Exponential Decay of Quasi-stationary States of Time-periodic Schrödinger Equation with Short Range Potentials

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Abstract

A solution $u(t, x) \in C(\boldsymbol{R}_t^1, L^2(\boldsymbol{R}_x^N))$ to a time dependent Schrödinger equation $i(\partial u/\partial t) = -\Delta u + V(t, x)u$ with time periodic potential $V(t+2\pi, x) = V(t, x)$ is called quasi-stationary state with quasi-level λ if it satisfies $u(t+2\pi, x) = \exp(-2\pi i\lambda)u(t, x)$. We show, under the condition $(1+|x|)^{1+\epsilon}V(t, x) \in C^1(\boldsymbol{R}_t^1, L^\infty(\boldsymbol{R}_x^N))$, that every non-threshold quasi-stationary state decays exponentially at infinity in the sense that $\exp(\alpha|x|)u(t, x) \in C(\boldsymbol{R}_t^1, L^2(\boldsymbol{R}_x^N))$ for $\alpha^2 < 1-(\lambda-[\lambda])$, if λ is its quasilevel, where $[\lambda]$ is the integral part of λ.

1. Introduction

We consider time dependent Schrödinger equations

$$(1.1) \qquad i\frac{\partial u}{\partial t} = -\Delta u + V(t, x)u, \qquad -\infty < t < \infty, \quad x \in \boldsymbol{R}^N$$

with real-valued potential $V(t, x)$ which is periodic in time:

$$(1.2) \qquad V(t, x) = V(t+2\pi, x), \qquad -\infty < t < \infty, \quad x \in \boldsymbol{R}^N.$$

We assume

$$(1.3) \qquad V(t, \cdot) \in C^1(\boldsymbol{R}, L_\rho^\infty(\boldsymbol{R}^N)), \qquad \rho > 1,$$

where $L_\gamma^p(\boldsymbol{R}^N)$ is the Banach space of weighted L^p-functions:

$$L_\gamma^p(\boldsymbol{R}^N) = \{f \in L_{loc}^p(\boldsymbol{R}^N) : \|\langle x\rangle^\gamma f\|_{L^p} < \infty\}, \qquad \langle x\rangle = (1+x^2)^{1/2}.$$

It is well-known ([9]), under the condition (1.3), that (1.1) generates a unique propagator, a family of strongly continuous unitary operators $\{U(t, s) : -\infty < t \leq s < \infty\}$ on $L^2(\boldsymbol{R}^N)$ such that for every $u_0 \in H^2(\boldsymbol{R}^N)$,

$$u(t)=U(t, s)u_0\in C^1(\boldsymbol{R}, L^2(\boldsymbol{R}^N))\cap C^0(\boldsymbol{R}, H^2(\boldsymbol{R}^N))$$

is a solution of (1.1) with the initial condition $u(s)=u_0$. Here $H^s(\boldsymbol{R}^N)$ is the standard Sobolev space of order s.

As in the Floquet theory for ordinary differential equations with periodic coefficients, the one period propagator $U=U(2\pi, 0)$ plays important role for analyzing the solutions of (1.1). We know that

(a) the Hibert space $L^2(\boldsymbol{R}^N)$ splits into the absolutely continuous subspace $L^2_{ac}(U)$ and the point spectral subspace $L^2_p(U)$ of U;

(b) $L^2_{ac}(U)$ consists of scattering states ([4], [7]): For every $u_0\in L^2_{ac}(U)$ there exists $u_\pm\in L^2(\boldsymbol{R}^N)$ such that

$$\|U(t, 0)u_0-\exp(it\Delta)u_\pm\| \longrightarrow 0 \qquad (t\to\pm\infty);$$

(c) if ϕ is an eigenfunction of U,

$$U\phi=e^{-2\pi i\lambda}\phi\,, \qquad \phi\in L^2(\boldsymbol{R}^N)\,, \qquad \lambda\in\boldsymbol{R} \tag{1.4}$$

then for the solution $u(t)=U(t, 0)\phi$, $e^{i\lambda t}(t, x)$ is periodic in t with period 2π.

For this last reason, ϕ or $u(t)$ is called quasi-stationary state for (1.1) and λ the quasi-level of ϕ, which is determined modulo integers only.

In this note, we shall show that all quasi-stationary states with the quasi-level $\lambda\notin\boldsymbol{Z}$ decay exponentially as $|x|\to\infty$, as do eigenfunctions of time independent Schrödinger operators. More precisely, we shall prove the following

THEOREM. *Let a real valued function $V(t, x)$ satisfy asssumptions* (1.2) *and* (1.3) *and let $\phi\in L^2(\boldsymbol{R}^N)$ satisfy the equation* (1.4) *with $\lambda\notin\boldsymbol{Z}$. Then for all $0\leq\alpha^2<1-(\lambda-[\lambda])$,*

$$\exp(\alpha\langle x\rangle)\phi(x)\in L^2(\boldsymbol{R}^N)$$

where $[\lambda]$ is the greatest integer not larger than λ.

The asymptotic behavior at infinity of eigenfunctions of (time independent) Schrödinger operators has been long studied, including general N-particle systems, and it has a large body of literature. Among others we mention only Agmon's lecture note [1] where the decay of eigenfunctions is studied in detail by PDE technique using the Agmon metric, and the work by Froese-Herbst [3] which applied Mourré's commutator estimate for finding the exponential decay rate of eigenfunctions. Technically we shall show in this paper that the commutator method employed by [3] is likewise effective for studying the exponential decay of quasi-stationary states of time periodic Schrödinger equations. We should remark here that the algebraic decay of ϕ has been known for a long time and $\langle x\rangle^l\phi\in L^2(\boldsymbol{R}^N)$ for all $l\geq 0$ (Nakamura [6], Kuwabara-Yajima [5]).

2. Preliminaries

Following [7] and [8], we introduce the extended phase space (or the grand Hilbert space)

$$\mathcal{K}=L^2(\boldsymbol{T}, L^2(\boldsymbol{R}^N)), \qquad \boldsymbol{T}=\boldsymbol{R}/2\pi\boldsymbol{Z}$$

and define a strongly continuous unitary group $\{\mathcal{U}(\sigma): -\infty<\sigma<\infty\}$ on $\mathcal{K}$ by

$$\mathcal{U}(\sigma)g(t)=U(t, t-\sigma)g(t-\sigma), \qquad g\in\mathcal{K}.$$

A simple computation shows the following

LEMMA 2.1. *Let* $\phi\in L^2(\boldsymbol{R}^N)$ *satisfy* (1.4) *and set*

$$f(t)=e^{i\lambda t}U(t, 0)\phi, \qquad -\infty<t<\infty. \tag{2.1}$$

Then $f\in\mathcal{K}$ *and*

$$\mathcal{U}(\sigma)f=e^{i\lambda\sigma}f, \qquad -\infty<\sigma<\infty. \tag{2.2}$$

Let K be the generator of $\{\mathcal{U}(\sigma)\}: \mathcal{U}(\sigma)=\exp(-i\sigma K)$, $-\infty<\sigma<\infty$. Then (2.2) implies

$$Kf=\lambda f. \tag{2.3}$$

For the generator K, we have the following

LEMMA 2.2. *The generator* K *is the maximal operator of* $-i\partial/\partial t-\Delta+V(t, x)$, *that is,*

$$\begin{cases} \mathcal{D}(K)=\{u\in\mathcal{K}: -\partial u/\partial t-\Delta u+V(t, x)u\in\mathcal{K}\}, \\ Ku=-i\partial u/\partial t-\Delta u+V(t, x)u, \qquad u\in\mathcal{D}(K), \end{cases} \tag{2.4}$$

where in the RHS of (2.4), *the derivatives are taken in the sense of distributions.*

In what follows $\|\ \|$ and $(\ ,\ )$ will stand for the norm and the inner product of $L^2(\boldsymbol{R}^N)$ and those of $\mathcal{K}$ will be distinguished by putting subscript $\mathcal{K}$. We shall denote the partial Fourier transform of $u\in\mathcal{S}'(\boldsymbol{T}\times\boldsymbol{R}^N)$ with respect to t (resp. (t, x)) as $\hat{u}(n, x)$ (resp. $\hat{u}(n, \xi)$). Formally

$$\hat{u}(n, x)=(2\pi)^{-1/2}\int_0^{2\pi} e^{-int}u(t, x)dt, \qquad n=0, \pm1, \pm2, \cdots$$

and

$$\hat{u}(n, \xi)=(2\pi)^{-(n+1)/2}\int_0^{2\pi} dt\int_{\boldsymbol{R}^N} dx\, e^{-int-ix\cdot\xi}u(t, x).$$

Proof of Lemma 2.2. We let $H_0=-\Delta$ with $\mathcal{D}(H_0)=H^2(\boldsymbol{R}^N)$ and set $U_0(t)=\exp(-itH_0)$, $-\infty<t<\infty$. Define a unitary group $\{\mathcal{U}_0(\sigma), -\infty<\sigma<\infty\}$ by

$$\mathcal{U}_0(\sigma)g(t)=\exp(-i\sigma H_0)g(t-\sigma), \qquad g\in\mathcal{K} \tag{2.5}$$

Taking the Fourier transform of (2.5) yields

$$(\mathcal{U}_0(\sigma)g)^\wedge(n, \xi)=e^{-i\sigma(\xi^2+n)}\hat{g}(n, \xi) \tag{2.6}$$

and the RHS is differentiable in $\mathcal{K}$ w.r.t. σ if and only if $\sum_{n=-\infty}^{\infty}\|(\xi^2+n)\hat{g}(n, \xi)\|_{L^2}^2<\infty$. Thus the generator K_0 of $\mathcal{U}_0(\sigma)$, $\mathcal{U}_0(\sigma)=\exp(-i\sigma K_0)$, is given as

$$\begin{cases} \mathcal{D}(K_0)=\{u\in\mathcal{K}, -i\partial u/\partial t-\Delta u\in\mathcal{K}\}, \\ K_0u=-i\partial u/\partial t-\Delta u. \end{cases} \tag{2.7}$$

Since $V(t, x)$ satisfies (1.2), we have by Duhamel's formula

$$U(t, s)u_0=U_0(t-s)u_0-i\int_s^t U_0(t-r)V(r)U(r, s)u_0ds,$$

where $V(r)$ is the multiplication by $V(r, x)$. It follows that

$$\begin{aligned} (\mathcal{U}(\sigma)u)(t)&=U(t, t-\sigma)u(t-\sigma) \\ &=U_0(\sigma)u(t-\sigma)-i\int_0^\sigma U_0(\sigma-r)V(t+r-\sigma)U(t+r-\sigma, t-\sigma)u(t-\sigma)dr \\ &=(\mathcal{U}_0(\sigma)u)(t)-i\int_0^\sigma (\mathcal{U}_0(\sigma-r)V\mathcal{U}(r)u)(t)dr, \end{aligned} \tag{2.8}$$

V being the multiplication by $V(t, x)$ in $\mathcal{K}$. Taking the Laplace transform of (2.8), we obtain

$$(K-z)^{-1}=(K_0-z)^{-1}-(K_0-z)^{-1}V(K-z)^{-1}, \qquad \operatorname{Im} z\neq 0. \tag{2.9}$$

which implies

$$\mathcal{D}(K)=\mathcal{D}(K_0), \qquad K=K_0+V. \tag{2.10}$$

The statement of the lemma follows from (2.7) and (2.10). ■

3. Proof of Theorem

We first show

$$\sup\{\alpha^2+\lambda: \alpha\geq 0, e^{\alpha\langle x\rangle}f\in\mathcal{K}\}=\tau\in\boldsymbol{Z} \tag{3.1}$$

by the method of reduction ad absurdum. We suppose $\tau=\alpha_0^2+\lambda\notin\boldsymbol{Z}$. Following Froese-Herbst [2, 3], we choose $0\leq\alpha_1$, $0<\gamma$ and a function $F_s(x)$, $0<s<1$ as follows and consider the function

$$f_s(t, x)=\exp(F_s(x))f(t, x) \tag{3.2}$$

where $f(t, x)$ is defined by (2.1):
If $\alpha_0>0$, then $0<\alpha_1<\alpha_0<\alpha_1+\gamma$ and if $\alpha_0=0$, then $\alpha_1=0$ and $\gamma>0$.

$$F_s(x)=\alpha_1\langle x\rangle+\gamma\rho_s(\langle x\rangle)\,, \tag{3.3}$$

$$\rho_s(\mu)=\int_0^\mu \langle s\tau\rangle^{-2}d\tau\,. \tag{3.4}$$

We write $\nabla_x F_x(x)=x\cdot g_s(x)$. A simple estimation shows the following

LEMMA 3.1 ([2]). *F_s and $g_s(x)$ satisfy the following statements.*
(1) *For every $s>0$,*

$$0\leq F_s(x)\leq\alpha_1\langle x\rangle+C_s$$

with a constant C_s. As $s\to 0$, $F_s(x)$ is increasing for every fixed $x\in\boldsymbol{R}^n$ and

$$\lim_{s\to 0} F_s(x)=(\alpha_1+\gamma)\langle x\rangle\,.$$

(2) *There exists a constant $C>0$ independent of s and $\gamma>0$ such that*

$$|\nabla F_s(x)|+\langle x\rangle|g_s(x)|+\langle x\rangle^2|\nabla g_s(x)|\leq C\,. \tag{3.4}$$

$$|x\cdot\nabla(\nabla F_s(x))^2|\leq C\{\gamma(\alpha_1+\gamma)+(\alpha_1+\gamma)^2\langle x\rangle^{-2}\}\,. \tag{3.5}$$

$$|(x\cdot\nabla)^2 g_s(x)|\leq C(\alpha_1+\gamma)\langle x\rangle^{-1}\,. \tag{3.6}$$

$$|(\nabla F_s(x))^2-\alpha_1^2|\leq 2\gamma(\alpha_1+\gamma)+(\alpha_1+\gamma)+(\alpha_1+\gamma)^2\langle x\rangle^{-2}\,. \tag{3.7}$$

LEMMA 3.2. *Let $f_s(t, x)$ be defined by* (3.2). *Then for $s>0$ $f_s\in\mathcal{K}$ and*

$$\hat{f}(n, x),\ \hat{f}_s(n, x)\in H^2(\boldsymbol{R}^N)\,,\qquad n=0, \pm 1, \pm 2, \cdots. \tag{3.8}$$

Proof: Since $F_s(x)$ is C^∞ with bounded derivatives, we have in $\mathcal{S}'(\boldsymbol{R}^{N+1})$,

$$-i\partial f_s/\partial t-\Delta_x f_s+V(t, x)f_s-(\nabla_x F_s)^2 f_s+\nabla_x(\nabla_x F_s(x)f_s)+\nabla_x F_s(x)\cdot\nabla_x f_s=\lambda f_s\,. \tag{3.9}$$

Taking the partial Fourier transform with respect to t, we find

$$\begin{aligned} n\hat{f}_s(n, x)-\Delta_x\hat{f}_s(n, x)+(Vf_s)^\wedge(n, x)-(\nabla_x F_s)^2(x)\hat{f}_s(n, x)&\\ +\nabla_x(\nabla_x F_s(x)\hat{f}_s(n, x))+\nabla_x F_s(x)\cdot\nabla_x\hat{f}_s(n, x)&=\lambda\hat{f}_s(n, x)\,, \end{aligned} \tag{3.10}$$

$n=0, \pm 1, \cdots$. Since $\hat{f}_s(n, x)\in L^2(\boldsymbol{R}^N)$, (3.10) and a simple elliptic estimate yield $\hat{f}_s(n, x)\in H^2(\boldsymbol{R}^N)$. The proof for $f(t, x)$ is similar. ■

We now use the following key identity ([3]): If $A=1/2(x\cdot\nabla+\nabla\cdot x)$ and ξ_s and G_s are multiplication operators by $\xi_s(x)=\exp F_s(x)$ and $G_s(x)=(x\cdot\nabla)^2 g_s(x)-x\cdot\nabla((\nabla F_s(x))^2)$, respectively, then for $\phi\in C_0^\infty(\boldsymbol{R}^N)$

(3.11) $$2Re(\xi_s A\xi_s\phi, \Delta\phi)=(\xi_s\phi, [A, -\Delta]\xi_s\phi)-4\|g_s^{1/2}A\xi_s\phi\|^2+(\xi_s\phi, G_s\xi_s\phi).$$

The identity (3.11) is a result of simple manipulations.

LEMMA 3.3. $\nabla_x f_s$ and $\sqrt{g_s}Af_s\in\mathcal{K}$ and

(3.12) $$\|\nabla_x f_s(t, x)\|_{\mathcal{K}}^2+\|\sqrt{g_s}Af_s(t, x)\|_{\mathcal{K}}^2$$
$$\leq\|G_s(x)f_s(t, x)\|_{\mathcal{K}}\|f_s(t, x)\|_{\mathcal{K}}+\|\langle x\rangle Vf_s\|_{\mathcal{K}}(\|\langle x\rangle Vf_s\|_{\mathcal{K}}+N\|\langle x\rangle^{-1}f_s\|_{\mathcal{K}}).$$

Proof: Since $\xi_s A\xi_s$, $-\Delta$ and $[A, -\Delta]=2\Delta$ are differential operators of order ≤ 2 with smooth coefficients, we easily see that (3.11) remains valid for $\phi\in H^2(\boldsymbol{R}^N)$ with compact support. Take $\chi\in C_0^\infty(\boldsymbol{R}^N)$ such that $\chi(x)=1$ for $|x|\leq 1$ and $\chi(x)=0$ for $|x|\geq 2$ and let $\chi_m(x)=\chi(x/m)$, $m=1, 2, \cdots$. Set $\phi(x)=\chi_m(x)\hat{f}(n, x)$ in (3.11) and add $-2Re(\xi_s A\xi_s\chi_m\hat{f}(n, x), (V\chi_m f)^\wedge(n, x))$ to both sides of the resulting equation. Since $\hat{f}(n, x)\in H^2(\boldsymbol{R}^N)$, we have in the LHS,

(3.13) $$-2Re(\xi_s A\xi_s\chi_m\hat{f}(n, x), (n-\Delta)\chi_m\hat{f}(n, x)+(V\chi_m f)^\wedge(n, x))$$
$$=-2Re(\xi_s A\xi_s\chi_m\hat{f}(n, x), \chi_m(Kf)^\wedge(n, x)-(\Delta\chi_m)\hat{f}(n, x)-2(\nabla\chi_m)\cdot\nabla\hat{f}(n, x))$$
$$=-2Re(\xi_s A\xi_s\chi_m\hat{f}(n, x), -(\Delta\chi_m)(x)\hat{f}(n, x)-2(\nabla\chi_m)(x)\cdot\nabla\hat{f}(n, x)).$$

Here we used the characterization (2.4) of K in the first step, and in the second (2.3) and the skew-symmetry of $\xi_s A\xi_s$. Note that the multiplications by $\langle x\rangle\nabla\chi_m(x)$ and $\langle x\rangle^2(\Delta\chi_m)(x)$ are vanishing as $m\to\infty$ in the strong topology of bounded operators in $L^2(\boldsymbol{R}^N)$. Hence

$$\lim_{m\to\infty}(\xi_s A\xi_s\chi_m\hat{f}(n, x), -(\Delta\chi_m)(x)\hat{f}(n, x)-2(\nabla\chi_m)(x)\cdot\nabla\hat{f}(n, x))$$
$$=\lim_{m\to\infty}(\langle x\rangle^{-1}A\chi_m\hat{f}_s(n, x), -\langle x\rangle\{(\Delta\chi_m)(x)-2\nabla\chi_m(x)\cdot\nabla F_s(x)\}\hat{f}_s(n, x)$$
$$-2\langle x\rangle(\nabla\chi_m)(x)\cdot\nabla\hat{f}_s(n, x))=0.$$

On the other hand, we have in the RHS

$$2(\chi_m\hat{f}_s(n, x), \Delta_x\chi_m\hat{f}_s(n, x))-4\|\sqrt{g_s}A\chi_m\hat{f}_s(n, x)\|^2$$
$$+(\chi_m\hat{f}_s(n, x), G_s\chi_m\hat{f}_s(n, x))-2Re(A\chi_m\hat{f}_s(n, x), (V\chi_m f_s)^\wedge(n, x))$$
$$\leq-2\|\chi_m\nabla\hat{f}_s(n, x)\|^2-2\|\chi_m\sqrt{g_s}A\hat{f}_s(n, x)\|^2+\|G_s(x)\hat{f}_s(n, x)\|\,\|\hat{f}_s(n, x)\|$$
$$+(2\|\chi_m\nabla\hat{f}_s(n, x)\|+N\|\langle x\rangle^{-1}\hat{f}_s(n, x)\|)\|\langle x\rangle(V\chi_m f_s)^\wedge(n, x)\|$$
$$+4\|(\nabla\chi_m)\hat{f}_s(n, x)\|\,\|\nabla\hat{f}_s(n, x)\|+4\|\sqrt{g_s}[A, \chi_m]\hat{f}_s(n, x)\|^2-2\|(\nabla\chi_m)\hat{f}_s(n, x)\|^2$$
$$+2\|[A, \chi_m]\hat{f}_s(n, x)\|\,\|(V\chi_m f_s)^\wedge(n, x)\|,$$

where we used the obvious estimate $\|\chi_m\langle x\rangle^{-1}A\hat{f}_s(n, x)\|\leq\|\chi_m\nabla\hat{f}_s(n, x)\|+(N/2)\|\langle x\rangle^{-1}\hat{f}_s(n, x)\|$. Using the fact that the multiplications by $\nabla\chi_m$, $\sqrt{g_s}[A, \chi_m]$ and $[A, \chi_m]$ are vanishing strongly in $L^2(\boldsymbol{R}^N)$ as $m\to\infty$, we see, therefore,

$$2\varlimsup_{m\to\infty}(||\chi_m\nabla\hat{f}_s(n,x)||^2+||\chi_m\sqrt{g_s}A\hat{f}_s(n,x)||^2)$$
$$\le||G_s(x)\hat{f}_s(n,x)||\,||\hat{f}_s(n,x)||$$
$$+(2||\nabla\hat{f}_s(n,x)||+N||\langle x\rangle^{-1}\hat{f}_s(n,x)||)||\langle x\rangle(Vf_s)^\wedge(n,x)||\,.$$

Thus the monotone convergence theorem implies $\sqrt{g_s}A\hat{f}_s(n,x)\in L^2(\boldsymbol{R}^N)$ and

$$||\nabla\hat{f}_s(n,x)||^2+2||\sqrt{g_s}A\hat{f}_s(n,x)||^2$$
$$\le||G_s(x)\hat{f}_s(n,x)||\,||\hat{f}_s(n,x)||$$
$$+||\langle x\rangle(Vf_s)^\wedge(n,x)||(||\langle x\rangle(Vf_s)^\wedge(n,x)||+N||\langle x\rangle^{-1}\hat{f}_s(n,x)||)\,.$$

Summing up both sides for $n\in\boldsymbol{Z}$ and using Plancherel identity, we obtain (3.12). ■

Note that the statement and proof of Lemma 3.2 and 3.3 remains valid for any F in place of F_s as long as $F(x)$ is rotationary symmetric, $||\nabla^k F(x)||\le C_k\langle x\rangle^{1-k}$, $k=1,2$, and $\exp(F(x))f(t,x)\in\mathcal{K}$. A similar and simpler proof yields the following

Corollary 3.4. *$\nabla_x f(t,x)\in\mathcal{K}$ and*

$$||\nabla_x f||^2_{\mathcal{K}}\le 2(N+||\langle x\rangle V||^2_\infty)||f||^2_{\mathcal{K}} \tag{3.14}$$

We set

$$h_s(t,x)=f_s(t,x)||f_s||^{-1}_{\mathcal{K}}\,. \tag{3.15}$$

Lemma 3.5. *If $p\in L^\infty(\boldsymbol{R}^N)$ and $p(x)\to 0$ as $|x|\to\infty$, then,*

$$\lim_{s\to 0}||ph_s||_{\mathcal{K}}=\lim_{s\to 0}||p\nabla_x h_s||_{\mathcal{K}}=\lim_{s\to 0}||p\sqrt{g_s}Ah_s||_{\mathcal{K}}=0\,. \tag{3.16}$$

Proof: As $s\to 0$, $F_s(x)\to(\alpha_1+\gamma)\langle x\rangle$ increasingly and $\nabla_x\cdot F_s(x)\to(\alpha_1+\gamma)x\cdot\langle x\rangle^{-1}$ boundedly. It follows by the assumption $\alpha_1+\gamma>\alpha_0$ that

$$||f_s(t,x)||_{\mathcal{K}}\to\infty\qquad(s\to\infty) \tag{3.17}$$

and by (3.14) and (3.17) that for compact $\Omega\subset\boldsymbol{T}\times\boldsymbol{R}^N$

$$\lim_{s\to 0}\int_\Omega|h_s(t,x)|^2dtdx=\lim_{s\to 0}\int_\Omega|\nabla h_s(t,x)|^2dtdx=0\,. \tag{3.18}$$

Since $p(x)\to 0$ as $|x|\to\infty$, (3.16) follows from (3.12) and (3.18). ■

Corollary 3.6. *As $s\to 0$*

$$||\nabla_x(\nabla_x F_s(x)\cdot h_s(t,x))+\nabla_x F_x(x)\cdot\nabla_x h_s(t,x)||_{\mathcal{K}}\to 0\,, \tag{3.19}$$

$$\|V(t,x)h_s(t,x)\|_{\mathcal{X}}+\|\langle x\rangle V(t,x)h_s(t,x)\|_{\mathcal{X}}\to 0\,. \tag{3.20}$$

Proof: Since $\nabla F_s(x)=x\cdot g_s(x)$, we have

$$\nabla_x(\nabla_x F_s\cdot h_s)+\nabla_x F_s\cdot\nabla_x h_s=2g_s\cdot Ah_s+(x\cdot\nabla_x g_s(x))h_s\,. \tag{3.21}$$

Lemma 3.5, (3.4) and (3.21) imply (3.19). (3.20) directly follows from (3.16). ■

LEMMA 3.7. *Let* $h_s(t,x)$ *be defined by* (3.15), *then*

$$\overline{\lim_{s\to 0}}\,\|-i\partial h_s/\partial t-\Delta h_s-(\alpha_1^2+\lambda)h_s\|_{\mathcal{X}}^2\le 2\alpha_1\gamma+\gamma^2\,. \tag{3.22}$$

Proof: Since $\nabla F_s(x)=(\alpha_1+\gamma\rho_s'(\langle x\rangle))x/\langle x\rangle$, we have

$$|(\nabla F_s)(x)^2-\alpha_1^2|\le\alpha_1^2\langle x\rangle^{-2}+2\alpha_1\gamma+\gamma^2\,.$$

Hence by (3.16),

$$\overline{\lim_{s\to 0}}\,\|((\nabla F_s)(x)^2-\alpha_1^2)h_s(t,x)\|_{\mathcal{X}}\le 2\alpha_1\gamma+\gamma^2\,.$$

Now we divide both sides of (3.9) by $\|f_s\|_{\mathcal{X}}$ and let $s\to 0$. Applying (3.19) and (3.20), we obtain (3.22). ■

LEMMA 3.8. *It follows that*

$$\overline{\lim_{s\to 0}}\,\|\nabla_x h_s(t,x)\|_{\mathcal{X}}^2\le C\gamma(\alpha_1+\gamma)\,. \tag{3.23}$$

Proof: We devide both sides of (3.12) by $\|f_s(t,x)\|_{\mathcal{X}}^2$ and take the limit $s\to 0$. Then by (3.20)

$$\overline{\lim_{s\to 0}}\,\|\nabla_x h_s(t,x)\|_{\mathcal{X}}^2\le\overline{\lim_{s\to 0}}\,\|G_s(x)h_s(t,x)\|_{\mathcal{X}}\,. \tag{3.24}$$

Remembering $G_s(x)=(x\cdot\nabla_x)^2 g_s(x)-x\cdot\nabla_x(\nabla_x F_s(x))^2$, we apply estimates (3.5) and (3.6) and Lemma 3.5 to the RHS of (3.24) to obtain (3.23). ■

Completion of the proof of (3.1): We have (3.22) and (3.23) with the constant C independent of γ and α_1 which satisfy the condition below (3.2). We shall show, however, that (3.22) and (3.23) contradict each other for small $\gamma>0$, if $\alpha_0^2+\lambda\notin\boldsymbol{Z}$. Let $\delta>0$. Then (3.23) implies

$$C\gamma(\alpha_1+\gamma)\ge\overline{\lim_{s\to 0}}\sum_{-\infty}^{\infty}\int_{|\xi|^2>\delta}|\xi|^2|\hat{h}_s(n,\xi)|^2d\xi\,,$$

hence

$$\underline{\lim_{s\to 0}}\sum_{-\infty}^{\infty}\int_{|\xi|^2\le\delta}|\hat{h}_s(n,\xi)|^2d\xi\ge 1-C\gamma(\alpha_1+\gamma)\delta^{-1}\,. \tag{3.25}$$

Now take $\alpha_1, \alpha_0<\alpha_1$, so close to α_0 that

$$\operatorname{dist}(\alpha_1^2+\lambda, \boldsymbol{Z})=\delta_0>0$$

and choose and fix $\delta<\delta_0/2$. (If $\alpha_0=0$, take $\alpha_1=\alpha_0$.) Then by (3.25)

$$\begin{aligned}
&\underline{\lim}_{s\to 0} \|-i\partial h_s/\partial t-\Delta h_s-(\alpha_1^2+\lambda)h_s\|_{\mathcal{K}}^2 \\
&=\underline{\lim}_{s\to 0} \sum_{-\infty}^{\infty} \|(\xi^2-(\alpha_1^2+\lambda-n))\hat{h}_s(n,\xi)\|^2 \\
&\geq \underline{\lim}_{s\to 0} \sum_{-\infty}^{\infty} \int_{|\xi|^2\leq\delta} |\xi^2-(\alpha_1^2+\lambda-n)|^2|\hat{h}_s(n,\xi)|^2 d\xi \\
&\geq (\delta_0^2/4)(1-C\gamma(\alpha_1+\gamma)\delta^{-1}).
\end{aligned}$$

This clearly contradicts with (3.22) for small γ and this proves (3.1).

Completion of the proof of the theorem: By (3.1), we have $e^{\alpha\langle x\rangle}f(t,x)\in\mathcal{K}$ for all $\alpha\geq 0$ such that $\alpha^2<[\lambda+1]-\lambda$. We need show that $e^{\alpha\langle x\rangle}f(t,x)\in L^2(\boldsymbol{R}^N)$ for every $t\in\boldsymbol{T}$. We write $F(x)=\alpha\langle x\rangle$ and $f_\alpha(t,x)=e^{\alpha\langle x\rangle}f(t,x)$. We first remark, as was noted below the proof of Lemma 3.3, that Lemma 3.2, 3.3 and estimates in their proof remain valid with $F(x)$, $g(x)=\alpha\langle x\rangle^{-1}$ and $f_\alpha(t,x)$ replacing $F_s(x)$, $g_s(x)$ and $f_s(t,x)$, respectively. In particular,

$$\nabla f_\alpha\in\mathcal{K}, \qquad \sqrt{g}\,A f_\alpha\in\mathcal{K}, \qquad \hat{f}_\alpha(n,x)\in H^2(\boldsymbol{R}^N), \tag{3.26}$$

$$-i\partial f_\alpha/\partial t-\Delta_x f_\alpha+Vf_\alpha-(\nabla F)^2 f_\alpha+\nabla_x(\nabla_x F(x) f_\alpha)+\nabla F\cdot\nabla_x f_\alpha=\lambda f_\alpha. \tag{3.27}$$

We take $\sigma(t)\in C_0^\infty(\boldsymbol{R}^1)$ and $\eta(x)\in C_0^\infty(\boldsymbol{R}^N)$ such that $\sigma(t)\geq 0$, $\eta(x)\geq 0$ and

$$\int\sigma(t)dt=\int\eta(x)dx=1, \tag{3.28}$$

and set, for $\varepsilon>0$

$$l_\varepsilon(t,x)=\int\sigma(s)\eta(y)l(t-\varepsilon s, x-\varepsilon y)dsdy \tag{3.29}$$

for $l\in\mathcal{S}'(\boldsymbol{T}\times\boldsymbol{R}^N)$. Multiplying $\chi_m(x)=\chi(x/m)$ to both sides of (3.27) and applying the operation (3.29) to the resulting equation, we have

$$\begin{aligned}
&-i\partial(\chi_m f_\alpha)_\varepsilon/\partial t-\Delta(\chi_m f_\alpha)_\varepsilon+([\Delta,\chi_m]f_\alpha)_\varepsilon+(\chi_m Vf_\alpha)_\varepsilon \\
&-(\chi_m(\nabla F)^2 f_\alpha)_\varepsilon+(\chi_m\nabla_x(\nabla_x F\cdot f_\alpha))_\varepsilon+(\chi_m\nabla F\cdot\nabla_x f_\alpha)_\varepsilon-\lambda(\chi_m f_\alpha)_\varepsilon=0.
\end{aligned} \tag{3.30}$$

We note all terms in (3.30) belong to $C_0^\infty(\boldsymbol{T}\times\boldsymbol{R}^N)$. Take inner product in $L^2(\boldsymbol{R}^N)$ of (3.30) and $(\chi_m f_\alpha)_\varepsilon$ and then take the imaginary parts in the resulting equation. We obtain

$$(3.31)\qquad -(1/2)\partial(\|(\chi_m f_\alpha)_\varepsilon(t)\|^2)/\partial t$$
$$=\mathrm{Im}\,\{(-([\Delta, \chi_m]f_\alpha)_\varepsilon-(\chi_m Vf_\alpha)_\varepsilon+(\chi_m(\nabla F)^2 f_\alpha)_\varepsilon$$
$$-(\chi_m\nabla_x(\nabla F\cdot f_\alpha))_\varepsilon-(\chi_m\nabla F\cdot\nabla_x f_\alpha)_\varepsilon, (\chi_m f_\alpha)_\varepsilon)\}\,.$$

Integrating (3.31) by t, we see

$$\|(\chi_m f_\alpha)_\varepsilon(t_0)\|^2-\|(\chi_m f_\alpha)_\varepsilon(t_1)\|^2$$
$$=2\,\mathrm{Im}\int_{t_0}^{t_1}((-2\nabla\chi_m\cdot\nabla f_\alpha-\Delta\chi_m\cdot f_\alpha-\chi_m Vf_\alpha+\chi_m(\nabla F)^2 f_\alpha$$
$$-2\chi_m gAf_\alpha-\chi_m(x\cdot\nabla g)f_\alpha)_\varepsilon, (\chi_m f_\alpha)_\varepsilon)dt\,.$$

Choose now t_0 so that $f_\alpha(t_0, x)\in L^2(\boldsymbol{R}^N)$ and let $\varepsilon\to 0$ and $m\to\infty$. Using (3.26), we see

$$\lim_{m\to\infty}\lim_{\varepsilon\to 0}\|\chi_m(f_\alpha)_\varepsilon(t_1)\|^2=\|f_\alpha(t_0)\|^2+4\,\mathrm{Im}\int_{t_0}^{t_1}(gAf_\alpha, f_\alpha)dt<\infty\,.$$

Thus $f_\alpha(t_1, x)\in L^2(\boldsymbol{R}^N)$ for every t_1 and

$$\|f_\alpha(t_1)\|^2=\|f_\alpha(t_0)\|^2+4\,\mathrm{Im}\int_{t_0}^{t_1}(gAf_\alpha, f_\alpha)dt\,.$$

This concludes the proof of the theorem.

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