The Topology of the Configuration of Projective Subspaces in a Projective Space II

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Abstract

We assume that there is given a locally finite family of euclidean subspaces in a euclidean space. In this paper we construct a locally finite regular cell complex which is simple homotopy equivalent to the complement of the union of the subspaces in question. The construction is done by generalizing the one employed by Deligne in the case when every member of the family is a real hyperplane in $\boldsymbol{C}^n$.

§ 1. Introduction

Let $\boldsymbol{F}$ be one of the following fields: $\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$. Now $\boldsymbol{F}^n$ is the right vector space over $\boldsymbol{F}$ consisting of vectors with n components in $\boldsymbol{F}$. Suppose given two vectors x, y in $\boldsymbol{F}^n$ such as ${}^t x=(x_1 \cdots x_n)$, ${}^t y=(y_1 \cdots y_n)$. Then the inner product $(x, y)_F$ is defined to be ${}^t\bar{x}\cdot y=\bar{x}_1\cdot y_1+\cdots+\bar{x}_n\cdot y_n$. Now we have the identity $(xa, yb)_F=\bar{a}\cdot(x, y)_F\cdot b$ for every a, b in $\boldsymbol{F}$.

Let W be a euclidean subspace of $\boldsymbol{F}^n$. Then we denote by W_R the real restriction of W. Now we write V for the standard vector space of W. Then we denote by $W^\perp$ the orthogonal complement of V over $\boldsymbol{F}$.

Let W be a euclidean subspace of $\boldsymbol{F}^n$. Then a *reference point* a of W is a point fixed in W. And a *system of normal vectors* R of the real restriction W_R is a sequence of vectors $\{r_\lambda \mid 0\leqq\lambda\leqq l-1\}$ spanning $W_R^\perp$.

Further throughout this paper, we promise to write $\boldsymbol{F}_3$ for $\{-1, 0, 1\}$.

Let $\mathscr{W}=\{W(i) \mid i\in I\}$ be a locally finite family of euclidean subspaces of $\boldsymbol{F}^n$.

Let i be an index in I.

Then the member $W(i)$ is assumed to be equipped with a reference point $a(i)$ and a system of normal vectors $R(i)$ of the real restriction $W(i)_R$ being a sequence of vectors $\{r_\lambda(i) \mid 0\leqq\lambda\leqq l(i)-1\}$.

For each λ satisfying $0\leqq\lambda\leqq l(i)-1$ and for each θ in $\boldsymbol{F}_3$, we define

$$L_\lambda^\theta(i)=\{x\in \boldsymbol{F}^n \mid \operatorname{sign} \operatorname{Re}(r_\lambda(i), x-a(i))_F=\theta\}.$$

If $\theta=0$, we write $L_\lambda(i)$ for the hyperplane $L_\lambda^0(i)$.

Now we consider the family $\mathscr{L}_0=\{L_0(i)\mid i\in I\}$ of hyperplanes of $\boldsymbol{F}^n_R$.

Appropriate choice of $R(i)$ for each i in I allows us to impose the next hypothesis without loss of generality.

H_0) $\mathscr{L}_0$ is locally finite.

Now $\mathscr{L}_0$ defines a cell decomposition of $\boldsymbol{F}^n$ of which cell is not necessarily closure compact. The resulting complex is denoted by $K[\mathscr{L}_0]$.

We denote by Θ the set of functions θ: $I\to\boldsymbol{F}_3$.

For each θ in Θ, we put

$$L_0^\theta=\cap\{L_0^{\theta(i)}(i)\mid i\in I\}\,.$$

The L_0^θ is a cell of $K[\mathscr{L}_0]$ if and only if it is nonvacuous.

Now for each θ in Θ we define $I(\theta)$ to be the set $\theta^{-1}(0)\subset I$.

If L_0^θ is nonvacuous, $i\in I(\theta)$ is equivalent to say $L_0^\theta\subset L_0(i)$.

Here we assume the following.

H_1) $K[\mathscr{L}_0]$ is a closure finite cell complex.

Let $'K[\mathscr{L}_0]$ denote the barycentric subdivision of $K[\mathscr{L}_0]$: that is, $'K[\mathscr{L}_0]$ is the simplicial complex whose set of vertices is the set of barycentres $\hat{L}_0^\theta$ taken once for each cell L_0^θ of $K[\mathscr{L}_0]$. For each sequence $L_0^{\theta_0}<\cdots<L_0^{\theta_q}$ of cells of $K[\mathscr{L}_0]$ the sequence of the corresponding barycentres $\hat{L}_0^{\theta_0},\cdots,\hat{L}_0^{\theta_q}$ is the set of vertices of a simplex of $'K[\mathscr{L}_0]$ and vice versa.

Let θ be an element of Θ such that L_0^θ is a cell of $K[\mathscr{L}_0]$.

We define $O(\theta)$ to be an open star of $\hat{L}_0^\theta$ in $'K[\mathscr{L}_0]$.

Let J be a subset of I. Then $(\Lambda\times E)|J$ denotes the set of functions (λ,ε): $J\to N\times\boldsymbol{F}_3^\times$ such that for each i in J we have $1\leqq\lambda(i)\leqq l(i)-1$.

For each (λ,ε) in $(\Lambda\times E)|J$, we define

$$L_\lambda^\varepsilon=\cap\{L_{\lambda(i)}^{\varepsilon(i)}(i)\mid i\in J\}\,.$$

Let θ be an element of Θ. We define $(\Lambda\times E)^\theta$ to be the set of indices $(\lambda,\varepsilon)^\theta$ taken once for each function (λ,ε) in $(\Lambda\times E)|I(\theta)$ satisfying the condition $L_0^\theta\cap L_\lambda^\varepsilon\neq\emptyset$.

For each index $(\lambda,\varepsilon)^\theta$ in $(\Lambda\times E)^\theta$, we define

$$N(\lambda,\varepsilon)^\theta=O(\theta)\cap L_\lambda^\varepsilon\,.$$

We define $\mathscr{N}$ to be an indexed family being a function which assigns to each $(\lambda,\varepsilon)^\theta$ in $(\Lambda\times E)^\theta$ with θ in Θ an open set $N(\lambda,\varepsilon)^\theta$.

We denote by $\tilde{Y}(\mathscr{W})$ the union $\cup\{W(i)\mid i\in I\}$.

Now we have the following.

PROPOSITION 2.6. *$\mathscr{N}$ is an open covering of $\boldsymbol{F}^n\backslash\tilde{Y}(\mathscr{W})$ whose each element is collapsible.*

We denote by $K(\mathscr{N})$ the complex of the nerves of $\mathscr{N}$ and by $|K(\mathscr{N})|$ the geometric realization of $K(\mathscr{N})$.

THEOREM 2.8. *The canonical map induces a simple homotopy equivalence*

$$F^n\backslash \tilde{Y}(\mathscr{W}) \underset{s}{\simeq} |K(\mathscr{N})| .$$

Now we consider the case, $F=C$ and for each i in I the member $W(i)$ is a hyperplane.

Let θ be an element of Θ such that L_0^θ is a cell of $K[\mathscr{L}_0]$. We define $D(\theta)$ to be the dual cell of L_0^θ in $'K[\mathscr{L}_0]$. We denote by $|D(\theta)|$ its geometric realization.

If L_0^θ is a face of $L_0^{\theta'}$, then $D(\theta')$ is a face of $D(\theta)$. And the converse is also true. Here $\iota_{\theta'}^\theta$: $D(\theta')\to D(\theta)$ denotes a natural injection.

Let θ be an element of Θ. We define E^θ to be the set of indices ε^θ taken once for each ε: $I(\theta)\to F_3^\times$ satisfying the condition $L_0^\theta \cap L_1^\varepsilon \neq \emptyset$. Here E^θ is assumed to have discrete topology.

If L_0^θ is a face of $L_0^{\theta'}$, then $I(\theta')$ is a subset of $I(\theta)$. We define $\pi_{\theta'}^\theta$: $E^\theta\to E^{\theta'}$, to be the projection sending ε^θ to $(\varepsilon')^{\theta'}$ with $\varepsilon'=\varepsilon|I(\theta')$.

Let θ be an element of Θ such that L_0^θ is a cell of $K[\mathscr{L}_0]$. We consider a topological space $|D(\theta)|\times E^\theta$.

If L_0^θ is a face of $L_0^{\theta'}$, then we have a diagramme

$$|D(\theta')|\times E^{\theta'} \xrightarrow{\iota_{\theta'}^\theta\times I} |D(\theta)|\times E^{\theta'} \xleftarrow{I\times\pi_{\theta'}^\theta} |D(\theta)|\times E^\theta .$$

Let $x\times\varepsilon^\theta$, $x'\times(\varepsilon')^{\theta'}$ be points of $|D(\theta)|\times E^\theta$, $|D(\theta')|\times E^{\theta'}$ respectively. We write $x'\times(\varepsilon')^{\theta'}\to x\times\varepsilon^\theta$ if L_0^θ is a face of $L_0^{\theta'}$ and $\iota_{\theta'}^\theta(x')=x$, $(\varepsilon')^{\theta'}=\pi_{\theta'}^\theta\varepsilon^\theta$.

We consider the disjoint union of topological spaces

$$\amalg\{\,|D(\theta)|\times E^\theta \mid \theta\in\Theta,\ L_0^\theta\in K[\mathscr{L}_0]\}\,.$$

Let us now define an equivalence relation as the weakest reflexive, symmetric and transitive relation including the property that $x\times\varepsilon^\theta$ and $x'\times(\varepsilon')^{\theta'}$ are equivalent whenever $x\times\varepsilon^\theta\to x'\times(\varepsilon')^{\theta'}$.

Let X be a topological space consisting of all equivalence classes. Clearly the family of the sets $|D(\theta)|\times\varepsilon^\theta$ defines the structure of a regular cell complex on X. The resulting cell complex is called the Deligne complex associated with $\mathscr{W}$ and denoted by $D[\mathscr{W}]$. Further we write as $X=|D[\mathscr{W}]|$.

Now we obtain the following.

THEOREM 6.3. *We have a piecewise linear homeomorphism*

$$|K(\mathscr{N})|\cong|D[\mathscr{W}]| .$$

In the special case, $\mathscr{W}$ is a family of real hyperplanes, the circumstances are rather simple.

In this case, above results were in essence obtained by Deligne [2] and were verified by the author [4] and independently by Salvetti [8].

The contents of Part II of this series of papers are arranged as follows.

In § 2, we describe the process to construct the open covering $\mathscr{N}$. In § 3, we form an another open covering $'\mathscr{N}$ and in § 4 we show that the complex of the nerves $K('\mathscr{N})$ of $'\mathscr{N}$ has the structure of a fibration over the complex $K[\mathscr{L}_0]$ which can be regarded as an analogy to the Deligne complex in the general case. In 5 we treat the case of configuration where its each member is a hyperplane. In 6 we aim at the construction of the Deligne complex.

§ 2. Open covering $\mathscr{N}$

Let W be a euclidean subspace of $\boldsymbol{F}^n$.

By definition, a *reference point* a of W is a point fixed in W and a *system of normal vectors* R of W_R is a sequence of vectors $\{r_\lambda \mid 0 \leqq \lambda \leqq l-1\}$ spanning $W_R^\perp$.

EXAMPLE. We define a system of normal vectors F of W to be a sequence of vectors $\{f_\mu \mid 0 \leqq \mu \leqq m-1\}$ spanning $W^\perp$.

Let us denote by E the canonical basis of $\boldsymbol{F}$ over $\boldsymbol{R}$, namely a sequence $\{e_\nu \mid 0 \leqq \nu \leqq d-1\}$ with $d = \dim_R \boldsymbol{F}$.

When once these are given, we can define a system of normal vectors R of W_R to be the set $\{r_\lambda \mid 0 \leqq \lambda \leqq l-1\}$ given by $r_{\mu d+\nu} = f_\mu e_\nu$ with $0 \leqq \mu \leqq m-1$, $0 \leqq \nu \leqq d-1$.

Let $\mathscr{W} = \{W(i) \mid i \in I\}$ be a locally finite family of euclidean subspaces of $\boldsymbol{F}^n$.

Let i be an index in I.

Now the member $W(i)$ has a reference point $a(i)$ and a system of normal vectors $R(i)$ of $W(i)_R$ being a sequence $\{r_\lambda(i) \mid 0 \leqq \lambda \leqq l(i)-1\}$.

For each λ satisfying $0 \leqq \lambda \leqq l(i)-1$ and each θ in $\boldsymbol{F}_3$, we have

$$L_\lambda^\theta(i) = \{x \in \boldsymbol{F}^n \mid \operatorname{sign} \operatorname{Re}(r_\lambda(i), x-a(i))_F = \theta\}.$$

We write $L_\lambda(i) = L_\lambda^0(i)$, in other words

$$L_\lambda(i) = \{x \in \boldsymbol{F}^n \mid \operatorname{Re}(r_\lambda(i), x-a(i))_F = 0\}.$$

Further we define

$$\begin{aligned} L_+(i) &= \cap \{L_\lambda(i) \mid 1 \leqq \lambda \leqq l(i)-1\} \\ &= \{x \in \boldsymbol{F}^n \mid 1 \leqq {}^\forall\lambda \leqq l(i)-1 \ \operatorname{Re}(r_\lambda(i), x-a(i))_F = 0\}. \end{aligned}$$

Obviously we have $W(i) = L_0(i) \cap L_+(i)$.

Now we obtain a family $\mathscr{L}_0 = \{L_0(i) \mid i \in I\}$ of hyperplane of $\boldsymbol{F}_R^n$.

Clearly for each i in I, $L_0(i)$ has a reference point $a(i)$ and a normal vector $r_0(i)$.

Further we define a family $\mathscr{L}_+ = \{L_+(i) \mid i \in I\}$.

Again for each i in I, $L_+(i)$ has a reference point $a(i)$ and a system of normal vectors $R_+(i)$ of $L_+(i)$ being the sequence $\{r_\lambda(i) \mid 1 \leqq \lambda \leqq l(i)-1\}$.

Appropriate choice of R, we can assume the next hypothesis.

H$_0$) L_0 is a locally finite family of hyperplanes of $\boldsymbol{F}_R^n$.

Now L_0 defines a cell decomposition of $\boldsymbol{F}^n$ of which cell is not necessarily closure compact. The resulting complex is $K[\mathscr{L}_0]$.

Let Θ be the set of all the functions $\theta\colon I \to \boldsymbol{F}_3$.

For each element θ of Θ, we define

$$L_0^\theta = \cap\{L_0^{\theta(i)}(i) \mid i \in I\}$$
$$= \{x \in F^n \mid \forall i \in I \ \mathrm{Re}(r_0(i), x-a(i))_F = \theta(i)\} .$$

Clearly L_0^θ is a cell of $K[\mathscr{L}_0]$ if and only if $L_0^\theta \neq \emptyset$. Alternatively we have

$$K[\mathscr{L}_0] = \{L_0^\theta \mid \theta \in \Theta, L_0^\theta \neq \emptyset\} .$$

For each element θ of Θ, we write $I(\theta)$ for $\theta^{-1}(0)$.

If L_0^θ is nonvacuous, $i \in I(\theta)$ is equivalent to say $L_0^\theta \subset L_0(i)$.

For each cell L_0^θ in $K[\mathscr{L}_0]$, we define $O(\theta)$ to be the open set

$$O(\theta) = \{x \in F^n \mid \mathrm{PO}(\theta)\}$$

where $\mathrm{PO}(\theta)$ denotes the set of properties as follows:

i) For each pair of i in $I(\theta)$ and j in $I \backslash I(\theta)$, we have

$$|\mathrm{Re}(r_0(i), x-a(i))_F| < |\mathrm{Re}(r_0(j), x-a(j))_F| .$$

ii) For each j in $I \backslash I(\theta)$, we have

$$\mathrm{sign}\,\mathrm{Re}(r_0(j), x-a(j))_F = \theta(j) .$$

We remark here that $O(\theta)$ is collapsible.

EXAMPLE 1. We consider the case where W is a finite family such that for each i in I the space $W(i)$ passes through 0.

Let S denote the unit sphere of F^n.

Then we can define the family $S\mathscr{W}$ to be $\{SW(i) \mid i \in I\}$ with $SW(i) = S \cap W(i)$.

Analogously we define $S\mathscr{L}_0$ to be $\{SL_0(i) \mid i \in I\}$ with $SL_0(i) = S \cap L_0(i)$.

Now $S\mathscr{L}_0$ defines a regular cell decomposition of S. The resulting complex is denoted by $K[S\mathscr{L}_0]$.

For each θ in Θ, we write $SL_0^\theta = S \cap L_0^\theta$.

Then we obtain $K[S\mathscr{L}_0] = \{SL_0^\theta \mid \theta \in \Theta, SL_0^\theta \neq \emptyset\}$.

Further by taking the restriction on S, we get an isomorphism from $K[\mathscr{L}_0]$ to the join $0 * K[S\mathscr{L}_0]$ where 0 denotes the origin L_0^0 of $K[\mathscr{L}_0]$.

Let L_0^θ be an nonvacuous cell in $K[\mathscr{L}_0]$ and let $L_0^\theta \neq 0$. Restricting on S, $O(\theta)$ corresponds to $SO(\theta) = S \cap O(\theta)$. From the definition we see that $SO(\theta)$ is nothing but an open star of the barycentre $\widehat{SL_0^\theta}$ of SL_0^θ in the barycentric subdivision $'K[S\mathscr{L}_0]$.

EXAMPLE 2. Next we consider the case when every cell L_0^θ of $K[\mathscr{L}_0]$ is closure compact.

Let L_0^θ be nonvacuous cell in $K[\mathscr{L}_0]$. Now $O(\theta)$ itself gives rise to an open star of the barycentre $\hat{L}_0^\theta$ of L_0^θ in the barycentric subdivision $'K[\mathscr{L}_0]$ of $K[\mathscr{L}_0]$.

Now we get back to the general case.

Here we assume another hypothesis.

H_1) $K[\mathscr{L}_0]$ is a closure finite, or equivalently locally finite cell complex.

Let $\hat{K}[\mathscr{L}_0]$ be the set of *barycentres* $\hat{L}_0^\theta$ taken once for each cell L_0^θ of $K[\mathscr{L}_0[$.

We define $\mathscr{O}$ to be the indexed family being a function which assigns to each $\hat{L}_0^\theta$ of $\hat{K}[\mathscr{L}_0]$ an open set $O(\theta)$.

The *derived complex* $'K[\mathscr{L}_0]$ of $K[\mathscr{L}_0]$ is by definition the complex of the nerves $K(\mathscr{O})$ of $\mathscr{O}$.

We define an order on the set of vertices $'K[\mathscr{L}_0]^0=\hat{K}[\mathscr{L}_0]$ by putting $\hat{L}_0^\theta<\hat{L}_0^{\theta'}$ if $L_0^\theta \prec L_0^{\theta'}$.

With respect this order, $'K[\mathscr{L}_0]$ becomes an ordered simplicial complex.

LEMMA 2.1. *A is a q-simplex of $'K[\mathscr{L}_0]$ with the set of vertices $A_0<\cdots<A_q$ if and only if there exists a sequence of cells $L_0^{\theta_0}\prec\cdots\prec L_0^{\theta_q}$ of $K[\mathscr{L}_0]$ and the corresponding sequence of barycentres $\hat{L}_0^{\theta_0},\cdots,\hat{L}_0^{\theta_q}$ is the sequence of vertices $A_0,\cdots,A_q$.*

Proof. We shall prove that $O(\theta_0)\cap\cdots\cap O(\theta_q)\neq\emptyset$ if and only if there exists a permutation ω such that $L_0^{\theta_{\omega(0)}}\prec\cdots\prec L_0^{\theta_{\omega(q)}}$.

First we prove the only if part.

We begin with showing that $O(\theta)\cap O(\theta')\neq\emptyset$ implies either $I(\theta)\subseteqq I(\theta')$ or $I(\theta')\subseteqq I(\theta)$.

Suppose not, then there exist i in $I(\theta)\cap(I\backslash I(\theta'))$ and i' in $I(\theta')\cap(I\backslash I(\theta))$. We choose a point x in $O(\theta)\cap O(\theta')$. Now we can conclude both

$$|\mathrm{Re}(r_0(i),\ x-a(i))_F|<|\mathrm{Re}(r_0(i'),\ x-a(i'))_F|$$

and

$$|\mathrm{Re}(r_0(i'),\ x-a(i'))_F|<|\mathrm{Re}(r_0(i),\ x-a(i))_F|\ .$$

This is contradiction.

Now we only need to verify that under the condition $I(\theta')\subseteqq I(\theta)$, $O(\theta)\cap O(\theta')\neq\emptyset$ assures $L_0^\theta\prec L_0^{\theta'}$.

Suppose there exists i in $I\backslash I(\theta)$. Let x be a point in $O(\theta)\cap O(\theta')$. Since $I\backslash I(\theta)\subseteqq I\backslash I(\theta')$, we can assert both $\operatorname{sign}\mathrm{Re}(r_0(i),\ x-a(i))_F=\theta(i)$ and $\operatorname{sign}\mathrm{Re}(r_0(i),\ x-a(i))_F=\theta'(i)$, and eventually $\theta(i)=\theta'(i)$.

Next we turn to the proof of the if part.

We assume that L_0^θ is a face of $L_0^{\theta'}$. Then we have $I(\theta)\supseteqq I(\theta')$ and $\theta|(I\backslash I(\theta))=\theta'|(I\backslash I(\theta))$.

Let $x,\ x'$ be points belonging to $L_0^\theta,\ L_0^{\theta'}$ respectively. Then there exists a segment c: $[0,1]\to L_0^\theta\cup L_0^{\theta'}$ such that $c(0)=x$, $c(1)=x'$.

If t is equal to 0, for each i in $I(\theta)$ we have $\mathrm{Re}(r_0(i),\ c(0)-a(i))_F=0$.

If t is in $[0,1]$, for each j in $I\backslash I(\theta)$ we have $\operatorname{sign}\mathrm{Re}(r_0(j),\ c(t)-a(j))_F=\theta(j)=\theta'(j)\neq 0$.

Therefore we can find a number ε in $(0,1]$ such that the point $c(t)$ belongs to $O(\theta)$ whenever t is in $[0,\varepsilon]$.

On the other hand, $L_0^{\theta'}$ is trivially contained in $O(\theta')$. Hence the point $c(t)$ always belongs to $O(\theta')$ if t is in $(0,1]$.

Consequently we have $O(\theta)\cap O(\theta')\neq\emptyset$.

Let J be a subset of I. We write $\mathscr{W}|J$, $\mathscr{L}_0|J$ and $\mathscr{L}_+|J$ for the family $\{W(j)\mid j\in J\}$, $\{L_0(j)\mid j\in J\}$ and $\{L_+(j)\mid j\in J\}$ respectively.

If J is a subset of I, we consider the union of the members of the corresponding family $\tilde{Y}(\mathscr{W}|J)=\cup\{W(j)\mid j\in J\}$, $\tilde{Y}(\mathscr{L}_0|J)=\cup\{L_0(j)\mid j\in J\}$, $\tilde{Y}(\mathscr{L}_+|J)=\cup\{L_+(j)\mid j\in J\}$.

First we note the following Lemma.

Lemma 2.2. *Let L_0^θ be a cell of $K[\mathscr{L}_0]$. Then we have*

$$L_0^\theta\cap\tilde{Y}(\mathscr{W})=L_0^\theta\cap\tilde{Y}(L_+|I(\theta)) .$$

Proof. From the definition follows

$$\begin{aligned} L_0^\theta\cap\tilde{Y}(\mathscr{W}) &= L_0^\theta\cap(\cup\{W(i)\mid i\in I\}) \\ &= \cup\{L_0^\theta\cap W(i)\mid i\in I\} \\ &= \cup\{L_0^\theta\cap L_0(i)\cap L_+(i)\mid i\in I\} . \end{aligned}$$

Further the definition implies

$$L_0^\theta\cap L_0(i)=\begin{cases} L_0^\theta & i\in I(\theta) , \\ \varnothing & i\in I\backslash I(\theta) . \end{cases}$$

Combining these results, we obtain

$$\begin{aligned} L_0^\theta\cap\tilde{Y}(\mathscr{W}) &= \cup\{L_0^\theta\cap L_+(i)\mid i\in I(\theta)\} \\ &= L_0^\theta\cap(\cup\{L_+(i)\mid i\in I(\theta)\}) . \end{aligned}$$

Let J be a subset of I. Then we set

$$(\Lambda\times E)|J=\{(\lambda,\varepsilon)\colon I\to\boldsymbol{N}\times\boldsymbol{F}_3^*\mid\forall i\in J\quad 1\leqq\lambda(i)\leqq l(i)-1\} .$$

For each $(\lambda,\varepsilon)\in(\Lambda\times E)|J$, we define

$$L_\lambda^\varepsilon=\cap\{L_{\lambda(i)}^{\varepsilon(i)}(i)\mid i\in J\} .$$

This set is convex and collapsible.

Lemma 2.3. *Let J be a subset of I. Then the family $\{L_\lambda^\varepsilon\mid(\lambda,\varepsilon)\in(\Lambda\times E)\mid J\}$ forms an open covering of $\boldsymbol{F}^n\backslash\tilde{Y}(\mathscr{L}_+|J)$.*

Proof. The definition induces the following:

$$\begin{aligned} \boldsymbol{F}^n\backslash\tilde{Y}(\mathscr{L}_+|J) &= \boldsymbol{F}^n\backslash(\cup\{L_+(i)\mid i\in J\}) \\ &= \boldsymbol{F}^n\backslash(\cup\{\cap\{L_\lambda(i)\mid 1\leqq\lambda\leqq l(i)-1\}\mid i\in J\}) \\ &= \cap\{\cup\{\boldsymbol{F}^n\backslash L_\lambda(i)\mid 1\leqq\lambda\leqq l(i)-1\}\mid i\in J\} \\ &= \cap\{\cup\{L_\lambda^{+1}(i)\cup L_\lambda^{-1}(i)\mid 1\leqq\lambda\leqq l(i)-1\}\mid i\in J\} \\ &= \cap\{\cup\{L_{\lambda(i)}^{\varepsilon(i)}(i)\mid(\lambda,\varepsilon)\in(\Lambda\times E)|J\}\mid i\in J\} \\ &= \cup\{\cap\{L_{\lambda(i)}^{\varepsilon(i)}(i)\mid i\in J\}\mid(\lambda,\varepsilon)\in(\Lambda\times E)|J\} \\ &= \cup\{L_\lambda^\varepsilon\mid(\lambda,\varepsilon)\in(\Lambda\times E)|J\} . \end{aligned}$$

COROLLARY 2.4. *Let J be a subset of I and let L_0^θ be a cell of $K[\mathscr{L}_0]$. Then the family* $\{L_0^\theta \cap L_\lambda^\varepsilon \mid (\lambda, \varepsilon) \in (\Lambda \times E)|J\}$ *forms an open covering of* $L_0^\theta \backslash (L_0^\theta \cap \tilde{Y}(\mathscr{L}_+|J))$.

For each θ in Θ, we define

$$(\Lambda \times E)^\theta = \{(\lambda, \varepsilon)^\theta \mid (\lambda, \varepsilon) \in (\Lambda \times E)|I(\theta),\ L_0^\theta \cap L_\lambda^\varepsilon \neq \emptyset\}.$$

COROLLARY 2.5. *Let L_0^θ be a cell of $K[\mathscr{L}_0]$. Then the family* $\{L_0^\theta \cap L_\lambda^\varepsilon \mid (\lambda, \varepsilon)^\theta \in (\Lambda \times E)^\theta\}$ *forms an open covering of* $L_0^\theta \backslash L_0^\theta \cap \tilde{Y}(\mathscr{W})$.

Let θ be an element of Θ. For each index $(\lambda, \varepsilon)^\theta$ in $(\Lambda \times E)^\theta$ we define

$$N(\lambda, \varepsilon)^\theta = O(\theta) \cap L_\lambda^\varepsilon$$

which is collapsible too.

We define

$$(\Lambda \times E)^K = \coprod\{(\Lambda \times E)^\theta \mid \theta \in \Theta,\ L_0^\theta \in K[\mathscr{L}_0]\}.$$

Finally we define $\mathscr{N}$ to be the indexed family being a function which assigns to each $(\lambda, \varepsilon)^\theta$ in $(K \times E)^K$ an open set $N(\lambda, \varepsilon)^\theta$.

PROPOSITION 2.6. *$\mathscr{N}$ is an open covering of $F^n \backslash \tilde{Y}(\mathscr{W})$ whose each member is collapsible.*

Let us denote by $K(\mathscr{N})$ the complex of the nerves of $\mathscr{N}$.

The following Lemma is trivial.

LEMMA 2.7. *C is a q-simplex of $K(\mathscr{N})$ with the set of vertices $C_0, \cdots, C_q$ if and only if the following conditions are satisfied.*

i) *There exists a simplex A of ${}'K[\mathscr{L}_0]$ with the sequence of vertices $A_0 \cdots A_q$: that is, there exists a sequence of cells $L_0^{\theta_0} \leqq \cdots \leqq L_0^{\theta_q}$ and the corresponding sequence of barycentres $\hat{L}_0^{\theta_0}, \cdots, \hat{L}_0^{\theta_q}$ is the sequence $A_0 \cdots A_q$.*

ii) *There exists a permutation ω and a sequence of indices $(\lambda_0, \varepsilon_0)^{\theta_0}, \cdots, (\lambda_q, \varepsilon_q)^{\theta_q}$ belonging to $(\Lambda \times E)^{\theta_0}, \cdots, (\Lambda \times E)^{\theta_q}$ respectively such that the corresponding intersection*

$$N(\lambda_0, \varepsilon_0)^{\theta_0} \cap \cdots \cap N(\lambda_q, \varepsilon_q)^{\theta_q} = O(\theta_0) \cap \cdots \cap O(\theta_q) \cap L_{\lambda_0}^{\varepsilon_0} \cap \cdots \cap L_{\lambda_q}^{\varepsilon_q}$$

is nonvacuous and the sequence of indices $(\lambda_0, \varepsilon_0)^{\theta_0}, \cdots, (\lambda_q, \varepsilon_q)^{\theta_q}$ is the sequence of vertices $C_{\omega(0)}, \cdots, C_{\omega(q)}$.

Let $|K(\mathscr{N})|$ denote the geometric realization of $K(\mathscr{N})$.

THEOREM 2.8. *The canonical map defines a simple homotopy equivalence*

$$F^n \backslash \tilde{Y}(\mathscr{W}) \underset{s}{\simeq} |K(\mathscr{N})|.$$

Proof. We may use the standard argument found in for instance Eilenberg-

Steenrod [3] or Spanier [7], and apply the sum formula of the Whitehead torsion treated in the papers such as Chapman [1].

§ 3. Open covering $'\mathscr{N}$

Let θ be an element of Θ with a cell L_0^θ of $K[\mathscr{L}_0]$.

Then $O(\theta)$ means an open set defined in § 2.

Now we consider a simplex A of $'K[\mathscr{L}_0]$ with vertices $A_0 < \cdots < A_q$. We can assume that we have a sequence $L_0^{\theta_0} \prec \cdots \prec L_0^{\theta_q}$ and $\hat{L}_0^{\theta_0}, \cdots, \hat{L}_0^{\theta_q}$ is $A_0, \cdots, A_q$. Here we should remark that we have $I(\theta_0) \supset \cdots \supset I(\theta_q)$.

For A as above, we define

$$O(A) = \cap\{O(\theta_i) \mid 0 \leqq i \leqq q\}\,.$$

In particular, for a vertex A being $\hat{L}_0^\theta$, $O(A)$ is nothing but $O(\theta)$. Hence if A is a simplex with vertices $A_0, \cdots, A_q$, we have

$$O(A) = \cap\{O(A_i) \mid 0 \leqq i \leqq q\}\,.$$

Let J be a subset of I. Then we have put

$$(\Lambda \times E)|J = \{(\lambda, \varepsilon)\colon J \to N \times F_3^\times \mid \forall i \in J \quad 1 \leqq \lambda(i) \leqq l(i) - 1\}\,.$$

Further for each (λ, ε) in $(\Lambda \times E)|J$ we have defined

$$L_\lambda^\varepsilon = \cap\{L_{\lambda(i)}^{\varepsilon(i)}(i) \mid i \in J\}\,.$$

Under these conventions we define

$$(\Lambda \times E)^A = \{(\lambda, \varepsilon)^A \mid (\lambda, \varepsilon) \in (\Lambda \times E)|I(\theta_0), \mathrm{P}(\theta_0)\}$$

where $\mathrm{P}(\theta_0)$ is the set of properties as follows:

0) There is a chain $({}_\rho\lambda, {}_\rho\varepsilon) \in (\Lambda \times E)|I(\theta_0)$ of functions with $0 \leqq \rho \leqq r$ satisfying the conditions:

i) For each $0 \leqq \rho \leqq r-1$, we have $L_{\rho\lambda}^{\rho\varepsilon} \cap L_{\rho+1\lambda}^{\rho+1\varepsilon} \neq \emptyset$.

ii) For $\rho = 0$, we have $({}_0\lambda, {}_0\varepsilon) = (\lambda, \varepsilon)$.

iii) For $\rho = r$, we have $L_0^{\theta_0} \cap L_{r\lambda}^{r\varepsilon} \neq \emptyset$.

Let A be a simplex of $'K[\mathscr{L}_0]$. For each index $(\lambda, \varepsilon)^A$ in $(\Lambda \times E)^A$, we define

$$N(\lambda, \varepsilon)^A = O(A) \cap L_\lambda^\varepsilon$$

which is collapsible.

Let A be a simplex of $'K[\mathscr{L}_0]$. Then we define $\mathscr{N}^A$ to be the indexed family being a function which assigns to each $(\lambda, \varepsilon)^A$ of $(\Lambda \times E)^A$ an open set $N(\lambda, \varepsilon)^A$.

Let $K(\mathscr{N}^A)$ denote the complex of the nerves of N^A.

LEMMA 3.1. *Let A be a simplex of $'K[\mathscr{L}_0]$. Then B is a q-simplex of $K(\mathscr{N}^A)$ with the set of vertices $B_0, \cdots, B_q$ if and only if there exists a sequence of indices $(\lambda_0, \varepsilon_0)^A, \cdots, (\lambda_q, \varepsilon_q)^A$ of $(\Lambda \times E)^A$ such that the corresponding intersection*

$$O(A)\cap L_{\lambda_0}^{\varepsilon_0}\cap\cdots\cap L_{\lambda_q}^{\varepsilon_q}$$

is nonvacuous and $(\lambda_i, \varepsilon_i)^A \neq (\lambda_j, \varepsilon_j)^A$ *if* $0\leqq i<j\leqq q$, *and the sequence of indices* $(\lambda_0, \varepsilon_0)^A, \cdots, (\lambda_q, \varepsilon_q)^A$ *is the sequence of vertices* $B_0, \cdots, B_q$.

Let A, A' be two simplices of $'K[\mathscr{L}_0]$.

If A' is a face of A, we can define a projection

$$\pi_{A'}^A\colon (\Lambda\times E)^A \longrightarrow (\Lambda\times E)^{A'}$$

as follows.

Let A be a q-simplex of $'K[\mathscr{L}_0]$ with vertices $A_0<\cdots<A_q$. Then we have a sequence $L_0^{\theta_0}<\cdots<L_0^{\theta_q}$ and $\hat{L}_0^{\theta_0}, \cdots, \hat{L}_0^{\theta_q}$ is $A_0, \cdots, A_q$. Let A' be a p-simplex of $'K[\mathscr{L}_0]$ with the vertices $A_0'<\cdots<A_p'$. As above we have a sequence $L_0^{\theta_0'}<\cdots<L_0^{\theta_p'}$ and $\hat{L}_0^{\theta_0'}, \cdots, \hat{L}_0^{\theta_p'}$ is $A_0', \cdots, A_p'$.

A' is a face of A if and only if there is a monotone increasing sequence of integers $0\leqq i_0<\cdots<i_p\leqq q$ such that $A_{i_0}=A_0', \cdots, A_{i_p}=A_p'$ and hence $\theta_{i_0}=\theta_0', \cdots, \theta_{i_p}=\theta_p'$.

Thus $A'\leqq A$ implies $O(A)\subseteqq O(A')$ and $I(\theta_0)\supseteqq I(\theta_{i_0})=I(\theta_0')$ implies $L_0^{\theta_0}\leqq L_0^{\theta_{i_0}}=L_0^{\theta_0'}$.

Let $(\lambda, \varepsilon)^A$ be an element of $(\Lambda\times E)^A$. Then the image $\pi_{A'}^A(\lambda, \varepsilon)^A=(\lambda', \varepsilon')^{A'}$ is defined by the formula $(\lambda', \varepsilon') = (\lambda, \varepsilon)|I(\theta_0')$.

If (λ, ε) is an element of $(\Lambda\times E)|I(\theta_0)$ satisfying $\mathrm{P}(\theta_0)$, then (λ', ε') gives rise to an element of $(\Lambda\times E)|I(\theta_0')$ satisfying $\mathrm{P}(\theta_0')$.

Thus the projection $\pi_{A'}^A$ is well-defined.

Further this map induces a simplicial map

$$\pi_{A'}^A\colon K(\mathscr{N}^A)\longrightarrow K(\mathscr{N}^{A'})\,.$$

For above remarks also imply that $N(\lambda, \varepsilon)^A\subseteqq N(\lambda', \varepsilon')^{A'}$ whenever $\pi_{A'}^A(\lambda, \varepsilon)=(\lambda', \varepsilon')^{A'}$.

We consider a function assigning to each injection $\iota_{A'}^A\colon A'\to A$ the projection $\pi_{A'}^A\colon K(\mathscr{N}^A)\to K(\mathscr{N}^{A'})$. Then we can easily show that the function defines a contravariant functor.

For later purpose, we consider the special case when A is a vertex $\hat{L}_0^\theta$.

Then we have

$$(\Lambda\times E)^A = \{\,(\lambda, \varepsilon)^A\mid (\lambda, \varepsilon)\in(\Lambda\times E)|I(\theta), \mathrm{P}(\theta)\,\}\,.$$

In this case, $\mathscr{N}^A$ is likewise defined.

Now we define

$$(\Lambda\times E)'^{K^0} = \amalg\{\,(\Lambda\times E)^A\mid A\in {}'K[L_0]^0\}\,.$$

We denote by $'\mathscr{N}$ the indexed family being a function which assigns to each $(\lambda, \varepsilon)^A$ of $(\Lambda\times E)'^{K^0}$ an open set $N(\lambda, \varepsilon)^A$.

Since $'\mathscr{N}$ is a subfamily of $\mathscr{N}$, we can easily deduce the following.

PROPOSITION 3.2. *$'\mathscr{N}$ is an open covering of $F^n\backslash\tilde{Y}(\mathscr{W})$ whose each element is collapsible.*

Let $K('\mathscr{N})$ denote the complex of the nerves of $'\mathscr{N}$.

LEMMA 3.3. *Let C be a q-simplex of $K('\mathscr{N})$ with the set of vertices $C_0, \cdots, C_q$ if and only if the following conditions are satisfied.*

i) *There exists a simplex A of $'K[\mathscr{L}_0]$ with the sequence of vertices $A_0 \leqq \cdots \leqq A_q$: that is, there exists a sequence $L_0^{\varrho_0} \leqq \cdots \leqq L_0^{\varrho_q}$ and the corresponding sequence of barycentres $\hat{L}_0^{\varrho_0}, \cdots, \hat{L}_0^{\varrho_q}$ is the sequence of vertices $A_0, \cdots, A_q$.*

ii) *There exists a permutation ω and a sequence of indices $(\lambda_0, \varepsilon_0)^{A_0}, \cdots, (\lambda_q, \varepsilon_q)^{A_q}$ belonging to $(\Lambda \times E)^{A_0}, \cdots, (\Lambda \times E)^{A_q}$ respectively such that the corresponding intersection*

$$N(\lambda_0, \varepsilon_0)^{A_0} \cap \cdots \cap N(\lambda_q, \varepsilon_q)^{A_q} = O(A) \cap L_{\lambda_0}^{\varepsilon_0} \cap \cdots \cap L_{\lambda_q}^{\varepsilon_q}$$

is nonvacuous and $(\lambda_i, \varepsilon_i)^{A_i} \neq (\lambda_j, \varepsilon_j)^{A_j}$ if $0 \leqq i < j \leqq q$, and the sequence of indices $(\lambda_0, \varepsilon_0)^{A_0}, \cdots, (\lambda_q, \varepsilon_q)^{A_q}$ is the sequence of vertices $C_{\omega(0)}, \cdots, C_{\omega(q)}$.

Proof. We only need to remark that $O(A_0) \cap \cdots \cap O(A_q) = O(A)$.

Each index of $(\Lambda \times E)'^{K^0}$ has the form $(\lambda, \varepsilon)^A$ with a vertex A of $'K[\mathscr{L}_0]$. We define the projection $'\pi$: $(\Lambda \times E)'^{K^0} \to 'K[\mathscr{L}_0]$ by the formula $'\pi(\lambda, \varepsilon)^A = A$.

Above arguments show that the map $'\pi$ defines a simplicial map

$$'\pi\colon K('\mathscr{N}) \longrightarrow 'K[\mathscr{L}_0] .$$

Let us denote by $|K('\mathscr{N})|$ the geometric realization of $K('\mathscr{N})$.

THEOREM 3.4. *The canonical map defines a simple homotopy equivaluence*

$$F^n \backslash \tilde{Y}(\mathscr{W}) \underset{s}{\simeq} |K('\mathscr{N})| .$$

§ 4. Structure of $K('\mathscr{N})$

The derived complex $'K[\mathscr{L}_0]$ is an ordered simplicial complex, hence the routine procedure provides us with the s.s. closure $'\bar{K}[\mathscr{L}_0]$ of $'K[\mathscr{L}_0]$.

A *q-simplex* of $'\bar{K}[\mathscr{L}_0]$ is a monotone nondecreasing sequence $\alpha = (\alpha_0, \cdots, \alpha_q)$ of vertices included among the vertices of some simplex of $'K[\mathscr{L}_0]$: that is, there exists a monotone nondecreasing sequence $L_0^{\varrho_0} \leqq \cdots \leqq L_0^{\varrho_q}$ such that the corresponding sequence $\hat{L}_0^{\varrho_0}, \cdots, \hat{L}_0^{\varrho_q}$ is the sequence $\alpha_0, \cdots, \alpha_q$. Let $'\bar{K}[\mathscr{L}_0]_q$ be the set of all the q-simplices. Then $'K[\mathscr{L}_0] = \coprod {}'\bar{K}[\mathscr{L}_0]_q$ is an s.s. complex under the face and degeneracy operators defined by

$$\partial_i(\alpha_0, \cdots, \alpha_q) = (\alpha_0, \cdots, \hat{\alpha}_i, \cdots, \alpha_q) ,$$
$$s_i(\alpha_0, \cdots, \alpha_q) = (\alpha_0, \cdots, \alpha_i, \alpha_i, \cdots, \alpha_q) .$$

Let $\alpha = (\alpha_0, \cdots, \alpha_q)$ be a simplex of $'\bar{K}[\mathscr{L}_0]$. We define its *carrier* $\{\alpha\}$ to be a simplex $\{\alpha_0, \cdots, \alpha_q\}$ of $'K[\mathscr{L}_0]$.

For each vertex A of $'K[\mathscr{L}_0]$, we have constructed a simplicial complex $K(\mathscr{N}^A)$. The s.s. closure $\bar{K}(\mathscr{N}^A)$ of this complex is defined as follows.

A *q-simplex* of $\bar{K}(\mathscr{N}^A)$ is a sequence $\beta = (\beta_0, \cdots, \beta_q)$ of vertices spanning some simplex of $K(\mathscr{N}^A)$: that is, there is a sequence of indices $(\lambda_0, \varepsilon_0)^A, \cdots, (\lambda_q, \varepsilon_q)^A$ of $(\Lambda \times E)^A$ such that $O(A) \cap L_{\lambda_0}^{\varepsilon_0} \cap \cdots \cap L_{\lambda_q}^{\varepsilon_q} \neq \emptyset$ and the sequence $(\lambda_0, \varepsilon_0)^A, \cdots, (\lambda_q, \varepsilon_q)^A$

gives rise to the sequence $\beta_0, \cdots, \beta_q$. The set of all the q-simplices $\bar{K}(\mathscr{N}^A)_q$ with the formulae

$$\partial_i(\beta_0, \cdots, \beta_q) = (\beta_0, \cdots, \hat{\beta}_i, \cdots, \beta_q)$$
$$s_i(\beta_0, \cdots, \beta_q) = (\beta_0, \cdots, \beta_i, \beta_i, \cdots, \beta_q)$$

defines $\bar{K}(\mathscr{N}^A)$.

Given $\beta=(\beta_0, \cdots, \beta_q)$, its carrier $\{\beta\}$ is the simplex $\{\beta_0, \cdots, \beta_q\}$.

Let A', A be two simplices of $'K[\mathscr{L}_0]$.

If A' is a face of A, then we can define the projection

$$\bar{\pi}^A_{A'}\colon\ \bar{K}(\mathscr{N}^A) \longleftarrow \bar{K}(\mathscr{N}^{A'})$$

after the manner in § 3.

The above procedure works well when we take up the complex $K('\mathscr{N})$. The existence of a simplical map $'\pi\colon K('\mathscr{N}) \to K[\mathscr{L}_0]$ defined in § 3 suggests to us the following definition.

A *q-simplex* of $\bar{K}'(\mathscr{N})$ is a sequence $\gamma=(\gamma_0, \cdots, \gamma_q)$ of vertices spanning some simplex of $K('\mathscr{N})$ satisfying the following conditions.

i) There exists a q-simplex of $'\bar{K}[\mathscr{L}_0]$ say $\alpha=(\alpha_0, \cdots, \alpha_q)$: that is, there exists a monotone nondecreasing sequence $L_0^{\theta_0} \leqq \cdots \leqq L_0^{\theta_q}$ such that the corresponding sequence $\hat{L}_0^{\theta_0}, \cdots, \hat{L}_0^{\theta_q}$ is the sequence $\alpha_0, \cdots, \alpha_q$. We write here $\{\alpha\}=A$, $\alpha_0=A_0, \cdots, \alpha_q=A_q$.

ii) There exists a sequence $(\lambda_0, \varepsilon_0)^{A_0}, \cdots, (\lambda_q, \varepsilon_q)^{A_q}$ of indices belonging to $(\Lambda\times E)^{A_0}, \cdots, (\Lambda\times E)^{A_q}$ respectively such that $O(A) \cap L^{\varepsilon_0}_{\lambda_0} \cap \cdots \cap L^{\varepsilon_q}_{\lambda_q} \neq \emptyset$ and the sequence $(\lambda_0, \varepsilon_0)^{A_0}, \cdots, (\lambda_q, \varepsilon_q)^{A_q}$ is a sequence $\gamma_0, \cdots, \gamma_q$. Again the set of q-simplices $\bar{K}('\mathscr{N})_q$ with the formulae

$$\partial_i(\gamma_0, \cdots, \gamma_q)=(\gamma_0, \cdots, \hat{\gamma}_i, \cdots, \gamma_q)$$
$$s_i(\gamma_0, \cdots, \gamma_q)=(\gamma_0, \cdots, \gamma_i, \gamma_i, \cdots, \gamma_q)$$

defines $\bar{K}('\mathscr{N})$.

For $\gamma=(\gamma_0, \cdots, \gamma_q)$, its carrier $\{\gamma\}$ is as before.

Here we remark that we have a natural projection

$$'\bar{\pi}\colon\ \bar{K}('\mathscr{N}) \longrightarrow {}'\bar{K}[\mathscr{L}_0]\ .$$

Under these conventions, we define a fibre product

$$\tilde{K}={}'\bar{K}[\mathscr{L}_0] \underset{'K[\mathscr{L}_0]}{\times} \amalg\{\bar{K}(\mathscr{N}^A) \mid A \in {}'K[\mathscr{L}_0]\}$$

over $'K[\mathscr{L}_0]$.

A *q-simplex* of $\tilde{K}$ is a pair $\alpha\times\beta$ with α being a q-simplex of $\bar{K}[\mathscr{L}_0]$ and β being a q-simplex of $\bar{K}(\mathscr{N}^A)$ where $A=\{\alpha\}$. Now the set of all the q-simplices $\tilde{K}_q$ with the face and degeneracy operators

$$\partial_i(\alpha\times\beta) = \partial_i\alpha\times\pi^{\{\alpha\}}_{\{\partial_i\alpha\}}\partial_i\beta$$
$$s_i(\alpha\times\beta) = s_i\alpha\times s_i\beta$$

forms an s.s. complex $\tilde{K}$.

Next we set up a relation on $\tilde{K}$. Let $\alpha\times\beta$ and $\alpha'\times\beta'$ be two q-simplices of $\tilde{K}$. Then $\alpha\times\beta\sim\alpha'\times\beta'$ if and only if $\alpha=\alpha'$, $\pi^{\{\alpha\}}_{\{(\partial_0)^i\alpha\}}\beta_i=\pi^{\{\alpha'\}}_{\{(\partial_0)^i\alpha'\}}\beta'_i$ for each i satisfying $0\leqq i\leqq q$. Clearly this relation is reflexive, symmetric and transitive.

Now the set of all equivalence classes is denoted by $K[\mathscr{W}]$. As easily checked $\alpha\times\beta\sim\alpha'\times\beta'$ implies $\partial_i(\alpha\times\beta)\sim\partial_i(\alpha'\times\beta')$, $s_i(\alpha\times\beta)\sim s_i(\alpha'\times\beta')$. Hence on $K[\mathscr{W}]$ the face and degeneracy operators are defined and $K[\mathscr{W}]$ satisfies the axiom of an s.s. complex.

Here we note that the natural projection π: $\tilde{K}\to K[\mathscr{W}]$ transforming $\alpha\times\beta$ to its equivalence class is an s.s. map.

THEOREM 4.1. *We have an s.s. isomorphism*

$$K[\mathscr{W}]\cong\bar{K}('\mathscr{N})\,.$$

Proof. First we construct an s.s. map

$$\tilde{\xi}\colon\ \tilde{K}\longrightarrow\bar{K}('\mathscr{N})\,.$$

Let $\alpha\times\beta$ be a q-simplex of $\tilde{K}$. We remark first that $\alpha=(\alpha_0,\cdots,\alpha_q)$ is an element of $'\bar{K}[\mathscr{L}_0]$: that is, we have $L_0^{\theta_0}\leqq\cdots\leqq L_0^{\theta_q}$ and $\alpha_i=\hat{L}_0^{\theta_i}$ for each i. Here we put $\{\alpha\}=A$, $\alpha_i=A_i$.

Next we consider $\beta=(\beta_0,\cdots,\beta_q)$ in $\bar{K}(\mathscr{N}^A)$. It is equivalent to say that we have $(\lambda_0,\varepsilon_0)^A,\cdots,(\lambda_q,\varepsilon_q)^A$ of $(\Lambda\times E)^A$ such that $O(A)\cap L_{\lambda_0}^{\varepsilon_0}\cap\cdots\cap L_{\lambda_q}^{\varepsilon_q}\neq\emptyset$ and $\beta_i=(\lambda_i,\varepsilon_i)^A$ for each i.

Above data allow us to define a q-simplex $\gamma=(\gamma_0,\cdots,\gamma_q)$ of $\bar{K}('\mathscr{N})$ as follows.

For each i satisfying $0\leqq i\leqq q$, we define $(\lambda'_i,\varepsilon'_i)^{A_i}$ in $(\Lambda\times E)^{A_i}$ to be $\pi^A_{A_i}(\lambda_i,\varepsilon_i)^A=\pi^{\{\alpha\}}_{\alpha_i}(\lambda_i,\varepsilon_i)^{\{\alpha\}}$, so we have $(\lambda'_i,\varepsilon'_i)=(\lambda_i,\varepsilon_i)|I(\theta_i)$. Above identities show $O(A_i)\supseteqq O(A)$ and that $L_{\lambda'_i}^{\varepsilon'_i}\supseteqq L_{\lambda_i}^{\varepsilon_i}$, hence we have $O(A_i)\cap L_{\lambda'_0}^{\varepsilon'_0}\cap\cdots\cap L_{\lambda'_q}^{\varepsilon'_q}\neq\emptyset$.

We put $\gamma_i=(\lambda'_i,\varepsilon'_i)^{A_i}$ for each i.

We set $\tilde{\xi}(\alpha\times\beta)=\gamma$.

Since we have $\pi^{\{\alpha\}}_{\alpha_i}=\pi^{\{(\partial_0)^i\alpha\}}_{\alpha_i}\cdot\pi^{\{\alpha\}}_{\{(\partial_0)^i\alpha\}}$, $\gamma_i=\pi^{\{\alpha\}}_{\alpha_i}\beta_i$ is shown to depend only on $\pi^{\{\alpha\}}_{\{(\partial_0)^i\alpha\}}\beta_i$. Thus $\tilde{\xi}$ induces a map ξ: $K[\mathscr{W}]\to\bar{K}('\mathscr{N})$ satisfying $\xi\circ\pi=\tilde{\xi}$.

Next we define an s.s. map

$$\tilde{\eta}\colon\ \bar{K}('\mathscr{N})\longrightarrow\tilde{K}\,.$$

Let $\gamma=(\gamma_0,\cdots,\gamma_q)$ be a q-simplex of $\bar{K}('\mathscr{N})$. γ is characterized by the next two conditions.

i) We have a q-simplex $\alpha=(\alpha_0,\cdots,\alpha_q)$: that is, we have $L_0^{\theta_0}\leqq\cdots\leqq L_0^{\theta_q}$ and $\alpha_i=\hat{L}_0^{\theta_i}$ for each i. We put $\{\alpha\}=A$, $\alpha_i=A_i$ for each i.

ii) We have $(\lambda_0,\varepsilon_0)^{A_0},\cdots,(\lambda_q,\varepsilon_q)^{A_q}$ belonging to $(\Lambda\times E)^{A_0},\cdots,(\Lambda\times E)^{A_q}$ respectively such that $O(A)\cap L_{\lambda_0}^{\varepsilon_0}\cap\cdots\cap L_{\lambda_q}^{\varepsilon_q}\neq\emptyset$ and $\gamma_i=(\lambda_i,\varepsilon_i)^{A_i}$ for each i.

Now we define a q-simplex $\beta=(\beta_0,\cdots,\beta_q)$ of $\bar{K}(\mathscr{N}^A)$ as follows.

For each i satisfying $0\leqq i\leqq q$, we define an index $(\hat{\lambda}_i,\hat{\varepsilon}_i)^A$ of $(\Lambda\times E)^A$ by the formulae

$$\begin{cases}(\hat{\lambda}_i,\hat{\varepsilon}_i)|I(\theta_i)=(\lambda_i,\varepsilon_i)\,,\\ (\hat{\lambda}_i,\hat{\varepsilon}_i)|(I(\theta_j)\backslash I(\theta_{j+1}))=(\lambda_j,\varepsilon_j)|(I(\theta_j)\backslash I(\theta_{j+1}))\quad 0\leqq j\leqq i-1\,.\end{cases}$$

From the definition follows $L^{\varepsilon_i}_{\lambda_i} \supseteqq L^{\hat{\varepsilon}_i}_{\hat{\lambda}_i} \supseteqq L^{\varepsilon_0}_{\lambda_0} \cap \cdots \cap L^{\varepsilon_i}_{\lambda_i}$, so if $0 \leqq j \leqq i$ $L^{\varepsilon_j}_{\lambda_j} \cap L^{\varepsilon_i}_{\lambda_i} \supseteqq L^{\hat{\varepsilon}_j}_{\hat{\lambda}_j} \cap L^{\hat{\varepsilon}_i}_{\hat{\lambda}_i} \supseteqq L^{\varepsilon_0}_{\lambda_0} \cap \cdots \cap L^{\varepsilon_i}_{\lambda_i}$. Thus for each i satisfying $0 \leqq i \leqq q$ we obtain $L^{\hat{\varepsilon}_0}_{\hat{\lambda}_0} \cap \cdots \cap L^{\hat{\varepsilon}_i}_{\hat{\lambda}_i} = L^{\varepsilon_0}_{\lambda_0} \cap \cdots \cap L^{\varepsilon_i}_{\lambda_i}$.

Since $(\lambda_0, \varepsilon_0)^A$ is an element of $(\Lambda \times E)^A$, there is a chain of indices $({}_\rho\lambda_0, {}_\rho\varepsilon_0)$ in $(\Lambda \times E)|I(\theta_0)$ with $0 \leqq \rho \leqq r$ satisfying the coditions $\mathrm{P}(\theta_0)$ as follows:

i) $L^{{}_\rho\varepsilon_0}_{{}_\rho\lambda_0} \cap L^{{}_{\rho+1}\varepsilon_0}_{{}_{\rho+1}\lambda_0} \neq \emptyset \qquad 0 \leqq \rho \leqq r-1$.

ii) $({}_0\lambda_0, {}_0\varepsilon_0) = (\lambda_0, \varepsilon_0)$.

iii) $L^{\theta_0}_0 \cap L^{{}_r\varepsilon_0}_{{}_r\lambda_0} \neq \emptyset$.

For each i satisfying $0 \leqq i \leqq q$ we define a chain of indices $({}_\rho\hat{\lambda}_i, {}_\rho\hat{\varepsilon}_i)$ in $(\Lambda \times E)|I(\theta_0)$ with $0 \leqq \rho \leqq r+1$ by

$$({}_0\hat{\lambda}_i, {}_0\hat{\varepsilon}_i) = (\hat{\lambda}_i, \hat{\varepsilon}_i)$$
$$({}_\rho\hat{\lambda}_i, {}_\rho\hat{\varepsilon}_i) = ({}_{\rho-1}\lambda_0, {}_{\rho-1}\varepsilon_i) \quad 1 \leqq \rho \leqq r-1 .$$

Then $L^{\hat{\varepsilon}_i}_{\hat{\lambda}_i} \cap L^{\varepsilon_0}_{\lambda_0} \supseteqq L^{\varepsilon_0}_{\lambda_0} \cap \cdots \cap L^{\varepsilon_q}_{\lambda_q}$ implies

i) $L^{{}_\rho\hat{\varepsilon}_i}_{{}_\rho\hat{\lambda}_i} \cap L^{{}_{\rho+1}\hat{\varepsilon}_i}_{{}_{\rho+1}\hat{\lambda}_i} \neq \emptyset \qquad 0 \leqq \rho \leqq r$,

ii) $({}_0\hat{\lambda}_i, {}_0\hat{\varepsilon}_i) = (\hat{\lambda}_i, \hat{\varepsilon}_i)$,

iii) $L^{\theta_0}_0 \cap L^{{}_{r+1}\hat{\varepsilon}_0}_{{}_{r+1}\hat{\lambda}_0} \neq \emptyset$.

Thus there is defined a sequence of indices $(\hat{\lambda}_i, \hat{\varepsilon}_i)^A$ in $(\Lambda \times E)^A$ such that $O(A) \cap L^{\hat{\varepsilon}_0}_{\hat{\lambda}_0} \cap \cdots \cap L^{\hat{\varepsilon}_q}_{\hat{\lambda}_q} \neq \emptyset$.

Now we define β by putting $\beta_i = (\hat{\lambda}_i, \hat{\varepsilon}_i)^A$ for each i.

Here we set $\tilde{\eta}(\gamma) = \alpha \times \beta$, and define $\eta = \pi \circ \tilde{\eta}$.

The composition $\eta \circ \xi$ is shown to be equal to the identity as follows.

Let $\alpha \times \beta$ be a q-simplex of $\tilde{K}$. We assume $\beta_i = (\lambda_i, \varepsilon_i)^A$. We define $(\lambda'_i, \varepsilon'_i)^A$ to be $\pi^A_{A_i}(\lambda_i, \varepsilon_i)^A$: that is, $(\lambda'_i, \varepsilon'_i) = (\lambda_i, \varepsilon_i)|I(\theta_i)$. We put $\gamma_i = (\lambda'_i, \varepsilon'_i)^{A_i}$.

Now we have $\tilde{\xi}(\alpha \times \beta) = \gamma$

Here we define $(\hat{\lambda}'_i, \hat{\varepsilon}'_i)^A$ by

$$\begin{cases} (\hat{\lambda}'_i, \hat{\varepsilon}'_i)|I(\theta_i) = (\lambda'_i, \varepsilon'_i) , \\ (\hat{\lambda}'_i, \hat{\varepsilon}'_i)|(I(\theta_j) \backslash I(\theta_{j-1})) = (\lambda'_j, \varepsilon'_j)|(I(\theta_j) \backslash I(\theta_{j+1})) \quad 0 \leqq j \leqq i-1 . \end{cases}$$

We put $\beta'_i = (\hat{\lambda}'_i, \hat{\varepsilon}'_i)^A$ and $\beta' = (\beta'_0, \cdots, \beta'_q)$.

We have $\tilde{\gamma}(\gamma) = \alpha \times \beta'$.

From the construction we have $(\hat{\lambda}'_i, \hat{\varepsilon}'_i)|I(\theta_i) = (\lambda'_i, \varepsilon'_i) = (\lambda_i, \varepsilon_i)|I(\theta_i)$. Therefore we obtain $\pi^{\{\alpha\}}_{\{(\partial_0)^i\alpha\}}(\hat{\lambda}'_i, \hat{\varepsilon}'_i)^{\{\alpha\}} = \pi^{\{\alpha\}}_{\{(\partial_0)^i\alpha\}}(\lambda_i, \varepsilon_i)^{\{\alpha\}}$.

This proves $\alpha \times \beta \sim \alpha \times \beta'$ and $\tilde{\eta}\tilde{\xi}(\alpha \times \beta) \sim \alpha \times \beta$.

In the same way we can also conclude that $\tilde{\xi} \circ \tilde{\eta}$ is the identity.

§ 5. Configuration of hyperplanes in $\boldsymbol{C}^n$

The family of complexified Weyl walls is the most important example of configuration in $\boldsymbol{C}^n$. In this section we specialize down to the case, hyperplane configuration.

Let $\mathscr{W} = \{ W(i) \mid i \in I \}$ denote the family of hyperplanes in $\boldsymbol{C}^n$.

We assume that each member $W(i)$ has a reference point $a(i)$ and a normal vector $f(i)$. Now we define a system of normal vectors $R(i)$ to be the sequence $\{r_0(i), r_1(i)\}$ given by $r_0(i) = f(i)$, $r_1(i) = f(i) \cdot \sqrt{-1}$.

For each θ in $\boldsymbol{F}_3$, we put

$$L_0^\theta(i)=\{x\in \boldsymbol{C}^n \mid \operatorname{sign}\operatorname{Re}(f(i),x-a(i))_C=\theta\}$$

$$\begin{aligned}L_1^\theta(i)&=\{x\in \boldsymbol{C}^n \mid \operatorname{sign}\operatorname{Re}(f(i)\sqrt{-1},x-a(i))_C=\theta\}\\ &=\{x\in \boldsymbol{C}^n \mid \operatorname{sign}\operatorname{Im}(f(i),x-a(i))_C=\theta\}\,.\end{aligned}$$

Further we abbreviate as $L_0(i)=L_0^0(i)$, $L_+(i)=L_1(i)=L_1^0(i)$.
We write $\mathscr{L}_0=\{L_0(i)\mid i\in I\}$, $\mathscr{L}_+=\mathscr{L}_1=\{L_1(i)\mid i\in I\}$.
We assume the next hypothesis.

H_0) $\mathscr{L}_0$ is locally finite.

$\mathscr{L}_0$ defines a cell decomposition of $\boldsymbol{C}^n$ which is denoted by $K[\mathscr{L}_0]$.
Let Θ be the set of all the functions θ: $I\to \boldsymbol{F}_3$.
For each element θ of Θ, we define

$$\begin{aligned}L_0^\theta&=\cap\{L_0^{\theta(i)}(i)\mid i\in I\}\\ &=\{x\in \boldsymbol{C}^n \mid {}^\forall i\in I \ \operatorname{sign}\operatorname{Re}(f(i),x-a(i))_C=\theta(i)\}\,.\end{aligned}$$

Trivially we have $K[\mathscr{L}_0]=\{L_0^\theta\mid \theta\in\Theta, L_0^\theta\neq\emptyset\}$.
As before we write $I(\theta)=\theta^{-1}(0)$.
For each element θ of Θ, we define

$$O(\theta)=\{x\in \boldsymbol{C}^n\mid \mathrm{P}O(\theta)\}\,.$$

where $\mathrm{P}O(\theta)$ denotes the set of the properties as follows:

i) For each pair of i in $I(\theta)$ and j in $I\backslash I(\theta)$, we have

$$|\operatorname{Re}(f(i),x-a(i))_C|<|\operatorname{Re}(f(j),x-a(j))_C|\,.$$

ii) For each j in $I\backslash I(\theta)$ we have

$$\operatorname{sign}\operatorname{Re}(f(j),x-a(j))_C=\theta(j)\,.$$

Clearly $O(\theta)$ is collapsible.
Now we assume the second hypothesis.

H_1) $K[\mathscr{L}_0]$ is closure finite.

Let $\hat{K}[\mathscr{L}_0]$ be the set of barycentres $\hat{L}_0^\theta$ taken once for each cell L_0^θ of $K[\mathscr{L}_0]$.

$\mathscr{O}$ is defined to be the indexed family which assigns to each $\hat{L}_0^\theta$ of $\hat{K}[\mathscr{L}_0]$ an open set $O(\theta)$.

The derived complex $'K[\mathscr{L}_0]$ of $K[\mathscr{L}_0]$ is by definition the complex of the neves $K(\mathscr{O})$ of $\mathscr{O}$.

LEMMA 5.1. *A is a q-simplex of $'K[\mathscr{L}_0]$ with the set of vertices $A_0<\cdots<A_q$ if and only if there exists a sequence of cells $L_0^{\theta_0}\prec\cdots\prec L_0^{\theta_q}$ of $K[\mathscr{L}_0]$ and the corresponding sequence of barycentres $\hat{L}_0^{\theta_0},\cdots,\hat{L}_0^{\theta_q}$ is the sequence of vertices $A_0,\cdots,A_q$.*

As before we write as $\tilde{Y}(\mathscr{W}) = \cup\{W(i) \mid i \in I\}$.

Let J be a subset of I. In our case, the set

$$(\Lambda\times E)|J = \{(\lambda, \varepsilon)\colon J \longrightarrow \boldsymbol{N}\times \boldsymbol{F}_3^\times \mid \forall i \in J \quad 1 \leqq \lambda(i) \leqq 1\}$$

is bijective to the one

$$E|J = \{\varepsilon\colon J \longrightarrow \boldsymbol{F}_3^\times\}\,.$$

For each ε in $E|J$, we define

$$\begin{aligned} L_1^\varepsilon &= \cap\{L_1^{\varepsilon(i)}(i) \mid i \in J\} \\ &= \{x \in \boldsymbol{C}^n \mid \forall i \in J \ \operatorname{sign} \operatorname{Im}(f(i), x-a(i))_C = \varepsilon(i)\}\,. \end{aligned}$$

This set is convex and collapsible.

For each θ in Θ, we define

$$E^\theta = \{\varepsilon^\theta \mid \varepsilon \in E|I(\theta),\, L_0^\theta \cap L_1^\varepsilon \neq \emptyset\}\,.$$

For each index ε^θ in E^θ, we define

$$N(\varepsilon^\theta) = O(\theta) \cap L_1^\varepsilon\,.$$

The is convex and collapsible.

We define

$$E^K = \amalg\{E^\theta \mid \theta \in \Theta,\, L_0^\theta \in K[\mathscr{L}_0]\}\,.$$

We define $\mathscr{N}$ to be the indexed family which assigns to each ε^θ of E^K an open set $N(\varepsilon^\theta)$.

PROPOSITION 5.2. *$\mathscr{N}$ is an open covering of $\boldsymbol{C}^n \backslash \tilde{Y}(\mathscr{W})$ whose each member is collapsible.*

Let us denote by $K(\mathscr{N})$ the complex of the nerves $\mathscr{N}$ and $|K(\mathscr{N})|$ its geometric realization.

THEOREM 5.3. *The canonical map induces a simple homotopy equivalence*

$$\boldsymbol{C}^n \backslash \tilde{Y}(\mathscr{W}) \underset{s}{\simeq} |K(\mathscr{N})|\,.$$

Let A be a q-simplex of $'K[\mathscr{L}_0]$ with vertices $A_0, \cdots, A_q$. We assume that we have $L_0^{\theta_0} < \cdots < L_0^{\theta_q}$ and $\hat{L}_0^{\theta_q}, \cdots, \hat{L}_0^{\theta_0}$ is $A_0, \cdots, A_q$. We remark here that we have $I(\theta_0) \supset \cdots \supset I(\theta_q)$.

Now we put

$$O(A) = \cap\{O(\theta_i) \mid 0 \leqq i \leqq q\}\,.$$

Further we define

$$E^A = \{\varepsilon^A \mid \varepsilon \in E|I(\theta_0),\ \mathrm{P}(\theta_0)\}$$

where $\mathrm{P}(\theta_0)$ stands for the set of properties as follows:

0) There is a chain ${}_\rho\varepsilon$ in $E|I(\theta_0)$ of functions with $0 \leqq \rho \leqq r$ satisfying the conditions:

i) For each $0 \leqq \rho \leqq r-1$, we have $L_1^{{}_\rho\varepsilon} \cap L_1^{{}_{\rho+1}\varepsilon} \neq \emptyset$.

ii) For $\rho=0$, ${}_0\varepsilon=\varepsilon$.

iii) For $\rho=r$, $L_0^\theta \cap L_1^{{}_r\varepsilon} \neq \emptyset$.

In the case where we are now concerned about, we can restate these properties in much simpler form. To interpret the circumstances, we need to prove some lemmas.

LEMMA 5.4. *Let J be a subset of I, ε an index in $E|J$. Further let J' be a subset of J, ε' an index in $E|J'$. Let $L_1^\varepsilon \neq \emptyset$. Then the following conditions are equivalent*:

i) $L_1^\varepsilon \cap L_1^{\varepsilon'} \neq \emptyset$;

ii) $\varepsilon'=\varepsilon|J'$;

iii) $L_1^\varepsilon \subseteqq L_1^{\varepsilon'}$.

Proof. Let $J \subseteqq I$ and $\varepsilon \in E|J$. If $J' \subseteqq J$, we denote by $\varepsilon|J'$ the restriction of ε on J'. Since $L_1^\varepsilon = \cap\{L_1^{\varepsilon(i)}(i) \mid i \in J\}$, it is easily seen that $L_1^\varepsilon = \cap\{L_1^{\varepsilon|J'} | J' \subseteqq J, {}^\sharp J'=1\}$.

Let $J \subseteqq I$, ${}^\sharp J=1$ and $\varepsilon, \varepsilon' \in E|J$. Then the following conditions are equivalent: i) $L_1^\varepsilon \cap L_1^{\varepsilon'} \neq \emptyset$; ii) $\varepsilon'=\varepsilon$; iii) $L_1^\varepsilon = L_1^{\varepsilon'}$.

Let $J \subseteqq I$, $\varepsilon \in E|J$ and let $J' \subseteqq J$, $\varepsilon' \in E|J'$. Then we have

$$\begin{aligned} L_1^\varepsilon \cap L_1^{\varepsilon'} = & \cap\{L_1^{(\varepsilon|J'')} \cap L_1^{(\varepsilon'|J'')} \mid J'' \subseteqq J', {}^\sharp J''=1\} \\ & \cap(\cap\{L_1^{(\varepsilon|J'')} \mid J'' \subseteqq J\backslash J', {}^\sharp J''=1\}. \end{aligned}$$

Let $J \subseteqq I$, $\varepsilon \in E|J$ and let $J' \subseteqq J$, $\varepsilon' \in E|J'$. Then $L_1^\varepsilon \cap L_1^{\varepsilon'} \neq \emptyset$ implies $L_1^{(\varepsilon|J'')} \cap L_1^{(\varepsilon'|J'')} \neq \emptyset$ for each $J'' \subseteqq J'$ with ${}^\sharp J''=1$ and hence $\varepsilon|J''=\varepsilon'|J''$ for each $J'' \subseteqq J'$ with ${}^\sharp J''=1$, that is $\varepsilon|J'=\varepsilon'$. The last condition yields $L_1^\varepsilon \subseteqq L_1^{\varepsilon'}$ and therefore $L_1^\varepsilon \cap L_1^{\varepsilon'} = L_1^\varepsilon$. Here we assume $L_1^\varepsilon \neq \emptyset$. Then $L_1^\varepsilon \subseteqq L_1^{\varepsilon'}$ shows $L_1^\varepsilon \cap L_1^{\varepsilon'} \neq \emptyset$.

COROLLARY 5.5. *Let $L_0^\theta, L_0^{\theta'}$ be cells of $K[\mathscr{L}_0]$ such that L_0^θ is a face of $L_0^{\theta'}$, so $I(\theta)$ contains $I(\theta')$. Let ε be an index in $E|I(\theta)$, ε' the one in $E|I(\theta')$. Let $L_1^\varepsilon \neq \emptyset$. Then the following conditions are equivalent*:

i) $L_1^\varepsilon \cap L_1^{\varepsilon'} \neq \emptyset$;

ii) $\varepsilon'=\varepsilon|I(\theta')$;

iii) $L_1^\varepsilon \subseteqq L_1^{\varepsilon'}$.

COROLLARY 5.6. *Let $L_0^\theta, L_0^{\theta'}$ be cells of $K[\mathscr{L}_0]$ such that L_0^θ is a face of $L_0^{\theta'}$. Let ε be an index in $E|I(\theta)$, ε' the one in $E|I(\theta')$. We assume that $L_0^\theta \cap L_1^\varepsilon \neq \emptyset$ and $L_1^\varepsilon \cap L_1^{\varepsilon'} \neq \emptyset$. Then we have $L_0^{\theta'} \cap L_1^{\varepsilon'} \neq \emptyset$.*

COROLLARY 5.7. $\mathrm{P}(\theta_0)$ *is equivalant to say that $L_0^{\theta_0} \cap L_1^\varepsilon \neq \emptyset$.*

Proof. We assume that $\mathrm{P}(\theta_0)$ holds. Then we have $L_1^{\rho_\varepsilon}\cap L_1^{\rho+1\varepsilon}\neq\emptyset$ if $0\leqq\rho\leqq r-1$. This implies ${}_\rho\varepsilon={}_{\rho+1}\varepsilon$ whenever $0\leqq\rho\leqq r-1$. On the other hand we have $\varepsilon_0=\varepsilon$, $L_0^{\theta_0}\cap L_1^{r\varepsilon}\neq\emptyset$. Hence we can conclude $L_0^{\theta_0}\cap L_1^{\varepsilon}\neq\emptyset$.

The converse is almost trivial, by putting $r=0$.

LEMMA 5.8. *Let A be a simplex of $'K[\mathscr{L}_0]$ as above. Then the natural injection sending ε^{θ_0} to ε^A induces a bijection $E^{\theta_0}\to E^A$*

Let A be a simplex of $'K[\mathscr{L}_0]$.

For each index ε^A in E^A, we define a collapsible open set

$$N(\varepsilon^A)=O(A)\cap L_1^{\varepsilon}\ .$$

For each simplex A of $'K[\mathscr{L}_0]$, we define $\mathscr{N}^A$ to be the indexed family which assigns to each ε^A of E^A an open set $N(\varepsilon^A)$.

LEMMA 5.9. *Let A be a simplex of $'K[\mathscr{L}_0]$. Then B is a q-simplex of $K(\mathscr{N}^A)$ with the set of vertices $B_0,\cdots,B_q$ if and only if there exists a sequence of indices $(\varepsilon_0)^A,\cdots,(\varepsilon_q)^A$ of E^A such that the corresponding intersection $O(A)\cap L_1^{\varepsilon_0}\cap\cdots\cap L_1^{\varepsilon_q}$ is nonvacuous and $(\varepsilon_i)^A\neq(\varepsilon_j)^A$ if $0\leqq i<j\leqq q$, and the sequence of indices $(\varepsilon_0)^A,\cdots,(\varepsilon_q)^A$ is the sequence of vertices $B_0,\cdots,B_q$.*

Let A', A be two simplices of $'K[\mathscr{L}_0]$.

If A' is a face of A we can define a projection

$$\pi_{A'}^A:\ E^A\longrightarrow E^{A'}$$

is defined as in §3 to be the map obtained by restricting the domain of definition.

Let A be a simplex of $'K[\mathscr{L}_0]$ with vertices $A_0<\cdots<A_q$. We assume that there exists $L_0^{\theta_0}<\cdots<L_0^{\theta_q}$ and we have $A_i=\hat{L}_i^{\theta_i}$ for each i. Let A' be a simplex of $'K[\mathscr{L}_0]$ with vertices $A_0'<\cdots<A_p'$. We assume that there exists $L_0^{\theta'_0}<\cdots<L_0^{\theta'_p}$ and we have $A_i'=\hat{L}_0^{\theta'_i}$ for each i.

If A' is a face of A, there is a monotone increasing sequence of integers $0\leqq i_0<\cdots<i_p\leqq q$ such that $A_{i_h}=A'_h$ so $\theta_{i_h}=\theta'_h$ for each h.

Let ε^A be an element of E^A. Then the image $\pi_{A'}^A(\varepsilon^A)=\varepsilon'_{A'}$ is defined by the formula $\varepsilon'=\varepsilon|I(\theta'_0)$.

This map induces a simplicial map

$$\pi_{A'}^A:\ K(\mathscr{N}^A)\longrightarrow K(\mathscr{N}^{A'})\ .$$

Here we treat the case, A is a vertex.

COROLLARY 5.10. *Let A be a vertex $\hat{L}_0^{\theta}$. Then the natural injection sending ε^θ to ε^A induces a bijection $E^\theta\to E^A$. Moreover we can identify $N(\varepsilon^\theta)$ with $N(\varepsilon^A)$.*

We define

$$E'^{K_0} = \coprod \{ E^A \mid A \in 'K[\mathscr{L}_0]^0 \} .$$

We define $'\mathscr{N}$ to be the indexed family which assigns to each ε^A of E^A an open set $N(\varepsilon^A)$.

We denote by $K('\mathscr{N})$ the complex of the nerves of $'\mathscr{N}$.

PROPOSITION 5.11. *The natural injection sending ε^θ to ε^A induces a bijection $E^\theta \to E^A$. Moreover we can identity $\mathscr{N}$ with $'\mathscr{N}$ via this map and hence we have an isomorphism $K(\mathscr{N}) \to K('\mathscr{N})$.*

LEMMA 5.12. *C is a q-simplex of $K(\mathscr{N})$ with the set of vertices $C_0, \cdots, C_q$ if and only if the following conditions are satisfied:*

i) *There exists a simplex A' of $'K[\mathscr{L}_0]$ with vertices $A'_0 \cdots A'_q$ such that there exists a sequence of cells $L_0^{\theta'_0} \leqq \cdots \leqq L_0^{\theta'_q}$ and the corresponding sequence of barycentres $\hat{L}_0^{\theta'_0}, \cdots, \hat{L}_0^{\theta'_q}$ is the sequence of vertices $A'_0, \cdots, A'_q$.*

ii) *There exists a permutation ω and a sequence of indices $(\varepsilon'_0)^{A'_0}, \cdots, (\varepsilon'_q)^{A'_q}$ belonging to $E^{A'_0}, \cdots, E^{A'_q}$ respectively such that $\varepsilon'_i = \varepsilon'_0 | I(\theta'_i)$ if $0 \leqq i \leqq q$ and $(\varepsilon'_i)^{A'_i} \neq (\varepsilon'_j)^{A'_j}$ if $0 \leqq i < j \leqq q$, and the sequence of indices $(\varepsilon'_0)^{A'_0}, \cdots, (\varepsilon'_q)^{A'_q}$ is the sequence of vertices $C_{\omega(0)}, \cdots, C_{\omega(q)}$.*

Proof. Let ω be a fixed permutation of the set $[q] = \{ i \in \boldsymbol{Z} \mid 0 \leqq i \leqq q \}$. Then we put $A_{\omega(i)} = A'_i$, $\varepsilon_{\omega(i)} = \varepsilon'_i$.

First we note that the definition induces

$$N(\varepsilon'_0)^{A'_0} \cap \cdots \cap N(\varepsilon'_q)^{A'_q} = O(A'_0) \cap \cdots \cap O(A'_q) \cap L_1^{\varepsilon'_0} \cap \cdots \cap L_1^{\varepsilon'_q} .$$

We only need to prove that this set is nonvacuous if and only if $\varepsilon'_0 = \varepsilon'_0 | I(\theta'_i)$ for each i.

We assume that

$$O(A'_0) \cap \cdots \cap O(A'_q)$$

is nonvacuous. Without loss of generality, this assumption can be restated in the form: there exists a monotone nondecreasing sequence $L_0^{\theta'_0} \leqq \cdots \leqq L_0^{\theta'_q}$ such that $A'_i = L_0^{\theta'_i}$ for every i.

We remark here that this assumption implies the following two assertions.

The first is $I(\theta'_0) \supseteqq \cdots \supseteqq I(\theta'_q)$.

Secondly $L_0^{\theta'_0}$ is shown to be contained in the closure of $O(A'_0) \cap \cdots \cap O(A'_q)$ in $|'K[\mathscr{L}_0]|$.

Since ε'_i belongs to $E^{A'_i} \subseteqq E | I(\theta'_i)$ for each i and $I(\theta'_0) \supseteqq \cdots \supseteqq I(\theta'_q)$, Corollary 5.5 shows that the set

$$L_1^{\varepsilon'_0} \cap \cdots \cap L_1^{\varepsilon'_q}$$

is nonvacuous if and only if $\varepsilon'_i = \varepsilon'_0 | I(\theta'_i)$ for each i, or equivalently $L_1^{\varepsilon'_0} \subseteqq L_1^{\varepsilon'_i}$ for every i, and then $L_1^{\varepsilon'_0} \cap \cdots \cap L_1^{\varepsilon'_q} = L_1^{\varepsilon'_0}$.

On the other hand, $(\varepsilon_0)^{A'_0}$ belongs to $E^{\theta'_0} = E^{A'_0}$, so we have $L_0^{\theta'_0} \cap L_1^{\varepsilon'_0}$ is nonvacuous. Since $L_1^{\varepsilon'_0}$ is open in $|'K[\mathscr{L}_0]|$ and $L_0^{\theta'_0}$ is contained in the closure of $O(A'_0)\cap\cdots\cap O(A'_q)$, $O(A'_i)\cap\cdots\cap O(A'_q)\cap L_1^{\varepsilon'_0}$ is nonvacuous.

Thus the assumption mentioned in the beginning induces that $O(A'_0)\cap\cdots\cap O(A'_q)\cap L_1^{\varepsilon'_0}\cap\cdots\cap L_1^{\varepsilon'_q}$ is nonvacuous if and only if $\varepsilon'_i=\varepsilon'_0|I(\theta'_i)$ for each i or equivalently $L_1^{\varepsilon'_0}\subseteqq L_1^{\varepsilon'_i}$ for each i.

$\alpha=(\alpha_0,\cdots,\alpha_q)$ is a q-simplex of $\bar{K}[\mathscr{L}_0]$ if and only if there exists a sequence of cells $L_0^{\theta_0}\leqq\cdots\leqq L_0^{\theta_q}$ and we have $\alpha_i=\hat{L}_0^{\theta_i}$ for each i.

Let $'\bar{K}[\mathscr{L}_0]_q$ denote the set of all q-simplices. Then $'\bar{K}[\mathscr{L}_0]=\coprod{}'\bar{K}[\mathscr{L}_0]_q$ denotes an s.s. complex with obvious definitions of the face and degeneracy operators.

$\beta=(\beta_0,\cdots,\beta_q)$ is a q-simplex of $\bar{K}(\mathscr{N}^A)$ if and only if there exists a sequence of indices $(\varepsilon_0)^A,\cdots,(\varepsilon_q)^A$ such that $O(A)\cap L_1^{\varepsilon_0}\cap\cdots\cap L_1^{\varepsilon_q}\neq\emptyset$ and we have $\alpha_i=(\varepsilon_i)^A$ for each i.

Let $\bar{K}(\mathscr{N}^A)_q$ denote the set of all q-simplices. Then $\bar{K}(\mathscr{N}^A)=\coprod\bar{K}(\mathscr{N}^A)_q$ denotes an s.s. complex with obvious definitions of the face and degeneracy operators.

LEMMA 5.13. *Let A be a simplex of $'K[\mathscr{L}_0]$. Then $\beta=(\beta_0,\cdots,\beta_q)$ is a q-simplex of $\bar{K}(\mathscr{N}^A)$ if and only if there exists one index $(\varepsilon_0)^A$ say β_0 in E^A such that $\beta=(s_0)^q\beta_0$.*

Proof. As observed above, the set $L_1^{\varepsilon_0}\cap\cdots\cap L_1^{\varepsilon_q}$ is nonvacuous if and only if $\varepsilon_i=\varepsilon_0$ for every i, and hence $\beta_i=\beta_0$ for every i.

$\gamma=(\gamma_0,\cdots,\gamma_q)$ is a q-simplex of $\bar{K}('\mathscr{N})$ if and only if the following conditions are satisfied.

i) There exists a q-simplex $\alpha=(\alpha_0,\cdots,\alpha_q)$ of $'\bar{K}[\mathscr{L}_0]$: that is, there exists a sequence of cells $L_0^{\theta_0}\leqq\cdots\leqq L_0^{\theta_q}$ and we have $\alpha_i=\hat{L}_0^{\theta_i}$ for each i. We write as $\{\alpha\}=A$, $\alpha_i=\hat{L}_0^{\theta_i}$ for each i.

ii) There exists a sequence of indices $(\varepsilon_0)^{A_0},\cdots,(\varepsilon_q)^{A_q}$ belonging to E^{A_0}, $\cdots,E^{A_q}$ respectively such that $O(A)\cap L_1^{\varepsilon_0}\cap\cdots\cap L_1^{\varepsilon_q}\neq\emptyset$ and we have $\gamma_i=(\varepsilon_i)^{A_i}$ for each i.

LEMMA 5.14. *$\gamma=(\gamma_0,\cdots,\gamma_q)$ is a q-simplex of $\bar{K}('\mathscr{N})$ if and only if the following conditions are satisfied.*

i) *There exists a q-simplex of $'\bar{K}[\mathscr{L}_0]$ say $\alpha=(\alpha_0,\cdots,\alpha_q)$: that is, there exists a sequence of cells $L_0^{\theta_0}\leqq\cdots\leqq L_0^{\theta_q}$ and we have $\alpha_i=\hat{L}_0^{\theta_i}$ for each i. We write as $\{\alpha\}=A$, $\alpha_i=A_i$ for each i.*

ii′) *There exists an index $(\varepsilon_0)^{A_0}$ of E^{A_0} and we have $\gamma_i=(\varepsilon_i)^A$ with $\varepsilon_i=\varepsilon_0|I(\theta_i)$ for each i.*

Proof. The proof is done in the same way as in the preceding Lemma.

A fibre product

$$\tilde{K}={}'\bar{K}[\mathscr{L}_0]\underset{'K[\mathscr{L}_0]}{\times}\coprod\{\bar{K}(\mathscr{N}^A)\mid A\in{}'K[\mathscr{L}_0]\}$$

is defined as follows:

A q-simplex of $\tilde{K}$ is a pair $\alpha\times\beta$ with α being a q-simplex of $'\bar{K}[\mathscr{L}_0]$ and β being a q-simplox of $\bar{K}(\mathscr{N}^A)$ where $A=\{\alpha\}$.

Now the face and degeneracy operators are defined by

$$\partial_i(\alpha\times\beta)=\partial_i\alpha\times\pi^{\{\alpha\}}_{\{\partial_i\alpha\}}\partial_i\beta$$
$$s_i(\alpha\times\beta)=s_i\alpha\times s_i\beta\ .$$

Let $\alpha\times\beta$, $\alpha'\times\beta'$ be two q-simplices of K. Then $\alpha\times\beta\sim\alpha'\times\beta'$ if and only if $\alpha=\alpha'$, $\pi^{\{\alpha\}}_{\{(\partial_0)^i\alpha\}}\beta_i=\pi^{\{\alpha'\}}_{\{(\partial_0)^i\alpha'\}}\beta'_i$.

LEMMA 5.15. *$\alpha\times\beta$ is a q-simplex of $\tilde{K}$ if and only if there exists an index $(\varepsilon_0)^A$ say β_0 in E^A such that $\beta=(s_0)^q\beta_0$ where $A=\{\alpha\}$.*

Let $\alpha\times(s_0)^q(\varepsilon_0)^A$, $\alpha'\times(s_0)^q(\varepsilon'_0)^A$ be two q-simplices of $\tilde{K}$. Then $\alpha\times(s_0)^q(\varepsilon_0)^A\sim\alpha'\times(s_0)^q(\varepsilon'_0)^A$ if and only if $\alpha=\alpha'$, $\varepsilon_0=\varepsilon'_0$.

Now we obtain the following.

PROPOSITION 5.16. *We have an s.s. isomorphism*

$$\tilde{K}\cong K('\mathscr{N})\ .$$

This result suggests us the existence of the complex as shown in the next section.

§6. The Deligne complex

As in the previous section we are only concerned with the family $\mathscr{W}=\{W(i)\mid i\in I\}$ of which each member $W(i)$ is a hyperplane in $\boldsymbol{C}^n$.

Already we know the existence of a collapsible open covering of $\boldsymbol{C}^n\backslash\tilde{Y}(\mathscr{W})$ which enables us to obtain a simple homotopy equivalence $\boldsymbol{C}^n\backslash\tilde{Y}(\mathscr{W})\underset{s}{\simeq}|K('\mathscr{N})|$. The aim of this section is to construct a regular cell complex $D[\mathscr{W}]$ called the Deligne complex associated with $\mathscr{W}$ whose derived complex $'D[\mathscr{W}]$ is isomorphic to $K(\mathscr{N})$.

Let θ be an element of Θ with a nonvacuous cell L^θ_0 of $K[\mathscr{L}_0]$. Then the dual cell $D(\theta)$ of L^θ_0 to be the subcomplex $\mathrm{Cl\,St}\,\hat{L}^\theta_0$ of $'K[\mathscr{L}_0]$, in other words

$$D(\theta)=\cup\{\{\hat{L}^{\theta_0}_0,\cdots,\hat{L}^{\theta_q}_0\}\in{}'K[\mathscr{L}_0]\mid L_0\leqq L^{\theta_0}_0\prec\cdots\prec L^{\theta_q}_0\}\ .$$

If L^θ_0 is a face of $L^{\theta'}_0$, then $D(\theta')$ becomes a subcomplex of $D(\theta)$. This being the case, $D(\theta')$ is called a face of $D(\theta)$ and is denoted by $D(\theta')\prec D(\theta)$. We use the notation $\iota^\theta_{\theta'}$: $D(\theta')\to D(\theta)$ to denote the natural inclusion.

For θ in Θ with a cell L^θ_0 of $K[\mathscr{L}_0]$, we define E^θ to be the set of indices ε^θ taken once for each function ε: $I(\theta)\to\boldsymbol{F}^\times_3$ satisfying the condition $L^\theta_0\cap L^\varepsilon_1\neq\emptyset$.

If L^θ_0 is a face of $L^{\theta'}_0$, then $I(\theta)$ contains $I(\theta')$. Hence we can define a projection $\pi^\theta_{\theta'}$: $E^\theta\to E^{\theta'}$ which sends ε^θ to $(\varepsilon')^{\theta'}$, with $\varepsilon'=\varepsilon|I(\theta')$.

For each θ with a cell L^θ_0 in $K[\mathscr{L}_0]$, we consider the cartesian produdt $D(\theta)\times E^\theta$.

If L^θ_0 is a face of $L^{\theta'}_0$, then we have a diagramme

$$D(\theta')\times E^{\theta'} \xrightarrow{\iota^{\theta}_{\theta'}\times 1} D(\theta)\times E^{\theta'} \xleftarrow{1\times\pi^{\theta}_{\theta'}} D(\theta)\times E^{\theta} .$$

Let $A\times\varepsilon^{\theta}$, $A'\times(\varepsilon')^{\theta'}$ be simplices of $D(\theta)\times E^{\theta}$, $D(\theta')\times E^{\theta'}$ respectively. We write $A'\times(\varepsilon')^{\theta'}\to A\times\varepsilon^{\theta}$ if L^{θ}_0 is a face of $L^{\theta'}_0$ and $\iota^{\theta}_{\theta'}A'=A$, $(\varepsilon')^{\theta'}=\pi^{\theta}_{\theta'}\varepsilon^{\theta}$.

We consider the disjoint union of simplicial complexes

$$\tilde{D}=\coprod\{D(\theta)\times E^{\theta}\mid\theta\in\Theta, L^{\theta}_0\in K[\mathscr{L}_0]\} .$$

Let us now define an equivalence relation as the weakest reflexive, that symmetric and transitive relation including the property that $A\times\varepsilon^{\theta}$ and $A'\times(\varepsilon')^{\theta'}$ are equivalent whenever $A\times\varepsilon^{\theta}\to A\times(\varepsilon')^{\theta'}$.

We denote by $D'[\mathscr{W}]$ the simplicial complex consisting of all equivalence classes.

Let us denote by χ: $\tilde{D}\to D'[\mathscr{W}]$ the natural projection.

Now we can verify that $\chi(D(\theta')\times(\varepsilon')^{\theta'})$ is a subcomplex of $\chi(D(\theta)\times\varepsilon^{\theta})$ if and only if the following conditions are satisfied:

i) $D(\theta')$ is a face of $D(\theta)$, or equivalently, L^{θ}_0 is a face of $L^{\theta'}_0$.

ii) $(\varepsilon')^{\theta'}=\pi^{\theta}_{\theta'}\varepsilon^{\theta}$ i.e. $\varepsilon'=\varepsilon|I(\theta')$.

When these conditions are satisfied $\chi(D(\theta')\times(\varepsilon')^{\theta'})$ is called a face of $\chi(D(\theta)\times\varepsilon^{\theta})$ and is denoted by $\chi(D(\theta')\times(\varepsilon')^{\theta'})\prec\chi(D(\theta)\times\varepsilon^{\theta})$.

Given a simplicial complex K, we employ the notation $|K|$ to denote the geometric realization of K.

Under this convention χ: $|\tilde{D}|\to|D'[\mathscr{W}]|$ stands for the corresponding natural projection.

From the definition, we can easily verify that the space $|D'[\mathscr{W}]|$ has the structure of CW complex admitting $\chi(|D(\theta)|\times\varepsilon^{\theta})$ as a regular cell given for each ε^{θ} in E^{θ} and θ in Θ with cell L^{θ}_0 in $K[\mathscr{L}_0]$.

The CW complex thus obtained is called the Deligne complex associated with $\mathscr{W}$ and denoted by $D[\mathscr{W}]$.

Let $'D[\mathscr{W}]$ denote the barycentric subdivision of $D[\mathscr{W}]$.

PROPOSITION 6.1. *We have a simplicial isomorphism*

$$K(\mathscr{N})\cong{}'D[\mathscr{W}] .$$

Proof. From Lemma 5.12, C is a q-simplex of $K(\mathscr{N})$ with vertices $C_0,\cdots,C_q$ if and only if the following conditions are satisfied:

i) There exists a simplex A of $'K[\mathscr{L}_0]$ with the sequence of vertices $A_0\leqq\cdots\leqq A_q$ such that there exists a sequence of cells $L^{\theta_0}_0\leqq\cdots\leqq L^{\theta_q}_0$ and the corresponding sequence of barycentres $\hat{L}^{\theta_0}_0,\cdots,\hat{L}^{\theta_q}_0$ is the sequence $A_0,\cdots,A_q$.

ii) There exists a permutation ω and a sequence of indices $(\varepsilon_0)^{A_0},\cdots,(\varepsilon_q)^{A_q}$ belonging to $E^{A_0},\cdots,E^{A_q}$ respectively such that $\varepsilon_i=\varepsilon_0|I(\theta_i)$ if $0\leqq i\leqq q$, $(\varepsilon_i)^{A_i}\neq(\varepsilon_j)^{A_j}$ if $0\leqq i<j\leqq q$ and the sequence of integers is the sequence $C_{\omega(0)},\cdots,C_{\omega(q)}$.

On the other hand, C' is a q-simplex of $'D[\mathscr{W}]$ with the sequence of vertices $C'_0,\cdots,C'_q$ if and only if there exists a permutation ω and a sequence of cells $\chi(D(\theta_0)\times(\varepsilon_0)^{\theta_0})\prec\cdots\prec\chi(D(\theta_q)\times(\varepsilon_q)^{\theta_q})$ and the corresponding sequence of barycentres $\chi(|D(\theta_0)|\times(\varepsilon_0)^{\theta_0})^\wedge,\cdots,\chi(|D(\theta_q)|\times(\varepsilon_q)^{\theta_q})^\wedge$ is the sequence $C'_{\omega(0)},\cdots,C'_{\omega(q)}$.

We now have an isomorphism transforming a vertex ε^A into a vertex $\chi(D(\theta)\times\varepsilon^\theta)^\wedge$ with $A=\hat{L}_0^\theta$.

COROLLARY 6.2. *We have a simplicial isomorphism*

$$K(\mathscr{N})\cong D'[\mathscr{M}]\ .$$

Proof. We only need to prove. $'D[\mathscr{W}]\cong D'[\mathscr{W}]$. This follows from the definition.

THEOREM 6.3. *We have a piecewise linear homeomorphism*

$$|K(\mathscr{N})|\cong|D[\mathscr{W}]|\ .$$

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