to 0 and study the limit of ψ_σ, where ψ_σ are period and satisfy the same boundary value on the bottom of the fluid. Under suitable assumptions we prove that there exists a subsequence ψ_{σ_i} which converges to some function ψ_0 in $W^{1,2}$ and that ψ_0 is a domain variational solution to the two dimensional steady water waves without surface tension.

The results in Chapter 2-4 are joint works with Professor Georg S. Weiss.

Chapter 2

Variation formulation

2.1 Notation

Throughout the paper let Ω be a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary in which to consider the combined problem for fluid and air. Since almost all of our results are completely local, we do not specify boundary conditions on $\partial\Omega$. Moreover, we denote by χ_A the characteristic function of the set A, by $\mathbf{x} = (x, y)$ a point in $\mathbb{R}^2$ by $\mathbf{x} \cdot \mathbf{y}$ the Euclidean inner product in $\mathbb{R}^2 \times \mathbb{R}^2$, by $|\mathbf{x}|$ the Euclidean norm in $\mathbb{R}^2$ and by $B_r(\mathbf{x}^0) := \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x} - \mathbf{x}^0| < r\}$ the ball of center $\mathbf{x}^0$ and radius r. By ν we will always refer to the outer normal on a given surface. We will use functions of bounded variation $BV(U)$, i.e. functions $f \in L^1(U)$ for which the distributional derivative is a vector-valued Radon measure. Here $|\nabla f|$ denotes the total variation measure (cf. [17]). Note that for a smooth open set $E \subset \mathbb{R}^2$, $|\nabla \chi_E|$ coincides with the surface measure on ∂E. Let us also define the surface divergence

$$\mathrm{div}_\Gamma \xi = \mathrm{div}\xi - \nu D\xi\nu,$$

where ν is the upward pointing unit normal on the free surface Γ.

Finally $\mathcal{L}^n$ shall denote the n-dimensional Lebesgue measure and $\mathcal{H}^s$ the s-dimensional Hausdorff measure.

2.2 Equations of motion

We consider two dimensional capillary gravity water waves. The only external force acting on the water is gravity. Let $(u(x,y,t), v(x,y,t))$ be the velocity field. Incompressibility implies that

$$u_x + v_y = 0.$$

Under the assumption that the water is inviscid, the equations of motion are Euler's equations:

$$\begin{aligned} u_t + uu_x + vu_y &= -P_x \\ v_t + uv_x + vv_y &= -P_y - g \end{aligned} \tag{2.1}$$

where $P(x,y,t)$ denotes the pressure and g is the gravitation constant. All our results hold for $g = 0$, too.

Although the present paper may be extended to the free surface being a Jordan curve, we confine ourselves here to the free surface being the graph of a function $\eta(t,x)$. The boundary conditions on the free boundary are then the dynamic condition

$$P = P_0 - \sigma\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_x \quad \text{on} \quad \{y = \eta(t,x)\}$$

where P_0 is the atmospheric pressure and $\sigma > 0$ is coefficient of surface tension, as well as the kinematic boundary condition

$$v = \eta_t + u\eta_x \quad \text{on} \quad \{y = \eta(t,x)\}.$$

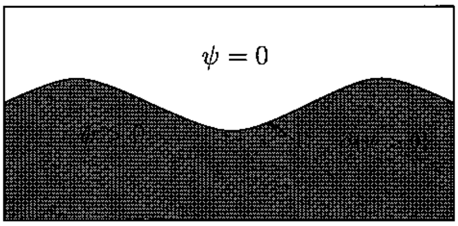
For waves traveling at constant speed $c > 0$, we define the stream function $\psi(x,y)$ by

$$\psi_x = -v \qquad \psi_y = u - c,$$

whereupon the flow may be described by the following free boundary problem:

$$\begin{aligned} &\Delta\psi = -\gamma(\psi) \quad \text{in} \quad \Omega \cap \{\psi > 0\} \\ &|\nabla\psi|^2 + 2gy - 2\sigma\kappa = \text{const} \quad \text{on} \quad \Gamma = \partial\{\psi > 0\}, \end{aligned} \tag{2.2}$$

where $\gamma(\psi) = v_x - u_y$ is the vorticity, Γ is the graph of η and $\kappa = \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_x$ is the curvature of Γ. By a translation, we may assume const $= 0$.



2.3 Variation formula

In this section we prove a variation formula for two dimensional water with surface tension.

Let

$$E(\psi) = \int_\Omega |\nabla\psi|^2 - \int_\Omega 2gy\chi_{\{\psi>0\}} - 2G(\psi)d\mathbf{x} - 2\mathcal{H}^1(\partial\{\psi > 0\}),$$

where $G(s) = \int_0^s \gamma(t)dt$.

Theorem 2.3.1. *Any critical point ψ of the functional E is a solution to the two dimensional capillary water waves if $\nabla\psi \neq 0$ in $\{\psi > 0\}$ and its free boundary $\partial\{\psi > 0\}$ is a C^2 curve.*

Proof. Let $\xi \in C_0^\infty(\Omega;\mathbb{R}^2)$, ψ be a critical point of E and $\psi_\epsilon(\mathbf{x}) = \psi(\mathbf{x} + \epsilon\xi(\mathbf{x}))$. Then

$$\delta E(\psi)(\xi) = \lim \frac{E(\psi_\epsilon) - E(\psi)}{\epsilon} = 0 \tag{2.3}$$

Since $\nabla\psi_\epsilon(\mathbf{x}) = \nabla\psi(\mathbf{x} + \epsilon\xi(\mathbf{x})) + D\xi(\mathbf{x})\nabla\psi(\mathbf{x} + \epsilon\xi(\mathbf{x}))$, direct calculation gives

$$\begin{aligned}
&\lim_{\epsilon\to 0} \tfrac{1}{\epsilon}\{\textstyle\int_\Omega |\nabla\psi_\epsilon|^2 d\mathbf{x} - \int_\Omega |\nabla\psi|^2 d\mathbf{x}\} \\
&= \lim_{\epsilon\to 0} \tfrac{1}{\epsilon}\{\textstyle\int_\Omega |(\nabla\psi + D\xi(\mathbf{x})\nabla\psi)(\mathbf{x} + \epsilon\xi(\mathbf{x}))|^2 d\mathbf{x} \\
&- \textstyle\int_\Omega |\nabla\psi(\mathbf{x} + \epsilon\xi(\mathbf{x}))|^2 d\,(\mathbf{x} + \epsilon\xi(\mathbf{x}))d\mathbf{x}\} \\
&= \textstyle\int_\Omega 2\nabla\psi D\xi\nabla\psi - |\nabla\psi|^2 \mathrm{div}\xi d\mathbf{x},
\end{aligned}$$

$$\begin{aligned}
&\lim_{\epsilon\to 0} \tfrac{1}{\epsilon}\{\textstyle\int_{\{\psi_\epsilon>0\}} 2gyd\mathbf{x} - \int_{\{\psi>0\}} 2gyd\mathbf{x}\} \\
&\lim_{\epsilon\to 0} \tfrac{1}{\epsilon}\{\textstyle\int_{\{\psi_\epsilon>0\}} 2gyd\mathbf{x} - \int_{\{\psi_\epsilon>0\}} 2g(y + \epsilon\xi(\mathbf{x}))d(\mathbf{x} + \epsilon(\xi(\mathbf{x}))\} \\
&= -\textstyle\int_{\{\psi>0\}} (2gy\mathrm{div}\xi + 2g\xi_2)d\mathbf{x},
\end{aligned}$$

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\tfrac{1}{\epsilon}\{\int_\Omega G(\psi_\epsilon)d\mathbf{x}-\int_\Omega G(\psi)d\mathbf{x}\}\\
&=\lim_{\epsilon\to 0}\tfrac{1}{\epsilon}\{\int_\Omega G(\psi(\mathbf{x}+\epsilon\xi(\mathbf{x})))d\mathbf{x}-\int_\Omega G(\psi)(\mathbf{x}+\epsilon\xi(\mathbf{x}))d\,(\mathbf{x}+\epsilon\xi(\mathbf{x}))\}\\
&=-\int_\Omega G(\psi)\mathrm{div}\xi d\mathbf{x}
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\tfrac{1}{\epsilon}\{\mathcal{H}^1(\partial\{\psi_\epsilon>0\})-\mathcal{H}^1(\partial\{\psi>0\})\}\\
&=-\int_\Gamma \mathrm{div}_\Gamma\xi(\mathbf{x})d\mathbf{x},
\end{aligned}
$$

where $\Gamma=\partial\{\psi>0\}$.

Therefore (2.3) implies

$$
\begin{aligned}
\int_\Omega(&|\nabla\psi|^2\mathrm{div}\xi-2\nabla\psi D\xi\nabla\psi-2G(\psi)\mathrm{div}\xi\\
&-2gy\chi_{\{\psi>0\}}\mathrm{div}\xi-2g\chi_{\{\psi>0\}}\xi_2)-2\sigma\int_\Gamma \mathrm{div}_\Gamma\xi=0.
\end{aligned}
$$

Since the free boundary $\partial\{\psi>0\}$ is a C^2 curve and

$$
\begin{aligned}
&\mathrm{div}(|\nabla\psi|^2\xi-2(\nabla\psi\cdot\xi)\psi-2G(\psi)\xi)\\
&=|\nabla\psi|^2\mathrm{div}\xi+\nabla(|\nabla\psi|^2)\cdot\xi-2(\nabla\psi\cdot\xi)\Delta\psi-2\nabla(\nabla\psi\cdot\xi)\nabla\psi\\
&-2G(\psi)\mathrm{div}\xi-2\gamma(\psi)(\nabla\psi\cdot\xi)\\
&=|\nabla\psi|^2\mathrm{div}\xi-2\nabla\psi D\xi\nabla\psi-2(\nabla\psi\cdot\xi)\Delta\psi\\
&-2G(\psi)\mathrm{div}\xi-2\gamma(\psi)(\nabla\psi\cdot\xi),
\end{aligned}
$$

we obtain that

$$
\begin{aligned}
0\;&=\int_\Omega(|\nabla\psi|^2\mathrm{div}\xi-2\nabla\psi D\xi\nabla\psi-2G(\psi)\mathrm{div}\xi\\
&-2gy\chi_{\{\psi>0\}}\mathrm{div}\xi-2g\chi_{\{\psi>0\}}\xi_2)-2\sigma\int_\Gamma\mathrm{div}_\Gamma\xi\\
&=\int_\Omega\mathrm{div}(|\nabla\psi|^2\xi-2(\nabla\psi\cdot\xi)\psi-2gy\xi-2G(\psi)\xi)\\
&+2(\nabla\psi\cdot\xi)\Delta\psi+2\gamma\psi\nabla\psi\cdot\xi+2\sigma\int_\Gamma\kappa\xi\cdot\nu\\
&=\int_\Gamma(-|\nabla\psi|^2-2gy+2\sigma\kappa)\xi\cdot\nu+2\int_\Omega(\Delta\psi+\gamma\psi)\nabla\psi\cdot\xi
\end{aligned}
$$

for any $\xi\in C_0^1(\Omega;\mathbb{R}^2)$.

Therefore

$$\Delta\psi=-\gamma(\psi)\;\;\text{in}\;\;\{\psi>0\}$$

and

$$|\nabla\psi|^2+2gy-2\sigma\kappa=0\;\;\text{on }\Gamma.$$

□

Remark 2.3.2. *This variational formula is a generalization of the result in [10] where surface tension is neglected. The proof in [10] relies on the hodograph transformation. Therefore they need the assumption that the free surface is a graph of some function and the stream function is monotone in the y direction. Our result holds for general simple curve.*

2.4 Domain variation solutions

Definition 2.4.1. *Let Ω be a domain in $\mathbb{R}^2$, $G(t) = \int_0^t \gamma(s)ds$. The function $\psi \in W^{1,2}_{\text{loc}}(\Omega)$ such that $\psi \geq 0$ and $\nabla\psi \neq 0$ in $\{\psi > 0\}$ is called a* domain variation solution of equation *(2.2), if $\psi \in C^0(\Omega) \cap C^2(\Omega \cap \{\psi > 0\})$ and*

$$\begin{aligned}\int_\Omega (|\nabla\psi|^2 \text{div}\xi - 2\nabla\psi D\xi\nabla\psi - 2G(\psi)\text{div}\xi \\ - 2gy\chi_{\{\psi>0\}}\text{div}\xi - 2g\chi_{\{\psi>0\}}\xi_2) - 2\sigma\int_\Gamma \text{div}_\Gamma\xi = 0\end{aligned} \tag{2.4}$$

for any $\xi(x) = (\xi_1(x), \xi_2(x)) \in C_0^\infty(\Omega; \mathbb{R}^2)$.

Remark 2.4.2. *A domain variation solution is a $W^{1,2}_{\text{loc}}$ weak solution of $\Delta\psi = -\gamma(\psi)$ in $\Omega \cap \{\psi > 0\}$. When $\Gamma \in C^{2,\alpha}$ and $\psi \in C^{2,\alpha}$, ψ is a classic solution of (2.2).*

Assumption: In this paper we assume that $\gamma \in L^\infty$, $\eta \in W^{1,1}_{\text{loc}}$ and that the curvature $\kappa = (\frac{\eta_x}{\sqrt{1+(\eta_x)^2}})_x$ is a Radon measure. We also assume that Γ can be touched at every point from below by a ball of fixed radius $\kappa_0 > 0$, which implies that $\kappa \geq -\kappa_0$ in the sense of measures. Both assumptions are justified by the expected free boundary condition (2.2).

Remark 2.4.3. *In the following two situations we can prove that for a domain variation solution the curvature of the free boundary is bounded from below in the sense of measure.*
Case 1: We assume that $u < c$ which is the same assumption as in [11]. If $\gamma \in C^{0,1}$, then ψ is a classic solution to equation $\Delta\psi = -\gamma(\psi)$ in $\Omega \cap \{\psi > 0\}$

and for each $\epsilon > 0$ the level set $\Gamma_\epsilon = \{\psi = \epsilon\}$ is the graph of some function. By Sard's theorem Γ_ϵ is a C^1 curve for a.e. $\epsilon > 0$. Let $\Omega_\epsilon = \{\psi > \epsilon\}$, ν_ϵ be the unit outward normal vector of Ω_ϵ, $0 \le f \in C_0^1(\Omega; \mathbb{R})$ and $\xi(\mathbf{x}) = f(\mathbf{x})\binom{0}{1}$. We calculate

$$\begin{aligned}
&\int_{\Omega_\epsilon} |\nabla\psi|^2 - 2\nabla\psi D\xi\nabla\psi - 2G(\psi)\mathrm{div}\xi \\
&= \int_{\Omega_\epsilon} \mathrm{div}(|\nabla\psi|^2\xi - 2(\nabla\psi\cdot\xi)\nabla\psi) - 2\int_{\Gamma_\epsilon} G(\epsilon)\xi\cdot\nu_\epsilon \\
&= \int_{\Gamma_\epsilon} -|\nabla\psi|^2\xi\cdot\nu_\epsilon - 2\int_{\Gamma_\epsilon} G(\epsilon)\xi\cdot\nu_\epsilon \\
&\le -2\int_{\Gamma_\epsilon} G(\epsilon)\xi\cdot\nu_\epsilon.
\end{aligned}$$

Letting $\epsilon \to 0$, we obtain $\int_\Omega |\nabla\psi|^2 - 2\nabla\psi D\xi\nabla\psi - 2G(\psi)\mathrm{div}\xi \le 0$. Therefore

$$\begin{aligned}
0 &\ge \int_\Omega gy\chi_{\{\psi>0\}}\mathrm{div}\xi + g\chi_{\{\psi>0\}}\xi_2 + \delta\int_\Gamma \mathrm{div}_\Gamma\xi \\
&= \int_{\{\psi>0\}} gy\frac{\partial}{\partial y}f(x,y) + gf(x,y) - \delta\int_\Gamma \kappa\xi\cdot\nu \\
&= \int_{\mathbb{R}} (g\eta(x) - \delta\kappa(x,\eta(x)))f(x,\eta(x)),
\end{aligned}$$

which implies that $\kappa(x,\eta(x)) \ge g\eta(x)\delta^{-1}$ in the sense of measure.
Case 2: Let $\Omega_0 = \{\psi > 0\}$, $X = \{u \in W^{1,2}(\Omega_0) : u \ge 0$ and $u - \psi = 0$ on $\partial\Omega_0\}$, $E(u) = \int_{\Omega_0} |\nabla u|^2 - 2G(u)$. If $E(\psi) = \inf_{u\in X} E(u)$, we choose the same test function ξ as in case 1 and then obtain that $E(\psi(\mathbf{x} + \epsilon\xi(\mathbf{x})) \ge E(\psi)$ for $\epsilon > 0$ which implies that $\int_\Omega |\nabla\psi|^2 - 2\nabla\psi D\xi\nabla\psi - 2G(\psi)\mathrm{div}\xi \le 0$. Consequently $\kappa(x,\eta(x)) \ge g\eta(x)\delta^{-1}$ in the sense of measure.

Remark 2.4.4. *Assuming that a general Jordan curve Γ can be touched at every point by a ball of fixed radius $\kappa_0^{-1} > 0$ contained in the water phase, all our results extend to the Jordan curve case, too. An extension of Remark 2.4.3, however, is not so obvious, and we leave that to future research.*

Chapter 3

Blow-up

3.1 A Bonnet type monotonicity formula

Suppose that $\psi \in W^{1,2}_{\mathrm{loc}} \cap C^0_{\mathrm{loc}}$ is a solution of $\Delta\psi = -\gamma(\psi)$ in $\Omega \cap \{\psi > 0\}$, $\partial B_r(\mathrm{x}^0) \cap \{\psi = 0\} \neq \emptyset$ for every $r \in (0, r_0)$. Let $\Phi^{\psi}_{\mathrm{x}^0}(r) = r^{-1}\int_{B_r(\mathrm{x}^0)} |\nabla\psi|^2$. Then $\frac{d}{dr}\Phi^{\psi}_{\mathrm{x}^0}(r) = r^{-2}(r\int_{\partial B_r(\mathrm{x}^0)} |\nabla\psi|^2 d\mathcal{H}^1 - \int_{B_r(\mathrm{x}^0)} |\nabla\psi|^2)$.
The well-known Wirtinger inequality states:

Lemma 3.1.1 (Wirtinger inequality). *Let $\eta \in W^{1,2}([0, 2\pi))$, $\eta(0) = \eta(2\pi) = 0$. Then $\int_0^{2\pi} \eta^2 \leq 4\int_0^{2\pi} (\eta')^2$. Equality holds if and only if $y = c\sin(\theta/2)$.*

Following the idea of [5, Theorem 3.1] we use this inequality to obtain

Proposition 3.1.2. *The limit $\lim_{r\to 0+} \Phi^{\psi}_{x^0}(r) \in [0, +\infty)$ exists.*

Proof. Using polar coordinates we calculate

$$
\begin{aligned}
& r\int_{\partial B_r(\mathbf{x}^0)} |\nabla\psi|^2 d\mathcal{H}^1 = r^2\int_0^{2\pi}(|\psi_r(r,\theta)|^2 + \tfrac{1}{r^2}|\psi_\theta(r,\theta)|^2)\,d\theta \\
& \geq 2r(\int_0^{2\pi}|\psi_r(r,\theta)|^2\,d\theta)^{1/2}(\int_0^{2\pi}|\psi_\theta(r,\theta)|^2\,d\theta)^{1/2} \\
& \geq 2r(\int_0^{2\pi}|\psi_r(r,\theta)|^2\,d\theta)^{1/2}(\tfrac{1}{4}\int_0^{2\pi}|\psi(r,\theta)|^2\,d\theta)^{1/2} \\
& \geq r\int_0^{2\pi}\psi(r,\theta)\psi_r(r,\theta)\,d\theta \\
& = \int_{\partial B_r(\mathbf{x}^0)}\psi\nabla\psi\cdot\nu\,d\mathcal{H}^1 \\
& = \int_{B_r(\mathbf{x}^0)}(|\nabla\psi|^2 - \psi\gamma(\psi)) \qquad (3.1)\\
& \geq \int_{B_r(\mathbf{x}^0)}|\nabla\psi|^2 - C_1\int_0^r s\int_0^{2\pi}\psi(s,\theta)\,d\theta ds \\
& \geq \int_{B_r(\mathbf{x}^0)}|\nabla\psi|^2 - C_2\int_0^r s(\int_0^{2\pi}\psi^2(s,\theta)\,d\theta)^{1/2}\,ds \\
& \geq \int_{B_r(\mathbf{x}^0)}|\nabla\psi|^2 - C_2 r^{1/2}(\int_0^r s^2\int_0^{2\pi}\psi^2(s,\theta)\,d\theta ds)^{1/2} \\
& \geq \int_{B_r(\mathbf{x}^0)}|\nabla\psi|^2 - C_3 r^{1/2}(\int_0^r s^2\int_0^{2\pi}\psi_\theta^2(s,\theta)\,d\theta ds)^{1/2} \\
& \geq \int_{B_r(\mathbf{x}^0)}|\nabla\psi|^2 - C_3 r^2(\int_{B_r(\mathbf{x}^0)}|\nabla\psi|^2)^{1/2}.
\end{aligned}
$$

Thus $\Phi' \geq -C_3 r^{1/2}\Phi^{1/2}$. Let $\Phi(r_0) = c_0 > 0$. Since all solutions of the ODE

$$
\begin{aligned}
&\Psi' = -C_3 r^{1/2}\Psi^{1/2}, \\
&\Psi(r_0) = c_0.
\end{aligned}
\qquad (3.2)
$$

are given by $\Psi(r) = (-\frac{C_3}{3}r^{3/2} + c_1)^2$, where $c_1 = c_0^{1/2} + \frac{C_3}{3}r_0^{3/2}$, the comparison theorem implies that $0 \leq \Phi \leq \Psi$. Consequently $0 \leq \Phi^{\psi}_{\mathbf{x}^0}(0+) = \lim_{r\to 0+}\Phi^{\psi}_{\mathbf{x}^0}(r) < +\infty$. □

Proposition 3.1.3. $\Phi^{\psi}_{\mathbf{x}}(0+)$ *is upper semi-continuous with respect to* $\mathbf{x}$.

Proof. Since $(\Phi^{\psi}_{\mathbf{x}})'(r) \geq -C_3 r^{1/2}(\Phi^{\psi}_{\mathbf{x}})^{1/2} \geq -C$, we obtain $\Phi^{\psi}_{\mathbf{x}}(0+) \leq \Phi^{\psi}_{\mathbf{x}}(r) + Cr$. By the continuity of $\Phi^{\psi}_{\mathbf{x}}(r)$ with respect to $\mathbf{x}$ for fixed r, we conclude that choosing r small and subsequently $|\mathbf{x}-\mathbf{x}^0|$ small,

$$
\begin{aligned}
\Phi^{\psi}_{\mathbf{x}}(0+) &\leq \Phi^{\psi}_{\mathbf{x}}(r) + Cr \leq \Phi^{\psi}_{\mathbf{x}^0}(r) + O(|\mathbf{x}-\mathbf{x}^0|) + Cr \\
&\leq \Phi^{\psi}_{\mathbf{x}^0}(0+) + O(|\mathbf{x}-\mathbf{x}^0|) + Cr + O(r).
\end{aligned}
$$

Thus $\Phi^{\psi}_{\mathbf{x}^0}(0+) \geq \limsup_{\mathbf{x}\to\mathbf{x}^0}\Phi^{\psi}_{\mathbf{x}}(0+)$. □

Let $\mathbf{x}^0 \in \Gamma$. By a translation we may assume that $\mathbf{x}^0$ is the origin. Let $\psi_r(\mathbf{x}) = r^{-1/2}\psi(r\mathbf{x})$. Then the free boundary of ψ_r is given by $\Gamma_r = \{\mathbf{x} : r\mathbf{x} \in$

$\Gamma\}$ and that Γ_r is the graph of $\eta_r(\mathbf{x}) = \eta(r\mathbf{x})/r$. Let κ_r be the curvature of Γ_r. Then $\kappa_r = r\kappa \geq -r\kappa_0$. In the new coordinates ψ_r is a domain variation solution in the sense

$$\int_{\Omega_r} (|\nabla\psi_r|^2 \mathrm{div}\xi - 2\nabla\psi_r D\xi\nabla\psi_r - 2rG(r^{1/2}\psi_r)\mathrm{div}\xi - 2g(y_0 + ry)r\chi_{\{\psi_r>0\}}\mathrm{div}\xi$$

$$- 2g(y_0 + ry)r\chi_{\{\psi_r>0\}}\xi_2) - 2\sigma\int_{\Gamma_r} \mathrm{div}_{\Gamma_r}\xi = 0, \tag{3.3}$$

where $\Omega_r = \{\mathbf{x} : \mathbf{x}^0 + r\mathbf{x} \in \Omega\}$.

Since $\int_{B_R} |\nabla\psi_r|^2 = r^{-1}\int_{B_{Rr}(\mathbf{x}^0)} |\nabla\psi|^2 = R\Phi^{\psi}_{\mathbf{x}^0}(Rr)$, $\{\psi_r\}$ is bounded in $W^{1,2}(B_R)$ for each $R > 0$. Therefore there exists a subsequence $\{\psi_{r_k}\}$ such that $\psi_{r_k} \to \psi_0$ weakly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ and strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$.

3.2 A growth estimate and blow-up sequences

Proposition 3.2.1. *There exists a $C_R < \infty$ such that $||\psi_r||_{C^{1/2}(\bar{B}_R)} \leq C_R$; here C_R is independent of r.*

Proof. We first show that $\int_{B_r(\mathbf{x})} |\nabla\psi|^2 < C_1 r$ for all $\mathbf{x}$ and small r. By Proposition 3.1.2 there is a constant C_2 such that $\int_{B_R(\mathbf{x})} |\nabla\psi|^2 < C_2 r$ for $\mathbf{x} \in \Gamma$.

For $\mathbf{x} \in \{\psi > 0\}$, let $d = \mathrm{dist}(\mathbf{x}, \Gamma)$.

If $r < d$, let $f_r(\mathbf{y}) = \psi(\mathbf{x}+r\mathbf{y})$ and $\bar{\psi}(s) = \frac{1}{2\pi s}\int_{\partial B_s(\mathbf{x})} \psi\, d\mathcal{H}^1 = \frac{1}{2\pi}\int_{\partial B_1} f_s\, d\mathcal{H}^1$ for $0 < s < r$. Then

$$\bar{\psi}'(s) = \frac{1}{2\pi s}\int_{\partial B_1} \nabla f_s(\mathbf{y}) \cdot \mathbf{y}\, d\mathcal{H}^1 = \frac{1}{2\pi s}\int_{B_1} \Delta f_s = \frac{1}{2\pi s}\int_{B_s(\mathbf{x})} -\gamma(\psi).$$

and

$$\bar{\psi}''(s) = \frac{1}{2\pi s}\int_{\partial B_s} -\gamma(\psi) - \frac{1}{2\pi s^2}\int_{B_s} -\gamma(\psi)$$

Since $\gamma \in L^\infty$, $|\bar{\psi}'(s)| \leq C_3 s$ for some $C_3 < \infty$ and $\bar{\psi}'' \in L^\infty$.

Replacing ψ in (3.1) by $\psi(\mathbf{y}) - \bar{\psi}(|\mathbf{y} - \mathbf{x}|)$, we obtain

$$\int_{B_r(\mathbf{x})} |\nabla(\psi(\mathbf{y}) - \bar{\psi}(|\mathbf{y} - \mathbf{x}|))|^2 \leq C_2 r.$$

On the other hand,

$$\begin{aligned}\int_{B_r(\mathbf{x})} |\nabla(\psi(\mathbf{y}) - \bar{\psi}(|\mathbf{y}-\mathbf{x}|))|^2 &\geq \int_{B_r(\mathbf{x})} (\tfrac{1}{2}|\nabla\psi|^2 - |\bar{\psi}'(s)|^2) \\ &\geq \tfrac{1}{2}\int_{B_r(\mathbf{x})} |\nabla\psi|^2 - C_3 r^2\end{aligned}$$

Consequently there exist $\delta > 0$ and $C_4 < \infty$ such that $\int_{B_r(\mathbf{x})} |\nabla\psi|^2 \leq C_4 r$ for $r \leq \delta$.

If $r \geq d$, $\int_{B_r(\mathbf{x})} |\nabla\psi|^2 \leq \int_{B_{2r}(\mathbf{y})} |\nabla\psi|^2 \leq 2C_2 r$.

By Morrey's Lemma (see, for example [16, Lemma 12.2]), $||\psi||_{C^{1/2}} \leq C$ for some $C < \infty$.

Noticing $\psi_r(\mathbf{x}) = \frac{\psi(r\mathbf{x})}{\sqrt{r}}$, we have

$$\begin{aligned}||\psi_r||_{C^{1/2}(\bar{B}_R)} &= \sup_{B_R} \tfrac{|\psi_r(\mathbf{x}^1)-\psi_r(\mathbf{x}^2)|}{|\mathbf{x}^1-\mathbf{x}^2|^{1/2}} \\ &= \sup_{B_R} \tfrac{|\psi(r\mathbf{x}^1)-\psi(r\mathbf{x}^2)|}{\sqrt{r}|\mathbf{x}^1-\mathbf{x}^2|^{1/2}} \\ &= ||\psi||_{C^{1/2}(\bar{B}_{rR})}\end{aligned}$$

which completes the proof. □

Consequently, passing to a subsequence if necessary, $\psi_{r_k} \to \psi_0$ in C^α for $\alpha < \frac{1}{2}$. we may prove the following strong convergence result for the gradient:

Proposition 3.2.2. *$\psi_{r_k} \to \psi_0$ strongly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$.*

Proof. It suffices to show that

$$\limsup_{k\to\infty} \int \zeta|\nabla\psi_{r_k}|^2 \leq \int \zeta|\nabla\psi_0|^2$$

for each non-negative $\zeta \in C_0^\infty(\mathbb{R}^2)$. We follow closely the proof of [9, Lemma 7.2]. The scaled ψ_{r_k} is a solution to equation

$$\Delta\psi_{r_k} = -r_k^{3/2}\gamma(\psi_{r_k} r_k^{1/2}) \text{ in } \{\psi_{r_k} > 0\}. \tag{3.4}$$

Multiplying this equation by $\zeta\psi_{r_k}$ and integrating by parts, we get

$$\int_{\mathbb{R}^2} \zeta|\nabla\psi_{r_k}|^2 + \psi_{r_k}\nabla\zeta\cdot\nabla\psi_{r_k} - \zeta r^{3/2}\psi_{r_k}\gamma(\psi_{r_k} r^{1/2}) = 0.$$

Letting $k \to \infty$ we have

$$\limsup_{k\to\infty} \int_{\mathbb{R}^2} \zeta |\nabla \psi_{r_k}|^2 = -\int_{\mathbb{R}^2} \psi_0 \nabla\zeta \cdot \nabla\psi_0.$$

Multiplying equation (3.4) by $\zeta\psi_0$ and integrating we get

$$\int_{\mathbb{R}^2} \zeta \nabla\psi_{r_k} \cdot \nabla\psi_0 + \psi_0 \nabla\zeta \cdot \nabla\psi_{r_k} - \zeta r^{3/2} \psi_0 \gamma(\psi_{r_k} r^{1/2}) = 0.$$

Letting $k \to \infty$,

$$\int_{\mathbb{R}^2} \zeta |\nabla\psi_0|^2 = -\int_{\mathbb{R}^2} \psi_0 \nabla\zeta \cdot \nabla\psi_0.$$

Thus

$$\limsup_{k\to\infty} \int_{\mathbb{R}^2} \zeta |\nabla\psi_{r_k}|^2 \le \int_{\mathbb{R}^2} \zeta |\nabla\psi_0|^2.$$

$\square$

Notice that

$$\begin{aligned} \Phi^{\psi_r}_{x^0}(T) &= T^{-1} \textstyle\int_{B_T(x^0)} |\nabla\psi_r|^2 \, dx \\ &= T^{-1} \textstyle\int_{B_T(0)} r |(\nabla\psi)(x^0 + rx)|^2 \, dx \\ &= (Tr)^{-1} \textstyle\int_{B_{Tr}(x^0)} |\nabla\psi|^2 \\ &= \Phi^{\psi}_{x^0}(Tr) \end{aligned}$$

Therefore, for every $T > 0$,

$$\begin{aligned} \Phi^{\psi_0}_{x^0}(T) &= T^{-1} \textstyle\int_{B_T(x^0)} |\nabla\psi_0|^2 \\ &= \lim_{k\to\infty} \Phi^{\psi_{r_k}}_{x^0}(T) \\ &= \lim_{k\to\infty} \Phi^{\psi}_{x^0}(Tr_k) \\ &= \Phi^{\psi}_{x^0}(0+) \end{aligned}$$

which implies

Proposition 3.2.3. $\psi_0(r,\theta) = c_1 r^{1/2} \cos(\frac{\theta}{2} + c_2)$.

Proof. Inequality (3.1) holds for ψ_0 with $\gamma = 0$. Since $\Phi^{\psi_0}_{\mathbf{x}^0}(r) \equiv \text{const} = \Psi^{\psi}_{\mathbf{x}^0}(0+)$, $(\Phi^{\psi_0}_{\mathbf{x}^0})' \equiv 0$. This implies that the inequality in (3.1) is in fact an equality for almost every r. Consequently the inequality in (3.1.1) is an equality and there are two functions $\alpha(r)$ and $\theta(r)$ such that for a.e. r and for $\theta \in [\theta(r), \theta(r) + 2\pi)$, ψ_0 is given by $\psi_0(r,\theta) = \alpha(r)\cos(\frac{\theta-\theta(r)}{2})$ and $|\frac{\partial}{\partial r}\psi_0|^2 = \frac{\int_0^{2\pi}|\frac{\partial}{\partial r}\psi_0|^2}{\int_0^{2\pi}|\psi_0|^2}\psi_0^2 = \frac{\int_0^{2\pi}|\frac{\partial}{\partial \theta}\psi_0|^2}{r^2\int_0^{2\pi}|\psi_0|^2}\psi_0^2 = \frac{1}{4r^2}\psi_0^2$. Therefore $\theta'(r) = 0$ and $|\alpha'(r)| = \frac{1}{2r}|\alpha(r)|$ for a.e. r. Consequently $\alpha(r) = \alpha_0 r^{1/2}$ and $\theta(r) = \theta_0$. □

3.3 Convergence of the curvature term

We will now study the convergence of $\int_{\Gamma_r} \operatorname{div}_{\Gamma_r}\xi$. Let us prove some lemmas first.

Lemma 3.3.1. *There exists an $L < \infty$ such that $\mathcal{H}^1(\Gamma_r \cap B_1) \le L$ for small r.*

Proof. The proof will be proceed by contradiction. We assume that

$$\limsup_{r\to 0} \mathcal{H}^1(\Gamma_r \cap B_1) = +\infty.$$

Then for every $\epsilon > 0$, there exist $r > 0$, $\mathbf{x}^1 = (x_1, y_1) \in \Gamma_r$, $\mathbf{x}^2 = (x_2, y_2) \in \Gamma_r$ such that $\mathbf{x}^2$ is a local maximum point of the graph, η_r is monotone increasing in (x_1, x_2), $|x_2 - x_1| < \epsilon$ and $|y_2 - y_1| \ge 2|x_2 - x_1|$ (see Figure 3.1).

Let $\nu_r = (\cos(\theta_r(x)), \sin(\theta_r(x)))$ be the unit normal of Γ_r. Then $\theta_r(x_2) = \frac{\pi}{2}$ since Γ_r can be touched by a ball from below. Moreover, there exists an x such that $\theta_r(x)$ exists and $\theta_r(x) > \frac{3\pi}{4}$ (otherwise $|y_2 - y_1| \le |x_2 - x_1|$).
Let $\mathbf{x}^3 = (x_3, y_3) \in \Gamma_r$ be the point such that $x_1 < x_3 < x_2$, $\theta_r(x_3) = \frac{3\pi}{4}$ and $\frac{\pi}{2} \le \theta_r(x) \le \frac{3\pi}{4}$ for $x_3 \le x \le x_2$. Then the arc length of Γ_r between $\mathbf{x}^2$ and $\mathbf{x}^3$ is bounded by

$$\int_{\mathbf{x}^3}^{\mathbf{x}^2} ds \le 2|x_2 - x_3| \le 2\epsilon.$$

Here $\int_{\mathbf{x}^3}^{\mathbf{x}^2}$ denotes the integral on Γ_r.

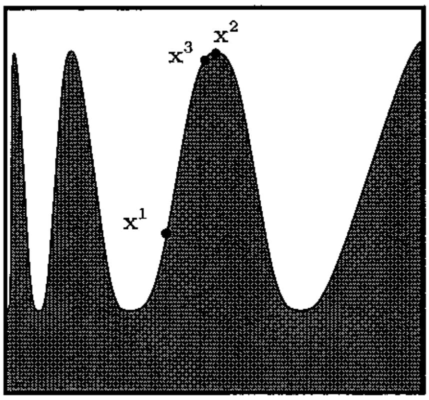


Figure 3.1: Controlling oscillations

Since $\kappa_r \geq -\kappa_0 r$,

$$-\frac{\pi}{4} = \theta_r(x_2) - \theta_r(x_3) = \int_{\mathbf{x}^3}^{\mathbf{x}^2} \frac{d\theta_r}{ds}\, ds = \int_{\mathbf{x}^3}^{\mathbf{x}^2} \kappa_r ds \geq -2\kappa_0 r\epsilon.$$

Thus we get a contradiction and the conclusion follows. $\square$

Remark 3.3.2. *Since $\mathcal{H}^1(\Gamma \cap B_r) = r\mathcal{H}^1(\Gamma_r \cap B_1)$, the above lemma implies $\mathcal{H}^1(\Gamma \cap B_r) \leq Lr$.*

Lemma 3.3.3. *Let $\nu = (\cos\theta(x), \sin\theta(x))$ be the unit outward normal vector of $\{\psi > 0\}$ at point $\mathbf{x} = (x, \eta(x))$ for a.e. $\mathbf{x}$. Then for each x_0, $\operatorname{esslim}_{x\to x_0+}\theta(x)$ and $\operatorname{esslim}_{x\to x_0-}\theta(x)$ exist.*

Proof. We suppose towards a contradiction that there exist sequences $\{x_k\}$ and $\{\tilde{x}_k\}$ such that $x_k \to x_0+$, $\tilde{x}_k \to x_0+$ and

$$\limsup_{k\to\infty} \theta(x_k) - \liminf_{k\to\infty} \theta(\tilde{x}_k) \geq \delta > 0. \tag{3.5}$$

Fix $r > 0$ and let $\tilde{\Gamma}$ be the connected component of $\Gamma \cap B_r(\mathbf{x}^0)$ that contains $\mathbf{x}^0 = (x_0, \eta(x_0))$. Let $\mathbf{x}^k = (x_k, \eta(x_k)) \in \tilde{\Gamma}$ and $\tilde{\mathbf{x}}^k = (\tilde{x}_k, \eta(\tilde{x}_k)) \in \tilde{\Gamma}$. By (3.5) there exist $i, j \in \mathbb{N}$ such that $x_i < \tilde{x}_j$ and $\theta(\tilde{x}_j) - \theta(x_i) < -\delta/2$. On the other hand,

$$-\delta/2 > \theta(\tilde{x}_j) - \theta(x_i) = \int_{\mathbf{x}^i}^{\tilde{\mathbf{x}}^j} \frac{d\theta}{ds}\, ds = \int_{\mathbf{x}^i}^{\tilde{\mathbf{x}}^j} \kappa\, ds \geq -Lr\kappa_0 \to 0$$

as $r \to 0$, a contradiction. Therefore $\operatorname{ess\,lim}_{x\to x_0+}\theta(x)$ exists. A similar argument shows that $\operatorname{ess\,lim}_{x\to x_0+}\theta(x)$ exists. $\square$

As a direct consequence we have

Lemma 3.3.4. *For each* $\mathrm{x} \in \Gamma$, *there exists an* $r_0 > 0$ *such that for* $r \le r_0$ $\Gamma \cap B_r(\mathrm{x})$ *is connected.*

Lemma 3.3.5. *Let* $R > 0$, $B_R^+ = B_R \cap \{x > 0\}$. *There exist a rotation* A *and* $\delta > 0$ *such that for* $r < \delta$, $A(\Gamma_r \cap B_R^+)$ *is the graph of a function* $\tilde{\eta}_r$ *and* $||\tilde{\eta}_r||_{W^{1,\infty}} \le C$, *where* C *is independent of* r.

Proof. It is sufficient to prove that there exists an $\epsilon > 0$ and a rotation such that $A(\Gamma \cap B_{R\delta}^+(\mathrm{x}^0))$ is the graph of a Lipschitz function. But that follows from the right continuity of $\theta(x)$. $\square$

Proposition 3.3.6. $\lim_{k\to\infty}\int_{\Gamma_{r_k}} \operatorname{div}_{\Gamma_{r_k}}\xi = (T(0-) - T(0+))\cdot\xi(0)$ *for* $\xi \in C_0^\infty(\Omega;\ \mathbb{R}^2)$, *where* $T(0+)$ *and* $T(0-)$ *denote the right and left unit tangent vector of* Γ *at* 0, *respectively.*

Proof. There exists an $R > 0$ such that $\operatorname{supp}\xi \subset B_R$. Notice that $\int_{\Gamma_{r_k}} \operatorname{div}_{\Gamma_{r_k}}\xi$ is independent of the choice of coordinates. By Lemma 3.3.5 $\Gamma_{r_k} \cap B_R^+$ is the graph of a Lipschitz function $\tilde{\eta}_k$ for each $k \in \mathbb{N}$, and $||\tilde{\eta}_k||_{W^{1,\infty}} \le C$, where C is independent of k.
Thus

$$\begin{aligned}\int_{\Gamma_{r_k}\cap B_R^+} \operatorname{div}_{\Gamma_{r_k}}\xi &= \int_0^R \sqrt{1+(\tilde{\eta}_k')^2}(\operatorname{div}\xi - \nu_k D\xi\nu_k)\\ &= \int_0^R (\tfrac{\partial\xi_1}{\partial x}(x,\tilde{\eta}_k(x))\tfrac{1}{\sqrt{1+(\tilde{\eta}_k')^2}} + \tfrac{\partial\xi_1}{\partial y}(x,\tilde{\eta}_k(x))\tfrac{\tilde{\eta}_k'}{\sqrt{1+(\tilde{\eta}_k')^2}}\\ &+\tfrac{\partial\xi_2}{\partial x}(x,\tilde{\eta}_k(x))\tfrac{\tilde{\eta}_k'}{\sqrt{1+(\tilde{\eta}_k')^2}} + \tfrac{\partial\xi_2}{\partial y}(x,\tilde{\eta}_k(x))\tfrac{(\tilde{\eta}_k')^2}{\sqrt{1+(\tilde{\eta}_k')^2}}).\end{aligned}$$

Recall that $\tilde{\eta}_k(x) = \frac{\tilde{\eta}(r_k x)}{r_k}$. By the right continuity of $\theta(x)$ we know that for any given $\epsilon > 0$ there exists a $k_0 \in \mathbb{N}$ such that $|\tilde{\eta}_k'(x) - \tilde{\eta}_k'(0+)| \le \epsilon$ for $0 \le x \le R$, $k \ge k_0$. Therefore $\frac{1}{\sqrt{1+(\tilde{\eta}_k')^2}} \to \frac{1}{\sqrt{1+(\tilde{\eta}'(0+))^2}}$, $\frac{\tilde{\eta}_k'}{\sqrt{1+(\tilde{\eta}_k')^2}} \to \frac{\tilde{\eta}'(0+)}{\sqrt{1+(\tilde{\eta}'(0+))^2}}$ and $\frac{(\tilde{\eta}_k')^2}{\sqrt{1+(\tilde{\eta}_k')^2}} \to \frac{(\tilde{\eta}'(0+))^2}{\sqrt{1+(\tilde{\eta}'(0+))^2}}$ in $L^1(0,R)$ respectively. Moreover,

$\nabla\xi(x, \tilde{\eta}_k(x)) \in L^\infty(0, R)$ and $\nabla\xi(x, \tilde{\eta}_k(x)) \to \nabla\xi(x, \tilde{\eta}'(0+)x)$ a.e. as $k \to \infty$. We conclude that

$$\begin{aligned}
&\lim_{k\to\infty} \int_{\Gamma_{r_k}\cap B_R^+} \mathrm{div}_{\Gamma_{r_k}}\xi \\
&= \int_0^R \big(\tfrac{\partial \xi_1}{\partial x}(x, \tilde{\eta}'(0+)x)\tfrac{1}{\sqrt{1+(\tilde{\eta}'(0+))^2}} + \tfrac{\partial \xi_1}{\partial y}(x, \tilde{\eta}'(0+)x)\tfrac{\tilde{\eta}'(0+)}{\sqrt{1+(\tilde{\eta}'(0+))^2}} \\
&\quad + \tfrac{\partial \xi_2}{\partial x}(x, \tilde{\eta}'(0+)x)\tfrac{\tilde{\eta}'(0+)}{\sqrt{1+(\tilde{\eta}'(0+))^2}} + \tfrac{\partial \xi_2}{\partial y}(x, \tilde{\eta}'(0+)x)\tfrac{(\tilde{\eta}'(0+))^2}{\sqrt{1+(\tilde{\eta}'(0+))^2}}\big) \\
&= -T(0+)\cdot\xi(0).
\end{aligned}$$

Similarly we have $\lim_{k\to\infty} \int_{\Gamma_{r_k}\cap B_R^-} \mathrm{div}_{\Gamma_{r_k}}\xi = T(0-)\cdot\xi(0)$. □

3.4 Characterization of blow-up limits

By the definition of domain variation solutions we infer that ψ_{r_k} is a solution in the sense

$$\begin{aligned}
&\int_{\Omega_{r_k}} (|\nabla\psi_{r_k}|^2 \mathrm{div}\xi - 2\nabla\psi_{r_k} D\xi \nabla\psi_{r_k} - 2r_k G(r_k^{1/2}\psi_{r_k})\mathrm{div}\xi \\
&\quad - 2g(y_0 + r_k y) y r_k \chi_{\{\psi_{r_k}>0\}} \mathrm{div}\xi - 2g(y_0 + r_k y)\chi_{\{\psi_{r_k}>0\}}\xi_2) \\
&\quad - 2\sigma \int_{\Gamma_{r_k}} \mathrm{div}_{\Gamma_{r_k}}\xi = 0.
\end{aligned} \tag{3.6}$$

Recalling that $\psi_{r_k} \to \psi_0$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ as $k \to \infty$, we obtain that ψ_0 is a domain variation solution in the sense

$$\int (|\nabla\psi_0|^2 \mathrm{div}\xi - 2\nabla\psi_0 D\xi \nabla\psi_0) + 2\sigma\xi(0)\cdot(T(0+) - T(0-)) = 0.$$

Lemma 3.4.1. *$v(r,\theta) = 2\sqrt{\frac{2\sigma}{\pi}}\sqrt{r}\cos\frac{\theta}{2}$ is a domain variation solution of*

$$\int_{\mathbb{R}^2} (|\nabla v|^2 \mathrm{div}\xi - 2\nabla v D\xi \nabla v) - 4\sigma\xi_1(0) = 0.$$

Here $\xi(\mathbf{x}) = (\xi_1(\mathbf{x}), \xi_2(\mathbf{x}))$ is a smooth vector-valued function with compact support. For any $c \neq 2\sqrt{\frac{2\sigma}{\pi}}$, $c\sqrt{r}\cos\frac{\theta}{2}$ is not a solution.

Proof. For each $\delta > 0$,

$$\begin{aligned}
I &= \int_{\mathbb{R}^2} (|\nabla v|^2 \mathrm{div}\xi - 2\nabla v D\xi \nabla v) - 4\sigma\xi_1(0) \\
&= o(1) + \int_{\mathbb{R}^2 \setminus B_\delta} (|\nabla v|^2 \mathrm{div}\xi - 2\nabla v D\xi \nabla v) - 4\sigma\xi_1(0) \\
&= o(1) + \int_{\mathbb{R}^2 \setminus B_\delta} \mathrm{div}(|\nabla v|^2 \xi) - \nabla|\nabla v|^2 \xi - 2\nabla v D\xi \nabla v - 4\sigma\xi_1(0).
\end{aligned}$$

Since v is harmonic in $\mathbb{R}^2 \setminus \{(r,\theta) \in \mathbb{R}^2 : \theta = \pi\}$,

$$\begin{aligned}
\mathrm{div}((\nabla v \cdot \xi)\nabla v) &= \Sigma_{i=1}^2 \tfrac{\partial}{\partial x_i}\left(\tfrac{\partial v}{\partial x_i}\left(\Sigma_{j=1}^2 \tfrac{\partial v}{\partial x_j}\xi_j\right)\right) \\
&= \Delta v\left(\Sigma_{i=1}^2 \tfrac{\partial v}{\partial x_i}\xi_i\right) + \Sigma_{i,j=1}^2 \tfrac{\partial v}{\partial x_i}\tfrac{\partial^2 v}{\partial x_j \partial x_i}\xi_j + \Sigma_{i,j=1}^2 \tfrac{\partial v}{\partial x_i}\tfrac{\partial v}{\partial x_j}\tfrac{\partial \xi_j}{\partial x_i} \\
&= \tfrac{1}{2}\nabla|\nabla v|^2 \cdot \xi + \nabla v D\xi \nabla v.
\end{aligned}$$

Therefore

$$\begin{aligned}
I &= o(1) - 4\sigma\xi_1(0) + \int_{\mathbb{R}^2 \setminus B_\delta} \mathrm{div}(|\nabla v|^2\xi) - 2\mathrm{div}((\nabla v \cdot \xi)\nabla v) \\
&= o(1) - 4\sigma\xi_1(0) + \int_{\partial B_\delta} -|\nabla v|^2 \xi \cdot \nu + 2\nabla v \cdot \xi \nabla v \cdot \nu.
\end{aligned}$$

Let $\tilde{\xi}(x) = \xi(\delta x)$. Since $\nabla v \cdot \nu = v/2$, we have

$$\begin{aligned}
I &= o(1) - 4\sigma\xi_1(0) + \int_{\partial B_1} -|\nabla v|^2 \tilde{\xi} \cdot x + 2\nabla v \cdot \tilde{\xi} \nabla v \cdot x \\
&= o(1) - 4\sigma\xi_1(0) + \int_{\partial B_1} -|\nabla v|^2 \tilde{\xi} \cdot x + v\nabla v \cdot \tilde{\xi}.
\end{aligned}$$

We use $\partial_x \theta = -\sin\theta$, $\partial_y \theta = \cos\theta$, $\partial_x r = \cos\theta$ and $\partial_y r = \sin\theta$ on S^1 to obtain in the case $v(r,\theta) = 2\sqrt{\frac{2\sigma}{\pi}}\sqrt{r}\cos\frac{\theta}{2}$ that

$$\begin{aligned}
I &= o(1) - 4\sigma\xi_1(0) - \tfrac{8\sigma}{\pi}\int_0^{2\pi} [(-\sin\theta(-\tfrac{1}{2}\sin\tfrac{\theta}{2}) + \tfrac{1}{2}\cos\theta\cos\tfrac{\theta}{2})^2 \\
&\quad + (\cos\theta(-\tfrac{1}{2}\sin\tfrac{\theta}{2}) + \tfrac{1}{2}\sin\theta\cos\tfrac{\theta}{2})^2](\tilde{\xi}_1\cos\theta + \tilde{\xi}_2\sin\theta \\
&\quad - (-\sin\theta(-\tfrac{1}{2}\sin\tfrac{\theta}{2}) + \tfrac{1}{2}\cos\theta\cos\tfrac{\theta}{2})\cos\theta\tilde{\xi}_1 \\
&\quad - (\cos\theta(-\tfrac{1}{2}\sin\tfrac{\theta}{2}) + \tfrac{1}{2}\sin\theta\cos\tfrac{\theta}{2})\cos\theta\tilde{\xi}_2 \\
&= o(1) - 4\sigma\xi_1(0) - \tfrac{8\sigma}{\pi}\int_0^{2\pi} \tfrac{1}{4}(\tilde{\xi}_1\cos\theta + \tilde{\xi}_2\sin\theta) \\
&\quad - \tfrac{1}{2}(\cos\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\tilde{\xi}_1 + \sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\tilde{\xi}_2) \\
&= o(1) - 4\sigma\xi_1(0) + \tfrac{2\sigma}{\pi}\int_0^{2\pi} \tilde{\xi}_1 \to 0
\end{aligned}$$

as $\delta \to 0$. We also obtain that $I \not\to 0$ in the case $v = c\sqrt{r}\cos\frac{\theta}{2}$ where $c \neq 2\sqrt{\frac{2\sigma}{\pi}}$. $\square$

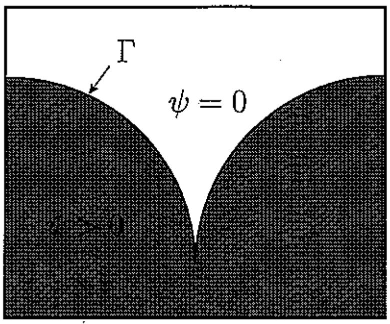


Figure 3.2: Do cusps exist?

By Lemma 3.4.1 and the fact that Γ_k is a graph in the y-direction we obtain

Proposition 3.4.2. *For each free boundary point* $\mathbf{x}$, *the blow-up limit* $\psi_0(r,\theta) = cr^{1/2}\cos(\frac{\theta}{2}+\frac{3\pi}{4})$ *where* $c = 2\sqrt{\frac{2\sigma}{\pi}}$ *or* $c = 0$.

Definition 3.4.3. *Let* $\mathbf{x}^0 \in \Gamma$. *We call* $\mathbf{x}^0$ *a cusp point if the blow-up limit at* $\mathbf{x}^0$ *is* $\psi_0(r,\theta) = 2\sqrt{\frac{2\sigma}{\pi}}r^{1/2}\cos(\frac{\theta}{2}+\frac{3\pi}{4})$. *We denote by* S *the set of cusp points.*

It follows that $\Phi^\psi_{\mathbf{x}}(0+) = \text{const} > 0$ for $\mathbf{x} \in S$ and $\Phi^\psi_{\mathbf{x}}(0+) = 0$ for $\mathbf{x} \notin S$. From the upper semi-continuity of $\Phi^\psi_{\mathbf{x}}(0+)$ with respect to $\mathbf{x}$ we obtain that S is closed.

Proposition 3.4.4. *Cusp points are isolated.*

Proof. Suppose towards a contradiction that there exists a sequence of cusp points $\{\mathbf{x}^k\}$ such that $\mathbf{x}^k \to \mathbf{x}^0$. Let $\psi_k(\mathbf{x}) = \frac{\psi(\mathbf{x}^0 + r_k\mathbf{x})}{\sqrt{r_k}}$, where $r_k = |\mathbf{x}^k - \mathbf{x}^0|$. We may assume that $\frac{\mathbf{x}^k - \mathbf{x}^0}{r_k} \to \zeta \in S^1$. By Proposition 3.2.2 and the closedness of S we know that $\psi_k \to \psi_0 = 2\sqrt{\frac{2\sigma}{\pi}}r^{1/2}\cos(\frac{\theta}{2}+\frac{3\pi}{4})$ in $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$. From the super semi-continuity of $\Phi^\psi_{\mathbf{x}}(0+)$ with respect to ψ and $\mathbf{x}$ we know that

$$\lim_{k\to\infty} \Phi^\psi_{\mathbf{x}^k}(0+) = \lim_{k\to\infty} \Phi^{\psi_k}_{\frac{\mathbf{x}^k-\mathbf{x}^0}{r_k}}(0+) \le \Phi^{\psi_0}_\zeta(0+) = 0,$$

which contradicts $\Phi^{\psi}_{\mathbf{x}^k}(0+) = \text{const} > 0$. $\square$

Chapter 4

Regularity

4.1 Regularity outside the set S

First we study the regularity of $\Gamma \setminus S$. Let $U \Subset \Omega$ be a domain such that $\overline{U} \cap S = \emptyset$.

4.1.1 C^1 regularity

Proposition 4.1.1. *$\Gamma \cap U$ is a C^1curve.*

Proof. It is sufficient to show that $\theta(x+) = \theta(x-)$ for $\mathbf{x} = (x, \eta(x)) \in \Gamma \setminus S$. We know that at $\mathbf{x}$ the blow-up limit $\psi_0 = 0$. Letting $r_k \to 0$ in equation (3.6), we obtain that $(T(x-) - T(x+)) \cdot \xi(\mathbf{x}) = 0$ for $\xi \in C_0^\infty(\Omega; \mathbb{R}^2)$. Consequently $\theta(x+) = \theta(x-)$. $\square$

4.1.2 $C^{1,\alpha}$ regularity

Let $1 < \alpha < 1$. We define $\Phi_{\mathbf{x}}^{\psi,\alpha+1}(r) = \frac{1}{r^{\alpha+1}} \int_{B_r(\mathbf{x})} |\nabla\psi|^2$. Following the method of A. Bonnet (see [5, Theorem 6.1]), we are going to show that there exist $r_0 > 0$ and $C < \infty$ such that $\Phi_{\mathbf{x}}^{\psi,\alpha+1}(r) \le C$ for $\mathbf{x} \in \Gamma \cap U$ and $r \le r_0$. By the C^1 regularity there exist $r_0 > 0$ and a cone K with opening angle $\frac{2\alpha\pi}{\alpha+1}$ and vertex at $\mathbf{x}$ such that $K \cap B_{r_0}(\mathbf{x}) \subset \{\psi = 0\}$. A straightforward change of variables in the Wirtinger inequality gives

$$\int_0^{2\pi} |\psi_\theta(r,\theta)|^2 \geq \frac{(\alpha+1)^2}{4} \int_0^{2\pi} \psi^2(r,\theta).$$

Similar to inequality (3.1) we obtain $\frac{d}{dr}\Phi_{\mathbf{x}}^{\psi,\alpha+1}(r) \geq -Cr^{(1-\alpha)/2}(\Phi_{\mathbf{x}}^{\psi,\alpha+1}(r))^{1/2}$. Therefore $\Phi_{\mathbf{x}}^{\psi,\alpha+1}(r)$ is bounded.

Proposition 4.1.2. *$\Gamma \cap U$ is a $C^{1,\alpha}$ curve.*

Proof. Let $\mathbf{x} \in \Gamma \cap U$. By a translation we may assume that $\mathbf{x} = 0$. Let ψ_r, Γ_r be the scaled ψ and Γ, respectively, as defined in Section 3.2. Moreover let $\nu_r(\mathbf{x}) = (\cos\theta_r(x), \sin\theta_r(x))$ be the unit outward normal vector on $\partial\{\psi_r > 0\}$. By the C^1 regularity $\theta_r(x) \to \theta(0)$ uniformly in $[1,1]$ as $r \to 0$. Recall that ψ_r is a domain variation solution in the sense

$$\int_{\Omega_r} (|\nabla\psi_r|^2 \mathrm{div}\xi - 2\nabla\psi_r D\xi\nabla\psi_r - 2rG(r^{1/2}\psi_r)\mathrm{div}\xi - 2g(y_0+ry)r\chi_{\{\psi_r>0\}}\mathrm{div}\xi$$
$$- 2g(y_0+ry)r_k\chi_{\{\psi_r>0\}}\xi_2) - 2\sigma\int_{\Gamma_r} \mathrm{div}_{\Gamma_r}\xi = 0. \tag{4.1}$$

We choose a test function $\xi \in C_0^1(B_1;\mathbb{R}^2)$ such that $\xi \cdot \nu_r \geq 0$ on $\Gamma_r \cap B_1$ and $\xi \cdot \nu_r \geq 1$ on $\Gamma_r \cap B_{1/2}$ for $r < r_0$. Then

$$\left|\int_{\Gamma_r\cap B_1} \mathrm{div}_{\Gamma_r}\xi\right| \leq C_1 r + C_2\int_{B_1} |\nabla\psi_r|^2 \leq C_1 r + C_2 r^{-1}\int_{B_r} |\nabla\psi|^2 \leq C_3 r^\alpha.$$

Let κ_r^+ be the positive part of κ. Since $\kappa_r \geq -\kappa_0 r$ and $\mathcal{H}^1(\Gamma_r \cap B_1) \leq L$, we have

$$-\int_{\Gamma_r\cap B_1} \mathrm{div}_{\Gamma_r}\xi = \int_{\Gamma_r\cap B_1} \xi\cdot\nu_r\kappa_r \geq -C_4 r + \int_{\Gamma_r\cap B_1} \xi\cdot\nu_r\kappa_r^+$$
$$\geq -C_4 r + \int_{\Gamma_r\cap B_{1/2}} \kappa_r^+.$$

Thus $\int_{\Gamma\cap B_{r/2}} \kappa^+ = \int_{\Gamma_r\cap B_{1/2}} \kappa_r^+ \leq C_4 r + C_3 r^\alpha$ for small r. For $\mathbf{x}^i \in \Gamma \cap U$, $i = 1,2$, we obtain that $|\theta(x_1) - \theta(x_2)| \leq \int_{\mathbf{x}^1}^{\mathbf{x}^2} |\kappa|\, ds \leq C_5(|\mathbf{x}^1 - \mathbf{x}^2| + |\mathbf{x}^1 - \mathbf{x}^2|^\alpha)$ which implies that ν is a C^α vector field on $\Gamma \cap U$. $\square$

By [16, Corollary 8.36] we have

Proposition 4.1.3. $\psi \in C^{1,\alpha}(\overline{\{\psi > 0\} \cap U})$.

4.1.3 Higher regularity

Let $\xi \in C_0^1(U;\mathbb{R}^2)$. We recall that ψ is a domain variation solution in the sense

$$\begin{aligned}\int_U (|\nabla\psi|^2 \mathrm{div}\xi - 2\nabla\psi D\xi\nabla\psi - 2G(\psi)\mathrm{div}\xi \\ - 2gy\chi_{\{\psi>0\}}\mathrm{div}\xi - 2g\chi_{\{\psi>0\}}\xi_2) - 2\sigma\int_\Gamma \mathrm{div}_\Gamma\xi = 0.\end{aligned} \tag{4.2}$$

For $\epsilon > 0$, let $U_\epsilon = U \cap \{\mathbf{x} : \psi(\mathbf{x}) > 0, \mathrm{dist}(\mathbf{x},\Gamma) > \epsilon\}$ and $\Gamma_\epsilon = U \cap \partial\Omega_\epsilon$. Then Γ_ϵ is a Lipschitz curve. We calculate

$$\begin{aligned}&\int_{U_\epsilon}(|\nabla\psi|^2\mathrm{div}\xi - 2\nabla\psi D\xi\nabla\psi - 2G(\psi)\mathrm{div}\xi \\ &-2gy\chi_{\{\psi>0\}}\mathrm{div}\xi - 2g\chi_{\{\psi>0\}}\xi_2) - 2\sigma\int_\Gamma \mathrm{div}_\Gamma\xi \\ &= \int_{U_\epsilon}\mathrm{div}(|\nabla\psi|^2\xi - 2(\nabla\psi\cdot\xi)\nabla\psi) + 2(\nabla\psi\cdot\xi)\Delta\psi \\ &+2\gamma(\psi)\nabla\psi\cdot\xi - 2g\mathrm{div}(y\xi) - \int_{\Gamma_\epsilon} G(\psi)\xi\cdot\nu + 2\sigma\int_{\Gamma\cap U}\kappa\xi\cdot\nu \\ &= \int_{U_\epsilon}\mathrm{div}(|\nabla\psi|^2\xi - 2(\nabla\psi\cdot\xi)\nabla\psi - 2gy\xi) - \int_{\Gamma_\epsilon} G(\psi)\xi\cdot\nu + 2\sigma\int_{\Gamma\cap U}\kappa\xi\cdot\nu.\end{aligned}$$

Since $\psi \in C^{1,\alpha}(\overline{\{\psi > 0\} \cap U})$, there exists a constant $C < \infty$ such that $|G(\psi)| < c\epsilon$ for $\mathbf{x} \in \Gamma_\epsilon$. Letting $\epsilon \to 0$ we obtain

$$\begin{aligned}0 &= \int_U(|\nabla\psi|^2\mathrm{div}\xi - 2\nabla\psi D\xi\nabla\psi - 2G(\psi)\mathrm{div}\xi \\ &\quad -2gy\chi_{\{\psi>0\}}\mathrm{div}\xi - 2g\chi_{\{\psi>0\}}\xi_2) - 2\sigma\int_\Gamma \mathrm{div}_\Gamma\xi \\ &= \int_{\{\psi>0\}\cap U}\mathrm{div}(|\nabla\psi|^2\xi - 2(\nabla\psi\cdot\xi)\nabla\psi - 2gy\xi) + 2\sigma\int_{\Gamma\cap U}\kappa\xi\cdot\nu \\ &= \int_{\Gamma\cap U}(-|\nabla\psi|^2 - 2gy + 2\sigma\kappa)\xi\cdot\nu.\end{aligned}$$

Hence $|\nabla\psi|^2 + 2gy - 2\sigma\kappa = 0$ on $\Gamma \cap U$ which implies $\kappa \in C^\alpha(\Gamma\cap U)$. We have therefore proved the following Theorem

Theorem 4.1.4. *$\Gamma \cap U$ is a $C^{2,\alpha}$ curve.*

If the vorticity function $\gamma \in C^\infty$, using [16, Theorem 9.19] we repeat the above procedure and get

Theorem 4.1.5. *If $\gamma \in C^\infty$, then $\Gamma \cap U$ is C^∞ and $\psi \in C^\infty(\overline{\{\psi > 0\}} \cap U)$.*

4.2 Cusps do not exist

Lemma 4.2.1. *Let $\mathbf{x}^0$ be a cusp point. There exist constants $\tau > 0$ and $\delta > 0$ such that $|\nabla\psi(\mathbf{z})| \geq \frac{\tau}{|\mathbf{z}-\mathbf{x}^0|^{1/2}}$ for all $\mathbf{z} \in \Gamma$ and $|\mathbf{z} - \mathbf{x}^0| < \delta$.*

Proof. We may assume $\mathbf{x}^0 = 0$. Let $\mathbf{z} \in \Gamma_r \cap B_1$ and $r_0 = 1/\kappa_0$. There exists a ball $B_{r_0}(\mathbf{y}) \subset \{\psi_r > 0\}$ such that $\mathbf{z} \in \partial B_{r_0}(\mathbf{y})$. Since $\psi_r \to \psi_0(r,\theta) = 2\sqrt{\frac{2\sigma}{\pi}}\sqrt{r}\cos(\frac{\theta}{2} + \frac{3\pi}{4})$ in C^α for $\alpha < \frac{1}{2}$, there exist $r_0 > 0$, $c > 0$ such that $\inf_{\mathbf{z}\in\partial B_{r_0/2}(\mathbf{y})} \psi_r \geq c$ for $r < r_0$, where c is independent of $\mathbf{z}$. We introduce an auxiliary function v in $A = B_{r_0}(\mathbf{y}) \setminus B_{r_0/2}(\mathbf{y})$ by

$$v(\mathbf{x}) = e^{-\beta r_0^2} - e^{-\beta\rho^2}$$

where $\rho = |\mathbf{y} - \mathbf{x}| > r_0/2$ and β is a positive constant yet to be determined. Direct calculation gives

$$\Delta v = -e^{-\beta\rho^2}(4\beta^2\rho^2 - 4\beta\rho).$$

We choose β large enough so that $\Delta v \leq -1$ in A. Since $\inf_{\mathbf{z}\in\partial B_{r_0/2}(\mathbf{y})} \psi(\mathbf{z}) > c > 0$, there is a constant $\epsilon > 0$ such that $\psi_r + \epsilon v \geq 0$ on $\partial B_{r_0/2}(\mathbf{y})$. This inequality also holds on ∂B_{r_0} where $\psi_r \geq 0$ and $v = 0$. Since $\Delta\psi_r(\mathbf{x}) = -r^{3/2}\gamma(\psi(r\mathbf{x})) \leq Cr^{3/2}$, there is a positive constant $r_1 < r_0$ such that $\psi_r + \epsilon v$ is superharmonic in A for $r < r_1$. The weak maximum principle now implies $\psi_r + \epsilon v \geq 0$ in A. Taking the normal derivative at $\mathbf{z}$, we obtain

$$\frac{\partial\psi_r}{\partial\nu}(\mathbf{z}) \leq \epsilon\frac{\partial v}{\partial\nu}(\mathbf{z}) = \epsilon v'(\kappa_0) = -\tau < 0$$

for $r < r_1$ and $\mathbf{z} \in \Gamma_r \cap B_1$. Combining this with $|\nabla\psi(r\mathbf{z})| = r^{-1/2}\frac{\partial\psi_r}{\partial\nu}(\mathbf{z})$ yields the desired result. □

Proposition 4.2.2. $S = \emptyset$.

Proof. Suppose towards a contradiction that there is a cusp $\mathbf{x}^0 \in S$. By a translation we may assume that $\mathbf{x}^0 = 0$. Noticing that $\theta(0+) = \pi$ and $\theta(x) \leq \pi$, there exists a sequence $\mathbf{x}^k = (x_k, y_k) \in \Gamma_k$ such that $x_k \to 0+$ and $\kappa(\mathbf{x}^k) \leq 0$. The boundary condition implies $|\nabla\psi(\mathbf{x}^k)|^2 = -2g\eta(x_k) + C_1 + 2\sigma\kappa(\mathbf{x}^k) \leq C_2$ which contradicts Lemma 4.2.1. $\square$

Thus we have proved our main theorem:

Theorem 4.2.3. *Let ψ be a domain variation solution satisfying the assumption in section 2.4. Then ψ is a classic solution to equation (2.2) and the free boundary $\Gamma = \partial\{\psi > 0\}$ is a $C^{2,\alpha}$ curve. Moreover, Γ is smooth if $\gamma \in C^\infty$.*

Chapter 5

Zero surface tension limit

In this section we study the zero surface tension limit two dimensional steady capillary gravity water waves. Let $0 < \sigma \leq 1$ be the coefficient of the surface tension and ψ_σ be the normalized stream functions. we assume ψ_σ are $2L$ periodic and satisfy the same boundary condition on the bottom $y = -d$.

Let $\Omega = (-L, L) \times (-d, 0)$, then ψ_σ be a solution to equations:

$$\begin{aligned}
&\Delta\psi_\sigma = -\gamma(\psi_\sigma) \quad \text{in} \quad \Omega \cap \{\psi > 0\} \\
&\psi_\sigma = 0 \quad \text{on} \quad y = \eta_\sigma(x) \\
&|\nabla\psi_\sigma|^2 + 2gy - 2\sigma\kappa_\sigma = 0 \quad \text{on} \quad y = \eta_\sigma(x) \\
&\psi_\sigma = p_0 \quad \text{on} \quad y = -d,
\end{aligned} \tag{5.1}$$

where $\gamma \in C^\infty((0, p_0)) \cap L^\infty([0, p_0])$ is the vorticity function, the free surface is the graph of some smooth function η_σ which is also $2L$ periodic, p_0 is a positive constant and $\kappa_\sigma = \frac{\eta_\sigma''}{(1+(\eta_\sigma')^2)^{3/2}}$ is the curvature of the free boundary $\partial\{\psi_\sigma > 0\}$.

In this Chapter we assume that $\gamma \geq 0$ and the arc length of the free boundaries $\partial\{\psi_\sigma > 0\}$ in one period is uniformly bounded.

In order to get the convergence of ψ_σ as $\sigma \to 0$, we need some uniform estimates.

Lemma 5.0.4. *$\{\psi_\sigma\}$ is uniformly bounded in $W^{1,2}(\Omega)$.*

Proof. Let $B = \{(x, -d), -L < x < L\}$ and $\Gamma_\sigma = \{(x, \eta_\sigma(x)), -L < x < L\}$. By Green's identity we have

$$\begin{aligned}\int_\Omega |\nabla\psi_\sigma|^2 d\mathbf{x} &= \int_\Omega |\nabla(\psi_\sigma - p_0)|^2 d\mathbf{x} \\ &= \int_\Omega (\psi_\sigma - p_0)\gamma(\psi_\sigma) d\mathbf{x} + \int_{\Gamma_\sigma} (\psi_\sigma - p_0)\tfrac{\partial\psi_\sigma}{\partial\nu} d\mathcal{H}^1 \\ &+ \int_B (\psi_\sigma - p_0)(-\tfrac{\partial\psi_\sigma}{\partial y}) dx.\end{aligned}$$

Since γ and ψ_σ are uniformly bounded,

$$\left|\int_\Omega (\psi_\sigma - p_0)\gamma(\psi_\alpha) d\mathbf{x}\right| \leq C_1 < \infty,$$

where C_1 is independent of σ. The boundary conditions give

$$\int_B (\psi_\sigma - p_0)(-\frac{\partial\psi_\sigma}{\partial y}) dx = 0$$

and

$$\begin{aligned}\left|\int_{\Gamma_\sigma} (\psi_\sigma - p_0)\tfrac{\partial\psi_\sigma}{\partial\nu} d\mathcal{H}^1\right| &\leq C_2(\mathcal{H}^1(\Gamma_\sigma))^{\frac{1}{2}}(\int_{\Gamma_\sigma} |\nabla\psi_\sigma|^2 d\mathcal{H}^1)^{\frac{1}{2}} \\ &= C_2(\mathcal{H}^1(\Gamma_\sigma))^{\frac{1}{2}}(\int_{\Gamma_\sigma} -2gy - 2\sigma\kappa_\sigma d\mathcal{H}^1)^{\frac{1}{2}}.\end{aligned}$$

Since psi_σ are $2L$ periodic, $\int_{\Gamma_\sigma} \kappa_\sigma = 0$. Thus

$$\begin{aligned}\left|\int_{\Gamma_\sigma} (\psi_\sigma - p_0)\tfrac{\partial\psi_\sigma}{\partial\nu} d\mathcal{H}^1\right| &\leq C_2(\mathcal{H}^1(\Gamma_\sigma))^{\frac{1}{2}}(\int_{\Gamma_\sigma} -2gy d\mathcal{H}^1)^{\frac{1}{2}} \\ &\leq C_3\mathcal{H}^1(\Gamma_\sigma) \leq C_4.\end{aligned}$$

Therefore there exists a constant $C < \infty$ such that $\int_\Omega |\nabla\psi_\sigma|^2 d\mathbf{x} \leq C$ for all $0 < \sigma \leq 1$. ☐

Let $\tilde{\Omega} = \mathbb{R} \times (-d-1, 1)$. We extend ψ_σ to $\tilde{\Omega}$ by defining $\psi_\sigma(x, y) = 0$ for $0 < y < 1$ and $\psi_\sigma(x, y) = p_0$ for $-d-1 < y < -d$.

Lemma 5.0.5. *There exists a constant $C < \infty$ such that $\|\psi_\sigma\|_{C^{\frac{1}{2}}(\Omega)} \leq C$, where C does not depend on σ.*

Proof. Similar to Proposition 3.1.2 and Proposition 3.2.1 there exist $\delta > 0$ and $C < \infty$ such that $|\frac{1}{r}\int_{B_r(\mathbf{x})} |\psi_\sigma|^2| < C$ for $\mathbf{x} \in \Omega$, $0 < r < \delta$. C does not depend on σ since $\int_\Omega |\nabla\psi_\sigma|^2$ which are uniformly bounded. By Morrey's Lemma we get a uniform $C^{\frac{1}{2}}$ estimate for ψ_σ. □

We may choose a subsequence $\{\psi_{\sigma_i}\}$ such that $\psi_{\sigma_i} \to \psi_0$ weakly in $W^{1,2}(\Omega)$ and strongly in $C^\alpha(\Omega)$ for some $\alpha < \frac{1}{2}$. Moreover we have the following strong convergence result:

Proposition 5.0.6. *$\psi_{\sigma_i} \to \psi_0$ strongly in $W^{1,2}_{\text{loc}}(\Omega)$*

Proof. It suffices to show that

$$\lim_{\sigma_i \to 0} \int_\Omega \zeta |\nabla\psi_{\sigma_i}|^2 = \int_\Omega \zeta |\nabla\psi_0|^2$$

for each $\zeta \in C_0^\infty(\Omega)$. Since ψ_{σ_i} satisfies

$$\Delta\psi_{\sigma_i} = -\gamma(\psi_{\sigma_i}) \ \text{ in } \ \{\psi_{\sigma_i} > 0\} \cdot \tag{5.2}$$

Multiplying this equation by $\zeta\psi_{\sigma_i}$ and $\zeta\psi_0$ separately and integrating by parts, we get

$$\int_\Omega \zeta|\nabla\psi_{\sigma_i}|^2 + \psi_{\sigma_i}\nabla\zeta \cdot \nabla\psi_{\sigma_i} - \zeta\psi_i\gamma(\psi_{\sigma_i})d\mathbf{x} = 0$$

and

$$\int_\Omega \zeta\nabla\psi_0 \cdot \nabla\psi_{\sigma_i} + \psi_0\nabla\zeta \cdot \nabla\psi_{\sigma_i} - \zeta\psi_0\gamma(\psi_{\sigma_i})d\mathbf{x} = 0.$$

The $W^{1,2}(\omega)$ weak convergence and $C^\alpha(\Omega)$ convergence imply that

$$\int_\Omega \zeta\nabla\psi_0 \cdot \nabla\psi_{\sigma_i} d\mathbf{x} - \int_\Omega \zeta|\nabla\psi_0|^2 d\mathbf{x} \to 0,$$

$$\int_\Omega \psi_{\sigma_i}\nabla\zeta \cdot \nabla\psi_{\sigma_i} - \psi_0\nabla\zeta \cdot \nabla\psi_{\sigma_i} d\mathbf{x} \to 0,$$

and

$$\int_\Omega \zeta\psi_{\sigma_i}\gamma(\psi_{\sigma_i}) - \zeta\psi_0\gamma(\psi_{\sigma_i})d\mathbf{x} \to 0$$

as $\sigma_i \to 0$. Therefore

$$\lim_{\sigma_i \to 0} \int_\Omega \zeta |\nabla \psi_{\sigma_i}|^2 d\mathbf{x} = \int_\Omega \zeta |\nabla \psi_0|^2 d\mathbf{x}$$

□

Lemma 5.0.7. *There exists a subsequence which is still denoted by ψ_{σ_i} that $\chi_{\{\psi_{\sigma_i}>0\}} \to \chi_{\{\psi_0>0\}}$ in L^1.*

Proof. Since the arc length $\partial\{\psi_\sigma\}$ is uniformly bounded, we know that $\{\chi_{\{\psi_\sigma>0\}}\}$ is uniformly bounded in the space of bounded variation functions. Thus there exists a subsequence ψ_{σ_i} and a Caccioppoli set A such that $\chi_{\{\psi_{\sigma_i}>0\}} \to \chi_A$ in $L^1(\Omega)$ and $\mathcal{H}^1(\partial A) < \infty$. By the uniformly convergence of ψ_{σ_i} it is easy to see that $\{\psi_0 > 0\} \subset A$. Therefore it is sufficient to show that $A^\circ \subset \{\psi_0 > 0\}$. We assume that towards a contradiction there exists a point $(x, y) = \mathbf{x} \in A^\circ \setminus \{\psi_0 > 0\}$, then there exists constant $r > 0$ and $\sigma_0 > 0$ such that $B_r(\mathbf{x}) \subset \{\psi_{\sigma_i} > 0\}$ for $\sigma_i < \sigma_0$. Let $\tilde{\Omega} = B_r(\mathbf{x}) \cup (x-r, x+r) \times (-d, y)$. Then $\tilde{\Omega} \subset \{\psi_{\sigma_i} > 0\}$. Therefore ψ_{σ_i} are supharmonic functions in $\tilde{\Omega}$ and consequently ψ_0 is also a supharmonic function in $\tilde{\Omega}$. On the other hand, we know that $\psi_0 \geq 0$ in $\tilde{\Omega}$ and $\psi_0(\mathbf{x}) = 0$ since $\mathbf{x} \notin \{\psi_0 > 0\}$. Thus we get a contradiction by the maximum principle. Therefore $\chi_{\{\psi_{\sigma_i}>0\}} \to \chi_{\{\psi_0>0\}}$ in L^1. □

It is easy to see that $G(\psi_{\sigma_i}) \to G(\psi_0)$ in L^1 since that $G(\psi_{\sigma_i})$ is uniformly bounded and $\psi_{\sigma_i} \to \psi_0$ a.e.

Recall that $\mathrm{div}_{\Gamma_\sigma}\xi = \mathrm{div}\xi - \nu_\sigma D\xi\nu_\sigma$, where ν is the unit normal vector the free surface Γ_σ. There exists a constant $C < \infty$ such that $\|\mathrm{div}_{\Gamma_\sigma}\xi\|_{L^\infty} < C$ for all σ. Therefore

$$|\sigma \int_{\Gamma_\sigma} \mathrm{div}_{\Gamma_\sigma}\xi| \leq \sigma C \mathcal{H}^1(\Gamma_\sigma) \to 0$$

as $\sigma \to 0$.

Notice that ψ_σ is a domain variation in the sense

$$\int_\Omega (|\nabla\psi_\sigma|^2 \mathrm{div}\xi - 2\nabla\psi_\sigma D\xi\nabla\psi_\sigma - 2G(\psi_\sigma)\mathrm{div}\xi \\ - 2gy\chi_{\{\psi_\sigma>0\}}\mathrm{div}\xi - 2g\chi_{\{\psi_\sigma>0\}}\xi_2) - 2\sigma\int_{\Gamma_\sigma} \mathrm{div}_{\Gamma_\sigma}\xi = 0.$$

Letting $\sigma_i \to$ we obtain the following theorem:

Theorem 5.0.8. *ψ_0 is a domain variation solution to the two-dimensional gravity water waves without surface tension, more precisely, ψ_0 satisfies*

$$\int_\Omega (|\nabla\psi_0|^2 \mathrm{div}\xi - 2\nabla\psi_0 D\xi\nabla\psi_0 - 2G(\psi_0)\mathrm{div}\xi \\ - 2gy\chi_{\{\psi_0>0\}}\mathrm{div}\xi - 2g\chi_{\{\psi_0>0\}}\xi_2) = 0.$$

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