On Harmonic Maps from CP^1 to $\Omega SU(n)$

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Abstract

Using the geometry of based loop group ΩG and some results on harmonic maps, we shall show that when $n \geq 3$ there exist harmonic but not $\pm$holomorphic maps from CP^1 to $\Omega SU(n)$ for each degree. Further we shall discuss what our result suggests from the point of view of the difference between Yang-Mills connections and instantons on S^4 in particular Atiyah-Jones conjecture.

§ 1. Introduction and statement of result

Let G be a connected compact Lie group and $\mathfrak{g}$ be its Lie algebra. Suppose that G is a simply-connected and simple Lie group. We define free loop group LG of G by the space of all maps $\alpha: S^1 \rightarrow G$ which are in the Sobolev class L_1^2 and based loop group ΩG of G by the subspace of LG in which element preserves a base point. Then LG and ΩG are infinite dimensional Hilbert Lie groups by $(\alpha \cdot \beta)(\theta) = \alpha(\theta) \cdot \beta(\theta)$ for any $\alpha, \beta \in LG$ and $\theta \in S^1$. The Lie algebra $Lie(LG)$ is given by $L\mathfrak{g}$ which are maps from S^1 to $\mathfrak{g}$ and $Lie(\Omega G)$ is by $\Omega \mathfrak{g} = \{\alpha \in L\mathfrak{g}: \alpha(0) = 0 \in \mathfrak{g}\}$.

The geometry of LG and ΩG is studied by many people, for example Atiyay, Pressley and Segal ([**A-P**] [**P**] [**P-S**]), and we know that they are equipped with fruitful nice geometric structures. In particular ΩG has a natural complex structure and a Kähler metric (see § 2). Furthermore S^1 acts on ΩG as rotation preserving the Kähler structure. (This is holomorphic action.) Then we have a moment map with respect to this S^1 action and Kählar structure and its critical manifold is $Hom(S^1, G)$.

Now in this paper we consider the adjoint action of G on ΩG and its adjoint decomposition, which also gives decomposition into connected components of $Hom(S^1, G)$. We describe the adjoint orbits and show that they lie in ΩG holomorphically and totally geodesic. Using this observation and some results on harmonic maps, we get the following theorem.

THEOREM. *Let* CP^1 *be a projective line with Fubini study metric and* $\Omega SU(n)$ *be with a natural Kählar metric* (§ 2).

(1) *If* $n \geq 3$, *there exist harmonic but non* $\pm$*holomorphic maps from* CP^1 *to* $\Omega SU(n)$ *for each degree.*

(2) *If $n=2$, there exist holomorphic and totally geodesic maps from* $\mathbf{C}P^1$ *to* $\Omega SU(2)$ *for each degree.*

REMARK. (1) For each map φ: $\mathbf{C}P^1 \to \Omega G$ we define the degree of φ by the following. Since G is a simply connected and simple compact Lie group, we have $\pi_1(G)=\pi_2(G)=1$ and $\pi_3(G)=\mathbf{Z}$. So $\pi_1(\Omega G)\cong\pi_2(G)=1$. Hurewitz theorem implies that $H_2(\Omega G; \mathbf{Z})\cong\pi_2(\Omega G)\cong\pi_3(G)\cong\mathbf{Z}$. We denote the generator of $H_2(\Omega G; \mathbf{Z})$ (resp. $H_2(\mathbf{C}P^1; \mathbf{Z})\cong\mathbf{Z}$) by $[\Omega G]$ (resp. $[\mathbf{C}P^1]$). Then the degree of φ is defined as an integer by $\varphi_*[\mathbf{C}P^1]=\deg\varphi[\Omega G]$.

(2) When $n=2$, Theorem dose not imply that all harmonic maps from $\mathbf{C}P^1$ to $\Omega SU(2)$ are holomorphic, but just that the harmonic maps which we construct are automatically holomorphic.

Harmonic or holomorphic maps from $\mathbf{C}P^1$ to ΩG are deeply related to the moduli spaces of Yang-Mills connections or instantons over S^4. From this point of view, we shall discuss above Theorem and our motivation in the last section.

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§ 2. Kähler structure and moment map

In this section we recall the geometric structures on ΩG, in particular complex Kähler structure and the existence of moment map of S^1 rotating action ([**A-P**], [**P-S**]).

Let $\langle\ ,\ \rangle_{\mathfrak{g}}=\langle\ ,\ \rangle$ be a G-invariant metric on $\mathfrak{g}$ and we fix this metric. We put for $\xi, \eta\in\Omega\mathfrak{g}$ and $\theta\in S^1$,

$$\omega(\xi, \eta)=\frac{1}{2\pi}\int_0^{2\pi}\langle\xi'(\theta), \eta(\theta)\rangle_{\mathfrak{g}}\,d\theta\ .$$

Here $\xi'(\theta)$ means the first derivation by θ of ξ: $S^1\to\mathfrak{g}$. Then direct calculation shows that ω becomes a left invariant closed 2-form and non-degenerate on ΩG. So this defines a symplectic structure on ΩG. Note that ω is also defined on LG and left invariant closed 2-form. However ω degenerates on constant loops.

Next we introduce almost complex structure J on ΩG. We have the following Fourier expansion of $\xi\in L\mathfrak{g}$

$$\xi(\theta)=\sum_{n=0}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\ , \qquad a_n, b_n\in\mathfrak{g}\ .$$

Now we see $L\mathfrak{g}$ as $L\mathfrak{g}=\Omega\mathfrak{g}\oplus\mathfrak{g}$ and this $\mathfrak{g}$-component corresponds to constant terms in the Fourier expansion. So for $\xi\in\Omega\mathfrak{g}$ we have

$$\xi(\theta)=\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\ .$$

Then we define J: $\Omega\mathfrak{g}\to\Omega\mathfrak{g}$ by

$$J: \sum_{n=1}^{\infty}(a_n \cos n\theta+b_n \sin n\theta)\to\sum_{n=1}^{\infty}(-b_n \cos n\theta+a_n \sin n\theta) .$$

and this gives almost complex structure (after moving by left transformations). Further it is easy to check that this is integrable. With this J, ΩG is a complex manifold. But we note that $(\Omega G, J)$ is not a complex Lie group.

Now let g be the Riemannian metric compatible with (ω, J) such that the followings hold

$$\omega(J\xi, J\eta)=\omega(\xi, \eta) , \qquad g(\xi, \eta)=\omega(\xi, J\eta) \text{ is positive definite.}$$

Then $(\Omega G, g, \omega, J)$ is a Kähler manifold. Here we point out a remark on the topology of ΩG. The topology by this Kähler metric g is different from a priori one by L_1^2-Sobolev loops. In fact $(\Omega G, g)$ is not complete. But we deal with $(\Omega G, g)$ because it fits well with Morse theory on ΩG as below.

We define the energy functional $\mathscr{E}$ on LG by

$$\mathscr{E}(\gamma)=\frac{1}{4\pi}\int_0^{2\pi}\langle\gamma(\theta)^{-1}\gamma'(\theta), \gamma(\theta)^{-1}\gamma'(\theta)\rangle_{\mathfrak{g}}\, d\theta \qquad \text{for} \quad \gamma\in LG .$$

where for γ: $S^1\to G$ we consider $\gamma'(\theta)$ as an element of $T_{\gamma(\theta)}G$ and pulling it back to the identity by $\gamma(\theta)^{-1}$ we consider $\gamma(\theta)^{-1}\gamma'(\theta)\in T_{id}G=\mathfrak{g}$. Since it is easy to see that $\mathscr{E}$: $LG\to\mathbf{R}$ is invariant under the multiplication by constant loops, $\mathscr{E}$ induces the functional on ΩG, say also $\mathscr{E}$: $\Omega G\to\mathbf{R}$.

On the other hand S^1 acts on ΩG by $(t\cdot\gamma)(\theta)=\gamma(t+\theta)\gamma^{-1}(t)$ for $t, \theta\in S^1$ and $\gamma\in\Omega G$. This S^1 action preserves that Kähler structure ω. Then we find out that $\mathscr{E}$ is the moment map with respect to above S^1 action. Namely $\mathscr{E}$ is S^1 invariant and $d\mathscr{E}=i(X_{\mathscr{E}})\omega$, where we denote by $X_{\mathscr{E}}$ the vector field along the S^1 action and $i(\mathscr{E}_{\mathscr{E}})$ means the interior product by the vector field $X_{\mathscr{E}}$. In particular since the critical manifold of a moment map is the fixed point set, we have $(t\cdot\gamma)(\theta)=\gamma(\theta)$ for any $t, \theta\in S^1$, that is $\gamma(t+\theta)=\gamma(t)\gamma(\theta)$. Therefore the critical manifold of $\mathscr{E}$ is just the set of all group homomorphisms from S^1 to G, $Hom(S^1, G)$. Further $Hom(S^1, G)$ is a complex submanifold of ΩG and in particular a compact complex Kähler manifold.

§ 3. Adjoint orbit decomposition

In this section we consider the adjoint action of G on ΩG and its adjoint orbit decomposition. In particular we describe the orbits in the critical manifold $Hom(S^1, G)$.

We have naturally the adjoint action of G on ΩG by

(3.1) $$(g\gamma)(\theta)=g\cdot\gamma(\theta)\cdot g^{-1} , \qquad \theta\in S^1$$

for $g\in G$ and $\gamma\in\Omega G$.

LEMMA 3.1. *The moment map in the previous section* $\mathscr{E}: \Omega G \to \mathbf{R}$ *is* G-*invariant under the adjoint action* (3.1).

Proof. For $g \in G$ and $\gamma \in \Omega G$ we have

$$\mathscr{E}(g\gamma)=\frac{1}{4\pi}\int_0^{2\pi}\langle(g\gamma)^{-1}(\theta)(g\gamma)'(\theta),\ (g\gamma)^{-1}(\theta)(g\gamma)'(\theta)\rangle_{\mathfrak{g}}\, d\theta\ .$$

Here we have

$$\begin{aligned}(g\gamma)^{-1}(\theta)(g\gamma)'(\theta)&=(g\gamma(\theta)g^{-1})^{-1}(g\gamma(\theta)g^{-1})'\\&=g\gamma(\theta)^{-1}g^{-1}g\gamma'(\theta)g^{-1}\\&=g(\gamma(\theta)^{-1}\gamma'(\theta))g^{-1}\ .\end{aligned}$$

Therefore since $\langle\ ,\ \rangle_{\mathfrak{g}}$ is G-invariant, we have $\mathscr{E}(g\gamma)=(\gamma)$. ■

This lemma enables us to decompose the critical manifold of $\mathscr{E}$ into the adjoint orbits. That is we have

$$Crit(\mathscr{E})=Hom(S^1, G)=\coprod_{\gamma} C_{\gamma} \tag{3.2}$$

where C_γ means the conjugacy class of $\gamma \in Hom(S^1, G)$.

Now we use the terminology of Lie algebra in order to describe the topology of the conjugacy class. Since for any $\gamma \in Hom(S^1, G)$ and $\theta \in S^1$ $\gamma(\theta)$ is contained in a torus of G, we can write as $\gamma(\theta)=\exp\theta X$ for some $X\in\mathfrak{g}$. Further since γ preserves a base point, we have $\exp X=\gamma(1)=\gamma(0)=id_G$. Conversely for any $X\in\mathfrak{g}$ with $\exp X=id_G$ we get an element $\gamma\in Hom(S^1, G)$ by $\gamma(\theta)=\exp\theta X$, $0\le\theta\le 1$. So we have

$$\mathfrak{h}:=\{X\in\mathfrak{g} \mid \exp X=id_G\}\simeq Hom(S^1, G)\ . \tag{3.3}$$

Then the adjoint action (3.1) of G induces the action on $\mathfrak{h}$, which is described by the following. Let X be an element of $\mathfrak{h}$ corresponding to γ, that is $\gamma(\theta)=\exp\theta X$. Then we have

$$\begin{aligned}\exp\theta(g\cdot X)&=(g\cdot\gamma)(\theta)=g\gamma(\theta)g^{-1}\\&=g(\exp\theta X)g^{-1}\\&=\exp\theta(Ad_g X)\ .\end{aligned} \tag{3.4}$$

Therefore we obtain that $(g\cdot X)=Ad_g(X)$ for $g\in G$ and $X\in\mathfrak{h}$, which means the action (3.1) induces adjoint representation on $\mathfrak{h}$. Corresponding to the decomposition (3.2), we have

$$\mathfrak{h}=\coprod_X C_X \tag{3.5}$$

where C_X means the adjoint orbit associated to the adjoint representation of $X\in\mathfrak{h}$.

Next we describe the orbit C_X. Let $G_X=\{g\in G \mid Ad_g(X)=X\}\subset G$ be the isotoropy subgroup of $X\in\mathfrak{h}$. Then we obtain the following lemma.

LEMMA 3.2. *As homogeneous spaces we have the following isomorphisms*

$$C_X \cong G/G_X \cong G/Z(T) .$$

Here $G_X=\exp\mathfrak{g}_X$ *when we denote by* $\mathfrak{g}_X$ *the set of the elements of* $\mathfrak{g}$ *which commute* X, $[Y, X]=0$. *We denote by* T *a torus of the closure of the one parameter subgroup generated by* $\{\exp tX\}$, *and* $Z(T)$ *means the centralizer of* T *in* G.

Proof. It is enough to show that $G_X=\exp\mathfrak{g}_X=Z(T)$.

First we shall show that $(G_X)_0(=$the identity component of $G_X)=\exp\mathfrak{g}_X$. Take any $y\in(G_X)_0$. Then we have $y=\exp Y$ for some $Y\in Lie(G_X)$. Since $\exp tY\in(G_X)_0\subset G_X$, we have $(Ad(\exp tY))(X)=X$. Differentiating the both hand sides by t and evaluating $t=0$, we get

$$0=\frac{d}{dt}X|_{t=0}=\frac{d}{dt}(Ad(\exp tY))(X)|_{t=0}=ad\,Y(X)=[Y, X] .$$

So $Y\in\mathfrak{g}_X$. It follows that $(G_X)_0\subset\exp\mathfrak{g}_X$.

Conversely take any $Y\in\mathfrak{g}_X$. Then since

$$(\exp tY)(\exp X)(\exp-tY)=\exp X ,$$

we have $(Ad(\exp tY))(X)=\exp X$. That is $\exp tY\in(G_X)_0$. Thus we obtain $(G_X)_0=\exp\mathfrak{g}_X$.

Secondly we shall show that $G_X=Z(T)$. Take any $g\in G_X$. Then $(Ad_g)(X)=X$ and this implies that $(Ad_g)(tX)=tX$ for all $t\in\mathbf{R}$. Namely $g\cdot\exp tX\cdot g^{-1}=\exp tX$ for all $t\in\mathbf{R}$. So we have $g\in Z(\overline{\exp tX})=Z(T)$. Converse is easily obtained by returning back above argument.

Finally since the centralizer of a torus is connected, $Z(T)_0=Z(T)$. Therefore we obtain $G_X=(G_X)_0=\exp\mathfrak{g}_X$. ■

Thus we have

$$Crit(\mathscr{E})=Hom(S^1, G)=\coprod_{\gamma} C_\gamma\cong\mathfrak{h}=\coprod_{X} C_X\cong\coprod_{X} G/G_X=\coprod_{X} G/Z(T) . \tag{3.6}$$

REMARK. We have a remark on the set of the index. An element $\gamma\in Hom(S^1, G)$ moves in the set of conjugacy classes of homomorphisms from S^1 to G. This set is identified with an integer lattice in $\mathfrak{h}$ modulo the action of Weyl group. For example when G is $SU(2)$, it is a non negative integers (see [A]).

The decomposition (3.6) gives also the decomposition of connected components of the critical manifold. In fact according to [A] we put $C^\gamma=\{f\in\Omega G \mid \gamma(\theta)^{-1}f(t)\gamma(\theta)=f(\theta+t)f(\theta)^{-1}$, for all $\theta, t\in S^1\}$ for $\gamma\in Hom(S^1, G)$. We put $h(\theta)=\gamma(\theta)f(\theta)$. Since γ is a homomorphism, $f\in C^\gamma$ implies $h(\theta)h(t)=h(\theta+t)$. Thus we have $h\in Hom(S^1, G)$ and $C^\gamma=\gamma^{-1}Hom(S^1, G)$. In particular when we denote by $(C^\gamma)_1$ the component of C^γ containing the base point $1\in\Omega G$, we have

$(C^{\gamma})_1=\gamma^{-1}C_\gamma$. Therefore C_γ is a connected component.

Now we consider a natural inclusion of a connected component $C_\gamma \cong C_X$ into ΩG. Then we have the following

LEMMA 3.3. *The natural inclusion i: $C_\gamma \hookrightarrow \Omega G$ is holomorphic and totally geodesic.*

Proof. Holomorphicity is directly followed by that C_γ is a connected component of the critical manifold and the critical manifold is a complex submanifold of ΩG.

We shall show that it is totally geodesic. By Proposition 8.8 in [K-N] it is enough to show that the inclusion i is auto-parallel. Namely take any point $\alpha \in C_\gamma \cong C_X$ and let $\tau=\tau_t$ be any curve in C_γ starting from α ($\tau_0=\alpha$, $0\leq t\leq 1$). Then it suffices to show that when any tangent vector $\mathfrak{X}\in T_\alpha(C_\gamma)$ transforms parallel along the curve τ, then it is also tangent to C_γ.

Since the action of S^1 to ΩG preserves the Riemannian metric g (see § 2) and C_γ is a fixed point set, $T_\alpha(C_\gamma)$ is identified with the subspace of $T_\alpha(\Omega G)$ on which the induced action on $T_\alpha(\Omega G)$ is identity. That is for any $\theta\in S^1$ we have $\theta\cdot(\mathfrak{X})=\mathfrak{X}$. Furthermore by that the action of S^1 to ΩG preserves the Riemannian metric g the action also preserves the Riemannian connection associated to the Riemannian metric g. So the S^1 action and the parallel transformation along τ commute. That is when we denote by $\tau(\mathfrak{X})$ the parallel transformation of the tangent vector $\mathfrak{X}$ along τ, we have $\theta\cdot\tau(\mathfrak{X})=\tau(\theta\cdot(\mathfrak{X}))=\tau(\mathfrak{X})$. Therefore the vector $\tau(\mathfrak{X})$ is invariant under the S^1 action. Thus we obtain $\tau(\mathfrak{X})\in T_{\tau_1}(C_\gamma)$ which implies $\tau(\mathfrak{X})$ is tangent to C_γ, too. It completes the proof. ■

In the rest of this section we shall deal with when $G=SU(n)$, $(n\geq 2)$. In that case we can describe explicitly some connected components C_γ of the critical manifold.

We define tori $T_{n,k}$ $(k=1, 2, \cdots, n-1)$ in $SU(n)$ by the following

$$T_{n,k}:=\left(\begin{array}{ccc|ccc} \multicolumn{3}{c}{\overbrace{\hphantom{e^{i(n-k)\theta}\quad\ddots\quad e^{i(n-k)\theta}}}^{k}} & \multicolumn{3}{c}{\overbrace{\hphantom{e^{-ik\theta}\quad\ddots\quad e^{-ik\theta}}}^{n-k}} \\ e^{i(n-k)\theta} & & & & & \\ & \ddots & & & O & \\ & & e^{i(n-k)\theta} & & & \\ & & & e^{-ik\theta} & & \\ & O & & & \ddots & \\ & & & & & e^{-ik\theta} \end{array}\right).$$

Then direct calculation shows that the centralizer $Z(T_{n,k})$ of $T_{n,k}$ in $SU(n)$ is given by

$$Z(T_{n,k})=\left\{\begin{pmatrix} A & O \\ O & B\end{pmatrix}\in SU(n)\mid A\in U(k),\ B\in U(n-k)\ \det A\ \det B=1\right\}.$$

From this we obtain the isomorphisms as homogeneous spaces

(3.7) $$G/Z(T_{n,k})=SU(n)/Z(T_{n,k})\cong Gr_{n,k}(\mathrm{C})$$

where $Gr_{n,k}(\mathrm{C})$ means a complex Grassmann manifold which is formed by k dimensional complex subspaces of C^n.

Therefore by Lemma 3.2, Lemma 3.3 and (3.7) we obtain

LEMMA 3.4. *When $G=SU(n)$, $(n\geq 2)$, there appear complex Grassmann manifolds $Gr_{n,k}(\mathrm{C})$ $(k=1, 2, \cdots, n-1)$ in connected components of the critical manifold of $\mathscr{E}$.*

In particular a natural inclusion i: $Gr_{n,k}(\mathrm{C})\to\Omega SU(n)$ is holomorphic and totally geodesic.

§ 4. Proof of Theorem

In this section we shall prove Theorem by using the observations in above sections and some results on harmonic maps. First of all we refer to the following Eells-Wood's theorem [**E-W 2**].

THEOREM 4.1 ([**E-W 2**]). *Let $\mathrm{C}P^n$ be a complex projective space with Fubini-study metric. Suppose $n\geq 2$. Then there exist harmonic and non$\pm$holomorphic maps from $\mathrm{C}P^1$ to $\mathrm{C}P^n$ for each degree.*

We denote the map in above Theorem by φ_l^n: $\mathrm{C}P^1\to\mathrm{C}P^n$ ($\deg\varphi_l^n=l$). Now we apply Lemma 3.4 in the previous section when $k=1$. Since $Gr_{n,1}(\mathrm{C})=\mathrm{C}P^{n-1}$, we have holomorphic and totally geodesic map

$$i\colon \mathrm{C}P^{n-1}\to\Omega SU(n)\ .$$

Then we consider the composition map $i\circ\varphi_l^{n-1}$: $\mathrm{C}P^1\to\mathrm{C}P^{n-1}\to\Omega SU(n)$, which is denoted by Φ_l^n.

LEMMA 4.2. *If $n\geq 3$, the map Φ_l^n: $\mathrm{C}P^1\to\Omega SU(n)$ is harmonic and non$\pm$ holomorphic.*

Proof. Harmonicity of Φ_l^n: We recall the composition property of harmonic maps. (See [**E-L**]). That is, when f_1: $(M, g)\to(N, h)$ and f_2: $(N, h)\to(L, k)$ are two C^∞-maps between Riemannian manifolds (g, h, k are Riemannian metrics), the trace of the second fundamental form of the composition map $f_2\circ f_1$ is given by

$$\begin{aligned}\tau(f_2\circ f_1)&\equiv\mathrm{trace}_g\,(\nabla d(f_2\circ f_1))\\&=df_2\circ\tau(f_1)+\mathrm{trace}\,\nabla df_2(df_1, df_1)\in C^\infty((f_2\circ f_1)^*TL)\ .\end{aligned}$$

Now if f_2 is a totally geodesic map, then $\nabla df_2=0$. Moreover if f_1 is a harmonic map, we have $\tau(f_1)=0$. So in that case it implies $\tau(f_2\circ f_1)=0$ that is harmonic. In our case when $n\geq 3$, φ_l^{n-1} is harmonic by Eells-Wood's Theorem 4.1 and i is totally geodesic by Lemma 3.4. Therefore the map $\Phi_l^n = i \circ \varphi_l^{n-1}$ is harmonic.

Non$\pm$holomorphicity of Φ_l^n: It suffices to show that the induced complex structure on the submanifold $C_\gamma \cong CP^{n-1}$ ($\gamma \in Hom(S^1, SU(n))$) of $\Omega SU(n)$ is the standard one. If so, the claim holds immediately because φ_l^{n-1}: $CP^1 \to CP^{n-1}$ is non$\pm$holomorphic but i: $C_\gamma \cong CP^{n-1} \to \Omega SU(n)$ is holomorphic. Now the former CP^{n-1} is equipped with Fubini-study metric which is Kähler. On the other hand the latter CP^{n-1} is equipped with the Kähler structure induced by that on $\Omega SU(n)$ (§2). According to the results of Kodaira-Hirzebruch-Yau [Y] the complex structure on CP^n which is Kähler is unique (standard one). Therefore two complex structures on CP^{n-1} we consider now is the same. It completes the proof. ∎

REMARK. It seems to be able to prove the second assertion directly, not by using the result of Kodaira-Hirzebruch-Yau.

Thus Lemma 4.2 implies the part (1) of Theorem.

Next we consider the case when $n=2$. We have a holomorphic and totally geodesic map i: $CP^1 \to \Omega SU(2)$ by Lemma 3.4. However in contrast to Theorem 4.1 we know by Eells-Wood [**E-W 1**] the following.

THEOREM 4.3 ([**E-W 1**]). *Let Σ_g be a closed Riemann surface with genus g. Then every harmonic map from Σ_g to CP^1 is holomorphic, if the degree of the map is greater than or equal to genus g.*

Now we apply this theorem to the case $g=0$. Then all harmonic maps from CP^1 to CP^1 with degree≥ 0 are holomorphic. (When the degree<0, it is anti-holomorphic.) Thus we can not construct harmonic but non$\pm$holomorphic CP^1 to $\Omega SU(2)$ by this method, but only holomorphic and totally geodesic maps.

REMARK. In §3 we find that not only CP^{n-1} but also $Gr_{n,k}(C)$ appear as connected components of the critical manifold. Now we may replace CP^{n-1} to $Gr_{n,k}(C)$ above argument. At this time corresponding to Eells-Wood's Theorem 4.1 and 4.3, we have some results on the holomorphicity of harmonic map from CP^1 to $Gr_{n,k}(C)$ by Erden-Wood [**Er-W**] and Wood [**W**]. Using these results might give us much more harmonic and non$\pm$holomorphic maps from CP^1 to $\Omega SU(n)$ when $n\geq 3$. But when $n=2$, we do not get more or sharp results because $Gr_{2,k}(C)=CP^1$.

By the way above argument when we showed the maps are non$\pm$holomorphic, we argued whether two projective spaces with Kähler structures are holomorphically isomorphic or not. In this context the following problem appears naturally.

PROBLEM. *Let M be a compact complex manifold with a Kähler metric. If M is homeomorphic to $Gr_{n,k}(C)$, then is it holomorphically isomorphic?*

This is a holomorphic characterization problem of Grassmann manifold, which is a Grassmann version of Kodaira-Hirzebruch-Yau's theorem.

§ 5. Instantons and Yang-Mills connections on S^4

In this section we shall discuss some relations between instantons or Yang-Mills connections on S^4 and holomorphic or harmonic maps from CP^1 to ΩG.

First of all we start with the following observation by Atiyah and Donaldson ([**A**], [**D**]).

THEOREM 5.1 (ATIYAH AND DONALDSON). *Let G be any classical compact Lie group and k be any positive integer. Then the following two spaces* (1) *and* (2) *are diffeomorphic.*

(1) *The moduli space $\mathcal{M}_k(G)_*$ of instantons on a principal G-bundle over S^4 with $c_2=k$ modulo based gauge group.*

(2) *The space $\mathcal{H}ol_k(CP^1, \Omega G)_*$ of based holomorphic maps from CP^1 to ΩG with degree k.*

Here the based gauge group means the group of automorphisms of the G-bundle which fix the fibre over a base point, and in (2) the holomorphic structure on ΩG is that introduced in § 2.

On the other hand we have (one of) Atiyah-Jones conjectures [**A-J**] concerning Yang-Mills connections and instantons on S^4, that states.

ATIYAH-JONES CONJECTURE. *All irreducible Yang-Mills connections on a principal G-bundle over S^4 are instantons.*

By Bianchi identity, instantons are Yang-Mills connections. Atiyah-Jones conjecture claims that the converse is true.

Now since CP^1 and ΩG are Kähler manifolds (§ 2), any holomorphic map from CP^1 to ΩG is harmonic with respect to the Kähler metrics. Thus we have

$$\mathcal{H}ol_k(CP^1, \Omega G)_* \subseteq \mathcal{H}ar_k(CP^1, \Omega G)_*$$

where by the right hand side we denote the space of based harmonic maps from CP^1 to ΩG with degree k. In this context our main result implies that when $G=SU(n)$ and $n \geq 3$, $\mathcal{H}ol_k(CP^1, \Omega G)_* \subsetneqq \mathcal{H}ar_k(CP^1, \Omega G)_*$. Therefore when we denote by $\mathcal{Y}\mathcal{M}_k(G)_*$ the quotient space of irreducible Yang-Mills connections on a principal G-bundle over S^4 with $c_2=k$ modulo based gauge group, we have the following diagram.

$$\begin{array}{ccc} \mathcal{H}ol_k(CP^1, \Omega SU(n))_* & \subsetneqq & \mathcal{H}ar_k(CP^1, \Omega SU(n))_* \\ \updownarrow \cong & & \\ \mathcal{M}_k(SU(n))_* & \subseteq & \mathcal{Y}\mathcal{M}_k(SU(n))_* . \end{array} \tag{5.1}$$

Although harmonic maps and Yang-Mills connections are both the critical points of certain functionals and conformal invariant, the relation is not so clear. But our result seems to suggest that Atiyah-Jones conjecture is negatively supported when $G=SU(n)$ and $n\geq 3$. Strictly speaking, even if it is proved that $\mathscr{H}ar_k(CP^1, \Omega SU(n))_*$ is diffeomorphic to $\mathscr{Y}\mathscr{M}_k(SU(n))_*$ corresponding to Atiyah-Donaldson Theorem 5.1, we can not claim only by our result that $\mathscr{M}_k(SU(n))_* \subsetneqq \mathscr{Y}\mathscr{M}_k(SU(n))_*$, namely Atiyah-Jones conjecture is false. We should describe an explicit correspondence between $\mathscr{H}ar_k(CP^1, \Omega SU(n))_*$ and $\mathscr{Y}\mathscr{M}_k(SU(n))_*$. It means that the diagram (5.1) should be commutative.

Recently Sibner-Sibner-Uhlenbeck in [**S-S-U**] show that there are Yang-Mills connections which are not instantons on trivial $SU(2)$-bundle (instanton number $k=0$) over S^4. Their method is purely analytic and using Taubes' machine [**T**]. However we do not know at all how the 'moduli' of non anti-self-dual but Yang-Mills connections on S^4 is.

ADDITIONAL COMMENT. While writing this note, the author heard that Sadun proved that there are Yang-Mills connections which are not instantons on a principal $SU(2)$-bundle over S^4 with $c_2\neq 1$.

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