A Generalization of Carlitz's Determinant

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0. Let

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}$$

be the Bernoulli polynomial of degree n and let $\bar{B}_n(x)$ be the Bernoulli periodic function defined by

$$\bar{B}_n(x)=B_n(x)\quad(0\leq x<1)\ ,\qquad \bar{B}_n(x+1)=\bar{B}_n(x)\ .$$

The following formula follows from a property of $B_n(x)$.

$$\bar{B}_n(-x)=(-1)^n\bar{B}_n(x)\ . \tag{1}$$

Let p be an odd prime number. For $(s, p)=1$ define s' by means of $ss'\equiv 1$ mod p. Carlitz has considered ([1]) the determinant

$$\det\left(p^n\bar{B}_n\left(\frac{rs'}{p}\right)\right)\quad(r, s=1, \cdots, (p-1)/2)$$

of order $(p-1)/2$. In the following, we shall consider two determinants, one of which is a generalization of the Carlitz's determinant.

1. Let $m=p^\alpha$ be the $\alpha(\geq 1)$-th power of an odd prime number p and let $g=\varphi(m)/2=(p^\alpha-p^{\alpha-1})/2$. We define two $g\times g$ determinants D_n and D'_n as follows:

$$D_n=\det\left(\bar{B}_n\left(\frac{ab}{m}\right)\right)$$

with (a, b)-entry $\bar{B}_n(ab/m)$ where $1\leq a,\ b<m/2$ and $(a, p)=(b, p)=1$.

$$D'_n=\det\left(\bar{B}_n\left(\frac{ab^{-1}}{m}\right)\right)$$

with (a, b)-entry $B_n(ab^{-1}/m)$ where $1\leq a,\ b<m/2,\ (a, p)=(b, p)=1$ and $\bar{B}_n(ab^{-1}/m)=\bar{B}_n(ab'/m)$ with $bb'\equiv 1$ mod $m\,(=p^\alpha)$.

The relation between two determinants D_n and D'_n is given by the following lemma.

Lemma 1.

i) $n=even\geq 2$.

$$D_n=\begin{cases}(-1)^{(\varphi(m)-4)/4}D'_n & (p\equiv 1 \bmod 4)\\ (-1)^{(\varphi(m)-2)/4}D'_n & (p\equiv 3 \bmod 4)\,.\end{cases}$$

ii) $n=odd\geq 1$.

$$D_n=\begin{cases}(-1)^{\varphi(m)/4}D'_n & (p\equiv 1 \bmod 4)\\ (-1)^{(\varphi(m)-2)/4}D'_n & (p\equiv 3 \bmod 4)\,.\end{cases}$$

Proof. We shall show the relation only in the case $n=\mathrm{odd}\geq 1$ and $p\equiv 1$ mod 4. The other cases are treated in similar ways.

Suppose $n=\mathrm{odd}\geq 1$ and $p\equiv 1$ mod 4.

The columns of determinant $D'_n=\det(\bar{B}_n(ab^{-1}/m))$ are, except for order, the same as those of determinant $D_n=\det(\bar{B}_n(ab/m))$. So, we shall consider the relation between the columns of these two determinants.

It is clear that the 1st column of D'_n coincides with that of D_n. We shall consider the other columns. As $(\boldsymbol{Z}/m\boldsymbol{Z})^\times$ $(m=p^\alpha)$ is a cyclic group whose order is divisible by 4, there exists exactly one $c\in\boldsymbol{Z}$ such that $1<c<m/2$, $c^2\equiv -1$ mod m. Then the c-th column of D'_n is

$$\begin{pmatrix}\vdots\\ \bar{B}_n\left(\dfrac{a(m-c)}{m}\right)\\ \vdots\end{pmatrix}=-\begin{pmatrix}\vdots\\ \bar{B}_n\left(\dfrac{ac}{m}\right)\\ \vdots\end{pmatrix}$$

$$=-(\text{the } c\text{-th column of } D_n)$$

(We have used formula (1)).

As for the $b\,(\neq 1, c)$-th column of D'_n

i) if $bb'\equiv 1$ mod m with $1<b'<m/2$, then the b-th column of $D'_n=$the b'-th column of D_n and the b'-th column of $D'_n=$the b-th column of D_n,

ii) if $bb'\equiv 1$ mod m with $m/2<b'<m$, then, by formula (1), the b-th column of $D'_n=-$(the $(m-b')$-th column of D_n) and the $(m-b')$-th column of $D'_n=-$(the b-th column of D_n).

Consequently, by interchanging $(\varphi(m)/2-1-1)/2$ times the columns of $-D'_n$, $-D'_n$ may be transformed to D_n. It accordingly follows that

$$\begin{aligned}D_n&=(-1)^{(\varphi(m)-4)/4}(-D'_n)\\ &=(-1)^{\varphi(m)/4}D'_n\,.\end{aligned}$$

2. It follows from the definition of $\bar{B}_n(x)$ that $\bar{B}_n(a/m)$ is a function of $\bar{a}=a \bmod m$ on $(\boldsymbol{Z}/m\boldsymbol{Z})^{\times}$. Furthermore, as a consequence of formula (1), it follows that

i) if n is even ≥ 2, then $\bar{B}_n(a/m)$ may be considered as a function on a finite abelian group $G=(\boldsymbol{Z}/m\boldsymbol{Z})^{\times}/\{\pm 1 \bmod m\}$, and

ii) if n is odd ≥ 1, then $\omega(a)\bar{B}_n(a/m)$ may be considered as a function on $G=(\boldsymbol{Z}/m\boldsymbol{Z})^{\times}/\{\pm 1 \bmod m\}$ where $\omega(a)=\omega(a \bmod m)$ is any (but fixed) odd Dirichlet character mod m (i.e., character on $(\boldsymbol{Z}/m\boldsymbol{Z})^{\times}$ such that $\omega(-1)=\omega(-1 \bmod m)=-1$).

Now, to compute determinant D_n', we shall make use of the following lemma ([3], Lemma 5.26).

Lemma 2. *Let G be a finite abelian group and let f be a function on G with values in $\boldsymbol{C}$ (=the field of complex numbers). Then*

$$\det (f(\sigma\tau^{-1}))_{\sigma,\tau\in G}=\prod_{\chi\in\hat{G}} \sum_{\sigma\in G} \chi(\sigma)f(\sigma)$$

where $\hat{G}$ denotes the group of all characters of G.

It is now convenient to treat separetely the cases n even and n odd.

i) n=even≥ 2. By Lemma 2,

$$D_n'=\det\left(\bar{B}_n\left(\frac{ab^{-1}}{m}\right)\right) \tag{2}$$
$$=\prod_{\chi(-1)=1} \sum_{\substack{1\leq a<m/2\\(a,p)=1}} \chi(a)\bar{B}_n\left(\frac{a}{m}\right)$$

where the product extends over all even Dirichlet characters mod m (i.e., characters of $(\boldsymbol{Z}/m\boldsymbol{Z})^{\times}$ such that $\chi(-1)=\chi(-1 \bmod m)=1$). Since $\chi(a)=0$ if $(a,p)>1$ and since

$$\chi(m-a)=\chi(a)\ , \qquad \bar{B}_n\left(\frac{m-a}{m}\right)=\bar{B}_n\left(\frac{a}{m}\right)$$

for all even χ and n even, we have

$$\sum_{\substack{1\leq a<m/2\\(a,p)=1}} \chi(a)\bar{B}_n\left(\frac{a}{m}\right)=\frac{1}{2}\sum_{a=1}^{m-1} \chi(a)\bar{B}_n\left(\frac{a}{m}\right)\ . \tag{3}$$

The n-th generalized Bernoulli number $B_{n,\chi}$ for a primitive Dirichlet character χ whose conductor is a divisor of $m=p^{\alpha}$ is expressed in the form ([3], Prop. 4.1)

$$B_{n,\chi}=m^{n-1}\sum_{a=1}^{m} \chi(a)B_n\left(\frac{a}{m}\right)\ . \tag{4}$$

Since $\chi(m)=0$ for $\chi\neq\chi_0$ (the principal character), we get

$$\sum_{a=1}^{m-1}\chi(a)\bar{B}_n\left(\frac{a}{m}\right)=\left(\frac{1}{m}\right)^{n-1}B_{n,\chi} \quad (\chi\neq\chi_0) \tag{5}$$

but, since $\chi(m)=1$ and $\bar{B}_n(1)=B_n=B_{n,\chi}$ for $\chi=\chi_0$, we get a slight different form for $\chi=\chi_0$

$$\sum_{a=1}^{m-1}\chi_0(a)\bar{B}_n\left(\frac{a}{m}\right)=\left(\frac{1}{m}\right)^{n-1}(1-m^{n-1})B_{n,\chi_0}\,. \tag{6}$$

Consequently, making use of (2), (3), (5) and (6) we obtain the following formula.

$$D'_n=\left(\frac{1}{2m^{n-1}}\right)^{\varphi(m)/2}(1-m^{n-1})\prod_{\chi(-1)=1}B_{n,\chi} \quad (m=p^{\alpha})\,.$$

ii) $n=\text{odd}\geq 1$. It is easy to see that

$$D'_n=\det\left(\bar{B}_n\left(\frac{ab^{-1}}{m}\right)\right)=\det\left(\omega(ab^{-1})\bar{B}_n\left(\frac{ab^{-1}}{m}\right)\right)\,.$$

Hence, by Lemma 2 we have

$$D'_n=\prod_{\chi(-1)=1}\sum_{\substack{1\leq a<m/2\\(a,p)=1}}\chi(a)\omega(a)\bar{B}_n\left(\frac{a}{m}\right)\,.$$

As ω is a fixed odd Dirichlet character mod m, the above formula may be written in the form

$$D'_n=\prod_{\chi(-1)=-1}\sum_{\substack{1\leq a<m/2\\(a,p)=1}}\chi(a)\bar{B}_n\left(\frac{a}{m}\right)$$

the product extending over all odd Dirichlet charaters mod m. Making use of (4), as in the case n even, the above formula reduces to

$$D'_n=\left(\frac{1}{2m}\right)^{\varphi(m)/2}\prod_{\chi(-1)=-1}B_{n,\chi}\,.$$

(We have used $\chi(m-a)=-\chi(a)$, $\bar{B}_n((m-a)/m)=-B_n(a/m)$, $\chi(m)=0$ for all odd χ and n odd.)

We summarize the above computations in the following

Proposition 1. *Let $m=p^{\alpha}$ (p=odd prime, $\alpha\geq 1$).*

i) *$n=\text{even}\geq 2$.*

$$\begin{aligned}D'_n&=\det\left(\bar{B}_n\left(\frac{ab^{-1}}{m}\right)\right)\quad\begin{pmatrix}a,b=1,\cdots,<m/2\\(a,p)=(b,p)=1\end{pmatrix}\\&=\left(\frac{1}{2m^{n-1}}\right)^{\varphi(m)/2}(1-m^{n-1})\prod_{\chi(-1)=1}B_{n,\chi}\,.\end{aligned}$$

ii) $n=odd\geq 1$.

$$D'_n=\left(\frac{1}{2m^{n-1}}\right)^{\varphi(m)/2}\prod_{\chi(-1)=-1}B_{n,\chi}\,.$$

Combining Prop. 1 with Lemma 1, we have the following

Proposition 2. *Let $m=p^\alpha$ (p=odd prime, $\alpha\geq 1$) and denote by $\bar{a}$ the least positive residue of $a\,(\in \boldsymbol{Z})$* mod m. *Let*

$$D_n=\det\left(\bar{B}_n\left(\frac{ab}{m}\right)\right)=\det\left(B_n\left(\frac{\overline{ab}}{m}\right)\right)\quad\begin{pmatrix}a, b=1,\cdots,<m/2\\(a,p)=(b,p)=1\end{pmatrix}.$$

Then,

i) $n=even\geq 2$.

$$D_n=\begin{Bmatrix}(-1)^{(\varphi(m)-4)/4}\\(-1)^{(\varphi(m)-2)/4}\end{Bmatrix}\left(\frac{1}{2m^{n-1}}\right)^{\varphi(m)/2}(1-m^{n-1})\prod_{\chi(-1)=1}B_{n,\chi}$$

$$\text{for}\quad\begin{cases}p\equiv 1 \bmod 4\\ p\equiv 3 \bmod 4\,.\end{cases}$$

ii) $n=odd\geq 1$.

$$D_n=\begin{Bmatrix}(-1)^{\varphi(m)/4}\\(-1)^{(\varphi(m)-2)/4}\end{Bmatrix}\left(\frac{1}{2m^{n-1}}\right)^{\varphi(m)/2}\prod_{\chi(-1)=-1}B_{n,\chi}\qquad\text{for}\quad\begin{cases}p\equiv 1 \bmod 4\\ p\equiv 3 \bmod 4\,.\end{cases}$$

3. Let $m=p^\alpha$ be the $\alpha(\geq 1)$-th power of an odd prime numbdr p. Let $K=\boldsymbol{Q}(\zeta)(\zeta=e^{2\pi i/m})$ denote the cyclotomic field of m-th roots of unity and $F=\boldsymbol{Q}(\zeta+\zeta^{-1})$ the maximal real subfield of K. If $\zeta_K(s)$, $\zeta_F(s)$ denote the Dedekind zeta function of K and F respectively, then (cf. [2], p. 135)

$$\zeta_F(n)=(-1)^{g(n/2+1)}\frac{(2\pi)^{gn}\sqrt{d_F}}{2^g d_F^n(n!)^g}\prod_{\chi(-1)=1}B_{n,\chi}\quad(n=\text{even}\geq 2)$$

$$\zeta_K(n)/\zeta_F(n)=(-1)^{g(n+1)/2}\frac{(2\pi)^{gn}\sqrt{d_K/d_F}}{2^g(d_K/d_F)^n(n!)^g}\prod_{\chi(-1)=-1}B_{n,\chi}\quad(n=\text{odd}\geq 1)$$

where $g=\varphi(m)/2$ and

$$d_F=p^{(\alpha p^\alpha-(\alpha+1)p^{\alpha-1}-1)/2}$$
$$d_K=(-1)^g p^{\alpha p^\alpha-(\alpha+1)p^{\alpha-1}}$$

denote the absolute discriminants of respective fields F and K.

Consequently, by Prop. 1 and 2, the calculation of rational determinants D'_n or D_n gives the explicit values of $\zeta_F(n)$ (n even) and $\zeta_K(n)/\zeta_F(n)$ (n odd).

4. Let $h^-(K)$ denote the relative class number (or the 1-st factor of the class number) of the cyclotomic field K of $m(=p^{\alpha})$-th roots of unity. Then, the class number formula ([3], Theorem 4.17) tells us that

$$h^{-}(K)=2m\prod_{\chi(-1)=-1}\left(-\frac{1}{2}B_{1,\chi}\right).$$

Combining this with formulas in Prop. 2, we obtain the following formula

$$h^{-}(K)=\begin{Bmatrix}(-1)^{\varphi(m)/4} \\ (-1)^{(\varphi(m)+2)/4}\end{Bmatrix}\cdot 2mD_1 \qquad \begin{matrix}(p\equiv 1 \bmod 4) \\ (p\equiv 3 \bmod 4)\end{matrix}$$

where

$$D_1=\det\left(\frac{\overline{ab}}{m}-\frac{1}{2}\right)\quad\begin{pmatrix}a, b=1, \cdots, <m/2 \\ (a, p)=(b, p)=1\end{pmatrix}$$

with (a,b)-entry $(B_1(\overline{ab}/m)=)\overline{ab}/m-1/2$, $\overline{ab}$ denoting the least positive residue of ab mod m $(=p^{\alpha})$.

References

[1] L. Carlitz, A generalization of Maillet's determinant and a bound for the first factor of the class number. *Proc. A.M.S.*, **12** (1961), 256–261.

[2] H. W. Leopoldt, Eine Verallgemeinerung der Bernoullischen Zahlen. *Abh. math. Sem. Univ. Hamburg*, **22** (1958), 131–140.

[3] L. C. Washington, *Introduction to Cyclotomic Fields*, Springer Verlag, New York, Heidelberg, Berlin, 1980.