On Pairs of Groups Having a Common 2-Subgroup of Odd Indices, II

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(Received February 21, 1990)

Abstract

We conclude our study of finite groups G and H with a common 2-subgroup S such that $|G:S|$ and $|H:S|$ are powers of odd primes q and r, respectively, and Sylow q-subgroups of G and Sylow r-subgroups of H are cyclic and nontrivial. The main objective is to obtain generators and relations of these groups in certain important cases.

Amalgams

In the first half [1] of this paper, we began to study the structure of finite groups G, H, and S satisfying the following conditions:

(A) S is a 2-subgroup both of G and of H.

(B) $|G:S|$ and $|H:S|$ are powers of odd primes q and r, respectively, and Sylow q-subgroups of G and Sylow r-subgroups of H are cyclic and nontrivial.

(C) No nonidentity subgroup of S is normal both in G and in H.

(D) $C_G(O_2(G))\leqq O_2(G)$ and $C_H(O_2(H))\leqq O_2(H)$.

We shall call the triple (G, S, H) an *amalgam*, identifying it with the amalgamated product of G and H over S. The main result of [1] may be stated as follows in the terminology which we shall define later.

THEOREM. *Let G, H, and S be finite groups satisfying the conditions (A), (B), (C), and (D). Assume without loss that $\Omega_1(Z(S))\nleqq Z(G)$. Then the amalgam (G, S, H) is a $GL_3(2)$-amalgam, an $Sp_4(2)$-amalgam, a $G_2(2)'$-amalgam, a $G_2(2)$-*

amalgam, an M_{12}-amalgam, an $Aut(M_{12})$-amalgam, a ${}^2F_4(2)'$-amalgam, or a ${}^2F_4(2)$-amalgam.

The purpose of this half of the paper is to discuss in further detail the structure of the groups G, H, and S in the above theorem. In particular, we shall describe the $G_2(2)'$-amalgams, $G_2(2)$-amalgams, M_{12}-amalgams, and $\mathrm{Aut}(M_{12})$-amalgams in terms of generators and relations, thereby establishing the uniqueness of each of the amalgams up to isomorphism. We shall briefly touch on the remaining four types of amalgams, for generators and relations of $\mathrm{GL}_3(2)$-amalgams and of $\mathrm{Sp}_4(2)$-amalgams, while easily obtained, do not seem to be needed in applying the above theorem, and Tanaka [2] has already studied ${}^2F_4(2)'$-amalgams and ${}^2F_4(2)$-amalgams for the same purpose. We remark that Tanaka [3] has recently generalized the above theorem and that all our results will have an important application to the study of simple finite groups [4].

From now on G, H, and S are finite groups satisfying (A), (B), (C), and (D),

$$\begin{aligned}&Q=O_2(G), \quad R=O_2(H),\\&S_*=[Q, O^2(G)][R, O^2(H)],\\&G_*=S_*O^2(G), \quad H_*=S_*O^2(H),\\&Q_*=O_2(G_*) \text{ and } R_*=O_2(H_*).\end{aligned}$$

By 3.3 of [1] the groups G_*, H_*, and S_* satisfy (A), (B), (C), and (D) with G_*, H_*, and S_* in the roles of G, H, and S, respectively. We now begin our discussion of the eight amalgams.

1. *$GL_3(2)$-amalgams* The amalgam (G, S, H) is called a $\mathrm{GL}_3(2)$-amalgam if $G/Q\cong H/R\cong D_6$ and $Q\cong R\cong E_4$.

In this case

$$G\cong H\cong S_4$$

and hence

$$S\cong D_8.$$

We leave the proof to the reader.

2. *$Sp_4(2)$-amalgams* The amalgam (G, S, H) is called an $\mathrm{Sp}_4(2)$-amalgam if $G/Q\cong H/R\cong D_6$ and $Q\cong R\cong E_8$.

In this case

$$G\cong H\cong S_4\times Z_2$$

by 3.9 of [1] and hence

$$S\cong D_8\times Z_2.$$

3. *$G_2(2)'$-amalgams* The amalgam (G, S, H) is called a $G_2(2)'$-amalgam if $G/Q \cong H/R \cong D_6$, $Q \cong Z_4 \times Z_4$, and $R \cong Z_4 * D_8$.

In this case S can be presented on three generators a, b, and c with the relations $a^4=b^4=c^2=1$, $ab=ba$, $a^c=a^{-1}b$, and $bc=cb$ as a set of defining relations.

A presentation of G is obtained by adjoining to the presentation of S one generator g and the relations $g^3=1$, $g^c=g^{-1}$, $a^g=a^{-1}b$, and $b^g=a^{-1}$.

A presentation of H is obtaind by adjoining to the presentation of S one generator h and the relations $h^3=1$, $h^{ac}=h^{-1}$, $b^h=b$, and $(a^2)^h=c$. (We shall refer to these presentations and generators as the standard presentations and standard generators, respectively.)

Proof. As $Q \neq R = \Omega_1(R)$, there is an involution $c \in R-Q$. Thus

$$S=\langle Q, c\rangle$$

and

$$c^2=1. \tag{3.1}$$

By the Baer-Suzuki theorem, there is an element $g \in G$ such that

$$g^3=1, \qquad g^c=g^{-1} \tag{3.2}$$

and

$$G=\langle S, g\rangle.$$

As $Z(R) \leqq Q$ by 4.1 of [5] and Q is abelian, $Z(R)$ centralizes $\langle Q, c\rangle = S$ and so, as $H=\langle S^H\rangle$,

$$Z(R) \leq Z(H).$$

Because of the condition (C), this gives $Z(R^g) \cap Z(R)=1$ as $G=\langle S^g, S\rangle$. Thus $Q=Z(R^g) \times Z(R)$. Let

$$\langle b\rangle = Z(R),$$

and $a=b^{-g}$. Then

$$S=\langle a, b, c\rangle,$$

and the following relations are satisfied:

$$a^4=b^4=1, \qquad ab=ba, \qquad bc=cb, \tag{3.3}$$

$$b^g=a^{-1}. \tag{3.4}$$

The structure of Q and the condition (D) show $C_Q(g)=1$ and hence $1=bb^g b^{g^2}=ba^{-1}a^{-g}$. Therefore

$$a^g=a^{-1}b. \tag{3.5}$$

We have $a^c=b^{-gc}=b^{-cgc}=b^{-g^2}=a^g$. Hence

(3.6) $$a^c=a^{-1}b\,.$$

The relations (3.1), (3.3), and (3.6) clearly show $|\langle a, b, c\rangle|\leqq 2^5$, while $|S|=2^5$. Thus these relations form a set of defining relations of S with respect to the generators a, b, and c. Similarly we see that the relations (3.1)—(3.6) form a set of defining relations of G. From (3.1) and (3.6), we have $(ac)^2=b\in Z(H)$, so by the Baer-Suzuki theorem, there is an element $h\in H$ such that

$$H=\langle S, h\rangle\,,$$

(3.7) $$h^3=1\,,$$

and $h^{ac}\equiv h^{-1} \bmod \langle b\rangle$. Since $b\in Z(H)$, we have

(3.8) $$b^h=b$$

and $\langle b, h\rangle=\langle b\rangle\times\langle h\rangle$, so

(3.9) $$h^{ac}=h^{-1}\,.$$

The condition (C) shows that Q^2, $(Q^2)^h$, and $(Q^2)^{h^{-1}}$ are all distinct. Further they intersect in $\langle b^2\rangle$ by (3.8). Hence, counting involutions in R, we see that involutions in R are contained in $Q^2\cup(Q^2)^h\cup(Q^2)^{h^{-1}}$. Hence, replacing h by h^{-1} if necessary, we may assume $c\in(Q^2)^h-\langle b\rangle$ and so $c^{h^{-1}}=a^2$ or a^2b^2. Accordingly $(a^2)^h=c$ or b^2c. We see that if we replace c and g by b^2c and b^2g, respectively, then (3.1)—(3.9) still hold. Moreover $S=\langle a, b, b^2c\rangle$ and $G=\langle S, b^2g\rangle$. Thus we may assume

(3.10) $$(a^2)^h=c\,.$$

Using (3.1), (3.3), (3.6), and (3.7)—(3.10), we further get $c^a=a^{-1}a^cc=a^2bc$ and $c^h=bc(a^2b^{-1}c)^h=bc(a^2bc)^{ch}=bcc^{ach}=bcc^{h^{-1}ac}=bc(a^2)^{ac}=a^2bc$. These relations together give $|\langle a, b, c, h\rangle|\leq 2^5 3$ and so (3.1), (3.3), (3.6), and (3.7)—(3.10) from a set of defining relations of H.

4. *$G_2(2)$-amalgams* The amalgam (G, S, H) is called a $G_2(2)$-amalgam if $G/Q\cong H/R\cong D_6$, $Q\cong D_8\sharp D_8=\langle\alpha, \beta, \gamma\,;\alpha^4=\beta^4=\gamma^2=1,\ \alpha\beta=\beta\alpha,\ \alpha^\gamma=\alpha^{-1},\ \beta^\gamma=\beta^{-1}\rangle$, and (G_*, S_*, H_*) is a $G_2(2)'$-amalgam.

In this case, if a, b, c, g, and h are the standard generators of S_*, G_*, and H_* obtained in (3), then presentations of S, G, and H are obtained by adjoining to the standard presentations of S_*, G_*, and H_* one common generator d, four common relations $d^2=1$, $a^d=a^{-1}$, $b^d=b^{-1}$, and $c^d=c$, the relation $d^g=d$ for G, and the relation $d^h=bcd$ for H.

Proof. Embed G, H, and S into the amalgamated product of G and H over S, and define

$$D=Q^{hg}\cap R^g\cap Q\cap R\cap Q^h.$$

Then as $|S:Q|=|S:R|=2$, we have $|D|\geqq 2$. From (3) we get $Q_*\cap R_*\cap Q_*^h=\langle b\rangle$ and so

$$Q_*^{hg}\cap R_*^g\cap Q_*\cap R_*\cap Q_*^h=(Q_*\cap R_*\cap Q_*^h)\cap(Q_*\cap R_*\cap Q_*^h)^g=1.$$

Hence, noticing $Q\cap S_*=Q_*$ and $R\cap S_*=R_*$, we have

$$\begin{aligned}D\cap S_*&=D\cap Q_*\cap R_*=D\cap S_*^g\cap Q_*\cap R_*\cap S_*^h\\&=D\cap R_*^g\cap Q_*\cap R_*\cap Q_*^h=D\cap S_*^{hg}\cap R_*^g\cap Q_*\cap R_*\cap Q_*^h\\&=D\cap Q_*^{hg}\cap R_*^g\cap Q_*\cap R_*\cap Q_*^h=1.\end{aligned}$$

As $|S:S_*|=2$, we conclude that $S=DS_*$ and $|D|=2$. Hence if we set $\langle d\rangle=D$, then

$$S=\langle d, S_*\rangle \quad\text{and}\quad d^2=1.$$

Consequently

$$G=\langle d, G_*\rangle \quad\text{and}\quad H=\langle d, H_*\rangle.$$

The structure of Q shows that Q_* is the unique abelian maximal subgroup of Q and that $Q-Q_*$ consists of involutions. Hence we have

$$a^d=a^{-1}, \qquad b^d=b^{-1}.$$

Further we see that $Q_*^2=Z(Q)$. Since $c=(a^2)^h\in(Q_*^2)^h$, we have

$$c^d=c.$$

Set $s=cg$. Then $s^2=1$, $R^g=R^s$, and $Q^{hg}=Q^{hs}$ as $c\in H\cap G^h$. This gives $D^s=D$ and so $d^s=d$. Since $d^c=d$, we conclude that

$$d^g=d.$$

Set $t=ach$. Then $t^2=b\in Q$ and $Q^h=Q^t$, so $t\in N_H(Q\cap R\cap Q^h)$. Now $|Q\cap R\cap Q^h|=8$ by 3.6 of [5] and so $Q\cap R\cap Q^h=\langle b, d\rangle$. Since $b^t=b$, we conclude that

$$d^t\equiv d \bmod \langle b\rangle.$$

In (3) we have shown $c^h=a^2bc$. Using this, we have $(a^2cd)^h\equiv a^2d^h\equiv a^2(d^{ca^{-1}})^t\equiv a^2(a^2d)^t\equiv a^2(a^2)^t d^t\equiv a^2cd \bmod \langle b\rangle$. This gives $(a^2cd)^h=a^2cd$ as $b^h=b$ and $h^3=1$. Hence it follows that

$$d^h=bcd.$$

We can now complete the proof by order considerations as in (3).

5. *M_{12}-amalgams* The amalgam (G, S, H) is called an M_{12}-amalgam if $G/Q\cong$

$H/R \cong D_6$, $Q \cong D_8 \sharp D_8$, $R \cong D_8 * D_8$ and $\langle Z(Q)^H \rangle$ is an abelian group (hence it is a 2-group contained in R).

In this case S can be presented on six generators a, b, c, u, v, and z with the following relations as a set of defining relations: $a^2=b^2=\cdots=z^2=1$, $[a,c]=v$, $[a,b]=cuvz$, $[u,a]=vz$, $[u,c]=z$, $[v,b]=z$ and $[x,y]=1$ for any other combinations x, y of elements of $\{a,b,\cdots,z\}$.

A presentation of G is obtained by adjoining to the presentation of S one generator g and the relations $g^3=1$, $g^b=g^{-1}$, $a^g=acuvz$, $c^g=c$, $u^g=a$, $v^g=vz$, and $z^g=v$.

A presentation of H is obtained by adjoining to the presentation of S one generator h and the relations $h^3=1$, $h^a=h^{-1}$, $b^h=bcz$, $c^h=b$, $u^h=uvz$, $v^h=u$, and $z^h=z$. (We shall refer to these presentations and generators as the standard presentations and standard generators, respectively.)

Proof. As $R \neq Q=\Omega_1(\Omega)$, there is an involution $a \in Q-R$. By the Baer-Suzuki theorem, there is an element $h \in H$ such that

$$h^3=1, \quad h^a=h^{-1},$$

and $H=\langle S,h \rangle$. Similarly there is an involution $b \in R-Q$, and there is an element $g \in G$ such that

$$g^3=1, \quad g^b=g^{-1},$$

and $G=\langle S,g \rangle$.

Before introducing further generators, we investigate the subgroup lattice of S. Set $Z=Z(S)$, $V=Z(Q)$, $U=\langle V^G \rangle$, and $W=\cap R^G$. The condition (D) shows $Z \leq V \cap Z(R)$ and the structures of Q and R show $V \cong E_4$ and $Z(R) \cong Z_2$, respectively. Hence $Z=Z(R)$ and U is an elementary abelian subgroup of R. The condition (C) forces $Z \neq Z^g$ and $V \neq V^h$. Hence $V=Z^g \times Z$ and, as $|U|=8$ by the structure of R, $U=VV^h$. Further $U \leqq Q \cap R$ by 3.8 of [5]. Since $Z=Z(R)$, $R^2=Z \leqq V \leqq R^g \cap R$ and similarly $(R^g)^2=Z^g \leqq R^g \cap R$. As $G=\langle R^g, R \rangle$ by 3.6 of [5], this gives $R^g \cap R \lhd G$, so $W=R^g \cap R=R^g \cap Q \cap R$. Consequently $|W|=8$ by 3.6 of [5]. Further $W^2 \leqq Z^g \cap Z=1$. Thus both U and W are elementary abelian subgroups of $Q \cap R$ of order 8. However $U \neq W$ by the condition (C), so $UW=Q \cap R$ and $U \cap W=V$. The condition (C) shows that U, U^g, $U^{g^{-1}}$ and W are all distinct and intersect in V. Hence, counting involutions in Q, we see that involutions in Q are contained in $U \cup U^g \cup U^{g^{-1}} \cup W$. Thus, replacing g by g^{-1}, if necessary, we may assume $a \in U^g$. Similarly we may assume $b \in W^h$. Finally as $C_V(g)=1$ by 3.8 of [5], we have $W=V \times C_W(g)$.

Let $\langle c \rangle=C_W(g)$, $\langle z \rangle=Z$, $\langle v \rangle=Z^g$, and $\langle u \rangle=Z^{gh}$. Then $S=\langle a,R \rangle=\langle a,b,Q \cap R \rangle=\langle a,b,c,U \rangle=\langle a,b,c,u,V \rangle=\langle a,b,c,u,v,z \rangle$. We now compute $[x,y]$ for the combinations x, y of elements of $\{a,b,\cdots,z\}$ except a, b. First $[U,W] \leqq [R,R]=Z$, but $[U,W] \neq 1$ as $Q \cap R$ is nonabelian by the structure of R. Thus $[U,W]=Z$. Hence $[U^g,W]=Z^g$ and $[U,W^h]=Z$. Now $U=\langle u,V \rangle$, $U^g=\langle a,V \rangle$, $W=\langle c,V \rangle$, and $W^h=\langle b,V^h \rangle$. Thus we conclude that

$$[u, c]=z, \quad [a, c]=v, \quad \text{and} \quad [v, b]=z.$$

Next $\langle[u, a]\rangle=[U, U^g]\leqq U\cap U^g=V$, while $[U, U^g]$ is contained neither in Z nor in Z^g as $C_H(U/Z)=R$ by the condition (C). Thus

$$[u, a]=vz.$$

Finally, since $C_W(g)^b=C_W(g^{-1})=C_W(g)$, we have

$$[b, c]=1.$$

We have $[x, y]=1$ for any other combinations x, y except a, b because U and W are abelian.

The definition of c, u, v, and z shows

$$z^g=v, \quad c^g=c, \quad z^h=z, \quad \text{and} \quad v^h=u.$$

Hence $u^h=v^{h^2}=v^{aha}=v^{ha}=u^a$. Thus

$$u^h=uvz.$$

Since $C_V(g)=1$, we have $1=zz^gz^{g^2}=zvv^g$, so

$$v^g=vz.$$

Hence $U^g=\langle vz, V^{hg}\rangle=\langle z, V^{hg}\rangle$ and so, replacing a by az, if necessary, we may assume $a\in V^{hg}$. (Notice here that $(az)^2=1$ and $h^{az}=h^{-1}$.) Hence $u^g=a$ or av. However, computations show $(vzh)^3=1$ and $(vzh)^{av}=(vzh)^{-1}$, so replacing a and h by av and vzh, respectively, if necessary, we may assume

$$u^g=a.$$

(Notice here that $V^h=V^{vzh}$, $W^h=W^{vzh}$, and $Z^{gh}=Z^{g(vzh)}$.) The condition (D) and the structure of Q yield that $[Q, G]\cong Z_4\times Z_4$ and $\Omega_1([Q, G])=V$. Hence $cu\in[Q, G]$ and $C_{[Q,G]}(g)=1$, so $cu(cu)^g(cu)^{g^2}=1$. From this we get

$$a^g=acuvz.$$

We have $a^b=u^{gb}=u^{bgb}=u^{g^2}=a^g$. Therefore

$$[a, b]=cuvz.$$

Since $C_H(U/Z)=C_H(R/U)=R$, we have $C_{R/Z}(h)=1$. Hence $cc^hc^{h^2}\in Z$, so $cc^h\in c^{h^2}Z$ and $[c, c^h]=(cc^h)^2=1$. As $c^h\in bV^h$, we conclude that $c^h=b$ or zb. However computations show $(zg)^3=1$, $(zb)^2=1$, and $(zg)^{zb}=(zg)^{-1}$. Hence, replacing g and b by zg and zb, respectively, if necessary, we may assume

$$c^h=b.$$

(Notice here that $Z^g=Z^{zg}$, $U^g=U^{zg}$, $C_W(g)=C_W(zg)$, and $V^{hg}=V^{hzg}$.) We have

$b^h uvz=(cv)^{h^2}=c^{ah^2}=c^{ha}=b^a=cuvzb=bcuv$. Hence

$$b^h=bcz.$$

We have verified all the required relations. It is easy to see that these relations form sets of defining relations of S, G, and H.

6. *Aut*(M_{12})-*amalgams* The amalgam (G, S, H) is called an Aut(M_{12})-amalgam if (G_*, S_*, H_*) is an M_{12}-amalgam and $|S:S_*|\geqq 2$.

In this case, if a, b, c, u, v, z, g, and h are the standard generators of S_*, G_*, and H_* obtained in (5), then presentations of S, G, and H are obtained by adjoining to the standard presentations of S_*, G_*, and H_* one common generator d, seven common relations $d^2=z$, $[a,d]=v$, $[b,d]=uz$, $[c,d]=vz$, $[u,d]=[v,d]=[z,d]=1$, the relation $d^g=dac$ for G, and the relation $d^h=duz$ for H. Consequently we have $|S:S_*|=2$.

Proof. Set $Z_*=Z(S_*)$, $V_*=Z(Q_*)$, and $U_*=\langle V_*^{H_*}\rangle$. Embed G, H, and S into the amalgamated product of G and H over S, and define

$$D=R^g\cap Q\cap R\cap Q^h\cap R^{gh}.$$

Then by the same argument as in (4), we have $|D:Z_*|\geqq 2$ and

$$D\cap S_*=Z_*.$$

For a while assume

$$(*) \qquad |S:S_*|=2.$$

Then it follows that

$$S=DS_* \quad \text{and} \quad |D:Z_*|=2.$$

Set $W=R^g\cap Q\cap R$. Then

$$W\cap W^h=D \quad \text{and} \quad |W|=16$$

by 3.6 of [5]. Hence $R=WW^h$. Suppose W is abelian. Then $D\leqq Z(R)$, while $R=DR_*$ and $Z(R_*)=Z_*$. Hence $Z(R)=D$ and in particular $D\lhd H$. Since $|D:Z_*|=2$, we conclude that $[D, O^2(H)]=1$. Further since $R=DR_*$ and $[R_*, R_*]=Z_*$, we have $[W, R]\leqq V_*$, and similarly $[W, R^g]\leqq V_*$. Hence $[W, G]\leqq V_*$ as $G=\langle R^g, R\rangle$ by 3.6 of [5]. Consequently $DV_*\lhd G$ and so, as $|DV_*|=8$ and $|D|=4$, we have $D\cap D^g\neq 1$. However $D\cap D^g\leqq Z(G)$ as $G=\langle R^g, R\rangle$, and hence $D\cap D^g\leqq Z(H)$ also, which contradicts (C). Therefore

$$W \text{ is nonabelian.}$$

If R/Z_* is abelian, then $[W, W]\leqq[R, R]\cap[R^g, R^g]=Z_*\cap Z_*^g=1$, a contradiction. Hence R/Z_* is nonabelian. Now $R_*\cong Q_8 * Q_8$ and so R_* has precisely two subgroups, say R_1 and R_2, which are isomorphic to Q_8. Using the standard generators of S_*, we may write $R_1=\langle bv, bcuz\rangle$ and $R_2=\langle cu, bcvz\rangle$, hence we see that $R_i \lhd H_*$ and h acts irreducibly on R_*/R_i, $i=1, 2$. Suppose $R_i \lhd H$ for some i. Then $R_i \lhd H$ for both i, and the action of h on R_*/R_i, $i=1, 2$, forces $[R, R]\leqq R_1 \cap R_2=Z_*$, which is a contradiction. Therefore, neither R_1 nor R_2 is normal in H.

We will now prove that (*) holds. Suppose $|S:S_*|>2$. Then there is a subgroup S_0 of $N_S(R_1)$ containing S_* with $|S_0:S_*|=2$. Set $G_0=S_0G_*$ and $H_0=S_0H_*$. By 3.3 of [5], (G_0, S_0, H_0) is an $\mathrm{Aut}(M_{12})$-amalgam satisfying (*) and $R_1 \lhd H_0$, which is a contradiction. Therefore (*) must hold.

Let $d\in D-Z_*$. Then $S=\langle d, S_*\rangle$, $G=\langle d, G_*\rangle$, and $H=\langle d, H_*\rangle$. As G_* acts irreducibly on V_*, we have

$$V_*\leqq Z(Q)$$

and so $U_*=V_*V_*^h\leqq Z(Q\cap R\cap Q^h)$. As $|Q\cap R\cap Q^h:U_*|=2$ by 3.6 of [5], we conclude that

$$Q\cap R\cap Q^h \text{ is abelian.}$$

Consequently we have

$$[u, d]=[v, d]=[z, d]=1\,.$$

Set $s=bg$. Then $s^2=1$ and $R^g=R^s$, and so

$$s \text{ normalizes } R^g\cap Q\cap R=W\,.$$

Since W is nonabelian and $W=\langle c, d, V_*\rangle$, we have $[W, W]=\langle [c, d]\rangle$ with $1\neq[c, d]\in V_*$. Hence $1\neq[c, d]\in C_{V_*}(s)$, and we conclude that

$$[c, d]=vz\,.$$

Similarly since $W^h=\langle b, d, V_*^h\rangle$, we have $\langle [b, d]\rangle=[W^h, W^h]=[W, W]^h=\langle vz\rangle^h$, so

$$[b, d]=uz\,.$$

Since $[R_*, R_*]=Z_*$, we have $(Q_*\cap R_*)/V_*\leqq C_{Q/V_*}(b)$. But $Q=\langle a, d, Q_*\cap R_*\rangle$ and we see that none of $[a, b]$, $[d, b]$, and $[ad, b]$ is contained in V_*. Hence $C_{Q/V_*}(b)=(Q_*\cap R_*)/V_*$. If $s\in N_G(\langle d, V_*\rangle)$, then $\langle d, V_*\rangle/V_*\leqq C_{Q/V_*}(s)$ and so, as $s^g=b$, we have $d^g\in Q_*\cap R_*$ and hence $d\in Q_*$, which is a contradiction. Therefore $\langle d, V_*\rangle^s\neq\langle d, V_*\rangle$ and hence $\langle c, V_*\rangle$, $\langle d, V_*\rangle$, and $\langle d, V_*\rangle^s$ form the set of all maximal subgroups of W containing V_*. Since W is not abelian, we conclude that $\langle d, V_*\rangle$ is not elementary abelian. Therefore

$$d^2=z\,.$$

As $Z_2\cong V_*/Z_* \lhd R/Z_*$, we have $V_*/Z_*\leqq Z(R/Z_*)$ and hence $U_*/Z_*\leqq Z(R/Z_*)$ also. This shows $[a, d]\in Z_*^g=\langle v\rangle$ as $a\in U_*^g$ and $d\in R^g$. Since $Q\cap R\cap Q^h$ is abelian, we have $C_W(U_*)=\langle d, V_*\rangle$ and hence $C_W(a)=C_W(U_*^g)=C_W(U_*^s)=\langle d, V_*\rangle^s\neq\langle d, V_*\rangle$. Therefore

$$[a, d]=v.$$

Since $\langle d, V_*\rangle^s\neq\langle d, V_*\rangle$, we have $d^s\equiv cd \bmod V_*$, while $d^s=d^{bg}=(duz)^g=d^g av$. Hence $d^g\equiv dac \bmod V_*$. Let $d^g=dacx$, $x\in V_*$. Then $duz=d^b=d^{gbg}=(dacx)^{bg}=(da)^g x^s=d^g acuvzx^s$, and hence $d^g=dacx^s$. Thus $x\in C_{V_*}(s)=\langle vz\rangle$. We see that $(d^{-1})^g=d^{-1}acxvz$. Therefore replacing d by d^{-1}, if necessary, we may assume

$$d^g=dac.$$

Let $t=ah$. Then $t^2=1$ and $D=W\cap W^t$, so $t\in N_H(D)$ and $d^t\equiv d \bmod Z_*$. Hence we have $(duv)^h\equiv duv \bmod Z_*$. This shows $(duv)^h=duv$ as $z^h=z$ and $h^3=1$. Hence it follows that

$$d^h=duz.$$

We have verified all the required relations. It is easy to see that these relations form sets of defining relations of S, G, and H.

7. *${}^2F_4(2)'$-amalgams* The amalgam (G, S, H) is called a ${}^2F_4(2)'$-amalgam if $G/Q\cong D_6$, $H/R\cong F_{20}$ (the Frobenius group of order 20), and there is an H-composition series $R=R_0>R_1>R_2>R_3=1$ of R such that the following seven groups Q_i, $i=0, 1, \cdots, 6$, from a G-composition series of $Q: Q_0=Q$, $Q_1=\langle R_1^G\rangle$, $Q_2=\cap R_0^G$, $Q_3=\langle (R_1\cap Q_2)^G\rangle$, $Q_4=\cap R_1^G$, $Q_5=\langle R_2^G\rangle$, and $Q_6=1$.

Here we record properties of this amalgam which will be needed in [4]. The following hold:

(i) $|Q|=2^{10}$, $|R|=2^9$, $|Q_1|=2^8$, and $|R_1|=2^5$.

(ii) $Q^2=Q_1$ and $R^2=R_1$.

(iii) $[Q, O^2(G)]=Q$ and $[R, O^2(H)]=R$.

Proof. (i) This has been proved in 1.2 of [5].

(ii) As Q_{i-1}/Q_i is a G-composition factor, we have $[Q_2, Q]Q_4=Q_3$ or Q_4. However if $[Q_2, Q]Q_4=Q_4$, then $[Q_2R_1, Q]\leqq R_1$, which contradicts (2.2.3) of [2] as $S=QR$ by 1.2.3 of [5]. Hence $[Q_2, Q]Q_4=Q_3$. Similarly we have $[Q_3, Q]Q_5=Q_4$ and $[Q_4, Q]=Q_5$. Thus $Q_3=[Q_2, Q]\leqq[Q, Q]$. From 1.2 of [5] we know $Q/Q_2\cong Z_4\times Z_4$. But as $[Q\cap R, Q]\nleqq Q_3$ by (2.2.3) of [2], Q/Q_3 is nonabelian. Therefore $[Q, Q]=Q_2$ and hence $Q^2=Q_1$. By the same argument and 1.2.4 of [5], we have $[R, R]=R_1$ and hence $R^2=R_1$.

(iii) The definition of Q_i shows $|Q_1R/R|=2$, so $|C_{R/R_1}(Q_1)|=4$ by (2.1.1) of [2] and then (2.2.3) of [2] implies that $C_{R/R_1}(Q_1)=Q_2R_1/R_1$. As $|Q_1: Q_2R_1|=2$, we conclude that Q_1/R_1 is abelian. Hence Q_1/Q_4 is abelian also. However as $[R_1, Q_1]\nleqq Q_5$ by (2.1.1) and (2.2.3) of [2], Q_1/Q_5 is nonabelian. Now 1.2.2 of [5] shows

$$(*) \qquad Q_{i-1} \leqq [Q, O^2(G)]Q_i\,, \qquad i=1,2,4,6\,.$$

Consequently $Q=[Q, O^2(G)]Q_2$. We have remarked in (ii) that Q/Q_2 is abelian while Q/Q_3 is not. If $Q_2 \nleqq [Q, O^2(G)]Q_3$, then as Q_2/Q_3 is a G-composition factor, $Q/Q_3=[Q, O^2(G)]Q_3/Q_3 \times Q_2/Q_3$ and so Q/Q_3 would be abelian, a contradiction. Thus $Q_2 \leqq [Q, O^2(G)]Q_3$ and so $Q=[Q, O^2(G)]Q_4$ by $(*)$. Hence a similar argument yields that $Q_1 \leqq [Q, O^2(G)]Q_5$ as Q_1/Q_4 is abelian while Q_1/Q_5 is not. Thus $Q=[Q, O^2(G)]$ by $(*)$. Finally, 1.2.1 and 1.2.4 of [5] show $R_{i-1} \leqq [R, O^2(H)]R_i$, $i=1,2,3$. Hence $R=[R, O^2(H)]$.

8. *$^2F_4(2)$-amalgams* The amalgam (G, S, H) is called a $^2F_4(2)$-amalgam if (G_*, S_*, H_*) is a $^2F_4(2)'$-amalgam and $|S:S_*|=2$.

At present we need no further comment on this amalgam.

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